

JOINT INVENTORY AND PRICING DECISIONS:
REFERENCE EFFECTS AND DELAY SENSITIVE CUSTOMERS

by

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for my beloved mother and friend, Ülker...

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ABSTRACT

JOINT INVENTORY AND PRICING DECISIONS: REFERENCE EFFECTS AND DELAY SENSITIVE CUSTOMERS

We consider joint inventory and pricing problem of a single product with stochastic demand in two different contexts. In the first one, we study a periodic review problem where the demand of the product is subject to reference effects. Randomness is introduced with additive and multiplicative random terms. The customers have different attitudes such as loss-aversion or loss-neutrality. For the demand models with an additive random term, we show that the problem can be decomposed into two subproblems and a steady state solution exists for the infinite horizon problem. Defining the modified revenue as revenue less production cost, we show that a state-dependent order-up-to policy is optimal for concave demand models with concave modified revenue functions. We also show that the optimal inventory level increases with the reference price. For the demand models with an additive and a multiplicative random term, we show that a state-dependent order-up-to policy is optimal for demand models and expected revenue functions which are concave after a transformation. We also provide an extensive computational study. For the second context, we consider a capacity constrained manufacturer who serves several classes of delay sensitive customers. We model the problem as an $M/M/1$ queueing system with non-preemptive priorities. We give closed form solutions for the inventory decisions. Using an approximation on to the problem we provide explicit solutions when there is a single customer type. We also show that the optimal prices are incentive compatible in the sense that they optimize the profit of the manufacturer even if the manufacturer does not have any information about an arriving customer and let the customer choose a price from a provided menu of prices.

ÖZET

BİRLEŞİK ENVANTER VE FİYATLANDIRMA KARARLARI: REFERANS ETKİSİ VE GECİKMEYE DUYARLI MÜŞTERİLER

Rassal talebe sahip bir ürünün birleşik envanter ve fiyatlandırma problemini iki farklı bağlamda değerlendiriyoruz. İlkinde, ürünün talebinin referans etkisine maruz kaldığı dönemsel gözden geçirmeli problemi incelemekteyiz. Rassallık toplamsal ve çarpımsal rassal terimlerle sağlanmakta. Müşteriler, zarardan kaçan ve zarar du-yarsız olmak üzere farklı davranışlar sergilemekteler. Sadece toplamsal rassal terim içeren talep modellerde problemin iki alt probleme ayrılabilceğini gösteriyoruz ve son-suz dönemli problemin kararlı durum çözümünün var olduğunu gösteriyoruz. Üretim maliyetini gelirden düşerek elde edilen değeri değiştirilmiş gelir olarak adlandırarak, konkav talep ve konkav değiştirilmiş gelir fonksiyonuna sahip modellerde durum deęiş-kenlerine baęlı hedef stok politikasının optimal olduğunu gösterebiliyoruz. Optimal hedef stoğun referans fiyatla yükseldiğini ispatlıyoruz. Sonrasında toplamsal ve çarpımsal rassal terimler içeren talep modellerini inceliyoruz. Bir dönüşümden sonra konkav olan talep ve gelir fonksiyonlarına sahip modeller için duruma baęlı hedef stok politikasının optimal olduğunu gösteriyoruz. İkinci bağlamda ise, çeşitli sınıflara ait gecikmeye du-yarlı müşterilere ürün satan kapasitesi sınırlı bir üreticinin sürekli gözden geçirmeli problemini göz önüne alıyoruz. Problemi tam öncelikli olmayan $M/M/1$ kuyruk sis-temine göre modelliyoruz. Envanter kararları için kapalı formda çözümler sunuyoruz. Problem için bir sürekli yaklaşı- lama sağlayıp, tek tip müşteri içeren problemler için basit ve açık çözümler veriyoruz. Optimal fiyatların teşvik uyumlu olduğunu, yani gelen her müşterinin özelliklerini üretici tarafından bilinmese de, gelen müşterilerin kendilerine sunulan fiyatlar arasından kendileri için hesaplanmış fiyatları seçeceğini gösteriyoruz. Ayrıca bazı sayısal örnekler sunuyoruz.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iv
ABSTRACT	v
ÖZET	vi
LIST OF FIGURES	x
LIST OF TABLES	xi
LIST OF SYMBOLS	xii
LIST OF ACRONYMS/ABBREVIATIONS	xv
1. INTRODUCTION	1
2. LITERATURE SURVEY	5
2.1. Dynamic Pricing	5
2.2. Pricing and Inventory Management in Queueing Systems	9
3. JOINT PRICING AND INVENTORY CONTROL UNDER REFERENCE EF- FECTS	13
3.1. Description of the Demand Model with Reference Effects	13
3.2. Dynamic Joint Inventory and Pricing Model	16
3.3. Demand Models with an Additive Random Term	17
3.3.1. Decomposition and Optimal Safety Stock	18
3.3.2. Optimality of the Order-up-to Policy	22
3.3.3. Characterization of the Optimal Policy Parameters	25
3.3.4. Customer Behaviour	28
3.4. Demand Models with an Additive and a Multiplicative Random Term .	29
3.4.1. Optimal Policy	33
3.5. Computational Study	36
3.5.1. An Illustrative Example	38
3.5.2. Effect of α , r_0 and Random Terms	40
3.5.3. Investigating the Value of Dynamic Programming	43
3.5.4. Investigating the Value of Reference Effect	46
3.5.4.1. Value of Modeling with Reference Effects	46
3.5.4.2. Value of Correctly Assessing Reference Effects	50

