AN APPLICATION OF ERGODIC THEORY TO GEOMETRIC RAMSEY THEORY

by

Sevim Şimşek

B.S., Mathematics, Boğaziçi University, 2006

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2009

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis advisor Assoc. Prof. Selçuk Demir for introducing me to Ergodic Theory, and for his guidance, motivation and encouragement throughout the process of writing this thesis.

I would like to thank Prof. Talin Budak, Prof. Nilgün Işık, Asst. Prof. Burak Gürel and Assoc. Prof. Emre Alkan for their participation in my thesis committee and their valuable instructions in Bogaziçi University.

I also want to thank my best friend Demet Vurgun for her endless support and company at all stages of this work, and my colleagues Handan Borluk, İlker Evrim Çolak and Erinç Özden for their support and motivation.

I don't know how to thank my fiance Oktay Ölmez for his love, assistance and understanding.

My special thanks are due to my family for their confidence, patience and support throughout my education.

ABSTRACT

AN APPLICATION OF ERGODIC THEORY TO GEOMETRIC RAMSEY THEORY

In this thesis we first present two examples from Geometric Ramsey theory in $\mathbb{R}^2$. Then we generalize these results to higher dimensions and construct our main theorem. Then we translate our geometric problem into dynamical form. Finally we prove the main theorem by using methods from ergodic theory.

ÖZET

ERGODİK TEORİNİN GEOMETRİK RAMSEY TEORİDE UYGULAMASI

Bu tezde öncelikle $\mathbb{R}^2$ de Geometrik Ramsey Teoriden iki örnek gösterilmiştir. Sonra daha yüksek boyutta bu sonuçlar genelleştirilmiş ve ana teorem oluşturulmustur. Daha sonra geometrik problem dinamik forma çevrilmiştir. Son olarak, ergodik teoriden metodlar kullanılarak ana teoremin ispatı yapılmıştır.

TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

$dist(u,v)$	The distance between u and v
$\overline{D}(E)$	Upper Density
$E(.\|Y)$	Conditional expectation
$\overline{f}$	Conjugate of f
$\\|f\\|$	Norm of f
$l(S)$	The length of the side of S
$L^2(X)$	Lebesgue integrable functions
$M(X)$	The space of regular probability measures on X
$M_k(\mathbb{R})$	Set of kxk matrices over $\mathbb{R}$
M^T	Transpose of M
$\mathbb{N}$	The set of natural numbers
P	Orthogonal projection
$\mathbb{R}$	The set of real numbers
$SO(k)$	Special orthogonal group
$T_PSO(k)$	The tangent space of $SO(k)$ at P
U	Unitary operator
U^*	Adjoint of U
$\overline{X}$	Closure of X
X^*	Dual space of X
$\mathbb{Z}$	The set of integers
1_A	Characteristic function of A
χ	Character on an abelian group
μ	Measure on a space
μ_y	Disintegration of μ with respect to the factor Y
$\square$	End of the proof
$\triangle$	Symmetric difference
$\\|.\\|_2$	Norm in L^2
$\\|.\\|_1$	Norm in L^1

$\forall$	For all
$\exists$	There exists
a.e.	Almost everywhere
s.t.	Such that

1. INTRODUCTION

Let E be a measurable subset of $\mathbb{R}^k$, and let S range over all cubes in the space. We set

$$\overline{D}(E) = \limsup_{l(S)\to\infty} \frac{m(S \cap E)}{m(S)},$$

where $l(S)$ denotes the length of the side of S. $\overline{D}(E)$ is the *upper density* of E. We are interested in configurations which are necessarily contained in E.

In 1985, a theorem of Furstenberg, Katznelson and Weiss [1] states that if $E \subset \mathbb{R}^2$, with $\overline{D}(E) > 0$, all large distances in E are attained. In 1986, this result was also proved, using methods of harmonic analysis, by Bourgain [2] and by Falconer and Marstrand[3]. It is natural to ask if the same is valid for larger configurations. Bourgain has shown by an example that this can not be done[2]. As some configurations may not be found in the set itself, it may be useful to weaken the condition, and we try to find the configurations arbitrarily close to the set. In the same paper Furstenberg, Katznelson and Weiss [1] show that with this weaker condition, one can find triangles in the plane.

In 1998, Tamar Ziegler generalizes this result to higher dimensions [4]. We are working on this theorem which states that if E is a measurable subset of $\mathbb{R}^k$, $k > 2$, with $\overline{D}(E) > 0$ and $V = \{0, v_1, v_2, ..., v_{k+1}\} \in \mathbb{R}^k$, where $v_1, v_2, ..., v_{k+1}$ are affinely independent, then for r large enough, we can find an isometric copy of rV in E_δ.

Working on this thesis requires a good understanding of measure theory, functional analysis and ergodic theory. In chapter 2, we will present the preliminary notions from these areas. In chapter 3, we will explain the necessary tools from ergodic theory. We will state the Mean Ergodic Theorem and prove it. We will also explain what a Kronecker factor of a dynamical system is.

Since the idea of the proof of the theorem is to translate the geometric problem to a dynamical problem, in chapter 4 we will translate the geometric problem to a dynamical form where E corresponds to some measurable set E, with positive measure, in a measure preserving system $(X, \mathcal{B}, \mu, \mathbb{R}^k)$. The statement that E_δ contains a certain configuration, corresponds to some recurrence condition on the set E. After the translation, our geometric problem takes the following form: Let $(X, \mathcal{B}, \mu, (T_u)_{u \in \mathbb{R}^k})$ be an $\mathbb{R}^k$ action, $u_1, u_2, ..., u_{k+1} \in \mathbb{R}^k$ affinely independent, and $A \subset X$, $\mu(A) > 0$. Then there exists t_0 such that for all $t > t_0$, there exists $P \in SO(k)$ such that

$$\mu(A \cap T_{tPu_1}^{-1} A \cap ... \cap T_{tPu_{k+1}}^{-1} A) > 0.$$

In the last chapter we will prove our main theorem by assuming the existence of the matrices $M \in M_k(\mathbb{R})$ and $P \in SO(k)$ satisfying some conditions. We will use Ergodic Theory and some Algebraic Geometric techniques in the theorem's proof. Then we will show the existence of these matrices by using techniques from Linear Algebra, Harmonic Analysis and Lie Groups Theory.

2. PRELIMINARIES

In the following sections we will give some of the measure theoretic and functional analytic preliminaries required for the understanding of the proof of our main theorem.

2.1. Measure Theoretic Background

Throughout in all chapters we shall be concerned with measure spaces by which we mean a triple $(X, \mathcal{B}, \mu)$ where X is a space, $\mathcal{B}$ a σ-algebra of sets in X, μ a non-negative σ-additive measure on X with $\mu(X) < \infty$. Now, we will give some important definitions;

Definition 2.1.1 *A probability space is any triple $(X, \mathcal{B}, \mu)$ where μ is a measure on a σ- algebra $\mathcal{B}$ with $\mu(X) = 1$.*

Definition 2.1.2 *The collection $\mathcal{B}$ of Borel sets is the smallest σ- algebra which contains all of the open sets. A Borel measure is any measure μ on the $\sigma-$algebra of the Borel sets.*

Definition 2.1.3 *A measure space $(X, \mathcal{B}, \mu)$ is regular if X is a compact metric space and $\mathcal{B}$ consists of all Borel sets in X.*

Definition 2.1.4 *A measure space $(X, \mathcal{B}, \mu)$ is separable if $\mathcal{B}$ is generated by a countable subset.*

Remark 2.1.5 *If X is a compact metric space, $\mathcal{B}$ the Borel $\sigma-$algebra on X, μ a Borel measure, $(X, \mathcal{B}, \mu)$ is separable.*

2.2. Functional Analytic Background

In this study, we are working with real numbers. So, let X be a Banach space over the field of real numbers. Let B be the closed unit ball of X. Let X^* be the Banach dual of X which contains continuous linear functionals on X and let B^* be the closed unit ball of X^*. We can use this dual space to create two useful topologies, the weak topology on X and the weak*-topology on X^*. On X we have two natural topologies; norm and weak(w) topologies. On X^* we have three topologies; norm, weak (w) and weak*- topologies(w^*).

Now, we will give some basic results in functional analysis without proofs, these results have proofs in [5].

Theorem 2.2.1 *(Alaoglu) The topological space* (B^*, w^*) *is compact.*

Let X be a compact space. $C(X)$ is defined as the space of continuous scalar functions on X. We equip $C(X)$ with the sup-norm. Then $C(X)$ is a Banach space. Then we have the following:

Theorem 2.2.2 *The Banach space* $C(X)$ *is separable iff* X *is metrizable.*

Theorem 2.2.2 has an important corollary;

Corollary 2.2.3 *Let* X *be a Banach space. Then* (B^*, w^*) *is metrizable iff* X *is separable.*

Now, we have an important theorem;

Theorem 2.2.4 *(Riesz Representation Theorem) Let* X *be a compact metric space and let* $C(X)^*$ *be the dual space of continuous functions on* X*. If* F *is positive in* $C(X)^*$*,* *then there exists a unique regular Borel measure* μ *on* X *such that* $F(f) = \int_X f d\mu$ *for all* $f \in C(X)$.

In fact, we can deduce from the Riesz Representation Theorem that there is one-to-one correspondence between the space of regular measures and the space of bounded linear functionals on the Banach space $C(X)$.

In this thesis, X will be a compact metric space. Hence, the space of continuous functions, $C(X)$ will be separable. By the corollary 2.2.3, the closed unit ball of the dual of $C(X)$ is metrizable with respect to the weak*- topology. Let $M(X)$ be the space of regular probability measures on X. It is a weak*-compact convex subset of $C(X)^*$, $(M(X), w^*)$ has the topology of a compact metric space. This is important for us in the following sense: Every sequence in $M(X)$ has a w^*- convergent subsequence.

