THEORETICAL FOUNDATIONS OF AXION PARTICLES

by

Cemile Ezer

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ABSTRACT

THEORETICAL FOUNDATIONS OF AXION PARTICLES

The Standard Model of particle physics has solved the problem of giving masses to quarks and leptons by introducing the Higgs particle which soon will be searched at the Large Hadron Collider at CERN. Another problem of the Standard Model is the strong CP violation problem which arises due to the interplay between the strong and weak sectors of the model. An understanding of this problem requires understanding the instantons and the quark mixing. In this thesis I present my understanding of these topics and consider how the strong CP violation problem can be solved by introducing the axion particle.

ÖZET

AXION PARÇACIKLARININ TEORİK TEMELLERİ

CERN'de bu sene başlayacak olan LHC deneyinde bulunması beklenen Higgs parcaçığı Standart Model'in açıklayamadığı fenomenlerden biri olan kuark ve leptonların kütle kazanım mekanizmasını açıklamak üzere öngörülmüştür. Standart Modelin bir başka açıklayamadığı fenomen ise bu modelde ki kuvvetli ve zayıf etkileşmelerin birleştirilmesinden doğan kuvvetli CP bozulumudur. Bu problemin anlaşılır olabilmesi için öncelikle instanton özelliklerinin ve kuark karışımı konusunun iyice anlaşılabilir olması gereklidir. Tezimde, anladığım şekliyle bu olaylara açıklık getirdim ve kuvvetli CP bozulumu probleminin axion parçacıklarıyla olan çözümüne yer verdim.

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1. INTRODUCTION

We know the term CP as the symmetry of the laws of the physics between matter and antimatter. In this content, charge conjugation (C) applies for matter-antimatter interchanging while parity operator (P) is only responsible for reversing their momenta. In strong and electromagnetic interactions, this symmetry is preserved. However, after discovery of the neutral kaon decays, it is proved that CP symmetry is violated in some weak interactions. Now, the most puzzling part is that Standard Model violates P and CP symmetries whereas those symmetries are conserved in strong interaction, which is a part of the Standard Model. Therefore, the challenge is to find new theories where this is not the case or at least where the effect of this phenomenon can be controlled [1].

One can begin with writing the QCD lagrangian as ,

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum \overline{\Psi}(iD_\mu\gamma^\mu - m)\Psi + \theta\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \qquad (1.1)$$

where F and $\tilde{F}$ represent gluon field strength tensor and self dual gluon field strength tensor, respectively and g is the strong interaction coupling constant. The sum runs over all quark flavours [2]. The final term on the right hand side is responsible for the CP violation. Let us make some explanations about this CP violating term in this lagrangian. First, we will show in instanton solutions that this term can be represented as full divergence.

$$\partial_\mu K^\mu = F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \qquad (1.2)$$

Since there are gauge configurations that satisfy the boundary conditions, this shows that $U(1)_A$ is not a symmetry of QCD. Thus, in the absence of θ value QCD would have a $U(1)_A$ symmetry. Second that, θ is cyclic. In other words, one should get the same results by changing θ to $\theta + 2\pi$. Finally, in a full theory, the complex phases of quark mass matrix appear in front of the $F\tilde{F}$ term and we know that these phases are also responsible for the CP violation. While dealing with CP violation, for simplicity one can redefine the quark fields and add these phases to θ term [3]. Therefore, the coefficient of $F\tilde{F}$ term takes its effective value as [4],

$$\bar{\theta} = \theta + \arg(detM) \tag{1.3}$$

This term violates C and CP symmetry. Since physics depend on $\bar{\theta}$, this value should be determined by experiments. The most striking experiment is the electric dipole moment of neutron. The neutron electric dipole is given as,

$$d_N \simeq \frac{e\theta m_q}{m_N^2} \tag{1.4}$$

The most recent value of this electric dipole moment is [5],

$$d_N < 6 \times 10^{-26} e.cm$$

and one can get the value of $\bar{\theta}$ as,

$$\bar{\theta} < 10^{-10}$$

If there were only strong interactions, one could set the value of θ zero and explain the conservation of P and CP symmetries. However, there are also weak interaction contributions and we know that weak interactions are responsible for CP violations. Therefore, we can not set θ value to zero. Moreover, since complex phases are arbitrary numbers, one can expect that $\bar{\theta}$ value should also be a random angle. This problem is known as *Strong CP Problem.* To explain this problem which occurs in the Standard Model, one needs a mechanism which is also consistent with the Standard Model.

Peccei-Quinn proposed the most elegant solution in which they introduced $U(1)_{PQ}$ symmetry, where PQ represents their initials, and the spontaneous broken of this symmetry gives rise to a dynamical field that is called axion [6, 7].

Axion physics as solution of Strong CP problem is highly non-trivial. In Part 2, we review the instantons. In Part 3, we discuss the quark mixing. In Part 4, we present Peccei Quinn axions and invisible axion models.

2. INSTANTONS

2.1. Instantons in one dimension

In this section, we will deal with the one dimensional motion of a spinless particle of unit mass in a potential $V(x)$.

$$H = \frac{p^2}{2} + V(x) \tag{2.1}$$

Although this is a well-known system and can be solved exactly, we are going to rederive some properties of this such system by using some unusual methods. However, these methods will help us to make generalization to the quantum field theory. Lagrangian of the system described above is,

$$L = \frac{1}{2}(\frac{dx}{dt})^2 - V(x) \tag{2.2}$$

Suppose that particle is initially at x_i in $-t_0/2$ and finally at the point x_f in $+t_0/2$. The transition amplitude of such system, is described by Feynman, can be found by using path integral method. That is to calculate that,

$$\langle x_f|e^{-iHt_0}|x_i\rangle = N\int[Dx]e^{iS[x(t)]} \tag{2.3}$$

where N is the normalization constant and $[Dx]$ denotes integration over all functions $x(t)$ satisfied the boundary conditions $x(-t_0/2) = x_i$ and $x(+t_0/2) = x_f$. Now, let us make some explanations for the equation (2.3). For the left hand side of the equation, if we expand in a complete set of energy states,

$$H|n\rangle = E_n|n\rangle \tag{2.4}$$

then, one can see that,

$$\langle x_f|e^{-iHt_0}|x_i\rangle = \sum_n e^{-iE_nt_0}\langle x_f|n\rangle\langle n|x_i\rangle \tag{2.5}$$

and finally we get a sum of oscillating exponentials. For the vacuum state which is the lowest energy state, it is appropriate to transform these oscillating exponentials to decreasing exponentials. To achieve this, we need to rotate the time variable from t to -iτ. Then, in the limit $\tau_0 \to \infty$, the remaining term will tell us energy and the wave function of the lowest level.

This is known as Wick's rotation. It yields us to work on Euclidean space. The physical importance of working in Euclidean space is that in quantum field theory in Minkowskian space one needs to compute path integrals which have to be analytically continued to be well defined. Therefore, we can define Euclidean space as an analytic continuation of Minkowskian space. The effect of this rotation in the action is that, in Minkowskian space,

$$S = \int Ldt \tag{2.6}$$

when we let t be imaginary, in other words make transformation from Minkowskian space to Euclidean space;

$$S = \int Ldt \longrightarrow -i\int d\tau L = iS_e \tag{2.7}$$

S_e is called Euclidean action. With this rotation, we convert the Minkowskian metric to the Euclidean one. For our system, Euclidean action is in the form of,

$$S_e = \int_{-\tau_0/2}^{+\tau_0/2} [\frac{1}{2}(\frac{dx}{d\tau})^2 + V(x)]d\tau \tag{2.8}$$

where $x(\tau)$ satisfies the boundary conditions, $x(-\tau_0/2) = x_i$ and $x(+\tau_0/2) = x_f$. The

Euclidean version of equation (2.3),

$$\langle x_f | e^{-H\tau_0} | x_i \rangle = N \int [Dx] e^{-S_e[x(\tau)]} \tag{2.9}$$

Since $S_e \geq 0$, it satisfies the decreasing exponentials demands for the vacuum state. Now, let us say that $X(\tau)$ satisfies the given boundary conditions and there is also an arbitrary function $x(\tau)$ that has the same boundary conditions [2],

$$x(\tau) = X(\tau) + \sum_n c_n x_n(\tau) \tag{2.10}$$

where $x_n(\tau)$ is a complete set of orthonormal functions that vanish at the boundaries.

$$\int d\tau x_n x_m = \delta_{mn} \qquad \text{and} \qquad x_n(\pm\tau_0/2) = 0 \tag{2.11}$$

And we can choose $[Dx]$ in the form,

$$[Dx] = \prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{2.12}$$

Now, let us consider the large value of the action for some reasons. For large value of S, the integral in (2.3) is dominated by the stationary points of S. That is known as semiclassical limit (small $\hbar\epsilon$) [8]. For simplicity, let us assume that there is only one stationary point that we called it as $X(\tau)$. If we denote $S_e(X(\tau)) = S_{eo}$, then the integral becomes,

$$N \int [Dx] e^{-S_e} \sim e^{-S_{eo}} \tag{2.13}$$

If there are several stationary points, then one needs to sum all the contributions from those points. In our assumption, since $X(\tau)$ is the only stationary point that makes action minimum, then from the least action principle,

$$\delta S_e = S_e[X(\tau) + \delta x(\tau)] - S_e[X(\tau)] = \int_{-\tau_0/2}^{+\tau_0/2} d\tau \delta x(\tau)[-\frac{d^2X}{d\tau^2} + V'(X)] \tag{2.14}$$

This gives us the equation of motion of unit mass particle in negative potential. The minus sign indicates that we are working on the Euclidean space.

$$\frac{d^2X}{d\tau^2} = \frac{dV(X)}{dx} \tag{2.15}$$

Now, let us return exponential factor in equation (2.13). It can be determined by the paths with action that differs by a little from S_{eo}. In other words, we need to choose the quadratic terms for,

$$S_e[X(\tau) + \delta x(\tau)] = S_{eo} + \int_{-\tau_0/2}^{+\tau_0/2} d\tau \delta x[-\frac{1}{2}\frac{d^2}{d\tau^2}\delta x + \frac{1}{2}V''(X)\delta x] \tag{2.16}$$

Suppose that we know a complete eigenfunctions and eigenvalues of the equation [2],

$$-\frac{d^2}{d\tau^2}x_n(\tau) + V''(X)x_n(\tau) = \varepsilon_n x_n(\tau) \tag{2.17}$$

