

OPTIMAL NONLINEAR CONTROLLER DESIGN
FOR FLEXIBLE ROBOT MANIPULATORS

by

Mustafa Doğan

B.S., Electronics-Telecommunication Engineering, İstanbul Technical University, 1988

M.S., Electrical and Electronics Engineering, Old Dominion University, 1999

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Graduate Program in Electrical and Electronics Engineering
Boğaziçi University

2006

To my wife, Müge

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to Prof. Yorgo Istefanopulos for his guidance and support throughout this research project. His great knowledge and innovative ideas made this thesis possible. It has been a privilege and a pleasure to work with him. His commitment to excellence will be the goal of my future career. I would like to thank Prof. Ömer Morgül for his excellent help. The context of this research has been enriched greatly by his profound knowledge, his valuable suggestions and guidance. His encouragement for the derivations of the proofs related with PDE approach of this thesis is greatly appreciated. I also sincerely thank Assoc. Prof. Eşref Eşkinat and Assoc. Prof. Yağmur Denizhan for serving on my thesis advisory committee and my doctoral jury, and also Prof. İbrahim Eksin and Prof. Kemal Cılız for serving as examiners in my thesis defense.

I am grateful to Prof. Emin Akata whose help and understanding made my teaching work endurable. I would also like to thank E. Deniz Diktaş for his indispensable help and Adem Özay for his technical support during the simulation phase of this project, and Bayram Ünal for his special help. I am obliged to my brother, İsmet Doğan who is an artist, for his valuable support that improved my imagination. I am very thankful to my wife, Müge, for her great support and belief in my ability. She has always been a constant source of love and encouragement while I pursued my education.

ABSTRACT

OPTIMAL NONLINEAR CONTROLLER DESIGN FOR FLEXIBLE ROBOT MANIPULATORS

Developing nonlinear adaptive and robust controllers for a two-link flexible robot arm is the main objective of this research. Two different modelling techniques are used to overcome model accuracy problems and these two are evaluated comparatively. These are FEM(Finite Element Method) as a reduced order-approximate model and PDE(Partial Differential Equations) approach as an exact model. Since FEM model needs modal truncation, unknown disturbances can excite neglected high-frequency modes generated by the nonlinearities of the plant. The second approach is modelling the flexible robot arms by PDE which are known to provide exactness. In order to improve the important features of flexible links such as low mass and moments of inertia and high natural frequencies, optimal shape design can result in nonuniform cross-section of links. Furthermore, a high fundamental frequency is desired since it implies a large bandwidth that will allow for fast motion without causing serious vibration problems and for stable endpoint control.

The main results of the study are robust regulation of the rigid modes and suppression of elastic vibrations of the flexible robot arm. The dynamic state feedback controller is used to achieve this goal in FEM approach. In the first part of this research the adaptive internal model approach, in parallel with a robust stabilizer, has been modified to manage totally unknown disturbances that can include neglected higher modes of the uniform flexible links as well as large parameter uncertainties such as tip mass changes. The stabilizer part of the controller which has been introduced for the first time in this research in conjunction with nonlinear systems, is optimized

successfully with a new efficient evolutionary algorithm. In the second approach (PDE) of this research, the control of a two-link flexible arm with nonuniform cross-section by design is improved by employing the Lyapunov method. LaSalle's invariance principle, extended for infinite dimension, is used in order to prove the asymptotic stability of the closed loop system without any modal truncation as opposed to former approaches in literature. Besides, large parameter uncertainties such as tip and hub mass changes in PDE approach are also handled effectively by the proposed nonlinear controller.

ÖZET

ESNEK ROBOT KOLLARI İÇİN DOĞRUSAL OLMAYAN OPTIMAL DENETLEYİCİ TASARIMI

Bu çalışmanın temel amacı, iki eklemlili esnek robot kolu için doğrusal olmayan, uyarlamalı ve dayanıklı denetleyiciler geliştirmektir. Modellemedeki doğruluk problemlerini aşmak ve daha sonra karşılaştırabilmek için iki farklı modelleme yöntemi kullanılmıştır. Bunlar sırasıyla, derecesi azaltılmış-yaklaşık FEM (Sonlu Elemanlar Yöntemi) ile eksiksiz tam bir modellemeye yatkın PDE (Kısmi Türevsel Denklemler) yöntemleridir. FEM yöntemi titreşim kiplerinin kırılmasını (modal truncation) gerektirdiği için bilinmeyen bozanetkenler, doğrusal olmayan fiziksel sistemin üretebileceği, ancak ihmal edilen titreşimin yüksek frekanslı kiplerini uyarabilir. İkinci yaklaşım ise esnek robot kollarını tam olarak modelleyebilmek için PDE yöntemini kullanmaktır. Esnek linklerin hafiflik, düşük eylemsizlik momenti ve yüksek öz frekans gibi önemli özelliklerini geliştirmek için, optimal biçim tasarımı sonucu kesit alanları birbiriyle uyumsuz olmayan linkler kullanılabilir. Ayrıca, esnek robot kolunun hızlı hareketinin ciddi titreşim sorunlarına neden olmaması ve kararlı uç-yük denetimi için gerekli olan büyük bant genişliğini; yüksek öz frekans sağlar.

Bu çalışmanın önemli sonuçlarını, esnek robot kolunun katı kiplerinin dayanıklı denetimi ve esnek titreşimlerinin sönmülendirilmesi olarak özetleyebiliriz. FEM yaklaşımında bu amacı gerçekleştirmek için dinamik durum geribeslemeli denetleyici kullanılmıştır. Araştırmanın ilk bölümünde, esnek kolların ihmal edilen yüksek frekanslı kipleri yanısıra uç-yük değişikliği gibi tamamıyla bilinmeyen bozanetkenlere rağmen eniyi denetim için Uyarlanır Dahili Model (UDM) yaklaşımı değiştirilerek uygulanmıştır. Ayrıca, kapalı-döngü sistemin kararlılığını sağlamak için, UDM ile paralel çalışan dayanıklı bir denetleyici de ek olarak tasarlanmıştır. Bu paralel denetleyici, UDM ile ilk kez bu çalışmada birlikte kullanılmış olup, sistemin doğrusal olmayan yapısı

da göz önüne alınarak, yeni ve verimli bir evrimsel algoritma yardımıyla başarılı bir şekilde eniyileştirilmiştir. Araştırmanın ikinci bölümünde ise (PDE), tasarım gereği düzgün olmayan kesit alanlı, iki eklemlili esnek robot kolunun denetimi, Lyapunov yöntemi kullanılarak geliştirilmiştir. Daha önce yayınlanmış diğer yaklaşımların aksine, titreşim kiplerinin kırılmasını (modal truncation) yapmaksızın, kapalı-döngü sistemin sonuçta kararlılığının kanıtında, sonsuz boyutlu sistemler için LaSalle değişmezlik ilkesi kullanılmıştır. Bununla birlikte, uç-yük yada omuz kütlesi değişmesi gibi büyük parametre değişiklikleri, doğrusal olmayan PDE denetleyici tarafından başarıyla dayanıklı bir şekilde dengelenmiştir.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iv
ABSTRACT	v
ÖZET	vii
LIST OF FIGURES	xi
LIST OF TABLES	xv
LIST OF SYMBOLS/ABBREVIATIONS	xvi
1. INTRODUCTION	1
1.1. Problem Definition	1
1.2. Objective	1
1.3. Methodology	1
2. ANALYTICAL MODEL WITH FEM	4
2.1. Calculation of Kinetic and Potential Energies of Elements	5
2.2. Total Energy of the Links for FEM Model	11
2.3. Equations of Motion for FEM Model	12
3. ANALYTICAL MODEL WITH PDE	15
3.1. Kinetic and Potential Energies for PDE Model	17
3.2. Equations of Motion for PDE Model	18
4. CONTROLLER DESIGN	20
4.1. Controller for Non-adaptive Case-FEM Model	20
4.2. Error-Feedback Controller for Adaptive Case-FEM Model	24
4.2.1. Adaptive Internal Model	24
4.2.2. Robust Stabilizer	29
4.3. PDE Controller Design	30
5. SIMULATION STUDIES	39
5.1. Non-Adaptive Case-FEM Model	39
5.2. Adaptive Case-FEM Model	48
5.3. PDE Approach	53
5.3.1. PDE Model Case 1	53
5.3.2. PDE Model Case 2	59

5.3.3. PDE Model Case 3	60
5.3.4. Validation of Lyapunov Theory	67
6. CONCLUSIONS	73
APPENDIX A: Numerical Examples for FEM	75
APPENDIX B: Derivation of Motion Equations for PDE	79
APPENDIX C: Proof of Decoupling for PDE	83
APPENDIX D: Derivation of Total Energy Rate for PDE	86
REFERENCES	91

LIST OF FIGURES

Figure 2.1.	Arm configuration for FEM Model	4
Figure 3.1.	Arm configuration for PDE Model	15
Figure 4.1.	Dynamic State Feedback Control Scheme - Inner Controlled System	21
Figure 4.2.	Error and Disturbance Dynamics	25
Figure 5.1.	Tip Mass Variation	40
Figure 5.2.	Θ_1 - Non-Adaptive Case	41
Figure 5.3.	Θ_2 - Non-Adaptive Case	41
Figure 5.4.	First Mode of Link 1- Non-Adaptive Case	42
Figure 5.5.	First Mode of Link 2- Non-Adaptive Case	42
Figure 5.6.	Control on Θ_1 - Non-Adaptive Case	43
Figure 5.7.	Control on Θ_2 - Non-Adaptive Case	43
Figure 5.8.	Θ_1 - Adaptive Case 1	44
Figure 5.9.	Θ_2 - Adaptive Case 1	44
Figure 5.10.	Error ($\dot{\Theta}_1$)- Adaptive Case 1	45
Figure 5.11.	Error ($\dot{\Theta}_2$)- Adaptive Case 1	45

Figure 5.12. First Mode of Link 1-Adaptive Case 1	46
Figure 5.13. First Mode of Link 2-Adaptive Case 1	46
Figure 5.14. Control on Θ_1 - Adaptive Case 1	47
Figure 5.15. Control on Θ_2 - Adaptive Case 1	47
Figure 5.16. Θ_1 - Adaptive Case 2	50
Figure 5.17. Θ_2 - Adaptive Case 2	50
Figure 5.18. First Mode of Link 1-Adaptive Case 2	51
Figure 5.19. First Mode of Link 2-Adaptive Case 2	51
Figure 5.20. Control on Θ_1 -Adaptive Case 2	52
Figure 5.21. Control on Θ_2 -Adaptive Case 2	52
Figure 5.22. Bending strain at the end of the link 1-PDE Model Case 1	55
Figure 5.23. Bending strain at the end of the link 2-PDE Model Case 1	56
Figure 5.24. Joint Angle of the link 1-PDE Model Case 1	56
Figure 5.25. Joint Angle of the link 2-PDE Model Case 1	57
Figure 5.26. Joint Angular Velocity of the link 1-PDE Model Case 1	57
Figure 5.27. Joint Angular Velocity of the link 2-PDE Model Case 1	58

Figure 5.28. Control Torque of Joint 1-PDE Model Case 1	58
Figure 5.29. Control Torque of Joint 2-PDE Model Case 1	59
Figure 5.30. Bending strain at the end of the link 1-PDE Model Case 2	61
Figure 5.31. Bending strain at the end of the link 2-PDE Model Case 2	61
Figure 5.32. Joint Angle of the link 1-PDE Model Case 2	62
Figure 5.33. Joint Angle of the link 2-PDE Model Case 2	62
Figure 5.34. Joint Angular Velocity of the link 1-PDE Model Case 2	63
Figure 5.35. Joint Angular Velocity of the link 2-PDE Model Case 2	63
Figure 5.36. Control Torque of Joint 1-PDE Model Case 2	64
Figure 5.37. Control Torque of Joint 2-PDE Model Case 2	64
Figure 5.38. Error for Joint Angle of the link 1-PDE Model Case 3	65
Figure 5.39. Error for Joint Angle of the link 2-PDE Model Case 3	65
Figure 5.40. Error for Bending Strain at end of the link 1-PDE Model Case 3	66
Figure 5.41. Error for Bending Strain at end of the link 2-PDE Model Case 3	66
Figure 5.42. Lyapunov function V for $K = 9$ - PDE Model Case 1	68
Figure 5.43. Time Derivative of Lyapunov function $\dot{V}$ for $K = 9$ - PDE Case 1	68

Figure 5.44. Approximate $\dot{V}$ for $K = 9$ - PDE Model Case 1	69
Figure 5.45. Approximation Error on $\dot{V}$ for $K = 9$ - PDE Model Case 1	69
Figure 5.46. Lyapunov function V for $K = 21$ - PDE Model Case 2	70
Figure 5.47. Enlarged Lyapunov function V for $K = 21$ - PDE Case 2	70
Figure 5.48. Time Derivative of Lyapunov function $\dot{V}$ for $K = 21$ - PDE Case 2	71
Figure 5.49. Approximate $\dot{V}$ for $K = 21$ - PDE Model Case 2	71
Figure 5.50. Approximation Error on $\dot{V}$ for $K = 21$ - PDE Model Case 2	72

LIST OF TABLES

Table 3.1.	Parameters for PDE model	16
Table 5.1.	Parameters of the Flexible Arm for FEM model.	39
Table 5.2.	Additional parameters for the adaptive cases for FEM model. . . .	40
Table 5.3.	Parameters of the Flexible Arm for PDE Case 1.	54
Table 5.4.	Tip and Hub Mass Variation for PDE Case 2.	60

LIST OF SYMBOLS/ABBREVIATIONS

E	Young's modulus
$\mathbf{C}$	Coriolis and Centrifugal matrix
$\mathbf{K}$	Stiffness matrix
$\mathbf{M}$	Mass matrix
$\mathbf{p}$	Generalized nodal coordinates vector
$\mathbf{q}$	Modal coordinates vector
T	Kinetic energy
U	Potential energy
$\mathbf{u}$	Control vector
V	Lyapunov function
V_s	Strain potential energy
W_{nc}	Non-conservative work
w	Flexural deflection
$\mathbf{y}$	Output vector
ρ	Density
τ	Input torque
θ_i	Angular position of the i^{th} joint
AIM	Adaptive Internal Model
BB-BC	Big Bang-Big Crunch evolutionary algorithm
FEM	Finite Element Method
PDE	Partial Differential Equations

1. INTRODUCTION

1.1. Problem Definition

Two main advantages of flexible robot arms are less weight and low energy consumption. However, the structural modelling and the control design of the flexible arm are much more complicated due to nonlinear coupling between elastic and rigid modes during the complex maneuvers especially with high angular velocities. The control of highly nonlinear multi-link flexible arms is subject to uncertainties caused by backlash, payload changes and external disturbances. Therefore, adaptive and robust control of multi-link flexible arms is a challenging problem.

1.2. Objective

Developing nonlinear adaptive and robust controllers for a two-link flexible robot arm is the main objective of this research. The dynamic state feedback controller is used to achieve robust regulation of the rigid modes as well as suppression of elastic vibrations. In this research, the internal model approach is adaptively tuned up for unknown disturbances, parallel with a robust stabilizer. The stabilizer part of the controller is optimized with a new evolutionary algorithm. Two different modelling techniques will be used in this work to overcome model accuracy problems and these two will be evaluated comparatively. These are FEM(Finite Element Method) as a reduced order-approximate model and PDE(Partial Differential Equations) approach as an exact model.

1.3. Methodology

Various methods have been proposed for control of flexible-link manipulators in the literature. Hybrid control of single flexible-link manipulator using feedback linearization and singular perturbation approach has been used in [1]. Adaptive feedback linearization has been applied successfully for nonlinear discrete-time model of a

single-link flexible manipulator [2]. Singular perturbation theory has also been used for position and force control in [3]. Strain feedback and active vibration control [4, 5] are other approaches different from the integrated structure-control in [6]. The feedback linearizable nonlinear systems that are subject to periodic disturbances are examined either for known disturbances [7] or for unknown disturbances [8]. In [7], the known disturbances are assumed not to excite zero-dynamics which is the dynamic system to constrain the output to remain identically zero for specific input and initial conditions. In [8], unknown disturbances included can excite high-frequency modes generated by the nonlinearities of the plant. In the first part of this research, adaptive internal model approach [8] with FEM, has been modified to manage totally unknown disturbances that can include neglected higher modes of the flexible links and also large parameter uncertainties such as tip mass changes.

Asymptotic tracking in control theory can be achieved essentially by three methods: tracking by dynamic inversion, adaptive tracking, tracking via internal models. Dynamic inversion requires exact knowledge of an initial state and a reference trajectory such that if the system is accordingly initialized and driven, it exactly follows the reference signal [9]. However, the computation of such control input requires perfect knowledge of the entire trajectory which is to be followed as well as perfect knowledge of the model of the plant. Thus, this method cannot be applied in the presence of large uncertainties on plant parameters as well as on the reference trajectory. Adaptive tracking can be expressed as tuning the controller parameters, in order to provide asymptotic convergence to zero of a tracking error. This approach can successfully manage parameter uncertainties, but still needs perfect knowledge of the entire trajectory which is to be followed. Therefore, it is not suitable for tracking uncertain trajectories. On the other hand, internal model based tracking can handle simultaneously parameter uncertainties as well as tracking unknown reference trajectories or rejecting unknown disturbances.

The second approach is modelling the flexible robot arms by PDE which are known to provide exactness. In order to improve the important features of flexible links such that low mass and moments of inertia and high natural frequencies [6, 10],

optimal shape design can be investigated. Furthermore, a high fundamental frequency is desired since it implies a large bandwidth that will allow for fast motion without causing serious vibration problems and stable endpoint control [11]. In this research, inspired by the last three approaches in [6, 10, 11], the control of a two-link flexible arm with variable cross-section by design is improved by employing the Lyapunov method. Different from the energy based multi-link flexible robot control proposed in [12, ch.2], LaSalle's invariance principle, extended for infinite dimension [13], will be used in order to prove the asymptotic stability of the closed loop system without any modal truncation. For the same PDE model which is a nonlinear distributed parameter system, boundary control [14, 15] can be improved as infinite dimensional extension of the adaptive internal model which is applied in the first approach rigorously.

The presentation of the research results is divided into six chapters. In chapters 3 and 4, analytical models of flexible robot arm and equations of motion are derived in detail for FEM and PDE approaches respectively. In chapter 5, nonlinear robust controllers are designed for non-adaptive and adaptive cases. Adaptive internal model has been introduced in parallel with a robust optimal stabilizer which is the main contribution for FEM approach. Infinite-dimensional, Lyapunov-based, nonlinear PDE controller for two nonuniform flexible links is developed, being the first in the literature. In chapter 6, simulation studies are presented which validate the proposed controller designs and demonstrate the necessity of the key assumptions. The final chapter summarizes the main conclusions of the research.

2. ANALYTICAL MODEL WITH FEM

A mathematical model of flexible-link arms can be obtained by either assumed modes [16] or finite element methods [6]. Based on the results presented in [17], in this research the finite element method has been used, since it gives more accurate results [18] and lends itself ideally to dynamic real-time controller implementations.

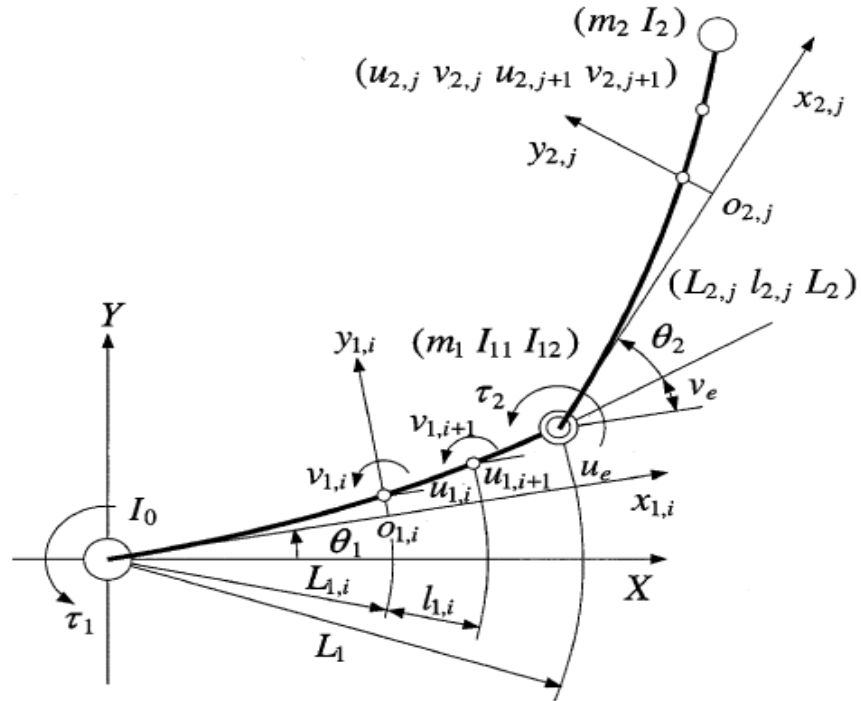


Figure 2.1. Arm configuration [6]

Since the ratio between the length of the beam and its thickness is sufficiently large such that $200 \gg 10$ as proposed in [19, 20], links can be modelled as Euler-Bernoulli beams, which can only be deformed in the flexural direction. Coulomb and viscous friction are also included. The links are modelled in clamped-free configuration, since natural modes of the separated clamped-free links agree very well with actual ones compared to pinned-free configuration [21].

Referring to Figure 2.1, OXY : global inertial system of coordinates; $o_{1,i}x_{1,i}y_{1,i}$: body-fixed system of coordinates attached to the element i of undeformed link 1, $o_{2,i}x_{2,i}y_{2,i}$: body-fixed system of coordinates attached to the element i of undeformed link 2, L_1, L_2 total lengths of links 1 and 2, $L_{1,i}$: accumulated length of first $(i - 1)$ elements of link 1, $L_{2,j}$: accumulated length of first $(j - 1)$ elements of link 2, $l_{1,i}$ and $l_{2,j}$: lengths of element i of link 1 and element j of link 2, θ_1, θ_2 : angular displacements of links 1 and 2, $(u_{1,i}, v_{1,i})$: flexural displacement and slope of node i of link 1, $(u_{2,j}, v_{2,j})$: flexural displacement and slope of node j of link 2, (u_e, v_e) : flexural displacement and slope at the end of link 1; I_0, I_2 : moments of inertia of joint 1 and tip; I_{11}, I_{12} : moments of inertia of joint 2 rotating with links 1 and 2; m_2 : tip mass; τ_1, τ_2 : motor torques at joints 1 and 2, C_f : coefficient of Coulomb friction, V_f : coefficient of viscous friction [6].

2.1. Calculation of Kinetic and Potential Energies of Elements

Each flexible link is represented by a finite number of Euler-Bernoulli beam elements. Derivation of energy terms for the finite element model of link 1 is briefly explained below [6]. Referring to Figure 2.1, the global position vector $\mathbf{r}_1$ shows an arbitrary point P_1 in body-fixed system of coordinates attached to element i of undeformed link 1 such that

$$\mathbf{r}_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{Bmatrix} L_{1,i} + x_{1,i} \\ y_{1,i} \end{Bmatrix} \quad (2.1)$$

where $(x_{1,i}, y_{1,i})$ are the local coordinates of P_1 . Then the kinetic energy of the element can be expressed as follows

$$T_{1,i} = \frac{1}{2} \int_0^{l_{1,i}} \rho A \left[\frac{\partial \mathbf{r}_1}{\partial t} \right]^T \cdot \frac{\partial \mathbf{r}_1}{\partial t} dx_{1,i} \quad (2.2)$$

where ρ is mass density and A is area of the uniform cross-section of links. The standard shape functions for the beam element are defined as [22]

$$\Omega_1^T = \begin{pmatrix} 0 \\ 1 - 3x_{1,i}^2/l_{1,i}^2 + 2x_{1,i}^3/l_{1,i}^3 \\ x_{1,i} - 2x_{1,i}^2/l_{1,i} + x_{1,i}^3/l_{1,i}^2 \\ 3x_{1,i}^2/l_{1,i}^2 - 2x_{1,i}^3/l_{1,i}^3 \\ -x_{1,i}^2/l_{1,i} + x_{1,i}^3/l_{1,i}^2 \end{pmatrix} \quad (2.3)$$

where each of the four shape functions that are well-known *Hermite cubics* interpolation functions, physically represents the deflection curve for the beam element produced by setting the corresponding degree of freedom to be one and the rest degrees of freedom to be zero [22]. The deflection $y_{1,i}$ in equation (2.1) can be approximated using lateral displacements ($u_{1,i}$) and slopes ($v_{1,i}$) of the nodes

$$y_{1,i} = \Omega_1 \cdot \mathbf{z}_{1,i}$$

where

$$\mathbf{z}_{1,i} = (\theta_1 \quad u_{1,i} \quad v_{1,i} \quad u_{1,i+1} \quad v_{1,i+1})^T$$

is the vector of nodal coordinates of element i . Therefore, $\mathbf{r}_1$ can be uniquely determined according to the values of vector $\mathbf{z}_{1,i}$ and the element kinetic energy in equation 2.2 becomes

$$T_{1,i} = \frac{1}{2} \dot{\mathbf{z}}_{1,i}^T M_{1,i} \dot{\mathbf{z}}_{1,i} \quad (2.4)$$

where $M_{1,i}$ is the 5×5 state-dependent element mass matrix. Then

$$M_{1,i} = \rho A \int_0^{l_{1,i}} \left[\frac{\partial \mathbf{r}_1}{\partial \mathbf{z}_{1,i}} \right]^T \cdot \frac{\partial \mathbf{r}_1}{\partial \mathbf{z}_{1,i}} dx_{1,i} \quad (2.5)$$

where

$$\frac{\partial \mathbf{r}_1}{\partial \mathbf{z}_{1,i}} = \begin{bmatrix} \frac{\partial \mathbf{r}_1}{\partial \theta_1} & \frac{\partial \mathbf{r}_1}{\partial u_{1,i}} & \frac{\partial \mathbf{r}_1}{\partial v_{1,i}} & \frac{\partial \mathbf{r}_1}{\partial u_{1,i+1}} & \frac{\partial \mathbf{r}_1}{\partial v_{1,i+1}} \end{bmatrix}$$

The 5×5 state and time-dependent symmetric mass matrix $M_{1,i}$ is calculated by symbolic derivation program. For details see Appendix A.

Assuming the manipulator can only rotate in the horizontal plane, the potential

energy of the element depends only on the flexural deflections. In the absence of gravity potential energy can be expressed as

$$U_{1,i} = \frac{1}{2} \mathbf{z}_{1,i}^T \begin{bmatrix} 0 & 0 \\ 0 & K_{1,i} \end{bmatrix} \mathbf{z}_{1,i}$$

where $K_{1,i}$ is the 4×4 basic beam element stiffness matrix for link 1, which can be expressed as [22]

$$K_{1,i} = EI_1 \begin{bmatrix} 12/l_{1,i}^3 & 6/l_{1,i}^2 & -12/l_{1,i}^3 & 6/l_{1,i}^2 \\ 6/l_{1,i}^2 & 4/l_{1,i} & -6/l_{1,i}^2 & 2/l_{1,i} \\ -12/l_{1,i}^3 & -6/l_{1,i}^2 & 12/l_{1,i}^3 & -6/l_{1,i}^2 \\ 6/l_{1,i}^2 & 2/l_{1,i} & -6/l_{1,i}^2 & 4/l_{1,i} \end{bmatrix}$$

where E is Young's Modulus and $I_1 = \frac{1}{12}bh^3$ (b is the width of the beam and h is the thickness of the beam) is the area moment of inertia for link 1.

Derivation of energy terms for the finite element model of link 2 is briefly explained below [6]. Referring to Figure 2.1, the global position vector $\mathbf{r}_2$ shows an arbitrary point P_2 in body-fixed system of coordinates attached to element j of undeformed link 2 such that

$$\begin{aligned} \mathbf{r}_2 = & \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \left(\begin{Bmatrix} L_1 \\ u_e \end{Bmatrix} \right) \\ & + \begin{bmatrix} \cos(\theta_2 + v_e) & -\sin(\theta_2 + v_e) \\ \sin(\theta_2 + v_e) & \cos(\theta_2 + v_e) \end{bmatrix} \left\{ \begin{matrix} L_{2,j} + x_{2,j} \\ y_{2,j} \end{matrix} \right\} \end{aligned} \quad (2.6)$$

where $(x_{2,j}, y_{2,j})$ are the local coordinates of P_2 . Then the kinetic energy of the element can be expressed as follows

$$T_{2,j} = \frac{1}{2} \int_0^{l_{2,j}} \rho A \left[\frac{\partial \mathbf{r}_2}{\partial t} \right]^T \cdot \frac{\partial \mathbf{r}_2}{\partial t} dx_{2,j} \quad (2.7)$$

where ρ is mass density and A is area of the uniform cross-section of links. The standard shape functions for the beam element are defined as [22]

$$\Omega_2^T = \left\{ \begin{array}{c} 0_{4 \times 1} \\ 1 - 3x_{2,j}^2/l_{2,j}^2 + 2x_{2,j}^3/l_{2,j}^3 \\ x_{2,j} - 2x_{2,j}^2/l_{2,j} + x_{2,j}^3/l_{2,j}^2 \\ 3x_{2,j}^2/l_{2,j}^2 - 2x_{2,j}^3/l_{2,j}^3 \\ -x_{2,j}^2/l_{2,j} + x_{2,j}^3/l_{2,j}^2 \end{array} \right\} \quad (2.8)$$

The deflection $y_{2,j}$ in equation (2.6) can be approximated using lateral displacements $(u_{2,j})$ and slopes $(v_{2,j})$ of the nodes

$$y_{2,j} = \Omega_2 \cdot \mathbf{z}_{2,j}$$

where

$$\mathbf{z}_{2,j} = (\theta_1 \quad u_e \quad v_e \quad \theta_2 \quad u_{2,j} \quad v_{2,j} \quad u_{2,j+1} \quad v_{2,j+1})^T$$

Therefore, $\mathbf{r}_2$ can be uniquely determined according to the values of vector $\mathbf{z}_{2,j}$ and the element kinetic energy in equation 2.7 becomes

$$T_{2,j} = \frac{1}{2} \dot{\mathbf{z}}_{2,j}^T M_{2,j} \dot{\mathbf{z}}_{2,j} \quad (2.9)$$

where $M_{2,j}$ is the 8×8 state-dependent element mass matrix. Then

$$M_{2,j} = \rho A \int_0^{l_{2,j}} \left[\frac{\partial \mathbf{r}_2}{\partial \mathbf{z}_{2,j}} \right]^T \cdot \frac{\partial \mathbf{r}_2}{\partial \mathbf{z}_{2,j}} dx_{2,j} \quad (2.10)$$

where

$$\frac{\partial \mathbf{r}_2}{\partial \mathbf{z}_{2,j}} = \begin{bmatrix} \frac{\partial \mathbf{r}_2}{\partial \theta_1} & \frac{\partial \mathbf{r}_2}{\partial u_e} & \frac{\partial \mathbf{r}_2}{\partial v_e} & \frac{\partial \mathbf{r}_2}{\partial \theta_2} & \frac{\partial \mathbf{r}_2}{\partial u_{2,j}} & \frac{\partial \mathbf{r}_2}{\partial v_{2,j}} & \frac{\partial \mathbf{r}_2}{\partial u_{2,j+1}} & \frac{\partial \mathbf{r}_2}{\partial v_{2,j+1}} \end{bmatrix}$$

The 8×8 state and time-dependent mass matrix $M_{2,j}$ is calculated by symbolic derivation program. For numerical examples see Appendix A.

The potential energy of the element j of link 2 can be expressed in a similar way as

$$U_{2,j} = \frac{1}{2} \mathbf{z}_{2,j}^T \begin{bmatrix} 0_{4 \times 4} & 0 \\ 0 & K_{2,j} \end{bmatrix} \mathbf{z}_{2,j}$$

where $K_{2,j}$ is the 4×4 basic beam element stiffness matrix for link 2 such as [22]

$$K_{2,j} = EI_2 \begin{bmatrix} 12/l_{2,j}^3 & 6/l_{2,j}^2 & -12/l_{2,j}^3 & 6/l_{2,j}^2 \\ 6/l_{2,j}^2 & 4/l_{2,j} & -6/l_{2,j}^2 & 2/l_{2,j} \\ -12/l_{2,j}^3 & -6/l_{2,j}^2 & 12/l_{2,j}^3 & -6/l_{2,j}^2 \\ 6/l_{2,j}^2 & 2/l_{2,j} & -6/l_{2,j}^2 & 4/l_{2,j} \end{bmatrix}$$

where E is Young's Modulus and I_2 is the area moment of inertia for link 2.

2.2. Total Energy of the Links for FEM Model

After assembling all components of both links the total kinetic and potential energies can be calculated as [22]

$$\begin{aligned} T &= \sum_{i=1}^{N_1} T_{1,i} + \sum_{j=1}^{N_2} T_{2,j} + T_{Hub} + T_{TipMass} \\ U &= \sum_{i=1}^{N_1} U_{1,i} + \sum_{j=1}^{N_2} U_{2,j} \end{aligned}$$

where N_1 and N_2 denote the number of finite elements in the links 1 and 2, respectively. For simplicity the kinetic energy terms related to the rotation and translation of the joint between links 1 and 2 are omitted.

The kinetic energies of the hub and tip mass are given by

$$T_{Hub} = \frac{1}{2} I_0 \dot{\theta}_1^2, \quad T_{TipMass} = \frac{1}{2} m_2 \dot{r}_t^2$$

where I_0 is the hub inertia and $\dot{r}_t$ is the tip velocity of link 2.