3. ERGODIC THEORY

3.1. Measure-Preserving Mappings

Definition 3.1.1 *Let $(X, \mathcal{B}, \mu)$ be a probability space. T is a measure preserving transformation of $(X, \mathcal{B}, \mu)$ if T is a transformation of X to itself with $T^{-1}B \in \mathcal{B}$ and $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.*

Lemma 3.1.2 *A measure μ in X is T-invariant iff*

$$\int f d\mu = \int f \circ T d\mu \quad \forall f \in L^1(X, \mathcal{B}, \mu).$$

Proof. If $\int f d\mu = \int f \circ T d\mu$ for all $f \in L^1(X, \mathcal{B}, \mu)$, then for any measurable set B we may take $f = 1_B$ to see that

$$\int 1_B d\mu = \mu(B) = \int 1_B \circ T d\mu = \int 1_{T^{-1}B} d\mu = \mu(T^{-1}B)$$

so T preserves μ.

Conversely, if T preserves μ i.e.$\mu(B) = \mu(T^{-1}B)$, then for any characteristic function 1_B holds

$$\int 1_B d\mu = \int 1_B \circ T d\mu$$

and hence for any simple function.

Let f be a nonnegative real-valued measurable function on X. Choose a sequence of simple functions (f_n) increasing to f. Then $f_n \circ T$ is a sequence of simple functions increasing to $f \circ T$, and so

$$\int f \circ T d\mu = \lim_{n\to\infty} \int f_n \circ T d\mu = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

showing that $\int f d\mu = \int f \circ T d\mu$ holds for f. □

3.2. Ergodicity

Definition 3.2.1 *A measure preserving transformation $T : X \to X$ of a probability space $(X, \mathcal{B}, \mu)$ is ergodic if all measurable sets $B \in \mathcal{B}$ with $T^{-1}B = B$ have measure 0 or 1.*

We have the following characteristic of ergodic transformation.

Proposition 3.2.2 *The following properties are equivalent for a measure preserving transformation T of $(X, \mathcal{B}, \mu)$.*

1. *T is ergodic.*
2. *For any $B \in \mathcal{B}$, $\mu(T^{-1}B \triangle B) = 0 \implies \mu(B) = 0$ or $\mu(B) = 1$.*
3. *For $A \in \mathcal{B}$, $\mu(A) > 0 \implies \mu(\bigcup_{n=1}^{\infty} T^{-n}A) = 1$.*
4. *For measurable $A, B \subseteq X$, $\mu(A)\mu(B) > 0$ implies that there exists some $n \geq 1$ with*

$$\mu(T^{-n}A \cap B) > 0.$$

5. *For measurable $f : X \longrightarrow \mathbb{C}$, $f = f \circ T$ a.e. implies that f is equal to a constant almost everywhere.*

Proof. (1. ⇒ 2.) : Assume that T is ergodic, and let B be an almost invariant measurable set that is, a measurable set B with $\mu(T^{-1}B \triangle B) = 0$. We wish to construct an invariant set from B, and this is achieved by means of the following *limsup* construction. Let

$$C = \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}B.$$

For any $N \geq 0$,

$$B \bigtriangleup \bigcup_{n=N}^{\infty} T^{-n}B \subseteq \bigcup_{n=N}^{\infty} B \bigtriangleup T^{-n}B$$

and $\mu(T^{-n}B \bigtriangleup B) = 0$ for all $n \geq 1$, since $T^{-n}B \bigtriangleup B$ is a subset of

$$\bigcup_{i=0}^{n-1} T^{-i}B \bigtriangleup T^{-(i+1)}B,$$

which has zero measure. Let $C_N = \bigcup_{n=N}^{\infty} T^{-n}B$; the sets C_N are nested,

$$C_0 \supseteq C_1 \supseteq ...,$$

and $\mu(C_N) = \mu(B)$ for each N. It follows that $\mu(C \bigtriangleup B) = 0$, so

$$\mu(C) = \mu(B).$$

Moreover,

$$T^{-1}C = \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-(n+1)}B = \bigcap_{N=0}^{\infty} \bigcup_{n=N+1}^{\infty} T^{-(n)}B = C.$$

Thus $T^{-1}C = C$, so by ergodicity $\mu(C) = 0$ or 1, so $\mu(B) = 0$ or 1.

(2. $\Rightarrow$ 3.) : Let A be a set with $\mu(A) > 0$, and let $B = \bigcap_{n=1}^{\infty} T^{-n}A$. Then $T^{-1} \subseteq B$; on the other hand $\mu(T^{-1}B) = \mu(B)$. so $\mu(T^{-1}B \bigtriangleup B) = 0$. It follows that $\mu(B) = 0$ or 1; since $T^{-1}A \subseteq B$ the former is impossible, so $\mu(B) = 1$ as required.

$(3. \Rightarrow 4.)$: Let A and B be sets of positive measure. By 3.,

$$\mu(\bigcap_{n=1}^{\infty} T^{-n}A) = 1,$$

So,

$$0 < \mu(B) = \mu(\bigcap_{n=1}^{\infty} B \cap T^{-n}A) \leq \sum_{n=1}^{\infty} \mu(B \cap T^{-n}A).$$

It follows that there must be some $n \geq 1$ with $\mu(B \cap T^{-n}A) > 0$.

$(4. \Rightarrow 1.)$: Let A be a set with $T^{-1}A = A$. Then

$$0 = \mu(A \cap X \backslash A) = \mu(T^{-n}A \cap X \backslash A)$$

for all $n \geq 1$ so, by (4), either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

$(2. \Rightarrow 5.)$: Assume that T is ergodic and $f : X \to \mathbb{C}$ is measurable and invariant under T in the stated sense. Since the real and the imaginary parts of f must also be invariant and measurable, we may assume without loss of generality that f is real-valued. Fix $k \in \mathbb{Z}$ and $n \geq 1$ and write

$$A_n^k = \{x \in X | f(x) \in [\frac{k}{n}, \frac{k+1}{n})\}.$$

Then $T^{-1}A_n^k \triangle A_n^k \subseteq \{x \in X | f \circ T(x) \neq f(x)\}$a null set, so by 2.

$$\mu(A_n^k) \in \{0, 1\}.$$

For each n, X is the disjoint union $\sqcup_{k \in \mathbb{Z}} A_n^k$. It follows that there must be exactly one $k = k(n)$ with $\mu(A_n^k) = 1$. Then f is constant on the set

$$Y = \bigcap n = 1^{\infty} A_n^{k(n)}$$

and $\mu(Y) = 1$, so f is constant almost everywhere.

$(5. \Rightarrow 2.)$: If $\mu(T^{-1}B\triangle B) = 0$ then $f = 1_B$ is a T -invariant measurable function, so by 5. 1_B is a constant almost everywhere. It follows that $\mu(B)$ is either 0 or 1. □

In the proof of the main theorem, $(X, \mathcal{B}, \mu, G)$ is a measure preserving system where X is an arbitrary space, $\mathcal{B}$ is a $\sigma-$algebra of subsets of X, μ is a $\sigma-$additive probability measure on the sets of $\mathcal{B}$, and G is a locally compact group acting on X by measure preserving transformations.
We may always assume that X is a compact metric space which the case for our purposes.

Definition 3.2.3 *The action of* G *is ergodic, if* $T^{-1}A = A, \quad \forall A \in \mathcal{B}, \quad T \in G$ *implies* $\mu(A) = 0$ *or* $\mu(A) = 1$.

3.3. The Mean Ergodic Theorem

We may study ergodic systems in two different ways with sets and with functions. For each measure preserving transformation we have an associated operator $U_T : L^2(X, \mathcal{B}, \mu, T) \to L^2(X, \mathcal{B}, \mu, T)$. This operator is defined by

$$U_T f = f \circ T.$$

Recall that $L^2(X, \mathcal{B}, \mu, T)$ is a Hilbert space, so an inner product is defined on it. For any functions $f, g \in L^2(X, \mathcal{B}, \mu, T)$ we have;

$$\begin{aligned}
\langle U_T f, U_T g \rangle &= \int f \circ T . \overline{g \circ T} d\mu \\
&= \int (f\overline{g}) \circ T d\mu \\
&= \int f\overline{g} d\mu \quad (since\ \mu\ is\ \ T - invariant) \\
&= \langle f, g \rangle.
\end{aligned}$$

Thus we can say that $U_T f = f \circ T$ is an isometry mapping of $L^2(X, \mathcal{B}, \mu, T)$ into $L^2(X, \mathcal{B}, \mu, T)$ whenever $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system. Moreover, U_T satisfies the relation

$$\langle U_T f, g \rangle = \langle f, U_T^* g \rangle$$

which defines an associated operator $U_T^* : L^2(X, \mathcal{B}, \mu, T) \to L^2(X, \mathcal{B}, \mu, T)$ called the adjoint of U_T. The operator U_T is an isometry if and only if

$$U_T^* U_T = I \tag{3.3.1}$$

is the identity operator on $L^2(X, \mathcal{B}, \mu, T)$ and

$$U_T U_T^* = P_{ImU_T} \tag{3.3.2}$$

is the projection operator onto image ImU_T.

Theorem 3.3.1 *(Mean Ergodic Theorem) Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system, and $f \in L^2(X, \mathcal{B}, \mu, T)$. Then*

$$\frac{1}{N} \sum_{k=1}^{N-1} U_T^k f \to P_T f$$

in $L^2(X, \mathcal{B}, \mu, T)$ norm, where P_T is the orthogonal projection of f onto

$$\Lambda = \{g \in L^2(X, \mathcal{B}, \mu, T) : U_T g = g\} \subseteq L^2(X, \mathcal{B}, \mu, T).$$

Proof. Λ is the set of all T-invariant functions.

Let $A = \{U_T g - g : g \in L^2(X, \mathcal{B}, \mu, T)\}$ and the orthogonal complement of A is $A^{\perp} = \{f \in L^2(X, \mathcal{B}, \mu, T) : \langle f, U_T g - g \rangle = 0 \;\; \forall \;\; U_T g - g \in A\}$.