3.6. Extensions	51
3.6.1. General Evolution Models	51
3.6.2. α as a Random Variable	52
3.6.3. α with Markovian Structure	52
3.6.4. Capacitated Problem	53
4. JOINT INVENTORY AND PRICING DECISIONS WITH DELAY SENSITIVE CUSTOMERS	54
4.1. Definiton of the Model	54
4.1.1. Problem Definition	54
4.1.2. Problem Formulation	57
4.2. Solution Approaches, Solutions and Analysis	59
4.2.1. Priority Rule Decision	60
4.2.2. Inventory Decision	61
4.2.3. Pricing Decision	63
4.2.4. Asymmetric Information: Incentive Compatibility	66
4.3. Continuous Approximation	68
4.3.1. If Inventory Holding is Expensive	71
4.4. Single Customer Type	71
4.4.1. Benefit of Holding Inventory	76
4.4.2. The Exact Problem	77
4.5. Numerical Illustrations	78
5. CONCLUSION	81
APPENDIX A: PROOFS OF THE RESULTS IN CHAPTER 3	85
A.1. Proof of Theorem 3.1	85
A.2. Proof of Theorem 3.4	85
A.3. Proof of Theorem 3.5	87
A.4. Proof of Theorem 3.6	88
A.5. Concavity Examples	89
A.6. Proof of Theorem 3.7	91
A.7. Supermodularity of $\hat{p}(d, r)$	92
A.8. Proof of Proposition 3.2	93

A.9. Example for a Concave Expected Demand and Revenue Function . . .	93
A.9.1. Example for a Concave Demand Function	94
A.9.2. Example for a Concave Revenue Function	95
A.10. On the Convexity of $\mathbb{D}$	96
A.11. Proof of Theorem 3.8	96
A.12. Proof of Theorem 3.11	99
APPENDIX B: PROOFS OF THE RESULTS IN CHAPTER 4	102
B.1. Proof of Proposition 4.1	102
B.2. Proof of Proposition 4.4	104
B.3. Proof of Lemma 4.2	106
B.4. The Derivatives for Single Customer Type	108
REFERENCES	111

LIST OF FIGURES

Figure 3.1.	Original Action Space.	32
Figure 3.2.	Transformed Action Space.	33
Figure 3.3.	Evolution of Optimal Decisions with r_0	39
Figure 3.4.	p^* and d^* with On Hand Inventory at Different Periods.	39
Figure 3.5.	Optimal Policy vs Reference Price at Different Periods.	40
Figure 3.6.	Total Expected Profit with r_0 and α	41
Figure 3.7.	Optimal Policy of First Period with α and r_0	42
Figure 3.8.	Evolution of Optimal Decisions with α and r_0	42
Figure 3.9.	Effect of Variances Over Optimal Decisions.	43
Figure 3.10.	Myopic and Optimal Decisions.	46
Figure 3.11.	Significance with Inventory Cost Share on Profit.	50
Figure 4.1.	C vs π^* and π^0	77
Figure 4.2.	Waiting Costs.	79
Figure 4.3.	Optimal Values of λ , p and λp	80