Taking into account the configuration of the links, the following boundary conditions are imposed on the nodal displacements

$$u_{1,1} = v_{1,1} = u_{2,1} = v_{2,1} = 0$$

In view of these boundary conditions, the vector of generalized coordinates or nodal coordinates can be expressed as

$$\begin{aligned} \mathbf{p} &= (\theta_1 \quad \theta_2 \quad \mathbf{p}_1^T \quad \mathbf{p}_2^T)^T \\ \mathbf{p}_1 &= (u_{1,2} \quad v_{1,2} \quad \cdots \quad u_{1,N_1} \quad v_{1,N_1} \quad u_e \quad v_e)^T \\ \mathbf{p}_2 &= (u_{2,2} \quad v_{2,2} \quad \cdots \quad u_{2,N_2} \quad v_{2,N_2} \quad u_{2,N_2+1} \quad v_{2,N_2+1})^T \end{aligned}$$

The total kinetic and potential energies can be written in terms of the vector $\mathbf{p}$ as follows,

$$T = \frac{1}{2} \dot{\mathbf{p}}^T M(\mathbf{p}) \dot{\mathbf{p}} \quad U = \frac{1}{2} \mathbf{p}^T K \mathbf{p}$$

where M and K are the overall mass and stiffness matrices, respectively, here M depends on $\mathbf{p}$. Both matrices that have the same dimension as $(2(N_1 + N_2) + 2) \times (2(N_1 + N_2) + 2)$, are symmetric and positive definite as well [19, 20].

2.3. Equations of Motion for FEM Model

The dynamic equations of motion can be obtained by employing the Lagrange's equations as follows

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\mathbf{p}}} \right] - \frac{\partial L}{\partial \mathbf{p}} = \mathbf{F}_{ext}$$

where $L = T - U$ is the Lagrangian and $\mathbf{F}_{ext}$ is the vector of external generalized forces. Performing the above calculations, the system dynamics can be expressed as

$$M(\mathbf{p})\ddot{\mathbf{p}} + C(\mathbf{p}, \dot{\mathbf{p}})\dot{\mathbf{p}} + K\mathbf{p} = \begin{Bmatrix} u_1 \\ u_2 \\ 0_{2(N_1+N_2) \times 1} \end{Bmatrix} = b\mathbf{u}$$

where $\mathbf{u} = (u_1 \ u_2)^T$ represents actuation and friction torques. Also $C(\mathbf{p}, \dot{\mathbf{p}})$ that represents coriolis and centrifugal effects, is a term obtained during differentiation of the nonlinear part of $M(\mathbf{p})$. In this work the equations of motion are generated by a general computer program implemented in MATLAB using Symbolic Toolbox.

Because of the high computational cost for simulating the system of equations and the complexity of the dynamic model, modal truncation is needed to select lower frequency modes of the flexible links. The essence of modal truncation is to neglect higher frequency components in the vibration. Since the lowest resonant frequencies of the neglected higher modes are almost tenfold of the retained first modes in the present work, and their corresponding amplitudes about $\frac{1}{100}$ of the amplitude of the first modes [23, ch. 4], therefore their contribution to the vibration can be negligible [19]. To proceed, transformation matrices have to be constructed from the linear parts of the mass and stiffness matrices in order to transform the nodal coordinates into modal coordinates [24]. Then

$$\mathbf{p}_1 = \varphi_1 \cdot \hat{\mathbf{q}}_1 \quad \mathbf{p}_2 = \varphi_2 \cdot \hat{\mathbf{q}}_2$$

where φ_1 and φ_2 are $2N_1 \times M_1$ and $2N_2 \times M_2$ transformation matrices, M_1 and M_2 denote the number of modes, $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ are the modal displacements of links 1 and 2, respectively. After transformation the equations of motions can be expressed as

$$\hat{M}(\hat{\mathbf{q}})\ddot{\hat{\mathbf{q}}} + \hat{C}(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}})\dot{\hat{\mathbf{q}}} + \hat{K}\hat{\mathbf{q}} = \hat{\mathbf{b}}\mathbf{u} \quad (2.11)$$

where $\mathbf{u} = (u_1 \ u_2)^T$, and $\hat{\mathbf{q}}$ is the vector of overall modal displacements

$$\hat{\mathbf{q}} = (\theta_1 \ \theta_2 \ \hat{\mathbf{q}}_1^T \ \hat{\mathbf{q}}_2^T)^T$$

In the present work the transformation is carried automatically by the symbolic derivation program and equations of motion are obtained readily in the form suitable for analysis and simulation, see Appendix A. For simulation and design of the controller only the first modes of both links are selected such that

$$\hat{\mathbf{q}} = (q_1 \ q_2 \ q_3 \ q_4 \ q_5)^T$$

where $q_1 = \theta_1 - \theta_{1d}$ and $q_2 = \theta_2 - \theta_{2d}$ are errors with respect to desired goal positions, q_3 and q_5 are the first modes of links 1 and 2, respectively, and $q_4 = v_e$, the slope at the end of link 1, which had to be retained during transformation, since it enters the equations of motion nonlinearly.

3. ANALYTICAL MODEL WITH PDE

In order to improve the important features of flexible links such that low mass and moments of inertia and high natural frequencies [6, 10], optimal shape design can be investigated. Furthermore, a high fundamental frequency is desired since it implies a large bandwidth that will allow for fast motion without causing serious vibration problems and stable endpoint control [11].

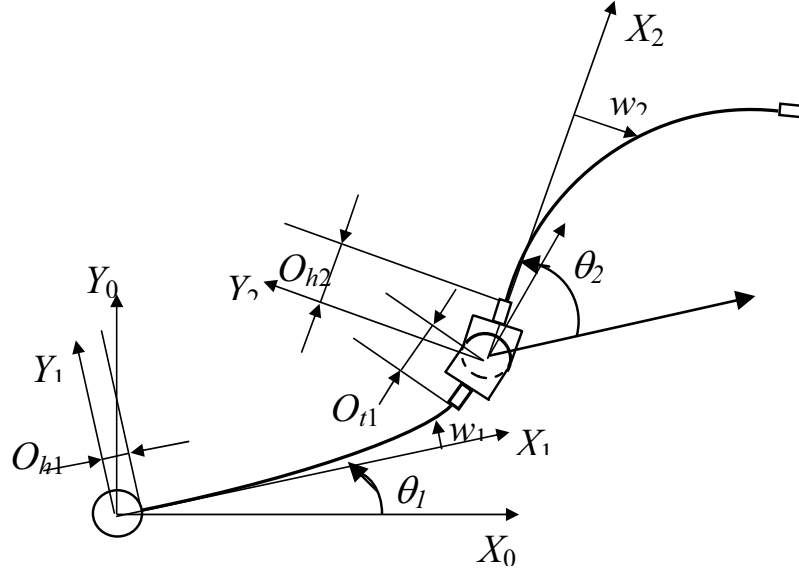


Figure 3.1. Arm configuration [4]

In this research, inspired by the last three approaches in [6, 10, 11], the control of a two-link flexible arm with variable cross-section by design is improved by employing the Lyapunov method. Different from the energy based multi-link flexible robot control proposed in [12, ch.2], LaSalle's invariance principle extended for infinite dimension [13] will be used in order to prove the asymptotic stability of the closed loop system without any modal truncation such that the higher order modes will not be ignored.

Referring to Figure 3.1 the various symbols represent the following; X_oY_o : global inertial system of coordinates; X_1Y_1 : body-fixed system of coordinates attached to undeformed link 1, X_2Y_2 : body-fixed system of coordinates attached to undeformed link 2, θ_1, θ_2 : angular displacements of links 1 and 2, w_1, w_2 : flexural displacements of links 1 and 2, O_{hi} : offset of the beam root to the center of the i^{th} input torque motor, O_{ti} : offset of the beam end to the center of tip mass [4].

Parameter	Description
E	Young's Modulus
$I_i(x_i)$	Variable beam cross section moment to z axis at the location x_i
I_{hi}	Inertia of i^{th} hub
I_{ti}	Tip inertia of i^{th} beam
l_i	Length of i^{th} link
m_{hi}	Mass of i^{th} hub
m_{ti}	Tip mass of i^{th} beam
x_i	Coordinate along the axial center of the i^{th} beam
$w_i(x_i, t)$	Transverse movement of point i at the location x_i of i^{th} beam
$\dot{w}_i(x_i, t)$	Time rate of transverse movement of point i at the location x_i
$w_{ix}(x_i, t)$	Axial rate of transverse movement of point i at the location x_i
$\rho_i(x_i)$	Variable density of the i^{th} link depends on the cross-sectional area
τ_i	Input torque at i^{th} motor
θ_1	Angular position of the first motor
$\hat{\theta}_2$	Angular position of the second motor
θ_2	$\theta_2 = \hat{\theta}_2 + w_{1x}(l_1, t)$

Table 3.1. Parameters for PDE model [4].

In this chapter, an exact model using partial differential equations with boundary conditions is derived using the Hamilton's Principle based on Zhang's work for uniform links [25]. Since the ratio between the length of the beam and its thickness is sufficiently large as proposed in [19, 20], links can be modelled as Euler-Bernoulli beams, which can only be deformed in the flexural direction. The links are modelled

in clamped-free configuration, since natural modes of the separated clamped-free links agree very well with actual ones compared to pinned-free configuration [21]. Assuming the manipulator moves in the horizontal plane, in the absence of gravity the potential energy depends only on the flexural deflections.

3.1. Kinetic and Potential Energies for PDE Model

In order to derive the PDE model, kinetic and potential energy expressions for the robot arm are given below [25], with the notation given in Table 3.1:

$$\begin{aligned}
T_1 &= \frac{1}{2} \int_0^{l_1} \rho_1(x_1) \left[x_1 \dot{\theta}_1 + \dot{w}_1 \right]^2 dx_1 \\
T_2 &= \frac{1}{2} \int_0^{l_2} \rho_2(x_2) \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \cos(\theta_2) + x_2 \dot{\theta}_2 + \dot{w}_2 \right\}^2 dx_2 \\
&\quad + \frac{1}{2} \int_0^{l_2} \rho_2(x_2) \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \sin(\theta_2) \right\}^2 dx_2 \\
T_3 &= \frac{1}{2} I_{h1} \dot{\theta}_1^2 \\
T_4 &= \frac{1}{2} I_{t1} \left(\dot{\theta}_1 + \dot{w}_{1x}(l_1, t) \right)^2 + \frac{1}{2} m_{t1} \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right]^2 \\
T_5 &= \frac{1}{2} I_{h2} \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 + \frac{1}{2} m_{h2} \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right]^2 \\
T_6 &= \frac{1}{2} I_{t2} \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{w}_{2x}(l_2, t) \right)^2 + \frac{1}{2} m_{t2} \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \sin(\theta_2) \right\}^2 \\
&\quad + \frac{1}{2} m_{t2} \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \cos(\theta_2) + l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t) \right\}^2
\end{aligned}$$

where T_1, T_2 are the kinetic energy of link 1 and 2 respectively; T_3 is the kinetic energy of Hub 1 or base joint; T_4 is the kinetic energy of tip mass at the end of link 1; T_5 is the kinetic energy of Hub 2; T_6 is the kinetic energy of tip mass at the end of link 2, and the total strain potential energy can be expressed as [25]

$$V_s = \frac{1}{2} \int_0^{l_1} EI_1(x_1) w_{1xx}^2 dx_1 + \frac{1}{2} \int_0^{l_2} EI_2(x_2) w_{2xx}^2 dx_2$$

where all offset values of the model (O_{hi}, O_{ti}) can be omitted by design.

3.2. Equations of Motion for PDE Model

The Extended Hamilton's Principle [19] is applied to obtain the governing equations such that

$$\int_{t_1}^{t_2} (\delta T - \delta V_s + \delta W_{nc}) dt = 0 \quad (3.1)$$

where $T = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$, and V_s are given in the previous section; δT and δV_s are the variation of *total* kinetic and potential energy respectively, δW_{nc} is the variation of non-conservative work which is the virtual work performed by the nonconservative forces e.g. control forces, the forces make the system dissipative [26, 16]; such as

$$\delta W_{nc} = \tau_1 \delta \theta_1 + \tau_2 \delta(\theta_2 - w_{1x}(l_1, t))$$

where τ_1, τ_2 represent the actuation torques for joint 1 and 2 respectively. Thus, equations for a two-link flexible arm with nonuniform cross-section are derived by the variational method and integration by parts, and are listed below with the notation given in Table 3.1. For details see Appendix B.

$$\ddot{w}_1 + \frac{(EI_1(x_1)w_{1xx})_{xx}}{\rho_1(x_1)} = -x_1 \ddot{\theta}_1 \quad (3.2)$$

$$\ddot{w}_2 + \frac{(EI_2(x_2)w_{2xx})_{xx}}{\rho_2(x_2)} + \cos(\theta_2) [\ddot{w}_1(l_1, t) + l_1 \ddot{\theta}_1] - \dot{\theta}_2 \sin(\theta_2) [\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1] = -x_2 \ddot{\theta}_2 \quad (3.3)$$

$$I_{h1} \ddot{\theta}_1 - EI_1(0)w_{1xx}(0, t) = \tau_1 \quad (3.4)$$

$$I_{t1}I_{h2} \ddot{\theta}_2 - I_{h2} EI_1(l_1)w_{1xx}(l_1, t) - I_{t1} EI_2(l_2)w_{2xx}(l_2, t) = (I_{t1} + I_{h2}) \tau_2 \quad (3.5)$$

$$\begin{aligned} & - \int_0^{l_2} \rho_2(x_2) x_2 dx_2 \dot{\theta}_2 \sin(\theta_2) [\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1] + \int_0^{l_2} x_2 (EI_2(x_2)w_{2xx})_{xx} dx_2 \\ & - I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] - I_{t2} [\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t)] - \sin(\theta_2) [\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1] \int_0^{l_2} \rho_2(x_2) \dot{w}_2 dx_2 \\ & - m_{t2} l_2 [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t) + \cos(\theta_2) [\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1]] - \\ & m_{t2} \sin(\theta_2) [\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1] \dot{w}_2(l_2, t) - I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] - EI_1(l_1)w_{1xx}(l_1, t) = 0 \end{aligned} \quad (3.6)$$

$$\begin{aligned}
& - \int_0^{l_2} \rho_2(x_2) \left\{ \left[\ddot{w}_1(l_1, t) + l_1 \ddot{\theta}_1 \right] + \cos(\theta_2) \left[\ddot{w}_2 + x_2 \ddot{\theta}_2 \right] - \dot{\theta}_2 \sin(\theta_2) \left[\dot{w}_2 + x_2 \dot{\theta}_2 \right] \right\} dx_2 \\
& - (m_{t1} + m_{h2} + m_{t2}) \left[\ddot{w}_1(l_1, t) + l_1 \ddot{\theta}_1 \right] + [(EI_1(x_1)w_{1xx})_x]_{x_1=l_1} \\
& - m_{t2} \left\{ \cos(\theta_2) \left[\ddot{w}_2(l_2, t) + l_2 \ddot{\theta}_2 \right] - \dot{\theta}_2 \sin(\theta_2) \left[\dot{w}_2(l_2, t) + l_2 \dot{\theta}_2 \right] \right\} = 0
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
& [(EI_2(x_2)w_{2xx})_x]_{x_2=l_2} \\
& - m_{t2} \left\{ \left[\ddot{w}_2(l_2, t) + l_2 \ddot{\theta}_2 \right] + \cos(\theta_2) \left[\ddot{w}_1(l_1, t) + l_1 \ddot{\theta}_1 \right] - \dot{\theta}_2 \sin(\theta_2) \left[\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1 \right] \right\} = 0
\end{aligned} \tag{3.8}$$

$$I_{t2} \left[\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t) \right] + EI_2(l_2)w_{2xx}(l_2, t) = 0 \tag{3.9}$$

$$w_1(0, t) = w_{1x}(0, t) = w_2(0, t) = w_{2x}(0, t) = 0 \tag{3.10}$$

where it is assumed that $\theta_2 \approx \hat{\theta}_2$ since the slope at the end of the first link ($w_{1x}(l_1, t)$) is relatively small. The equations (3.2) and (3.3) are the main PDEs(balance of forces) and the equations (3.4) and (3.5) are ODEs (conservation of momentum) for rigid coordinates. Equations (3.6 - 3.10) are the boundary conditions where equation (3.10) includes the four boundary conditions at the *clamped* end of the links [23]. We first define the following variables for $i = 1, 2$

$$\tau_{bi} = -EI_i(0)w_{ixx}(0, t) \tag{3.11}$$

$$\tau_{ei} = -EI_i(l_i)w_{ixx}(l_i, t) \tag{3.12}$$

where τ_{b1}, τ_{b2} represent the base strain measured torques and τ_{e1}, τ_{e2} represent the end strain measured torques for the first and second beams respectively. Boundary conditions (3.6), (3.8) and (3.9) can be simplified further to obtain the following one

$$\begin{aligned}
& \sin(\theta_2) \left[\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1 \right] \left\{ -\dot{\theta}_2 \int_0^{l_2} \rho_2(x_2) x_2 dx_2 \right. \\
& \left. - l_2 \dot{\theta}_2 m_{t2} - \int_0^{l_2} \rho_2(x_2) \dot{w}_2 dx_2 - m_{t2} \dot{w}_2(l_2, t) \right\} \\
& - \tau_{b2} + \tau_{e1} - I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] - I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] = 0
\end{aligned} \tag{3.13}$$

Consequently the new set of boundary conditions will be (3.13), (3.7), (3.8) and (3.10).

where $f_i(q)$ is a nonlinear function of the modal coordinates. See Appendix A for details. Since the open-loop system has not full relative degree and also nonminimum phase by its nature as explained in the sequel, the dynamic extension algorithm [27, ch. 5] has been applied twice as shown in Figure 4.1,

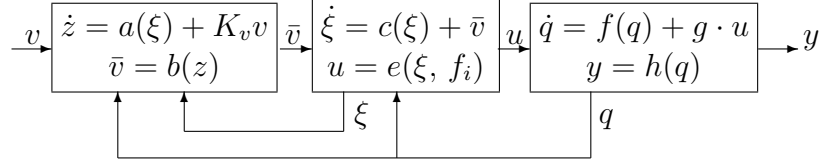


Figure 4.1. Dynamic State Feedback Control Scheme - Inner Controlled System

where $a(0) = b(0) = c(0) = d(0) = e(0, 0) = 0$, and $q(0) = 0_{10 \times 1}$, $z(0) = 0_{6 \times 1}$, $\xi(0) = d(z)$. Also $a, b, c, d \in \mathbf{R}^6$, and $e \in \mathbf{R}^2$ are linear vector functions in their arguments. Since the auxiliary states (z_i 's) are dummy, the dimension of the closed-loop system became 16. In order to obtain a linear and controllable system from the nonlinear system (4.1) and to follow the steps of exact feedback linearization as defined in [27], the relative degree of the system has to be determined such that

$$L_g h(\mathbf{q}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \forall \mathbf{q} \quad (4.3)$$

$$L_f h_1(\mathbf{q}) = q_6$$

$$L_f h_2(\mathbf{q}) = q_7$$

$$L_f^2 h_1(\mathbf{q}) = f_6$$

$$L_f^2 h_2(\mathbf{q}) = f_7$$

$$L_g L_f h(\mathbf{q}) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \quad (4.4)$$

where $L_{(\cdot)}(\cdot)$ denotes the Lie¹ derivative and the relative degree r is exactly equal to the number of times that the output $y(t)$ must be differentiated until the r^{th} order derivative of $y(t)$ depends explicitly on the input $u(t)$. Specifically, the multi-variable

¹The Lie derivatives are defined as follows: $L_f h(q) = \left[\frac{\partial h(q)}{\partial q} \right]^T \cdot f(q)$. The multiple Lie derivative yields: $L_f^k h(q) = \left[\frac{\partial L_f^{k-1} h(q)}{\partial q} \right]^T \cdot f(q)$

nonlinear system has vector relative degree $\{r_1, r_2\}$ due to the 2×1 output vector $\mathbf{y}$, if the $L_g L_f h(\mathbf{q})$ matrix of equation (4.4) which is expressed symbolically in equation (4.5) as well, must be nonsingular while achieving the $L_g h(\mathbf{q})$ matrix of equation (4.3) to be zero matrix [27, ch.5].

$$L_g L_f h(\mathbf{q}) = \begin{bmatrix} L_{g1} L_f^{r_1-1} h_1 & L_{g2} L_f^{r_1-1} h_1 \\ L_{g1} L_f^{r_2-1} h_2 & L_{g2} L_f^{r_2-1} h_2 \end{bmatrix} \quad (4.5)$$

Observe that the upper first two rows of the transpose of the modal transformation matrix φ^T that is given in Appendix A, have only two nonzero entries and since the inertia matrix M is positive definite and symmetric [20] so is M^{-1} . Thus, one can prove that $D_{12} = D_{21}$ and D_{11}, D_{22} must be nonzero positive real numbers ($\forall \mathbf{q} \in \mathbf{R}$ including the origin). Consequently, the matrix given in the left side of equation (4.4) is nonsingular, hence the global relative degree of the system (4.1) is $\{r_1, r_2\} = \{2, 2\}$ while the dimension of the open-loop system (4.1) is $n = 10$. Since $r = 2 + 2 = 4 < 10$, dynamic state feedback should be applied to obtain the linear and controllable system. The dynamic extension algorithm [27, ch. 5] has been applied twice as shown in Figure 4.1, such that the dimension of the composite system becomes $n_c = 16$ and its global relative degree $\{r_1, r_2\} = \{8, 8\}$ is achieved by making the 8th derivative of the output y depend on the reference input vector $v = (v_1 \ v_2)^T$. Therefore the condition $r_1 + r_2 = n_c$ is also fulfilled, so that close-loop system including auxiliary states ($\xi = [\xi_1, \dots, \xi_6]^T$) can be transformed into a linear and controllable system in normal coordinates [27]. Noninteracting control with stability is determined with the following control laws

$$\begin{aligned} u_1 &= (-f_6 + \xi_1 - D_{12}u_2)/D_{11} \\ u_2 &= (-f_7 + \xi_4 - D_{21}u_1)/D_{22} \end{aligned} \quad (4.6)$$

where $D_{12} = D_{21}$ and $D_{11}, D_{22} \neq 0$ implies that u_1, u_2 can be defined uniquely as

follows

$$\begin{aligned} u_1 &= \frac{(-D_{22} f_6 + D_{22} \xi_1 + D_{12} f_7 - D_{12} \xi_4)}{(D_{11} D_{22} - D_{12}^2)} \\ u_2 &= \frac{(-D_{11} f_7 + D_{11} \xi_4 + D_{12} f_6 - D_{12} \xi_1)}{(D_{11} D_{22} - D_{12}^2)} \end{aligned}$$

The reference inputs $\mathbf{v} = [v_1 \ v_2]^T$ in Figure 4.1 are designed as

$$v_1 = -800(\theta_1 - \theta_{1d}) - 60q_6 - \dot{q}_6, \quad v_2 = -800(\theta_2 - \theta_{2d}) - 60q_7 - \dot{q}_7 \quad (4.7)$$

where the gains are determined by trial and error to get less overshoot in time responses and to minimize the control energy. In the next section, these controller gains will be optimized. The zero dynamics(ξ and z) of system 4.1 (basically built by a chain of integrators with appropriate feedback law) is stable at the origin. Define $ad_f^k g = [f, ad_f^{k-1} g]$ for $k \geq 1$, $ad_f^0 g = g$ and also $ad_f^1 g = [f, g]$ called well-known Lie product. In the light of [27, Th. 7.4.4], we have

Lemma 4.1.1 *If $[ad_{\tilde{f}}^k \tilde{g}_1, ad_{\tilde{f}}^h \tilde{g}_2] = 0$ for all $k \geq 0$ and all $h \geq 0$ then the necessary condition for asymptotic stability of the closed-loop system via dynamic feedback is fulfilled.*

Proof: After applying nonlinear *dynamic* state feedback control laws given by equation 4.6, and rearranging the state equations, we have the following subsystem

$$\dot{\mathbf{q}} = \tilde{f}(\mathbf{q}, \xi) + [\tilde{g}_1 \ \tilde{g}_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.8)$$

where $\tilde{g}_1 = 0$ and $\tilde{g}_2 = 0$. Besides the actual input torques are applied only to rigid coordinates and affect the equations of $\dot{q}_6$ and $\dot{q}_7$. For the system 4.8 calculations prove that $[ad_{\tilde{f}} \tilde{g}_1, ad_{\tilde{f}} \tilde{g}_2] = 0$. $\diamond$

Asymptotic stability of the modal coordinates is guaranteed such that

$$\lim_{t \rightarrow \infty} \mathbf{q}(t) = 0$$

which has been verified by simulations in the present study. For further details , [27, Th. 9.6.2] will be helpful for its proof see also [28, section 12.3].

4.2. Error-Feedback Controller for Adaptive Case-FEM Model

4.2.1. Adaptive Internal Model

Designing an internal-model based control scheme can generate the control law C_σ that renders the output-zeroing manifold invariant. This method corresponds to placement of transmission zeroes in the transfer function from disturbance to output in linear systems. The internal model part of the controller is augmented with a robust stabilizer as a second part of the controller later on, see Figure 4.2. Based on the set-up in the previous section, the controller architecture has been extended to include the adaptive case. The adaptive internal model approach in [8, 29] has been applied in this study, yielding the error and disturbance dynamics as follows

$$\begin{aligned}
 \dot{\mathbf{z}} &= f_0(\mathbf{z}, q_1, q_2, \mathbf{w}, m_2) \\
 \dot{q}_1 &= q_6 \\
 \dot{q}_2 &= q_7 \\
 e_1 &= q_6 \\
 \dot{e}_1 &= \xi_1(v_1) + \rho_1(w_1) \\
 e_2 &= q_7 \\
 \dot{e}_2 &= \xi_4(v_2) + \rho_2(w_2)
 \end{aligned} \tag{4.9}$$

where $\mathbf{z} = [q_8 \ q_9 \ q_{10}]^T$; m_2 is the varying tip-mass; v_1, v_2 are the control inputs; $\rho_1(w_1) = w_1$, $\rho_2(w_2) = w_2$ and $\mathbf{w} = [w_1 \ w_2]^T$ represents *neutrally stable* exosystem or disturbances with the following dynamics:

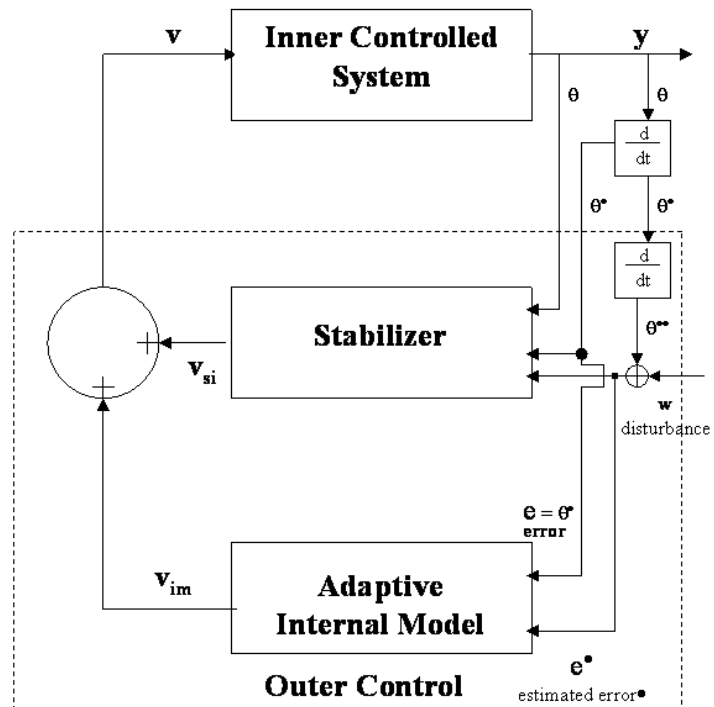


Figure 4.2. Error and Disturbance Dynamics

$$\begin{aligned}\dot{\mathbf{w}} &= S \cdot \mathbf{w} \\ S &= \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} \\ w_1 &= r \cos(\sigma t + \alpha_1) \\ w_2 &= r \cos(\sigma t + \alpha_2)\end{aligned}\tag{4.10}$$

where $r = \sqrt{w_1^2(0) + w_2^2(0)}$; $\alpha_1 = -\arctan(w_2(0)/w_1(0))$; $\alpha_2 = \frac{\pi}{2} + \alpha_1$, and σ is unknown exosystem parameter. Thus, unknown external disturbances can be modelled within the exosystem. Besides, initial conditions for the exosystem and the system (4.9) have been assumed to lie in a compact set. Since the zero dynamics(ξ) is

stable, and also the exosystem (4.10) is neutrally stable as explained above; the zero dynamics augmented with the exosystem have a compact attractor. Thus, the substantially weaker assumption of [29] is satisfied; this provides a design advantage when compared with the assumption in the earlier literature that the zero dynamics of the plant have to be globally asymptotically stable. The rather strong assumption in [29] that the set of all feedforward inputs needed to keep the *error* identically zero should be solutions of a suitable *linear* differential equation is relaxed as solutions of a *nonlinear* differential equation in [30], as well. Consequently, the easiness-to-design and robustness of the proposed controller are improved significantly by the work in [30, 29].

In Figure 4.2, $\theta = (\theta_1 - \theta_{1d} \quad \theta_2 - \theta_{2d})^T$, represents the output of the closed-loop system such as $\mathbf{y} = (q_1 \quad q_2)^T$. All derivative blocks will be realized by linear observers. Since the error is given as time derivative of joint angular velocity $\mathbf{e} = \dot{\theta}$, if the error goes to zero, then highly nonlinear system become a linear and controllable by virtue of exact feedback linearization theory as defined in [27]. Also the desired goal positions for joints achieved and retained as nonlinear steady state solutions. The error derivatives are obtained by inner control, see Figure 4.1 and 4.2, added with the external disturbances which act as control input torques in outer loop at the same time. By making the error($\dot{\theta}$) zero, the error derivative($\ddot{\theta}$) will be zero as well. Consequently, the effects of the disturbances are removed with the stable zero dynamics. The outer control is adaptively tuned up for unknown disturbances(unknown frequency and amplitude) and large parameter uncertainties.

In order to ensure the existence of linear observable internal model, the nonadaptive control laws, C_σ , should be calculated such as

$$\begin{aligned} C1_\sigma &= L_S^2 \rho_1(w_1) = -\sigma^2 \cdot w_1 \\ C2_\sigma &= L_S^2 \rho_2(w_2) = -\sigma^2 \cdot w_2 \end{aligned}$$

where $L_S^k \rho_i$ is the multiple Lie derivative as explained in the previous section and S given by equation (4.10). In the light of [9, Lemma 1.8.1] the following equation can be constructed

$$L_S^4 C_\sigma = a_0 C_\sigma + a_1 L_S C_\sigma + a_2 L_S^2 C_\sigma + a_3 L_S^3 C_\sigma$$

where all a_i 's depend on σ . After solution of the last equation for the coefficients $a_i(\sigma)$, we have the linear observable system for the part of internal model of the controller as follows :

$$\begin{aligned} \dot{\eta} &= \Phi(\sigma) \cdot \eta \\ v_{im} &= \Gamma \cdot \eta \end{aligned} \tag{4.11}$$

where η is the state vector and v_{im} is the output of the dynamic controller formulated as Adaptive Internal Model in Figure 4.2 and

$$\Phi(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_0(\sigma) & a_1(\sigma) & a_2(\sigma) & a_3(\sigma) \end{bmatrix}$$

$$\Gamma = (1 \ 0 \ 0 \ 0)$$

Note that the spectrum of $\Phi(\sigma)$ must contain the distinct eigenvalues of the exosystem (4.10) that also depend on unknown exosystem parameter, σ . Higher order harmonics of these sinusoids may be generated by the nonlinearities of the plant, thus the spectrum of $\Phi(\sigma)$ must as well contain multiples of the eigenvalues of the exosystem [8].