If $f \in \Lambda$ *i.e.* $U_T f = f$, then by the isometry property of U_T,

$$\langle f, U_T g - g\rangle = \langle U_T f, U_T g\rangle - \langle f, g\rangle = 0 \quad \text{so} \quad f \in A^{\perp}.$$

Hence $\Lambda \subseteq A^{\perp}$.

If $f \in A^{\perp}$, then $\langle U_T g - g, f\rangle = \langle U_T g, f\rangle - \langle g, f\rangle = 0$ and $\langle U_T g, f\rangle = \langle g, f\rangle$ for all $g \in L^2(X, \mathcal{B}, \mu, T)$. By the definition of U_T^*, we have$\langle U_T g, f\rangle = \langle g, U_T^* f\rangle = \langle g, f\rangle$, so $U_T^* f = f$. By (3.3.1) and (3.3.2), we have $f = U_T f$ and $f \in \Lambda$, so $A^{\perp} \subseteq \Lambda$. Therefore,

$$A^{\perp} = \Lambda.$$

It follows that $L^2(X, \mathcal{B}, \mu, T) = \Lambda \oplus \overline{A}$, where $\overline{A}$ is the closure of A, so any $f \in L^2(X, \mathcal{B}, \mu, T)$ can be written as

$$f = P_T f + h \tag{3.3.3}$$

with $h \in \overline{A}$.

We claim that $\dfrac{1}{N}\sum_{k=1}^{N-1} U_T^k h \longrightarrow 0$ with $L^2(X, \mathcal{B}, \mu, T)$ norm.

This is clear, if $h = U_T g - g \in A$, then

$$\begin{aligned} &\left\| \frac{1}{N}\sum_{k=1}^{N-1} U_T^k (U_T g - g) \right\|_2 \\ &= \left\| \frac{1}{N}\big((U_T g - g) + (U_T^2 g - U_T g) + ... + (U_T^N g - U_T^{N-1} g)\big) \right\|_2 \\ &= \frac{1}{N}\left\|(U_T^N g - g)\right\|_2 \longrightarrow 0 \qquad as \quad N \longrightarrow \infty. \end{aligned} \tag{3.3.4}$$

If $h \in \overline{A}$, then there exists a sequence (g_i) in $L^2(X, \mathcal{B}, \mu, T)$ with the property that $h_i = U_T g_i - g_i \to h$ as $i \to \infty$. Then for any $i \geq 1$,

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k h \right\|_2 \leq \left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k (h - h_i) \right\|_2 + \left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k h_i \right\|_2 .$$

Fix $\varepsilon > 0$ and choose, by the convergence of (3.3.4), i and N so large that

$$\|h - h_i\| \leq \varepsilon \tag{3.3.5}$$

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k h_i \right\|_2 = \left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k (U_T g_i - g_i) \right\|_2 < \varepsilon. \tag{3.3.6}$$

By (3.3.5) and (3.3.6), we have

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} U_T^k h \right\|_2 \leq 2\varepsilon.$$

So, $\frac{1}{N} \sum_{k=0}^{N-1} U_T^k h \to 0$ as $N \to \infty$ for any $h \in \overline{A}$ in $L^2(X, \mathcal{B}, \mu, T)$ norm. Since $h = f - P_T f$ by equation (3.3.3)

$$\frac{1}{N} \sum_{k=0}^{N-1} U_T^k (f - P_T f) = \frac{1}{N} \sum_{k=0}^{N-1} U_T^k (f) - \frac{1}{N} \sum_{k=0}^{N-1} U_T^k (P_T f) \to 0$$

therefore

$$\frac{1}{N} \sum_{k=0}^{N-1} U_T^k f \to P_T f$$

in $L^2(X, \mathcal{B}, \mu, T)$ norm. $\square$

Corollary 3.3.2 *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system, and $f \in L^2(X, \mathcal{B}, \mu, T)$. Then*

$$\frac{1}{N}\sum_{k=1}^{N-1} U_T^k f \to P_T f$$

in $L^2(X, \mathcal{B}, \mu, T)$ norm, where $P_T f$ is the constant function $\int f d\mu$.

We refer to [6] for the proof of this corollary.

3.4. Factors

Definition 3.4.1 *Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ be measure preserving systems on probability spaces. The system $(Y, \mathcal{D}, \nu, S)$ is a factor of $(X, \mathcal{B}, \mu, T)$, if there are sets X' in $\mathcal{B}$ and Y' in $\mathcal{D}$ with $\mu(X') = 1$, $\nu(Y') = 1$, $\quad TX' \subseteq X'$, $\quad SY' \subseteq Y'$ and a measure preserving map $\Phi : X' \to Y'$ with $\Phi \circ T(x) = S \circ \Phi(x)$ for all $x \in X'$. The system $(X, \mathcal{B}, \mu, T)$ will also be described as an extension of $(Y, \mathcal{D}, \nu, S)$.*

Remark 3.4.2 *A homomorphism $\alpha : (X, \mathcal{B}, \mu, G) \longrightarrow (Y, \mathcal{D}, \nu, G)$ of measure preserving system is a homomorphism of measure spaces which commutes with the action of G. In our case we say that $(Y, \mathcal{D}, \nu, G)$ is a factor of $(X, \mathcal{B}, \mu, G)$, and $(X, \mathcal{B}, \mu, G)$ is an extension of $(Y, \mathcal{D}, \nu, G)$. The two measure preserving systems are equivalent if the homomorphism of one to the other is invertible.*

3.4.1. The Kronecker Factor

Definition 3.4.3 *An action of a locally compact abelian group G by measure preserving transformations T_g on a measure space $(X, \mathcal{B}, \mu)$ is a Kronecker action if X is a compact abelian group, μ is the Haar measure on X, and there is a homomorphism τ, $\tau : G \longrightarrow X$ with $\tau(G)$ a dense subgroup of X, and*

$$T_g(x) = x + \tau(g).$$

In this case, the system $(X, \mathcal{B}, \mu, G)$ is called a Kronecker system.

Definition 3.4.4 *A character on a compact abelian group G is a continuous homomorphism*

$$\chi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

The set of all continuous characters on G, denoted by $\widehat{G}$, forms a group under pointwise multiplication. This means the operation on $\widehat{G}$ is defined by

$$(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$$

for all $g \in G$, and the trivial character $\chi(g) = 1$ is the identity.

Remark 3.4.5 *All characters on $\mathbb{R}$ are of the form $\chi(x) = e^{2\pi i \langle y,x \rangle}$ for some $y \in \mathbb{R}$.*

For the proof of the following theorem, we refer to [6].

Theorem 3.4.6 *Let $(X, \mathcal{B}, \mu, T_g)$ be an ergodic measure preserving action of an abelian group G, then there is a map $\pi : X \longrightarrow Z$ where Z is a compact abelian group, and a Kronecker action T_g on Z such that $T_g\pi(x) = \pi(T_g(x))$ for a.e. $x \in X$.*

Remark 3.4.7 *For every character χ on Z the function $\chi'(x) = \chi(\pi(x))$ satisfies*

$$\begin{aligned}\chi'(T_g x) &= \chi(\pi(T_g x)) = \chi(T_g \pi(x)) \\ &= \chi(\tau(g) + \pi(x)) = \chi(\tau(g))\chi(x) = \chi(\tau(g))\chi'(x)\end{aligned}$$

and so it is an eigenfunction of the $G-$action, and moreover, every eigenfunction of the $G-$action comes about this way.

Definition 3.4.8 *The factor system (Z, D, m, G) defined in the theorem (3.4.6), where D is the algebra of Borel sets, and m the Haar measure, is unique up to isomorphism and is called the Kronecker factor of $(X, \mathcal{B}, \mu, G)$.*

In the following two sections, the propositions are proved in [6].

3.5. Conditional Expectation

Let $\alpha : (X, \mathcal{B}, \mu, G) \longrightarrow (Y, \mathcal{D}, \nu, G)$ be a homomorphism. Using this homomorphism we can lift measurable functions f on Y to X by $f \rightarrow f^\alpha = f \circ \alpha$. This map identifies $L^2(Y, \mathcal{D}, \nu)$ with a closed subspace $L^2(X, \mathcal{B}, \mu)^\alpha \subset L^2(X, \mathcal{B}, \mu)$. Let P is an orthogonal projection of $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{D}, \nu)^\alpha$. Now, we define an operator as *conditional expectation* $E(f|Y)$ for $f \in L^2(X, \mathcal{B}, \mu)$ by

$$E(f|Y) \in L^2(Y, \mathcal{D}, \nu) \quad \text{and} \quad E(f|Y)^\alpha = Pf.$$

Proposition 3.5.1 *The conditional expectation; $E(f|Y)$ defined for $f \in L^2(X, \mathcal{B}, \mu)$ has the following properties:*

1. *$f \rightarrow E(f|Y)$ is a linear operator of $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{D}, \nu)$.*
2. *If $f \geq 0$, $E(f|Y) \geq 0$.*
3. *If $f \in L^2(Y, \mathcal{D}, \nu)$, $E(f^\alpha|Y) = f$.*
4. *In particular $\int f d\mu = \int E(f|Y) d\nu$.*

Remark 3.5.2 *For each $f \in L^1(X, \mathcal{B}, \mu)$, and action T of G, we have $E(Tf|Y) = TE(f|Y)$ which means that the operator $E(.|Y)$ commutes with the action of G.*

3.6. Disintegration of Measure

Let $(X, \mathcal{B}, \mu)$ be a regular measure space, and let $\alpha : (X, \mathcal{B}, \mu) \longrightarrow (Y, \mathcal{D}, \nu)$ be a homomorphism. Suppose α is induced by a map $\varphi : X \rightarrow Y$ which defines a homomorphism with respect to some structure on X and Y. In this case the measure μ has a disintegration in terms of fiber measures μ_y, where μ_y is concentrated on the fiber $\varphi^{-1}(y) = X_y$.