LIST OF TABLES

Table 3.1.	Random Terms.	37
Table 3.2.	Cost Parameters.	38
Table 3.3.	Experiment Values for α and r_0	40
Table 3.4.	Average Profit Improvement(%) with T , c , h and b	45
Table 3.5.	Average Profit Improvement(%) with Variances of Random Terms.	45
Table 3.6.	Average Profit Improvement(%) with r_0	45
Table 3.7.	Average Profit Improvement(%) with α	48
Table 3.8.	Average Profit Improvement(%) with r_0	48
Table 3.9.	Significance(%) for Variances of Random Terms.	49
Table 3.10.	Significance(%) for T and α	49
Table 3.11.	Significance(%) for r_0	49
Table 3.12.	Significance(%) for Production Cost.	50
Table 4.1.	Approximation Values.	69
Table 4.2.	Parametric Analysis.	74

LIST OF SYMBOLS

a_i	Coefficients of the mean demand function
b_t	Backorder cost per unit per unit time
c_t	Production cost
C_i	Waiting cost of type- i customer per unit time
$D(\cdot)$	Random demand function
$\hat{d}(\cdot)$	Mean demand function
d	Mean demand, decision variable
$F(\cdot)$	Reference effect on demand function
$G_t(\cdot), \hat{G}_t(\cdot)$	Expected inventory related cost
$\hat{G}_t^\gamma(\cdot)$	Modifiend inventory related cost
h_t	Inventory holding cost per unit per unit time
$H_t(\cdot)$	Inventory related cost
I_a^*	The system that is subject to stochastic demand under refer-
	ence effects with additive random term
I^0	The relaxed version of the system I_a^* that can return the prod-
	ucts.
$\hat{I}^*$	The capacity constrained system that is subject to stochastic
	demand with delay sensitive customers
$\tilde{I}^0$	An $\hat{I}^*$ system that holds no inventory
$J_t(\cdot)$	Value function of the I^* system
$\hat{J}_t(\cdot)$	Value function of the $\hat{I}^*$ system
L_i	Number of type- i customers in the $\hat{I}^*$ system
M	Number of delay sensitive customer types
N	Number of outstanding orders in the $\hat{I}^*$ system
$\mathbb{N}$	The set of natural numbers
p	Price, decision variable
$\hat{p}(\cdot)$	Price function
Q_i	Number of type- i customers in the $\hat{I}^*$ queue
r	Reference price, state variable

t	Time index
S_i	Service time of a type- i customer in the $\hat{I}^*$ system
T	Length of the planning horizon
u	The priority rule, $u \in \mathbf{U}$
$\mathbf{U}$	The set of all permutations of the set $\{1, \dots, M\}$
$V_t(\cdot)$	Profit-to-go function of the I^* system
$\hat{V}_t(\cdot)$	Profit-to-go function of the $\hat{I}^*$ system
W_i	Soujourn time of a type- i customer in the $\hat{I}^*$ system
W_i^q	Waiting time of a type- i customer in the $\hat{I}^*$ queue
x	Inventory level before ordering
y	Inventory level after ordering
α	Memory parameter
γ	Discount rate
Δ	Safety Stock
ϵ	Additive random term, random variable
θ	Multiplicative random term, random variable
$\kappa(\cdot)$	A function of reference price
λ_i	Arrival rate of a type- i customer in the the $\hat{I}^*$ system
μ	Service rate of the $\hat{I}^*$ system
$\nu(\cdot)$	Base demand function
$\Pi_t(\cdot)$	Modified revenue function
$\Pi_t^p(\cdot)$	Transformed revenue function
ρ	The utilization of the $\hat{I}^*$ system
$\phi_t(\cdot)$	Value function of I_a^* system
$\phi(\cdot)$	Limit function of $\phi_t(\cdot)$
$\phi_t^*(\cdot)$	Auxiliary value function of I_a^* system
$\phi^*(\cdot)$	Limit function of $\phi_t^*(\cdot)$
$\Phi_t(\cdot)$	Profit-to-go function of I_a^* system
$\Phi(\cdot)$	Limit function of $\Phi_t(\cdot)$
$\Phi_t^*(\cdot)$	Auxiliary profit-to-go function of I_a^* system