The canonical parametrization of the internal model should be improved in order to develop the solution extensively. Given any Hurwitz matrix F and any vector G such

that the pair (F, G) is chosen to be controllable, then we can construct the Sylvester equation [8, 31] such that

$$M_\sigma \Phi(\sigma) - F M_\sigma = G \Gamma$$

where $\Phi(\sigma)$ and F have disjoint spectra by assumption. Thus, the matrix, M_σ is *unique* [32], and is nonsingular due to observability of $(\Phi(\sigma), \Gamma)$ and controllability of (F, G) [8, Lemma 4.1]. Applying a *linear similarity transformation* to the system of equation (4.11) we have the canonical internal model as follows

$$\begin{aligned}\dot{\eta} &= (F + G\Psi_\sigma) \cdot \eta \\ v_{im} &= \Psi_\sigma \cdot \eta\end{aligned}$$

where $M_\sigma \Phi(\sigma) M_\sigma^{-1} = F + G\Psi_\sigma$, and $\Psi_\sigma = \Gamma M_\sigma^{-1}$. Therefore, $\Phi(\sigma)$ is similar to $F + G\Psi_\sigma$. Note that, since (F, G) is controllable and G has just one column, the poles of $\Phi(\sigma)$ can be assigned uniquely by the row vector Ψ_σ for adaptation. Since the derivatives of the errors will be used in the controller implementation, the errors and their derivatives should be estimated by a high-speed observer. Then the linear observer can be chosen as

$$\dot{\hat{q}} = M_g \cdot \hat{q} + L_g \cdot e_i \quad i = 1, 2$$

where

$$\begin{aligned}M_g &= \begin{bmatrix} -g \cdot c_1 & 1 \\ -g^2 \cdot c_0 & 0 \end{bmatrix} \\ L_g &= \begin{bmatrix} g \cdot c_1 \\ g^2 \cdot c_0 \end{bmatrix}\end{aligned}$$

Here, $g > 0$ is a design parameter, and the polynomial $p_c(\lambda) = \lambda^2 + c_1 \cdot \lambda + c_0$ is chosen to be Hurwitz. In order to prevent the occurrence of finite escape times for large value of the parameter g , the estimates need to be saturated. The controller is

then implemented using the estimates instead of the measured states. In order to solve the problem of semi-global robust output regulation for system (4.9) we need to define $\hat{\vartheta}$ from the estimates to use it in the feedback path of adaptation and stabilization [8, Th. VII.1]. Consider the new dynamic error feedback controller :

$$\begin{aligned}
 \hat{\vartheta} &= \hat{q}_2 + k\hat{q}_1 \\
 \dot{\zeta} &= (F + G\hat{\Psi})\zeta - KG \cdot \text{sat}(l, \hat{\vartheta}) \\
 \frac{d}{dt}\hat{\Psi} &= -\gamma \cdot \text{sat}(l, \hat{\vartheta})\zeta^T \quad (\text{Adaptation Rule}) \\
 v_{im} &= \hat{\Psi}\zeta \\
 v &= v_{im} + v_{si}
 \end{aligned}$$

where $\hat{\Psi}$ is the estimate for Ψ_σ ; the adaptive rule is based on model reference adaptive control and true convergence of the estimates is proved in [8]; F, G are defined in the previous section; k, K, γ are proportional gains; l is the saturation level; the v_{s1}, v_{s2} are as defined below and $\text{sat}(\cdot, \cdot)$ is a saturation function defined as

$$\text{sat}(l, s) = \begin{cases} s, & |s| \leq l \\ \frac{s}{|s|}, & |s| > l. \end{cases}$$

The adaptive law is robust with respect to the drift phenomena with this saturation function [9, section 3.4].

4.2.2. Robust Stabilizer

In this research to stabilize the system v_{si} control inputs given below have been used, different than the approach in [8] such that

$$\begin{aligned}
v_{s1} &= -K_{c1}(\theta_1 - \theta_{1d}) - K_{c2} q_6 - \dot{q}_6 \\
v_{s2} &= -K_{c1}(\theta_2 - \theta_{2d}) - K_{c2} q_7 - \dot{q}_7
\end{aligned}$$

where K_{c1} and K_{c2} are optimized by a novel optimization method, an evolutionary algorithm termed as "Big Bang - Big Crunch", BB-BC [33]. This new method relies on one of the theories of the *evolution* of the universe; namely, the Big Bang and Big Crunch Theory. In the Big Bang phase, energy dissipation produces disorder and randomness is the main feature of this phase; whereas, in the Big Crunch phase, randomly distributed particles are drawn into an order. Inspired by this theory, an optimization algorithm is developed in [33]. The algorithm generates gaussian-distributed random points in the Big Bang phase and shrinks those points to a single representative point via a center of mass (the value of inverse of the fitness function which is equal to $[K_{c2} | \theta_1 - \theta_{1d} |]^2 + [K_{c1} | \theta_2 - \theta_{2d} |]^2$) approach in the Big Crunch phase. Both phases follow each other iteratively with certain constraints. It is shown in [33] that the performance of the new BB-BC method outperforms the classical Genetic Algorithm (GA) for many benchmark test functions by its fast convergence and accuracy to find the global optimum.

4.3. PDE Controller Design

The total energy of the system (3.2 - 3.10) is calculated to get the total energy rate of the system. Then with this insight, the control laws are obtained by Lyapunov approach as given below

$$\tau_{11} = \tau_{b1} + I_{h1} \left(\frac{-\ddot{w}_1(l_1, t)}{l_1} - K_1(\alpha \dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) \right) \quad (4.12)$$

$$\begin{aligned}
\tau_{21} &= (I_{h2} \tau_{e1} + I_{t1} \tau_{e2} + I_{t1} I_{h2} \left(\frac{-\ddot{w}_2(l_2, t)}{l_2} \right. \\
&\quad \left. - K_2(\alpha \dot{\theta}_2 + \beta(\theta_2 - \theta_{2d})) \right)) / (I_{t1} + I_{h2})
\end{aligned} \quad (4.13)$$

where K_1, K_2, α, β are positive constants and θ_{1d}, θ_{2d} are the desired positions. Besides, $\ddot{w}_i(l_i, t)$ signals in the control laws can be obtained after filtering the output of the accelerometers which are known to be vibration measurement tools. On the other hand, the control laws (4.12 - 4.13) should be augmented with a parallel controller to ensure asymptotic stability of the closed loop system. In addition to (4.12 - 4.13), we also propose the following control laws

$$\tau_{12} = \tau_{b2} - \tau_{e2} - K \dot{\theta}_1 \quad (4.14)$$

$$\tau_{22} = \left(I_{t1} I_{h2} (\ddot{\theta}_1 + \ddot{\theta}_2) + I_{t1} \tau_{e2} \right) / (I_{t1} + I_{h2}) \quad (4.15)$$

$$\tau_1 = \tau_{11} + \gamma_1 \tau_{12} \quad (4.16)$$

$$\tau_2 = \tau_{21} + \gamma_2 \tau_{22} \quad (4.17)$$

where $K, \gamma_1, \gamma_2 > 0$ are proportional gains that are adjusted to make the control laws τ_{11}, τ_{21} dominant. This is achieved by choosing small γ_1 and γ_2 . The main controller (4.16 - 4.17) consists of the parallel connection of two separate controllers. Since the system (3.2 - 3.10) became decoupled and linear by the dominant part of the controller for details see the Appendix C, then additivity and homogeneity properties of linear systems can be applied for the closed-loop system. Thus, the effects of the control laws τ_{12}, τ_{22} should be considered independently. Consequently, the robustness of the proposed controller is improved significantly by decoupling of links 1 and 2, and also rigid and flexible dynamics. Besides the decoupled system, see Appendix C, become robust with respect to large uncertainties of the system parameters such as hub and tip mass changes. The latter has been verified by simulations, see next chapter. Then, the main theorem can be introduced as follows

Theorem 4.3.1 *The control laws (4.16 - 4.17) achieve set-point regulation of the rigid modes and asymptotic stability of the two-link flexible arm with variable cross-section.*

Proof:

The total energy V of the system is a good candidate for Lyapunov function and can be expressed in terms of kinetic and potential energy terms with additional correction term such that

$$V = (T_1 + T_2 + T_3) + (T_4 + T_5 + T_6) + V_s + \dot{\theta}_1^2 \geq 0 \quad (4.18)$$

where T_i s are the kinetic energy terms of beams, tips and hubs for link 1 and 2 [25]. The total strain potential energy for both links is represented as V_s . After applying the *dominant* control laws (4.12 - 4.13) to the system (3.2 - 3.10) and after some lengthy but straightforward calculations for details, see the Appendix D, the total energy rate of the system is obtained as follows

$$\dot{V} < -K' \dot{\theta}_1^2 - K'' \ddot{\theta}_1^2 < 0 \quad . \quad (4.19)$$

Note that $K' > 0$ and $K'' > 0$ are given in Appendix D, and the equation (4.19) only assures the stability of the closed loop system but does not prove the asymptotic stability. The latter can be shown by using LaSalle's invariance principle extended to infinite dimensional spaces [13]. In order to initiate this part of the proof, equations (3.2)- (3.5) should be rewritten in the new coordinates such that

$$\begin{aligned} \mathbf{z} &= [z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \quad z_7 \quad z_8]^T \\ &= [w_1 \quad \dot{w}_1 \quad w_2 \quad \dot{w}_2 \quad \theta_1 - \theta_{1d} \quad \dot{\theta}_1 \quad \theta_2 - \theta_{2d} \quad \dot{\theta}_2]^T \end{aligned}$$

The system (3.2)- (3.5) can be decomposed to linear and nonlinear parts separately, see [34], such as

$$\dot{\mathbf{z}} = A \mathbf{z} + f(\mathbf{z}) \quad (4.20)$$

where A represents the infinite dimensional linear operator and f represents the non-linear operator. For the linear part of the system (3.2)- (3.3) which is dissipative by feedback controls (4.16 - 4.17), will be considered in the energy Hilbert space $\mathbf{H}_i = H_E^2(0, l_i) \times L^2(0, l_i)$, $H_E^2(0, l_i) = \{w_i \in H^2(0, l_i) \mid w_i(0) = w_{ix}(0) = 0\}$, in which the inner product induced norm is defined by [35]

$$\|(w_i, g_i)\|_{\mathbf{H}_i}^2 = \int_0^{l_i} [\rho_i(x)|g_i(x)|^2 + EI_i(x)|w_{ixx}(x)|^2] dx$$

for $i = 1, 2$ and $\forall (w_i, g_i) \in \mathbf{H}_i$, $0 < x < l_i$. Define operator $A_i : D(A_i) (\subset \mathbf{H}_i) \rightarrow \mathbf{H}_i$ as

$$A_i(w_i, g_i) = (g_i, \frac{-1}{\rho_i(x)}(EI_i(x)w_{ixx}(x))_{xx}) \quad (4.21)$$

$$D(A_i) = \{(w_i, g_i) \in (H_E^2 \cap H^4) \times H_E^2\}$$

where A_i^{-1} is compact on $\mathbf{H}_i$ [35, Lemma 2.1]. We also define the function space

$$\hat{\mathbf{H}} = \mathbf{H}_1 \times \mathbf{H}_2 \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$$

Using the set of equations (3.2 - 3.10) and integration by parts, we have the linear operator $A : \hat{\mathbf{H}} \rightarrow \hat{\mathbf{H}}$ and the nonlinear operator $f : \hat{\mathbf{H}} \rightarrow \hat{\mathbf{H}}$ are given in equation (4.20) such as:

$$A = \begin{bmatrix} A_1 & 0_{2 \times 2} & 0_{1 \times 4} \\ 0_{2 \times 2} & A_2 & B_1 \\ 0_{4 \times 2} & 0_{4 \times 2} & B_2 \end{bmatrix} \quad (4.22)$$

$$f(\mathbf{z}) = \begin{bmatrix} 0 \\ -x_1 W_1 - \frac{x_1}{I_{t1}} (\tau_{e1} - \tau_{b2} + \tau_{e2}) \\ 0 \\ -x_2 W_2 + \frac{x_2}{I_{t1}} (\tau_{e1} - \tau_{b2}) + x_2 \frac{I_{h2} + I_{t1}}{I_{h2} I_{t1}} \tau_{e2} + f_0 \\ 0 \\ W_1 + \frac{1}{I_{t1}} (\tau_{e1} - \tau_{b2} + \tau_{e2}) \\ 0 \\ W_2 - \frac{1}{I_{t1}} (\tau_{e1} - \tau_{b2}) - \frac{I_{h2} + I_{t1}}{I_{h2} I_{t1}} \tau_{e2} \end{bmatrix}$$

where A_1, A_2 are given in equation (4.21); τ_{ei}, τ_{bi} are defined by equations (3.11)-

(3.12), and B_1, B_2, W_1, W_2, f_0 are given below

$$\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 0 & \frac{x_1 I_{h2} \tilde{\beta}}{I_{h2} + I_{t1}} & \frac{x_1 I_{h2} \tilde{\alpha}}{I_{h2} + I_{t1}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x_2 \tilde{\beta} & -x_2 \tilde{\alpha} \end{bmatrix} \\
B_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-I_{h2} \tilde{\beta}}{I_{h2} + I_{t1}} & \frac{-I_{h2} \tilde{\alpha}}{I_{h2} + I_{t1}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{\beta} & \tilde{\alpha} \end{bmatrix} \\
W_1 &= \frac{I_{h2}}{\gamma_2 I_{t1}} \frac{(EI_2(x_2)w_{2xx})_{xx}}{l_2 \rho_2(x_2)} \Big|_{x_2=l_2} \\
W_2 &= \frac{I_{h2} + I_{t1}}{-\gamma_2 I_{t1}} \frac{(EI_2(x_2)w_{2xx})_{xx}}{l_2 \rho_2(x_2)} \Big|_{x_2=l_2} \\
f_0 &= \text{Cos}\theta_2 \left[K_1 l_1 (\alpha \dot{\theta}_1 + \beta (\theta_1 - \theta_{1d})) \right. \\
&\quad \left. - \frac{\gamma_1 l_1}{I_{h1}} (\tau_{b2} - \tau_{e2} - K \dot{\theta}_1) \right] + \dot{\theta}_2 \text{Sin}\theta_2 \left[\dot{w}_1(l_1, t) + l_1 \dot{\theta}_1 \right] \\
\tilde{\beta} &= \frac{I_{h2} + I_{t1}}{\gamma_2 I_{t1}} K_2 \beta \\
\tilde{\alpha} &= \frac{I_{h2} + I_{t1}}{\gamma_2 I_{t1}} K_2 \alpha
\end{aligned}$$

where B_1, B_2 are finite dimensional linear bounded operators, $f(\mathbf{0}) = 0$ and $f(\mathbf{z})$ is differentiable. Since $(\lambda I - A)^{-1}$ is compact for $\lambda > \tilde{\alpha}$, (see e.g. [35]), then it follows that the solutions of (4.20) locally exists in; moreover if $\mathbf{z}(0) \in D(A)$, then $\mathbf{z}(t) \in D(A)$ as well. Since the solutions are bounded, see (4.19), it can easily be shown that local solutions can be extended globally as well. In order to complete the proof, we should show that $\dot{V} = 0$ implies $\mathbf{z} = 0$. If $\dot{V} = 0$ then $\delta W_{nc} = 0$ since the power associated with nonconservative forces is equal to the time rate of change of the total energy [19]. For the system (3.2 - 3.10),

$$\delta W_{nc} = \tau_1 \delta \theta_1 + \tau_2 \delta (\theta_2 - w_{1x}(l_1, t))$$

where $\delta\theta_1$ and $\delta(\theta_2 - w_{1x}(l_1, t))$ are arbitrary nonzero variations by definition then $\tau_1 = 0$ and $\tau_2 = 0$. Besides, $\dot{V} = 0$ implies $\dot{\theta}_1 = \ddot{\theta}_1 = 0$ directly and since $\tau_1 = 0$ by (3.4) and (3.11) we obtain $\tau_{b1} = 0$ as well. Using the boundary condition (3.8) we get $W_1 = W_2 = 0$ for $f(\mathbf{z})$, see equations (C.6), (C.8), and (C.11) in the Appendix C. Thus, $W_1 = W_2 = 0$, $\ddot{\theta}_1 = 0$ and equations (C.9), (4.20) implies that $\tau_{bi} = \tau_{ei} = 0$. Using these results and the control laws (4.16 - 4.17) for the boundary condition (3.6) with equation (C.9), we have $\tau_{b2} = -I_{h2}\ddot{\theta}_2$. Equations (3.5) and (3.9) give respectively

$$\tau_{e2} = -I_{h2}\ddot{\theta}_2 = I_{t2}\ddot{\theta}_2 \quad (4.23)$$

that verifies $\tau_{e1} = \tau_{b2} = \tau_{e2} = \ddot{\theta}_2 = 0$. Thus, the main PDEs (3.2 - 3.3) become homogeneous, see equations (C.5 - C.6) in the Appendix C such as

$$\rho_1(x_1)\ddot{w}_1 + (EI_1(x_1)w_{1xx})_{xx} = 0 \quad (4.24)$$

$$\rho_2(x_2)\ddot{w}_2 + (EI_2(x_2)w_{2xx})_{xx} = 0 \quad (4.25)$$

where flexural deflection $w_i(x, t)$ can be expressed by separation of variables such that [36]

$$w_i(x, t) = \phi_i(x)e^{\lambda_i t} \quad (4.26)$$

where $i = 1, 2$; λ_i 's are nonzero complex eigenvalues and ϕ_i 's are eigenfunctions for equations (4.24 - 4.25) with the above boundary conditions that are derived for $\dot{V} = 0$. Thus, equations (4.24 - 4.25) become variable coefficient, ordinary differential

equations of order four, such as [36]

$$\lambda_1^2 \rho_1(x_1) \phi_1(x_1) + \frac{d}{dx_1^2}(EI_1(x_1) \frac{d}{dx_1^2} \phi_1(x_1)) = 0 \quad (4.27)$$

$$\lambda_2^2 \rho_2(x_2) \phi_2(x_2) + \frac{d}{dx_2^2}(EI_2(x_2) \frac{d}{dx_2^2} \phi_2(x_2)) = 0 \quad (4.28)$$

For nonuniform i^{th} link, $\phi_i(x)$ can be defined as

$$\begin{aligned} \phi_i(x) &= p_i(x)r_i(x) \\ \phi_{ix}(x) &= p_{ix}(x)r_i(x) + p_i(x)r_{ix}(x) \\ \phi_{iix}(x) &= p_{iix}(x)r_i(x) + 2p_{ix}(x)r_{ix}(x) + p_i(x)r_{iix}(x) \\ \phi_{iixx}(x) &= p_{iixx}(x)r_i(x) + 3p_{iix}(x)r_{ix}(x) + 3p_{ix}(x)r_{iix}(x) + p_i(x)r_{iixx}(x) \\ \phi_{iixxx}(x) &= p_{iixxx}(x)r_i(x) + 4p_{iixx}(x)r_{ix}(x) + 6p_{iix}(x)r_{iix}(x) + 4p_{ix}(x)r_{iixx}(x) \\ &\quad + p_i(x)r_{iixxx}(x) \end{aligned}$$

where $p_i(x) = a_i x^4 + b_i x^3 + c_i x^2 + d_i x + e_i$ and $r_i(x)$ is possibly nonlinear function which is the fourth order differentiable at least and satisfies the conditions such that $r_i(0) \neq 0$, $r_i(l_i) \neq 0$. Then the coefficients of the polynomial $p_i(x)$ are obtained by using the boundary conditions such as

$$\phi_i(0) = 0 \rightarrow e_i = 0 \quad from \quad (3.10)$$

$$\phi_{ix}(0) = 0 \rightarrow d_i = 0 \quad from \quad (3.10)$$

$$\phi_i(l_i) = 0 \quad from \quad (C.7) - (C.8)$$

$$\phi_{ix}(l_i) = 0 \quad see \quad Remark \quad 4.3.2$$

$$\phi_{iix}(0) = 0 \rightarrow c_i = 0 \quad from \quad \tau_{bi} = 0$$

$$\phi_{iixx}(l_i) = 0 \rightarrow 12a_i l_i^2 + 6b_i l_i = 0 \quad from \quad \tau_{ei} = 0$$

$$\phi_{iixxx}(l_i) = 0 \rightarrow 24a_i l_i + 6b_i = 0 \quad from \quad equations \quad (C.10) - (C.11)$$

$$\phi_{iixxxx}(l_i) = 0 \rightarrow 24a_i = 0 \quad from \quad equations \quad (C.5) - (C.8)$$

thus $a_i = b_i = 0$ while $\rho_i(x_i)$, $EI_i(x_i)$, $EI_{ix}(x_i)$ and $EI_{ixx}(x_i)$ are assumed to be nonzero at $x_i = 0$, and $x_i = l_i$. Consequently, $p_i(x) = 0$, $\phi_i(x) = 0$, then $w_i = \dot{w}_i = 0$ for $\dot{V} = 0$. On the other hand, using the dominant control laws (4.12 - 4.13) and the above results that include ($\tau_{bi} = 0$, $\tau_{ei} = 0$), and equations (C.7), (C.8) in the Appendix C yields $\ddot{w}_i(l_i, t) = 0$, we also have $\dot{\theta}_1 = \dot{\theta}_2 = 0$ and $\theta_1 = \theta_{1d}$, $\theta_2 = \theta_{2d}$. Consequently, $\dot{V} = 0$ really implies $\mathbf{z} = 0$.

In the light of [13, Theorem 3.64 and 3.65], the closed-loop system (3.2 - 4.17) is asymptotically stable. $\diamond$

Alternate proof for the main theorem:

Examining the decoupled linear system (C.5 - C.11) which is obtained in Appendix C, and assuming $\tau_{e1} = \tau_{b2}$, local convergence of all coordinates can be shown. In order to prove the local stability of the system, closed-loop equation (4.20) which includes the effects of all control laws, is investigated thoroughly. Using the boundary condition (3.8) we get $W_1 = W_2 = 0$ for $f(\mathbf{z})$, see equations (C.6), (C.8), and (C.11) in the Appendix C. Thus, $W_1 = W_2 = 0$, $\ddot{\theta}_1 = 0$ and equations (C.9), (4.20) implies that $\tau_{bi} = \tau_{ei} = 0$. Using these results and the control laws (4.16 - 4.17) for the boundary condition (3.6) with equation (C.9), we have $\tau_{b2} = -I_{h2} \ddot{\theta}_2$. Equations (3.5) and (3.9) give respectively

$$\tau_{e2} = -I_{h2} \ddot{\theta}_2 = I_{t2} \ddot{\theta}_2 \quad (4.29)$$

that verifies $\tau_{e1} = \tau_{b2} = \tau_{e2} = \ddot{\theta}_2 = 0$.

Remark 4.3.2 *Since we have already neglected w_{1x} , we have also omitted $\ddot{w}_{1x}$ in the final calculations. Note that such terms are also omitted in [25]. Note that although in [25] two damping terms are introduced in the simulations, we do not use any damping term in our case.*

5. SIMULATION STUDIES

5.1. Non-Adaptive Case-FEM Model

Parameter	Value
Length of links, $L_1 = L_2$	0.5 m
Young's Modulus, E	200 GPa
Density, ρ	7800 kgm^{-3}
Cross-sectional area	$9e - 5$ m^2
Hub inertia, I_0	0.02 kgm^2
Centroidal inertia, $I_1 = I_2$	$4.7e - 11$ m^4
Tip mass, m_2	0.2, 1.2, 0.5, 1.2 kg
θ_{1d} (desired)	$\pi/2$ rad
θ_{2d} (desired)	$-\pi/2$ rad
Coef. of Viscous fric. V_f	0.001
Coef. of Coulomb fric. C_f	0.07

Table 5.1. Parameters of the Flexible Arm for FEM model.

The proposed control scheme is tested with the simulation program implemented in MATLAB. The parameters used in the model for system 4.1 given in [6] are listed in Table 5.1. The simulation results are presented in the Figures 5.1 - 5.7. Tip-mass has been varied as can be seen in Fig. 5.1. Non-adaptive case with tip-mass variation is simulated first without any disturbance. No significant changes can be observed in time responses compared to the case of constant tip-mass. This shows the robustness of the proposed controller. The flexible hub-arm system is a non-minimum phase in nature. Therefore, the initial torque may result in undesired vibration at the beginning of the motion, can be seen in Figures 5.2 and 5.3.

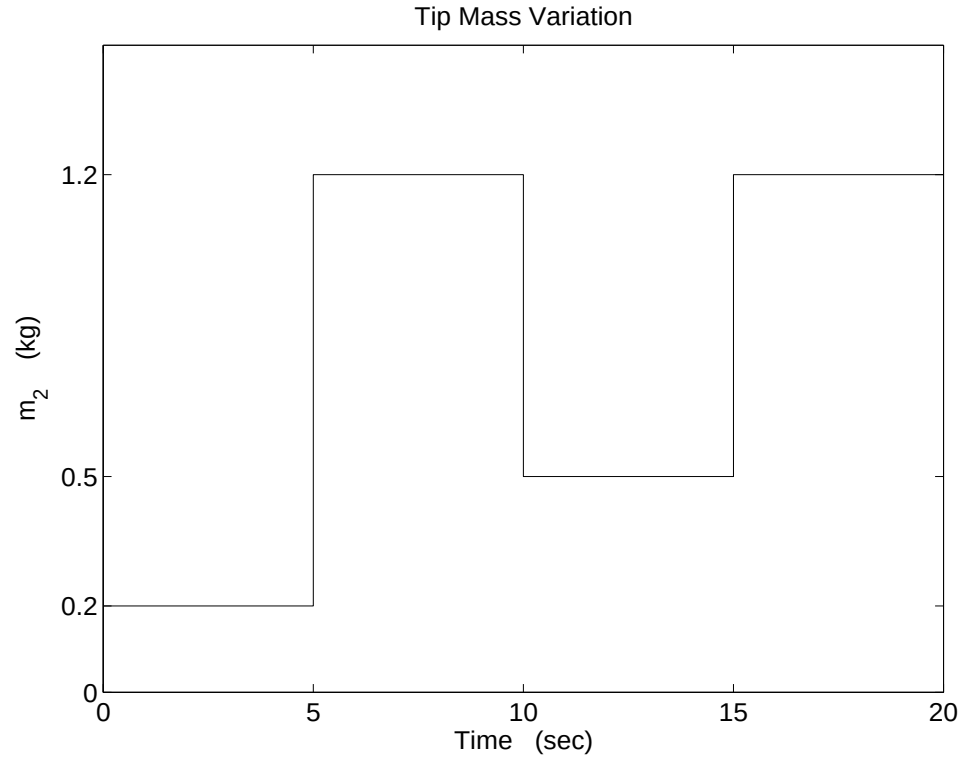
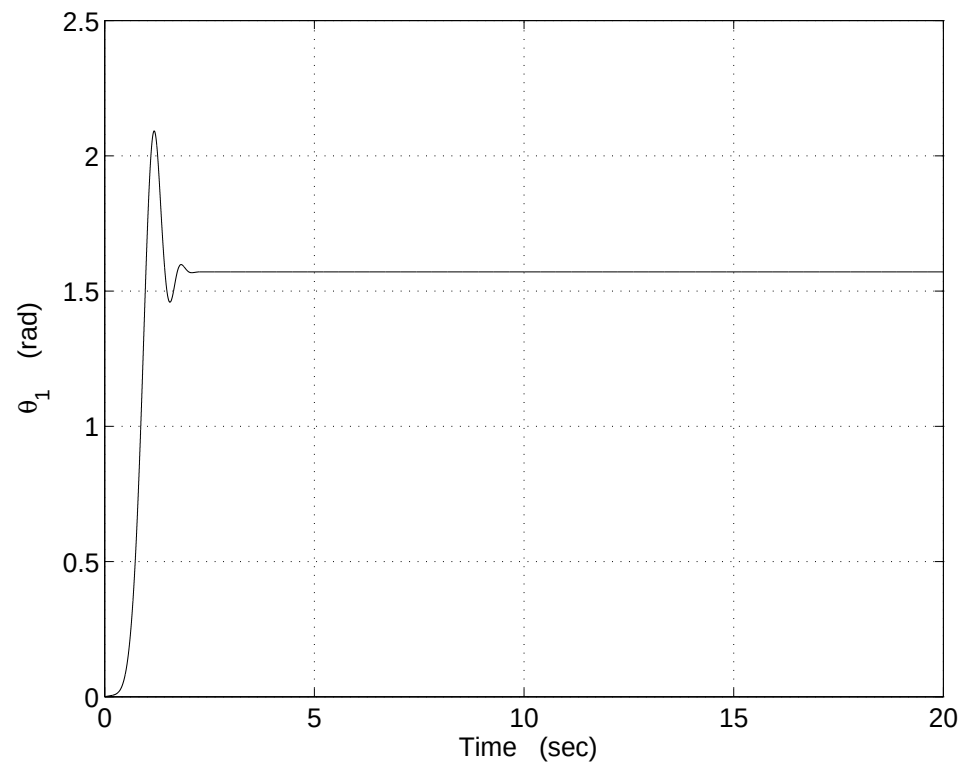
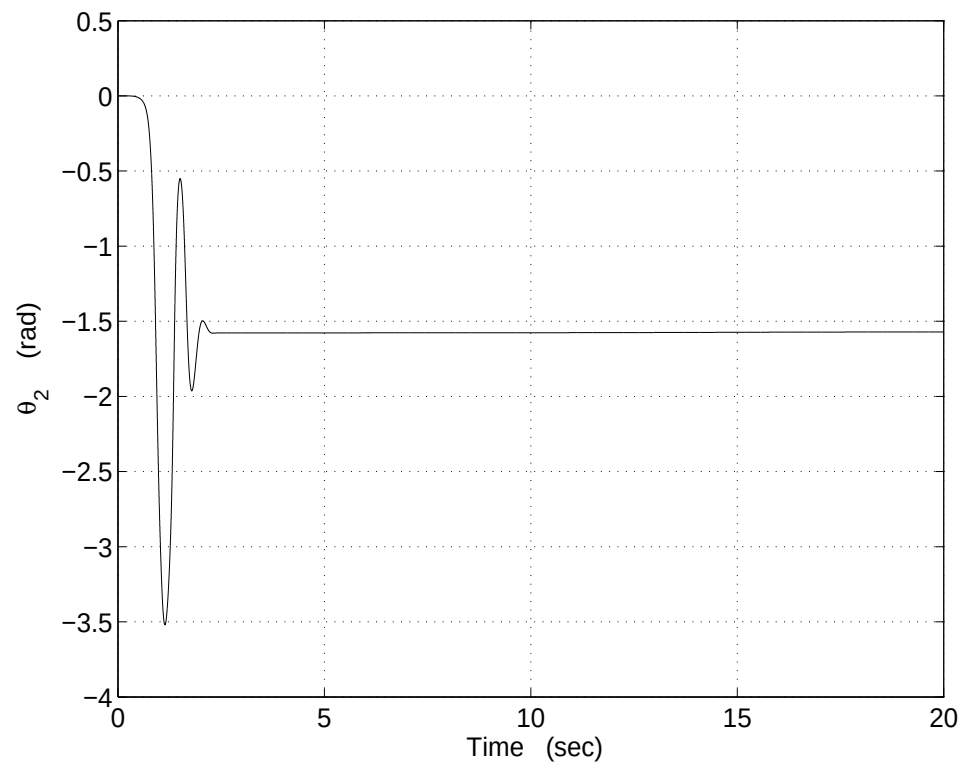


Figure 5.1. Tip Mass Variation

Parameter	Value
Coef. of Viscous fric. V_f	0.001
Coef. of Coulomb fric. C_f	0.0
σ_0 to initiate Adaptation	1
σ exosystem @ case1	1 and 2 after 7 th second
σ exosystem @ case2	10 and 40 after 7 th second
$g = 10, \gamma = 1 \quad k = 0.5$	$K = 0.001 \quad l = 100$
a_0 for $\Phi(\sigma)$	$-100\sigma^4$
a_2 for $\Phi(\sigma)$	$-101\sigma^2$
$a_1 = a_3 = 0$ for $\Phi(\sigma)$	$c_0 = 2, \quad c_1 = 3$
$w_1(0)$ exosystem	$\frac{\pi}{20}$
$w_2(0)$ exosystem	$\frac{-\pi}{20}$

Table 5.2. Additional parameters for the adaptive cases for FEM model.