Theorem 3.6.1 *There exits a measurable map from Y to $M(X)$, $y \rightarrow \mu_y$ which satisfies:*

1. *For every $f \in L^1(X, \mathcal{B}, \mu)$, $f \in L^1(X, \mathcal{B}, \mu_y)$ for a.e. $y \in Y$, and* $E(f|Y) = \int f d\mu_y$ *for a.e. $y \in Y$,*
2. $\int \{\int f d\mu_y\} d\nu(y) = \int f d\mu$ *for every $f \in L^1(X, \mathcal{B}, \mu)$.*

The map $y \rightarrow \mu_y$ is characterized by condition (1). We shall write $\mu = \int \mu_y d\nu$ and refer to this as the disintegration of measure μ with respect to the factor Y.

Remark 3.6.2 *If $(X, \mathcal{B}, \mu, G)$ is a measure preserving system, $\mathcal{D}$ the algebra of all $G-$invariant sets, $\mu = \int \mu_x d\mu(x)$ the disintegration of μ with respect to $\mathcal{D}$, then μ_x is $G-$invariant and ergodic, for a.e. x. [7].*

By using this result, we can consider our systems as ergodic.

4. TRANSLATION OF THE GEOMETRIC PROBLEM TO A DYNAMICAL PROBLEM

In our theorems, we will deal with subsets of $\mathbb{R}^k$ that have the *Positive Upper Density* property. Let E be a measurable subset of $\mathbb{R}^k$,we set

$$\overline{D}(E) = \limsup_{l(S)\to\infty} \frac{m(S\cap E)}{m(S)},$$

where S range over all cubes in $\mathbb{R}^k$, and $l(S)$ denotes the length of the side of S. $\overline{D}(E)$ is the *upper density* of E. First, we will present here two results from Geometric Ramsey Theory which are valid for the sets E having $\overline{D}(E) > 0$.

Theorem 4.0.3 *(Furstenberg-Katznelson-Weiss) If $E \in \mathbb{R}^2$ with $\overline{D}(E) > 0$, there exists l_0 such that for any $l > l_0$ one can find a pair of points $x, y \in E$ with $\|x-y\| = l$.*

Theorem 4.0.4 *(Furstenberg-Katznelson-Weiss) Let $E \in \mathbb{R}^2$ with $\overline{D}(E) > 0$, and let E_δ denotes the points at distance less than δ from E. Let $u, v \in \mathbb{R}^2$, then there exists l_0 such that for $l > l_0$ there exists a triple $\{x, y, z\} \subset E_\delta$ forming a triangle congruent to $\{0, lu, lv\}$.*

Now, we generalize the result of Theorem 4.0.4 to higher dimensions.

Theorem 4.0.5 *(Main Theorem) Let $E \in \mathbb{R}^k$ for $(k > 2)$ have positive upper density, and let E_δ denotes the points of distance $< \delta$ from E. If $u_1, u_2, ..., u_{k+1}$ are $k+1$ points which are affinely independent, then there exists l_0 such that for any $l > l_0$ and for any $\delta > 0$ there exists $\{x_1, x_2, ..., x_{k+2}\} \subset E_\delta$ forming a configuration congruent to $\{0, lu_1, ..., lu_{k+1}\}$.*

Our Main theorem means that for l large enough, we can find an isometric copy of $\{0, lu_1, ..., lu_{k+1}\}$ in E_δ. Since we will solve this problem using methods from ergodic theory, we would like to translate it to a dynamical problem.

Let Γ be the set of bounded and continuous functions on $\mathbb{R}^k$.

$$\Gamma = \{f : \mathbb{R}^k \longrightarrow \mathbb{R}; f \ \text{ is bounded and continuous}\},$$

For each bounded subset $B_N = [-N, N]$ of $\mathbb{R}^k$, we define the semi-norms on Γ by

$$\|f\|_{B_N} = sup\{|f(x)| : x \in B_N\},$$

For any $f \in \Gamma$, $f \neq 0$, there is $N \in \mathbb{N}$ such that $\|f\|_{B_N} \neq 0$ then $\{\| \ \ \|_{B_N} : B_N \subset \mathbb{R}^k\}$ is a sufficient family of seminorms and $f_n \longrightarrow f$ means uniform convergence over bounded sets B_N.

Let $E \subset \mathbb{R}^k$ with $\overline{D}(E) > 0$. Define a function

$$\varphi(u) = min\{1, dist(u, E)\}$$

where $dist(u, E) = inf\{\| \ u - v \ \| : v \in E\}$, $\|u - v\|$ denotes the metric in $\mathbb{R}^k$ and $\varphi \in \Gamma$. The functions $\varphi_v(u) = \varphi(u + v)$ form an equicontinuous, uniformly bounded family, then by Arzela-Ascoli Theorem [**?**] the family has a compact closure in the topology of uniform convergence over bounded sets in $\mathbb{R}^k$. Denote this closure by X.

Let $\mathbb{R}^k$ act on X by $T_v\Psi(u) = \Psi(u + v)$ for $\Psi \in X$, $u, v \in \mathbb{R}^k$. X is a compact metrizable space and we can identify Borel measures on X with positive functionals on $C(X)$ by Riesz Representation Theorem.

Since $\overline{D}(E) > 0$, there exists a sequence of cubes S_n such that

$$\frac{m(S_n \cap E)}{m(S_n)} \longrightarrow \overline{D}(E),$$

with $\overline{D}(E) > 0$.

Using the sequence of cubes $\{S_n\}$ we wish to define a probability measure on X. Let F_n are positive functionals on $C(X)$ for $f \in C(X)$ defined by

$$F_n(f) = \frac{1}{m(S_n)} \int\limits_{S_n} f(T_v(\varphi))dm(v).$$

By Riesz Representation theorem we have

$$F_n(f) = \int_X f d\mu_n = \frac{1}{m(S_n)} \int\limits_{S_n} f(T_v(\varphi))dm(v).$$

Since $M(X)$-the set of all probability measures is compact with weak*topology and every sequence in $M(X)$ has weak*convergent subsequence, we have a convergent subsequence $\{n_k\}$ s.t.

$$\mu_{n_k} \longrightarrow \mu$$

in weak*topology and μ is the desired probability measure such that

$$\int\limits_X f d\mu = \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int\limits_{S_{n_k}} f(T_v(\varphi))dm(v). \tag{4.0.1}$$

Set $f_0(\Psi) = \Psi(0)$, then f_0 is a continuous function by the definition of topology on X. Now, define the subset $\tilde{E}$ of X by

$$\tilde{E} = \{\Psi \in X : f_0(\Psi) = \Psi(0) = 0\}.$$

$\tilde{E}$ is closed subset of X.

Lemma 4.0.6 *The measure of $\tilde{E}$ is positive.*

Proof. Since $\mu(\tilde{E}) = \lim_{l\to\infty} \int\limits_X (1 - f_0(\Psi))^l d\mu(\Psi)$, it suffices to show that for any l

$$\int\limits_X (1 - f_0(\Psi))^l d\mu(\Psi) \geq \overline{D}(E).$$

By (4.0.1) we have;

$$\begin{aligned}\int\limits_X (1 - f_0(\Psi))^l d\mu(\Psi) &= \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int\limits_{S_{n_k}} (1 - f_0(T_v(\varphi)))^l dm(v) \\ &= \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int\limits_{S_{n_k}} (1 - \varphi(v))^l dm(v).\end{aligned}$$

Since $\varphi(v) = 0$ for $v \in E$, the last expression is at least

$$\lim_{k\to\infty} \frac{m(S_{n_k} \cap E)}{m(S_{n_k})} = \overline{D}(E),$$

So,

$$\int\limits_X (1 - f_0(\Psi))^l d\mu(\Psi) \geq \overline{D}(E).$$

□

We now establish the correspondence between E and $\tilde{E}$.

Proposition 4.0.7 *Let $E \subset \mathbb{R}^k$ and $\tilde{E} \in X$ as above. If for $u_1, u_2, ..., u_{k+1} \in \mathbb{R}^k$ we* *have*

$$\mu(\tilde{E} \cap T_{u_1}^{-1}\tilde{E} \cap ... \cap T_{u_{k+1}}^{-1}\tilde{E}) > 0 \tag{4.0.2}$$

then for all $\delta > 0$,

$$E_\delta \cap (E_\delta - u_1) \cap ... \cap (E_\delta - u_{k+1}) \neq \emptyset.$$

Proof. Define the function g on X by

$$g(\Psi) = \begin{cases} \delta - f_0(\Psi), & \text{if} \quad f_0(\Psi) < \delta, \\ 0, & \text{if} \quad f_0(\Psi) \geq \delta. \end{cases}$$

By (4.0.2) we have

$$\int 1_{\tilde{E}}(\Psi)1_{\tilde{E}}(T_{u_1}\Psi)...1_{\tilde{E}}(T_{u_{k+1}}\Psi)d\mu > 0.$$

Since g is positive on $\tilde{E}$, there is a subset of positive measure F of $\tilde{E}$ and a constant $c > 0$ such that on the subset $F \quad g > c$. The integral on this set

$$\int g(\Psi)g(T_{u_1}\Psi)...g(T_{u_{k+1}}\Psi)d\mu > 0.$$

In particular for some $\Psi = T_w\varphi$ the integrand is positive. Since

$$\int g(T_w\varphi)g(T_{u_1}T_w\varphi)...g(T_{u_{k+1}}T_w\varphi) > 0$$

then $g(T_{u_i}T_w\varphi) > 0$ for all $i = 0, ..., k+1$

$$\begin{aligned} g(T_{u_i+w}\varphi) > 0 &\Longleftrightarrow \delta - f_0(T_{u_i+w}\varphi) > 0 \\ &\Longleftrightarrow f_0(T_{u_i+w}\varphi) < \delta \\ &\Longleftrightarrow T_{u_i+w}\varphi(0) < \delta \\ &\Longleftrightarrow \varphi(u_i + w) < \delta \\ &\Longleftrightarrow u_i + w \in E_\delta. \end{aligned}$$