$\Phi^*(\cdot)$	Limit function of $\Phi_t^*(\cdot)$
$\phi_t^0(\cdot)$	Value function of I^0 system
$\hat{\phi}_t^0(\cdot)$	Value function of I^0 system when d is the decision variable
$\Phi_t^0(\cdot)$	Profit-to-go function of I^0 system
χ_-	Left derivative of the reference effect function
χ_+	Right derivative of the reference effect function
χ	Derivative of the reference effect function
$\tilde{\chi}$	Index of the loss-aversion

LIST OF ACRONYMS/ABBREVIATIONS

ADRE	Absolute Difference Reference Effect
RDRE	Relative Difference Reference Effect
SDO	State-Dependent Order-up-to

1. INTRODUCTION

In recent years, companies have developed strategies for choosing the optimal price and quantity simultaneously. Although pricing has been considered in the context of the inventory problems in the early literature, it is the last fifteen years that pricing has emerged as a tool for maximizing profit. This is due to three factors: (i) An increased availability of demand data, (ii) ease of changing prices due to new technologies and (iii) an availability of decision-support tools for analyzing demand data and for dynamic pricing (Elmaghraby and Keskinocak, 2003). Current technology enables us to infer information about the product-customer relations using available data. This information can be used to develop strategies to increase profit using techniques such as dynamic pricing.

The literature is emerging parallel to technological progress and adoption of new technology. Although between 1955 and 1975 there were studies which take pricing into consideration, the main results were found in late 90s due to pricing technology becoming widely available. A firm has to decide both on the price and the inventory replenishment quantities simultaneously in order to maximize its profit. However the price is not the only factor that affects the demand. As customers buy a particular product frequently, they develop their own price expectations, which is called the *reference price*. The reference price is an internal standard against which observed prices are compared (Kalyanaram and Winer, 1995). If the current price is greater or less than the reference price, then it is perceived as *high* or *low*, respectively. The customers perceive the difference between the price and the reference price as a loss or a gain. Following excerpt is about the iPad from Apple (Darlin, 2010):

‘The new series’ most expensive version, which includes 64 gigabytes of storage and connect to AT&T’s cellphone network costs \$830. That gives consumers a *reference point*, which marketers call a *price anchor*. It helps convince people that they will *save* by spending less than \$830.’

In marketing, the reference effect on demand have been accepted as an empirical

generalization. The concept of reference price asserts that consumers make decisions based on both actual and reference prices (Kalyanaram and Winer, 1995). Hence the inventory and pricing policy is affected by this reference price. Although a firm can increase its profit in a period with a discount, these discounts lower the reference price of customers and they may judge the ‘normal’ price as ‘expensive’ which results in a loss of profit in future periods. Likewise increasing the price may have the reverse effect for the profits of the current and the future periods. Therefore a company faces a trade-off between the current and future benefits.

Chapter 3 aims to identify optimal joint inventory and pricing policy of a firm who faces a random demand depending on the price and the price history of the firm that is captured by the reference price. Our goals are (i) to find the optimal inventory and pricing policy for finite and infinite planning horizons, (ii) to find the effects of cost parameters and reference effect mechanism on the profit and (iii) to investigate the value of the dynamic programming and the reference effect.

In particular we make four contributions in the context of joint inventory and pricing under reference effects. First, *for the demand models with the additive random term*, we show that by employing the safety stock as the decision variable, the problem can be decomposed into two easier subproblems if the initial inventory is low: pricing under deterministic demand with reference effects and a newsvendor problem. This result is valid for *all* demand models with an additive random term. These two subproblems are well studied in the literature and by the decomposition, the steady state solutions are easily characterized.

Second, we show the optimality of a state-dependent order-up-to (SDO) policy. For demand models with an additive random term, we define the *modified revenue* as revenue less production cost. We show that an SDO policy is optimal for concave demand models with concave modified revenue functions. We also provide the solutions for the variables under the optimal policy. When randomness is due to *the additive and the multiplicative random terms*, by employing an inverse demand function and using the mean demand as the decision variable, the expected inventory cost function

can be shown to be convex. However the action space that is defined by a lower and an upper bound on the price is no more convex after the transformation. We deal with this problem by defining a new and a more convenient action space which preserves convexity after the transformation. Using this new action space, we show that an SDO policy is optimal for concave demand models and concave transformed expected revenue functions.