Figure 5.2. Θ_1 - Non-Adaptive CaseFigure 5.3. Θ_2 - Non-Adaptive Case

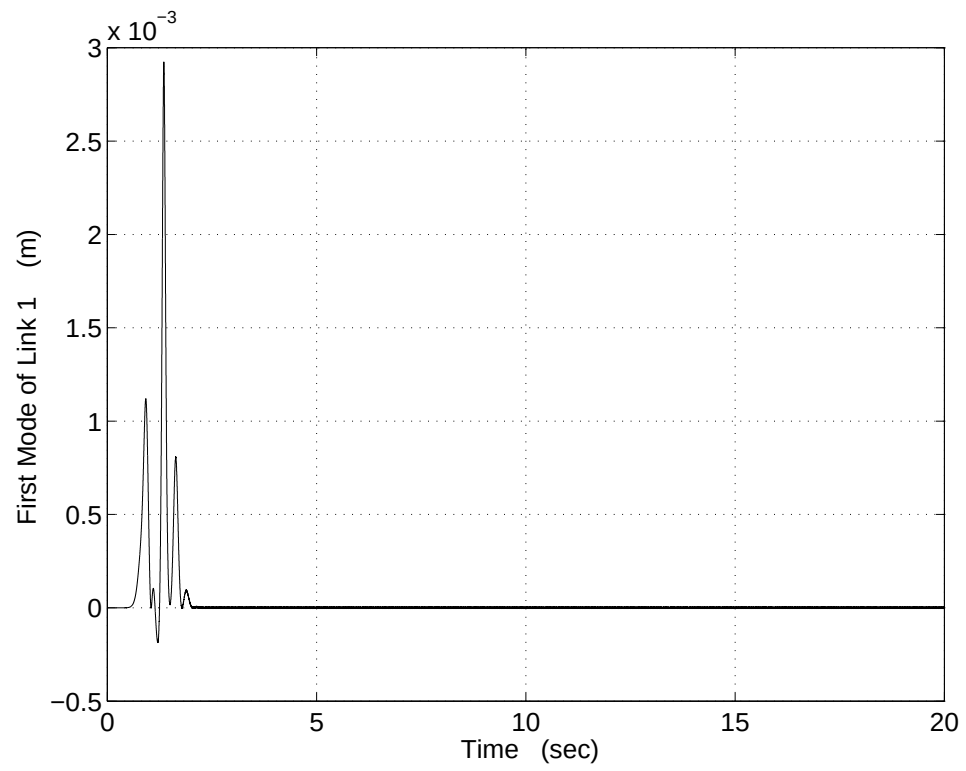


Figure 5.4. First Mode of Link 1- Non-Adaptive Case

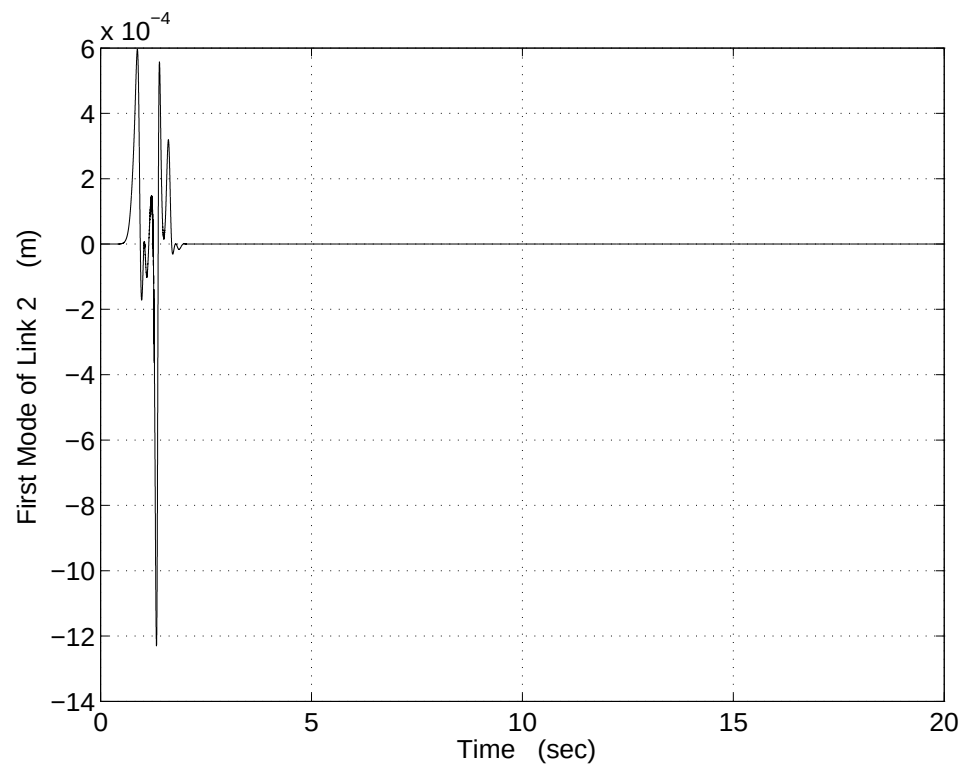


Figure 5.5. First Mode of Link 2- Non-Adaptive Case

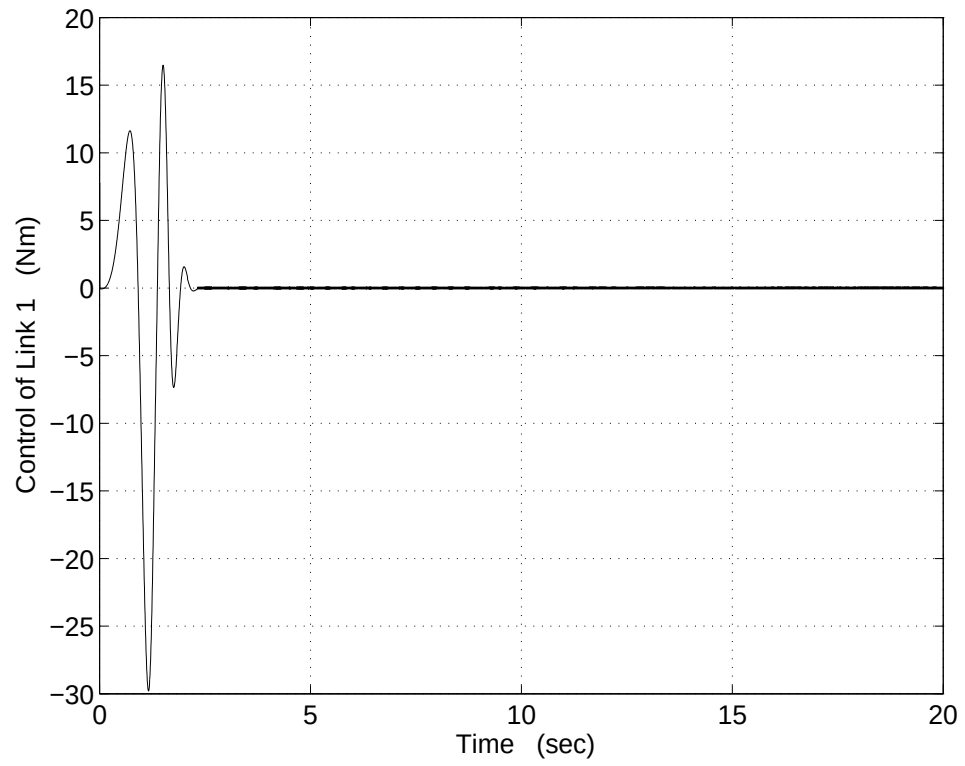


Figure 5.6. Control on Θ_1 - Non-Adaptive Case

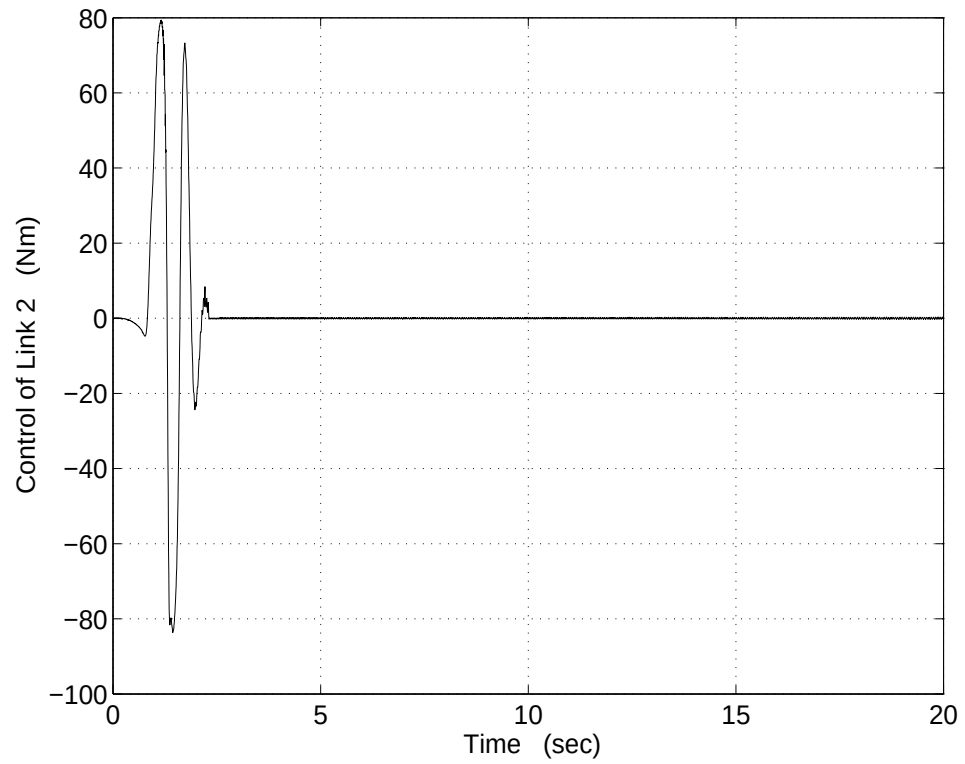
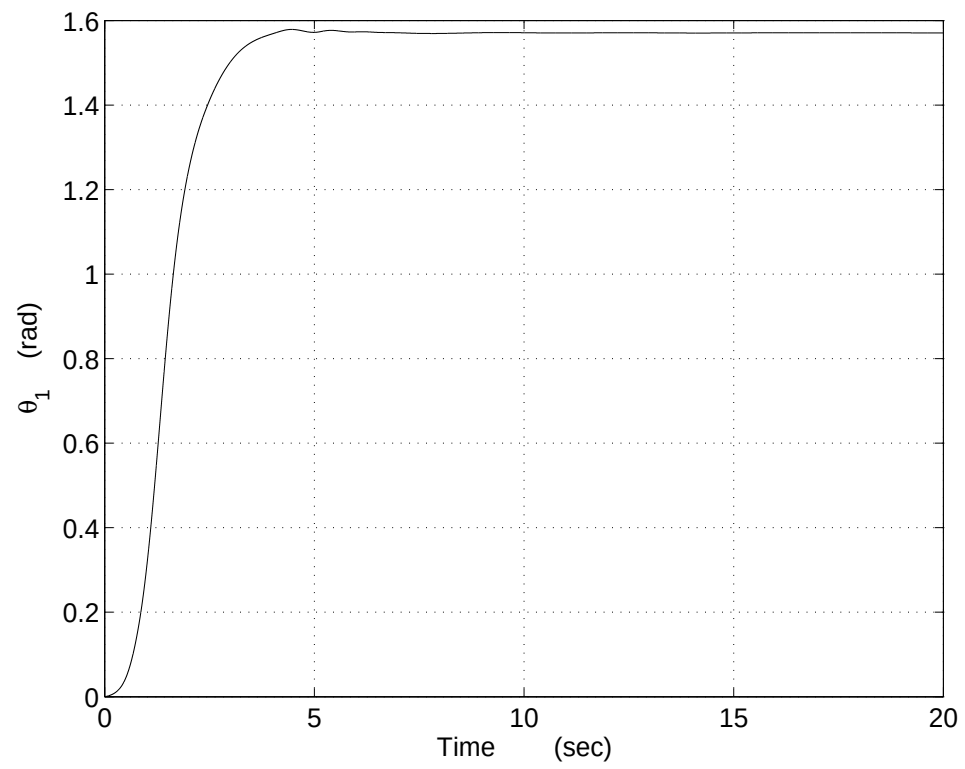
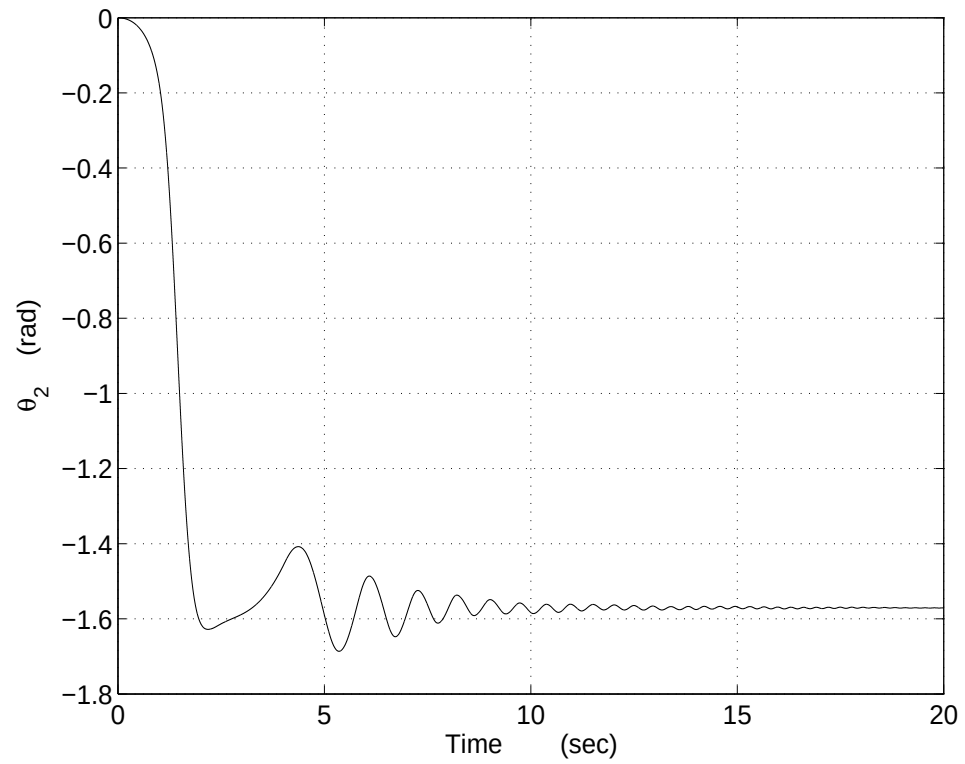
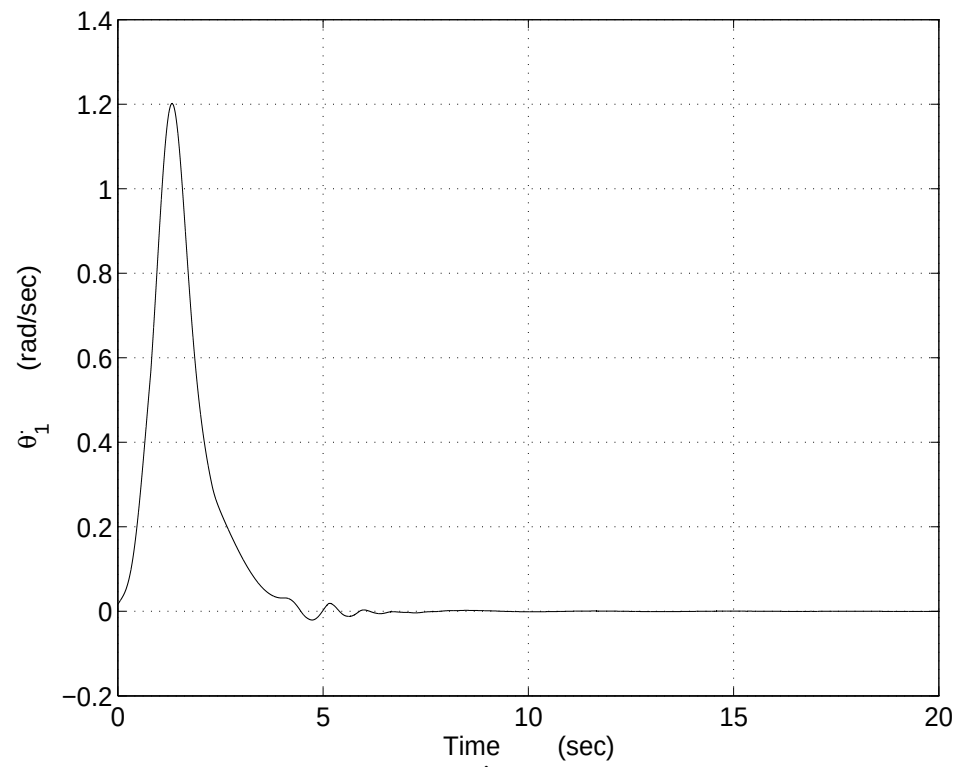
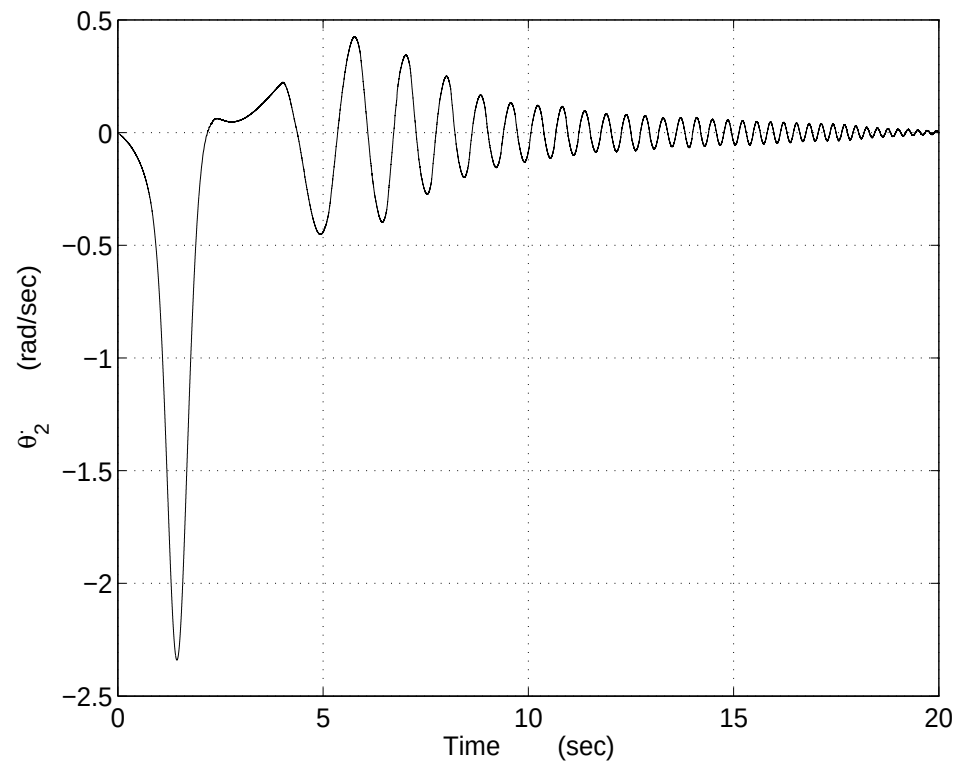


Figure 5.7. Control on Θ_2 - Non-Adaptive Case

Figure 5.8. Θ_1 - Adaptive Case 1Figure 5.9. Θ_2 - Adaptive Case 1

Figure 5.10. Error ($\dot{\theta}_1$)- Adaptive Case 1Figure 5.11. Error ($\dot{\theta}_2$)- Adaptive Case 1

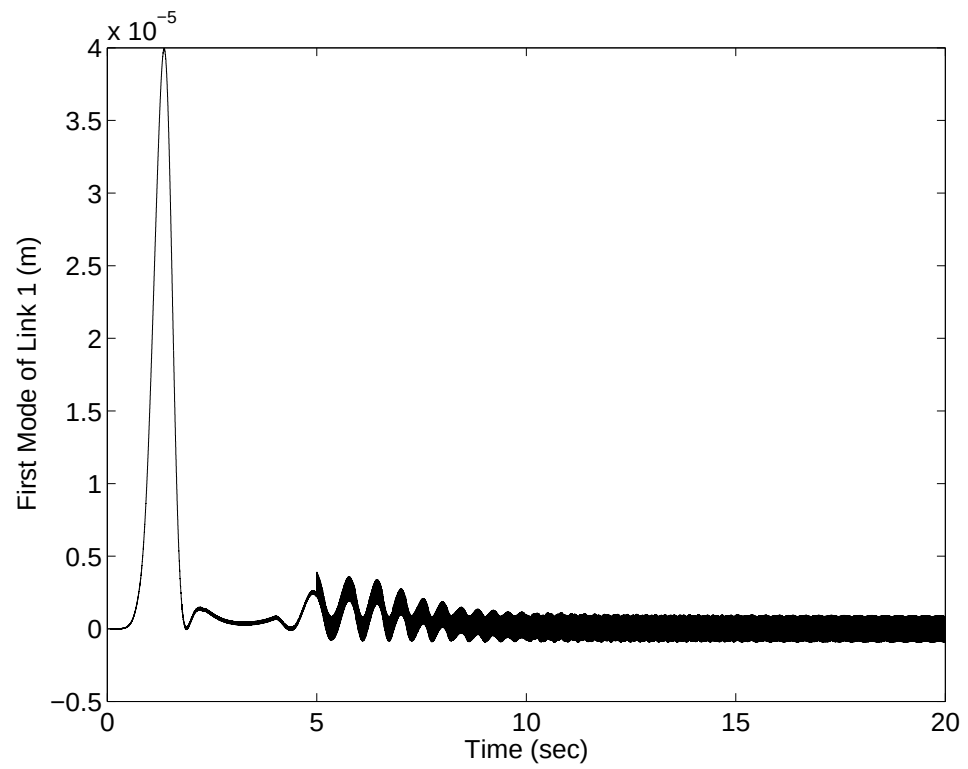


Figure 5.12. First Mode of Link 1-Adaptive Case 1

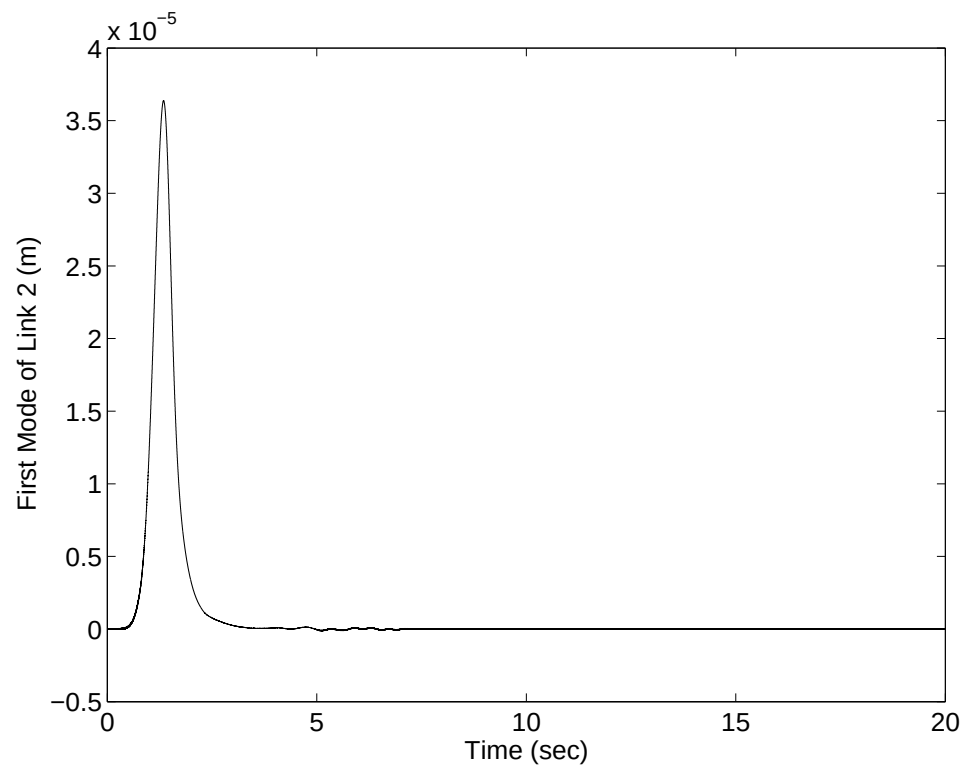


Figure 5.13. First Mode of Link 2-Adaptive Case 1

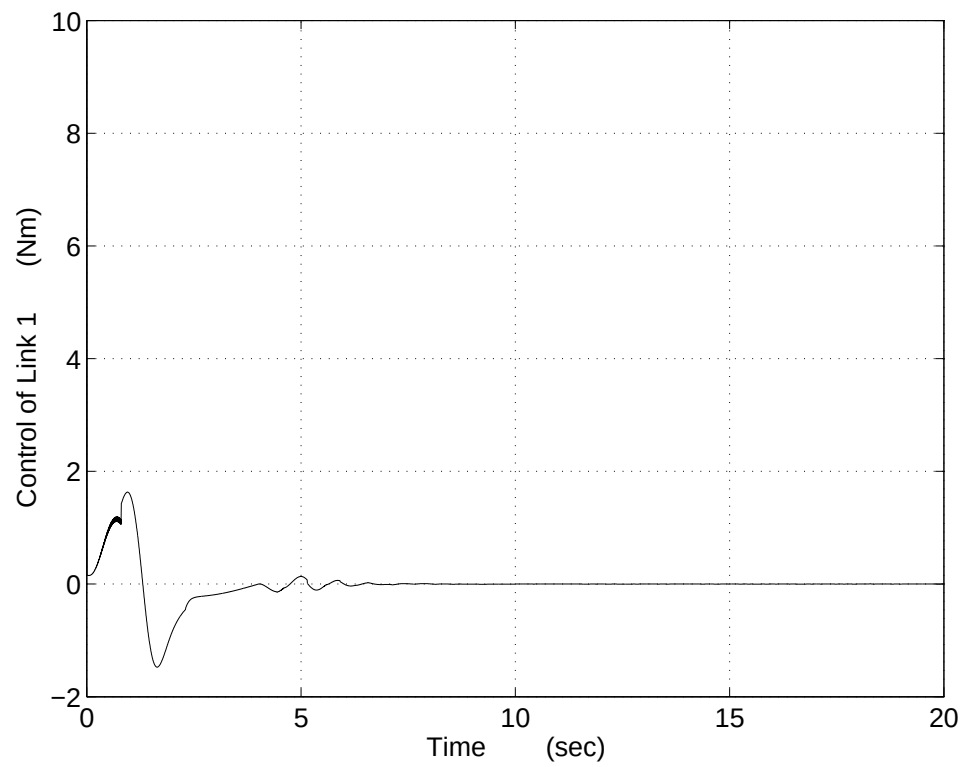


Figure 5.14. Control on Θ_1 - Adaptive Case 1

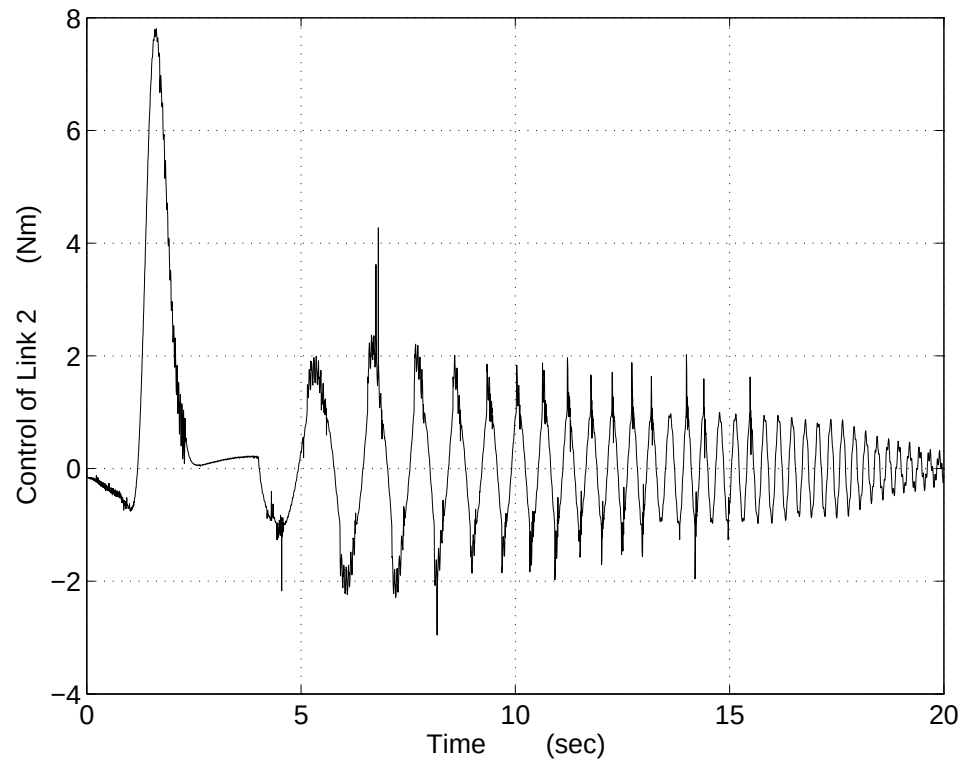


Figure 5.15. Control on Θ_2 - Adaptive Case 1

5.2. Adaptive Case-FEM Model

The parameters used in the model for system 4.9 are listed in Table 5.2. The simulation results for two different cases are presented in Figures 5.8 - 5.21. In both cases adaptation is turned on after 4 seconds and never turned off later on. In the first case that has low-frequency disturbance, we start with $\sigma = 1$ until $t = 7$ secs and then change to $\sigma = 2$ for the rest of the simulation. Fortunately, possible undesired vibration at the beginning of the motion (especially for link 2) is prevented very well in the first case by optimization of the controller parameters. In the second case that has high-frequency disturbance, we start with $\sigma = 10$ until $t = 7$ secs and then change to $\sigma = 40$ for the rest of the simulation. In particular, $\sigma = 40$ corresponds to lowest resonant frequency of neglected higher modes (first link: 344 rad/sec, second link: 245 rad/sec) of vibration, but the proposed controller is still good for handling high frequency disturbances and also adaptation is quick as well. In order to avoid the chattering in the control signals, see the results in [45], adaptive low pass pre-filter is used for the controls. This pre-filtering resulted in acceptable small oscillations on the time responses. The adaptive case 1 has much less overshoot (almost none) and more smooth control history compared with the case 2. In order to initiate the adaptive update rule Ψ_0 has been calculated by

$$\Phi(\sigma_0) = F + G\Psi_0.$$

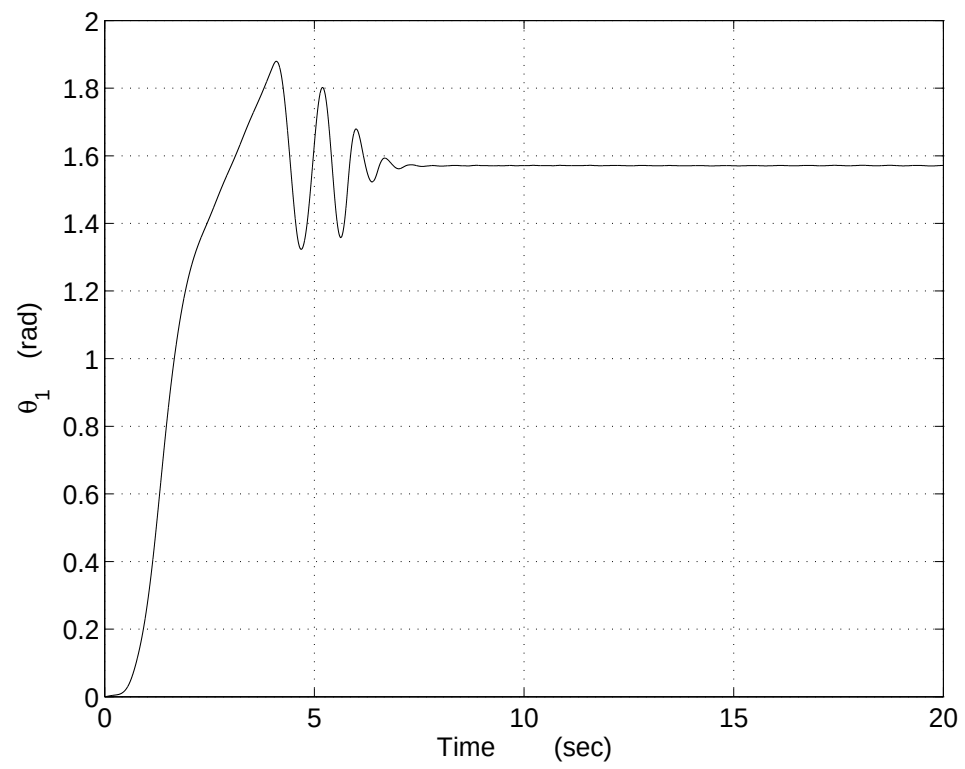
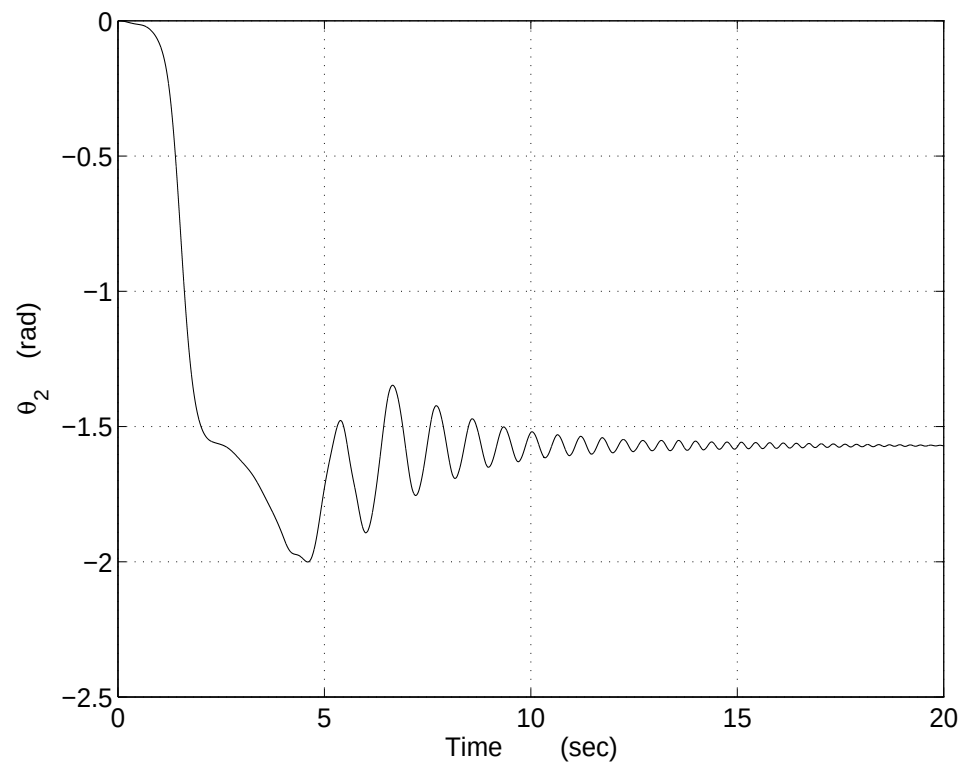
where $\sigma_0 = 1$, $G = [0 \quad 0 \quad 0 \quad 1]^T$ and

$$\Phi(\sigma_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 0 & -101 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8640 & -3624 & -566 & -39 \end{bmatrix}$$

Hence, $\Psi_0 = [8540 \quad 3624 \quad 465 \quad 39]$ is obtained. Then, the triplet (F, G, Ψ_σ) has been recalculated by MATLAB balanced realization routine from controllability canonical form. As oppose to other canonical forms, the balanced realization is numerically well-conditioned [8]. In the first part of the simulation in both adaptive cases, initial conditions for the exosystem have been chosen in the $[-\pi/20, \pi/20]$ interval. As can be seen in figures 5.8 to 5.15, the adaptation mechanism worked very well. Particularly, the errors in Figure 5.10 and 5.11 are recovered fast by adaptation after the 4th second. However, in the second part of the simulation initial conditions for the exosystem have been chosen $[-\pi/3, \pi/3]$ interval and as can be seen in [44], it took longer to adapt to disturbances with a significant overshoot compared to the first part, since the domain of attraction in the second part has been expanded largely, and also the amplitude of the external disturbances increased from 0.2 to 1.5. This shows the robustness of the adaptive controller and also the *boundary of basin of attraction* is close to boundary of the interval $[-\pi/3, \pi/3]$. Thus, the initial conditions for the exosystem have been assumed to lie in the compact set $\mathcal{K}_w = \{w_1^2 + w_2^2 \leq 2(\pi/3)^2\}$ as stated in the previous chapter. Finally, the compact set of initial conditions for the system (4.9) can be chosen as the closed neighborhood of the origin, since the robot arm will be at rest initially.