If we insert this equation into the equation (2.16), we will get

$$S_e[X(\tau) + \delta x(\tau)] = S_{eo} + \frac{1}{2}\sum_m c_m c_n \varepsilon_n \int_{-\tau_0/2}^{+\tau_0/2} d\tau x_m(\tau) x_n(\tau) \tag{2.18}$$

By using the identity that we define in equation (2.11), we will find that

$$S_e = S_{eo} + \frac{1}{2}\sum_n \varepsilon_n c_n^2 \tag{2.19}$$

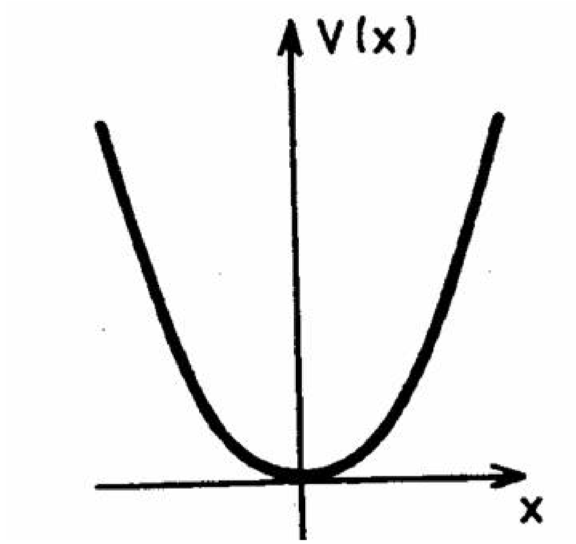

Figure 2.1. One dimensional potential in Minkowskian space

Recalling the equation (2.12) and with the help of the Gaussian integral,

$$\langle x_f|e^{-H\tau_0}|x_i\rangle = \prod_n \frac{N}{\sqrt{2\pi}} e^{-S_{eo}} \int_{-\infty}^{+\infty} dc_n \exp(-\frac{1}{2}\varepsilon_n c_n^2) \tag{2.20}$$

We obtain,

$$\langle x_f|e^{-H\tau_0}|x_i\rangle = N e^{-S_{eo}} \prod_n \varepsilon_n^{-\frac{1}{2}} \tag{2.21}$$

If we go back to the equation (2.17) and consider left hand side as a second order operator, then from the finite dimensional rule we can use determinant form as,

$$\prod_n \varepsilon_n^{-\frac{1}{2}} = [det(-\frac{d^2}{d\tau^2} + V''(X))]^{-\frac{1}{2}} \tag{2.22}$$

It should be emphasized that we assume that all eigenvalues are positive to avoid the existence of zero modes. We shall shortly see if this is not the case. To close this section, let us show the evaluation of the euclidean transition amplitude for the harmonic oscillation potential. Therefore, we write $V(x)$ as $\frac{1}{2}\omega^2 x^2$. Our potential will be like the function shown in the Fig(2.1).

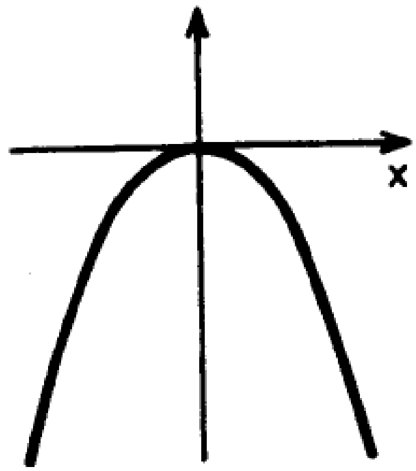

Figure 2.2. One dimensional potential in Euclidean space

As the initial conditions we choose $x_i = x_f = 0$. We are dealing with this problem because we need physical arguments to fix N and harmonic oscillator will be our reference to find the ratio of determinants when we go to more complicated systems. Since our potential is $V(x) = \frac{1}{2}x^2\omega^2$, so that $V'' = \omega^2$. For the inverted potential, that is the Euclidean form of the potential, it will be as shown in Fig(2.2).

The only condition that obeys the boundary condition for the equation,

$$\frac{d^2X}{d\tau^2} = \frac{dV(X)}{dx}$$

$$\frac{d^2X}{d\tau^2} = \omega^2 X \tag{2.23}$$

is $X(\tau) = 0$. Otherwise, the particle goes to plus or minus infinity. Since our Euclidean action only depends on $X(\tau)$ in the harmonic oscillator case, for $X(\tau) = 0$ solution it will be zero. Therefore the equation (2.21) reduces to,

$$\langle x_f = 0|e^{-H_{ho}\tau_0}|x_i = 0\rangle = N[det(-\frac{d^2}{d\tau^2} + \omega^2)]^{-\frac{1}{2}} \tag{2.24}$$

For the explicit results of (2.26), we need eigenfunctions and corresponding eigenvalues of the operator in the square brackets. For example, the lowest eigenfunction is $\cos(\frac{\pi\tau}{\tau_o})$ and the next is $\sin(\frac{2\pi\tau}{\tau_0})$ [2]. Both satisfy the boundary conditions. So these eigenfunctions give us eigenvalues as,

$$\varepsilon_n = \frac{n^2\pi^2}{\tau_0^2} + \omega^2 \tag{2.25}$$

Now, we come to the crucial part. Since the product of eigenvalues equals determinant of this operator, to extract the physical information from eigenvalues, we need to split the determinant in two parts [2]. In other words,

$$N[det(-\frac{d^2}{d\tau^2} + \omega^2)]^{-\frac{1}{2}} = N[\prod_n \frac{n^2\pi^2}{\tau_0^2}]^{-\frac{1}{2}} \times [\prod_n (1 + \frac{\omega^2\tau_0^2}{n^2\pi^2})]^{-\frac{1}{2}} \tag{2.26}$$

One can see that first term on the right hand side gives us the free particle transition amplitude, namely

$$N[\prod_n \frac{n^2\pi^2}{\tau_0^2}]^{-\frac{1}{2}} = \langle x_f = 0|e^{-H_0\tau_0}|x_i = 0\rangle \tag{2.27}$$

where $H_0 = p^2/2$. By using this, we can obtain that

$$N[\prod_n \frac{n^2\pi^2}{\tau_0^2}]^{-\frac{1}{2}} = \langle x_f = 0|e^{-p^2\tau_0/2}|x_i = 0\rangle = \int \frac{dp}{2\pi} e^{-p^2\tau_0/2} = \frac{1}{\sqrt{2\pi\tau_0}} \tag{2.28}$$

The last term in the equation (2.26), we can use the formula given in ([9], on pg 45, Eq(1.431.2)),

$$\prod_n (1 + \frac{\omega^2\tau_0^2}{n^2\pi^2}) = \frac{\sinh(\omega\tau_0)}{\omega\tau_0} \tag{2.29}$$

When we put all these together, the final result will be,

$$\langle x_f = 0|e^{-H_{ho}\tau_0}|x_i = 0\rangle = \frac{1}{\sqrt{2\pi\tau_0}}(\frac{\sinh(\omega\tau_0)}{\omega\tau_0})^{-1/2} = (\frac{\omega}{\pi})^{1/2}(2\sinh(\omega\tau_0))^{-1/2} \quad (2.30)$$

When we expand this equation for the limit $\tau_0 \to \infty$, we obtain

$$\langle x_f = 0|e^{-H_{ho}\tau_0}|x_i = 0\rangle = (\frac{\omega}{\pi})^{1/2}e^{-\omega\tau_0/2}(1 + \frac{1}{2}e^{-2\omega\tau_0} + ...) \quad (2.31)$$

It follows that for the lowest state $E_0 = \omega/2$ and $[\Psi(0)]^2 = (\frac{\omega}{\pi})^{1/2}$. The next term will contribute to the $n = 2$ state. One can see that odd terms do not have any contributions. Now we will go to the more relevant systems where working on the Euclidean space is the most relevant tool to study tunneling phenomenon.

2.2. Double well potential and tunneling

Once we know the calculus of instantons in one dimensional potential, we can examine the method in a more concrete example. For this reason, we need a particle moving in a potential which has a set of degenerate minima. Let us consider a unit mass particle moving in a one dimensional potential shown in Fig(2.3). Now our potential is,

$$V(x) = \lambda(x^2 - \eta^2)^2 \quad (2.32)$$

To fix the parameters we write it as,

$$8\lambda\eta^2 = \omega^2 \quad (2.33)$$

where ω is the frequency introduced in the previous chapter. Therefore, for $x = \pm\eta$ minima, the curve is the same in the previous potential. For $x = 0$ point, $V(x) = \frac{\omega^4}{64\lambda}$ and if $\frac{\lambda}{\omega^3} << 1$ then this means that the wall which separates these two minima is

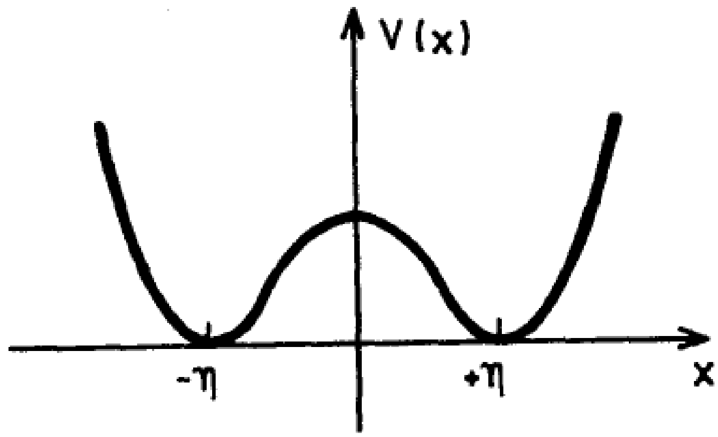

Figure 2.3. The double well potential in Minkowskian space

very high. For a moment, let us say that it equals to infinity. Thus, we have twofold degeneracy. Qualitatively speaking, particle can live in the right well or in the left well while it executes small oscillations near $x = +\eta$ or $x = -\eta$. Therefore, the ground state expectation value for our problem becomes,

$$\langle x \rangle_0 = +\eta(1 + corrections) \qquad \text{or} \qquad \langle x \rangle_0 = -\eta(1 + corrections) \tag{2.34}$$

So the original symmetry of the system $x \to -x$ is broken, however we know that expectation value should be zero and there is no degeneracy. From classical point of view, one might say if in the initial time particle is in one of these minima, then it is impossible for it to feel existence of the other minimum. However, we know from quantum mechanic that for a finite barrier, the particle in the one of these minima has non vanishing transition amplitude to tunnel to the other well so that wave function of these two individual harmonic oscillators will mix.