Thus

$$w \in E_\delta, w + u_1 \in E_\delta, ..., w + u_{k+1} \in E_\delta.$$

□

Thus we obtain the dynamical form of the geometric problem;

Theorem 4.0.8 *(Dynamical version of the Main Theorem) Let $(X, \mathcal{B}, \mu, (T_u)_{u\in\mathbb{R}^k})$ be an $\mathbb{R}^k$ action, $u_1, u_2, ..., u_{k+1} \in \mathbb{R}^k$ affinely independent, and $A \subset X$ with $\mu(A) > 0$. Then there exists t_0 such that for all $t > t_0$ and there exists $P \in SO(k)$ such that*

$$\mu(A \cap T_{tPu_1}^{-1}A \cap ... \cap T_{tPu_{k+1}}^{-1}A) > 0.$$

5. THE MAIN THEOREM

Throughout this section $(X, \mathcal{B}, \mu, (T_u)_{u\in\mathbb{R}^k})$ is an ergodic action of $\mathbb{R}^k$, $(Z, D, m, (T_u)_{u\in\mathbb{R}^k})$ the corresponding Kronecker factor, $\tau : \mathbb{R}^k \longrightarrow Z$ the homomorphism inducing the $\mathbb{R}^k$ action of Z. We have a map $\pi : X \longrightarrow Z$ which defines a disintegration of measure μ to measures μ_z, $z \in Z$, with μ_z supported by $\pi^{-1}(z)$ for a.e. z. Let $\widehat{f}(z)$ be the projection of $f \in L^2(X, \mathcal{B}, \mu, (T_u)_{u\in\mathbb{R}^k})$ to $L^2(Z, D, m, (T_u)_{u\in\mathbb{R}^k})$, i.e

$$\widehat{f}(z) = \int f d\mu_z.$$

We can lift functions $\widehat{f}(z)$ on $L^2(Z, D, m, (T_u)_{u\in\mathbb{R}^k})$ to $L^2(X, \mathcal{B}, \mu, (T_u)_{u\in\mathbb{R}^k})$ by $\widehat{f}^\pi(x) = \widehat{f} \circ \pi(x)$.

5.1. Reduction to the Kronecker Factor

Definition 5.1.1 *Let Z be a compact abelian group, $\tau : \mathbb{R}^k \longrightarrow Z$ a homomorphism. We say $u_1, u_2, ..., u_l$ are $\tau-$independent if given $\{\chi_i\}_{i=1}^l \in \widehat{Z}$, $\prod_i \chi_i(\tau(u_i)) \neq 1$ unless χ_i are all trivial characters on Z.*

The following lemma and propositions are valid for Kronecker factor Z of X. We know that every ergodic measure preserving system has Kronecker factor by the theorem (3.4.6), so this Kronecker factor lifts the results of lemma and propositions to the ergodic measure preserving system $(X, \mathcal{B}, \mu)$.

Lemma 5.1.2 *If $u_1, u_2, ..., u_l$ are $\tau-$independent, then for any $f_1, ..., f_l \in L^\infty$*

$$\frac{1}{N}\sum_{n=1}^{N} T_{nu_1+a_1} f_1 ... T_{nu_l+a_l} f_l \quad \xrightarrow{L^2(Z)} \int f_1 ... \int f_l$$

uniformly in $a_1, ..., a_l$.

Proof. It is enough to show this for characters on Z, take $f_i = \chi_i$.

$$\int \Big| \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{l} T_{nu_i + a_i} \chi_i(z) \Big|^2 dm(z)$$
$$= \int \Big| \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{l} \chi_i(z + \tau(nu_i + a_i)) \Big|^2 dm(z)$$
$$= \int \Big| \prod_{i=1}^{l} \chi_i(\tau(a_i)) \chi_i(z) \frac{1}{N} \sum_{n=1}^{N} (\chi_i(\tau(u_i)))^n \Big|^2 dm(z)$$
$$= \Big| \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{l} (\chi_i(\tau(u_i)))^n \Big|^2.$$

By the Weyl theorem we have

$$\Big| \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{l} (\chi_i(\tau(u_i)))^n \Big|^2 \longrightarrow 0,$$

if each χ_i is not trivial. □

Lemma 5.1.3 *Let H be a Hilbert space, $\xi \in \Xi$ some index set, and let $u_n(\xi) \in H$ be uniformly bounded in n, ξ. Assume that for each m the limit*

$$\gamma_m(\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle u_n(\xi), u_{n+m}(\xi) \rangle$$

exists uniformly, and

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \gamma_m(\xi) = 0$$

uniformly. Then

$$\frac{1}{N} \sum_{n=1}^{N} u_n(\xi) \underset{\rightarrow}{H} \ 0$$

uniformly in ξ.

Proof. Let M be large enough so that $\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\gamma_m(\xi)$ is small uniformly in ξ. Let N be large enough with respect to M so that the two expressions

$$\frac{1}{N}\frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{n+m}(\xi), \quad \frac{1}{N}\sum_{n=1}^{N}u_n(\xi)$$

are close uniformly in ξ. Then we have:

$$\begin{aligned}
\Big\|\frac{1}{N}\frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{n+m}(\xi)\Big\|^2 &\leq \frac{1}{N}\sum_{n=1}^{N}\Big\|\frac{1}{M}\sum_{m=1}^{M}u_{n+m}(\xi)\Big\|^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M^2}\Big\|\sum_{m=1}^{M}u_{n+m}(\xi)\Big\|^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M^2}\|u_{n+1}(\xi)+u_{n+2}(\xi)+...+u_{n+M}(\xi)\Big\|^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M^2}\langle u_{n+1}(\xi)+...+u_{n+M}(\xi), u_{n+1}(\xi)+...+u_{n+M}(\xi)\rangle \\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M^2}\sum_{m_1,m_2=1}^{M}\langle u_{n+m_1}, u_{n+m_2}\rangle \\
&= \frac{1}{M^2}\sum_{m_1,m_2=1}^{M}(\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+m_1}, u_{n+m_2}\rangle) \\
&\xrightarrow{N\to\infty} \quad \frac{1}{M^2}\sum_{m_1,m_2=1}^{M}\gamma_{m_1-m_2} \quad \xrightarrow{M\to\infty} \quad 0.
\end{aligned}$$

□

Lemma 5.1.4 *Let $(X,\mathcal{B},\mu,(T_u)_{u\in\mathbb{R}^k})$ be an ergodic action of $\mathbb{R}^k$. Suppose for some $v\in\mathbb{R}^k$, T_v acts ergodically on $(X,\mathcal{B},\mu)$. Then every eigenfunction of T_v is an eigenfunction of the $\mathbb{R}^k$ action.*

Proof. Let $\varphi(x)$ is an eigenvector of T_v with the eigenvalue λ.

$$T_v\varphi(x) = \varphi(T_v x) = \lambda\varphi(x).$$

For each T_u,

$$T_u T_v \varphi(x) = \varphi(T_u T_v x) = \varphi(T_v T_u x) = \lambda\varphi(T_u x).$$

Now, apply T_v to $\varphi(T_u x)\overline{\varphi(x)}$ to get

$$\begin{aligned} T_v(\varphi(T_u x)\overline{\varphi(x)}) &= \varphi(T_v T_u x)\overline{\varphi(T_v x)} \\ &= \lambda\varphi(T_u x)\overline{\lambda}\ \ \overline{\varphi(x)} \\ &= \varphi(T_u x)\overline{\varphi(x)}. \end{aligned}$$

Since $\lambda\overline{\lambda} = |\lambda| = 1$, $\varphi(T_u x)\overline{\varphi(x)}$ is T_v-invariant. By ergodicity of T_v, $\varphi(T_u x)\overline{\varphi(x)}$ is constant. Moreover, $|\varphi(x)|$ is constant by the equation $|\varphi(T_v x)| = |\lambda||\varphi(x)| = |\varphi(x)|$ and the ergodicity of T_v. So that,

$$\overline{\varphi(x)}\varphi(x) = c$$

and we conclude that

$$\varphi(T_u x)\overline{\varphi(x)} = \varphi(T_u x)\frac{c}{\varphi(x)} = c'$$

$$\text{and} \qquad \varphi(T_u x) = \text{constant}\varphi(x)$$

therefore, $\varphi(x)$ is an eigenfunction of $\mathbb{R}^k$ action. □

Proposition 5.1.5 *Let $(X, \mathcal{B}, \mu, (T_u)_{u\in\mathbb{R}^k})$ be an ergodic action of $\mathbb{R}^k$, and let $u_1, u_2, ..., u_l \in \mathbb{R}^k$, $l \leq k+1$ be s.t. for all $i \leq l$, $\{u_j - u_i\}_{j\neq i, j=1}^{l}$ are $\tau-$ independent, and*

$\{T_{u_j-u_i}\}_{j\neq i,j=1}^{l}$ *act ergodically , and assume that* $\{T_{u_i}\}_{i=1}^{l}$ *also act ergodically. Let* $f_1,...,f_l$ *be bounded measurable functions on* X. *Then*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{l}T_{nu_i+a_i}f_i(x)-\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{l}T_{nu_i+a_i}\widehat{f_i^{\pi}}(x)\quad \xrightarrow{L^2(X)}\; 0$$

uniformly in $a_1,...,a_l$.