Third, for the demand models with an additive random term, by employing an inverse demand function and using the mean demand as the decision variable, we show that the optimal order-up-to level increases with the reference price if the inverse demand function is supermodular.

Fourth, we provide an extensive computational study showing the effects of random terms, value of dynamic programming and value of correctly estimating the demand model with reference effects. We also show computational results for the optimal policies in a period and through periods. All of the results for the demand models with an additive random term holds when the planning horizon is finite or infinite.

The customers of a firm need not to be homogeneous, they may have different characteristics. Some customers are very sensitive to delays and do not want to wait. However some customers do not have too much concern about the time. A capacity constrained firm can determine its inventory and pricing policies according to the characteristics of its customers. It can reduce the waiting times of the customers by holding inventory and can give priorities to the delay-sensitive customers in exchange for a premium. Hence it can generate revenue by utilizing the attributes of his customers.

In Chapter 4 our aim is to identify joint inventory and pricing policy of a capacity constrained manufacturer. It produces, stocks and sells a single product to several customer classes who differ in their sensitivities for waiting and in their valuation of the product. The manufacturer's objective is to maximize its profit by deciding on its inventory, priority and pricing policy.

In particular we make following contributions in the context of joint inventory and pricing with delay sensitive customers. First, we analyze the case where the manufacturer knows types of arriving customers. We provide the optimal priority scheduling rule among the customers. We explicitly give the optimal order-up-to level when the arrival rates are given. We also show that some customer types are not-profitable, hence they are charged with high prices to prevent them entering the system. Then we show that these solutions are also optimal when the manufacturer knows only the attributes of the customer types but not types of the arriving customers by showing that the prices are incentive compatible, i.e. they are tailor made in the sense that a customer can minimize his costs by choosing the price that the manufacturer charges for him. We make a continuous approximation and provide the optimal order-up-level for a given arrival rate. The approximation enables one to explore the case where there is a single customer type. We provide exact and explicit solutions for the optimal values of the variables and then show the value of the ability of holding inventory. We also provide numerical analysis for two customer types.

The rest of the thesis is outlined as follows. Chapter 2 provides the literature survey. We study joint pricing and inventory problem under reference effects in Chapter 3. The problem of a capacity constrained firm with delay sensitive customers is analyzed in Chapter 4. We conclude in Chapter 5. We provide an appendix for the proofs.

2. LITERATURE SURVEY

We can classify the literature into two groups. The first group studies the joint inventory and pricing in the context of dynamic pricing. The problems are formulated using dynamic programming and the decision maker has to decide on the inventory and prices simultaneously at *every period*. This group of papers is related to the joint inventory and pricing under reference effects. We survey this literature in Section 2.1. In Section 2.2 we survey the second group related to the pricing problems with delay sensitive customers or inventory rationing problems. These problems are formulated using queueing models.

2.1. Dynamic Pricing

The literature can be classified according to several aspects such as demand model (deterministic/stochastic), horizon length (single period/multi period), review type for multi period problems (continuous/periodic), modeling the unsatisfied orders (back-order/lost sales), setup cost (positive/zero) and modeling the dependence of demand (only current price/current and past prices).

In stochastic models, demand is generally assumed to be a function of price and random perturbation(s). A general model used commonly in the literature is to incorporate an additive and a multiplicative random term to a deterministic demand function (mean demand function) which is contingent on the price.

(Petruzzi and Dada, 1999) is a pioneering work in joint pricing and inventory replenishment problem. They show that for the single period joint pricing and inventory decision problems, optimal prices for both additive and multiplicative random term models consist of a premium added to the optimum riskless price. (Yao *et al.*, 2006) provide a survey over the single period problems and show the necessary conditions over the mean demand and the random term for the single period profit function to be unimodal.