Let us consider this phenomenon in Euclidean space. The solutions easily come up when the potential is turned upside down in Fig(2.4).

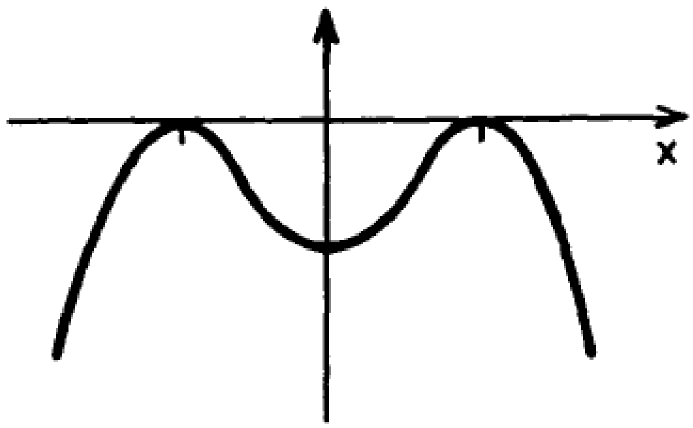

Figure 2.4. The double well potential in Euclidean space

In doing so the double well potential becomes two-humped potential. At this time, I want to emphasize on instantons. Instantons are the solution of the Euclidean equation of motion and they make connection between these minima. They appear only when we turn potential upside down. Now, this is appropriate to calculate transition amplitude between $x_i = -\eta$ and $x_f = +\eta$. That is to find that,

$$\langle -\eta | e^{-H\tau_0} | +\eta \rangle \qquad \text{and} \qquad \langle +\eta | e^{-H\tau_0} | -\eta \rangle \tag{2.35}$$

A finite action in the limit $\tau_0 \to \infty$ is obtained when the particle stays at the top of a hump, hence we have two stationary points, i.e. $X(\tau) = +\eta$ and $X(\tau) = -\eta$. Since we only consider the lowest energy level, working in more trivial $\tau_0 \to \infty$ limit will make calculations simpler. For such a path, i.e at $\tau_0/2$ at the point η and $-\tau_0/2$ at the point $-\eta$, since the Euclidean energy is zero,

$$E = \frac{1}{2}\dot{X}^2 - V(X) = 0$$

If we insert the potential in equation (2.32), the result will be,

$$\frac{dX}{(X^2 - \eta^2)} = \sqrt{2\lambda} d\tau$$

Solution of this equation in the limit above is,

$$X(\tau) = \eta \tanh \frac{\omega(\tau - \tau_c)}{2} \tag{2.36}$$

This is so called instanton and τ_c is the center of it. The reverse path that beginning from $+\eta$ going to $-\eta$ will give us anti-instantons. That is possible by replacing τ with $-\tau$ in the equation (2.36). Now, we can compute action associated with $X(\tau_c)$ by using equation (2.8).

$$S_{eo} = \int_{-\infty}^{+\infty} d\tau \dot{X}^2 = \int_{-\eta}^{+\eta} (-\sqrt{2\lambda})(X^2 - \eta^2) dx = \frac{\omega^3}{12\lambda} \tag{2.37}$$

We can conclude this solution that the action of the instanton has got nothing to do with the center of it. So it can be any point that the symmetry of the original problem is preserved. In other words, Lagrangian of this system is invariant under shifts in time. We also need to note that there are infinitely many solutions distributed arbitrarily with respect to origin. Now, let us calculate one instanton contribution to the transition amplitude. As we know that the transition amplitude is,

$$\langle -\eta | e^{-H\tau_0} | +\eta \rangle = N.[det(-\frac{d^2}{d\tau^2} + \omega^2)]^{-1/2} \times [\frac{det(-\frac{d^2}{d\tau^2} + V''(X))}{det(-\frac{d^2}{d\tau^2} + \omega^2)}]^{-1/2} e^{-S_{eo}} \tag{2.38}$$

where we have divided and multiplied by the determinant of the harmonic oscillator [2]. When we insert $X(\tau)$ in the equation above, we will immediately come up with Schrodinger equation with Posch-Teller potential. However, we are very lucky that it is a solvable problem and we can use this solution to find the eigenvalues of the equation,

$$-\frac{d^2}{d\tau^2} x_n(\tau) + [\omega^2 - \frac{3\omega^2/2}{\cosh^2(\frac{\omega\tau}{2})}] x_n(\tau) = \varepsilon_n x_n(\tau) \tag{2.39}$$

If we rearrange Landau and Lifshitz solution of this kind of potential ([10], pp. 73 and

80), the eigenvalues become,

$$\varepsilon_n = \omega^2(1 - \frac{(4-2n)^2}{16}) \tag{2.40}$$

Among all the eigenvalues, x_o stands as zero mode with zero eigenvalue. One can think that it challenges the evaluation of the determinants. However, it manifests the translational invariance of the system. By using this property,

$$S[X(\tau,\tau_c)] - S[X(\tau,\tau_c+\delta\tau_c)] = 0 \tag{2.41}$$

Therefore, the zero mode (i.e $\epsilon_0 = 0$) should be proportional to $X(\tau,\tau_c) - X(\tau,\tau_c+\delta\tau_c)$. Final result for normalized zero mode is [2],

$$x_o = -\frac{1}{\sqrt{S_{eo}}}\frac{d}{d\tau_c}X(\tau,\tau_c) \tag{2.42}$$

that is the same as,

$$x_o = \frac{1}{\sqrt{S_{eo}}}\frac{d}{d\tau}X(\tau) \tag{2.43}$$

and $\frac{1}{\sqrt{S_{eo}}}$ is the normalization constant. For zero mode dependence, from equation (2.10),

$$\Delta x(\tau) = x_o(\tau)\Delta c_o \tag{2.44}$$

Under a shift in $\Delta\tau_c$, by using our solutions, we can conclude that;

$$\Delta x(\tau) = \Delta X(\tau) = \frac{dx}{d\tau_c}\Delta\tau_c = -\sqrt{S_{eo}}x_o(\tau)\Delta\tau_c \tag{2.45}$$

Identification from equation (2.44) and (2.46),

$$\Delta c_o = \sqrt{S_{eo}}\Delta\tau_c \tag{2.46}$$

where we don't insert the minus sign because they are varying in the same interval. With all these partial results at hand, the final determinant form becomes,

$$[\frac{det(-\frac{d^2}{d\tau^2}+V''(X))}{det(-\frac{d^2}{d\tau^2}+\omega^2)}]^{-1/2} = [\frac{det'(-\frac{d^2}{d\tau^2}+V''(X))}{det(-\frac{d^2}{d\tau^2}+\omega^2)}]^{-1/2}\sqrt{\frac{S_{eo}}{2\pi}}\omega d\tau_c \tag{2.47}$$

where det' stands for the reduced determinant form since we remove the zero mode inside it. In doing this, we have learned that how zero mode should be replaced by some collective coordinates. This is the method of finding Jacobian of the transformation. Now, it is easy to evaluate the other eigenvalues. I want to write the final result. For the exact calculations, one can learn from Shifman ([2], pp. 220 and 223). One instanton contribution,

$$\langle -\eta|e^{-H\tau_o}|+\eta\rangle_{one-instanton} = (\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_o/2})(\sqrt{\frac{6}{\pi}}\sqrt{S_{eo}}e^{-S_{eo}})\omega d\tau_c \tag{2.48}$$

At large τ_o, one should give importance to the paths constructed from many instanton and anti-instanton solutions. The first term in parenthesis corresponds to the simple harmonic oscillator. For the second term in bracket, we can call instanton density.

$$d = \sqrt{\frac{6}{\pi}}\sqrt{S_{eo}}e^{-S_{eo}} \tag{2.49}$$

where $\sqrt{S_{eo}}$ comes from zero mode. In QCD, each zero mode leads this pre-exponential factor and integration over collective coordinates.

Now, for the paths, one should consider the distance between two instantons. Suppose we have i instantons and anti-instantons centered at points like shown in

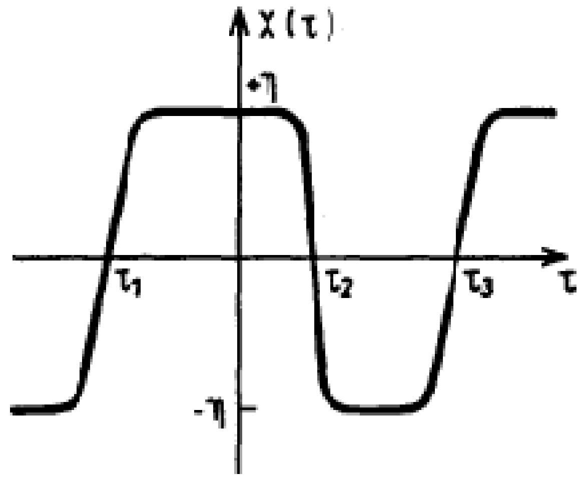

Figure 2.5. i separated instantons or anti-instantons

Fig(2.5).

$$-\frac{\tau_o}{2} < \tau_1 < \tau_2 < \tau_3 < ..... < \tau_i < +\frac{\tau_o}{2}$$

If the distance between two centers is large enough, i.e. ($|\tau_n - \tau_m| >> \omega^{-1}$), the action is iS_o where S_o is the one-instanton action. The transition amplitude becomes,

$$(\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_o/2}) \rightarrow (\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_o/2})d^i \int_{-\tau_o/2}^{+\tau_o/2} \omega d\tau_i \int_{-\tau_o/2}^{+\tau_i} \omega d\tau_{i-1} \int_{-\tau_o/2}^{+\tau_2} \omega d\tau_1 \tag{2.50}$$

Finally, we have

$$(\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_o/2})(\frac{(d\omega\tau_o)^i}{i!}) \tag{2.51}$$

From $-\eta$ to $+\eta$ transition, only odd number terms contribute. We need to sum over i. Thus we have,

$$\langle -\eta|e^{-H\tau_o}| + \eta\rangle = \sum_{1,3,..}(\frac{\omega}{\pi})^{1/2}e^{-\omega\tau_o/2}(\frac{(d\omega\tau_o)^i}{i!}) = (\frac{\omega}{\pi})^{1/2}e^{-\omega\tau_o/2}\sinh(d\omega\tau_o) \tag{2.52}$$

From $+\eta$ to $+\eta$ transition, only even number terms contribute. Therefore,

$$\langle +\eta | e^{-H\tau_o} | +\eta \rangle = \sum_{2,4,..} (\frac{\omega}{\pi})^{1/2} e^{-\omega\tau_o/2} (\frac{(d\omega\tau_o)^i}{i!}) = (\frac{\omega}{\pi})^{1/2} e^{-\omega\tau_o/2} \cosh(d\omega\tau_o) \tag{2.53}$$

Taking to the limit $\tau_o \to \infty$ preserves the symmetry between right and left well. For $\langle -\eta | 0 \rangle = \langle 0 | \eta \rangle = (\frac{\omega}{4\pi})^{1/4}$ one can show that the symmetry of the system is not broken. Since the distance between two is large enough and the terms in the sum are converging well, we can ignore the terms of $i >> d\omega\tau_o$ and define the constraint as $|\tau_n - \tau_m| >> d^{-1}\omega^{-1}$. One can achieve two instantons far from each other by taking $d \to 0$ or equivalently taking $\lambda \to 0$. Thus, for $\lambda << 1$ limit instantons and anti-instantons are acting like non-interacting particles. In other words, they are all far from each other so that they don't feel the existence of the each other. Such an approximation is known as dilute gas approximation. Unfortunately, we don't have free parameter like λ in QCD. Thus, it is not appropriate approximation for QCD.

