SPECTRAL AND SCATTERING PROPERTIES OF POINT INTERACTIONS

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ABSTRACT

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Point interactions in all relevant dimensions are important to model short range interactions in quantum mechanics. They are exactly solvable and play an important role in solid state, nuclear and atomic physics. In this thesis, we investigate point interactions on flat space and on a general compact manifold in two and three dimensions. The problem contains divergences and they can be renormalized using the coupling constant renormalization method. We derive the properties of the bound states, prove that the resolvent operator that we have found defines a Hamiltonian, and calculate scattering cross sections. We repeat some of our calculations on a general compact Riemannian manifold and solve the same problem on a sphere and hyperbolic planes H^2 and H^3. At the end we give some numerical examples.

ÖZET

NOKTASAL ETKİLEŞİMLERİN TAYF VE SAÇILMA ÖZELLİKLERİ

Kuantum mekaniğinde, anlamlı bütün boyutlarda, noktasal etkileşimler kısa mesafeli etkileşimleri modellemede önemlidirler. Tam olarak çözülebilirler ve katı hal, çekirdek ve atom fiziğinde onemli rol oynarlar. Bu tezde, düz uzayda ve genel bir kompakt manifold üzerinde, iki ve üç boyutlarda noktasal etkileşimleri araştırıyoruz. Problem sonsuzluklar içeriyor ve "coupling constant renormalizasyonu" yöntemiyle renormalize edilebiliyor. "Bound state"lerin ozelliklerini ve bulduğumuz resolvent operatörünün bir Hamiltonyen tanımladığını gösteriyoruz ve tesir kesitlerini hesaplıyoruz. Hesaplarımızın bazılarını genel kompakt bir Riemannyen manifold üzerinde tekrar ediyoruz ve problemi küre üzerinde ve hiperbolik düzlemler H^2 ve H^3'te çözüyoruz. En son olarak da bazı nümerik örnekler veriyoruz.

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1. INTRODUCTION

When a particle's Compton wavelength is large compared to the typical range of a potential, we may approximately describe the system as a particle interacting with a point center. Most of the interactions in atomic, nuclear and solid state physics are good candidates for such an approximation scheme. In one dimension, this problem has a simple exponentially decaying solution which can be found in many quantum mechanics textbooks. It is also possible to extend this solution in various directions keeping its one dimensional structure [1].

Historically, this approximation was first used in one dimension by Kronig and Penney in 1931 [2] to describe a nonrelativistic electron moving in a fixed crystal lattice and eventually the Kronig-Penney model became a standard model in solid state physics. A few years later Bethe and Peierls [3] and Thomas [4] studied point interactions to describe a nonrelativistic quantum mechanical particle interacting with a very short range potential in three dimensions and Thomas realized that renormalization of the coupling constant is necessary to make the Hamiltonian bounded. In 1936 Fermi introduced the so called Fermi-pseudopotentials [5] which made explicit by Breit 10 years later and which can be identified with point interactions for less than three dimensions..

The first rigorous mathematical work is done by Berezin and Faddeev [6] in 1961 on the definition of the point interaction Hamiltonians in three dimensions as self adjoint operators in $L^2(\mathbf{R}^3)$. It turned out that there is a one parameter family of self-adjoint operators for two and three dimensions leading to point interactions which was put on a mathematical basis by Berezin and Faddeev using Krein's theory of self adjoint extensions. It also turned out that in one dimension free Hamiltonian exhibits a four parameter family of self adjoint extensions in $L^2(\mathbf{R})$ which leads to additional types of point interactions [7].

By 1950s, zero-range potentials also began to be studied in many body theories and quantum statistical mechanics. The study of N particles interacting through delta function potentials in one dimension has been a valuable example [9, 10, 11]. The many body problem was studied as a model for quantum chromodynamics in [12] and a detailed analysis is given in the unpublished thesis of J. Hoppe [13]. An interesting and truly inspiring discussion of the many body version of this problem, which is a non-relativistic $\lambda\phi^4$ theory, is given in Rajeev's article [14]. In this work, we will not go into details of many body theories.

When we look at the delta potentials in more than one dimension we immediately run into a difficulty: the ground state energy is not bounded from below. The solution of this puzzle comes from controlling the short range behavior of the potential, that is we do not allow the potential to become so strong suddenly, but reduce its strength as we come closer and closer to the center. This is accomplished in the physics terminology by introducing a cut-off and assuming the strength of each source being dependent on this coupling. Adjusting the coupling to remove this divergence gives us the renormalized problem. This physical approach can be turned into a mathematically sound expression and we can study this well defined problem to gain more insight into more complex problems which require renormalization. In fact, this type of behavior is well-known in the theory of strong interaction in which the coupling strength goes down as one increases the momentum scale.

In this work we will concentrate on this model problem, hoping that its scrutiny will provide us some valuable insights into more complex problems which require renormalization. (The importance of this problem to understand renormalization is also emphasized in an article by R. Jackiw [15]). A complete and mathematically rigorous study of multiple delta potentials in all relevant dimensions with a very good historical introduction is presented in the book by Albeverio *et al* [7]. Most of our results are essentially available in this source, but their presentation is very general and it is from the point of view of self-adjoint extensions. We rather try to use elementary and physically motivated methods.

2. POINT INTERACTIONS ON FLAT SPACE

2.1. Bound States

In one dimension, we use Dirac's delta functions to describe point interactions. Generalization to higher dimensions requires careful attention. It turns out that point interaction problem for higher dimensions is rather problematic and the heuristic definition, that we generalize from the one dimensional case, causes some unphysical results like infinities. To clearly see this let us formally define the potential by using delta functions and try to solve the Schrodinger equation.

So the Hamiltonian is just the sum of free Hamiltonian and delta potentials centered at points $a_i \in \mathbb{R}^D$:

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \sum_{i=1}^{N} g_i \, \delta^D(x - a_i), \tag{2.1}$$

where $D \geq 2$ is the dimension of the space, ∇^2 denotes the Laplacian in D dimensions, m is the mass of the particle and $x \in \mathbb{R}^D$.

Let us suppose that bound states exist. Actually one can easily see that in two dimensions the Hamiltonian is not bounded from below if there exist even a single bound state. For example, if $\psi(x)$ is an eigenfunction with energy E, then $\psi(\alpha x)$ is also an eigenfunction, but this time with energy $\alpha^2 E$ as one can see by using the scaling property of the delta function. Hence existence of a negative energy state implies that the Hamiltonian is unbounded from below. One intuitively expect that the situation is even worst for higher dimensions.

We will first demonstrate the divergence problem which leads to an unbounded Hamiltonian. Then we will introduce the technique which will eventually cure the divergence. For, let us define a new parameter ν for the negative energy states via

$E = -\nu^2$, so the Schrodinger equation can be written as $H\psi = -\nu^2\psi$ and we have these inverse Fourier relations:

$$\psi(x) = \int \frac{d^D k}{(2\pi)^D}\, e^{ikx}\, \tilde{\psi}(k), \tag{2.2}$$

$$\delta^D(x - a_i) = \int \frac{d^D k}{(2\pi)^D}\, e^{ik(x-a_i)}, \tag{2.3}$$

which we insert into the Schrodinger equation to get

$$\int \frac{d^D k}{(2\pi)^D}\, e^{ikx} \left\{ \frac{\hbar^2}{2m} k^2\, \tilde{\psi}(k) + \nu^2\, \tilde{\psi}(k) - \sum_{i=1}^{N} g_i A_i\, e^{-ika_i} \right\} = 0, \tag{2.4}$$

where $A_i = \psi(a_i)$ and we used the fact that $\delta^D(x - a_i)\psi(x) = \delta^D(x - a_i)\psi(a_i)$.

Equation 2.4 is valid for all $x \in \mathbb{R}^D$, so the expression inside the curly parentheses must be identically zero. This gives us the Fourier transform of the wave function as

$$\tilde{\psi}(k) = \sum_{i=1}^{N} g_i\; A_i\; \frac{e^{-ik.a_i}}{\frac{\hbar^2}{2m} k^2 + \nu^2}, \tag{2.5}$$

and the wave function is therefore

$$\psi(x) = \sum_{i=1}^{N} g_i\; A_i \int \frac{d^D k}{(2\pi)^D}\, \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 + \nu^2}. \tag{2.6}$$

To find the energy eigenvalues, we will use the definition of A_i's. Hence we have

$$A_i = \psi(a_i) = \sum_{j=1}^{N} g_j\; A_j \int \frac{d^D k}{(2\pi)^D}\, \frac{e^{ik.(a_i-a_j)}}{\frac{\hbar^2}{2m} k^2 + \nu^2}. \tag{2.7}$$

Let us group the terms with the same indices and then we obtain

$$\left\{ g_i^{-1} - \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{\frac{\hbar^2}{2m} k^2 + \nu^2} \right\} A_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{g_j}{g_i}\; A_j \int \frac{d^D k}{(2\pi)^D}\, \frac{e^{ik.(a_i-a_j)}}{\frac{\hbar^2}{2m} k^2 + \nu^2} = 0. \tag{2.8}$$

This is a linear equation in terms of A_i's, so we may express it as a matrix equation by $\Phi(\nu)A = 0$, where $\Phi(\nu)$ is defined as

$$\Phi_{ij}(\nu) = \begin{cases} g_i^{-1} - \displaystyle\int \frac{d^D k}{(2\pi)^D} \, \frac{1}{\frac{\hbar^2}{2m} k^2 + \nu^2} & \text{if } i = j \\ \\ -\displaystyle\frac{g_j}{g_i} \int \frac{d^D k}{(2\pi)^D} \, \frac{e^{ik.(a_i - a_j)}}{\frac{\hbar^2}{2m} k^2 + \nu^2} & \text{if } i \neq j. \end{cases} \tag{2.9}$$

Remember that the volume element in D dimensions contains the factor k^{D-1}. So for $D \geq 2$, the integral in the diagonal part of $\Phi(\nu)$ is divergent. This is not acceptable for a physical theory.

There are several methods to get rid off this type of divergences and to get physically acceptable results. These methods are widely used in Quantum Field Theory , and which have the common name "renormalization". Here we will use the so called "coupling constant renormalization" to remove this divergence. In QFT the coupling constant renormalization is done perturbatively, i.e each term in the perturbative expansion of the coupling constant is renormalized separately. Here we will use a non-perturbative approach.

In this scheme, we redefine the coupling constants (or the strength of delta potentials) as functions of some parameter Λ, put this new parameter as a cut-off to the Fourier integrals and look for a meaningful $g^{-1}(\Lambda)$ which would make the diagonal part of Φ finite in the $\Lambda \to \infty$ limit, i.e.

$$\left| g_i^{-1}(\Lambda) - \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{1}{\frac{\hbar^2}{2m} k^2 + \nu^2} \right| < \infty \quad \text{as } \Lambda \to \infty. \tag{2.10}$$

We may now extend the definition of the Φ matrix as:

$$\Phi_{ij}(\Lambda,\nu) = \begin{cases} g_i^{-1}(\Lambda) - \displaystyle\int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{1}{\frac{\hbar^2}{2m}\, k^2 + \nu^2} & \text{if } i = j \\ \\ -\dfrac{g_j(\Lambda)}{g_i(\Lambda)} \displaystyle\int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{e^{ik.(a_i - a_j)}}{\frac{\hbar^2}{2m}\, k^2 + \nu^2} & \text{if } i \neq j. \end{cases} \tag{2.11}$$

To get a finite difference, $g_i^{-1}(\Lambda)$ must have the same divergence as $\Lambda \to \infty$ which means that $g_i(\Lambda)$ must go to zero in this limit. So the strengths of delta potentials must go down to zero to make the problem physically meaningful. This shows that delta potentials are too strong to be a model of point interaction potential for dimensions higher than one and they must be smoothed out by means of the coupling constant renormalization. Unfortunately, by doing this, we seem to loose the information about the strengths of deltas. Even though the coupling constants go down to zero, we should be able to determine the strengths of the interactions. Therefore it is necessary to introduce new parameters μ_i, which will eventually take the place of strengths of point potentials. So from now on $g_i^{-1} \equiv g_i^{-1}(\Lambda, \mu_i)$. Intuitively the most natural and simple choice of $g_i^{-1}(\Lambda, \mu_i)$ is

$$g_i^{-1}(\Lambda, \mu_i) = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{1}{\frac{\hbar^2}{2m}\, k^2 + \mu_i^2}, \tag{2.12}$$

which is just the divergent integral with ν replaced by μ_i. One can easily see that this choice removes the divergence for $D = 2, 3$, but unfortunately not for higher dimensions. Moreover it is impossible to remove the divergence with other choices of $g_i^{-1}(\Lambda, \mu_i)$. To clearly see this, let us consider the divergent integral more carefully. If we rescale the integrand by doing the change of variable $u^2 = \frac{\hbar^2}{2m}\, k^2$, we get

$$\begin{aligned} \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{1}{\frac{\hbar^2}{2m}\, k^2 + \nu^2} &= C_D \int_0^\Lambda \frac{k^{D-1}}{\frac{\hbar^2}{2m}\, k^2 + \nu^2} \, dk \\ &= C_D \left(\frac{\hbar^2}{2m}\right)^{-D/2} \int_0^{\sqrt{\frac{\hbar^2}{2m}}\,\Lambda} \frac{u^{D-1}}{u^2 + \nu^2} \, du, \end{aligned} \tag{2.13}$$

where C_D is a constant depending on the dimension. The integrand, by division algorithm, can be written as

$$\frac{u^{D-1}}{u^2+\nu^2} = \begin{cases} u^{D-3} - u^{D-5}\nu^2 + u^{D-7}\nu^4 - \dots \\ \qquad +(-1)^{\frac{D+1}{2}}\nu^{D-3} \quad -(-1)^{\frac{D+1}{2}}\dfrac{\nu^{D-1}}{u^2+\nu^2} & \text{if } D \text{ is odd} \\ \\ u^{D-3} - u^{D-5}\nu^2 + u^{D-7}\nu^4 - \dots \\ \qquad +(-1)^{\frac{D}{2}}u\nu^{D-4} \quad -(-1)^{\frac{D}{2}}\dfrac{u\,\nu^{D-2}}{u^2+\nu^2} & \text{if } D \text{ is even} \end{cases} \tag{2.14}$$

$$= \begin{cases} \displaystyle\sum_{n=1}^{\frac{D-1}{2}}(-1)^{n+1}\nu^{2n-2}u^{D-(2n+1)} - (-1)^{\frac{D+1}{2}}\frac{\nu^{D-1}}{u^2+\nu^2} & \text{if } D \text{ is odd} \\ \\ \displaystyle\sum_{n=1}^{\frac{D-2}{2}}(-1)^{n+1}\nu^{2n-2}u^{D-(2n+1)} - (-1)^{\frac{D}{2}}\frac{u\,\nu^{D-2}}{u^2+\nu^2} & \text{if } D \text{ is even.} \end{cases} \tag{2.15}$$

Hence the asymptotic expansion of the integral, in which we neglect the terms which remain finite as $\Lambda \to \infty$, is just

$$\int_{|k|<\Lambda}\frac{d^Dk}{(2\pi)^D}\,\frac{1}{\frac{\hbar^2}{2m}k^2+\nu^2} \approx \begin{cases} \displaystyle\sum_{n=1}^{\frac{D-1}{2}}\frac{(-1)^{n+1}}{D-2n}\left(\frac{\hbar^2}{2m}\right)^{\frac{D-2n}{2}}\nu^{2n-2}\Lambda^{D-2n} & \text{if } D \text{ is odd} \\ \\ \displaystyle\sum_{n=1}^{\frac{D-2}{2}}\frac{(-1)^{n+1}}{D-2n}\left(\frac{\hbar^2}{2m}\right)^{\frac{D-2n}{2}}\nu^{2n-2}\Lambda^{D-2n} \\ \qquad -(-1)^{\frac{D}{2}}\nu^{D-2}\ln\left(\sqrt{\frac{\hbar^2}{2m}}\,\Lambda\right) & \text{if } D \text{ is even.} \end{cases} \tag{2.16}$$

Consider also the asymptotic expansion of $g_i^{-1}(\Lambda,\mu_i)$. In order to get rid off the divergence it should be exactly equal to the asymptotic expansion of the above integral. So both of them can only be a function of Λ and they should not contain any ν or μ_i term. But this is not the case when $D > 3$. *So the coupling constant renormalization works only for $D = 2, 3$ for our problem.*

Now, let us extend one of our definitions and define a new matrix μ to simplify the notation:

$$\mu_{ij} = \begin{cases} \mu_i & \text{if } i = j \\ \\ \sqrt{\dfrac{\hbar^2}{2md_{ij}^2}} & \text{if } i \neq j. \end{cases} \tag{2.17}$$

Finally the renormalized $\Phi(\nu)$ becomes

$$\Phi_{ij}(\nu) = \frac{1}{2\pi\left(\frac{\hbar^2}{2m}\right)} \begin{cases} \ln\left(\dfrac{\nu}{\mu_{ij}}\right) & \text{if } i = j \\ \\ -K_0\left(\dfrac{\nu}{\mu_{ij}}\right) & \text{if } i \neq j \end{cases} \tag{2.18}$$

for $D = 2$ and

$$\Phi_{ij}(\nu) = \frac{\nu}{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}} \begin{cases} 1 - \dfrac{\mu_{ij}}{\nu} & \text{if } i = j \\ \\ -\dfrac{e^{-\nu/\mu_{ij}}}{\nu/\mu_{ij}} & \text{if } i \neq j \end{cases} \tag{2.19}$$

for $D = 3$.