In both adaptive cases, the controller gains have been optimized by BB-BC [33]. This optimization improved on the results reported in [44]. Comparing figures 5.2 to 5.7 which represent the non-optimized and non-adaptive case with figures 5.8 to 5.21 which represent the optimized-adaptive cases, is observed that overshoot decreased enormously (almost none at case 1), and total control energy is reduced since the amplitude of the control signal reduces to the one tenth of the values in the non-optimized cases as given in [44]. Overall vibration control for both adaptive cases where the first modes of links are in the order of magnitude of 10^{-5} , is improved

Figure 5.16. Θ_1 - Adaptive Case 2Figure 5.17. Θ_2 - Adaptive Case 2

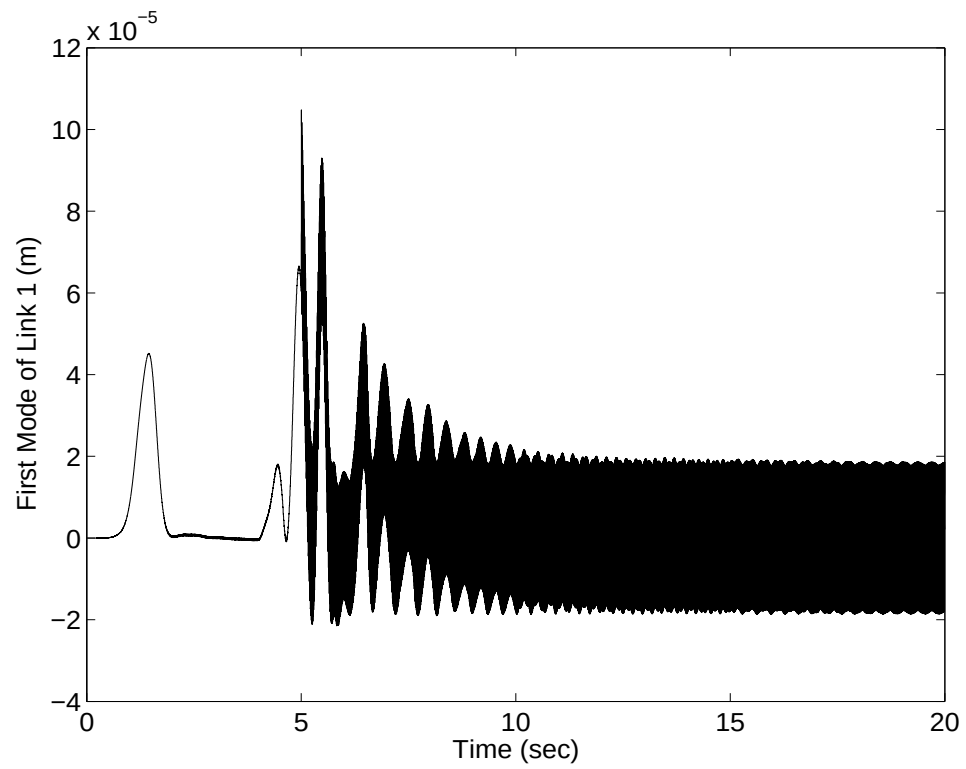


Figure 5.18. First Mode of Link 1-Adaptive Case 2

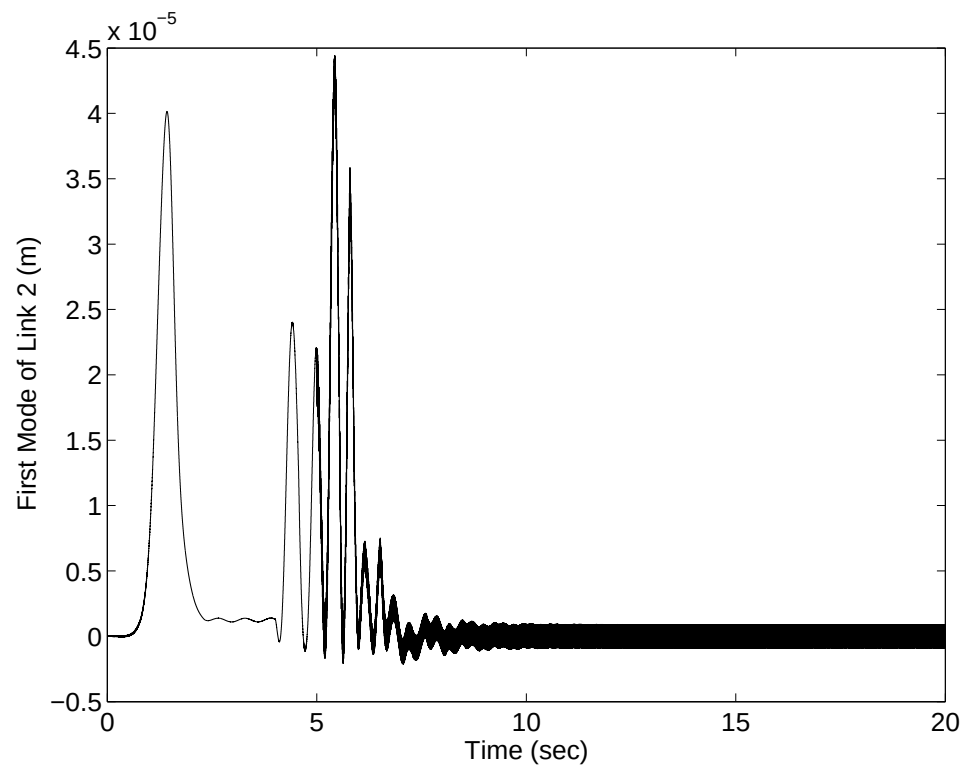


Figure 5.19. First Mode of Link 2-Adaptive Case 2

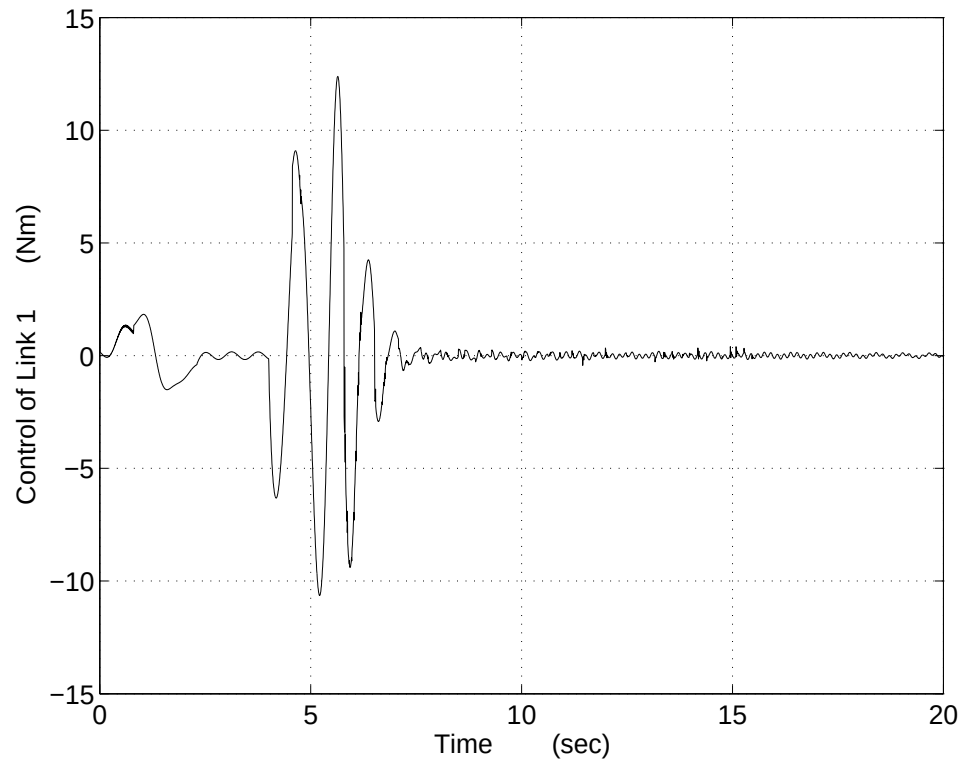


Figure 5.20. Control on Θ_1 -Adaptive Case 2

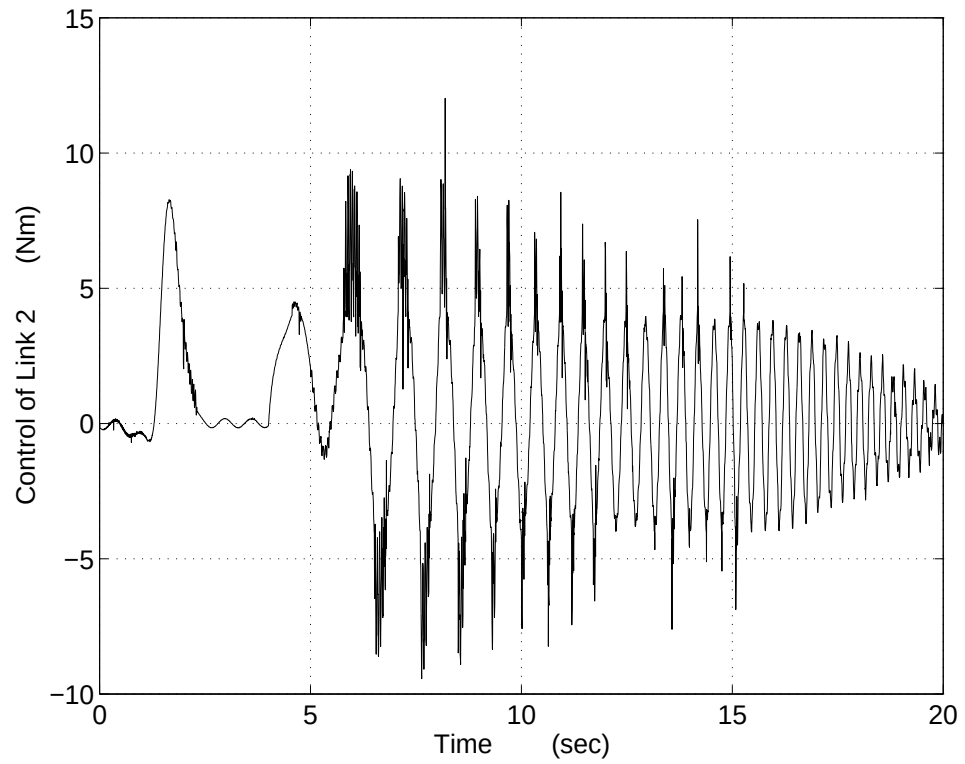


Figure 5.21. Control on Θ_2 -Adaptive Case 2

significantly as compared with the non-adaptive case where the first modes of links are in the order of magnitude of 10^{-3} . This result emphasizes the robustness of the proposed controller especially with respect to high frequency disturbances as shown in the adaptive case 2.

5.3. PDE Approach

5.3.1. PDE Model Case 1

The proposed control scheme is tested with the simulation program implemented in MATLAB. The PDEs are discretized in the space domain by the finite difference method, to obtain ODEs at each of the nodes. Then, ODEs are solved numerically by MATLAB algorithm, *ode45* that is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair [40]. Since the nonlinear boundary conditions required a smaller time-step for accurate results, the explicit finite difference scheme that requires very small time-steps as well and easy to implement efficiently [41], is adapted from [43]. Chaotic vibrations of a modified Euler-Bernoulli beam that are difficult to catch, have been solved successfully by the same scheme in [43]. The fifth-order accuracy of the numerical scheme is achieved by the Dormand-Prince pair [40] that is widely accepted as the best fifth-order pair [42]. On the other hand, robust numerical stability is succeeded by making the ratio $\Delta t / \Delta x^2 \leq 0.5$ as low as 0.048, see Table 5.3 [43, 41]. Instead of dealing with complexity of the fourth order derivative approximation, the second order derivative approximation that allows a simpler stability analysis, has been used with the help of auxiliary states. Those states are more meaningful in a real problem as well since they correspond to physical variables such as deflections(w), velocity(U) and bending moments(M) [43] such that

$$\begin{aligned} U_i &= \dot{w}_i(x, t) & \dot{U}_i &= \ddot{w}_i \\ M_i &= w_{ixx} & \dot{M}_i &= U_{xx} \end{aligned}$$

Parameter	Value
Length of links,	$l_1 = 0.5\ m, l_2 = 0.6\ m$
Time step	$\Delta t = 3e - 5\ \text{sec}$
Spatial steps	$\Delta x_1 = l_1/20,$
	$\Delta x_2 = l_2/20$
Young's Modulus, E	$70\ \text{GPa}$
Density, ρ	$2742\ \text{kgm}^{-3}$
Thickness of links (m)	$c_1 = 0.003175,$
	$c_2 = 0.00238$
Maximum height for tapering at the root of the link	$b_o = 0.0654\ m$
Linear slope for tapering	$a_1 = 0.04$
Hub inertias (kgm^2)	$I_{h1} = 0.0055,$
	$I_{h2} = 0.0068$
Tip inertias, (kgm^2)	$I_{t1} = 0.0139,$
	$I_{t2} = 0.00024$
Hub mass (kg)	$m_{h2} = 0.678$
Tip mass, (kg)	$m_{t1} = 0.981,$
	$m_{t2} = 0.204$
θ_{1d} (desired)	$\pi/2\ \text{rad}$
θ_{2d} (desired)	$-\pi/2\ \text{rad}$
$K_1 = .007, \quad K_2 = 0.01$	$\alpha = 600, \quad \beta = 800$
$\gamma_1 = 1e - 3, \quad \gamma_2 = 1e - 4$	$K = 9$

Table 5.3. Parameters of the Flexible Arm for PDE Case 1.

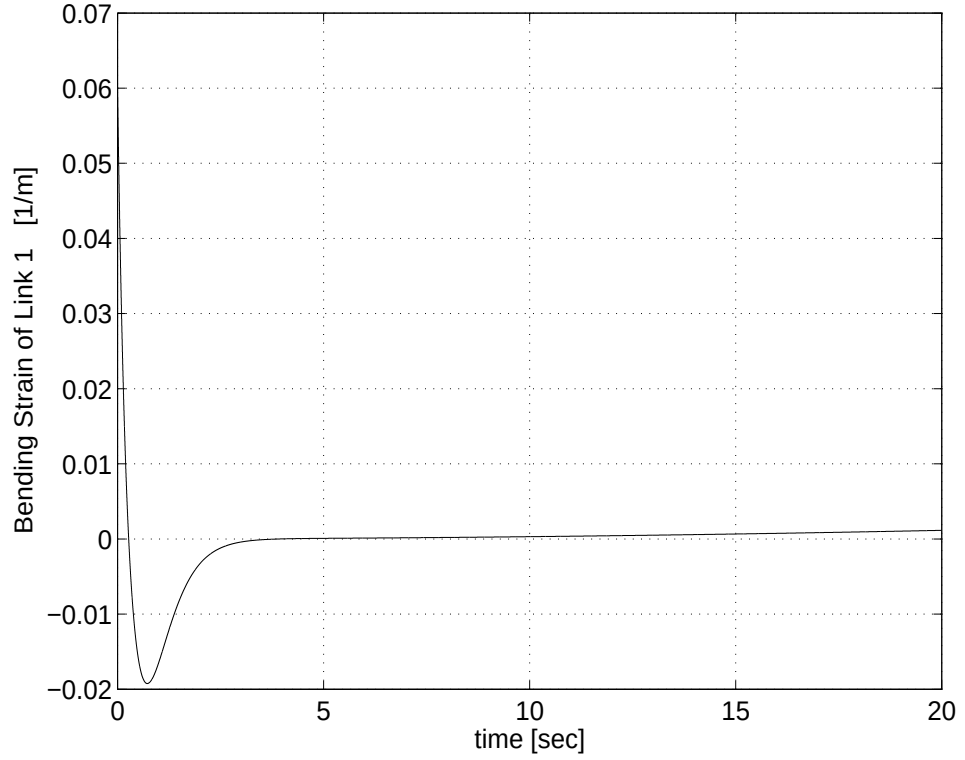


Figure 5.22. Bending strain at the end of the link 1-PDE Model Case 1

where $i = 1, 2$ and $\ddot{\theta}_i$'s are obtained from equations (3.4 - 3.5). However, the number of ODEs to solve and the computation time are increased in return of the robust stability of the numeric scheme. The parameters used in the model for system (3.2 - 3.10), partially given in [25] are listed in Table 5.3.

Although the control setup in the previous section is given for links that can have any kind of variable cross-section; in this particular simulation, rectangular cross-sections of given uniform thickness are used. For small values of tip mass and tip inertia moment relative to the ones for beam, the optimum shape is approximately a linearly *tapered* beam [11]. Therefore, the height $b_i(x)$, density $\rho_i(x)$ and cross-section area moment $I_i(x)$ at any point can be calculated with the parameters given in Table 5.3 such as

$$b_i(x) = b_o - 2l_i a_1 + 2(l_i - x)a_1$$

$$\rho_i(x) = \rho b_i(x) c_i$$

$$I_i(x) = b_i(x) c_i^3 / 12.$$

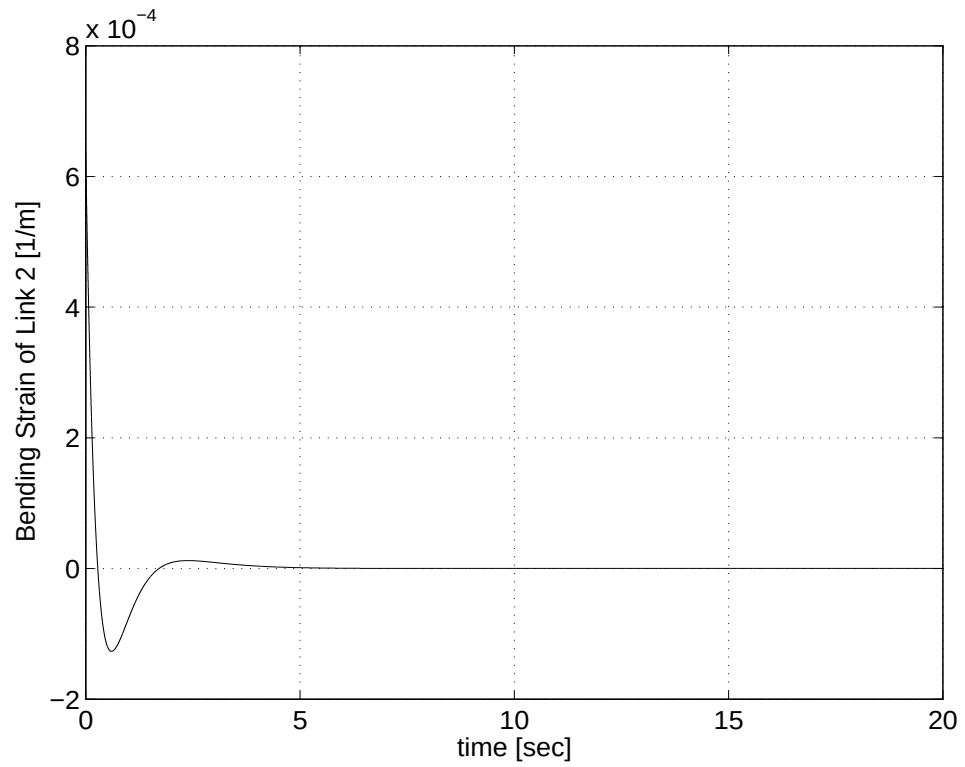


Figure 5.23. Bending strain at the end of the link 2-PDE Model Case 1

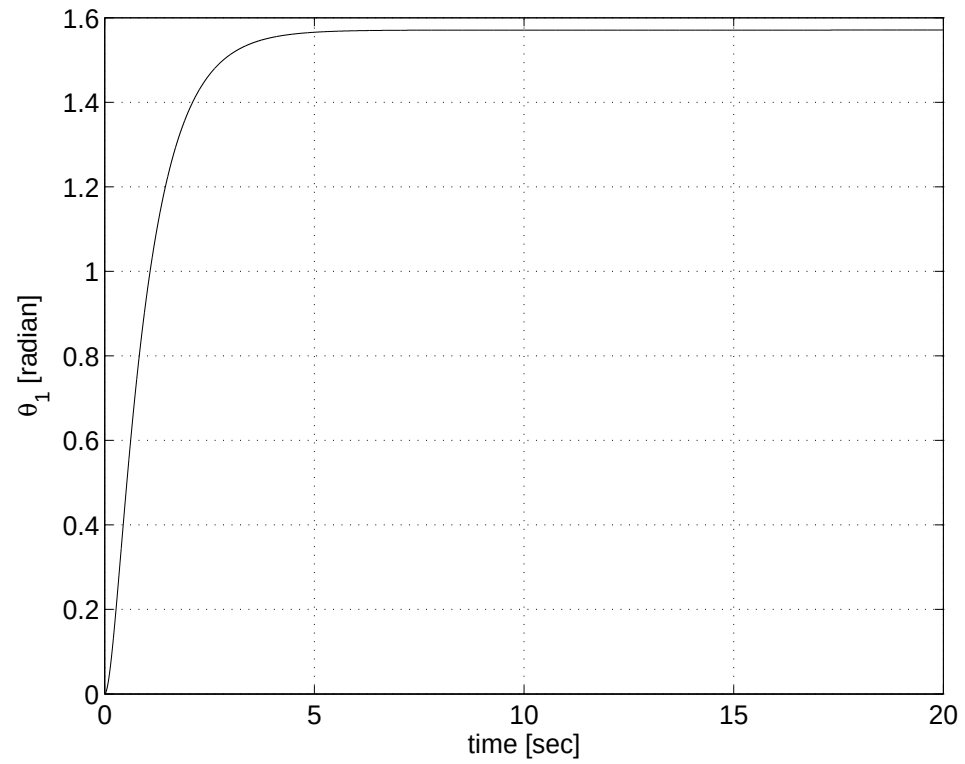


Figure 5.24. Joint Angle of the link 1-PDE Model Case 1

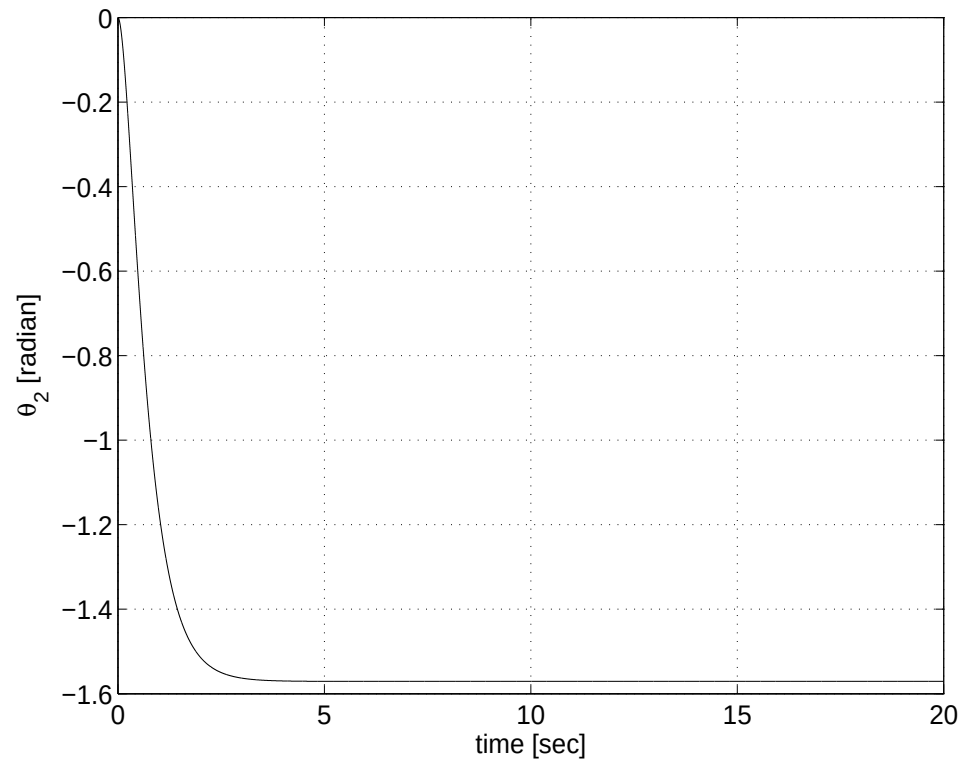


Figure 5.25. Joint Angle of the link 2-PDE Model Case 1

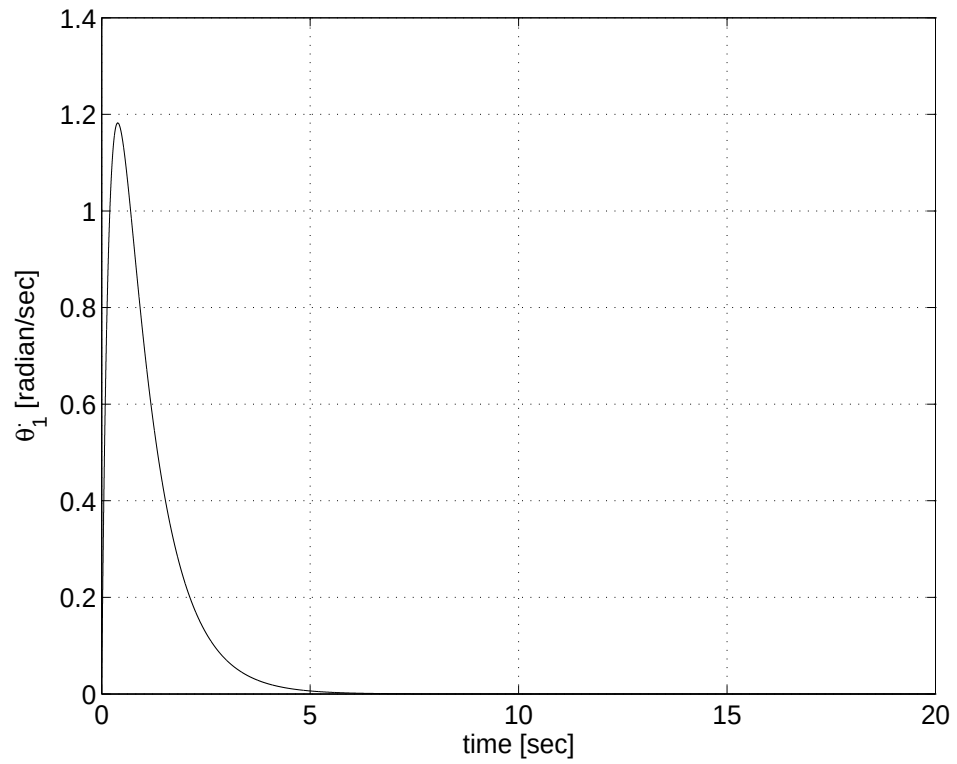


Figure 5.26. Joint Angular Velocity of the link 1-PDE Model Case 1

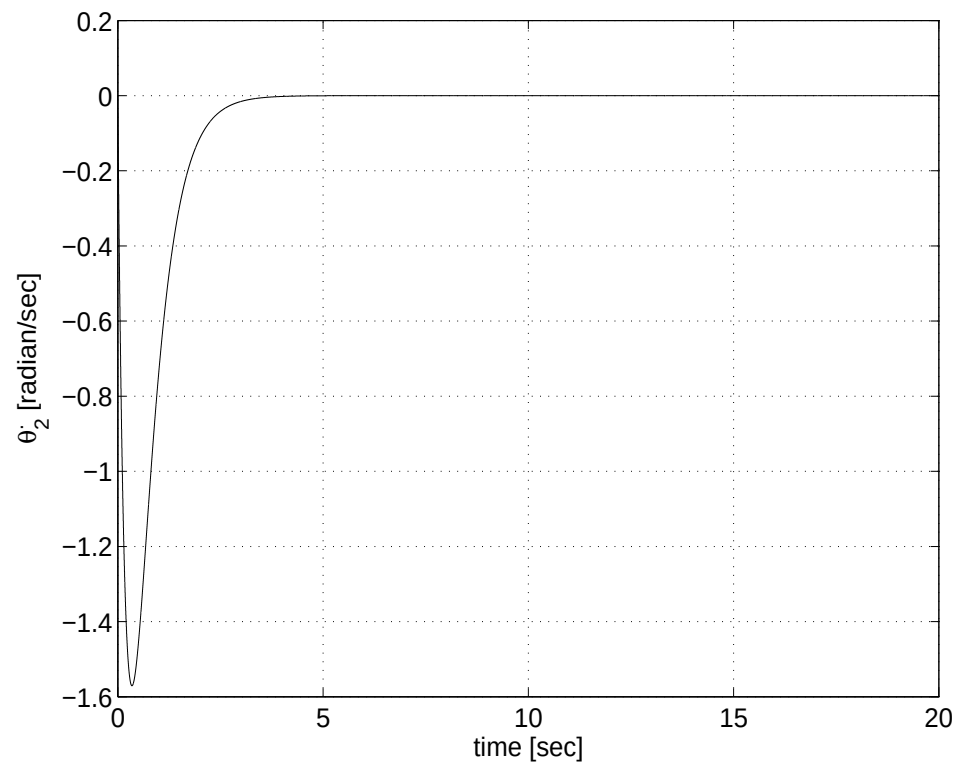


Figure 5.27. Joint Angular Velocity of the link 2-PDE Model Case 1

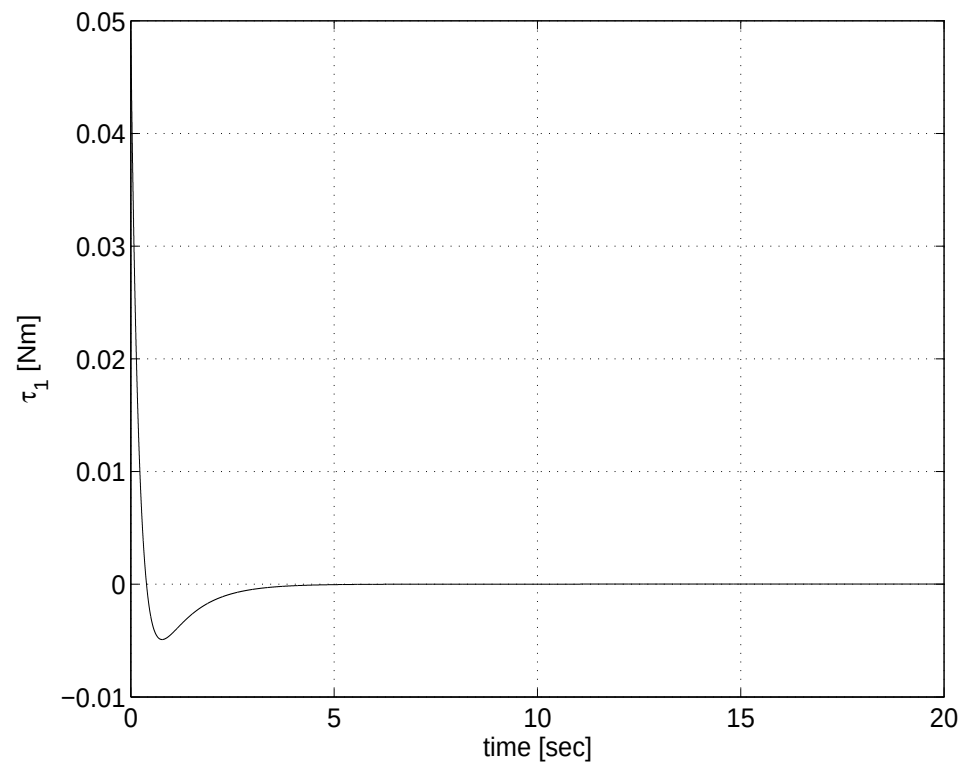


Figure 5.28. Control Torque of Joint 1-PDE Model Case 1

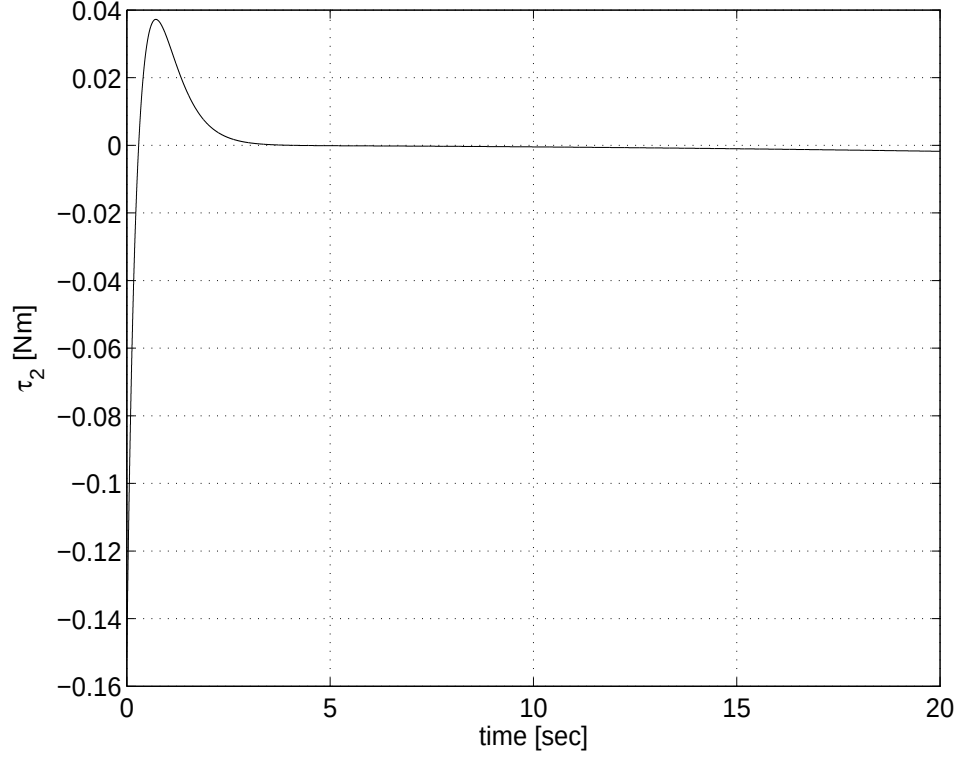


Figure 5.29. Control Torque of Joint 2-PDE Model Case 1

The simulation results for PDE Case 1, are presented in Figures (5.22 - 5.29). After the 10th second, small deviations from zero in Figure (5.22) and (5.29) could be the accumulated numerical error caused by truncation and round-off especially for long simulation run (20 seconds) with a small time-step (3e-5 second). In Figures (5.24 - 5.27), isolation of rigid coordinates from the flexible ones and decoupling as explained in Appendix C result in fast and smooth convergence. These results prove robustness of the proposed controller. Smooth time histories of all variables of interest without overshoot show the effectiveness of the controller performance especially with such demanding desired positions ($\theta_{1d} = \pi/2$ and $\theta_{2d} = -\pi/2$).