2.3. Instantons in gauge field theories

2.3.1. Instantons as tunneling effect in quantum chromodynamics

The Lagrangian of QCD is defined in Minkowski space as,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum \overline{\Psi}(iD_\mu \gamma^\mu - m)\Psi \tag{2.54}$$

$F_{\mu\nu}$ is matrix-valued gluon field strength as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{i}[A_\mu, A_\nu] \tag{2.55}$$

and we can find $F^a_{\mu\nu}$ by using relations;

$$F_{\mu\nu} = g F^a_{\mu\nu} \lambda^a / 2$$

$$[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}] = if^{abc}\frac{\lambda^c}{2}$$

hence three vector field strength, $a = 1, 2, 3$, becomes,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2.56}$$

g is so called gauge coupling constant and f^{abc} is the structure constant of SU(3). The group that we need to deal with is SU(3) so quark fields can be transformed by using SU(3) representation matrices. However, for simplicity, we are going to discuss this matter without taking into account the particular choice of the group and the quark fields. Thus, we can begin by the simplest non-Abelian group, SU(2). For above relations, we need to know that for SU(2) group, representation matrices change as $\frac{\sigma^a}{2}$ and structure constant equals to ϵ^{abc}.

In our previous double well potential example, we only have one degree of freedom so that tunneling between two minima was obvious. In the case of QCD, first we need to find out gauge field configurations that minimize action and also from where to where will the system tunnel. Since the space of fields is infinitely dimensional, we need to single out the degree(s) of freedom that tunnel and delocalized from in the space of field. This is very similar in the case of the double well potential. Let us illustrate the direction of this degree of freedom in infinitely dimensional space of QCD fields. We can make analogy with a particle living in a closed circle and feeling the gravitation.

Classically, it lives in the lowest state so that never feels the existence of the upper side. Quantum mechanically, it oscillates around the origin. For such a way ,it feels that it can wind around the circle. To find out the degree of freedom, we'd better consider the Hamiltonian of the system. For a vector potential $A_\mu = (A_o, \mathbf{A})$ Electric

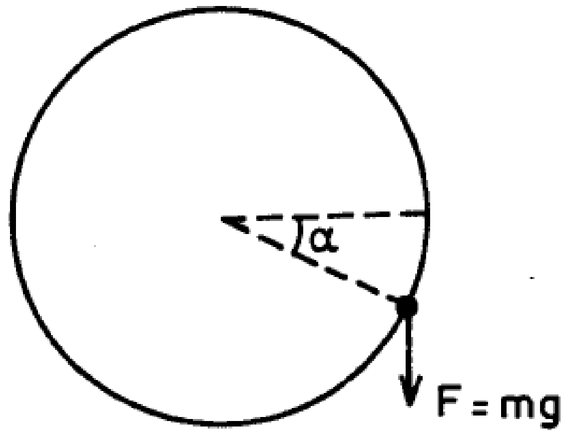

Figure 2.6. Particle living on a one dimensional manifold

and magnetic field is defined as,

$$E = \frac{\partial \mathbf{A}}{\partial t} - \nabla A_o$$

$$B = -(\nabla \times \mathbf{A})$$

Now, we can see that right hand sides of the equations are components of the field strength tensor. We can redefine them as,

$$F_{oi} = F^{oi} = E_i$$

$$F_{ij} = -F^{ij} = -\epsilon_{ijk} B_k$$

By using the lagrangian that we define in the equation (2.54), one can find that,

$$\mathcal{L} = \frac{1}{2}[(E^a)^2 - (B^a)^2]$$

from hamiltonian description where the dynamical momentum variable equals to E^a

in the case of $A_o = 0$ condition,

$$H = \frac{1}{2} \int d^3x (E^a E^a + B^a B^a) \tag{2.57}$$

where one can make analogy with kinetic energy. The state of the system at any given momenta in time is described by the gauge field configuration that is A^a_μ. Under a gauge transformation, potential gets the value,

$$A_\mu \rightarrow U A_\mu U^{-1} + iU\partial_\mu U^{-1} \tag{2.58}$$

In the zero energy state, one can see that,

$$A^a_i \rightarrow \frac{i}{g} U \partial_i U^{-1} \tag{2.59}$$

This is known as *pure gauge* condition. U, is the matrix representation of SU(2), allows us make transformation from Euclidean space into the gauge group. Ultimately, we are interested in zero energy states that may have tunneling connection between each other and as a boundary condition, when x goes to infinity, we need vanishing gauge field configurations. We can achieve this by introducing spatial independent U matrix. Therefore, it can be any constant.

$$U(x \rightarrow \infty) = 1 \tag{2.60}$$

With this boundary condition, our three dimensional space becomes topologically equivalent to three dimensional sphere.

The group space of SU(2) is also a three dimensional sphere. All matrix representation of SU(2) can be written as ,

$$M = \alpha + i\vec{\beta}\vec{\sigma} \qquad \text{where } \alpha \text{ and } \beta \text{ are real parameters}$$

$\vec{\sigma}$ is well-known Pauli matrices. From group conditions, $M^{\dagger}M = 1$ and $detM = 1$, imply that

$$\alpha^2 + \vec{\beta}^2 = 1$$

Since our U(x) matrix belongs to SU(2), the space of all x coordinates is topologically equivalent to S_3. U(x) is for a mapping of the sphere in the coordinate space onto a sphere in the group space. Such a mapping,

$$\Pi_3(S_3) = Z$$

where Z is integer. Since Z can be $0, \pm 1, \pm 2, ...$, plus and minus signs correspond to clockwise and anticlockwise mapping respectively. This is known as the winding number. Examples of different maps in different classes are [8],

- $U_o(x) = 1$ has the winding number 0.
- $U_1(x) = exp(i\pi \frac{\vec{\sigma}\vec{x}}{(\mathbf{x}^2+a^2)^{1/2}})$ has winding number 1.
- Finally, $U_n(x) = [U_1(x)]^n$ has winding number n.

Now, we can define the degree of freedom corresponding to the motion along this circle [2].

$$K_\mu = 2\epsilon_{\mu\nu\lambda\sigma}(A^a_\nu \partial_\lambda A^a_\sigma + \frac{g}{3} f^{abc} A^a_\nu A^b_\lambda A^c_\sigma) \tag{2.61}$$

where K_μ is called $Chern - Simons$ current. From the definition of the field strength tensor (2.56), one can conclude that,

$$\partial_\mu K^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma} \tag{2.62}$$

The dual field strength is,

$$\tilde{F}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} F^a_{\lambda\sigma} \tag{2.63}$$

and hence,

$$\partial_\mu K^\mu = F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \tag{2.64}$$

Now, we are ready to define related $Chern - Simons$ charge.

$$\mathcal{K} = \frac{g^2}{32\pi^2}\int K_o(x) d^3x \tag{2.65}$$

In the equation above, $K_o(x)$ is given by,

$$K_o(x) = 2\epsilon^{ijk}(A^a_i \partial_j A^a_k + \frac{g}{3} f^{abc} A^a_i A^b_j A^c_k) \tag{2.66}$$

For simplicity, we can write the equations in the trace form.

$$\mathcal{K} = \frac{1}{8\pi^2}\int d^3x Tr(AdA - \frac{2i}{3}A^3) \tag{2.67}$$

Under a gauge transformation,

$$A' = UAU^{-1} + iUdU^{-1} \tag{2.68}$$

Chern-Simons charge takes the form as,

$$\mathcal{K}' = \mathcal{K} + \frac{1}{24\pi^2}\int d^3x Tr(UdU^{-1} \wedge UdU^{-1} \wedge UdU^{-1}) \tag{2.69}$$

Since vector potential A is odd under parity, the first term on the right hand side evaluates to zero. The second term is thoroughly a homotopy invariant. Mainly, it

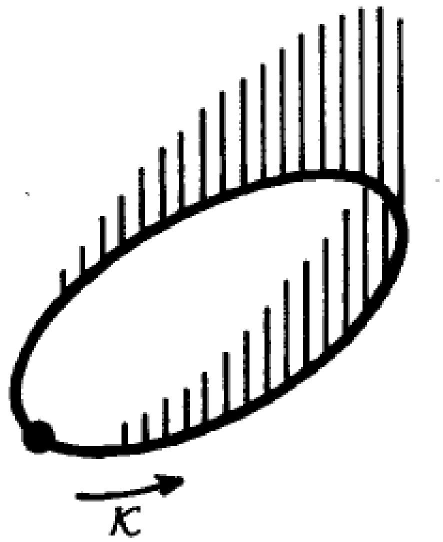

Figure 2.7. Moving in $\mathcal{K}$ direction has the topology of the circle in the space of gauge fields

allows continuous deformation between two functions. As an example, for infinitesimal transformations it totally vanishes and for U_1 it equals to one ([8], pp. 285 and 291).