(Federgruen and Heching, 1999) study the periodic review multi-period problem without a setup cost where the unsatisfied orders are backordered. They show the optimality of a base-stock list-price policy when the mean demand function is linear. This policy states that there exists an optimum inventory level and price pair at each period and if on-hand inventory is below the optimal inventory level, then the firm gives an order to reach that level and sets the optimum price. Otherwise it orders nothing and goes to a price discount. (Chen and Simchi-Levi, 2004a) and (Chen and Simchi-Levi, 2004b) introduce a setup cost to this setting while (Feng and Chen, 2003) and (Chen and Simchi-Levi, 2006) study continuous review version of this problem with a setup cost. (Polatoglu and Sahin, 2000) and (Chen *et al.*, 2006) analyze the case where unsatisfied orders are lost. (Huh and Janakiraman, 2008) study a periodic review system with stationary parameters and with a setup cost. There is a set of levers to control the demand which is assumed to be concave over these levers. They study models with backordering (for both additive and multiplicative random terms) and also for lost sales with additive random term under the assumptions of (Chen *et al.*, 2006). (Song *et al.*, 2009) analyze the models with lost sales and a multiplicative random term. In all of the studies including setup cost, the optimal policy turns out to be a variant of an (s, S, p) policy. This policy states that if on hand inventory is below s , then the firm places an order to bring its inventory level to S and announces the price p . Otherwise it orders nothing and announces a state dependent price.

The references cited above consider the models where demand in a period depends only on the price in that period. (Ahn *et al.*, 2007) analyze a finite horizon model with deterministic demand which depends linearly on price under a periodic review setting. Their demand model is different from previously mentioned studies. It is assumed that some customers arriving during early periods wait in the system for a lower price to make their purchase. They show that cyclic policies where price sequences are non-increasing is either optimal or perform well. (Ahn *et al.*, 2009) study a finite horizon uncapacitated random demand model with a given price set including two prices, a regular price and a sale price, under periodic review. Both lost sales and backordering cases are analyzed. They show that the optimal policy is a threshold policy.

Customers who are waiting for a future discount are said to act strategically. They decide to buy a product in anticipation of the future prices. Reference price, on the other hand, is the standard against which the purchase price of a product is judged (Monroe, 1973). The theoretical basis of reference price comes from adaptation level theory (Helson, 1964) which states that expectation-based reference price is the adaptation level against which current prices are judged (Monroe, 1973). (Kalyanaram and Winer, 1995) provide three generalizations for reference price based on their review of empirical studies. First, reference prices have a consistent and significant impact on consumer demand. This is known as the reference effect on demand (Helson, 1964). Second, reference prices utilize past prices as part of the consumer's information set. There are several models for the evolution of reference price (Mazumdar *et al.*, 2005) such as exponential smoothing which is introduced by the adaptive expectation model (Nerlove, 1958). Third, consumers react differently to price increases and price decreases relative to the reference price. If the price is higher (lower) than the reference price, they perceive it as a loss (gain). Customers can be classified into three groups. If customers are more responsive to discounts than surcharges, they are called *loss-seeking*; otherwise they are *loss-averse*. If they give the same response to both, then they are *loss-neutral*. Prospect theory (Kahneman and Tversky, 1979) also provides a theoretical basis for reference prices and states that customers are more sensitive to losses and hence they are loss-averse.

(Fibich *et al.*, 2003) study dynamic pricing under deterministic linear demand with reference effects in a continuous time framework. (Popescu and Wu, 2007) study the same problem when time is discrete for more general demand models. They explicitly calculate the steady state prices when customers are loss-neutral or loss-averse and show that a firm should adopt a skimming or a penetration strategy depending on the initial reference price.

The reference effect is modeled to be dependent on the difference (absolute difference reference effect [ADRE] model) or on the relative difference (relative difference reference effect [RDRE] model) of the price and the reference price. The former model has the drawback that if the difference is same then the change in demand remains

same at every price level (Popescu and Wu, 2007). (Gimpl-Heersink, 2008), (Zhang, 2010) and (Chen *et al.*, 2011) study the linear demand models. The linearity implies loss-neutrality and ADRE. Although (Gimpl-Heersink, 2008) show the optimality of an SDO policy for two periods, by using a transformed profit-to-go function, (Zhang, 2010) and (Chen *et al.*, 2011) show that an SDO policy is optimal for the linear demand when the time horizon is finite or infinite. In an SDO policy there is an optimal order-up-to level and price pair which depends on the state, i.e. the reference price. (Chen *et al.*, 2011) show that the optimal order-up-to level increases with the reference price. They prove that the reference price converges to a steady state and provide characterizations of the steady state solutions.