Proof. We will prove the theorem by induction on l. In basis step; for $l=1$:

$$\frac{1}{N}\sum_{n=1}^{N}T_{nu+a}f=\frac{1}{N}\sum_{n=1}^{N}T_{nu}\circ T_a f=\frac{1}{N}\sum_{n=1}^{N}T_a\circ T_{nu}f.$$

By the Mean Ergodic Theorem,

$$\frac{1}{N}\sum_{n=1}^{N}T_a\circ T_{nu}f\longrightarrow\int T_a f d\mu$$

uniformly in a. Since $\int T_a f d\mu=\int f d\mu$ by ergodicity of T_a,

$$\frac{1}{N}\sum_{n=1}^{N}T_{nu+a}f\longrightarrow\int f d\mu$$

uniformly in a. In Induction hypothesis, assume that for $l-1<k+1$ it is true, and suppose $u_1,...,u_l$ satisfy the conditions above. Now, we apply Lemma (5.1.3) with $\xi=(a_1,...,a_l)$, $H=L_2(X,\mathcal{B},\mu)$ and

$$v_n=\prod_{i=1}^{l}T_{nu_i+a_i}f_i.$$

We have

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle v_n, v_{n+m}\rangle\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int\prod_{i=1}^{l}T_{nu_i+a_i}f_iT_{(n+m)u_i+a_i}\overline{f_i}d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int\prod_{i=1}^{l}T_{nu_i+a_i}f_i(T_{nu_i+a_i}\circ T_{mu_i})\overline{f_i}d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int\prod_{i=1}^{l}T_{nu_i+a_i}f_iT_{nu_i+a_i}(T_{mu_i}\overline{f_i})d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int\prod_{i=1}^{l}T_{nu_i+a_i}(f_iT_{mu_i}\overline{f_i})d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int\prod_{i=1}^{l}(T_{nu_1+a_1}\circ T_{nu_i-nu_1+a_i-a_1})(f_iT_{mu_i}\overline{f_i})d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int T_{nu_1+a_1}(f_1T_{mu_1}\overline{f_1})\prod_{i=2}^{l}T_{nu_i-nu_1+a_i-a_1}(f_iT_{mu_i}\overline{f_i})d\mu\\
&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int f_1T_{mu_1}\overline{f_1}\prod_{i=2}^{l}T_{n(u_i-u_1)+(a_i-a_1)}(f_iT_{mu_i}\overline{f_i})d\mu\\
&=\int f_1T_{mu_1}\overline{f_1}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\prod_{i=2}^{l}T_{n(u_i-u_1)+(a_i-a_1)}(f_iT_{mu_i}\overline{f_i})d\mu\\
&=\int f_1T_{mu_1}\overline{f_1}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\prod_{i=2}^{l}T_{n(u_i-u_1)+(a_i-a_1)}\widehat{(f_iT_{mu_i}\overline{f_i})}^{\pi}d\mu && (5.1.1)\\
&=\int f_1T_{mu_1}\overline{f_1}d\mu\prod_{i=2}^{l}\int f_iT_{mu_i}\overline{f_i}d\mu && (5.1.2)\\
&=\prod_{i=1}^{l}\int f_iT_{mu_i}\overline{f_i}d\mu=\gamma_m.
\end{aligned}$$

Equation (5.1.1) is obtained by the induction hypothesis and equation (5.1.2) is obtained by using Lemma (5.1.2) with

$$\{(u_j-u_1)-(u_i-u_1)\}_{j\neq i,j=1}^{l+1}=\{u_j-u_i\}_{j\neq i,j=1}^{l+1}$$

are $\tau-$ independent. Finally,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\gamma_m=\int...\int f_1(x_1)...f_l(x_l)F(x_1,...,x_l)d\mu(x_1)...d\mu(x_l)$$

where

$$F(x_1,...,x_l)=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{l}\overline{f_i}(T_{mu_i}x_i)=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{l}\overline{f_i}(T_{u_i}^{m}x_i)$$

which is well defined by the Mean Ergodic Theorem. Now, suppose f_i is orthogonal to all eigenfunctions of the $\mathbb{R}^k$- action, then by Lemma (5.1.4) it is orthogonal to all eigenfunctions of T_{u_i}. Clearly

$$\begin{aligned}F(T_{u_1}x_1,...,T_{u_l}x_l)&=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{l}\overline{f_i}(T_{u_i}^{m+1}x_i)\\&=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\prod_{i=1}^{l}\overline{f_i}(T_{u_i}^{m}x_i)\\&=F(x_1,...,x_l).\end{aligned}$$

So, $F(x_1,...,x_l)$ is an eigenfunction of T_u and f_j is orthogonal to $F(x_1,...,x_l)$, we conclude that

$$\int f_j(x_j)F(x_1,...,x_l)d\mu(x_j)=0.$$

By Fubini's Theorem and the previous result, we have

$$\int...\int f_2(x_2)\int f_1(x_1)F(x_1,...,x_l)d\mu(x_1)d\mu(x_2)...=0$$

and hence

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\gamma_m=0.$$

Then in this case we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{l}T_{nu_i+a_i}f_i=0$$

in $L^2(X)$ uniformly in $a_1,...,a_l$. Now set,

$$f_j=\widehat{f_j}^{\pi}+\widehat{f_j}^{\pi^{\perp}}.$$

Since $f-\widehat{f}\circ\pi$ is orthogonal to the subspace of $L^2(X)$ spanned by eigenfunction [1], we have the result:

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{l}T_{nu_i+a_i}f_i(x)-\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{l}T_{nu_i+a_i}\widehat{f_i^{\pi}}(x)\quad\underrightarrow{L^2(X)}\;\;0$$

uniformly in $a_1,...,a_l$. □

5.2. Proof of the Main Theorem

Without loss of generality, we may assume by disintegration of μ with respect to Kronecker factor Z of X i.e. μ_z , that the action of $\mathbb{R}^k$ is ergodic. Let $f=1_A$ be the characteristic function of A, and $\widehat{f}$ the projection of f in $L^2(Z)$, and $\widehat{f}(z)=\displaystyle\int fd\mu_z$. Since $f\geq 0$, we have $\widehat{f}\geq 0$. In our main theorem we have a subset A of positive measure $\mu(A)>0$, then we have

$$\begin{aligned}\mu(A)&=\int fd\mu\\&=\int\Big\{\int fd\mu_z\Big\}dm\quad(by\;\; the\;\; Theorem\;\;(3.6.1))\\&=\int\widehat{f}dm>0.\end{aligned}$$

We know Z is a topological group and a neighborhood of zero is a neighborhood of the identity element. There is a neighborhood of the identity element W in Z, such that if $w_1, w_2, ..., w_{k+1} \in W$, then

$$\int_Z \widehat{f}(z)\widehat{f}(z+w_1)\widehat{f}(z+w_2)...\widehat{f}(z+w_{k+1})dm(z) > a > 0 \tag{5.2.1}$$

for some a.
Now define a homomorphism $\varphi : M_k(\mathbb{R}) \to Z^{k+1}$ such that $\varphi(M) = (\tau(Mu_1), ..., \tau(Mu_{k+1}))$ where $M_k(\mathbb{R})$ is the set of all $k \times k$ matrices over $\mathbb{R}$, and let Ω be the closure of the image of $M_k(\mathbb{R})$ in Z^{k+1}. i.e. $\Omega = \overline{\{\varphi(M) : M \in M_k(\mathbb{R})\}}$.

Now, we will assume that there exist $M \in M_k(\mathbb{R})$ and $P \in SO(k)$ such that $\{Mu_1, ..., Mu_{k+1}\}$ satisfying the condition of proposition (5.1.5), and $\{\varphi(nM)\}_{n=1}^{\infty}$ is dense in Ω, and M^tP is antisymmetric matrix. In the next section, we will show the existence of these matrices.
Since Z^{k+1} is a compact metrizable group and Ω is closed, Ω is compact.
Claim1: There exists an $L \in \mathbb{N}$, such that for each $U \in M_k(\mathbb{R})$ there exists an $0 \leq n \leq L$ s.t. $\varphi(nM + U) \in W^{k+1}$.

W^{k+1} is an open neighborhood of zero in Z^{k+1}. Let consider the set of open neighborhoods of 0 in Z^{k+1}: $V^o(0)$. We have this property that

$$\forall V' \in V^o(0) \;\; \exists V \in V^o(0) \text{ s.t. } \; V = -V, \;\; V + V \subseteq V'.$$

If we choose $V' = W$, we have

$$V + V \subseteq W^{k+1}.$$

Let $z_0 = 0, z_1, ..., z_N \in Z^{k+1}$. Since Ω is compact,

$$\Omega \subseteq (z_0 + V) \cup (z_1 + V) \cup ... \cup (z_N + V).$$

Since $\overline{\{\varphi(nM) : n \in \mathbb{N}\}} = \Omega$, there exists $\{n_0, n_1, ..., n_N\}$ s.t.

$$\varphi(n_0 M) \in z_0 + V, ..., \varphi(n_N M) \in z_N + V.$$

Let choose $L > n_0 n_1 ... n_N$, so $\forall i \in [0, N], \quad \exists n < L \quad s.t. \quad \varphi(nM) \in z_i + V$
Since $\Omega \subseteq \bigcup_{i=0}^{N}(z_i + V)$ and $\Omega = \overline{\varphi(M_k(\mathbb{R}))}$, for any $U \in M_k(\mathbb{R})$

$$-\varphi(U) \in z_i + V \quad \text{for} \quad \text{some} \quad i \in [0, N].$$

So $\exists i \in [0, N]$ s.t.