In the help of these knowledge, one can conclude that this charge corresponds to the winding number. Therefore, it should be an integer.

$$\mathcal{K} = n$$

We can say that $\mathcal{K}$ has a topology of a circle and integer numbers of $\mathcal{K}$ correspond to zero potential energy. We can show this non-trivial topology like Fig(2.7). If we cut this circle and map it onto the straight line, we will get periodic potential as shown in (Fig.2.8). One can see that any integer value winding number corresponds to the zero energy and for the other values, field strength tensor is non-vanishing and energy of the system is positive. However, since the topology of our problem implies a circle, going from circle to a straight line requires the quasiperiodic Bloch boundary condition [2].

$$\Psi(\mathcal{K} + 1) = e^{i\theta}\Psi(\mathcal{K})$$

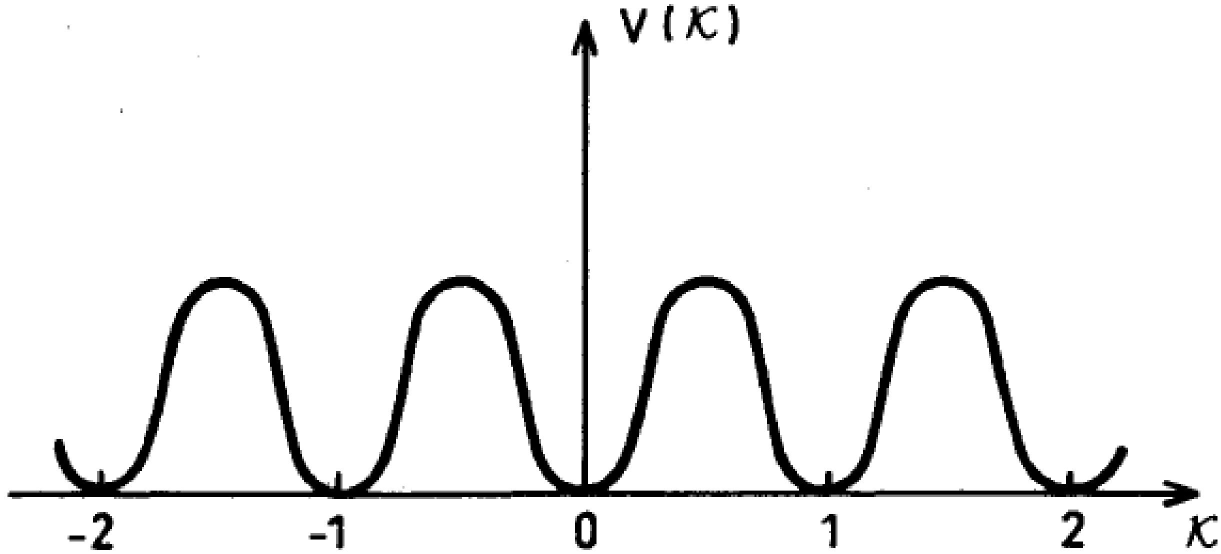

Figure 2.8. Periodic potential versus $\mathcal{K}$ as winding number

where θ is the vacuum angle. We will return the vacuum angle and related strong CP problem in the last section. Let us assume that at $t = -\infty$ our system is in $\mathcal{K} = n$ and at $t = +\infty$ in $\mathcal{K} = n \pm 1$. Therefore, we can say that our system tunnels under the hump of the potential. These tunnellings are described as instantons. Now, we are going to deal with instantons as solutions of the Euclidean equation of motion.

2.3.2. Instanton solutions

Before we go into Euclidean action, let us investigate how the fields are affected under this transformation. In transition to Euclidean space, the spatial coordinates do not change, whereas the time coordinate does.

$$\hat{x}_4 = ix_o \qquad \text{and} \qquad \hat{x}_i = x^i$$

Since the time coordinate is changed, we can redefine field strength tensor as,

$$\hat{F}_{ij} = F_{ij} \qquad \hat{F}_{4j} = -iF_{0j} \tag{2.70}$$

The Euclidean action can be obtained from Minkowskian lagrangian Eq.(2.54) by using these substitutions.

$$S_e = \frac{1}{4}\int d^4x \hat{F}^a_{\mu\nu}\hat{F}^a_{\mu\nu} \tag{2.71}$$

For simplicity, we omit upper index notation. However, for exact calculations one should keep them in mind. If we recall the dual field strength tensor definition Eq.(2.63), we can form Euclidean action as,

$$S_e = \int d^4x[\frac{1}{4}\hat{F}^a_{\mu\nu}\tilde{\hat{F}}^a_{\mu\nu} + \frac{1}{8}(\hat{F}^a_{\mu\nu} - \tilde{\hat{F}}^a_{\mu\nu})^2] \tag{2.72}$$

The first term on the rhs is known as topological charge and it equals to,

$$Q = \frac{g^2}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \tag{2.73}$$

From our definition of the Chern-Simons charge (2.65), we can see that,

$$Q = \mathcal{K}' - \mathcal{K} \tag{2.74}$$

where $\mathcal{K}'$ refers to the future while $\mathcal{K}$ to the past. Thus, Q can only take integer values. To get the explicit solutions of instantons, one needs to recall that instantons are bound to minima of the Euclidean action between the space of charge Q fields. The only way to obtain the minima of the action is,

$$S_e = Q\frac{8\pi^2}{g^2} + \frac{1}{8}(F^a_{\mu\nu} - \tilde{F}_{\mu\nu})^2] \tag{2.75}$$

For positive Q values;

$$F^a_{\mu\nu} = +\tilde{F}^a_{\mu\nu} \tag{2.76}$$

and related minimum Euclidean action becomes,

$$S_e = Q\frac{8\pi^2}{g^2} \tag{2.77}$$

For negative Q values; with the substitution of $x_i \to -x_i$ and $F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \to -F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$,

$$F^a_{\mu\nu} = -\tilde{F}^a_{\mu\nu} \tag{2.78}$$

and related minimum Euclidean action becomes,

$$S_e = |Q|\frac{8\pi^2}{g^2} \tag{2.79}$$

While doing these calculations, we need to keep our mind that only the fields that have the same topological charge can deform one to another continuously. Therefore, arrangements for positive Q values are for instanton solutions, whereas negative value dependents are for anti-instanton solutions. If we recall the Yang Mills condition,

$$D_\mu F_{\mu\nu} = 0 \tag{2.80}$$

For self dual and anti-self dual fields, this condition is fulfilled directly so that,

$$D_\mu F^a_{\mu\nu} = D_\mu \tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} D_\mu F^a_{\lambda\sigma} = \frac{1}{6}\epsilon_{\mu\nu\lambda\sigma}(D_\mu F^a_{\lambda\sigma} + D_\sigma F^a_{\mu\lambda} + D_\lambda F^a_{\sigma\mu}) = 0 \tag{2.81}$$

where the last term in parenthesis is known as Bianchi identity.

$$(D_\mu F^a_{\lambda\sigma} + D_\sigma F^a_{\mu\lambda} + D_\lambda F^a_{\sigma\mu}) = 0$$

The method that we will use for instanton solutions makes it possible for us to get one and many instanton solutions in a more compact way. The gauge field with the following ansatz is [11],

$$A_\mu(x) = \frac{1}{2}\bar{\eta}_{a\mu\nu}\sigma^a\partial_\nu[ln\phi(x)] \tag{2.82}$$

where $\bar{\eta}_{a\mu\nu}$ is known as t'Hooft symbol. If we try to write it in a more compact way,

$$\bar{\Sigma}_{\mu\nu} = \frac{1}{2}\bar{\eta}_{a\mu\nu}\sigma^a \tag{2.83}$$

$\bar{\Sigma}_{\mu\nu}$ are the components of a matrix constructed from Pauli matrices.

$$\bar{\Sigma}_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & +\sigma^3 & -\sigma^2 & -\sigma^1 \\ -\sigma^3 & 0 & +\sigma^1 & -\sigma^2 \\ +\sigma^2 & -\sigma^1 & 0 & -\sigma^3 \\ +\sigma^1 & +\sigma^2 & +\sigma^3 & 0 \end{pmatrix} \tag{2.84}$$

Now it is better for the later calculations to define commutator relations of this quantity.

$$[\bar{\Sigma}_{\mu\sigma}, \bar{\Sigma}_{\nu\rho}] = i(\delta_{\mu\nu}\bar{\Sigma}_{\sigma\rho} + \delta_{\rho\sigma}\bar{\Sigma}_{\mu\nu} - \delta_{\mu\rho}\bar{\Sigma}_{\sigma\nu} - \delta_{\nu\sigma}\bar{\Sigma}_{\mu\rho})$$

$$\epsilon_{\mu\nu\alpha\beta}\bar{\Sigma}_{\beta\sigma} = (\delta_{\mu\sigma}\bar{\Sigma}_{\nu\alpha} + \delta_{\nu\sigma}\bar{\Sigma}_{\alpha\mu} + \delta_{\alpha\sigma}\bar{\Sigma}_{\mu\nu})$$

and

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\Sigma}_{\alpha\beta} = -\overline{\Sigma}_{\mu\nu}$$

If we use these quantities on the field strength tensors and demand on the condition $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, we will have,

$$\partial_\sigma\partial_\sigma(ln\phi(x)) + (\partial_\sigma(ln\phi))^2 = 0 \tag{2.85}$$

In compact form,

$$\frac{1}{\phi}\Box\phi = 0 \qquad \text{where} \qquad \Box = \partial_\sigma\partial_\sigma \qquad \text{in Euclidean metric} \tag{2.86}$$

According to the equation above, if ϕ is non-singular, this reduces $\Box\phi = 0$ which means that $A_\mu = 0$. Thus let us consider singular $\phi(x)$. For instance,

$$\text{For } \phi(x) = \tfrac{1}{|x^2|} \rightarrow \frac{1}{\phi}\Box\phi = \frac{1}{\phi}\partial_\sigma(-\frac{2x_\sigma}{|x^4|}) = 0 \qquad \text{at } x \neq 0$$

At $x = 0$,

$$\Box(\frac{1}{|x^2|}) = -4\pi^2\delta^4(x) \rightarrow \int d^4x\Box(\frac{1}{|x^2|}) = -4\pi^2$$

Therefore $\frac{1}{|x^2|}$ satisfies our demand and allow us to rewrite it more generally as,

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\rho^2}{|x_\mu - b_{i\mu}|^2} \tag{2.87}$$

To get single instanton solution, we need to take $N = 1$ and for simplicity we can also take $y_\mu = (x - b_1)_\mu$.