So we have the linear equation $\Phi(\nu)A = 0$, but obviously we must have $\det \Phi(\nu) = 0$ not to get all A_i's equal to zero. Therefore this determinant equation gives us *the energy equation*. In general, it is not easy to give a general solution, since this is a *nonlinear equation* for ν, so one should use numerical methods to find energy eigenvalues for a given configuration.

There is an obvious and important question that appears with this new formalism. What is the meaning of the new parameter μ_i? To understand this let us consider a single center case. We have only the diagonal element times A equal to zero as the matrix equation. We do not want A to be zero hence the matrix element must be zero. This shows that for both two and three dimensions $-\mu_i^2$ is just the bound state energy of the single center located at a_i. *So we measure the strength of the interaction with the bound state energy in this new formalism.* This is a kind of dimensional transmutation.

There was a freedom in choosing $g_i^{-1}(\mu_i, \Lambda)$. What if we choose it as a different function of μ_i and Λ? Is the renormalization still possible? What is the meaning of the parameter μ_i in this new picture? First of all we showed that both divergent term must have exactly the same divergence which should be independent of ν and μ_i's. So ν and μ_i dependence come from the finite terms which we have neglected in the asymptotic expansion of both divergent terms. So after the regularization, we may have an expression of the type $f(\mu_i) - h(\nu)$ as the diagonal term of $\Phi(\nu)$, where h should be a one-to-one function. So in the general picture $-[h^{-1}(f(\mu_i))]^2$ is just the bound state energy of the single delta located at the point a_i.

We may also find the normalization for bound state wave function, but we have to be careful. We must find the normalized wave function in terms of Λ and then take the limit $\Lambda \to \infty$ not to get a vanishing wave function. So the normalization constant is

$$\begin{aligned}
|C(\Lambda)|^{-2} &= \sum_{i,j=1}^{N} g_i(\Lambda)\, g_j(\Lambda)\, A_i(\Lambda)\, A_j(\Lambda) \times \\
&\qquad \int d^D x \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 + \nu^2} \int_{|q|<\Lambda} \frac{d^D q}{(2\pi)^D} \frac{e^{iq.(x-a_j)}}{\frac{\hbar^2}{2m} q^2 + \nu^2} \\
&= \sum_{i,j=1}^{N} g_i(\Lambda)\, g_j(\Lambda)\, A_i(\Lambda)\, A_j(\Lambda) \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(a_j-a_i)}}{\left(\frac{\hbar^2}{2m} k^2 + \nu^2\right)^2} \\
&= \frac{1}{2\nu} \sum_{i,j=1}^{N} g_i(\Lambda)\, g_j(\Lambda)\, A_i(\Lambda) \frac{\partial \Phi(\Lambda, \nu)}{\partial \nu} A_j(\Lambda).
\end{aligned} \tag{2.20}$$

The properly normalized wave function of nth state is therefore

$$\psi_n(x) = \sqrt{2\nu_n} \left\{ \sum_{k,l=1}^{N} A_k(\nu_n) \left. \frac{\partial \Phi_{kl}(\nu)}{\partial \nu} \right|_{\nu=\nu_n} A_l(\nu_n) \right\}^{-\frac{1}{2}} \times \\ \sum_{i=1}^{N} A_i(\nu_n) \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 + \nu_n^2}, \tag{2.21}$$

where ν_n is the nth root of the energy equation $\det \Phi(\nu) = 0$. Notice that $A_k(\nu)$'s normalization is unimportant since if we scale them by some number, the result does not change.

2.2. Resolvent

Our renormalized problem is defined by a limiting process over the coupling constants and this limit eventually makes the coupling constants zero. In this situation, it is not easy to give an explicit expression for the Hamiltonian. We need something else in order to understand the system and find out the properties of it. The resolvent operator is a good candidate for this.

The resolvent operator or the Green's operator is the inverse operator of the Hamiltonian subtracted by a complex number, i.e. $R(z) = (H - z\mathbf{1})^{-1}$, where $z \in \mathbb{C}$. This is an analytic family of bounded operators defined over an open set in the complex plane. It has poles around the point spectrum of the Hamiltonian and a branch cut along the continuous spectrum, and every properties of the system can be derived in terms of it.

The integral kernel $R(x, y|z)$ of the resolvent operator $R(z)$ can be defined as the solution to the inhomogeneous equation

$$\left\{ -\frac{\hbar^2}{2m} \nabla^2 - \sum_{i=1}^{N} g_i \delta^D(x - a_i) - z \right\} \psi(x) = \chi(x), \tag{2.22}$$

where $\psi(x)$ is in the specific form

$$\psi(x) = \int d^D y \, R(x, y|z)\chi(y). \tag{2.23}$$

Or if we write it in Fourier space we get the following:

$$\tilde{\psi}(k) = \int \frac{d^D q}{(2\pi)^D} \, \tilde{R}(k, q|z)\tilde{\chi}(q). \tag{2.24}$$

If we transform equation (2.22) to the Fourier space, we find

$$\tilde{\psi}(k) = \frac{\tilde{\chi}(k)}{\frac{\hbar^2}{2m} k^2 - z} + \sum_{i=1}^{N} g_i \; A_i \; \frac{e^{-ik.a_i}}{\frac{\hbar^2}{2m} k^2 - z}, \tag{2.25}$$

where again $A_i = \psi(a_i)$. If we insert (2.25) into the definition of A_i's we again encounter divergent terms. Therefore we apply the same renormalization procedure, and obtain

$$A_i = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{\tilde{\chi}(k) \, e^{ik.a_i}}{\frac{\hbar^2}{2m} k^2 - z} + \sum_{j=1}^{N} g_j(\Lambda, \mu_j) A_j \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{e^{ik.(a_i - a_j)}}{\frac{\hbar^2}{2m} k^2 - z}. \tag{2.26}$$

If we divide both sides by $g_i(\Lambda, \mu_i)$, put previously found $g_i(\Lambda, \mu_i)^{-1}$, and cancel similar terms, we get

$$\left[\Phi(\Lambda, \sqrt{-z}) A\right]_i = g_i(\Lambda)^{-1} B_i(\Lambda), \tag{2.27}$$

where B_i's are defined as

$$B_i(\Lambda) = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} \, \frac{\tilde{\chi}(k) \, e^{ik.a_i}}{\frac{\hbar^2}{2m} k^2 - z}. \tag{2.28}$$

We can solve the equation (2.27) for A in terms of Φ^{-1}, and put back the result into (2.25) and then take the limit $\Lambda \to \infty$, at the end we get

$$\tilde{\psi}(k) = \frac{\tilde{\chi}(k)}{\frac{\hbar^2}{2m} k^2 - z} + \sum_{i,j=1}^{N} \Phi^{-1}_{ij}(\sqrt{-z}) \int \frac{d^D q}{(2\pi)^D} \, \frac{\tilde{\chi}(q) \, e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 - z} \, \frac{e^{-ik.a_i}}{\frac{\hbar^2}{2m} k^2 - z}. \tag{2.29}$$

Therefore $\psi(x)$ is:

$$\begin{aligned}
\psi(x) &= \int \frac{d^D k}{(2\pi)^D} \frac{\tilde{\chi}(k)\, e^{ik.x}}{\frac{\hbar^2}{2m} k^2 - z} \\
&+ \sum_{i,j=1}^{N} \Phi^{-1}_{ij}(\sqrt{-z}) \int \frac{d^D q}{(2\pi)^D} \frac{\tilde{\chi}(q)\, e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 - z} \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 - z} \qquad (2.30) \\
&= \int d^D y\, \chi(y) \Bigg\{ \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-y)}}{\frac{\hbar^2}{2m} k^2 - z} \\
&+ \sum_{i,j=1}^{N} \Phi^{-1}_{ij}(\sqrt{-z}) \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq.(a_j-y)}}{\frac{\hbar^2}{2m} q^2 - z} \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 - z} \Bigg\}. \qquad (2.31)
\end{aligned}$$

And finally the resolvent can be found as

$$\begin{aligned}
R(x,y|z) &= \left[\int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-y)}}{\frac{\hbar^2}{2m} k^2 - z} \right] + \\
& \sum_{i,j=1}^{N} \left[\int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(x-a_i)}}{\frac{\hbar^2}{2m} k^2 - z} \right] \Phi^{-1}_{ij}(\sqrt{-z}) \left[\int \frac{d^D k}{(2\pi)^D} \frac{e^{ik.(a_j-y)}}{\frac{\hbar^2}{2m} k^2 - z} \right] \qquad (2.32)
\end{aligned}$$

or in Fourier space, it is:

$$\tilde{R}(p,q|z) = \left[\frac{(2\pi)^D \delta^D(p-q)}{\frac{\hbar^2}{2m} p^2 - z} \right] + \sum_{i,j=1}^{N} \left[\frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m} p^2 - z} \right] \Phi^{-1}_{ij}(\sqrt{-z}) \left[\frac{e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 - z} \right]. \qquad (2.33)$$

Note that the terms inside parentheses are related to the free resolvents. We may express the resolvent of the full Hamiltonian as

$$R(x,y|z) = R_0(x,y|z) + \sum_{i,j=1}^{N} R_0(x,a_i|z)\, \Phi^{-1}_{ij}(\sqrt{-z})\, R_0(a_j,y|z), \qquad (2.34)$$

or in Fourier space as

$$\begin{aligned}
\tilde{R}(p,q|z) &= \tilde{R}_0(p,q|z) \\
&+ \iint \frac{d^D p'}{(2\pi)^D} \frac{d^D q'}{(2\pi)^D}\, \tilde{R}_0(p,p'|z) \left[\sum_{i,j=1}^{N} e^{-ip'.a_i}\, \Phi^{-1}_{ij}(\sqrt{-z})\, e^{iq'.a_j} \right] \tilde{R}_0(q',q|z). \qquad (2.35)
\end{aligned}$$

Our renormalized problem is defined through its resolvent -or its Green function in the physics terminology- not by its Hamiltonian. But the knowledge of the resolvent is enough to determine all the physical information about the system as we will see.

Let us make a digression on the meaning of these types of formulas for the resolvent. Let us assume that we have a Hamiltonian given as $H = H_0 - \sum_{i=1}^{N} g_i |f_i\rangle\langle f_i|$, where $|f_i\rangle$ represents an element of the Hilbert space. We can work out the formula for the resolvent of H in terms of H_0. Let us define the kets $|\psi\rangle$ and $|\chi\rangle$ via $(H-z)|\psi\rangle = |\chi\rangle$. So if we explicitly write the Hamiltonian, we get

$$\left[H_0 - z - \sum_{j=1}^{N} g_j |f_j\rangle\langle f_j| \right] |\psi\rangle = |\chi\rangle, \tag{2.36}$$

$$(H_0 - z)\,|\psi\rangle = |\chi\rangle + \sum_{j=1}^{N} g_j |f_j\rangle\langle f_j|\psi\rangle, \tag{2.37}$$

$$|\psi\rangle = (H_0 - z)^{-1}\,|\chi\rangle + \sum_{j=1}^{N} g_j\,(H_0 - z)^{-1}\,|f_j\rangle\langle f_j|\psi\rangle, \tag{2.38}$$

assuming that z is not in the spectrum of the H_0. Take the f_i component and we have

$$\langle f_i|\psi\rangle = \langle f_i|\,(H_0 - z)^{-1}\,|\chi\rangle + \sum_{j=1}^{N} g_j \langle f_i|\,(H_0 - z)^{-1}\,|f_j\rangle\langle f_j|\psi\rangle. \tag{2.39}$$

If we group ψ terms on one side, we find

$$\begin{aligned} \left\{ g_i^{-1} - \langle f_i|\,(H_0 - z)^{-1}\,|f_i\rangle \right\} \langle f_i|\psi\rangle \quad - \quad & \sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{g_j}{g_i}\, \langle f_i|\,(H_0 - z)^{-1}\,|f_j\rangle\langle f_j|\psi\rangle = \\ & g_i^{-1}\,\langle f_i|\,(H_0 - z)^{-1}\,|\chi\rangle. \end{aligned} \tag{2.40}$$

Let us define a new matrix $A_{ij}(z)$ as

$$A_{ij}(z) = \begin{cases} g_i^{-1} - \langle f_i|\,(H_0 - z)^{-1}\,|f_i\rangle & \text{if } i = j \\ -\dfrac{g_j}{g_i}\,\langle f_i|\,(H_0 - z)^{-1}\,|f_j\rangle & \text{if } i \neq j. \end{cases} \tag{2.41}$$

So in terms of this matrix $A_{ij}(z)$, we may write the previous expression as

$$\sum_{j=1}^{N} A_{ij}(z)\langle f_j|\psi\rangle = g_i^{-1} \langle f_i| (H_0 - z)^{-1} |\chi\rangle. \tag{2.42}$$

Then let us take the inverse of $A_{ij}(z)$, so we have

$$\langle f_i|\psi\rangle = \sum_{j=1}^{N} A_{ij}^{-1}(z)\langle f_j| (H_0 - z)^{-1} |\chi\rangle. \tag{2.43}$$

Now, let us put this back into (2.39) and we get

$$|\psi\rangle = (H_0 - z)^{-1} |\chi\rangle + (H_0 - z)^{-1} \sum_{i,j=1}^{N} |f_i\rangle A_{ij}^{-1}(z)\langle f_j| (H_0 - z)^{-1} |\chi\rangle, \tag{2.44}$$

from which we can easily conclude that

$$(H - z)^{-1} = (H_0 - z)^{-1} + (H_0 - z)^{-1} \left\{ \sum_{i,j=1}^{N} |f_i\rangle A_{ij}^{-1}(z)\langle f_j| \right\} (H_0 - z)^{-1}. \tag{2.45}$$

Notice that the term in parenthesis is essentially the operator Φ. Indeed, we can approximate our delta functions by properly chosen set of functions $f_i(\Lambda)$, which depend on a parameter Λ, and as we now know we should allow g_i to depend on Λ as to cancel out a possible divergence (since these formulae are quite general, for a different choice of H_0 one may not encounter a divergence). We can even choose Λ as a discrete set and as $\Lambda \to \infty$, $f_i(\Lambda)$ sequence converges to the delta functions in an appropriate topology. This incidentally gives another way to prove the existence of the Hamiltonian. Such formulae were extensively discussed in problems associated with self-adjoint extensions of operators, notably by Krein and his school. (For a beautiful discussion of these and related problems in connection to physics see the book by Albeverio and Kurasov [16] and also Albeverio *et al* [7]). Therefore, our problem is also a kind of self-adjoint extension of the free Hamiltonian. It is defined through regulating (or controlling) the behavior of the wave function in the vicinity of these interaction points. This point of view will be useful when we discuss point interactions on manifolds.