5.3.2. PDE Model Case 2

In the second part of the simulations as PDE Case 2, the proportional gain, K in the stabilizer part of the controller, see equation 4.14, that is directly proportional to the negativity of the total energy rate of the closed-loop system, has been increased to $K = 21$ from $K = 9$. In the mean time, tip mass for link 1, m_{t1} ; tip and hub mass for link 2, m_{t2} , m_{h2} have been changed at the fifth second, see Table 5.4. The simulation

Parameter			Value
Hub mass	(0-5 sec.)	(kg)	$m_{h2} = 0.678$
Tip mass	(0-5 sec.)	(kg)	$m_{t1} = 0.981$
	(0-5 sec.)		$m_{t2} = 0.204$
Hub mass	(5-10 sec.)	(kg)	$m_{h2} = 0.378$
Tip mass	(5-10 sec.)	(kg)	$m_{t1} = 0.481$
	(5-10 sec.)		$m_{t2} = 0.704$
Proportional Gain			$K = 21$

Table 5.4. Tip and Hub Mass Variation for PDE Case 2.

results for PDE Case 2, are presented in Figures (5.30 - 5.37). The step changes in masses did not affect the performance of the controller at all. We observe that tip deflections of links are getting smaller by increasing K as time responses exhibit smoother transition than the ones for the PDE Case 1. Finally, maximum tip deflection for PDE Case 2 has been decreased to $4 \cdot 10^{-7}$ m. that is almost the amplitude of the accumulated numeric noise.

5.3.3. PDE Model Case 3

The dominance of the control laws (4.12 - 4.13) has been verified with this case. The parameters for PDE Case 3 are the same as ones for PDE case 1 with the exception $\gamma_1 = \gamma_2 = 0$, in order to activate only the dominant laws. The errors between PDE case 1 and 3, are recovered well in steady-state, see Figures (5.38 - 5.41). Since the control laws (4.14 - 4.15) are designed only for asymptotic stability of the closed-loop system, the performance of the error regulation in *steady state* proves evidently the robustness and efficiency of the proposed controller.

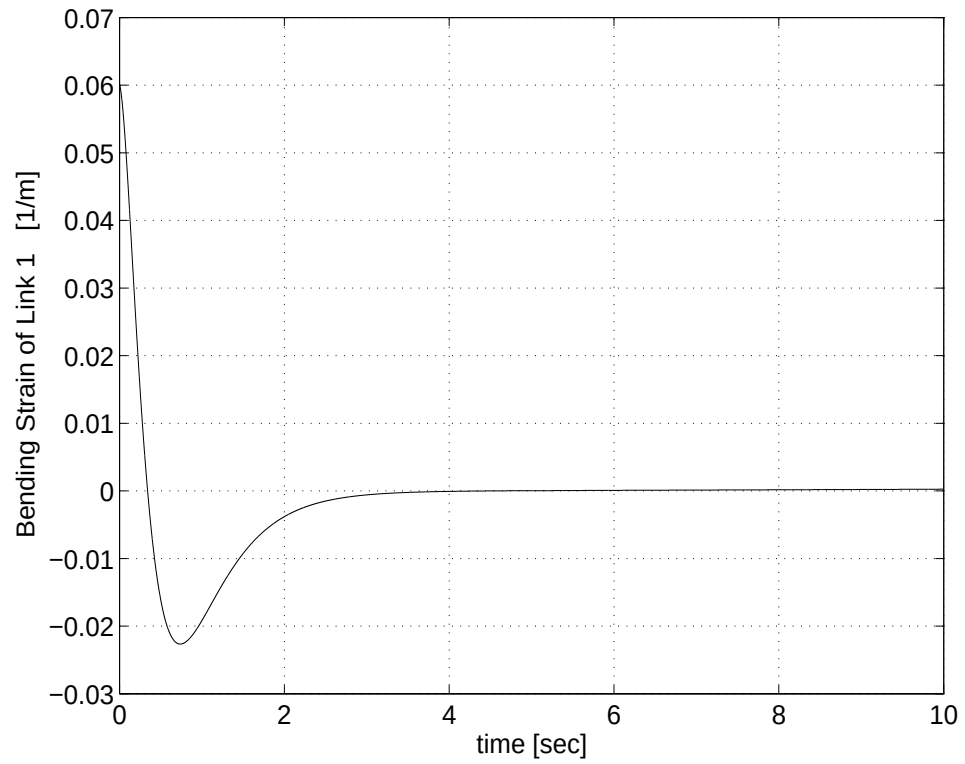


Figure 5.30. Bending strain at the end of the link 1-PDE Model Case 2

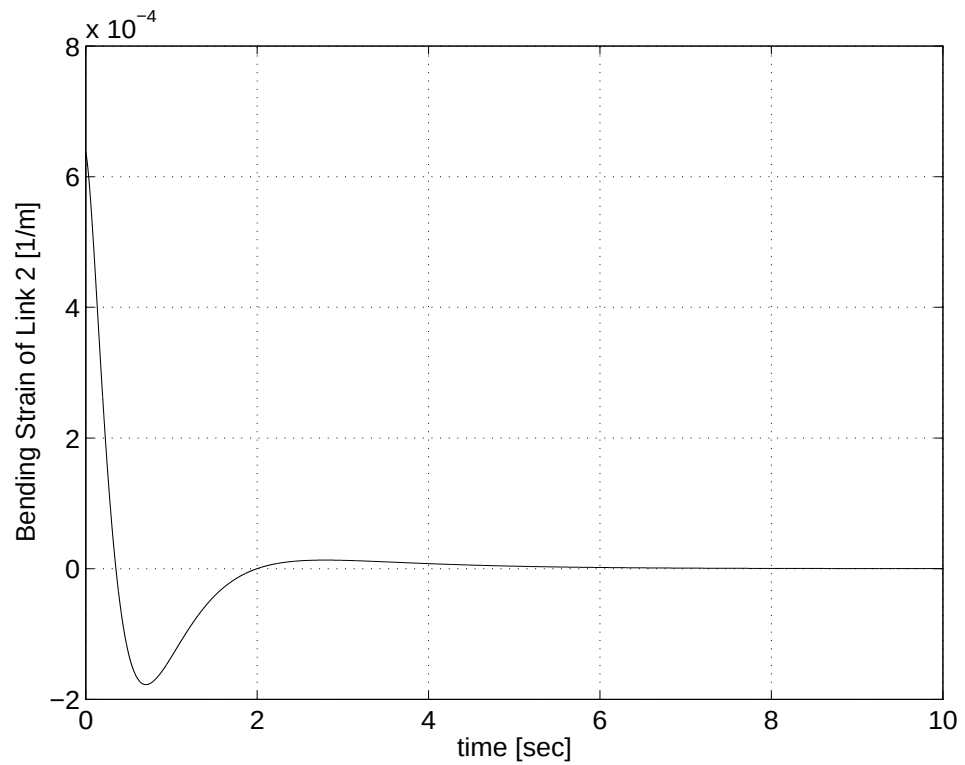


Figure 5.31. Bending strain at the end of the link 2-PDE Model Case 2

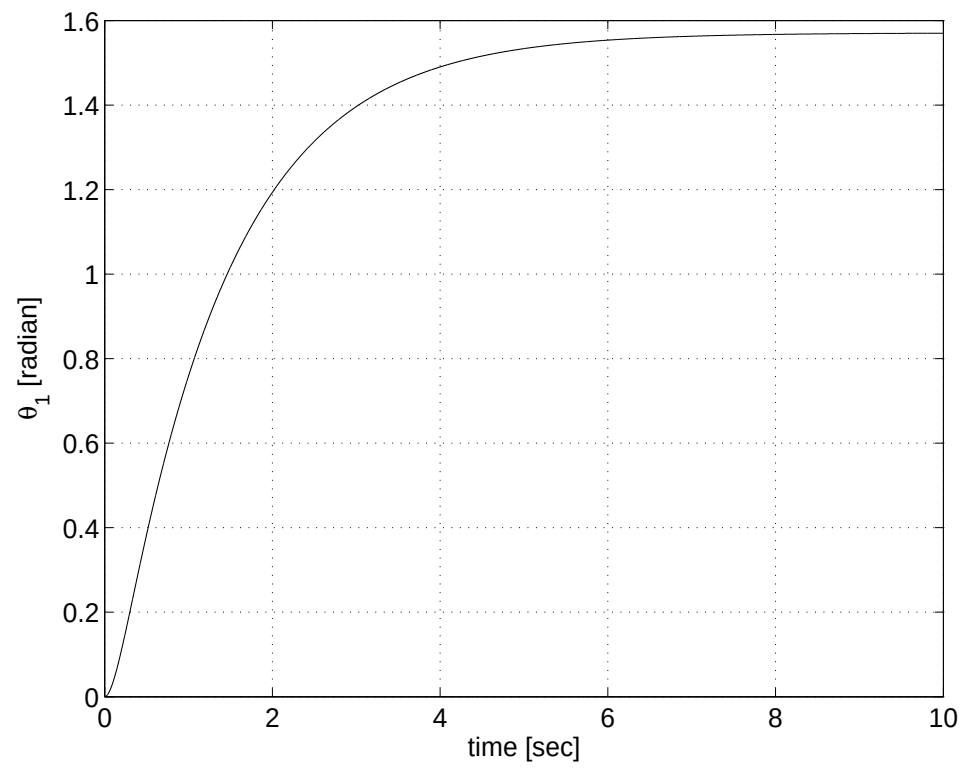


Figure 5.32. Joint Angle of the link 1-PDE Model Case 2

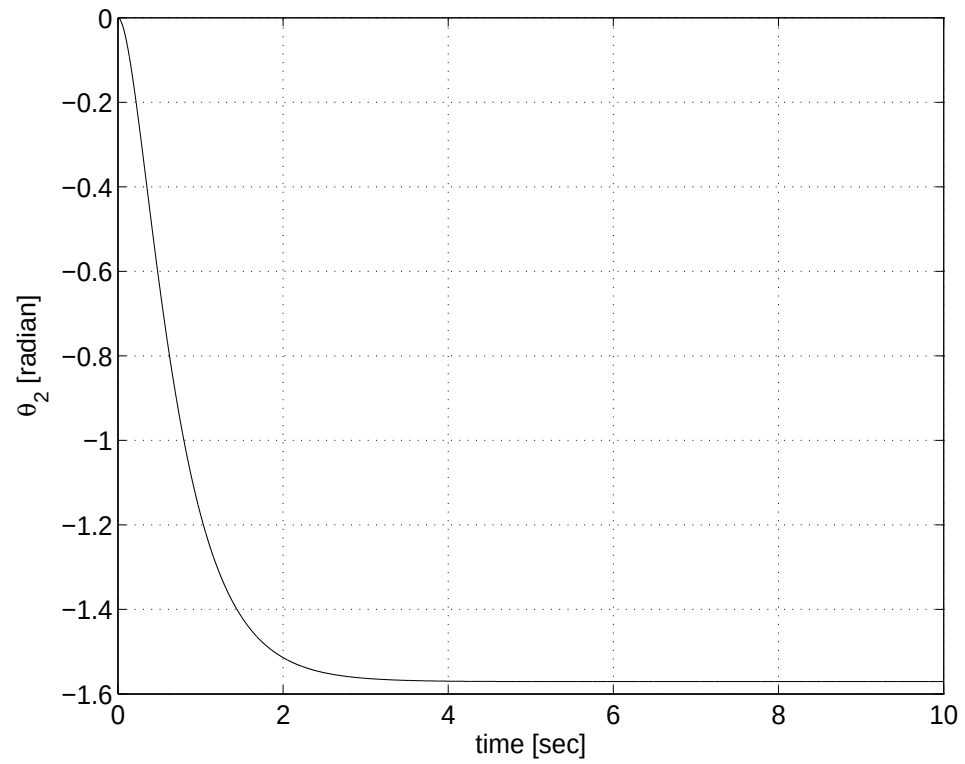


Figure 5.33. Joint Angle of the link 2-PDE Model Case 2

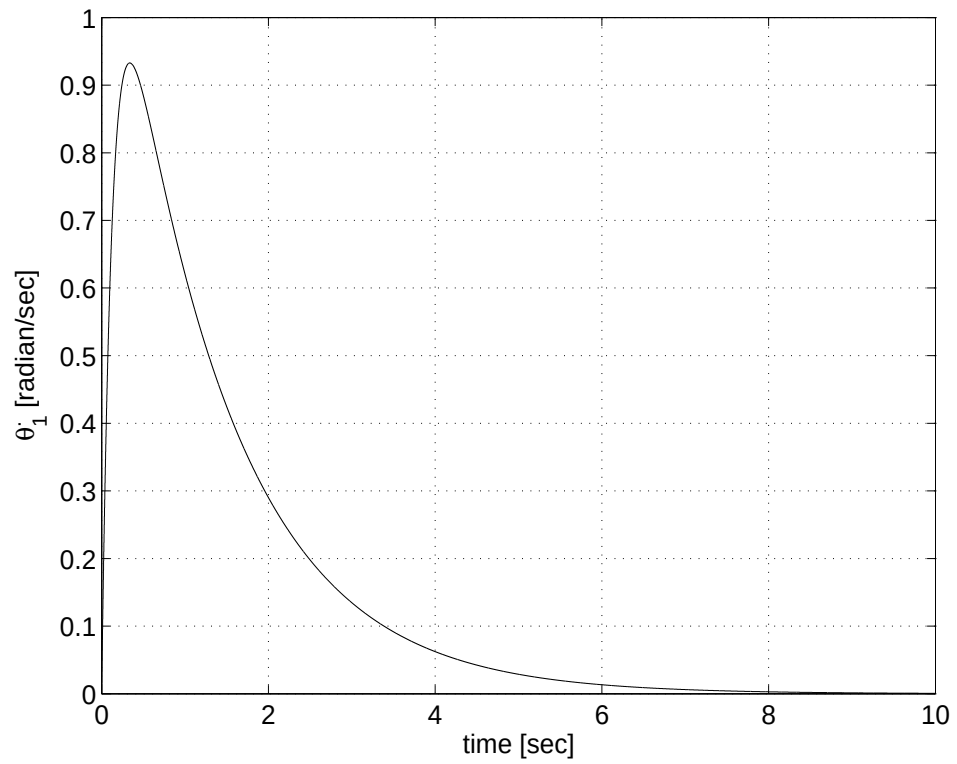


Figure 5.34. Joint Angular Velocity of the link 1-PDE Model Case 2

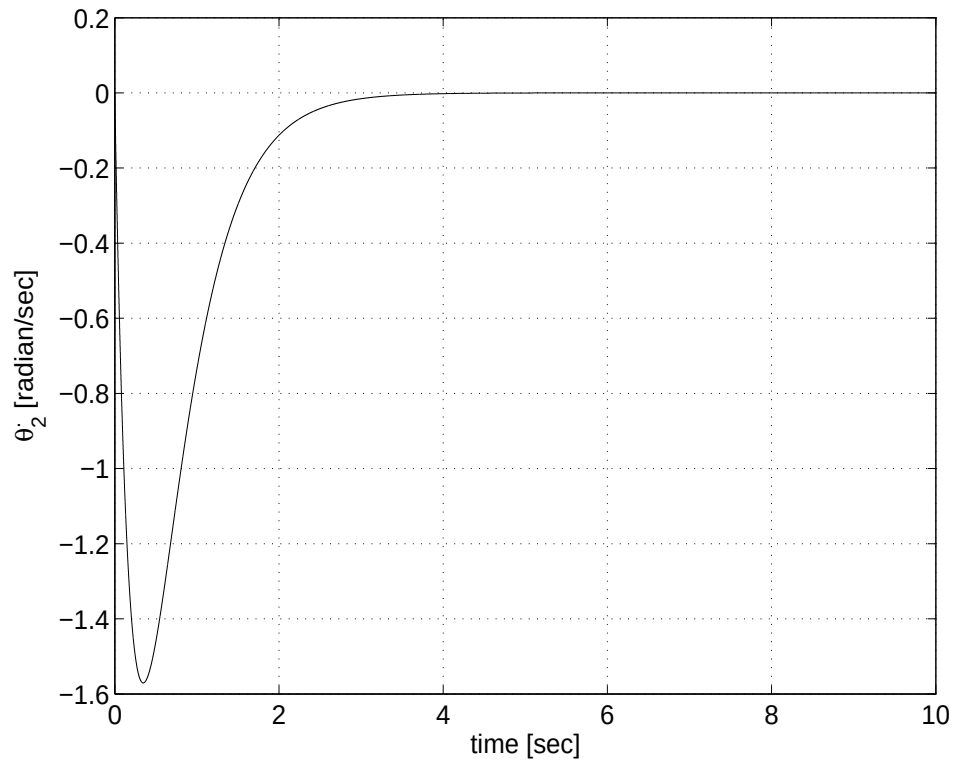


Figure 5.35. Joint Angular Velocity of the link 2-PDE Model Case 2

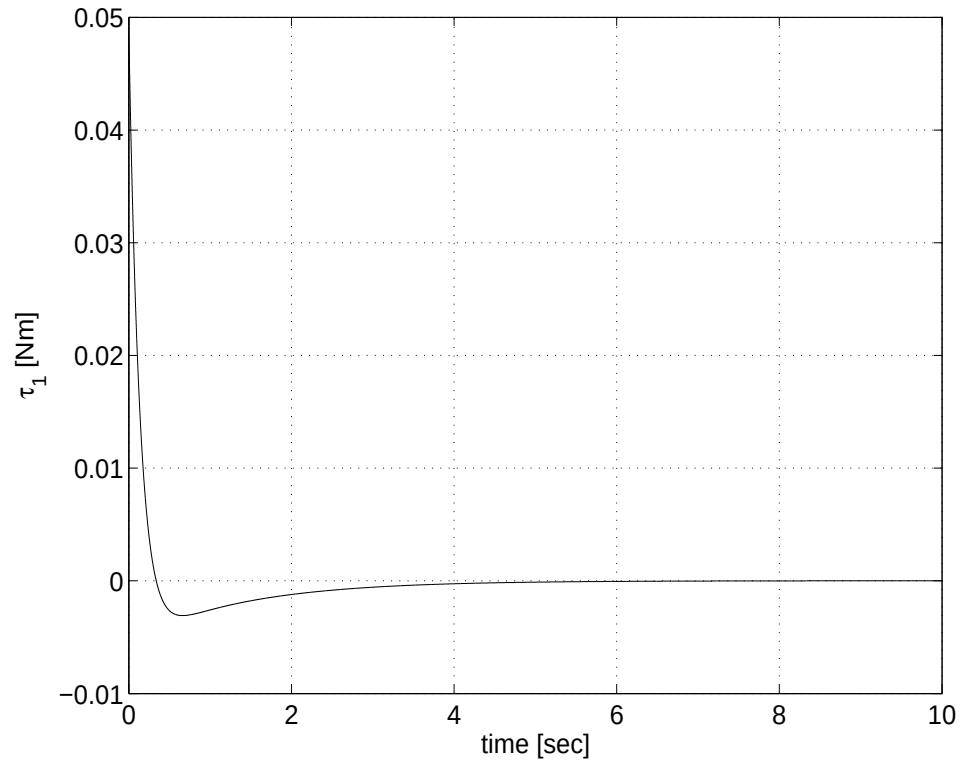


Figure 5.36. Control Torque of Joint 1-PDE Model Case 2

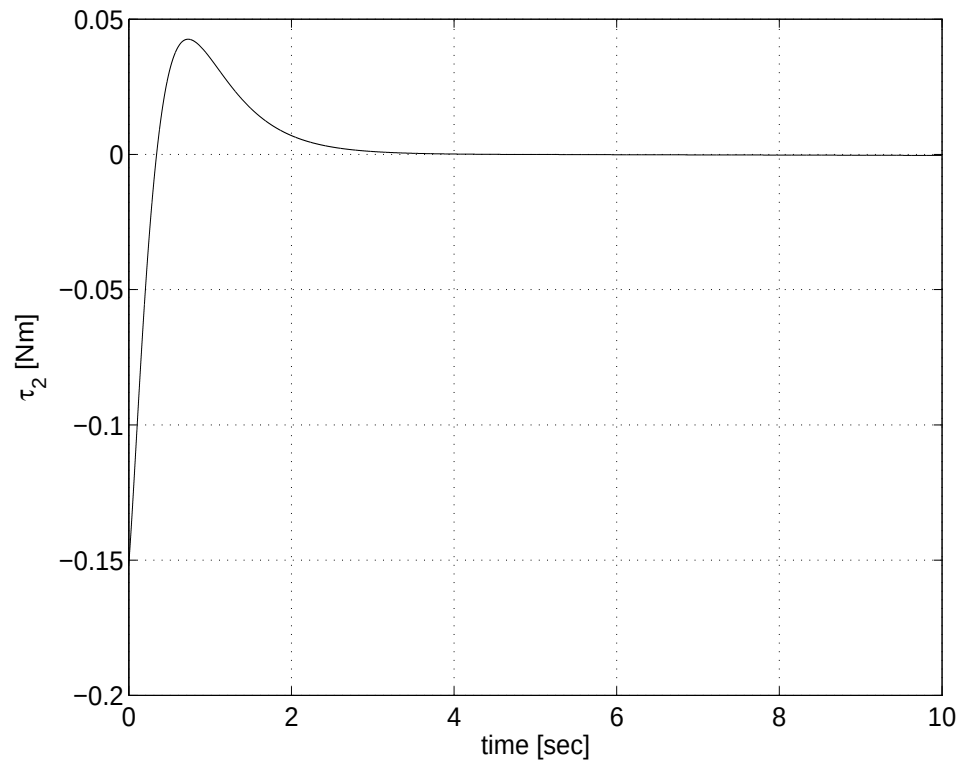


Figure 5.37. Control Torque of Joint 2-PDE Model Case 2

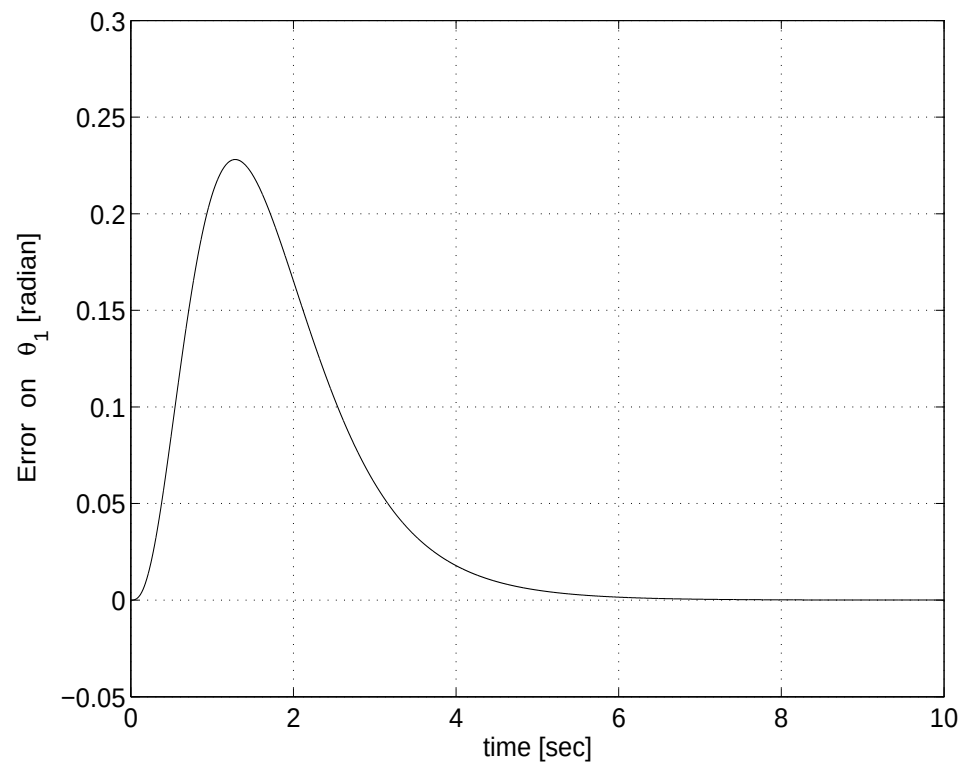


Figure 5.38. Error for Joint Angle of the link 1-PDE Model Case 3

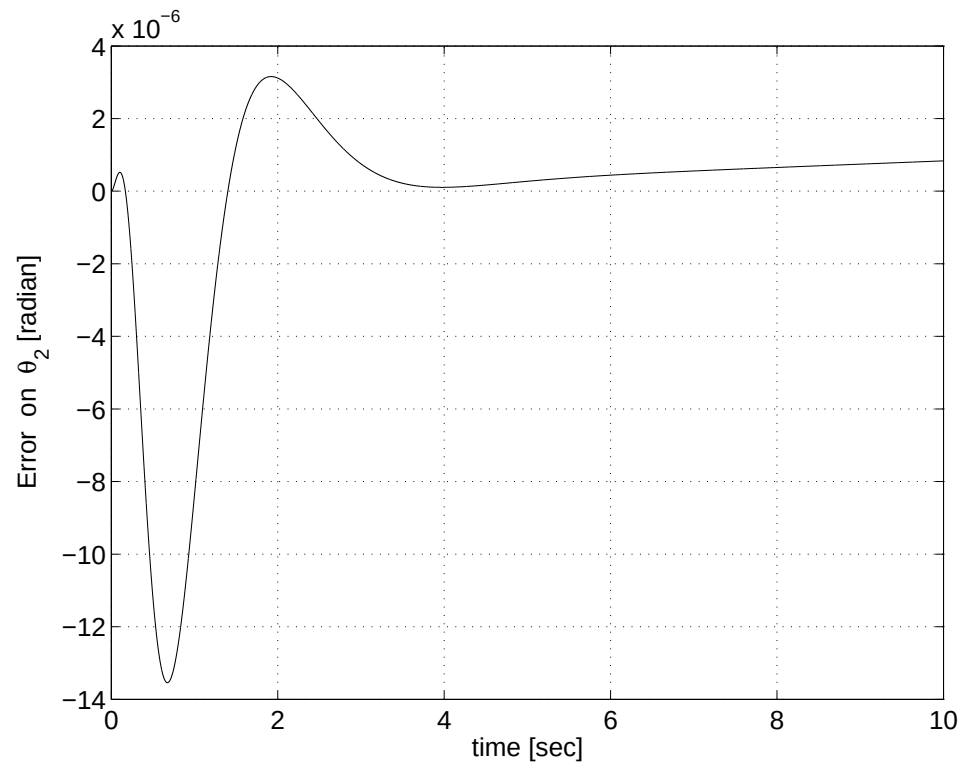


Figure 5.39. Error for Joint Angle of the link 2-PDE Model Case 3

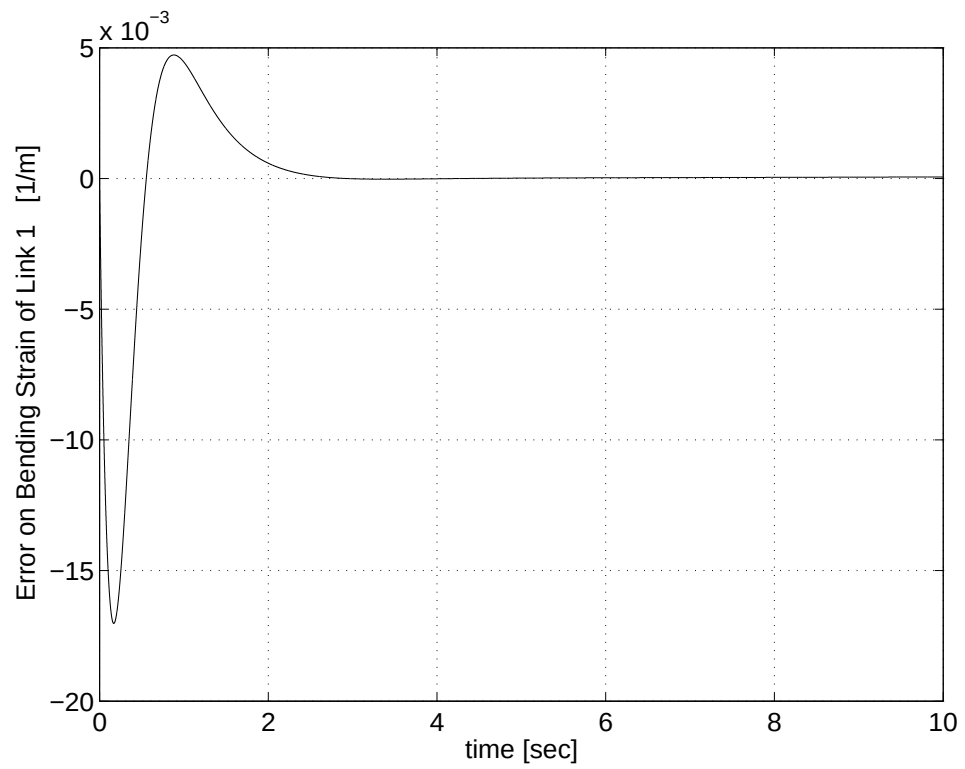


Figure 5.40. Error for Bending Strain at end of the link 1-PDE Model Case 3

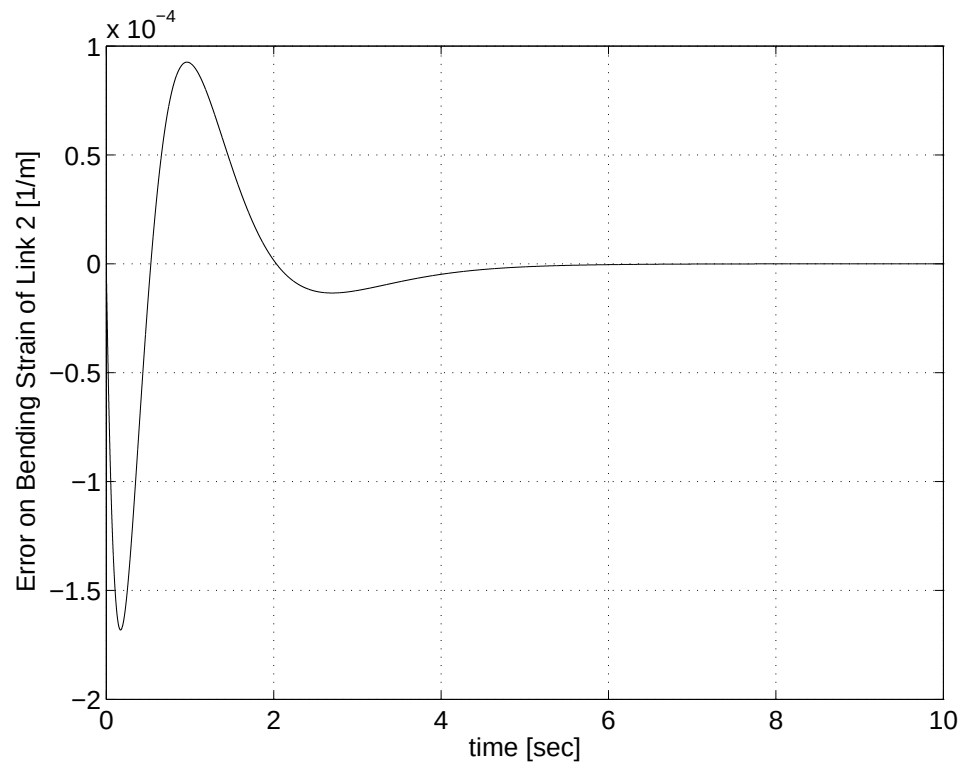


Figure 5.41. Error for Bending Strain at end of the link 2-PDE Model Case 3

5.3.4. Validation of Lyapunov Theory

The constraints of Lyapunov theory have been verified by post-processing of the data that is produced in PDE cases 1 and 2. The Lyapunov function, V that is defined by equation (4.18), should be non-negative and non-increasing while its time derivative, $\dot{V}$ should be negative that is proved in Appendix D. All assumptions and omissions that have been done for proving negativity of the derivative of the Lyapunov function (4.18) in Appendix D, are included in Figures 5.44 and 5.49 as approximate $\dot{V}$ for two different PDE cases. In order to be able to compare the exact $\dot{V}$ evolutions which are represented by Figures 5.43 and 5.48 with the approximate $\dot{V}$ ones, the errors between them have been shown in Figures 5.45 and 5.50. These results prove that the approximation for $\dot{V}$ is quite acceptable in this research. We also observe that the Lyapunov function in PDE case 1 for $K = 9$ is just start to increase for very short time in Figure 5.42. This proves the critical value of K is close to $K = 9$. However, the Lyapunov function in PDE case 2 for $K = 21$ satisfies the all constraints well in Figures 5.46 and 5.48. Compare the error in Figure 5.45 with the diminished error in Figure 5.50, is observed that increasing K to 21 improved the approximation error performance of the controller. Consequently, the robustness of the proof for sufficiently large K values has been shown by figures for PDE case 2.