$$\phi(x) = 1 + \frac{\rho^2}{y^2} \tag{2.88}$$

This is leading to,

$$A_\mu(x) = -2\rho^2\bar{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2(y^2+\rho^2)} \tag{2.89}$$

One can notice that this solution is singular at $y = 0$ or equivalently $x = b_1$. However, singularity can be removed by a correspondingly singular gauge transformation.

$$iU_1^\dagger \partial_\mu U_1 = -2\bar{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2} \tag{2.90}$$

where $U_1 = \frac{x_4 + i\vec{\sigma}\vec{x}}{\sqrt{x^2}}$. Thus,

$$A_\mu = iU_1^\dagger \partial_\mu U_1 \frac{\rho^2}{y^2+\rho^2} \tag{2.91}$$

Under gauge transformation,

$$A'_\mu \rightarrow U_1(A_\mu + i\partial_\mu)U_1^\dagger \tag{2.92}$$

When we insert Eq.(2.91) into the equation above, we get

$$A'_\mu = -iU_1^\dagger \partial_\mu U_1 \frac{y^2}{y^2+\rho^2} \tag{2.93}$$

By using the definition of y_μ, we can conclude that,

$$A'_\mu = 2\bar{\Sigma}_{\mu\nu}\frac{(x-b_1)_\nu}{(x-b_1)^2+\rho^2} \tag{2.94}$$

For gauge transformed solution, we can use the self dual and antisymmetric notation as [11],

$$\Sigma_{\mu\nu} = \frac{1}{2}\eta_{a\mu\nu}\sigma^a \tag{2.95}$$

where $\eta_{a\mu\nu}$ is defined as,

$$\eta_{a\mu\nu} = -\eta_{a\nu\mu} = \begin{cases} \epsilon_{a\nu\mu} & \text{for} \quad \mu,\nu = 1,2,3 \\ \delta_{a\mu} & \text{for} \quad \nu = 4 \end{cases} \tag{2.96}$$

Introducing this self dual notation and replacing anti self dual one in Eq(2.90) yields us to have gauge transformed and non-singular field as,

$$A'_\mu = 2\Sigma_{\mu\nu}\frac{(x-b_1)_\nu}{(x-b_1)^2+\rho^2} \tag{2.97}$$

where b_1 is the center and ρ represents radius of the instanton. When we substitute this solution into the gluon field strength equation,

$$F'_{\mu\nu} = -4\Sigma_{\mu\nu}\frac{\rho^2}{[(x-b_1)^2+\rho^2]^2} \tag{2.98}$$

In terms of matrix element, we can write this solutions in the form of,

$$\begin{aligned}(A')^a_\mu &= \tfrac{2}{g}\eta_{a\mu\nu}\tfrac{(x-b_1)_\nu}{(x-b_1)^2+\rho^2}\\ (F')^a_{\mu\nu} &= -\tfrac{4}{g}\eta_{a\mu\nu}\tfrac{\rho^2}{[(x-b_1)^2+\rho^2]^2}\end{aligned} \tag{2.99}$$

These solutions belong to $Q = 1$ sector. One can construct solutions of $Q = -1$ sector, this is known as anti-instanton solutions, by changing self dual t'Hooft symbol with anti self dual one. The anti-instanton solutions of $Q = -1$ sector are,

$$\begin{aligned}(A')^a_\mu &= \tfrac{2}{g}\bar{\eta}_{a\mu\nu}\tfrac{(x-b_1)_\nu}{(x-b_1)^2+\rho^2}\\ (F')^a_{\mu\nu} &= -\tfrac{4}{g}\bar{\eta}_{a\mu\nu}\tfrac{\rho^2}{[(x-b_1)^2+\rho^2]^2}\end{aligned} \tag{2.100}$$

Now, we are ready to find out many instanton solutions. For N instanton solutions,

$$A_\mu(x) = \bar{\Sigma}_{\mu\nu}\partial_\nu[ln(1+\sum_{i=1}^{N}\frac{\rho^2}{|y_i|^2})] \tag{2.101}$$

When we take the derivative, we obtain the equation as,

$$A_\mu(x) = -2\bar{\Sigma}_{\mu\nu} \sum_i \frac{\rho^2 y_{i\nu}}{|y_i|^4} (1 + \sum_j \frac{\rho^2}{|y_j|^2})^{-1} \tag{2.102}$$

Using the identity given in the equation (2.90) for each y_i value;

$$A_\mu(x) = i \sum_i U_1^\dagger \partial_\mu U_1 f_i(x) \tag{2.103}$$

where $f(x) = \frac{\rho^2}{|y_i|^4}(1 + \sum_j \frac{\rho^2}{|y_j|^2})^{-1}$. In one instanton solution, there was only one singularity to be removed. However, in this case there are N singularities that one needs to deal with. As $x = a_i$ limit,

$$f_i(x) = \delta_{ij}$$

$$A_\mu(x) = iU_1^\dagger(x - b_i)\partial_\mu U_1(x - b_i)$$

Therefore, near any of the singular point N-instanton solution will behave like a pure gauge. Thus, N-instanton solution can be regarded as a collection of N individual solutions. Their shapes will be dominated by the remaining ones. However, at the center ($x_i \simeq b_i$), the effect of the other is negligible and the N-instanton solution is dominated by that single instanton [11]. The related action will be,

$$S_e = N\frac{8\pi^2}{g^2} \tag{2.104}$$

which is N times the single instanton action.

Finally, we tried to find out instanton solutions as the solutions of highly non-trivial and non-linear field equations. There are many models for instanton solutions. Actually, the term instanton implies the finite Euclidean action solutions of any models. In all the models, instantons are used for describing vacuum tunneling and related phenomena. They are very powerful even in classical levels. However, their effectiveness in QCD is the most important part for us. We will discuss them as the bases of axion particles in the last section.

3. QUARK FLAVOUR MIXING

In particle physics, flavour is a quantum number that is preserved in the strong and electromagnetic interactions. Elementary particles carry many of these quantum numbers and characterization of these particles and related interactions depend on the conservation of these numbers. However, experiments based on weak interactions showed us that flavour changing exists in nature. The most crucial example of flavour changing is neutron decay. It is well known that neutron is an unstable particle and by emitting one electron it can decay into a proton. The reaction is,

$$udd \rightarrow uud + e^{-} + \overline{\nu}_e \tag{3.1}$$

and related Feynman diagram is shown as in Fig(3.1). Precisely, the more massive down-quark(d) changes its flavour into up-quark(u) by emitting one W boson. Since the transition between families are not possible, the transition of the lightest u quark can not be expected. Therefore, proton decays have not been observed in nature and consequently proton as the lightest hadron remains stable. We can summarize that transition is possible between quarks of different flavours in charged weak interactions.

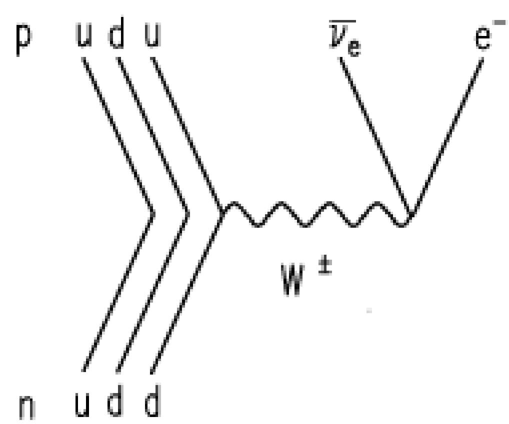

Figure 3.1. The Feynman diagram of neutron beta decay

The early history of weak interaction theory begins with Fermi in 1933. It is known that all fundamental particles are paired as (e^-,ν_e),(μ^-,ν_μ),(u,d) and so forth. With the definition of Fermi constant,

$$\frac{G_F}{\sqrt{2}} = \frac{{g_W}^2}{8W^2} \tag{3.2}$$

one can conclude that neutron life time is related with the Fermi coupling constant and with the help of the experimental results, Fermi coupling constant is precisely $G_F = 1,166392(2) \times 10^{-5} GeV^{-2}$ for $\hbar = c = 1$ [12]. There are other interactions dependent on the charged moderator W. One of them is muon decay. From the experimental results, the relation between neutron Fermi coupling constant and muon Fermi coupling constant is,

$$G_n/G_\mu = 0,9740 \pm 0.001 \tag{3.3}$$

It seems that these two values are approximately the same. The difference is the result of the quantum corrections and nucleus dependency of beta decays. However, for charged Kaon, experiments showed that the decay rate is very different than either neutron or muon. In other words, it required totally different effective coupling constant. This was the result of new quantum number called strangeness. Kaon decay process can be reduced as,

$$s \rightarrow u + e^- + \overline{\nu}_e \tag{3.4}$$

and related Fermi coupling constant is,

$$G_K/G_\mu \simeq 0.22 \tag{3.5}$$

From these cases, we have at least three different Fermi coupling constants that are G_n , G_μ and G_K. It is very understandable for someone to question the universality of Fermi

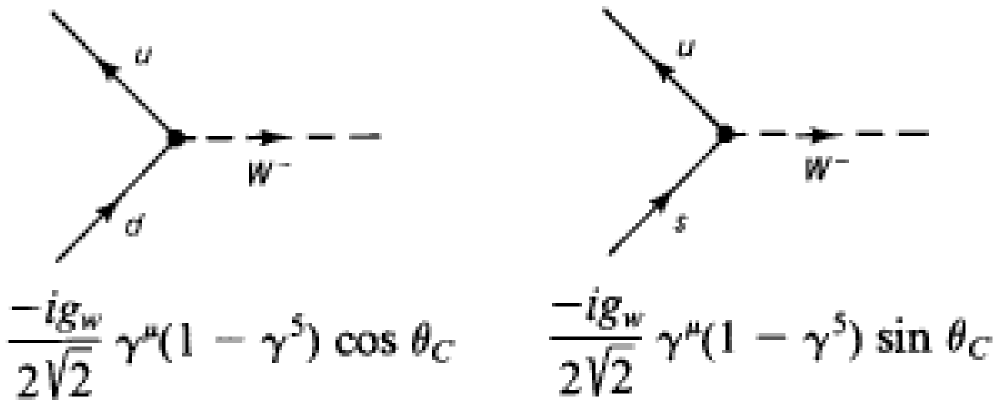

Figure 3.2. Cabibbo angle as addition to propagators

coupling constant for weak interactions. With this right question, the solution arises from Nicola Cabbibo in 1963 when the only known quarks were u,d and s. According to this theory, propagators take some additional terms [13]. They took the form as shown in Fig(3.2). Ultimately, if we accept muon decay as the main measurement for Fermi coupling constant, the other two become,

$$G_n = G_F \cos\theta_C$$

$$G_K = G_F \sin\theta_C$$

The strangeness-changing process is rather weaker than strangeness-conserving one. Thus, Cabibbo angle θ_C should be small. It has the value $\theta_C = 13,1^o$ which is determined from experiments [14]. With this redefinition, one could save the universality of weak interactions if it could explain K^0 decay into $\mu\mu^-$. With Cabibbo theory, K^0 decay amplitude equals to $\sin\theta_C\cos\theta_C$. However, the calculated rate exceeds the experimental limit. To solve this paradox, Glashow, Iliopoulos and Maiani introduced the charm quark, c. The couplings of c quark to s and d are given as in Fig(3.3). Now, K^0 decay loop can also be obtained from $u \to c$ with the amplitude $-\sin\theta_C\cos\theta_C$. Therefore, these two cancel each other. The Cabibbo-GIM mechanism implies that quantum eigenstates of weak interactions are rotated in quark flavour space with respect to the their mass eigenstates. In other words, for weak interactions in the Cabibbo theory