2.3. Analysis of the Bound States

Now, we should answer a natural question, how do we know that the ground state energy of this system has a lower bound, i.e. did we really cure the original problem? The answer to this question is in the operator Φ. Essentially all the information about bound states can be found from this matrix. There are no poles of the resolvent coming from the other terms which are basically the resolvents of the free Hamiltonian. The lower bound for the ground state energy is easy to find by considering the inverse of Φ. For, let us define the matrix A as the diagonal part of Φ, and $-B$ as the remaining part. A is obviously invertible for $\nu \neq \mu_i$, hence if we choose $\nu > \mu$, where $\mu = \max_i \mu_i$, we can safely write Φ as

$$\Phi = A - B = A(1 - A^{-1}B). \tag{2.46}$$

Therefore Φ is invertible if and only if $(1 - A^{-1}B)$ has an inverse and $(1 - A^{-1}B)$ has an inverse if the norm $\|A^{-1}B\| < 1$, in which case we write the inverse of Φ as

$$\Phi^{-1} = (1 - A^{-1}B)^{-1}A^{-1}. \tag{2.47}$$

From the definition of the norm we have

$$\|A^{-1}B\| = \sup_{|f|=1} \left|(A^{-1}B)f\right|, \tag{2.48}$$

where f is a unit vector in N dimensional space. We may explicitly write the norm $|(A^{-1}B)f|$ in terms of the components of f and give an upper bound for this norm, by keeping in mind that $|f_i| \leq 1$, A is diagonal and monotonic increasing (decreasing) with respect to ν (μ_{ij}), B is also monotonic increasing (decreasing) with respect to ν

(μ_{ij}) and A and $-B$ are both positive for positive ν. So we have

$$
\begin{aligned}
\|A^{-1}B\| &= \sup_{|f|=1} \left|(A^{-1}B)f\right| \\
&= \sup_{|f|=1} \left\{ \sum_{i=1}^{N} [\Phi_{ii}]^{-2} \left[\sum_{\substack{j=1\\ j\neq i}}^{N} (-\Phi_{ij}) f_j \right]^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{i=1}^{N} [\Phi_{ii}]^{-2} \left[\sum_{\substack{j=1\\ j\neq i}}^{N} (-\Phi_{ij}) \right]^2 \right\}^{1/2} \\
&\leq \sum_{i=1}^{N} [\Phi_{ii}]^{-1} \sum_{\substack{j=1\\ j\neq i}}^{N} (-\Phi_{ij}) \\
&\leq N(N-1) \frac{\max_{\substack{i,j=1\\ j\neq i}}^{N} (-\Phi_{ij})}{\min_{i=1}^{N} \Phi_{ii}}.
\end{aligned}
\tag{2.49}
$$

Hence, we finally get

$$
\|A^{-1}B\| \leq N(N-1) \begin{cases} \ln^{-1}\left(\dfrac{\nu}{\mu}\right) \, K_0\left(\dfrac{\nu}{\mu_d}\right) & \text{for } D=2 \\[2ex] \left(1-\dfrac{\mu}{\nu}\right)^{-1} \dfrac{e^{-\nu/\mu_d}}{\nu/\mu_d} & \text{for } D=3, \end{cases}
\tag{2.50}
$$

where $\mu_d = \max_{i\neq j} \mu_{ij} = \sqrt{\frac{\hbar^2}{2md^2}}$ and $d = \min_{i\neq j} d_{ij}$, similarly $\mu = \max_i \mu_{ii}$. Remember that K_0 has the following integral representation:

$$
K_0(x) = \int_0^\infty e^{-x\cosh t} \, dt,
\tag{2.51}
$$

and we have $t < \cosh t \ \ \forall t \in \mathbb{R}$, so $e^{-x\cosh t} < e^{-xt} \ \ \forall t \in \mathbb{R}$, therefore

$$
K_0(x) < \frac{1}{x}.
\tag{2.52}
$$

Also $e^{-t} \leq 1 \;\; \forall t \geq 0$, so $e^{-t}/t \leq 1/t \;\; \forall t \geq 0$. Therefore we have

$$\|A^{-1}B\| \leq N(N-1)\frac{\mu_d}{\nu} \begin{cases} \ln^{-1}\left(\dfrac{\nu}{\mu}\right) & \text{for } D = 2 \\ \\ \left(1 - \dfrac{\mu}{\nu}\right)^{-1} & \text{for } D = 3. \end{cases} \tag{2.53}$$

We want this to be less than one, so we must have:

$$N(N-1)\frac{\mu_d}{\mu} \leq \begin{cases} \dfrac{\nu}{\mu}\ln\left(\dfrac{\nu}{\mu}\right) & \text{for } D = 2 \\ \\ \dfrac{\nu}{\mu} - 1 & \text{for } D = 3. \end{cases} \tag{2.54}$$

This inequality equation indicates that there exist a $\nu^* > \mu$ (so our previous choice $\nu > \mu$ was appropriate) for a given d and N for which $\Phi(\nu)$ is invertible when $\nu > \nu^*$. So the ground state energy cannot be less than $-\nu^{*2}$. Hence we can give the bound as

$$E_{gr} \geq -N^2(N-1)^2\, W^{-2}\left(N(N-1)\frac{\mu_d}{\mu}\right)\mu_d^2 \tag{2.55}$$

for $D = 2$, where $W(z)$ is the product log function (Lambert W-Function) and

$$E_{gr} \geq -\left(N(N-1)\,\mu_d + \mu\right)^2 \tag{2.56}$$

for $D = 3$.

Now consider two centers with the same bound state energies (or strengths), we have the following energy equations:

$$\begin{aligned} \ln\frac{\nu}{\mu} &= \pm K_0\left(\frac{\nu}{\mu_d}\right) \quad && \text{for } D = 2, \\ 1 - \frac{\mu}{\nu} &= \pm\frac{e^{-\nu/\mu_d}}{\nu/\mu_d} && \text{for } D = 3. \end{aligned} \tag{2.57}$$

Let us look for small d solutions. We can expand the Bessel function and the exponential for small d, so we get the energy levels as:

$$E = -\nu^2 \simeq \begin{cases} -\dfrac{e^\gamma}{2}\,\mu\,\mu_d & \text{for } D = 2 \\ \\ -\left(\dfrac{\mu_d + \mu}{2}\right)^2 & \text{for } D = 3, \end{cases} \tag{2.58}$$

where γ is the Euler's gamma constant. One common feature of these ground states energies is that they both change with $1/d$. So as $d \to 0$, the ground state energies go down to $-\infty$. This is because of the renormalization that we introduced. The functional structure of g_i's makes addition of two delta a divergent process.

A better bound for ground state energy is produced if we use the following method. We previously found that we can write the inverse of Φ as

$$\begin{aligned} \Phi^{-1} &= (1 - A^{-1}B)^{-1}A^{-1} \\ &= \big(1 + A^{-1}B + (A^{-1}B)^2 + \dots\big)A^{-1}. \end{aligned} \tag{2.59}$$

We again choose $\nu > \mu$, so that A has an inverse. A^{-1}_{ij} and B_{ij} are bounded from below with zero for this range. We may also give an upper bound for these matrix elements. For two dimensions we have

$$\begin{aligned} A^{-1}_{ij} &= 2\pi\left(\frac{\hbar^2}{2m}\right)\ln^{-1}\frac{\nu}{\mu_i}\,\delta_{ij} \;\leq\; 2\pi\left(\frac{\hbar^2}{2m}\right)\ln^{-1}\frac{\nu}{\mu} \\ (A^{-1}B)_{ij} &= \sum_{l=1}^{N}\ln^{-1}\frac{\nu}{\mu_i}\,\delta_{il}\,B_{lj} \;\leq\; \ln^{-1}\frac{\nu}{\mu}\,K_0\left(\frac{\nu}{\mu_d}\right). \end{aligned} \tag{2.60}$$

Similarly, for the mth power we get

$$\big(A^{-1}B\big)^m_{ij} \leq N^{m-1}\left[\ln^{-1}\frac{\nu}{\mu}\,K_0\left(\frac{\nu}{\mu_d}\right)\right]^m. \tag{2.61}$$

Since this is a geometric series, we have

$$
\begin{aligned}
0 \leq \Phi_{ij}^{-1}(\nu) \ &\leq \ 2\pi \left(\frac{\hbar^2}{2m}\right) \ln^{-1}\frac{\nu}{\mu}\,\frac{1}{N}\,\sum_{m=0}^{\infty}\left[N \ln^{-1}\frac{\nu}{\mu}\,K_0\left(\frac{\nu}{\mu_d}\right)\right]^m \\
&\leq \ 2\pi \left(\frac{\hbar^2}{2m}\right) \frac{N^{-1}\ln^{-1}\frac{\nu}{\mu}}{1 - N\ln^{-1}\frac{\nu}{\mu}\,K_0\left(\frac{\nu}{\mu_d}\right)}, \qquad (2.62)
\end{aligned}
$$

and the radius of convergence is simply

$$
N \ln^{-1}\frac{\nu}{\mu}\,K_0\left(\frac{\nu}{\mu_d}\right) < 1. \qquad (2.63)
$$

If we use the bound (2.52) we see that the above inequality is always true if we have

$$
N \ln^{-1}\frac{\nu}{\mu}\,\frac{\mu_d}{\nu} < 1 \qquad (2.64)
$$

from which we can conclude that

$$
E_{gr} \geq -N^2\,W^{-2}\left(\frac{N\mu_d}{\mu}\right)\mu_d^2. \qquad (2.65)
$$

Obviously this has a better N dependence than our previous bound.

Now let us look at the three dimensional case. We have

$$
\begin{aligned}
A_{ij}^{-1} \ &= \ \frac{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}}{\nu}\left(1-\frac{\mu_i}{\nu}\right)^{-1}\delta_{ij} \ \leq \ \frac{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}}{\nu}\left(1-\frac{\mu}{\nu}\right)^{-1} \qquad (2.66) \\
(A^{-1}B)_{ij} \ &= \ \sum_{l=1}^{N}\frac{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}}{\nu}\left(1-\frac{\mu_i}{\nu}\right)^{-1}\delta_{il}\,B_{lj} \\
&\leq \ \left(1-\frac{\mu}{\nu}\right)^{-1}\frac{e^{-\nu/\mu_d}}{\nu/\mu_d}. \qquad (2.67)
\end{aligned}
$$

Similarly, for the mth power we get

$$
(A^{-1}B)_{ij}^{m} \leq N^{m-1}\left[\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{e^{-\nu/\mu_d}}{\nu/\mu_d}\right]^m. \qquad (2.68)
$$

Since this is a geometric series, we have

$$
\begin{aligned}
0 \leq \Phi_{ij}^{-1}(\nu) &\leq \frac{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}}{\nu}\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{1}{N}\sum_{m=0}^{\infty}\left[N\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{e^{-\nu/\mu_d}}{\nu/\mu_d}\right]^m \\
&\leq \frac{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}}{\nu}\frac{N^{-1}\left(1-\frac{\mu}{\nu}\right)^{-1}}{1-N\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{e^{-\nu/\mu_d}}{\nu/\mu_d}}
\end{aligned}
\tag{2.69}
$$

and the radius of convergence is simply

$$
N\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{e^{-\nu/\mu_d}}{\nu/\mu_d} < 1. \tag{2.70}
$$

If we use the bound (2.52) we see that the above inequality is always true if we have

$$
N\left(1-\frac{\mu}{\nu}\right)^{-1}\frac{1}{\nu/\mu_d} < 1 \tag{2.71}
$$

from which we can conclude that

$$
E_{gr} \geq -\left(N\mu_d+\mu\right)^2. \tag{2.72}
$$

Obviously this one also has a better N dependence than our previous bound.

Next we will show that there are always bound states, this is physically the desired result since we are modelling a set of attractive centers. To see this we will use an inductive argument. Recall that the solution is found by setting $\det\Phi(\nu) = 0$, and then solving for ν values. We may also write $\det\Phi(\nu) = \prod_i \lambda_i(\nu)$ where $\lambda_i(\nu)$'s are the eigenvalues of $\Phi(\nu)$. So $\det\Phi(\nu) = 0$ is the same as $\lambda_k(\nu) = 0$. To show that we always have a bound state we use a kind of comparison theorem: we know that the single attractor case has always a bound state solution, then we assume that the ground state energy of the N centers case has a ground state energy which is negative. To proceed further we make a slight digression, we show that as functions of the parameter

ν the eigenvalues are increasing functions. To show this we write $\lambda_k(\nu)$ as

$$\begin{aligned}\lambda_k(\nu) &= \langle x_k|\Phi|x_k\rangle \\ &= \sum_{i,j=1}^{N}(x_k)_i\Phi_{ij}(x_k)_j,\end{aligned} \tag{2.73}$$

where $x_k \equiv x_k(\nu)$'s are eigenvectors corresponding to the eigenvalue $\lambda_k(\nu)$. Let us assume for simplicity that the eigenvalues are non-degenerate - these type of matrices are dense. Hence we have

$$\begin{aligned}\frac{\partial\lambda_k(\nu)}{\partial\nu} &= \sum_{i,j=1}^{N}(x_k)_i\frac{\partial\Phi_{ij}}{\partial\nu}(x_k)_j \\ &= \sum_{i,j=1}^{N}(x_k)_i\int\frac{d^Dk}{(2\pi)^D}\,\frac{e^{ik.(a_j-a_i)}}{\left(\frac{\hbar^2}{2m}k^2+\nu^2\right)^2}\,(x_k)_j \\ &= \sum_{i=1}^{N}\int\frac{d^Dk}{(2\pi)^D}\,\frac{\left|e^{ik.(a_j-a_i)}(x_k)_i\right|^2}{\left(\frac{\hbar^2}{2m}k^2+\nu^2\right)^2} \\ &\geq 0,\end{aligned} \tag{2.74}$$

which is clearly true since the integrant is positive. So the eigenvalues are monotonic increasing functions of ν. Let us now recall Cauchy's Interlacing Theorem: if we have symmetric matrices Φ and Φ', where Φ' is constructed by deleting one row and the corresponding column of Φ, then the eigenvalues of these matrices are in the order below:

$$\lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \lambda_2' \leq ... \leq \lambda_{N-1} \leq \lambda_{N-1}' \leq \lambda_N. \tag{2.75}$$

Let therefore the eigenvalues of $\Phi_N(\nu)$, the N center case, be denoted as λ_k'. The interlacing theorem says that the $N+1$ case has its eigenvalues interlaced by λ_k' sequence and thus the smallest eigenvalue of $\Phi_{N+1}(\nu)$, $\lambda_1(\nu)$, is dominated by $\lambda_1(\nu)'$. If we know that the zero of $\lambda_1(\nu)'$ corresponds to ν_N, then the zero root of the equation $\lambda_1(\nu) = 0$ should be found at $\nu_{N+1} > \nu_N$ since the eigenvalues are monotonic increasing functions of ν so that we need to increase ν to increase the value of $\lambda_1(\nu)$. This means that the ground state energy of the $N+1$ center case is smaller than the N center case as expected (recall that $E = -\nu^2$). Moreover we find a similar ordering among the (possible) zeros of the other eigenvalues. This is the analog of Sturm comparison theorem of eigenvalues for our problem. This result is physically not surprising, but the point of our discussion is that once we have a finite formulation of the problem, what one expects from the regularized formalism is in fact still true and we can rigorously prove all our intuitive guesses. Incidentally the induction hypothesis now allows us to conclude that the ground state energy is always negative. (A general proof for the degenerate case can also be found, but it takes more work.)