$$\varphi(U) + z_i \in V. \tag{5.2.2}$$

<u>Claim2:</u> $z_i + V \subseteq W^{k+1} - \varphi(U)$

Let $v \in V$, $z_i + v \in z_i + V$. We know that $V + V \subseteq W^{k+1}$ and by (5.2.2) $\varphi(U) + z_i \in V$, then

$$\varphi(U) + z_i + v \in V + V \subseteq W^{k+1},$$

for any $v \in V$. So we have $\varphi(U) + z_i + V \subseteq W^{k+1}$. Now, there exists $n < L$, $\varphi(nM) \in z_{i+V}$ and $\varphi(nM) \in W^{k+1} - \varphi(U)$ by (claim2).
Hence for all $U \in M_k(\mathbb{R})$, $\varphi(nM + U) \in W^{k+1}$. Then by (5.2.1) ,

$$\frac{1}{N}\sum_{n=1}^{N}\int_{Z} \widehat{f}(z)\widehat{f}(z + \tau(nMu_1 + Uu_1))...\widehat{f}(z + \tau(nMu_{k+1} + Uu_{k+1}))dm(z) > \frac{a}{2L}$$

for all N greater than L.
Now apply the proposition (5.1.5) to deduce that there exists an N_0 so that if $N > N_0$

$$\frac{1}{N}\sum_{n=1}^{N}\int_{Z} f(x)T_{nMu_1+Uu_1}f(x)...T_{nMu_{k+1}+Uu_{k+1}}f(x)d\mu > \frac{a}{4L}$$

for all U. Hence

$$\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-1}_{(nM+U)u_1}A\cap ...\cap T^{-1}_{(nM+U)u_{k+1}}A) > \frac{a}{4L}$$

for all $N > N_0$, for all U. Now take $U = tP$. For each t, there exists an $n < N_0$ s.t.

$$\mu(A\cap T^{-1}_{(nM+tP)u_1}A\cap ...\cap T^{-1}_{(nM+tP)u_{k+1}}A) > \frac{a}{4L}.$$

Since M^TP is antisymmetric, $M \in T_P(SO(k))$- the tangent space of $SO(k)$ at P.[8]. We have for any $\varepsilon > 0$

$$P' := Pexp(\varepsilon nP^{-1}M) = P(I + \varepsilon nP^{-1}M + o(\varepsilon)) = P + \varepsilon nM + o(\varepsilon);$$

hence

$$(\frac{1}{\varepsilon}P + nM) - \frac{1}{\varepsilon}P' = o(1).$$

Our action T_u is continuous in the sense that for any measurable set $A \subset X$, $\forall\varepsilon > 0\ \ \exists\delta$ s.t.

$$\|u - u'\| \leq \delta \Longrightarrow |\mu(A\cap T_u^{-1}A) - \mu(A\cap T_{u'}^{-1}A)| \leq \varepsilon.$$

For $t = \dfrac{1}{\varepsilon}$ large enough, P' as above, and if we take $u = (tP + nM)u_i$ and $u' = (tP')u_i$ for $i = 1, ..., 1 + k$

$$\|u - u'\| \leq \delta \text{ implies that}$$

$$\mu(A\cap T^{-1}_{(nM+tP)u_1}A\cap ...\cap T^{-1}_{(nM+tP)u_{k+1}}A) \text{ is very close to}$$

$\mu(A \cap T^{-1}_{tP'u_1} A \cap ... \cap T^{-1}_{tP'u_{k+1}} A)$. Therefore

$$\mu(A \cap T^{-1}_{tP'u_1} A \cap ... \cap T^{-1}_{tP'u_{k+1}} A) > \frac{a}{8L} > 0.$$

□

5.3. The Existence of the Matrices M and P in The Proof of The Main Theorem

We first show the following proposition which is proved by Pugh and Shub [9]. If $\mathbb{R}^k$ acts ergodically, then the set of points which do not act ergodically is very small.

Proposition 5.3.1 *If $(X, \mathcal{B}, \mu, T_u)$ is an ergodic action of $\mathbb{R}$, then but for a countable set of $u \in \mathbb{R}$, T_u acts ergodically. If $(X, \mathcal{B}, \mu, T_u)$ is an ergodic action of $\mathbb{R}^l$, then but for a countable set of $l-1$ dimensional hyperplanes, all $l-1$ dimensional hyperplanes through the origin act ergodically.*

We now prove the main proposition.

Proposition 5.3.2 *Let φ, Ω be as in the previous section. There exist matrices $M \in M_k(\mathbb{R})$, $P \in SO(k)$ satisfying the following conditions:*

1. *$\{\varphi(nM)\}_{n=1}^{\infty}$ is dense in Ω*
2. *$\{M(u_j - u_i)\}_{j=1,(j \neq i)}^{k+1}$ are τ-independent, for $i = 1, ..., k+1$*
3. *$\{T_{Mu_i}\}_{i=1}^{k+1}$, $\{T_{M(u_j-u_i)}\}_{i,j=1,(j\neq i)}^{k+1}$ act ergodically.*
4. *$M^T P$ is an antisymmetric matrix.*

For the proof we will need the following lemmas:

The following lemma is proved in [4].

Lemma 5.3.3 *The group $SO(k)$ $(k \geq 3)$ linearly spans $M_k(\mathbb{R})$.*

Lemma 5.3.4 *Let $\{s_i\}_{i=1}^k, \{v_i\}_{i=1}^k \subset \mathbb{R}^k$, and define a linear map s.t.*

$$f(M) = \sum_{i=1}^{k} \langle s_i, M v_i \rangle$$

with $f \not\equiv 0$. Let $\mathcal{D} = \{D \in SO(k) : \ f(D) = c \ \}$ for some constant $c \in \mathbb{R}$. Then

$$dim\mathcal{D} < dimSO(k).$$

Proof. By the way of contradiction, assume $f(P) = c$ for all $P \in SO(k)$. Let O be some matrix in $SO(k)$, then define $f(M-O) = f(M) - f(O) = f(M) - c$ since f is linear. Since $f \not\equiv 0, \quad \exists M \in M_k(\mathbb{R})$ *s.t.* $f(M - O) \neq 0$.

By the Lemma (5.3.3), we can write each element of $M_k(\mathbb{R})$ by the linear combination of vectors in $SO(k)$. So, if $M \in M_k(\mathbb{R})$, then there exist $\{P_1, P_2, ..., P_n\}$ affinely independent vectors of $SO(k)$ and

$$a_1, a_2, ..., a_n \in \mathbb{R} \quad \text{with} \quad \sum_{i=1}^{n} a_i = 1$$

s.t. $M = a_1P_1 + a_2P_2 + ... + a_nP_n$.

Now,

$$\begin{aligned}
&f(M - O) = f(a_1P_1 + a_2P_2 + ... + a_nP_n - O)\\
&= f(a_1(P_1 - O) + a_2(P_2 - O) + ... + a_n(P_n - O) + (a_1 + a_2 + ... + a_n - 1)O)\\
&= a_1 f(P_1 - O) + a_2 f(P_2 - O) + ... + a_n f(P_n - O)\\
&+ (a_1 + a_2 + ... + a_n - 1) f(O) \ \text{ since } \ f \ \text{ is } \ \text{linear}\\
&= (a_1 + a_2 + ... + a_n - 1) f(O) \ \text{ since } \ \forall P_i \in SO(k) \ \ f(P_i) = c, \ f(P_i - O) = 0\\
&= (a_1 + a_2 + ... + a_n - 1)c = 0 \ \text{ since } \ a_1 + a_2 + ... + a_n = 1.
\end{aligned}$$

But we can write all of the matrices in $M - O$ form, so we have found $\forall M \in M_k(\mathbb{R})$, $f(M) \equiv 0$ which is a contradiction to the property of f. Therefore there is a point

$P \in SO(k)$ which is not in $\mathcal{D}$. Hence $\mathcal{D}$ is a proper subvariety of $SO(k)$. Since $SO(k)$ is irreducible, by [10] $dim\mathcal{D} < dimSO(k)$. □

Lemma 5.3.5 *Let φ be the homomorphism*

$$\varphi : M_k(\mathbb{R}) \to Z^{k+1}$$
$$M \to (\tau(Mu_1), ..., \tau(Mu_{k+1}))$$

Let Ω be the closure of the image of $M_k(\mathbb{R})$ in Z^{k+1}. Then for all matrices but a countable number of hyperplanes in $M_k(\mathbb{R}) = \mathbb{R}^{k^2}$, the image of $\{(nM)\}_{n=1}^{\infty}$ is dense in Ω.

Proof. Since Z^{k+1} is a compact abelian group , so Ω is compact, metrizable and abelian gropu. Then $L^2(\Omega)$ is seperable unitary representations of Ω. The characters of Ω are irreducible finite dimesional representations of Ω. By Peter Weyl Theorem [**?**], characters of Ω form an orthonormal basis for $L^2(\Omega)$. So, $\widehat{\Omega}$ is countable. As $\chi \circ \varphi$ is a character on $M_k(\mathbb{R})$ it is of the form $\chi \circ \varphi(M) = e^{2\pi i <N,M>}$ for some $N \in M_k(\mathbb{R})$. Since $\widehat{\Omega}$ is countable, the set of $M \in M_k(\mathbb{R})$ for which $\exists \chi \in \widehat{\Omega} \quad \chi \circ \varphi(M) = 1$ is countable union of hyperplanes which has empty interior. And also, we have If $\chi \circ \varphi(M) = 1$, then

$$\begin{aligned}\chi(\varphi(2M)) &= \chi(2\varphi(M)) \\ &= \chi(\varphi(M) + \varphi(M)) \\ &= \chi(\varphi(M))\chi(\varphi(M)) = 1.\end{aligned}$$

So , for all $n \geq 1 \quad \chi(\varphi(nM)) = 1$ Since $M_k(\mathbb{R})$ is complete metric space, by Baire Cathegory Theorem the countable union of hyperplanes can not be equal to $M_k(\mathbb{R})$. If M is in the complement of this union, then

$$\forall \chi \in \widehat{\Omega} - \{1\}, \quad \chi(\varphi(M)) \neq 1.$$

Then for all matrices but a countable number of hyperplanes in $M_k(\mathbb{R})$, the image of $\{(nM)\}_{n=1}^{\infty}$ is dense in Ω. □

Lemma 5.3.6 *For each $j = 1, ..., \infty$, let $\{v_{ij}\}_{i=1}^{k} \subset \mathbb{R}^k$ be a linearly independent set, and $\{s_{ij}\}_{i=1}^{k} \subset \mathbb{R}$, such that $(s_{1j}, ..., s_{kj}) \neq \overrightarrow{0}$. There exists an antisymmetric matrix $B \subset M_k(\mathbb{R})$ s.t.*

$$\forall j : \qquad f_{j,B}(M) := \sum_{i=1}^{k} \langle s_{ij}, MBv_{ij} \rangle \not\equiv 0.$$