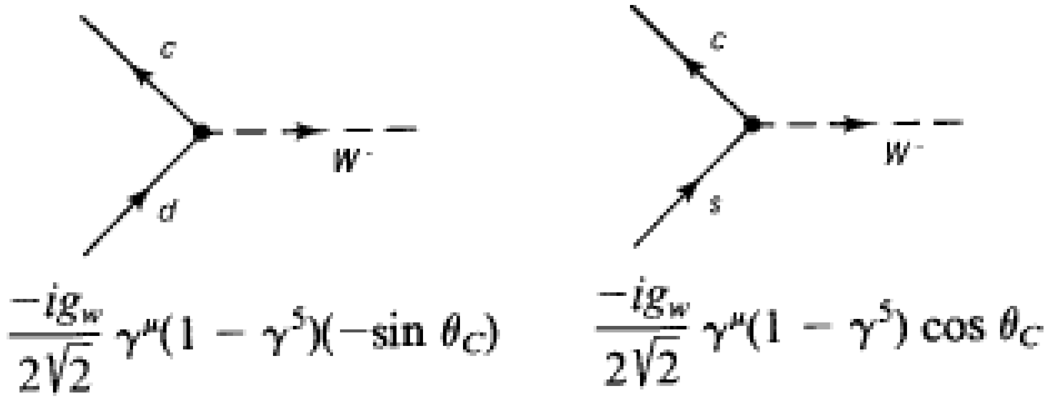

Figure 3.3. c coupling to s and d quarks

with GIM modification, we need a unitary matrix to determine the transition amplitude between different flavours. This 2x2 matrix have 4 real parameters because for $U_1 \times SU(2) = U_2$ group, there are 1+3 = 4 parameters [15]. Since each of these four quarks possesses arbitrary phase factors and three of them are relative to each other, U depends only 4-3=1 real parameter, θ_C. Finally, we may write it as,

$$U = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \tag{3.6}$$

and it yields us to have,

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \tag{3.7}$$

Lagrangian of charged quark weak current to $W^{\pm}$ is given by [16],

$$L_{int} = \frac{-g}{2\sqrt{2}}\overline{u}\gamma_\mu(1-\gamma_5)d'W^\mu + h.c \tag{3.8}$$

However, we know that there are not four but six quarks in Standard Model. Discovery of third generation led to investigate their behaviours through weak interactions. It could be better to discuss the flavour mixing of six quark theory in following section.

3.1. The Cabibbo-Kobayashi-Maskawa Matrix

The beauty quark (b) was discovered in the form of $\Upsilon(b\bar{b})$ at Fermilab in 1977. To complete the list of quarks, we waited until 1995 when CDF and D0 experiment groups in Fermilab found it in the process $p\overline{p} \to t\bar{t}X$ and consequently $t \to b + W^{+}$. For six quark theory, the two dimensional rotational matrix should be replaced by a three dimensional one and symbolically can be written as,

$$V_3 = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{3.9}$$

It yields us to have,

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{3.10}$$

For 2×2 unitary matrix, it was just a rotation in quark flavour space. Now, for three generations we need to define 3×3 unitary matrix. Since unitary matrix can include complex parameters, this will lead some CP violating effects. After parametrization of CKM matrix, we can discuss the effects of it on CP violating phenomena.

For $U(1) \times SU(3)$, there are $1 + 8 = 9$ parameters. For six quarks, there are five arbitrary phases. Therefore, this unitary matrix has $9 - 5 = 4$ real parameters.There are several parametrization of the Cabibbo-Kobayashi-Maskawa matrix. These possible parameterizations differ both in the choice of mixing angles and the positioning of the phases. We are going to use the one advocated by Particle Data Group. We can choose these parameters as three mixing angles, θ_{12}, θ_{13}, θ_{23} and a phase δ_{13}. The unitary matrix can be obtained from two rotation matrices $R_{ij}(\theta_{ij})$ and one special unitary matrix $S_{ij}(\theta_{ij}, \delta_{ij})$ [17].

$$V_3 = R_{23} S_{13} R_{12} \tag{3.11}$$

Explicit form of these matrices are,

$$R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.12}$$

$$R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{3.13}$$

$$S_{13}(\theta_{13}, \delta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \tag{3.14}$$

where $c_{ij} = \cos\theta ij$, $s_{ij} = \sin\theta ij$ and θ_{ij} is the mixing angle between ith and jth quarks.

If we take dot products of these given matrices, final version of CKM matrix will be,

$$V_3 = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{3.15}$$

Since θ_{12} is the Cabibbo angle, we can use it to determine the value of the matrix elements. Actually, quark masses and and mixing angles are the fundamental quantities whose values must be determined by experiments. The present status of this matrix is [19],

$$V_{CKM} = \begin{pmatrix} 0.97383^{+0.00024}_{-0.00023} & 0.2272^{+0.0010}_{-0.0010} & 3.96^{+0.09}_{-0.09} \times 10^{-3} \\ \\ 0.2271^{+0.0010}_{-0.0010} & 0.97296^{+0.00024}_{-0.00024} & 42.21^{+0.10}_{-0.80} \times 10^{-3} \\ \\ 8.14^{+0.32}_{-0.64} \times 10^{-3} & 41.61^{+0.12}_{-0.78} \times 10^{-3} & 0.999100^{+0.000034}_{-0.000004} \end{pmatrix}$$

By using quantities of this matrix element, one can confirm the unitarity of CKM matrix. Unitarity requires that the sum of the squares of the each elements in the same row must equal to unity. For present status,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999803905$$

$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.000007256$$

$$|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 0.9999984617$$

it satisfies the unitarity condition.

3.2. Unitarity Triangle

The unitarity of CKM matrix allows us to have various relations between its elements. Particular property is that due to the unitarity any row and any column must be orthogonal to each other. This orthogonality condition can be shown as,

$$\sum_i V_{ij}V_{ik}^* = \delta_{jk} \qquad \text{and} \sum_j V_{ij}V_{kj}^* = \delta_{ik} \tag{3.16}$$

This relation can be represented as *Unitarity Triangle* in the complex plane. For this representation, we can use Wolfenstein parameters. In this method, CKM matrix elements are dependent only four real parameters; A, λ, ρ and η [18]. Some standard parameters' representations in terms of Wolfenstein parameters are,

$$\begin{aligned} s_{12} &= \lambda \\ s_{23} &= A\lambda^2 \\ s_{13}e^{-i\delta_{13}} &= A\lambda^3(\rho - i\eta) \end{aligned} \tag{3.17}$$

If we neglect fourth order of λ, the CKM matrix is formed as,

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \tag{3.18}$$

The widely used unitarity triangle arises from the equation below,

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \tag{3.19}$$

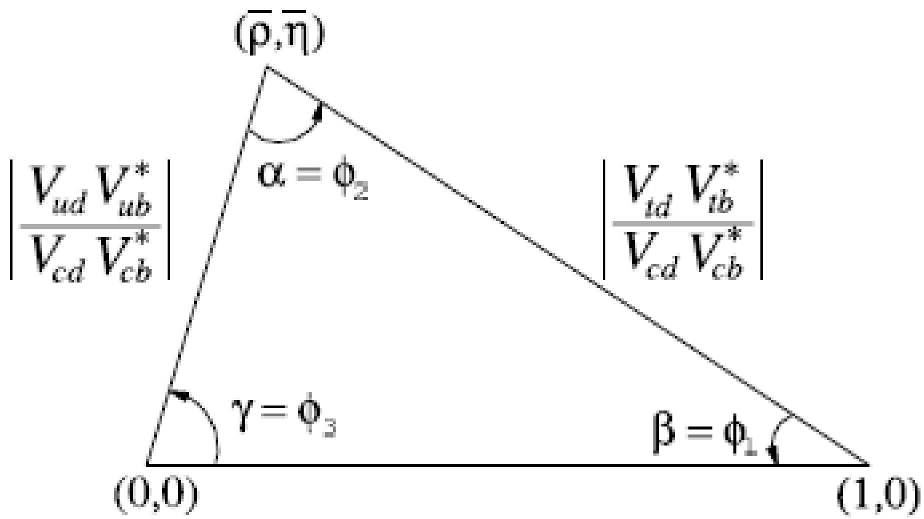

Figure 3.4. The Unitarity Triangle

The length of one of the sides can be normalized to the real value 1. Dividing the equation above by $V_{cd}V_{cb}^*$, one can obtain the Unitarity triangle relation as,

$$1+\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}+\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}=0 \tag{3.20}$$

In Fig(3.4), one can find the so called unitarity triangle scheme. The vertex of the triangle is the complex vector,

$$\begin{gathered} 1+\tfrac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}=(\rho+i\eta)(1-\tfrac{\lambda^2}{2})+O(\lambda^4) \\ \overline{\rho}=\rho(1-\tfrac{\lambda^2}{2}) \qquad \overline{\eta}=\eta(1-\tfrac{\lambda^2}{2}) \end{gathered} \tag{3.21}$$

It can be seen that the angle $\gamma(\phi_3)$ equals to the phase angle of the standard parametrization.

$$\begin{gathered} \tan\gamma=\tfrac{\overline{\eta}}{\overline{\rho}}=\tfrac{\eta}{\rho} \\ \gamma=\arctan\tfrac{\eta}{\rho}=\delta_{13} \end{gathered} \tag{3.22}$$