The well-known result about the degeneracy of eigenvalues under symmetry can be shown here as well, that is if there is a finite group acting on the locations of the delta attractors, the eigenvalues will be split according to this symmetry group. This is easy to see since the the commutation of the group generators with the Hamiltonian is equivalent to the commutation of the resolvent with the generators.

2.4. Existence of the Hamiltonian

Since we are not able to write a formula for the Hamiltonian after the renormalization, it is not so obvious that our formula for the resolvent actually defines a Hamiltonian. Since we can find the renormalized resolvent as limits of resolvents for finite operators, one can use this to prove the existence. Yet we will use a different approach to test our ideas. The basic idea is that, if we are given this formula, is it possible to prove the existence without knowing that it comes from such limits. Although it is not possible to write the Hamiltonian explicitly, we can still prove that it exists. For this, we must first show that the resolvent operator satisfies the resolvent equation, which is

$$(z_1 - z_2)R(z_1)R(z_2) = R(z_1) - R(z_2) \tag{2.76}$$

or in terms of the resolvent kernel

$$(z_1 - z_2) \int d^D w \; R(x, w|z_1) R(w, y|z_2) = R(x, y|z_1) - R(x, y|z_2). \tag{2.77}$$

Remember that our resolvent was in the form

$$R(x, y|z) = R_0(x, y|z) + \sum_{i,j=1}^{N} R_0(x, a_i|z) \, \Phi^{-1}_{ij}(\sqrt{-z}) \, R_0(a_j, y|z). \tag{2.78}$$

Free resolvent obviously satisfies the resolvent equation, so we have

$$(z_1 - z_2) \int d^D w \; R_0(x, w|z_1) R_0(w, y|z_2) = R_0(x, y|z_1) - R_0(x, y|z_2). \tag{2.79}$$

We can easily see this by considering the integral representation of R_0, i.e:

$$\begin{aligned}
(z_1 - z_2) \int d^D w \int \frac{d^D k}{(2\pi)^D} \; \frac{e^{ik.(x-w)}}{\frac{\hbar^2}{2m} k^2 - z_1} \int \frac{d^D k}{(2\pi)^D} \; \frac{e^{ik.(w-y)}}{\frac{\hbar^2}{2m} k^2 - z_2} \\
= (z_1 - z_2) \int \frac{d^D k}{(2\pi)^D} \; \frac{e^{ik.(x-y)}}{(\frac{\hbar^2}{2m} k^2 - z_1)(\frac{\hbar^2}{2m} k^2 - z_2)} \\
= \int \frac{d^D k}{(2\pi)^D} \; \frac{e^{ik.(x-y)}}{\frac{\hbar^2}{2m} k^2 - z_1} - \int \frac{d^D k}{(2\pi)^D} \; \frac{e^{ik.(x-y)}}{\frac{\hbar^2}{2m} k^2 - z_2}.
\end{aligned} \tag{2.80}$$

We also have the identity

$$(z_1 - z_2) \int d^D w \; R_0(a_i, w|z_1) R_0(w, a_j|z_2) = \Phi_{ij}(\sqrt{-z_1}) - \Phi_{ij}(\sqrt{-z_2}). \tag{2.81}$$

We may also use this result for the full resolvent. Now If we insert the full resolvent into the resolvent equation, we get

$$
\begin{aligned}
&(z_2 - z_1) \int d^D w \Big\{ R_0(x, w|z_1) R_0(w, y|z_2) \\
&\qquad + \sum_{l,m=1}^{N} R_0(x, w|z_1) R_0(w, a_l|z_2)\, \Phi^{-1}_{lm}(\sqrt{-z_2})\, R_0(a_m, y|z_2) \\
&\qquad + \sum_{i,j=1}^{N} R_0(x, a_i|z_1)\, \Phi^{-1}_{ij}(\sqrt{-z_1})\, R_0(a_j, w|z_1) R_0(w, y|z_2) \\
&\qquad + \sum_{i,j,l,m=1}^{N} R_0(x, a_i|z_1)\, \Phi^{-1}_{ij}(\sqrt{-z_1})\, R_0(a_j, w|z_1) R_0(w, a_l|z_2) \times \\
&\qquad \Phi^{-1}_{lm}(\sqrt{-z_2}) R_0(a_m, y|z_2) \Big\} \\
= &\quad \sum_{l,m=1}^{N} [R_0(x, a_l|z_1) - R_0(x, a_l|z_2)]\, \Phi^{-1}_{lm}(\sqrt{-z_2})\, R_0(a_m, y|z_2) \\
&+ \sum_{i,j=1}^{N} R_0(x, a_i|z_1)\, \Phi^{-1}_{ij}(\sqrt{-z_1})\, [R_0(a_j, y|z_1) - R_0(a_j, y|z_2)] \\
&+ \sum_{i,j,l,m=1}^{N} R_0(x, a_i|z_1)\, \Phi^{-1}_{ij}(\sqrt{-z_1})\, [\Phi_{jl}(\sqrt{-z_1}) - \Phi_{jl}(\sqrt{-z_2})] \times \\
&\Phi^{-1}_{lm}(\sqrt{-z_2}) R_0(a_m, y|z_2) \\
= &\quad R_0(x, y|z_1) + \sum_{i,j=1}^{N} R_0(x, a_i|z_1)\, \Phi^{-1}_{ij}(\sqrt{-z_1})\, R_0(a_j, y|z_1) \\
&\qquad - R_0(x, y|z_2) - \sum_{i,j=1}^{N} R_0(x, a_i|z_2)\, \Phi^{-1}_{ij}(\sqrt{-z_2})\, R_0(a_j, y|z_2). \qquad (2.82)
\end{aligned}
$$

Now, we quote the following theorem taken from [17]: Let $R(z)$ be a family of bounded operators defined on an open set Δ, which extends to infinity in some direction on the complex plane. Let us further assume that $R(z)$ satisfies the resolvent equation on Δ. If there is a sequence $\lambda_n \in \Delta$, such that $|\lambda_n| \to \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} -\lambda_n R(\lambda_n) f = f \quad \text{for all normalized } f \in \mathcal{H}, \qquad (2.83)$$

where $\mathcal{H}$ is the Hilbert space under consideration, then $R(z)$ is the resolvent family of a unique densely defined closed operator.
Now, to show this we must verify that

$$\lim_{n\to\infty} \| \lambda_n R(\lambda_n) f + f \| \to 0 \quad \text{as } n \to \infty. \tag{2.84}$$

We can choose λ_n as $-n^2 E_0$ because we know that the resolvent is nonsingular for $(-\infty, -E_0)$, where E_0 is the absolute value of the bound that we have found for the ground state energy E_{gr}. So we must have

$$\lim_{n\to\infty} \Big\| \, n^2 E_0 \, R(-n^2 E_0) f - f \Big\| \to 0 \quad \text{as } n \to \infty. \tag{2.85}$$

Which implies, if we work on the phase space:

$$\begin{aligned}
&\lim_{n\to\infty} \left\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D} \, \tilde{R}(p, q| - n^2 E_0) f(q) - f(p) \right\| \\
&= \lim_{n\to\infty} \Bigg\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D} \Bigg\{ \tilde{R}_0(p, q| - n^2 E_0) + \iint \frac{d^D p'}{(2\pi)^D} \frac{d^D q'}{(2\pi)^D} \, \tilde{R}_0(p, p'| - n^2 E_0) \times \\
&\left[\sum_{i,j=1}^{N} e^{-ip'.a_i} \, \Phi_{ij}^{-1}(n\sqrt{E_0}) \, e^{iq'.a_j} \right] \, \tilde{R}_0(q', q| - n^2 E_0) \Bigg\} f(q) - f(p) \Bigg\| \\
&\leq \lim_{n\to\infty} \left\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D} \, \tilde{R}_0(p, q| - n^2 E_0) f(q) - f(p) \right\| \\
&+ \lim_{n\to\infty} \Bigg\| n^2 E_0 \iiint \frac{d^D q}{(2\pi)^D} \frac{d^D p'}{(2\pi)^D} \frac{d^D q'}{(2\pi)^D} \, \tilde{R}_0(p, p'| - n^2 E_0) \times \\
&\qquad \left[\sum_{i,j=1}^{N} e^{-ip'.a_i} \, \Phi_{ij}^{-1}(n\sqrt{E_0}) \, e^{iq'.a_j} \right] \, \tilde{R}_0(q', q| - n^2 E_0) f(q) \Bigg\| .
\end{aligned} \tag{2.86}$$

The first terms comes from the resolvent of the free Hamiltonian, so it obviously goes to zero in the $n \to \infty$ limit, but let us also prove this:

$$\begin{aligned}
&\lim_{n\to\infty} \left\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D}\ \tilde{R}_0(p,q|-n^2E_0) f(q) - f(p) \right\| \\
&= \lim_{n\to\infty} \left\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D} \frac{(2\pi)^D \delta^D(p-q)}{\frac{\hbar^2}{2m} p^2 + n^2 E_0} f(q) - f(p) \right\| \\
&= \lim_{n\to\infty} \left\| \frac{\frac{\hbar^2}{2m}\, p^2\, f(p)}{\frac{\hbar^2}{2m}\, p^2 + n^2 E_0} \right\| \\
&= \lim_{n\to\infty} \left[\int \frac{d^D p}{(2\pi)^D} \left| \frac{\frac{\hbar^2}{2m}\, p^2\, f(p)}{\frac{\hbar^2}{2m}\, p^2 + n^2 E_0} \right|^2 \right]^{1/2} \\
&= \lim_{n\to\infty} \left[\int_{\frac{\hbar^2}{2m} p^2 \le n E_0} \frac{d^D p}{(2\pi)^D} \left| \frac{f(p)}{1 + \frac{n^2 E_0}{\frac{\hbar^2}{2m} p^2}} \right|^2 \right]^{1/2} + \lim_{n\to\infty} \left[\int_{\frac{\hbar^2}{2m} p^2 > n E_0} \frac{d^D p}{(2\pi)^D} \left| \frac{f(p)}{1 + \frac{n^2 E_0}{\frac{\hbar^2}{2m} p^2}} \right|^2 \right]^{1/2} \\
&\le \lim_{n\to\infty} \frac{1}{1+n} \left[\int \frac{d^D p}{(2\pi)^D} |f(p)|^2 \right]^{1/2} + \lim_{n\to\infty} \left[\int_{\frac{\hbar^2}{2m} p^2 > n E_0} \frac{d^D p}{(2\pi)^D} |f(p)|^2 \right]^{1/2} = 0.
\end{aligned} \tag{2.87}$$

This is evident because f is square integrable, hence it must decay to 0 as $p \to \infty$, so does $\int_{\frac{\hbar^2}{2m} p^2 > n E_0} \frac{d^D p}{(2\pi)^D} |f(p)|^2$. Therefore the term coming from the free resolvent goes to 0 as expected. Now let us look at the remaining terms:

$$\begin{aligned}
&\lim_{n\to\infty} \Bigg\| n^2 E_0 \iiint \frac{d^D q}{(2\pi)^D} \frac{d^D p'}{(2\pi)^D} \frac{d^D q'}{(2\pi)^D}\ \tilde{R}_0(p,p'|-n^2E_0) \times \\
&\qquad \left[\sum_{i,j=1}^{N} e^{-ip'.a_i}\ \Phi^{-1}_{ij}(n\sqrt{E_0})\ e^{iq'.a_j} \right] \tilde{R}_0(q',q|-n^2E_0) f(q) \Bigg\| \\
&= \lim_{n\to\infty} \left\| n^2 E_0 \int \frac{d^D q}{(2\pi)^D} \sum_{i,j=1}^{N} \left[\frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m} p^2 + n^2 E_0} \right] \Phi^{-1}_{ij}(n\sqrt{E_0}) \left[\frac{e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 + n^2 E_0} \right] f(q) \right\| \\
&\le \lim_{n\to\infty} \sum_{i,j=1}^{N} n^2 E_0 \left\| \int \frac{d^D q}{(2\pi)^D} \left[\frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m} p^2 + n^2 E_0} \right] \Phi^{-1}_{ij}(n\sqrt{E_0}) \left[\frac{e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 + n^2 E_0} \right] f(q) \right\| \\
&\le \lim_{n\to\infty} \sum_{i,j=1}^{N} n^2 E_0 \left| \Phi^{-1}_{ij}(n\sqrt{E_0}) \right| \times \left| \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq.a_j}}{\frac{\hbar^2}{2m} q^2 + n^2 E_0} f(q) \right| \times \left\| \frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m} p^2 + n^2 E_0} \right\|.
\end{aligned} \tag{2.88}$$

Now, let us look at the last term. We have

$$
\begin{aligned}
\left\| \frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m}\, p^2 + n^2 E_0} \right\| &= \left[\int \frac{d^D p}{(2\pi)^D} \left| \frac{e^{-ip.a_i}}{\frac{\hbar^2}{2m}\, p^2 + n^2 E_0} \right|^2 \right]^{1/2} \\
&= \left[\int \frac{d^D p}{(2\pi)^D} \left| \frac{1}{\frac{\hbar^2}{2m}\, p^2 + n^2 E_0} \right|^2 \right]^{1/2} \\
&\leq \frac{1}{n^2 E_0} \left[\int \frac{d^D q}{(2\pi)^D} \left| \frac{1}{\frac{\frac{\hbar^2}{2m}}{n^2 E_0}\, q^2 + 1} \right|^2 \right]^{1/2} \\
&\leq \frac{1}{n^2 E_0} \left[n^D \left(\frac{E_0}{\frac{\hbar^2}{2m}} \right)^{D/2} \right]^{1/2} \left[\int \frac{d^D r}{(2\pi)^D} \left| \frac{1}{r^2 + 1} \right|^2 \right]^{1/2} \\
&\leq C_D\; E_0^{D/4-1} \left(\frac{\hbar^2}{2m} \right)^{-D/4} \frac{1}{n^{2-D/2}},
\end{aligned}
\tag{2.89}
$$

where C_D is a real number depending only on the dimension D of the space. For the other term we need to find a more careful bound:

$$
\left| \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq.a_j}}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} f(q) \right| \leq \int \frac{d^D q}{(2\pi)^D} \frac{|f(q)|}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0}. \tag{2.90}
$$