Proof. Let $\mathcal{B}$ be the space of antisymmetric matrices. Let $\{E_{11}, E_{21}, ..., E_{kk}\}$ be a k^2 standard basis matrices for $M_k(\mathbb{R}) = \mathbb{R}^{k^2}$ where $(E_{ij})_{mn} = \begin{cases} 1, & \text{m=i and n=j,} \\ 0, & \text{otherwise.} \end{cases}$
Let write for any $M \in M_k(\mathbb{R})$ by linear combination of k^2 standard basis, where $C_{ij} \in \mathbb{R}$

$$M = C_{11}E_{11} + C_{21}E_{21} + ... + C_{kk}E_{kk}.$$

Since $f_{j,B}(M)$ is linear in M, then we have

$$\begin{aligned} f_{j,B}(M) &= \sum_{i=1}^{k} \langle s_{ij}, (C_{11}E_{11} + C_{21}E_{21} + ... + C_{kk}E_{kk})Bv_{ij} \rangle \\ &= \sum_{i=1}^{k} \langle s_{ij}, C_{11}E_{11}Bv_{ij} \rangle + ... + \sum_{i=1}^{k} \langle s_{ij}, C_{kk}E_{kk}Bv_{ij} \rangle \end{aligned}$$

and if we have $f_{j,B}(M) \equiv 0$, i.e.

$$\sum_{i=1}^{k} \langle s_{ij}, C_{11}E_{11}Bv_{ij} \rangle + ... + \sum_{i=1}^{k} \langle s_{ij}, C_{kk}E_{kk}Bv_{ij} \rangle \equiv 0$$

then B satisfies the k^2 linear equations given by the standard basis for the $\mathbb{R}^{k^2}$. For each j, define sets $E_j = \{B \in \mathcal{B} : f_{j,B}(M) \equiv 0\}$. The sets E_j's form a linear subspace of $\mathcal{B}$. Since we have only a countable number of inequalities, it suffices to show that this linear subspace is a proper subspace of $\mathcal{B}$. So without loss of generality, we have only one inequality. Assume

$$\forall B \in \mathcal{B} \quad : \sum_{i=1}^{k} \langle M s_i, B v_i \rangle \equiv 0.$$

Without loss of generality $s_{11} \neq 0$. Let $M = E_{11}$ and $M = E_{21}$, then we get

$$\sum_{l=1}^{k} b_{1l} (\sum_{j=1}^{k} s_{j1} v_{jl}) = s_{11} (\sum_{j=1}^{k} b_{1j} v_{1j}) + ... + s_{k1} (\sum_{j=1}^{k} b_{1j} v_{kj}) = 0$$
$$\sum_{l=1}^{k} b_{2l} (\sum_{j=1}^{k} s_{j1} v_{jl}) = s_{11} (\sum_{j=1}^{k} b_{2j} v_{1j}) + ... + s_{k1} (\sum_{j=1}^{k} b_{2j} v_{kj}) = 0.$$

As this is true for all $b_{12}, ..., b_{1k}, b_{23}, ..., b_{2k} \in \mathbb{R}$,

$$s_{11} v_{1j} + ... + s_{k1} v_{kj} = 0$$

for $1 \leq j \leq k$. Hence

$$s_{11} v_1 + ... + s_{k1} v_k = 0$$

which is a contradiction to the linear independence of the v_i. Therefore, there exists an antisymmetric matrix B which makes the map f identically zero. $\square$

Proof of the Proposition 5.3.2

First condition is satisfied by Lemma (5.3.5), for the second condition we will show the $\tau-$ independence requirement of $\{M(u_j - u_i)\}_{j=1,(j \neq i)}^{k+1}$, for $i = 1, ..., k+1$. Suppose $w_1, ...w_k \in \mathbb{R}^k$ are $\tau-$ independent, if for characters $\{\chi_i\}_{i=1}^{k}$ in $\hat{Z}$

$$\prod_i \chi_i(\tau(w_i))$$
$$= \chi_1 \circ \tau(w_1) ... \chi_k \circ \tau(w_k) \neq 1.$$

As $\chi \circ \tau$ is a character on $\mathbb{R}^k$, it is of the form $\chi \circ \tau(w) = e^{2\pi i \langle s,w \rangle}$, so the condition above is equivalent to

$$e^{2\pi i \langle s_{m1}, w_1 \rangle} e^{2\pi i \langle s_{m2}, w_2 \rangle} ... e^{2\pi i \langle s_{mk}, w_k \rangle} \neq 1.$$

Then we have

$$\sum_{j=1}^{k} 2\pi i \langle s_{mj}, w_j \rangle \neq 0 \quad \text{and} \quad \sum_{j=1}^{k} \langle s_{mj}, w_j \rangle \neq 1.$$

Hence the M we are seeking satisfy the inequalities

$$g_m(M) = \sum_{(j \neq i), j=1}^{k+1} \langle s_{mj}, M(u_j - u_i) \rangle \neq 1 \tag{5.3.1}$$

$$\text{for} \quad i = 1, ..., k+1.$$

Since $u_1, ... u_{k+1}$ are affinely independent, for each $i = 1, ..., k+1$ the $\{(u_j - u_i)\}_{(j \neq i), j=1}^{k+1}$ are linearly independent, so by the Lemma (5.3.6) $g_m \not\equiv 0$ and $\{M(u_j - u_i)\}_{j=1,(j \neq i)}^{k+1}$ are $\tau-$ independent.

By the Lemma (5.3.6), there exists an antisymmetric matrix B, such that

$$f_{j,B}(M) := \sum_{i=1}^{k} \langle s_{ij}, MBv_{ij} \rangle \not\equiv 0.$$

For each j the set of M with $f_{j,B}(M) = c_j$ is an hyperplane in $M_k(\mathbb{R})$. This space intersects with $SO(k)$ in a proper algebraic subvariety of $SO(k)$ by the Lemma (5.3.4). Therefore for almost all $P \in SO(k)$, the matrix $M = PB$ satisfy the (5.3.1). Since $P \in SO(k)$, $B = M^T P$ is antisymmetric matrix. Finally, we need to show the ergodicity of the $T_{M(u_j - u_i)}$ and T_{Mu_i}. By the Proposition (5.3.1), for almost all $M = PB$, $T_{M(u_j - u_i)}$ and T_{Mu_i} act ergodically for $i = 1, ..., k+1$ and $j = 1, ..., k+1$. We have found the desired matrices M and P for the proof of our main theorem. □

6. CONCLUSION

This thesis is studied the proof of a geometric problem by using ergodic theory. Our problem is the multidimensional form of the theorem of Furstenberg , Katznelson and Weiss [1]. We followed the paper of Tamar Ziegler[4]. In our exposition, there are some changes though. For example, we observed that the theorem of Furstenberg , Katznelson and Weiss [1] directly applies to our situation as well. At some other points, Tamar Ziegler claims more than what is really needed. We don't think the claim is true as stated. Although we don't have a counter example. We tried to understand what is necessary for the proof and we supplied a proof for that.

REFERENCES

1. Furstenberg, H., Y. Katznelson and B. Weiss, "Ergodic Theory and Configurations in Sets of Positive Density", *Mathematics of Ramsey Theory*, Springer Verlag, Vol. 5, pp. 184-198, 1990.

2. Bourgain, J., "A Szemerédi Type Theorem For Sets Of Positive Density in $\mathbb{R}^k$", *Israel Journal of Mathematics*, Vol. 54, pp. 307-316, 1986.

3. Falconer, K.J. and J.M. Marstrand "Plane Sets With Positive Density at Infinity Contain All Large Distances", *Bulletin London Mathematical Society*, Vol. 18, pp. 471-474, 1986.

4. Ziegler, T., "An application of Ergodic Theory to a problem in Geometric Ramsey Theory", Israel Journal of Mathematics, Vol. 114, pp. 271-288, 1999.

5. Fabian, M. and P. Habala, *Functional Analysis and Infinite-dimesional Geometry*, Canadian Mathematical Society, 2001.

6. Furstenberg, H. *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, 1981.

7. Einsiedler, M. and T. Ward, *Ergodic Theory: with a view towards Number Theory*, http://www.mth.uea.ac.uk/ergodic, 2009.

8. Rossman, W., *Lie Groups: An Intorduction Through Linear Gorups*, Oxford University Press, New York, 2002.

9. Pugh, C. and M. Shub, "Ergodic Elements of Ergodic Actions", Composito Mathematica, Vol. 23, No. 1, pp. 115-122, 1971.

10. Kraft, H., *Algebraic Transformation Groups, Part 6: Basic Algebraic Geometry*, http://www.math.unibas.ch/ kraft, 2006.

REFERENCES NOT CITED

Marsden, J. E. and M. J. Hoffman, *Elementary Classical Analysis*, W. H. Freeman, 1993.

Parry, W., *Topics In Ergodic Theory*, Cambridge University Press, 1981.

Petersen, K., *Ergodic Theory*, Cambridge University Press, 1983.

Rokhlin, V. A., "Selected Topics from the Metric Theory of Dynamical Sysytem", Uspehi Mat. Nauk, No. 4, pp. 57-128, American Mathematical Society Translation Series, No. 49, pp. 171-240, 1966.

Royden, H. L., *Real Analysis*, Macmillan Publishing, 1968.

Ziegler, T., "Nilfactors of $\mathbb{R}^m-$ Actions and Configurations in Sets of Positive Upper Density in $\mathbb{R}^m$", J. Anal. Math., Vol. 99, pp. 249-266, 2006.

Zimmer, R., *Essential Results of Functional Analysis*, The Univesity of Chicago Press, 1990.