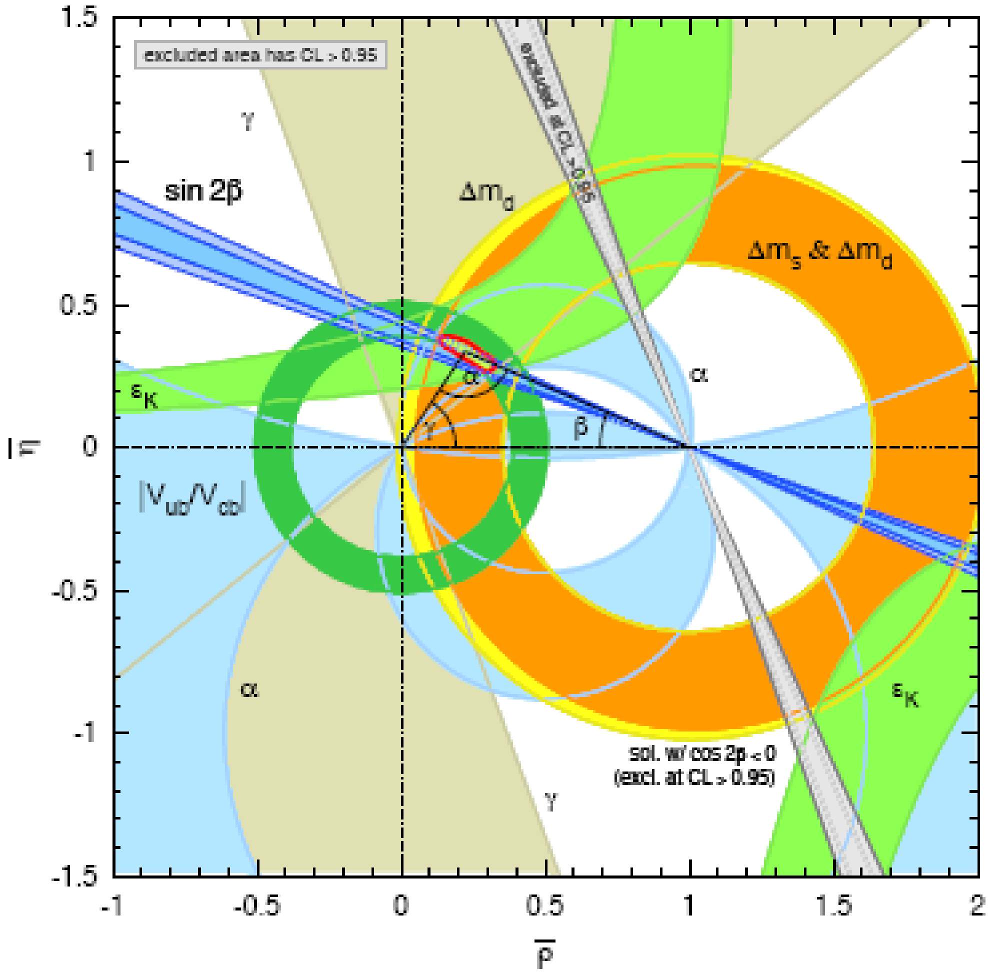

Figure 3.5. The present status of the unitarity triangle

For $\alpha(\phi_1)$ and $\beta(\phi_2)$,

$$\tan\alpha = \frac{\eta}{\eta^2-\rho(1-\rho)}$$

$$\tan\beta = \frac{\eta}{(1-\rho)} \tag{3.23}$$

The present status of unitarity triangle given by Particle Data Group is shown in Fig(3.5) [19].

4. AXIONS

4.1. Axions as the solution of Strong CP problem

The most appealing explanation for Strong CP problem comes from the spontaneously broken $U(1)_{PQ}$ symmetry and axions as a result of this symmetry breaking. This symmetry is spontaneously broken and results us to replace static $\bar{\theta}$ CP violating angle with CP conserving fields, axions [3]. In the presence of $U(1)_{PQ}$ symmetry, axion fields transform as,

$$a(x) \rightarrow a(x) + \alpha f_a$$

where f_a is the order parameter associated with breaking $U(1)_{PQ}$ and it is called *axion decay constant* [4]. With this additional symmetry, lagrangian takes the form as,

$$L = L_{SM} + \bar{\theta}\frac{g^2}{32\pi^2}F^b_{\mu\nu}\tilde{F}^{b\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a + L_{int}(\frac{\partial_\mu a}{f_a}, \Psi) + \frac{a}{f_a}\frac{g^2}{32\pi^2}F^b_{\mu\nu}\tilde{F}^{b\mu\nu} \tag{4.1}$$

The last term gives the effective potential and related axial current. The axial current is,

$$\partial_\mu J^\mu_{PQ} = \frac{g^2}{32\pi^2}F^b_{\mu\nu}\tilde{F}^{b\mu\nu} \tag{4.2}$$

and the minimum of the effective potential is obtained as,

$$\langle\frac{\partial V_{eff}}{\partial a}\rangle = -\frac{g^2}{f_a 32\pi^2}\langle F^b_{\mu\nu}\tilde{F}^{b\mu\nu}\rangle|_{<a>=-\bar{\theta}f_a} \tag{4.3}$$

As we can see that the minimum occurs at $<a>=-\bar{\theta}f_a$. QCD anomaly serves us to generate a periodic potential for the axion field [4].

$$V_{eff} \sim \cos(\bar{\theta} + \frac{<a>}{f_a}) \tag{4.4}$$

If we rewrite Eq(4.1) in terms of $a_{phys} = a - <a>$, then CP violating terms are cancelled and Strong CP problem is solved.

4.2. Invisible Axion Models

Expanding effective potential around its minimum again gives us the mass term of the axion particles. The known axion mass is,

$$m_a \simeq 6eV \frac{10^6 GeV}{f_a} \tag{4.5}$$

f_a in the equation below is totally an arbitrary parameter. Therefore, one can construct any axion models by taking f_a in different values. The constraint of f_a and axion mass are given in the Fig(4.1).

After lots of trial, $f_a \sim \upsilon$ where $\upsilon = (\sqrt{2}G_F)^{-1/2}$ the scale of electroweak symmetry breaking is ruled out. However, introducing a new scale as $f_a >> \upsilon$ can save the theory [20]. This is called *Invisible Axion Model* and it is still viable. The life time of the axion is very large and therefore it can be considered as a stabile particle. The vacuum state of the axion can be visualized as particle on a bullet [21]. This vacuum state stays there for a long time and then begins to oscillate when the Hubble time $(1/H)$ is larger than the oscillation period $(1/m_a)$. This occurs at 1 GeV. For $T < T_1$, classical axion field begins to feel the existence of the potential and rolls down and this leads to have a conserved mass term like $m_a A^2$ where A is the coefficient of the classical axion field. This coherent axion field contributes to the cosmic energy and

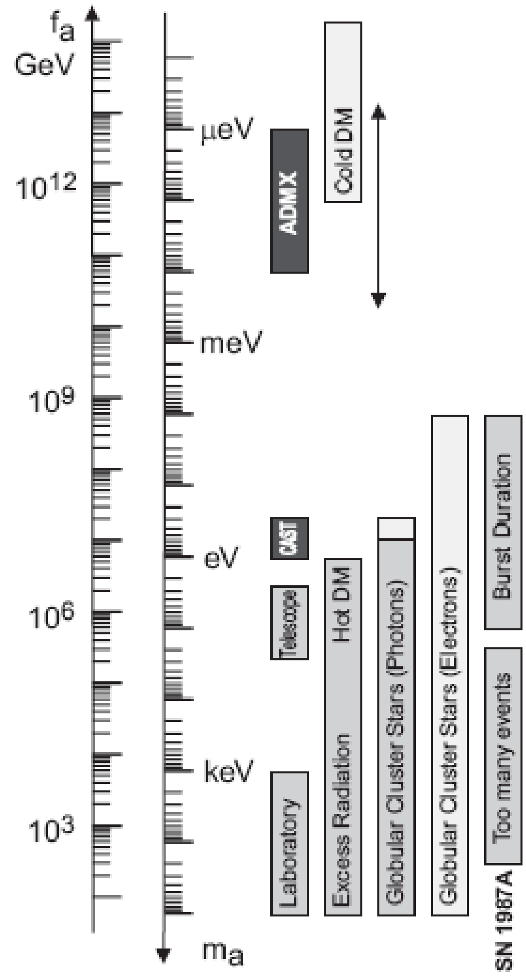

Figure 4.1. The ranges of axion decay constant, or equivalently axion mass [22]

these considerations constraint on f_a value as [21],

$$10^9 GeV < f_a < 10^{12} GeV \tag{4.6}$$

5. CONCLUSION

In this thesis, we have built up theoretical foundations of axions. The mathematics behind this method is the same as that of employed in Higgs mechanism in GSW (Glashow-Weinberg-Salam) electroweak theory. Like Higgs particles, axions have not been detected yet. There are many attempts to find these particles via the Primakoff effect. Primakoff effect allows axion-photon conversion and vice versa in the presence of a strong magnetic field. It can be shown like in Fig(5.1).

Solar axions are expected to produce in the core of the Sun via this effect. CAST (Cern Axion Solar Telescope) aims to detect these solar axions. The experiment uses 10 m long, decommissioned LHC magnet producing 9T magnetic field. The magnet is mounted on a platform that allows it to move horizontally $\pm 40^o$ and vertically $\pm 8^o$. This flexibility allows us to track the sun during sunrise and sunset in these ranges. There are two types of detectors, CCD and MicroMEGAS. CCD and MicroMEGAS are used for sunrise axions while another MicroMEGAS is used for sunset axion observations.

Figure 5.1. Primakoff effect

During 2003 and 2004 (Phase I), experiment run with the vacuum inside of the magnetic pipes and searched for the axion mass up to 0.02 eV. No signal was detected. However, CAST Phase II extends its potential to higher axion masses by introducing a buffer gas in the cold bore. During 2005 and 2006, ^4He was used as a refractive gas and allowed us to reach the mass 0.39 eV. Now, CAST is implementing ^3He as a buffer gas and with this implementation the estimated limit is up to 1.2 eV [23].

Finally, CAST is running with higher sensitivity that enables it to enter the axion mass range and this is the first time for a laboratory experiment to reach this scale. Therefore, all the results from the CAST experiment have the crucial impacts on the discovery of these axion particles.

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