If we split this integral into two parts, as we did for the free resolvent, and apply the Cauchy-Schwarz inequality to both parts we get

$$
\begin{aligned}
\int \frac{d^D q}{(2\pi)^D} \frac{|f(q)|}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} \leq & \int_{\frac{\hbar^2}{2m} q^2 \leq n E_0} \frac{d^D q}{(2\pi)^D} \frac{|f(q)|}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} \\
& + \int_{\frac{\hbar^2}{2m} q^2 > n E_0} \frac{d^D q}{(2\pi)^D} \frac{|f(q)|}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} \\
\leq & \left[\int_{\frac{\hbar^2}{2m} q^2 \leq n E_0} \frac{d^D q}{(2\pi)^D} \; |f(q)|^2 \right]^{1/2} \left[\int_{\frac{\hbar^2}{2m} q^2 \leq n E_0} \frac{d^D q}{(2\pi)^D} \left| \frac{1}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} \right|^2 \right]^{1/2} \\
& + \left[\int_{\frac{\hbar^2}{2m} q^2 > n E_0} \frac{d^D q}{(2\pi)^D} \; |f(q)|^2 \right]^{1/2} \left[\int_{\frac{\hbar^2}{2m} q^2 > n E_0} \frac{d^D q}{(2\pi)^D} \left| \frac{1}{\frac{\hbar^2}{2m}\, q^2 + n^2 E_0} \right|^2 \right]^{1/2} .
\end{aligned}
\tag{2.91}
$$

Let us consider each term separately. We have

$$\left[\int_{\frac{\hbar^2}{2m}q^2\le nE_0}\frac{d^Dq}{(2\pi)^D}\ |f(q)|^2\right]^{1/2}\le\left[\int\frac{d^Dq}{(2\pi)^D}\ |f(q)|^2\right]^{1/2}\le 1, \tag{2.92}$$

and

$$\begin{aligned}\left[\int_{\frac{\hbar^2}{2m}q^2\le nE_0}\frac{d^Dq}{(2\pi)^D}\left|\frac{1}{\frac{\hbar^2}{2m}q^2+n^2E_0}\right|^2\right]^{1/2}&\\ =E_0^{D/4-1}\left(\frac{\hbar^2}{2m}\right)^{-D/4}\frac{1}{n^{2-D/2}}&\left[\int_{r^2\le\frac{1}{n}}\frac{d^Dr}{(2\pi)^D}\left|\frac{1}{r^2+1}\right|^2\right]^{1/2}\\ =E_0^{D/4-1}\left(\frac{\hbar^2}{2m}\right)^{-D/4}\frac{1}{n^{2-D/2}}&\ T_n,\end{aligned} \tag{2.93}$$

where $T_n\to 0$ as $n\to\infty$.

$$\left[\int_{\frac{\hbar^2}{2m}q^2>nE_0}\frac{d^Dq}{(2\pi)^D}\ |f(q)|^2\right]^{1/2}=T_n', \tag{2.94}$$

where $T_n'\to 0$ again as $n\to\infty$.

$$\begin{aligned}\left[\int_{\frac{\hbar^2}{2m}q^2>nE_0}\frac{d^Dq}{(2\pi)^D}\left|\frac{1}{\frac{\hbar^2}{2m}q^2+n^2E_0}\right|^2\right]^{1/2}&\\ =E_0^{D/4-1}\left(\frac{\hbar^2}{2m}\right)^{-D/4}\frac{1}{n^{2-D/2}}&\left[\int_{r^2>\frac{1}{n}}\frac{d^Dr}{(2\pi)^D}\left|\frac{1}{r^2+1}\right|^2\right]^{1/2}\\ \le C_D\ E_0^{D/4-1}\left(\frac{\hbar^2}{2m}\right)^{-D/4}&\frac{1}{n^{2-D/2}}.\end{aligned} \tag{2.95}$$

At the end we can conclude that

$$\int\frac{d^Dq}{(2\pi)^D}\ \frac{|f(q)|}{\frac{\hbar^2}{2m}q^2+n^2E_0}\le E_0^{D/4-1}\left(\frac{\hbar^2}{2m}\right)^{-D/4}\frac{1}{n^{2-D/2}}\ C_n, \tag{2.96}$$

where $C_n\to 0$ again as $n\to\infty$.

So if we combine these, we have:

$$\lim_{n\to\infty} \left\| n^2 E_0 \iiint \frac{d^Dq}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}\frac{d^Dq'}{(2\pi)^D}\, \tilde{R}_0(p,p'|-n^2E_0) \times \right.$$
$$\left. \left[\sum_{i,j=1}^{N} e^{-ip'.a_i}\ \Phi^{-1}_{ij}(n\sqrt{E_0})\ e^{iq'.a_j}\right] \tilde{R}_0(q',q|-n^2E_0)f(q) \right\| \tag{2.97}$$
$$\leq \lim_{n\to\infty} C_D\, E_0^{D/2-1}\left(\frac{\hbar^2}{2m}\right)^{-D/2} n^{D-2}\, C_n \sum_{i,j=1}^{N}\left|\Phi_{ij}(n\sqrt{E_0})\right|.$$

Now let us look at Φ^{-1} term. We have previously found bounds for the matrix elements of Φ^{-1}. So we have

$$\left|\Phi^{-1}_{ij}\left(n\sqrt{E_0}\right)\right| \leq \frac{1}{N}\begin{cases} \dfrac{(2\pi)\left(\frac{\hbar^2}{2m}\right)}{\ln\frac{n\sqrt{E_0}}{\mu}}\left[\dfrac{1}{1-N\ln^{-1}\frac{n\sqrt{E_0}}{\mu}\ K_0\left(\frac{n\sqrt{E_0}}{\mu_d}\right)}\right] & \text{for } D=2 \\[2ex] \dfrac{(4\pi)\left(\frac{\hbar^2}{2m}\right)^{3/2}}{n\sqrt{E_0}}\left[\dfrac{\left(1-\frac{\mu}{n\sqrt{E_0}}\right)^{-1}}{1-N\left(1-\frac{\mu}{n\sqrt{E_0}}\right)^{-1}\frac{e^{-n\sqrt{E_0}/\mu_d}}{n\sqrt{E_0}/\mu_d}}\right] & \text{for } D=3. \end{cases} \tag{2.98}$$

This shows that

$$\sum_{i,j=1}^{N}\left|\Phi_{ij}(n\sqrt{E_0})\right| \approx N\begin{cases} \dfrac{(2\pi)\left(\frac{\hbar^2}{2m}\right)}{\ln\frac{n\sqrt{E_0}}{\mu}} & \text{for } D=2 \\[2ex] \dfrac{(4\pi)\left(\frac{\hbar^2}{2m}\right)^{3/2}}{n\sqrt{E_0}} & \text{for } D=3, \end{cases} \tag{2.99}$$

as $n\to\infty$. So by remembering also $C_n\to 0$ as $n\to\infty$ one can conclude that

$$\lim_{n\to\infty}\left\|\, n^2E_0\, R(-n^2E_0)f - f\right\| \to 0 \quad \text{as } n\to\infty. \tag{2.100}$$

Thus R is the resolvent family of a unique densely defined, closed, self adjoint Hamiltonian.

2.5. Scattering Properties

Once we have an explicit formula for the resolvent we can compute the scattering amplitudes for this problem. Essentially, the knowledge of the operator $T(z)$ in standard scattering formalism [18], where z is a complex parameter, is enough to find the scattering matrix.

Let us find the cross sections. The main equation for the operator $T(z)$ can be written as

$$R(z) = R_0(z) + R_0(z)T(z)R_0(z), \tag{2.101}$$

from which we can read off the matrix elements $\langle p\,|T(z)|q\rangle$ directly as

$$\langle p\,|T(z)|q\rangle = \sum_{i,j=1}^{N} e^{-ip.a_i}\ \Phi_{ij}^{-1}\left(\sqrt{-z}\right)\ e^{iq.a_j}. \tag{2.102}$$

The on-shell T matrix is defined as $t(E_p) = \lim_{\epsilon\to 0^+} T(E_p + i\epsilon)$. Since $\Phi^{-1}(\nu)$ is an algebraic function of its arguments and we are away from the poles of it (assuming that there are no eigenvalues embedded into the continuum) we may change the order of the limit and inverse operations. So it is useful to define a new matrix $\Omega(E_p)$ as

$$\Omega_{ij}(E_p) = \frac{1}{2\pi\left(\frac{\hbar^2}{2m}\right)} \begin{cases} \ln\dfrac{\sqrt{E_p}}{\mu_{ii}} - i\dfrac{\pi}{2} & \text{if } i = j \\[2ex] \dfrac{\pi}{2}\left[Y_0\left(\dfrac{\sqrt{E_p}}{\mu_{ij}}\right) - i\,J_0\left(\dfrac{\sqrt{E_p}}{\mu_{ij}}\right)\right] & \text{if } i \neq j, \end{cases} \tag{2.103}$$

for two dimensions and

$$\Omega_{ij}(E_p) = \frac{1}{4\pi\left(\frac{\hbar^2}{2m}\right)^{3/2}} \begin{cases} -\mu_{ii} - i\sqrt{E_p} & \text{if } i = j \\ \\ \mu_{ij}\left[\cos\left(\dfrac{\sqrt{E_p}}{\mu_{ij}}\right) - i\sin\left(\dfrac{\sqrt{E_p}}{\mu_{ij}}\right)\right] & \text{if } i \neq j, \end{cases} \tag{2.104}$$

for three dimensions, which is simply formed from $\lim_{\epsilon\to 0^+}\Phi\left(\sqrt{-(E_p+i\epsilon)}\right)$ where $E_p = \hbar^2 p^2/(2m)$. Hence we have

$$\langle p\,|t(E_p)|q\rangle = \sum_{i,j=1}^{N} e^{-ip.a_i}\;\Omega_{ij}^{-1}(E_p)\;e^{iq.a_j} \quad [E_p = E_q]. \tag{2.105}$$

The S matrix can be found in terms of the t matrix as

$$\begin{aligned} \langle p\,|S|q\rangle &= \delta^D(p-q) - 2\pi i\delta(E_p - E_q)\langle p\,|t(E_p)|q\rangle \\ &= \delta^D(p-q) + 2\pi i\,\frac{m}{p}\;\delta(|p|-|q|)\sum_{i,j=1}^{N} e^{-ip.a_i}\;\Omega_{ij}^{-1}(E_p)\;e^{iq.a_j}. \end{aligned} \tag{2.106}$$

The scattering amplitude $f(p\to q)$ is

$$\begin{aligned} f(p\to q) &= -(2\pi)^2\frac{m}{\hbar^2}\langle p\,|t(E_p)|q\rangle \\ &= -(2\pi)^2\frac{m}{\hbar^2}\sum_{i,j=1}^{N} e^{-ip.a_i}\;\Omega_{ij}^{-1}(E_p)\;e^{iq.a_j} \quad [E_p = E_q]. \end{aligned} \tag{2.107}$$

We may express differential and total cross sections in terms of the scattering amplitude. In two dimensions, one can see that the differential cross section formula and optical theorem change slightly and take the following form:

$$\frac{d\sigma}{d\alpha} = \begin{cases} \frac{1}{(2\pi)^5 p}\,|f(p \to q)|^2 & \text{for } D=2 \\ |f(p \to q)|^2 & \text{for } D=3, \end{cases} \tag{2.108}$$

$$\sigma = \begin{cases} \frac{2}{(2\pi)^2 p}\,\Im f(p \to p) & \text{for } D=2 \\ \frac{p}{4\pi}\,\Im f(p \to p) & \text{for } D=3. \end{cases} \tag{2.109}$$

So for a single center, the total cross section is

$$\sigma(E) = \begin{cases} \frac{4\pi^2\hbar}{\sqrt{2mE}}\left[\frac{1}{\pi^2+\ln^2\frac{E}{\mu^2}}\right] & \text{for } D=2 \\ \\ 32\pi^4\frac{\hbar^2}{2m}\frac{1}{E+\mu^2} & \text{for } D=3, \end{cases} \tag{2.110}$$

which is consistent with [19]. Similarly for two delta centers of same strengths (or bound state energies) with a distance d apart we have

$$\sigma(E) = \begin{cases} \frac{8\hbar}{\sqrt{2mE}}\frac{J_0\left(\frac{\sqrt{E}}{\mu_d}\right)+1}{\left[J_0\left(\frac{\sqrt{E}}{\mu_d}\right)+1\right]^2+\left[Y_0\left(\frac{\sqrt{E}}{\mu_d}\right)+\frac{1}{\pi}\log\left(\frac{E}{\mu^2}\right)\right]^2} & \text{for } D=2 \\ \\ 64\pi^4 d^2\,\frac{1+\frac{\mu_d}{\sqrt{E}}\sin\frac{\sqrt{E}}{\mu_d}}{1+\frac{\mu^2}{\mu_d^2}+\frac{E}{\mu_d^2}+\frac{2}{\mu_d}\left[\sqrt{E}\sin\frac{\sqrt{E}}{\mu_d}-\mu\sin\frac{\sqrt{E}}{\mu_d}\right]} & \text{for } D=3, \end{cases} \tag{2.111}$$

where p (incoming wave vector) is orthogonal to the line joining two centers. Note that as $d \to \infty$, the total cross section for two centers becomes twice the total cross section for a single center as expected and as $d \to 0$, the total cross section for two centers vanishes.

3. POINT INTERACTIONS ON A MANIFOLD

3.1. On a General Manifold

In this section we will treat the point interaction problem on a general Riemannian manifold. Suppose M is a D dimensional Riemannian manifold, we define the kinetic energy by the Laplace-Beltrami operator which is the generalization of the Laplace operator for curved space. Laplace-Beltrami operator has the coordinate expression

$$\nabla_g^2 = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{D} \partial_i (g^{ij} \sqrt{g}\, \partial_j), \tag{3.1}$$

where g_{ij} is the metric tensor and $g = \det(g_{ij})$. (Note that we defined the operator with a minus sign to simplify the calculations.)