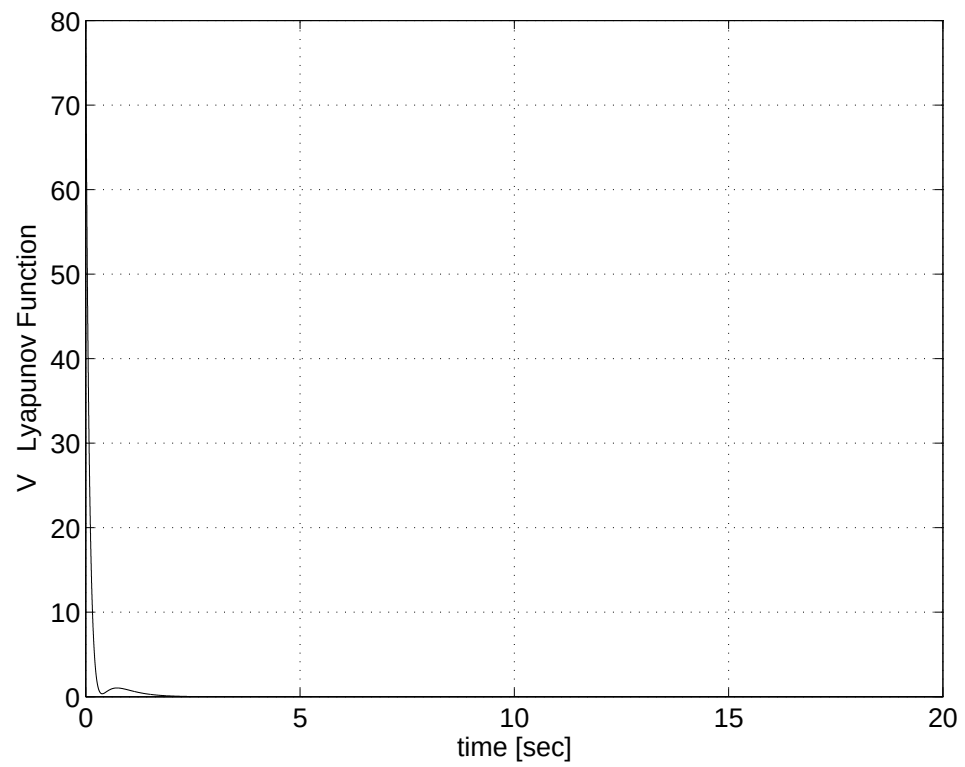


Figure 5.42. Lyapunov function V for $K = 9$ - PDE Model Case 1

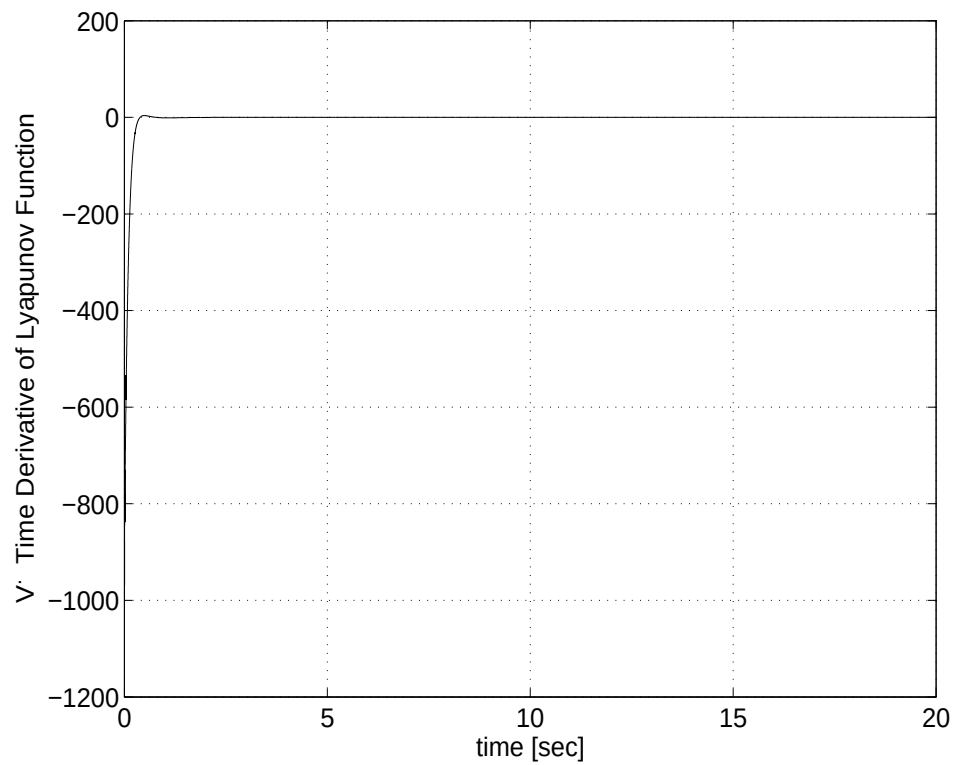


Figure 5.43. Time Derivative of Lyapunov function $\dot{V}$ for $K = 9$ - PDE Case 1

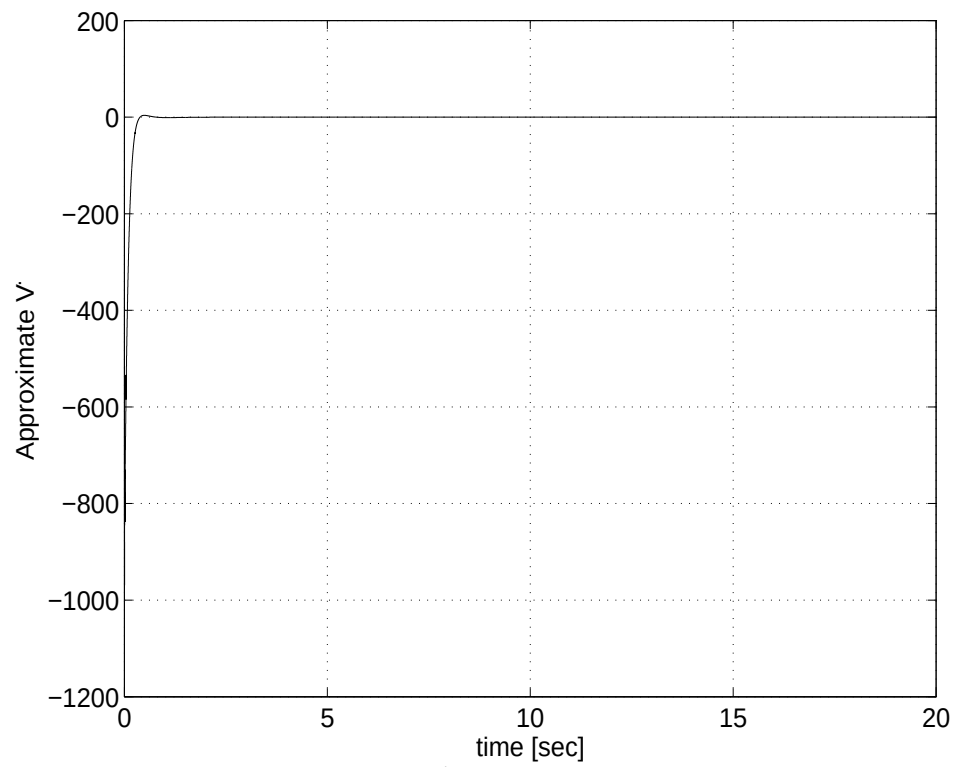


Figure 5.44. Approximate $\dot{V}$ for $K = 9$ - PDE Model Case 1

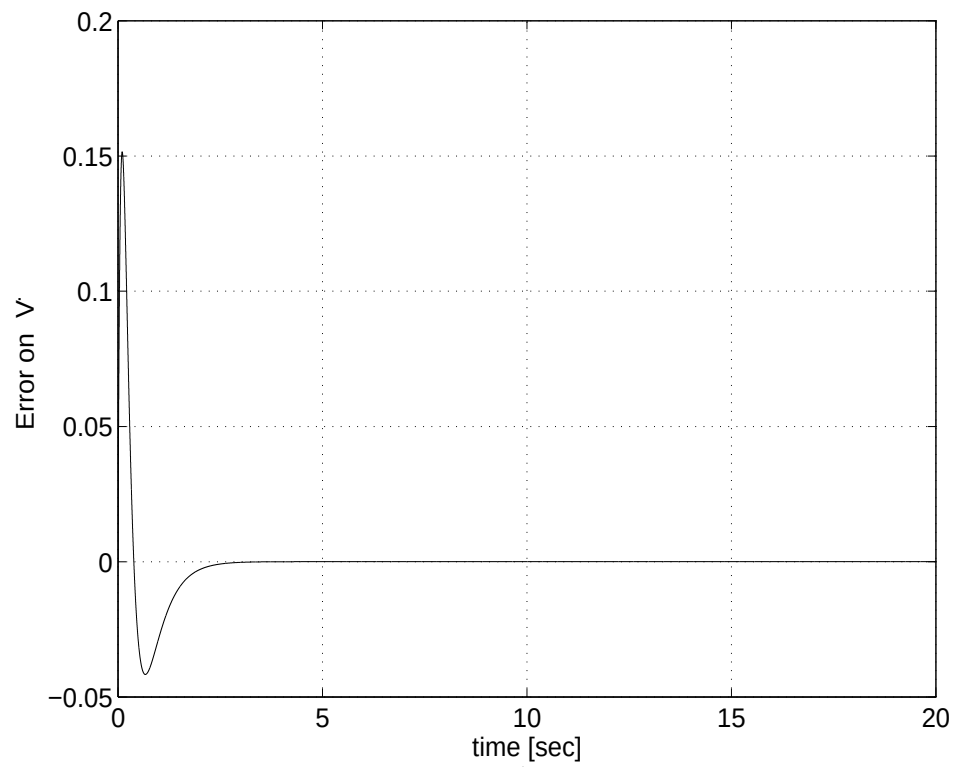


Figure 5.45. Approximation Error on $\dot{V}$ for $K = 9$ - PDE Model Case 1

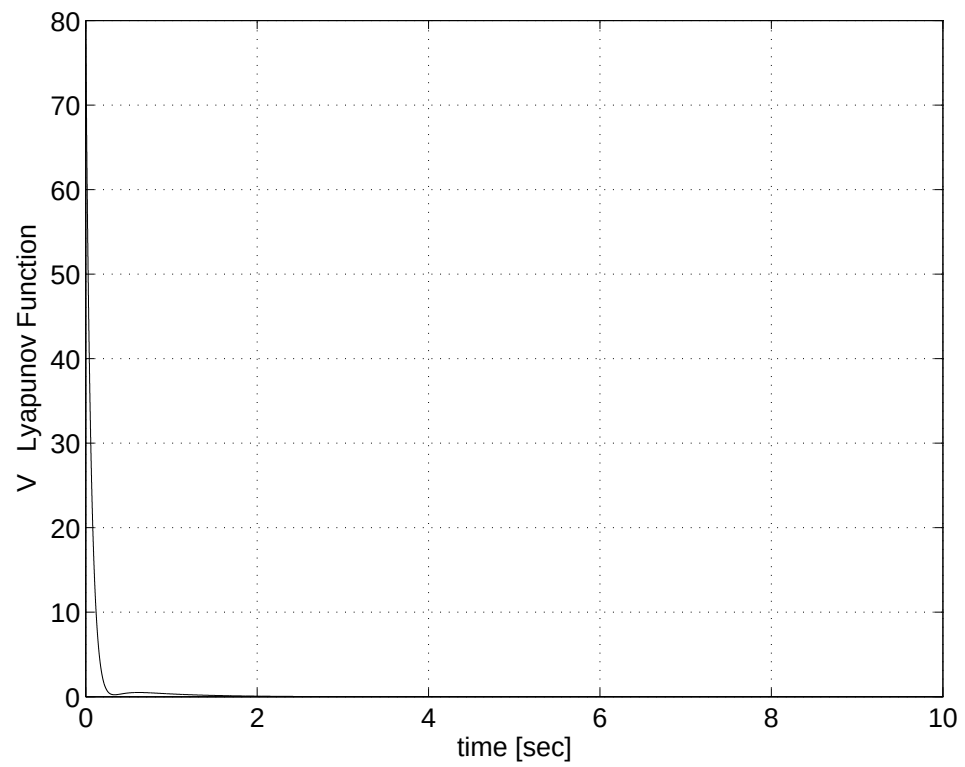


Figure 5.46. Lyapunov function V for $K = 21$ - PDE Model Case 2

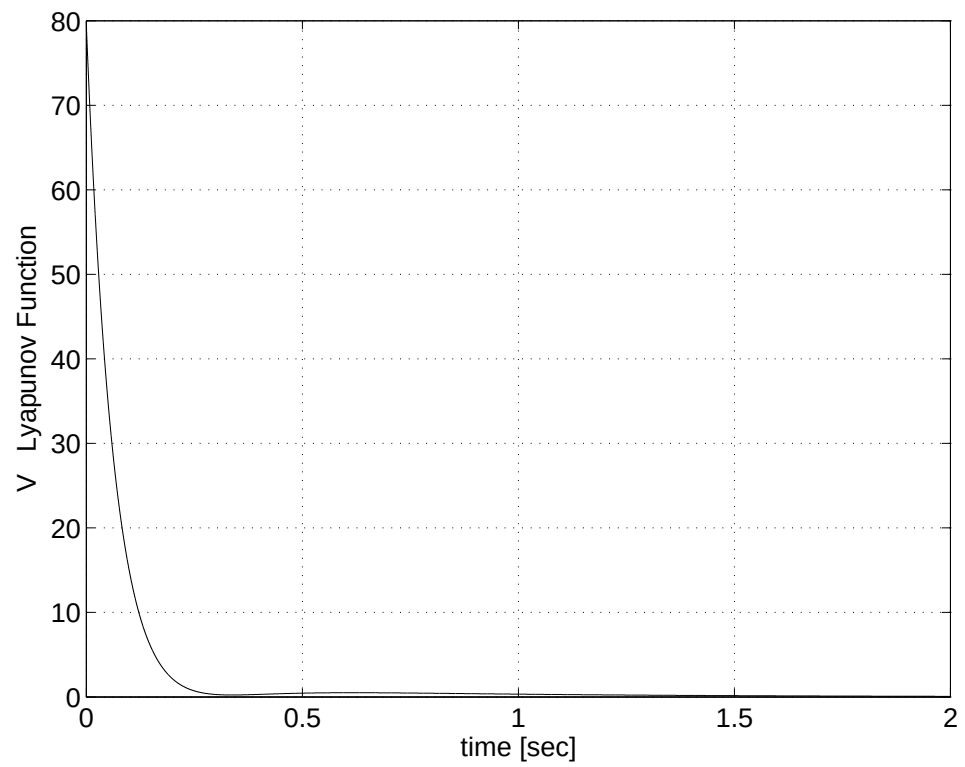


Figure 5.47. Enlarged Lyapunov function V for $K = 21$ - PDE Case 2

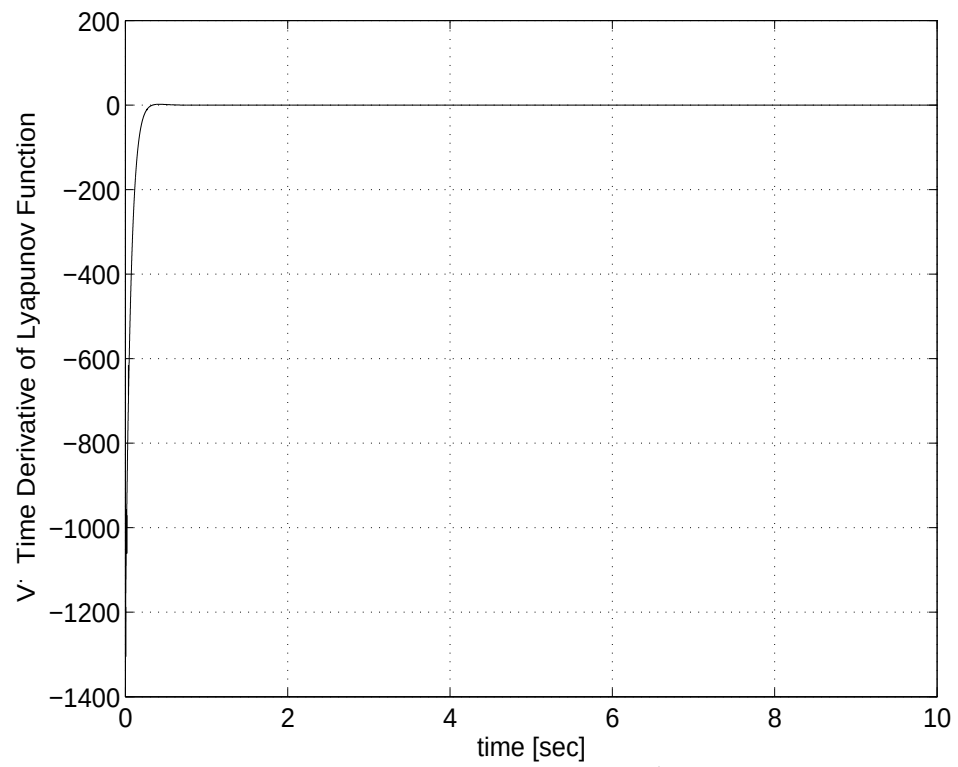


Figure 5.48. Time Derivative of Lyapunov function $\dot{V}$ for $K = 21$ - PDE Case 2

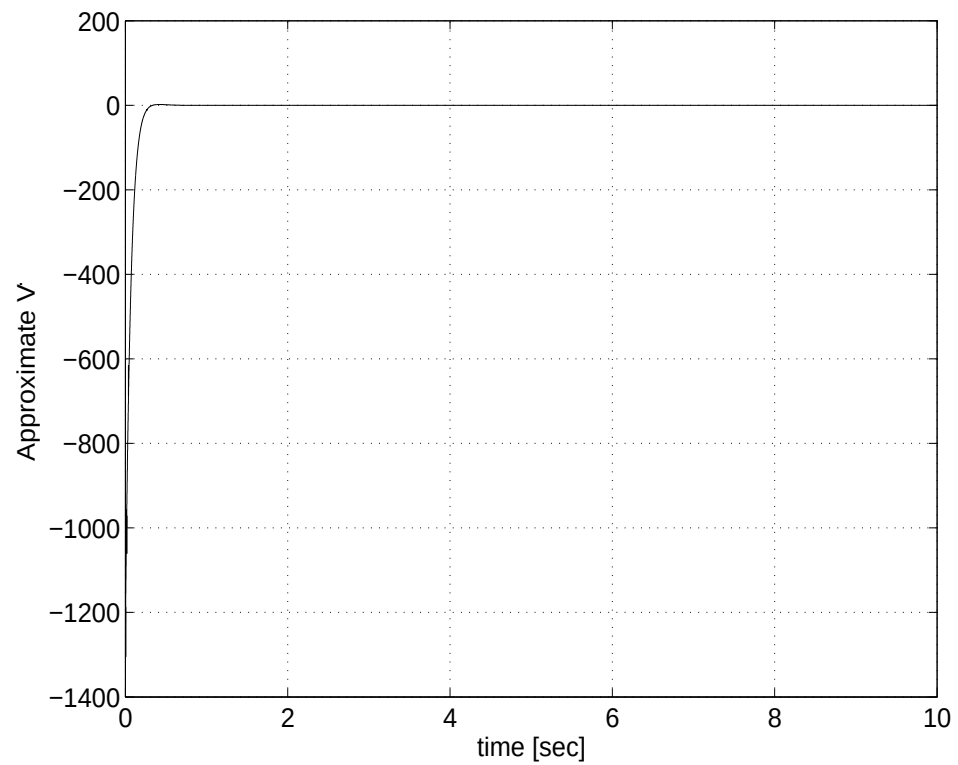


Figure 5.49. Approximate $\dot{V}$ for $K = 21$ - PDE Model Case 2

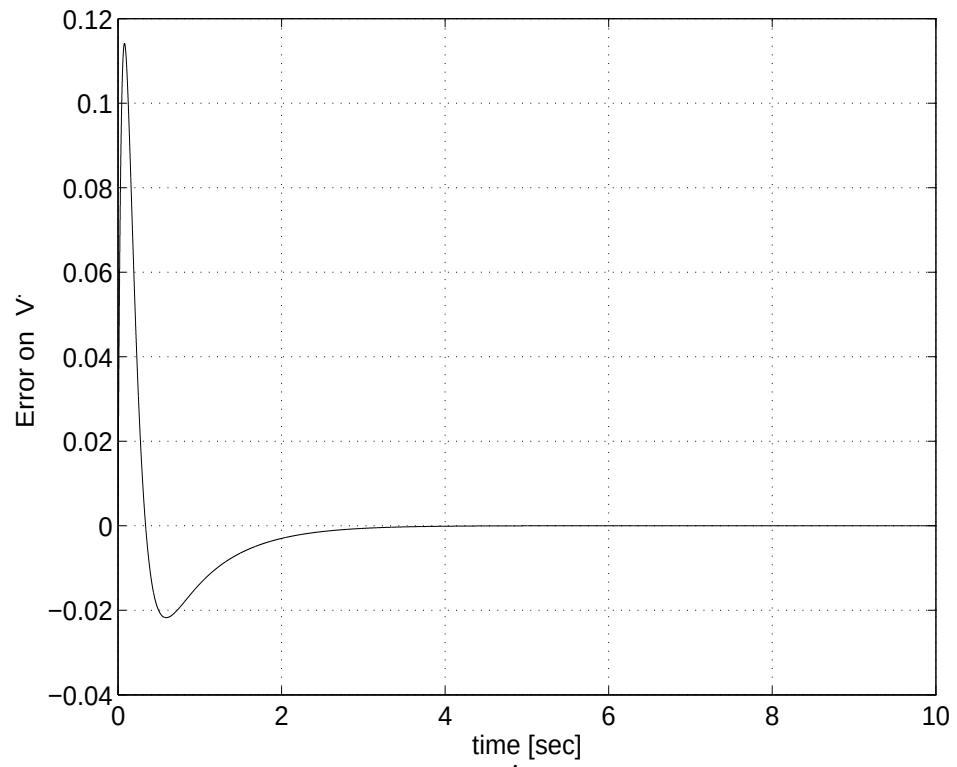


Figure 5.50. Approximation Error on $\dot{V}$ for $K = 21$ - PDE Model Case 2

6. CONCLUSIONS

It has been shown by the simulation results that the new design method can provide asymptotic stability of flexural modes and set-point regulation of rigid modes simultaneously. Also, the nonadaptive controllers have robustness with respect to large parameter uncertainties such as tip mass changes, verified by simulations for both methods, FEM and PDE. Dynamic state-feedback, strain-feedback and acceleration-feedback are used to design the control laws. The controllers for both approaches can be easily implemented with accelerometers, strain gauges and encoders. These realizations should be completed with linear observers.

In the FEM adaptive case; the domain of attraction for the exosystem states is important for fast adaptation and smooth time history. The proposed controller has also robustness with respect to external disturbances, since the controller achieved good performance with Adaptive Internal Model when the amplitude of the disturbance increased more than sevenfold of its original value. Decoupling by separation of the controller tasks such as inner and outer control loops improved the robustness of the proposed controller, being the main contribution for the adaptive internal model. The stabilizer part of the nonlinear controller has been introduced for the first time in this research. The fast convergence and accuracy for determining global optimum of the new evolutionary algorithm, BB-BC, is ideal for real-time controller implementations. It has found the optimum controller gains in just 0.8 seconds. This optimization of the stabilizer gains improved the performance significantly and removed the necessity for saturated control inputs that was also verified by simulation results. Since the unknown external disturbances deteriorate markedly the controller performance, the robustness of the controller has been proven once more in the FEM adaptive cases.

The derived PDE(Partial Differential Equations) infinite dimensional model, is used elaborately to design new Lyapunov-based controllers. In the proof of the main theorem, the infinite dimensionality of the problem has been retained as opposed to other energy-based approaches for multi-link robot arms in the literature. In the second

part of this research, derivation of the PDE model for nonuniform multi-link robot arm and the proofs in detail, is an effort and also a major progress. In future, adaptation of the controller with respect to external unknown disturbances would be an extension to the proposed controller that already manages the nonuniform flexible robot arms successfully.

Comparing the time responses for FEM with the simulation results for PDE, it is observed that the required control energy in PDE cases is much less than the one in FEM due to the exactness of PDE and different rigidity of the beam material. The convergence rate of the states in PDE is relatively slow. However, PDE responses are smoother than the ones in FEM approach, and have no overshoot or oscillations for all states. Even though the PDE approach as an exact model has certain advantages over FEM which ignores the high frequency dynamics, resulting in control and observation spillovers, the performance of the proposed controller with FEM is quite acceptable mainly due to the success of the robust optimal stabilizer which is applied to multi-link robot arms for the first time in this work. Besides, dynamic real-time controller implementation is quite easy for FEM approach.

The PDE controller can be optimized with the evolutionary algorithm for future research. Both approaches, Adaptive Internal Model (AIM) and PDE, can also be combined to design a new controller by applying the nonlinear semigroup theory with AIM. In future research, dynamic controllers for boundary controlled distributed parameter systems can be improved to utilize the infinite dimensional Adaptive Internal Model.

APPENDIX A: Numerical Examples for FEM

Overall mass matrix can be given for just a fixed time point, since its nonlinear part is state dependent. For simplicity, $\hat{M}$ is given below implicitly as an example for time $t = 1 \text{ sec.}$ and tip mass $m_2 = 0.2 \text{ kg}$ values such that

$$\varphi^T \cdot \hat{M} = \begin{bmatrix} 0.4534 & 0.1728 & 0.1316 & 0.1721 & -0.0707 \\ 0.1728 & 0.0793 & 0.0469 & 0.0793 & -0.0334 \\ 0.1316 & 0.0469 & 0.0430 & 0.0466 & -0.0187 \\ 0.1721 & 0.0793 & 0.0466 & 0.0793 & -0.0334 \\ -0.0707 & -0.0334 & -0.0187 & -0.0334 & 0.0143 \end{bmatrix}$$

where the 14×5 coordinate transformation matrix φ is given for tip mass $m_2 = 0.2 \text{ kg}$ value, below

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.0665 & 0 & 0 \\ 0 & 0 & 0.6795 & 0 & 0 \\ 0 & 0 & 0.1876 & 0 & 0 \\ 0 & 0 & 0.6601 & 0 & 0 \\ 0 & 0 & 0.2509 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.0344 \\ 0 & 0 & 0 & 0 & -0.3835 \\ 0 & 0 & 0 & 0 & -0.1183 \\ 0 & 0 & 0 & 0 & -0.5957 \\ 0 & 0 & 0 & 0 & -0.2246 \\ 0 & 0 & 0 & 0 & -0.6576 \end{bmatrix}$$

The vectors φ_1 , and φ_2 mentioned in section 2.3, consist of the nonzero elements of the third and the fifth columns of φ respectively. Since we took only the first modes of both links and also v_e is excluded for the link 1, φ_1 is a 5×1 vector and φ_2 is a 6×1 vector. For simplicity, overall stiffness matrix $\hat{K}$ is given below implicitly as an

example for tip mass $m_2 = 0.2 \text{ kg}$ value such that

$$\varphi^T \cdot \hat{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 56.7041 & -53.8813 & 0 \\ 0 & 0 & -53.8813 & 225.0000 & 0 \\ 0 & 0 & 0 & 0 & 11.3754 \end{bmatrix}$$

The symbolic derivation program prepares first the equations for nodal coordinates. These equations that includes coriolis and centrifugal matrix $\hat{C}$ implicitly, are too long to be listed here. Therefore, a truncated version of them is given below

$$\begin{aligned} dxdt(1,1) &= -117/1000 * q(6) * t(5) * q(3) * t(10) * q(10) * c + \dots \\ dxdt(2,1) &= (-1/4 * m2 * q(6)^2 - 351/2000 * t(5) * q(8) * q(6) - \dots \\ dxdt(3,1) &= -1/100880 * q(3) * (-8767968042535 * q(6)^2 * t(1) - \dots \\ dxdt(4,1) &= 1/295147 * q(3) * (-68516477988 * q(6)^2 * t(4) + \dots \\ dxdt(5,1) &= -1391740959132551/23058432 * q(6)^2 * q(4) + \dots \\ dxdt(6,1) &= -685164779/2951479 * q(6)^2 * q(4) - \dots \\ dxdt(7,1) &= -24300 * t(5) * q(3) + 24300 * t(3) * q(3) + \dots \\ dxdt(8,1) &= 2025 * t(5) * q(3) - 2025 * t(3) * q(3) - \dots \\ dxdt(9,1) &= (-117/2 * q(6)^2 - 117/5 * t(5) * q(8) * q(6)) * s + \dots \\ dxdt(10,1) &= 1/295147905179352825856 * q(5) * (-332 * t(9) + \dots \\ dxdt(11,1) &= -2025 * t(11) * q(5) + 243 * t(10) * q(5) - \dots \\ dxdt(12,1) &= 1/2951479 * q(5) * (-59767450 * t(6) - \dots \\ dxdt(13,1) &= 2025 * t(11) * q(5) - 24300 * t(10) * q(5) + \dots \\ dxdt(14,1) &= -225 * t(11) * q(5) + 2025 * t(10) * q(5) - \dots \end{aligned} \tag{A.1}$$

where $s = \sin(\theta_2 + v_e)$, $c = \cos(\theta_2 + v_e)$, and m_2 is the variable tip mass. Since all nodal coordinates are replaced with modal ones via transformation, in the above equation (A.1), $t(i)$ $i = 1$ to 5 consists of the components of φ_1 and $t(i)$ $i = 6$ to 11 consists of the components of φ_2 . The linear part of overall mass matrix depends on the variable tip mass. Therefore φ_1 , φ_2 and overall transformed stiffness matrix are varying with tip mass m_2 . Finally, the matrix $D = \varphi^T M^{-1}$ can be expressed implicitly for time $t = 1$ sec. and tip mass $m_2 = 0.2$ kg values as follows

$$D \cdot \varphi = \begin{bmatrix} 49.957 & -6.297 & -57.23 & -45.01 & -2.28 \\ -6.297 & 4.06e5 & -38457 & -2.03e5 & 3.32e5 \\ -57.23 & -38457 & 35252 & 29662 & -14958 \\ -45.01 & -2.03e5 & 29662 & 1.30e5 & -1.22e5 \\ -2.28 & 3.32e5 & -14958 & -1.22e5 & 4.08e5 \end{bmatrix}$$

where D and the equation (A.1) are multiplied to obtain f_i 's such as

$$\begin{bmatrix} f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{bmatrix} = D \cdot dxdt$$

where $dxdt$ is a 14×1 vector whose entries are given above.