From the spectral theorem [20] we know that the eigenvalue problem $\nabla_g^2 \phi = \lambda \phi$ on a *compact connected Riemannian manifold* M without boundary has a complete orthonormal system of C^∞ eigenfunctions $\phi_0, \phi_1, \ldots$ in $L^2(M)$ with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and $\lambda_n \to \infty$ as $n \to \infty$. This means that we can expand any function on M in terms of these eigenfunctions. Hence we have these expansions:

$$\psi(x) = \sum_{l \geq 0} C_l\, \phi_l(x) \tag{3.2}$$

$$\delta^D(x - a_i) = \sqrt{g} \sum_{l \geq 0} \phi_l(x) \phi_l^*(a_i), \tag{3.3}$$

where C_l's are expansion coefficients and δ^D is the D dimensional delta function for which the volume integral is normalized. Let us put these into the Schrodinger equation

$$\left\{ \frac{\hbar^2}{2m} \nabla_g^2 - \sum_{i=1}^{N} g_i\, \delta^D(x - a_i) \right\} \psi = -\nu^2 \psi, \tag{3.4}$$

and we get

$$\sum_{l\geq 0}\left\{\frac{\hbar^2}{2m}\,\lambda_l\,C_l-\sqrt{g}\sum_{i=1}^{N}A_i\,g_i\,\phi_l^*(a_i)+\nu^2 C_l\right\}\phi_l(x)=0, \tag{3.5}$$

where $A_i=\psi(a_i)$. ϕ_l's form a complete orthonormal system so we have

$$C_l=\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}\,\sqrt{g}\sum_{i=1}^{N}A_i\,g_i\,\phi_l^*(a_i). \tag{3.6}$$

Let us put this back into the definition of A_i and we get

$$A_i=\sum_{j=1}^{N}A_j\,g_j\,\sqrt{g}\sum_{l\geq 0}\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}\,\phi_l(a_i)\phi_l^*(a_j), \tag{3.7}$$

and if we group the A_i terms we have

$$\left[g_i^{-1}-\sqrt{g}\sum_{l\geq 0}\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}|\phi_l(a_i)|^2\right]A_i-\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\sqrt{g}\,\frac{g_j}{g_i}\,\sum_{l\geq 0}\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}\,\phi_l(a_i)\phi_l^*(a_j)\right]A_j=0. \tag{3.8}$$

This naturally defines the Φ matrix as

$$\Phi_{ij}(\nu)=\begin{cases} g_i^{-1}-\sqrt{g}\sum_{l\geq 0}\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}|\phi_l(a_i)|^2 & \text{if } i=j \\ \\ -\sqrt{g}\,\frac{g_j}{g_i}\,\sum_{l\geq 0}\frac{1}{\frac{\hbar^2}{2m}\,\lambda_l+\nu^2}\,\phi_l(a_i)\phi_l^*(a_j) & \text{if } i\neq j, \end{cases} \tag{3.9}$$

from which we can extract all the useful information such as bound state energies and scattering properties.

Now let us consider the problem of point interactions on a sphere as an example and let us repeat the above calculations for this special case for clarity. We have N

center point interaction potential located at $(\theta_i, \phi_i)_{i=1}^N$ on a sphere. So the Schrodinger equation can be written as

$$\left\{ \frac{\hbar^2}{2m} \nabla_s^2 - \sum_{i=1}^{N} g_i \, \delta_R^2(\theta - \theta_i, \phi - \phi_i) \right\} \psi = -\nu^2 \psi, \tag{3.10}$$

where ∇_s^2 is Laplace-Beltrami operator on the sphere and $\delta_R^2(\theta - \theta_i, \phi - \phi_i)$ is the two dimensional delta function on the sphere centered at (θ_i, ϕ_i) for which the surface integral is normalized. Spherical Harmonics form a complete orthonormal basis for S^2, i.e. they are eigenfunctions of the Laplace-Beltrami operator with eigenvalues $l(l+1)/R^2$, where R is the radius of the sphere. So we can expand any function in terms of them. Hence we have these expansions:

$$\psi(\theta, \phi) = \sum_{l \geq 0} \sum_{m=-l}^{l} C_{lm} Y_l^m(\theta, \phi) \tag{3.11}$$

$$\delta_R^2(\theta - \theta_i, \phi - \phi_i) = \frac{1}{R^2} \sum_{l \geq 0} \sum_{m=-l}^{l} Y_l^m(\theta, \phi) Y_l^{m*}(\theta_i, \phi_i), \tag{3.12}$$

where C_{lm}'s are expansion coefficients. If we put these back into the Schrodinger equation we get

$$\sum_{l \geq 0} \sum_{m=-l}^{l} \left\{ \frac{\hbar^2}{2mR^2} \, l(l+1) C_{lm} + \nu^2 C_{lm} - \frac{1}{R^2} \sum_{i=1}^{N} g_i \, A_i \, Y_l^{m*}(\theta_i, \phi_i) \right\} Y_l^m(\theta, \phi) = 0, \tag{3.13}$$

where $A_i = \psi(\theta_i, \phi_i)$. C_{lm} can be found from this expression as

$$C_{lm} = \frac{1}{\frac{\hbar^2}{2mR^2} \, l(l+1) + \nu^2} \, \frac{1}{R^2} \sum_{i=1}^{N} g_i \, A_i \, Y_l^{m*}(\theta_i, \phi_i). \tag{3.14}$$

If we put this back into the definition of A_i we get

$$A_i = \sum_{j=1}^{N} g_j \, A_j \frac{1}{R^2} \sum_{l \geq 0} \sum_{m=-l}^{l} \frac{Y_l^m(\theta_i, \phi_i) Y_l^{m*}(\theta_j, \phi_j)}{\frac{\hbar^2}{2mR^2} \, l(l+1) + \nu^2}. \tag{3.15}$$

Let us group i and j terms together, we get

$$A_i \left\{ g_i^{-1} - \frac{1}{R^2} \sum_{l\geq 0} \sum_{m=-l}^{l} \frac{|Y_l^m(\theta_j,\phi_j)|^2}{\frac{\hbar^2}{2mR^2}\, l(l+1)+\nu^2} \right\} - \sum_{\substack{j=1 \\ i\neq j}}^{N} \frac{g_j}{g_i} A_j \frac{1}{R^2} \sum_{l\geq 0} \sum_{m=-l}^{l} \frac{Y_l^m(\theta_i,\phi_i) Y_l^{m*}(\theta_j,\phi_j)}{\frac{\hbar^2}{2mR^2}\, l(l+1)+\nu^2} = 0. \quad (3.16)$$

We have the following identity:

$$\begin{aligned} \sum_{m=-l}^{l} Y_l^m(\theta_i,\phi_i) Y_l^{m*}(\theta_j,\phi_j) &= \frac{2l+1}{4\pi} P_l\left(\cos\theta_i \cos\theta_j + \cos(\phi_j-\phi_i)\sin\theta_i \sin\theta_j\right) \\ &= \frac{2l+1}{4\pi} P_l \left(1 - \frac{{d_{ij}}^2}{2}\right), \end{aligned} \quad (3.17)$$

where $d_{ij} = |\hat{r}_i - \hat{r}_j|$, distance between interaction centers normalized with the radius R of the sphere. We can easily see that the coefficient of the A_i term is infinite. So again we define g_i's as a function of some parameter Λ, do the summation up to Λ and choose $g_i^{-1}(\Lambda)$'s such that we get a finite difference when we take the $\Lambda \to \infty$ limit. The simplest choice is obviously

$$g_i^{-1}(\Lambda) = \frac{1}{4\pi R^2} \sum_{l=0}^{\Lambda} \frac{2l+1}{\frac{\hbar^2}{2mR^2}\, l(l+1)+\mu_i^2}, \quad (3.18)$$

which clearly removes the divergence. We again put an extra parameter μ into the definition of $g_i^{-1}(\Lambda)$ to keep track of the strength of the interaction. We will see in a moment what it means. Hence we get the matrix equation $\Phi(\nu)A = 0$, where we may define $\Phi(\nu)$ as

$$\Phi(\nu) = \frac{1}{4\pi R^2 \mu_R^2} \begin{cases} \phi\left(\frac{\nu}{\mu_R}\right) - \phi\left(\frac{\mu_i}{\mu_R}\right) & \text{if } i=j \\ \\ -\sum_{l\geq 0} \frac{2l+1}{l(l+1)+\frac{\nu^2}{\mu_R^2}}\, P_l\left(1-\frac{d_{ij}^2}{2}\right) & \text{if } i\neq j. \end{cases} \quad (3.19)$$

The function ϕ is defined as

$$\phi(x) = -\frac{1}{x^2} + H_{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}} + H_{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2}}, \tag{3.20}$$

where H's are Harmonic Numbers. Harmonic Numbers are commonly defined on integers as $H_n = \sum_{k=1}^{n} k^{-1}$ and has the analytical continuation on entire complex plane as $H_z = \psi(z+1) + \gamma$, where $\psi(z) = \frac{\Gamma'[z]}{\Gamma[z]}$ is the digamma function and $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant. Harmonic Numbers have the mirror symmetry $H_{\bar{z}} = \bar{H}_z$, which shows that $\phi(x) \in \mathbb{R}$ for $x > 0$.

Note that we can easily see that the non diagonal terms converge to a finite value by using the following bound for the Legendre polynomials:

$$|P_l(x)| \leq \sqrt{\frac{2}{\pi l(1-x^2)}}. \tag{3.21}$$

Remember that $-1 \leq 1 - \frac{d_{ij}^2}{2} \leq 1$, $d_{ij} \neq 0$ if $i \neq j$ and $\frac{\nu^2}{\gamma^2} > 0$, so we have

$$\begin{aligned}
\left| \sum_{l\geq 0} \frac{2l+1}{l(l+1)+\frac{\nu^2}{\mu_R^2}} \; P_l\left(1-\frac{d_{ij}^2}{2}\right) \right| &\leq \sum_{l\geq 0} \frac{2l+1}{l(l+1)+\frac{\nu^2}{\mu_R^2}} \; \left| P_l\left(1-\frac{d_{ij}^2}{2}\right) \right| \\
&\leq \frac{\mu_R^2}{\nu^2} + \sum_{l\geq 1} \frac{2l+1}{l(l+1)+\frac{\nu^2}{\mu_R^2}} \; \left| P_l\left(1-\frac{d_{ij}^2}{2}\right) \right| \\
&\leq \frac{\mu_R^2}{\nu^2} + \sqrt{\frac{8}{\pi}} \, \frac{1}{d_{ij}\sqrt{4-d_{ij}^2}} \sum_{l\geq 1} \frac{2l+1}{\sqrt{l}\left[l(l+1)+\frac{\nu^2}{\mu_R^2}\right]} \\
&\leq \frac{\mu_R^2}{\nu^2} + \sqrt{\frac{8}{\pi}} \, \frac{1}{d_{ij}\sqrt{4-d_{ij}^2}} \sum_{l\geq 1} \frac{2l+1}{l^{5/2}} \\
&\leq \frac{\mu_R^2}{\nu^2} + \sqrt{\frac{8}{\pi}} \, \frac{2\zeta(3/2)+\zeta(5/2)}{d_{ij}\sqrt{4-d_{ij}^2}} \\
&< \infty,
\end{aligned} \tag{3.22}$$

where ζ is the Riemann Zeta function and $\zeta(3/2) \approx 2.61238$, $\zeta(5/2) \approx 1.34149$.

Now we may look for the meaning of the new parameter μ_i. We again consider a single center case. We have the energy equation $\phi\left(\frac{\nu^2}{\mu_R^2}\right) - \phi\left(\frac{\mu^2}{\mu_R^2}\right) = 0$. One might easily see that ϕ is a monotonic increasing function by considering its properties around the origin and at ∞. Around the origin, Harmonic Numbers are negligible and $\phi(x)$ goes like $-1/x^2$, at greater values $-1/x^2$ decays out and Harmonic Numbers go like logarithm. Hence $\phi\left(\frac{\nu^2}{\mu_R^2}\right) - \phi\left(\frac{\mu^2}{\mu_R^2}\right) = 0$ implies $-\nu^2 = -\mu^2$. Therefore $-\mu^2$ is just the bound state energy of a single center as previously.

3.2. Heat Kernel

Now let us go back to the general result. Remember that we have found Φ as

$$\Phi_{ij}(\nu) = \begin{cases} g_i^{-1} - \sqrt{g} \displaystyle\sum_{l \geq 0} \frac{1}{\frac{\hbar^2}{2m}\lambda_l + \nu^2} |\phi_l(a_i)|^2 & \text{if } i = j \\ \\ -\sqrt{g}\, \dfrac{g_j}{g_i} \displaystyle\sum_{l \geq 0} \frac{1}{\frac{\hbar^2}{2m}\lambda_l + \nu^2}\, \phi_l(a_i)\phi_l^*(a_j) & \text{if } i \neq j. \end{cases} \tag{3.23}$$

We do not know anything about the convergence of these sums. For special cases as our previous example one can find a closed form expression or at least remove the divergence. What happens for the general case? Can we still remove the divergence? We can attack this problem more easily by using the heat kernel method.

Heat kernel $K_t(x, y)$ is the unique fundamental solution to the heat equation $\nabla^2\phi = -\partial_t\phi$. It has the symmetry $K_t(x, y) = K_t(y, x)$ and goes to a delta function when $t \to 0^+$. The reason we want to use the heat kernel is that there are well known asymptotic expansions and bounds for the heat kernel which will be useful to understand the nature of the divergences.

Now remember the Krein's resolvent formula for the Hamiltonian $H = H_0 -$

$\sum_{i=1}^{N} g_i|f_i\rangle\langle f_i|$. We previously showed that

$$(H-z)^{-1} = (H_0-z)^{-1} + (H_0-z)^{-1}\left\{\sum_{i,j=1}^{N} |f_i\rangle A_{ij}^{-1}(z)\langle f_j|\right\}(H_0-z)^{-1}. \tag{3.24}$$

Let us project it to the coordinate components and we have the resolvent kernel as

$$\begin{aligned} R(x,y|z) &= R_0(x,y|z) + \int dx'dy' R_0(x,x'|z)\left\{\sum_{i,j=1}^{N} f_i(x')A_{ij}^{-1}(z)f_j(y')\right\}R_0(y',y|z) \\ &= R_0(x,y|z) + \sum_{i,j=1}^{N}\left[\int dx' R_0(x,x'|z)f_i(x')\right]A_{ij}^{-1}(z)\left[\int dy' R_0(y',y|z)f_j(y')\right]. \end{aligned} \tag{3.25}$$

Now let us choose the functions $f_i(x)$'s as bump functions centered at $x = a_i$ which in the limit will go to delta functions, i.e $f_i(x) \to \delta^D(x-a_i)$. So we have

$$R(x,y|z) = R_0(x,y|z) + \sum_{i,j=1}^{N} R_0(x,a_i|z)A_{ij}^{-1}(z)R_0(a_j,y|z). \tag{3.26}$$

Remember that we defined $A_{ij}(z)$ as

$$A_{ij}(z) = \begin{cases} g_i^{-1} - \langle f_i|\left(H_0-z\right)^{-1}|f_i\rangle & \text{if } i=j \\ -\dfrac{g_j}{g_i}\,\langle f_i|\left(H_0-z\right)^{-1}|f_j\rangle & \text{if } i\neq j. \end{cases} \tag{3.27}$$

The free Hamiltonian is bounded from below so we can write the free resolvent operator as an integral for $Re[z] < 0$, hence we have

$$(H_0-z)^{-1} = \frac{1}{\hbar}\int_0^{\infty} e^{-\frac{t}{\hbar}\ \frac{\hbar^2}{2m}\nabla_g^2 - z}\ dt, \tag{3.28}$$

which we can also extend it analytically to the entire complex plane. So the free resolvent kernel is just

$$\begin{aligned}\langle f_i|\left(H_0-z\right)^{-1}|f_j\rangle &= \frac{1}{\hbar}\langle f_i|\int_0^\infty e^{-\frac{t}{\hbar}\ \frac{\hbar^2}{2m}\nabla_g^2-z}\ dt|f_j\rangle \\ &= \frac{1}{\hbar}\int_0^\infty e^{\frac{zt}{\hbar}}\langle f_i|e^{-\left[\frac{t}{\hbar}\right]\frac{\hbar^2}{2m}\nabla_g^2}|f_j\rangle\, dt. \end{aligned} \tag{3.29}$$

And in the limit $f_i(x) \to \delta^D(x-a_i)$, we will get

$$\langle f_i|\left(H_0-z\right)^{-1}|f_j\rangle \to \int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i,a_j)\,\frac{dt}{\hbar}, \tag{3.30}$$

where $K_t(a_i,a_j)$ is the heat kernel, and the operator $e^{-\left[\frac{t}{\hbar}\right]\ \frac{\hbar^2}{2m}\nabla_g^2}$ is the solution to the heat equation. From the spectral theorem on a compact manifold [21] we know that the heat kernel has the following series expansion:

$$K_t(x,y) = \sum_{n\geq 0} e^{-\frac{\hbar^2}{2m}\lambda_n\left[\frac{t}{\hbar}\right]}\phi_n(x)\phi_n^*(y), \tag{3.31}$$

which converges uniformly on $M\times M$ for each $t>0$. Hence we have

$$\begin{aligned}\langle f_i|\left(H_0-z\right)^{-1}|f_j\rangle &\to \int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i,a_j)\,\frac{dt}{\hbar} \\ &= \sum_{n\geq 0}\phi_n(a_i)\phi_n^*(a_j)\int_0^\infty e^{-\ \frac{\hbar^2}{2m}\lambda_n - z\ \left[\frac{t}{\hbar}\right]}\,\frac{dt}{\hbar} \\ &= \sum_{n\geq 0}\frac{1}{\frac{\hbar^2}{2m}\lambda_n - z}\,\phi_n(a_i)\phi_n^*(a_j), \end{aligned} \tag{3.32}$$

which is the same result that we already obtained. Now heat kernel has the following asymptotic expansion in D dimension

$$K_t(x,y) \sim \frac{e^{-\frac{2md(x,y)^2}{\hbar^2 4\left[\frac{t}{\hbar}\right]}}}{(4\pi\left[\frac{t}{\hbar}\right])^{D/2}}\sum_{n\geq 0} f_n(x,y)\left[\frac{t}{\hbar}\right]^n \quad \text{as } t\to 0^+, \tag{3.33}$$

where the series coefficients $f_n(x,y)$ depend on the metric, and $f_0(x,y)=1$. As $x\to y$, this allows us to estimate the divergence of the eigenvalue sums in 3.32. We expect naively

$$g_i^{-1}(\epsilon) \sim \int_{\epsilon}^{\infty} \frac{e^{-\mu_i^2\left[\frac{t}{\hbar}\right]}}{\left(4\pi\left[\frac{t}{\hbar}\right]\right)^{D/2}}, \tag{3.34}$$

in order to cure the divergence. It is very hard to find an exact expression for the coupling constant without specifying a metric, nevertheless we can still try our ideas on some examples. But before the examples let us prove an important result by using the heat kernel method.

Remember the resolvent equation:

$$R(z) - R(z') = (z - z')R(z) \bullet R(z'), \tag{3.35}$$

where $\bullet$ denotes the operator product. And our resolvent was in the form

$$R(z) = R_0(z) + R_0(z)\Phi^{-1}(z)R_0(z). \tag{3.36}$$

Let us try to find the necessary condition for $R(z)$ to satisfy the resolvent equation. Remember that free resolvent operator $R_0(z)$ obviously satisfies the resolvent equation, therefore

$$R_0(z) - R_0(z') = (z - z')R_0(z) \bullet R(z'). \tag{3.37}$$

So we have

$$R(z) - R(z') = R_0(z) + R_0(z)\Phi^{-1}(z)R_0(z) - R_0(z') - R_0(z')\Phi^{-1}(z')R_0(z') \tag{3.38}$$

$$
\begin{aligned}
&= \Big[R_0(z) - R_0(z')\Big] \\
&\quad + R_0(z)\Phi^{-1}(z)\Big[R_0(z) - R_0(z')\Big] + \Big[R_0(z) - R_0(z')\Big]\Phi^{-1}(z')R_0(z') \\
&\quad + R_0(z)\Phi^{-1}(z)R_0(z') - R_0(z)\Phi^{-1}(z')R_0(z')
\end{aligned}
$$

$$
\begin{aligned}
&= (z - z')R_0(z) \bullet R(z') + (z - z')R_0(z)\Phi^{-1}(z)R_0(z) \bullet R(z') \\
&\quad + (z - z')R_0(z) \bullet R_0(z')\Phi^{-1}(z')R_0(z') \\
&\quad + R_0(z)\left[\Phi^{-1}(z) - \Phi^{-1}(z')\right] R_0(z')
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
&= (z - z')R(z) \bullet R(z') + \Big[R_0(z)\left[\Phi^{-1}(z) - \Phi^{-1}(z')\right] R_0(z') \\
&\qquad - (z - z')R_0(z)\Phi^{-1}(z)R_0(z) \bullet R_0(z')\Phi^{-1}(z')R_0(z')\Big].
\end{aligned}
$$

Hence $R(z)$ satisfies the resolvent equation if the last term is zero, i.e.

$$
R_0(z)\left[\Phi^{-1}(z) - \Phi^{-1}(z')\right] R_0(z') = (z - z')R_0(z)\Phi^{-1}(z)R_0(z) \bullet R_0(z')\Phi^{-1}(z')R_0(z'),
\tag{3.40}
$$

which implies

$$
\begin{aligned}
\Phi^{-1}(z) - \Phi^{-1}(z') &= (z - z')\Phi^{-1}(z)R_0(z) \bullet R_0(z')\Phi^{-1}(z') \\
\Phi(z) - \Phi(z') &= -(z - z')R_0(z) \bullet R_0(z').
\end{aligned}
\tag{3.41}
$$

We can write $\Phi(z)$ using the heat kernel as

$$
\Phi_{ij}(z) = \lim_{\epsilon \to 0}
\begin{cases}
g_i^{-1}(\epsilon, \mu_i) - \displaystyle\int_\epsilon^\infty K_t(a_i, a_i) e^{z\frac{t}{\hbar}} \frac{dt}{\hbar} & \text{if } i = j \\[2ex]
-\dfrac{g_j^{-1}(\epsilon, \mu_j)}{g_i^{-1}(\epsilon, \mu_i)} \displaystyle\int_0^\infty K_t(a_i, a_j) e^{z\frac{t}{\hbar}} \frac{dt}{\hbar} & \text{if } i \neq j,
\end{cases}
\tag{3.42}
$$

hence we have

$$\Phi_{ij}(z) - \Phi_{ij}(z') = \begin{cases} \int_0^\infty K_t(a_i, a_i) \left[e^{z' \frac{t}{\hbar}} - e^{z \frac{t}{\hbar}} \right] \frac{dt}{\hbar} & \text{if } i = j \\ \\ \int_0^\infty K_t(a_i, a_j) \left[e^{z' \frac{t}{\hbar}} - e^{z \frac{t}{\hbar}} \right] \frac{t}{\hbar} & \text{if } i \neq j, \end{cases}$$

$$= \int_0^\infty K_t(a_i, a_j) \left[e^{z' \frac{t}{\hbar}} - e^{z \frac{t}{\hbar}} \right] \frac{t}{\hbar}$$

$$= -(z - z') \left[R_0(z) \bullet R_0(z') \right]_{ij} , \tag{3.43}$$

because $g^{-1}(\epsilon)$ have the same divergences so they cancel out and the remaining parts give the desired result.

3.3. Examples on Hyperbolic Planes H^2 and H^3

Hyperbolic geometry is a non-Euclidean geometry, also called Lobachevsky-Bolyai-Gauss geometry, having constant sectional curvature $-\frac{1}{R^2}$. This geometry satisfies all of Euclid's postulates except the parallel postulate, which says: Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended. For the hyperbolic geometry the "only one straight lines" condition is extended to "many straight line". Minkowski space-time which describes four dimensional mass free universe is an example to an hyperbolic space that we encounter in physics.

3.3.1. Hyperbolic Plane H^3

We define the hyperbolic plane H^3 as

$$H^3 = \{x \in \mathbb{R}^3 | x_3 > 0\}, \tag{3.44}$$

and the hyperbolic distance is given by

$$\cosh \frac{d(x,y)}{R} = 1 + \frac{|x-y|^2}{2\, x_3\, y_3}, \tag{3.45}$$

where R is the scaling parameter which is used to construct the space.
We have an explicit formula [20] for the heat kernel of the three dimensional hyperbolic plane H^3 as

$$K_t(x,y) = \frac{\frac{d(x,y)}{R}\; e^{-\frac{\hbar}{2mR^2}t - \frac{m d(x,y)^2}{2\hbar t}}}{\left(4\pi \left[\frac{\hbar}{2mR^2}\right] t\right)^{3/2} \sinh \frac{d(x,y)}{R}}. \tag{3.46}$$

Hence we have the free resolvent kernel as

$$\begin{aligned}
\langle a_i|\,(H_0 - z)^{-1}\,|a_j\rangle &= \frac{1}{\hbar}\int_0^\infty e^{z\frac{t}{\hbar}}\, \frac{\frac{d_{ij}}{R}\; e^{-\frac{\hbar}{2mR^2}t - \frac{m d_{ij}^2}{2\hbar t}}}{\left(4\pi\left[\frac{\hbar}{2mR^2}\right] t\right)^{3/2} \sinh\frac{d_{ij}}{R}}\, dt \\
&= \frac{\frac{d_{ij}}{R}}{(4\pi)^{3/2}\mu_R^2 \sinh \frac{d_{ij}}{R}} \int_0^\infty u^{-3/2} e^{-\ 1-\frac{z}{\mu_R^2}\ u - \frac{\mu_R^2}{4\mu_{d_{ij}}^2}\frac{1}{u}} \\
&= \left[\frac{1}{4\pi}\, \frac{\frac{d_{ij}}{R}}{\sinh \frac{d_{ij}}{R}} e^{-\sqrt{\frac{\mu_R^2}{\mu_{d_{ij}}^2} - \frac{z}{\mu_{d_{ij}}^2}}}\right] \frac{\mu_{d_{ij}}}{\mu_R^3},
\end{aligned} \tag{3.47}$$

where $d_{ij} = d(a_i, a_j)$, $\mu_R^2 = \frac{\hbar^2}{2mR^2}$, $\mu_{d_{ij}}^2 = \frac{\hbar^2}{2md_{ij}^2}$. Obviously for $i = j$ we have a divergence. But this time the divergence of the integral comes from $t = 0$. Hence we renormalize the divergent term by introducing a lower cut-off ϵ to the integral and by defining the coupling constant as a function of this cut-off, i.e

$$A_{ii}(z) = \lim_{\epsilon \to 0^+} \left[g_i^{-1}(\epsilon) - \frac{1}{(4\pi)^{3/2}\mu_R^2} \int_\epsilon^\infty u^{-3/2} e^{-\ 1-\frac{z}{\mu_R^2}\ u} \right]. \tag{3.48}$$

The natural choice for $g_i^{-1}(\epsilon)$ is simply

$$g_i^{-1}(\epsilon) = \frac{1}{(4\pi)^{3/2}\mu_R^2} \int_\epsilon^\infty u^{-3/2} e^{-\ 1+\frac{\mu_i^2}{\mu_R^2}\ u}, \tag{3.49}$$

where again we introduced a new parameter μ_i to keep track of the strength of point interactions. So in the $\epsilon \to 0$ limit we have

$$A_{ij}(z) = \frac{1}{4\pi} \frac{1}{\mu_R^2} \begin{cases} \sqrt{1 - \frac{z}{\mu_R^2}} - \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} & \text{if } i = j \\ \\ -\frac{\mu_{d_{ij}}}{\mu_R} \frac{\frac{d_{ij}}{R}}{\sinh \frac{d_{ij}}{R}} \, e^{-\sqrt{\frac{\mu_R^2}{\mu_{d_{ij}}^2} - \frac{z}{\mu_{d_{ij}}^2}}} & \text{if } i \neq j. \end{cases} \tag{3.50}$$

Now let us consider the two center case on the hyperbolic plane H^3 as in the flat space case and let us assume again that their strengths (or bound state energies of each center) are the same. So we have the energy equation

$$\sqrt{1 + \frac{\nu^2}{\mu_R^2}} - \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} = \pm \frac{\mu_{d_{ij}}}{\mu_R} \frac{\frac{d_{ij}}{R}}{\sinh \frac{d_{ij}}{R}} \, e^{-\sqrt{\frac{\mu_R^2}{\mu_{d_{ij}}^2} - \frac{z}{\mu_{d_{ij}}^2}}}. \tag{3.51}$$

If we expand it for small d we have

$$\sqrt{1 + \frac{\nu^2}{\mu_R^2}} - \sqrt{1 + \frac{\mu_i^2}{\mu_R^2}} = \pm \frac{\mu_{d_{ij}}}{\mu_R} \left\{ 1 - \sqrt{\frac{\mu_R^2}{\mu_{d_{ij}}^2} + \frac{\nu^2}{\mu_{d_{ij}}^2}} \right\}. \tag{3.52}$$

From which we can conclude

$$E_{gr} = -\nu^2 \simeq \frac{3}{4} \mu_R^2 - \frac{\mu^2}{4} - \frac{\mu_{d_{ij}}^2}{4} - \frac{\mu_{d_{ij}} \mu_R}{2} \sqrt{1 + \frac{\mu^2}{\mu_R^2}}. \tag{3.53}$$

Note that this has again an inverse quadratic singularity in terms of d as in the flat space problem.