APPENDIX B: Derivation of Motion Equations for PDE

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T_1 dt &= \int_{t_1}^{t_2} \int_0^{l_1} \rho_1(x_1) [x_1 \dot{\theta}_1 + \dot{w}_1] \delta [x_1 \dot{\theta}_1 + \dot{w}_1] dx_1 dt \\
&= - \int_{t_1}^{t_2} \int_0^{l_1} \rho_1(x_1) [x_1 \ddot{\theta}_1 + \ddot{w}_1] [x_1 \delta \theta_1 + \delta w_1] dx_1 dt \\
\int_{t_1}^{t_2} \delta T_2 dt &= \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \delta [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] dx_2 dt \\
&\quad + \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) [x_2 \dot{\theta}_2 + \dot{w}_2] \delta [x_2 \dot{\theta}_2 + \dot{w}_2] dx_2 dt \\
&\quad + \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \cos(\theta_2) [x_2 \dot{\theta}_2 + \dot{w}_2] \delta [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] dx_2 dt \\
&\quad + \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \cos(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \delta [x_2 \dot{\theta}_2 + \dot{w}_2] dx_2 dt \\
&\quad - \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] [x_2 \dot{\theta}_2 + \dot{w}_2] \delta \theta_2 dx_2 dt \\
&= - \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \left\{ [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + \cos(\theta_2) [x_2 \ddot{\theta}_2 + \ddot{w}_2] \right. \\
&\quad \left. - \dot{\theta}_2 \sin(\theta_2) [x_2 \dot{\theta}_2 + \dot{w}_2] \right\} [l_1 \delta \theta_1 + \delta w_1(l_1, t)] dx_2 dt \\
&\quad - \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \left\{ [x_2 \ddot{\theta}_2 + \ddot{w}_2] + \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \right. \\
&\quad \left. - \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \right\} \delta w_2 dx_2 dt \\
&\quad - \int_{t_1}^{t_2} \int_0^{l_2} \rho_2(x_2) \left\{ \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] - \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \right. \\
&\quad \left. + [x_2 \ddot{\theta}_2 + \ddot{w}_2] \right\} x_2 + \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] [x_2 \dot{\theta}_2 + \dot{w}_2] \delta \theta_2 dx_2 dt \\
\int_{t_1}^{t_2} \delta T_3 dt &= -I_{h1} \int_{t_1}^{t_2} \ddot{\theta}_1 \delta \theta_1 dt
\end{aligned}$$

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T_4 dt &= I_{t1} \int_{t_1}^{t_2} [\dot{\theta}_1 + \dot{w}_{1x}(l_1, t)] \delta [\dot{\theta}_1 + \dot{w}_{1x}(l_1, t)] dt \\
&\quad + m_{t1} \int_{t_1}^{t_2} [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \delta [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] dt \\
&= - \int_{t_1}^{t_2} \left\{ I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] + m_{t1} l_1 [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \right\} \delta \theta_1 dt \\
&\quad - \int_{t_1}^{t_2} \left\{ I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] \right\} \delta w_{1x}(l_1, t) dt - m_{t1} \int_{t_1}^{t_2} [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \delta w_1(l_1, t) dt \\
\int_{t_1}^{t_2} \delta T_5 dt &= I_{h2} \int_{t_1}^{t_2} [\dot{\theta}_1 + \dot{\theta}_2] \delta [\dot{\theta}_1 + \dot{\theta}_2] dt
\end{aligned}$$

$$\begin{aligned}
& + m_{h2} \int_{t_1}^{t_2} [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \delta [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] dt \\
& = - \int_{t_1}^{t_2} \{ I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] + m_{h2} l_1 [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \} \delta \theta_1 dt \\
& - \int_{t_1}^{t_2} \{ I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] \} \delta \theta_2 dt - m_{h2} \int_{t_1}^{t_2} [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \delta w_1(l_1, t) dt \\
\int_{t_1}^{t_2} \delta T_6 dt & = -I_{t2} \int_{t_1}^{t_2} [\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t)] \delta [\theta_1 + \theta_2 + w_{2x}(l_2, t)] dt \\
& - m_{t2} \int_{t_1}^{t_2} \{ \cos(\theta_2) [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t)] + [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \} \delta [l_1 \theta_1 + w_1(l_1, t)] dt \\
& - m_{t2} \int_{t_1}^{t_2} \{ \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t)] \} \delta [l_2 \theta_2 + w_2(l_2, t)] dt \\
& + m_{t2} \int_{t_1}^{t_2} \dot{\theta}_2 \sin(\theta_2) [l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t)] \delta [l_1 \theta_1 + w_1(l_1, t)] dt \\
& + m_{t2} \int_{t_1}^{t_2} \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \delta [l_2 \theta_2 + w_2(l_2, t)] dt \\
& - m_{t2} \int_{t_1}^{t_2} \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] [l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t)] \delta \theta_2 dt
\end{aligned}$$

$$\begin{aligned}
- \int_{t_1}^{t_2} \delta V_s dt & = - \int_{t_1}^{t_2} \int_0^{l_1} EI_1(x_1) w_{1xx} \delta w_{1xx} dx_1 dt \\
& - \int_{t_1}^{t_2} \int_0^{l_2} EI_2(x_2) w_{2xx} \delta w_{2xx} dx_2 dt \\
& = \int_{t_1}^{t_2} [-EI_1(x_1) w_{1xx} \delta w_{1x} + (EI_1(x_1) w_{1xx})_x \delta w_1]_0^{l_1} dt \\
& + \int_{t_1}^{t_2} [-EI_2(x_2) w_{2xx} \delta w_{2x} + (EI_2(x_2) w_{2xx})_x \delta w_2]_0^{l_2} dt \\
& + \int_{t_1}^{t_2} \int_0^{l_1} [-(EI_1(x_1) w_{1xx})_{xx} \delta w_1] dx_1 dt \\
& + \int_{t_1}^{t_2} \int_0^{l_2} [-(EI_2(x_2) w_{2xx})_{xx} \delta w_2] dx_2 dt
\end{aligned}$$

After combining equation (3.1) with above equations we get

$$\int_{t_1}^{t_2} \int_0^{l_1} H_1 \delta w_1 dx_1 dt + \int_{t_1}^{t_2} \int_0^{l_2} H_2 \delta w_2 dx_2 dt + \int_{t_1}^{t_2} H_3 \delta \theta_1 dt$$

$$\begin{aligned}
& + \int_{t_1}^{t_2} H_4 \delta\theta_2 dt + \int_{t_1}^{t_2} H_5 \delta w_1(l_1, t) dt + \int_{t_1}^{t_2} H_6 \delta w_2(l_2, t) dt \\
& + \int_{t_1}^{t_2} H_7 \delta w_{1x}(l_1, t) dt + \int_{t_1}^{t_2} H_8 \delta w_{2x}(l_2, t) dt \\
& + \int_{t_1}^{t_2} [-EI_1(0) w_{1xx}(0, t) \delta w_{1x}(0, t) + (EI_1(x_1) w_{1xx})_x|_{x_1=0} \delta w_1(0, t)] dt + \\
& \int_{t_1}^{t_2} [-EI_2(0) w_{2xx}(0, t) \delta w_{2x}(0, t) + (EI_2(x_2) w_{2xx})_x|_{x_2=0} \delta w_2(0, t)] dt = 0
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= -(EI_1(x_1) w_{1xx})_{xx} - \rho_1(x_1) [x_1 \ddot{\theta}_1 + \ddot{w}_1] \\
H_2 &= -(EI_2(x_2) w_{2xx})_{xx} - \rho_2(x_2) \left\{ [x_2 \ddot{\theta}_2 + \ddot{w}_2] + \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \right. \\
&\quad \left. - \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \right\}
\end{aligned}$$

$$\begin{aligned}
H_3 &= - \int_0^{l_1} \rho_1(x_1) [x_1 \ddot{\theta}_1 + \ddot{w}_1] x_1 dx_1 \\
&- l_1 \int_0^{l_2} \rho_2(x_2) \left\{ [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + \cos(\theta_2) [x_2 \ddot{\theta}_2 + \ddot{w}_2] - \dot{\theta}_2 \sin(\theta_2) [x_2 \dot{\theta}_2 + \dot{w}_2] \right\} dx_2 \\
&- I_{h1} \ddot{\theta}_1 - \left\{ I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] + m_{t1} l_1 [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \right\} \\
&- \left\{ I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] + m_{h2} l_1 [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] \right\} - I_{t2} [\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t)] + \tau_1 \\
&- m_{t2} l_1 \left\{ \cos(\theta_2) [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t)] + [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] - \dot{\theta}_2 \sin(\theta_2) [l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t)] \right\} \\
H_4 &= - \int_0^{l_2} \rho_2(x_2) \left\{ \left\{ \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] - \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \right\} \right. \\
&\quad \left. + [x_2 \ddot{\theta}_2 + \ddot{w}_2] \right\} x_2 + \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] [x_2 \dot{\theta}_2 + \dot{w}_2] dx_2 \\
&- I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] - I_{t2} [\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t)] \\
&- m_{t2} l_2 \left\{ \cos(\theta_2) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t)] - \dot{\theta}_2 \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] \right\} \\
&- m_{t2} \sin(\theta_2) [l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t)] [l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t)] + \tau_2 \\
H_5 &= - \int_0^{l_2} \rho_2(x_2) \left\{ [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + \cos(\theta_2) [x_2 \ddot{\theta}_2 + \ddot{w}_2] - \dot{\theta}_2 \sin(\theta_2) [x_2 \dot{\theta}_2 + \dot{w}_2] \right\} dx_2 \\
&- (m_{t1} + m_{h2} + m_{t2}) [l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t)] + (EI_1(x_1) w_{1xx})_x|_{x_1=l_1} \\
&- m_{t2} \left\{ \cos(\theta_2) [l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t)] - \dot{\theta}_2 \sin(\theta_2) [l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t)] \right\}
\end{aligned}$$

$$\begin{aligned}
H_6 &= (EI_2(x_2) w_{2xx})_x \big|_{x_2=l_2} \\
&\quad -m_{t2} \left\{ \cos(\theta_2) \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] - \dot{\theta}_2 \sin(\theta_2) \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] + \left[l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t) \right] \right\} \\
H_7 &= -I_{t1} \left[\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t) \right] - EI_1(l_1) w_{1xx}(l_1, t) - \tau_2 \\
H_8 &= -I_{t2} \left[\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t) \right] - EI_2(l_2) w_{2xx}(l_2, t)
\end{aligned}$$

After some algebraic manipulations with setting $H_i = 0$ for $i = 1$ to 8, we get equations
(3.2 - 3.10) . $\diamond$

APPENDIX C: Proof of Decoupling for PDE

If the dominant part of the control laws (4.12 - 4.13) are substituted into equations (3.4 - 3.5) then we get

$$\ddot{\theta}_i = -\frac{\ddot{w}_i(l_i, t)}{l_i} - K_i(\alpha\dot{\theta}_i + \beta(\theta_i - \theta_{id})) \quad (\text{C.1})$$

where $i = 1, 2$. Define $\theta_e = \theta_i - \theta_{id}$, and then observe $\dot{\theta}_e = \dot{\theta}_i$, $\ddot{\theta}_e = \ddot{\theta}_i$. Applying the Laplace transform to equation (C.1) with the zero initial conditions then we have

$$\begin{aligned} \theta_i(s) - \frac{\theta_{id}}{s} &= -\frac{\mathcal{L}\{\ddot{w}_i(l_i, t)\}}{K_i\beta l_i} \frac{K_i\beta}{(s^2 + K_i\alpha s + K_i\beta)} \quad \text{or} \quad (\text{C.2}) \\ \theta_i(s) - \frac{\theta_{id}}{s} &= -\frac{\mathcal{L}\{\ddot{w}_i(l_i, t)\}}{K_i\beta l_i} L_p(s) \end{aligned}$$

where $i = 1, 2$ and $L_p(s)$ is well-known the second order Lowpass filter transfer function with the natural frequency $w_{ni} = \sqrt{K_i\beta}$ and the damping ratio $\zeta_i = \alpha\sqrt{K_i/4\beta}$. Then, the bandwidth of the Lowpass filter can be calculated [37, 38] such that

$$w_{Bi} = w_{ni} \left[(1 - 2\zeta_i^2) + \sqrt{4\zeta_i^4 - 4\zeta_i^2 + 2} \right]^{\frac{1}{2}}$$

where w_{Bi} can be adjusted to a small value less than $\frac{1}{5}$ of the fundamental vibration frequency by choosing the convenient controller parameters (K_i, α, β) . The minimum fundamental vibration frequency for both links can be approximated [35, 23] such as

$$w_{fmin} = \sqrt{\frac{3.03EI_{min}}{l_2^3 m_{t2}}}$$

where this formula has been verified in FEM approach for nonlinear cases and e.g. $w_{fmin} = 9.7 \text{ rad/sec}$, $w_{B1} = 1.8 \text{ rad/sec}$, and $w_{B2} = 1.6751 \text{ rad/sec}$ are chosen for simulation studies with the parameters in Table 5.3. Since $\ddot{w}_i(l_i, t)$ will have the same harmonics with the vibration modes due to *undamped* equations (3.2 - 3.3), see Remark 4.3.2, its frequency spectrum cannot include lower frequency harmonics than the fundamental one and starts with w_{fmin} [35, Proposition 2.3] and also see [19, ch. 8]. Therefore, the amplitude of the right hand side of equation (C.2) become zero. Thus, we get $\theta_i - \theta_{id} = 0$ by inverse Laplace transform of the left side of equation (C.2) and hence $\dot{\theta}_i = 0$ as well. Consequently, equation (C.1) can be simplified furthermore such as

$$\ddot{\theta}_i = -\frac{\ddot{w}_i(l_i, t)}{l_i} \quad (\text{C.3})$$

Since $\ddot{\theta}_i$, $\ddot{w}_i$, $\dot{\theta}_i$, and $\dot{w}_i$ are continuous in time with zero initial conditions, time integration of equation (C.3) will give the following result [39]

$$\dot{\theta}_i = -\frac{\dot{w}_i(l_i, t)}{l_i} \quad (\text{C.4})$$

Applying equations (C.3 - C.4) to the system (3.2 - 3.10) we get

$$\ddot{w}_1 + \frac{(EI_1(x_1)w_{1xx})_{xx}}{\rho_1(x_1)} = -x_1 \ddot{\theta}_1 \quad (\text{C.5})$$

$$\ddot{w}_2 + \frac{(EI_2(x_2)w_{2xx})_{xx}}{\rho_2(x_2)} = -x_2 \ddot{\theta}_2 \quad (\text{C.6})$$

$$l_1 \ddot{\theta}_1 = -\ddot{w}_1(l_1, t) \quad (\text{C.7})$$

$$l_2 \ddot{\theta}_2 = -\ddot{w}_2(l_2, t) \quad (\text{C.8})$$

$$-\tau_{b2} + \tau_{e1} - I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] - I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] = 0 \quad (\text{C.9})$$

$$[(EI_1(x_1)w_{1xx})_x]_{x_1=l_1} = 0 \quad (\text{C.10})$$

$$[(EI_2(x_2)w_{2xx})_x]_{x_2=l_2} = 0 \quad (\text{C.11})$$

where the main equations (3.2 - 3.5) are replaced with equations (C.5 - C.8) and the new set of boundary conditions will be equations (C.9), (C.10), (C.11), and (3.10). This new system (C.5 - C.8) with very simple boundary conditions, is linear and decoupled as shown above. $\diamond$

APPENDIX D: Derivation of Total Energy Rate for PDE

$$\begin{aligned}
\dot{T}_1 &= \int_0^{l_1} \rho_1(x_1) \left[x_1 \dot{\theta}_1 + \dot{w}_1 \right] \left[x_1 \ddot{\theta}_1 + \ddot{w}_1 \right] dx_1 \\
\dot{T}_2 &= \int_0^{l_2} \rho_2(x_2) \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] + \left[x_2 \dot{\theta}_2 + \dot{w}_2 \right] \left[x_2 \ddot{\theta}_2 + \ddot{w}_2 \right] \right. \\
&\quad \left. - \dot{\theta}_2 \sin(\theta_2) \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[x_2 \dot{\theta}_2 + \dot{w}_2 \right] + \cos(\theta_2) \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] \left[x_2 \dot{\theta}_2 + \dot{w}_2 \right] \right. \\
&\quad \left. + \cos(\theta_2) \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[x_2 \ddot{\theta}_2 + \ddot{w}_2 \right] \right\} dx_2 \\
\dot{T}_3 &= I_{h1} \ddot{\theta}_1 \dot{\theta}_1
\end{aligned}$$

$$\begin{aligned}
\dot{T}_4 &= I_{t1} \left[\dot{\theta}_1 + \dot{w}_{1x}(l_1, t) \right] \left[\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t) \right] + m_{t1} \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] \\
\dot{T}_5 &= I_{h2} \left[\dot{\theta}_1 + \dot{\theta}_2 \right] \left[\ddot{\theta}_1 + \ddot{\theta}_2 \right] + m_{h2} \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] \\
\dot{T}_6 &= I_{t2} \left[\dot{\theta}_1 + \dot{\theta}_2 + \dot{w}_{2x}(l_2, t) \right] \left[\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{w}_{2x}(l_2, t) \right] + m_{t2} \left\{ \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] \right. \\
&\quad \left. + \left[l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t) \right] \left[l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t) \right] - \dot{\theta}_2 \sin(\theta_2) \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t) \right] \right. \\
&\quad \left. + \cos(\theta_2) \left[l_1 \ddot{\theta}_1 + \ddot{w}_1(l_1, t) \right] \left[l_2 \dot{\theta}_2 + \dot{w}_2(l_2, t) \right] + \cos(\theta_2) \left[l_1 \dot{\theta}_1 + \dot{w}_1(l_1, t) \right] \left[l_2 \ddot{\theta}_2 + \ddot{w}_2(l_2, t) \right] \right\}
\end{aligned}$$

The potential energy rate can be calculated with the notation of equations (3.11 - 3.12)

$$\begin{aligned}
\dot{V}_s &= \int_0^{l_1} EI_1(x_1) w_{1xx} \dot{w}_{1xx} dx_1 + \int_0^{l_2} EI_2(x_2) w_{2xx} \dot{w}_{2xx} dx_2 \\
&= -\tau_{e1} \dot{w}_{1x}(l_1, t) - (EI_1(x_1) w_{1xx})_x \dot{w}_1 |_{x_1=l_1} + \int_0^{l_1} \dot{w}_1 (EI_1(x_1) w_{1xx})_{xx} dx_1 \\
&\quad -\tau_{e2} \dot{w}_{2x}(l_2, t) - (EI_2(x_2) w_{2xx})_x \dot{w}_2 |_{x_2=l_2} + \int_0^{l_2} \dot{w}_2 (EI_2(x_2) w_{2xx})_{xx} dx_2 \tag{D.1}
\end{aligned}$$

After applying the dominant control laws (4.12 - 4.13) to the system (3.2 - 3.10) and using equations (3.2 - 3.3) with integration, the new expressions for $\dot{T}_1$, $\dot{T}_2$ are obtained to be able to cancel out the terms in equation (D.1) as follows

$$\begin{aligned}
\dot{T}_1 &= - \int_0^{l_1} [x_1 \dot{\theta}_1 + \dot{w}_1] (EI_1(x_1) w_{1xx})_{xx} dx_1 \\
&= (EI_1(x_1) w_{1xx})_x \dot{w}_1 |_{x_1=l_1} + \dot{\theta}_1 (\tau_{b1} - \tau_{e1}) - \int_0^{l_1} \dot{w}_1 (EI_1(x_1) w_{1xx})_{xx} dx_1 \\
\dot{T}_2 &= - \int_0^{l_2} [x_2 \dot{\theta}_2 + \dot{w}_2] (EI_2(x_2) w_{2xx})_{xx} dx_2 \\
&= (EI_2(x_2) w_{2xx})_x \dot{w}_2 |_{x_2=l_2} + \dot{\theta}_2 (\tau_{b2} - \tau_{e2}) - \int_0^{l_2} \dot{w}_2 (EI_2(x_2) w_{2xx})_{xx} dx_2
\end{aligned}$$

After some cancellations, $\dot{V}$ can be determined without any approximation as

$$\begin{aligned}
\dot{V} &= \dot{T}_1 + \dot{T}_2 + \dot{T}_3 + \dot{T}_4 + \dot{T}_5 + \dot{T}_6 + \dot{V}_s + 2\ddot{\theta}_1 \frac{d\dot{\theta}_1}{dt} \\
&= [I_{h2}(\ddot{\theta}_1 + \ddot{\theta}_2) + I_{t1}(\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)) + I_{h1}\ddot{\theta}_1 - \tau_{e1} + \tau_{b1} + \tau_{e2}] \dot{\theta}_1 \\
&\quad + [I_{t1}(\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)) - \tau_{e1}] \dot{w}_{1x}(l_1, t) + [I_{h2}(\ddot{\theta}_1 + \ddot{\theta}_2) + \tau_{b2}] \dot{\theta}_2 \\
&\quad + 2\ddot{\theta}_1 \frac{d\dot{\theta}_1}{dt}.
\end{aligned}$$

The equation (3.13) can be simplified further with the dominant control laws (4.12 - 4.13) and by using equation (C.4), then equation (C.9) is obtained. Thus, we get

$$-\tau_{b2} + \tau_{e1} - I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] - I_{t1} [\ddot{\theta}_1 + \ddot{w}_{1x}(l_1, t)] = 0 \quad (\text{D.2})$$

Consequently, the total energy rate of the system can be expressed simply by using equation (D.2) and Remark 4.3.2 such as

$$\begin{aligned} \dot{V} = & \left(\tau_{e2} - \tau_{b2} + \tau_{b1} + I_{h1} \ddot{\theta}_1 \right) \dot{\theta}_1 + \left(I_{t1} \ddot{\theta}_1 - \tau_{e1} \right) \dot{w}_{ix}(l_1, t) + \left(\tau_{e1} - I_{t1} \ddot{\theta}_1 \right) \dot{\theta}_2 \\ & + 2\ddot{\theta}_1 \frac{d\dot{\theta}_1}{dt} \end{aligned} \quad (\text{D.3})$$

Further simplifications can be elaborated by assuming $\tau_{e1} = \tau_{b2}$ and using Remark 4.3.2 in equation (C.9) such as

$$\begin{aligned} I_{t1} \ddot{\theta}_1 &= -I_{h2} [\ddot{\theta}_1 + \ddot{\theta}_2] \\ \ddot{\theta}_2 &= \frac{I_{h2} + I_{t1}}{-I_{h2}} \ddot{\theta}_1 \end{aligned} \quad (\text{D.4})$$

Note that $\ddot{\theta}_i$, and $\dot{\theta}_i$ are continuous in time with zero initial conditions, time integration of equation (D.4) implies $\dot{\theta}_2 = \frac{I_{h2} + I_{t1}}{-I_{h2}} \dot{\theta}_1$. Thus, we are left with the only $\dot{\theta}_1$ terms in equation (D.3). Since $\dot{w}_{ix}(l_1, t)$ is neglected, see Remark 4.3.2, the total energy rate of the system reduces to

$$\dot{V} = \left(\tau_{e2} - \tau_{b2} + \tau_{b1} + I_{h1} \ddot{\theta}_1 \right) \dot{\theta}_1 + (\tau_{b2} - I_{t1} \ddot{\theta}_1) \frac{I_{h2} + I_{t1}}{-I_{h2}} \dot{\theta}_1 + 2\ddot{\theta}_1 \frac{d\dot{\theta}_1}{dt} \quad (\text{D.5})$$

then by equations (3.4), (D.4), (C.1), and (4.16) that is given explicitly below

$$\tau_{b1} + I_{h1} \ddot{\theta}_1 = \tau_1 \quad (\text{D.6})$$

$$\tau_1 = \tau_{b1} + I_{h1} \left(\frac{-\ddot{w}_1(l_1, t)}{l_1} - K_1(\alpha \dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) \right) + \gamma_1 (\tau_{b2} - \tau_{e2} - K \dot{\theta}_1) \quad (\text{D.7})$$

and also assuming $\tau_{b2} - \tau_{e2} - K \dot{\theta}_1 = -K \dot{\theta}_1$ for sufficiently large K values, then we have

$$\ddot{\theta}_1 = \frac{-\ddot{w}_1(l_1, t)}{l_1} - K_1(\alpha \dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) + \frac{\gamma_1}{I_{h1}} (-K \dot{\theta}_1)$$

$$\frac{d\ddot{\theta}_1}{dt} = \frac{-1}{l_1} \frac{d\ddot{w}_1(l_1, t)}{dt} - K_1(\alpha\ddot{\theta}_1 + \beta\dot{\theta}_1) + \frac{\gamma_1}{I_{h1}} (-K\ddot{\theta}_1) \quad (\text{D.8})$$

Since $\frac{-\ddot{w}_1(l_1, t)}{l_1}$ asymptotically approaches to $\ddot{\theta}_1$ in short time, see Appendix C, we can assume the following inequality such as

$$\frac{-\ddot{w}_1(l_1, t)}{l_1} > 2 \left[K_1(\alpha\dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) + \frac{\gamma_1}{I_{h1}} (K\dot{\theta}_1) \right]$$

using this inequality and also neglecting the term, $\frac{-1}{l_1} \frac{d\ddot{w}_1(l_1, t)}{dt}$, in equation D.8, since forced undamped response of Euler-Bernoulli beam has a cosine term in time-domain, see [19, section 8.10], we have

$$\begin{aligned} \ddot{\theta}_1 &> K_1(\alpha\dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) + \frac{\gamma_1}{I_{h1}} (K\dot{\theta}_1) \\ \frac{d\ddot{\theta}_1}{dt} &= -K_1(\alpha\ddot{\theta}_1 + \beta\dot{\theta}_1) + \frac{\gamma_1}{I_{h1}} (-K\ddot{\theta}_1) \end{aligned}$$

Since $|\theta_1 - \theta_{1d}| < M |\dot{\theta}_1|$ for $M > 0$ by Mean Value Inequality [39], and $K_1\beta > 0$, we get

$$\begin{aligned} -K_1\beta\ddot{\theta}_1 &< -K_1\beta \left[K_1(\alpha\dot{\theta}_1 + \beta(\theta_1 - \theta_{1d})) + \frac{\gamma_1}{I_{h1}} (K\dot{\theta}_1) \right] \\ -K_1\beta\ddot{\theta}_1 &< -K_1\beta\dot{\theta}_1 \left[K_1(\alpha + \beta M) + \frac{K\gamma_1}{I_{h1}} \right] \end{aligned}$$

then

$$\begin{aligned} \ddot{\theta}_1 \frac{d\ddot{\theta}_1}{dt} &= -(K_1\alpha + \frac{K\gamma_1}{I_{h1}}) \ddot{\theta}_1^2 - K_1\beta\ddot{\theta}_1\dot{\theta}_1 \\ \ddot{\theta}_1 \frac{d\ddot{\theta}_1}{dt} &< -(K_1\alpha + \frac{K\gamma_1}{I_{h1}}) \ddot{\theta}_1^2 - K_1\beta \left[K_1(\alpha + \beta M) + \frac{K\gamma_1}{I_{h1}} \right] \dot{\theta}_1^2 \end{aligned} \quad (\text{D.9})$$

Comparing equation (D.5) with (D.9), $2 K_1 \beta \gg I_{h1} + I_{t1}(I_{h2} + I_{t1})/I_{h2}$ can be achieved easily by design. Finally, a bound on total energy rate of the system obtained in the shortest form with the help of the previous results such as

$$\dot{V} < -K' \dot{\theta}_1^2 - K'' \ddot{\theta}_1^2$$

where

$$K' = 2 \left(K_1 \beta \left[K_1 (\alpha + \beta M) + \frac{K \gamma_1}{I_{h1}} \right] \right)$$

$$K'' = 2 \left(K_1 \alpha + \frac{K \gamma_1}{I_{h1}} \right)$$

and also τ_{e2} , τ_{b2} , τ_{b1} terms in equation (D.5), are neglected by choosing K large enough. This result has been verified by simulations, see the last section in simulation studies.

◇

REFERENCES

1. Vandegrift, M. W., F. L. Lewis, and S. Q. Zhu, 'Flexible-Link robot arm control by a feedback linearization/Singular perturbation approach', *Journal of Robotic Systems*, vol. 11(7), pp. 591-603, 1994.
2. Rokui, M. R. and K. Khorasani, 'Experimental results on discrete-time nonlinear adaptive tracking control of a Flexible-link Manipulator', *IEEE Transactions on systems, man, and cybernetics*, vol.30/1, pp. 151-164, 2000.
3. Siciliano, B. and L. Villani, 'Two-Time scale force and position control of flexible manipulators', *Proceedings of the IEEE International Conference on Robotics and Automation*, pp. 2729-2734, Seoul, 2001.
4. Zhang, X., W. Xu and S. S. Nair, 'PDE modeling and control of a flexible two-link manipulator', *Proceedings of the American Control Conference*, pp. 3796-3801, Anchorage, 2002.
5. Raab, F. J. and D. J. Trudnowski, 'Experiments in two-axis vibration damping using inertial torques through momentum wheel control', *Proceedings of the American Control Conference*, pp. 3477-3481, Philadelphia, 1998.
6. Zhu, Y., J. Qiu and J. Tani, 'Simultaneous Optimization of a Two-link Flexible Robot Arm', *Journal of Robotic Systems*, vol. 18(1), pp. 29-38, 2001.
7. Alleyne, A. and M. Pomykalski, 'Control of a Class of Nonlinear Systems subject to Periodic Exogenous Signals', *IEEE Transactions on Control Systems Technology*, vol.8/2, pp.279-287, 2000.
8. Serrani, A., A. Isidori and L. Marconi, 'Semiglobal Nonlinear Output Regulation with Adaptive Internal Model', *IEEE Transactions on Automatic Control*, vol.46/8, pp.1178-1194, 2001.

9. Isidori, A., L. Marconi and A. Serrani , *Robust Autonomous Guidance, An Internal Model Approach* , Springer-Verlag, London, 2003.
10. Moallem, M., R. V. Patel and K. Khorasani, ‘Optimization of an Actuated Flexible Arm for Improved Control Properties’, *Proceedings of the IEEE International Conference on Control Applications*, pp. 709-714, Anchorage, 2000.
11. Wang, F. Y. and J. L. Russell, ‘Optimum Shape Construction of Flexible Manipulators with Tip Loads’, *Proceedings of the 31st IEEE Conference on Decision and Control*, pp.311-316, Tucson, 1992.
12. Wang, F. Y. and Y. Gao , *Advanced Studies of Flexible Robotic Manipulators*, World Scientific, New Jersey, 2003.
13. Luo, Z. H., B. Z. Guo and Ö. Morgül, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London, 1999.
14. Byrnes, C. I., D. S. Gilliam, A. Isidori and V. I. Shubov, ‘Set Point Boundary Control for a Nonlinear Distributed Parameter System’, *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp.312-317, Hawaii, 2003.
15. Byrnes, C. I., D. S. Gilliam, A. Isidori and V. I. Shubov, ‘Static and Dynamic Controllers for Boundary Controlled Distributed Parameter Systems’, *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp.3324-3325, Bahamas, 2004.
16. Meirovitch, L., *Dynamics and Control of Structures*, John Wiley, New York, 1990.
17. Theodore, R. J. and A.Ghosal, ‘Comparison of the Assumed Modes and Finite Element Models for Flexible Multilink Manipulators’, *Int Journal of Robotics Research*, vol. 14/2, pp. 91-111, 1995.
18. Moallem, M., R. V. Patel and K. Khorasani, *Flexible-link Robot Manipulators:Control Techniques and Structural Design*, Springer-Verlag, London, 2000.

19. Meirovitch, L., *Fundamentals of Vibrations*, McGraw-Hill, New York, 2001.
20. Reddy, J. N., *An Introduction to the Finite Element Method*, McGraw-Hill, New York, 1993.
21. Hastings, G. G. and W. J. Book, 'A Linear Dynamic Model for Flexible Robotic Manipulators', *IEEE Control System Magazine*, vol.7, pp.61-64, 1987.
22. Yang, T. Y., *Finite Element Structural Analysis*, Prentice Hall, New Jersey, 1986.
23. Den Hartog, J. P., *Mechanical Vibrations*, Dover, New York, 1985.
24. Shabana, A. A., *Dynamics of the Multibody Systems 2nd edition*, Cambridge University Press, Cambridge, 1998.
25. Zhang, X., 'Multi-scale System Dynamics and Control-Two Case Studies', Ph.D. Thesis, University of Missouri-Columbia, May 2001.
26. Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970.
27. Isidori, A., *Nonlinear Control Systems, 3rd edition*, Springer-Verlag, London, 1995.
28. Isidori, A., *Nonlinear Control Systems II*, Springer-Verlag, London, 1999.
29. Byrnes, C. I. and A. Isidori, 'Limit Sets, Zero Dynamics, and Internal Models in the Problem of Nonlinear Output Regulation', *IEEE Transactions on Automatic Control*, vol.48/10, pp.1712-1723, 2003.
30. Byrnes, C. I. and A. Isidori, 'Nonlinear Internal Models for Output Regulation', *IEEE Transactions on Automatic Control*, vol.49/12, pp.2244-2247, 2004.
31. Åström, K. J. and B. Wittenmark, *Adaptive Control, 2nd edition*, Addison-Wesley, New York, 1995.

32. Ogata, K., *Discrete-Time Control Systems, 2nd edition*, Prentice-Hall, London, 1995.
33. Erol, O.K. and I. Eksin, 'A New Optimization Method: Big Bang - Big Crunch ', *Advances in Engineering Software*, vol. 37, pp. 106-111, 2006.
34. Morgül, Ö., 'Orientation and Stabilization of a Flexible Beam Attached to a Rigid Body: Planar Motion', *IEEE Transactions on Automatic Control*, vol.36, No.8, pp.953-962, 1991.
35. Guo, B. Z., 'Riesz Basis Property and Exponential Stability of Controlled Euler-Bernoulli Beam Equations with Variable Coefficients', *SIAM Journal of Control Optim.*, vol.40, No.6, pp.1905-1923, 2002.
36. Baruh, H., *Analytical Dynamics*, McGraw-Hill, Boston, 1999.
37. Kuo, B. C., *Digital Control Systems, 2nd edition*, New York: Oxford University Press, 1992.
38. Kuo, B. C., *Automatic Control Systems, 6th edition*, Prentice-Hall, New Jersey, 1991.
39. Marsden, J. E. and M. J. Hoffman, *Elementary Classical Analysis, 2nd edition*, W. H. Freeman, New York, 1993.
40. Dormand, J. R. and P. J. Prince, 'A family of embedded Runge-Kutta formulae', *Journal of Comp. Appl. Math.*, vol. 6, pp 19-26, 1980.
41. Chari, M. V. K. and S. J. Salon, *Numerical Methods in Electromagnetism*, Academic Press, San Diego, 2000.
42. Papakostas, S. N. and G. Papageorgiou, 'A Family of Fifth-Order Runge-Kutta Pairs', *AMS Mathematics of Computation*, vol. 65, pp. 1165-1181, 1996.

43. Abhyankar, N.S., E.K. Hall II and S.V. Hanagud, 'Chaotic Vibrations of Beams:Numerical Solution of Partial Differential Equations', *ASME Journal of Applied Mechanics*, vol.60, pp.167-174, 1993.
44. Doğan, M., Y. Istefanopulos and E.D. Diktaş, 'Nonlinear Control of two-link Flexible Arm with Adaptive Internal Model, *Proc. IEEE International Conference On Mechatronics*, pp. 292-298, Istanbul, 2004.
45. Doğan, M. and Y. Istefanopulos, 'Optimal Nonlinear Control of Two-Link Flexible Arm with Adaptive Internal Model, *Proc. 44th IEEE Conference On Decision and Control, and the European Control Conference*, pp. 771-776, Seville, Spain, 2005.