We may also write the resolvent kernel explicitly as

$$R(x, y|z) = R_0(x, y|z) + \sum_{i,j=1}^{N} R_0(x, a_i|z) \, A_{ij}^{-1}(\sqrt{-z}) \, R_0(a_j, y|z), \tag{3.54}$$

where $R_0(x,y|z)$ is again the free resolvent kernel and given by

$$R_0(x,y|z) = \frac{1}{4\pi}\,\frac{1}{\mu_R^2}\,\frac{\mu_{d(x,y)}}{\mu_R}\;\frac{\frac{d(x,y)}{R}}{\sinh\frac{d(x,y)}{R}}\; e^{-\sqrt{\frac{\mu_R^2}{\mu_{d(x,y)}^2}-\frac{z}{\mu_{d(x,y)}^2}}} \tag{3.55}$$

3.3.2. Hyperbolic Plane H^2

The hyperbolic distance for the hyperbolic plane H^2 is defined via the following equality:

$$\cosh\frac{d(x,y)}{R} = 1 + \frac{|x-y|^2}{2\,x_2\,y_2}. \tag{3.56}$$

And we have an explicit formula [20] for the heat kernel as

$$K_t(x,y) = \frac{\sqrt{2}}{(4\pi\left[\frac{\hbar}{2mR^2}\right]t)^{3/2}}\; e^{-\frac{\hbar}{2mR^2}\frac{t}{4}} \int_{\frac{d(x,y)}{R}}^{\infty} \frac{r e^{-\frac{r^2}{4}\frac{2mR^2}{\hbar}\frac{1}{t}}}{\sqrt{\cosh r - \cosh\frac{d(x,y)}{R}}}\,dr. \tag{3.57}$$

Hence we have the free resolvent kernel as

$$\begin{aligned}
\langle a_i|\,(H_0-z)^{-1}|a_j\rangle &= \frac{1}{\hbar}\int_0^{\infty} e^{z\frac{t}{\hbar}}\frac{\sqrt{2}}{(4\pi\left[\frac{\hbar}{2mR^2}\right]t)^{3/2}}\; e^{-\frac{\hbar}{2mR^2}\frac{t}{4}} \\
&\qquad\times\left[\int_{\frac{d_{ij}}{R}}^{\infty}\frac{r e^{-\frac{r^2}{4}\frac{2mR^2}{\hbar}\frac{1}{t}}}{\sqrt{\cosh r-\cosh\frac{d_{ij}}{R}}}\,dr\right]dt \\
&= \frac{1}{(4\pi)^{3/2}\mu_R^2}\int_0^{\infty} du\; u^{-3/2} e^{-\left(\frac{1}{4}-\frac{z}{\mu_R^2}\right)u}\int_{\frac{d_{ij}}{R}}^{\infty}\frac{r e^{-\frac{r^2}{4}\frac{1}{u}}}{\sqrt{\cosh r-\cosh\frac{d_{ij}}{R}}}\,dr \\
&= \frac{1}{(4\pi)^{3/2}\mu_R^2}\int_{\frac{d_{ij}}{R}}^{\infty} dr\;\frac{r}{\sqrt{\cosh r-\cosh\frac{d_{ij}}{R}}}\int_0^{\infty} du\; u^{-3/2} e^{-\left(\frac{1}{4}-\frac{z}{\mu_R^2}\right)u-\frac{r^2}{4}\frac{1}{u}} \\
&= \frac{1}{4\pi\,\mu_R^2}\int_{\frac{d_{ij}}{R}}^{\infty}\frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r-\cosh\frac{d_{ij}}{R}}}\,dr.
\end{aligned}$$

Again, we have a divergence for $i = j$. Therefore we again apply the same renormalization procedure and we get

$$\begin{aligned} A_{ii}(z) &= \lim_{\epsilon \to 0^+} \left[g_i^{-1}(\epsilon) - \frac{1}{4\pi\,\mu_R^2} \int_{\epsilon}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - 1}}\, dr \right] \\ &= \lim_{\epsilon \to 0^+} \left[g_i^{-1}(\epsilon) - \frac{\sqrt{2}}{4\pi\,\mu_R^2} \int_{\frac{\epsilon}{2}}^{\infty} \frac{e^{-u\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sinh u}\, du \right]. \end{aligned} \tag{3.58}$$

Now, the natural choice for $g_i^{-1}(\epsilon)$ is simply

$$g_i^{-1}(\epsilon) = \frac{\sqrt{2}}{4\pi\,\mu_R^2} \int_{\frac{\epsilon}{2}}^{\infty} \frac{e^{-u\sqrt{1+\frac{4\mu_i^2}{\mu_R^2}}}}{\sinh u}\, du. \tag{3.59}$$

So in the $\epsilon \to 0$ limit we have

$$A_{ij}(z) = \frac{1}{4\pi}\frac{1}{\mu_R^2} \begin{cases} \sqrt{2}\left[\psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{z}{\mu_R^2}}\right) - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu_i^2}{\mu_R^2}}\right)\right] & \text{if } i = j \\ -\int_{\frac{d_{ij}}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh\frac{d_{ij}}{R}}}\, dr & \text{if } i \neq j, \end{cases} \tag{3.60}$$

where ψ is the digamma function.

Again we may write the resolvent kernel as in the H^3 example. Now the free resolvent kernel is just

$$R_0(x, y|z) = \frac{1}{4\pi}\frac{1}{\mu_R^2} \int_{\frac{d(x,y)}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh\frac{d(x,y)}{R}}}\, dr. \tag{3.61}$$

4. NUMERICAL EXAMPLES

4.1. Two Dimensional Flat Space

For simplicity we choose all centers to have equal bound state energy of $-\mu^2 = -\hbar^2/(2md^2)$ where d is the lattice distance for bound state example and distance between adjacent centers for scattering examples.

4.1.1. Bound States

We consider an arrangement of 25 attractive centers located on a square lattice of dimensions 5×5.

First of all, we need to solve the energy equation ($\det \Phi(\nu) = 0 \Leftrightarrow \lambda_k(\nu) = 0$) to find the energy eigenvalues, which can be done by using numerical root finding methods. If we plot the eigenvalues with respect to ν (figure 4.1), we see that we have 8 eigenvalues which cross the vertical axis, but actually 3 of them are degenerate so that we have 11 states.

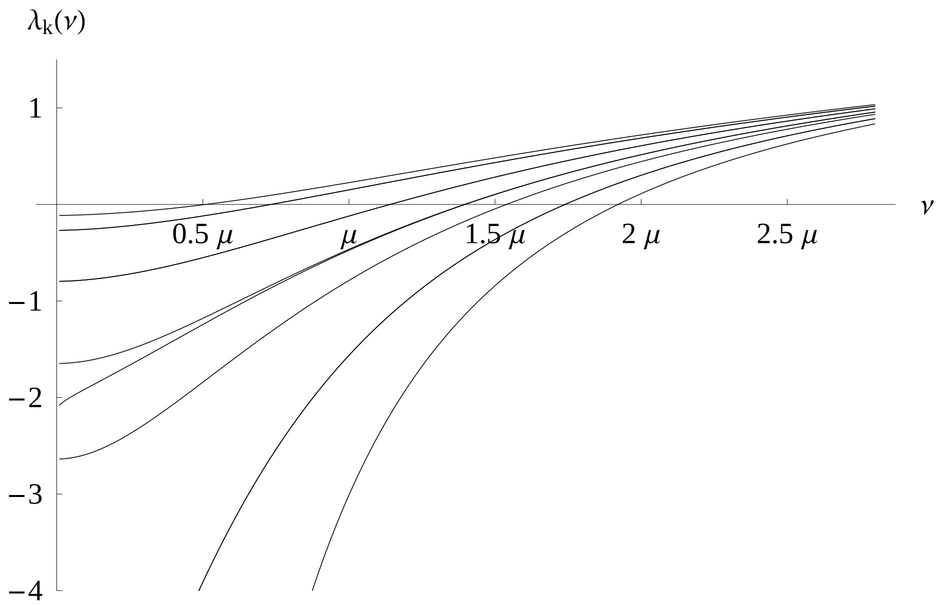

Figure 4.1. Eigenvalues of the matrix Φ.

Energy levels can be found to be (in μ^2 units): -3.689, -3.026, -2.368, -1.954, -1.947, -1.301, -0.537, -0.260.

Note that one can find the number of bound states including the degenerate ones by counting the negative eigenvalues of the matrix $\Phi(\nu)$ as $\nu \to 0^+$, because as we have shown previously, eigenvalues are monotonic increasing functions of ν, so we only have a root when $\lambda_k(0^+) < 0$, otherwise we might have a resonance.

By using the energy eigenvalues we can find the properly normalized wave functions and therefore the probability distributions for each state (figure 4.2).

4.1.2. Scattering Properties

We give two examples for the scattering states. First, we consider the total cross section for multiple delta attractors located on a line (figure 4.3). As energy increases, the total cross section decays to zero as expected. We have also some peaks around some fixed energy values which corresponds to interference effects due to the geometrical properties of the system.

Next we consider two centers and investigate the correlation between the total cross section and separation distance (figure 4.4). For both of the examples we choose initial wave vector orthogonal to the line joining centers.

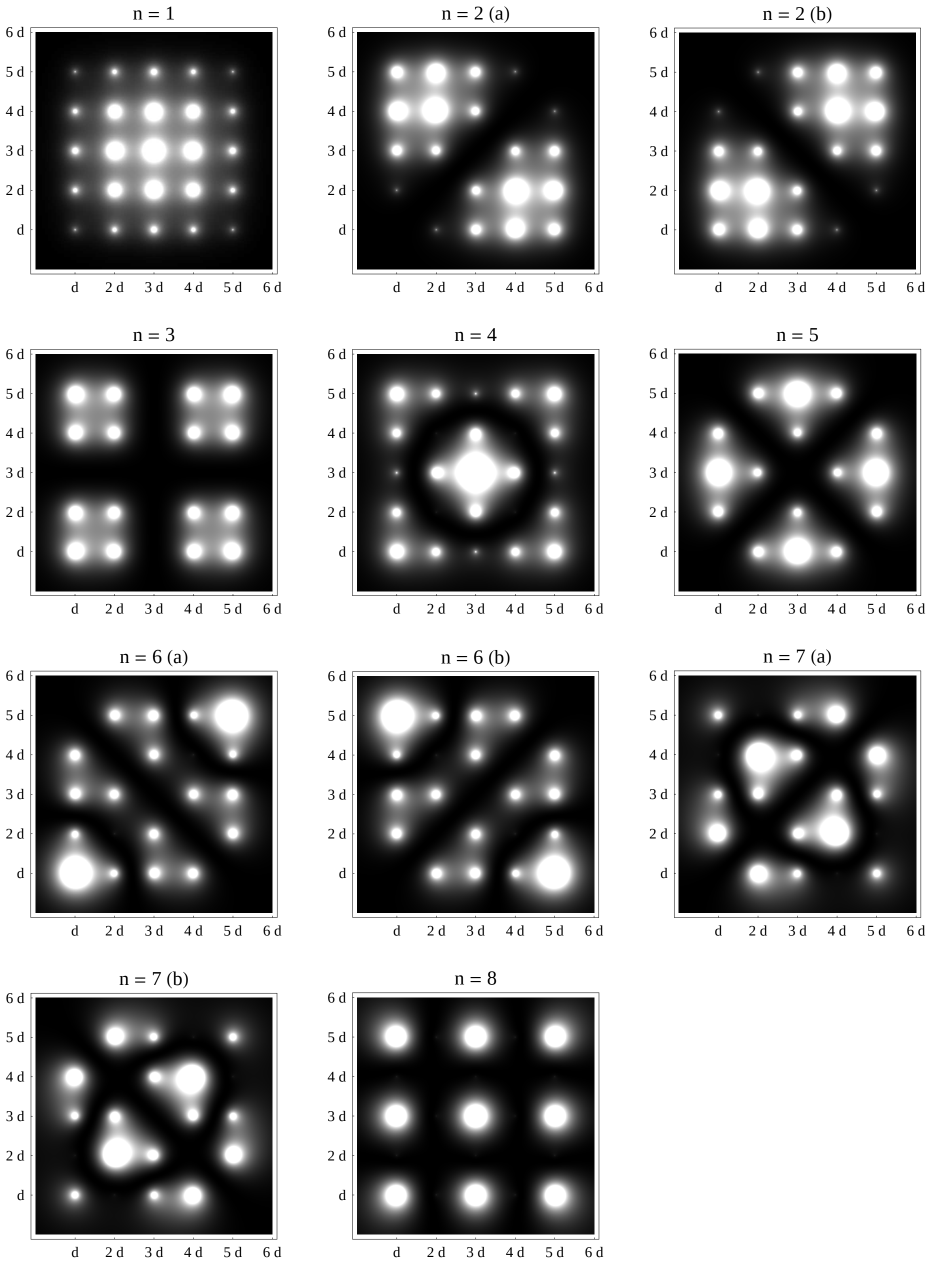

Figure 4.2. Density plot of the probability distributions for the bound states. From black to white the probability increases.

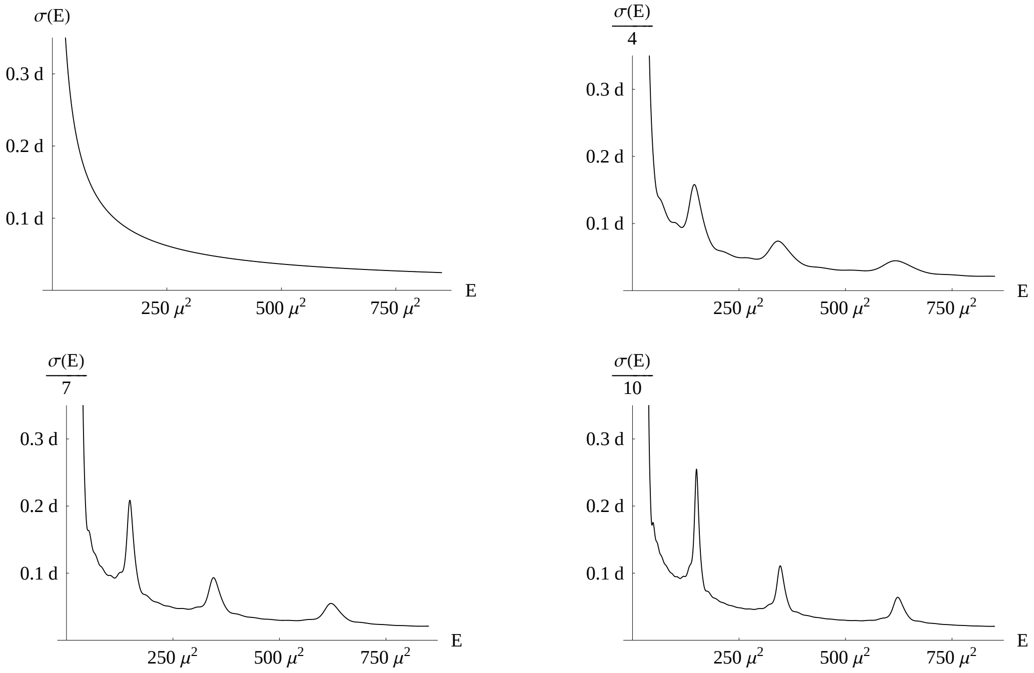

Figure 4.3. Total cross section normalized with the number of point interactions versus energy for 1, 4, 7 and 10 centers located on a line.

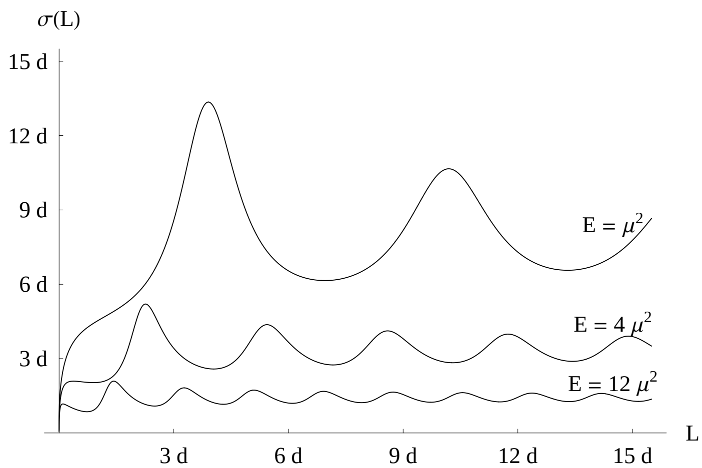

Figure 4.4. Total cross section versus distance for two point interactions corresponding to different energies.

5. Conclusion

In this thesis, we investigated point interactions on flat space and on a general compact manifold in two and three dimensions. The problem contains divergences and they have to be renormalized. We introduced the coupling constant renormalization method which removes the divergences. By using the resolvent kernel we derived some properties of the bound states and found bounds for the ground state energy. We proved that the resolvent operator that we have found defines a Hamiltonian, and calculated scattering cross sections. We repeated some of our calculations on a general compact Riemannian manifold and solved the same problem on a sphere and hyperbolic planes H^2 and H^3.

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