(Elmaghraby and Keskinocak, 2003), (Chan *et al.*, 2004), and (Yano and Gilbert, 2005) review joint replenishment and pricing problems and (Kalyanaram and Winer, 1995), (Mazumdar *et al.*, 2005), (Popescu and Wu, 2007) and (Arslan and Kachani, 2010) provide an extensive reference price discussion.

(Popescu and Wu, 2007) study deterministic demand models, hence inventory level is determined endogenously by the pricing decisions. However we study pricing and inventory decisions jointly under stochastic demand. (Gimpl-Heersink, 2008) extends her results for optimality of an SDO policy under restrictive assumptions and to the best of our knowledge there are no demand models that can satisfy these assumptions. We present less restrictive conditions and provide demand models which satisfy our assumptions. (Zhang, 2010) and (Chen *et al.*, 2011) study linear demand models. We show the optimality of such a policy for concave demand models and show that our result works for models with additive and multiplicative random terms, RDRE or ADRE models, and loss-neutral or loss-averse customers when the planning horizon is finite or infinite. Hence our result applies to a different domain of demand functions not covered in (Zhang, 2010) and (Chen *et al.*, 2011).

2.2. Pricing and Inventory Management in Queueing Systems

Much of the initial work on pricing problems with multiple classes of customers has been done in the context of a queueing network, where heterogeneous customers arrive with different sensitivities to delay, and pricing determines the allocation of the priorities. Our work is closely related to the priority pricing problem for the $M/M/1$ queueing system. In this problem there is a service facility like a communication network or a production line which is modeled as a $M/M/1$ queueing system. The users of the service, or customers, have different levels of *urgencies* and hence different types of customers have different sensitivities for waiting. The service facility announces different levels of priorities to be chosen by the customers according to their urgency levels. The priorities can be preemptive or non-preemptive. A job with a high priority preempts the job in the service in a queueing system with preemptive priority. In a system with non-preemptive priorities, the service cannot be preempted and the job with the high priority waits for the service to be completed. Since every customer prefers a high priority, the aim is to construct priority schedules and pricing of these priorities which allocates service capacity and determines the priorities optimally, i.e. it maximizes the expected net value of the system. Such a *priority pricing* scheme should be incentive compatible in order to make customers choose the right price in the sense that these prices are not only optimal from their own view, but optimal for the entire system; the service facility and the customers.

For a system of homogeneous customers, i.e. there is a single customer type, (Mendelson, 1985) shows that a price that captures the externality cost of congestion is necessary to induce users to submit jobs at a rate that is socially optimal. The *externality cost* is the expected delay cost per unit time, inflicted on the rest of the system by an infinitesimal unit increase in the job flow. (Mendelson and Whang, 1990) are among the first to study incentive compatible priority pricing strategies in queueing systems with multiple user classes. They analyze a system that is modeled as an $M/M/1$ queueing system with several user classes which differ in their valuation of the service and their waiting per unit time costs. When service requirements are homogeneous, they show that the optimal prices which maximize the net value of the

system are also incentive compatible. For the non-homogeneous service requirement case, they show that these prices fail to be incentive compatible and provide an optimal pricing scheme which depends on priorities and service times which are shown to be incentive compatible. (Mandjes, 2004) analyze a communication network where there are two types of customers, delay sensitive or delay tolerant. He assumes that the customers incur the waiting cost in a non-linear way rather than incurring per unit time. He explicitly provides the optimal prices.

The objective of these three papers is to optimize the net value of the whole system, which is also called the *social optimization*. The socially optimal mechanism is of interest for an operation that serves customers within an organization. However, the objective of a company who deals with external customers is to maximize its own profit. (Katta and Sethuraman, 2005) study the problem of a profit maximizing firm in the context of an $M/M/1$ queueing system. The aim is to find the optimal pricing and scheduling rule. They are able to characterize the optimal policy by making some simplifying assumptions and find that pooling the customer types into a single priority class is optimal. (Aféche, 2004) analyzes the same problem with two types of customers. The aim is to find the optimal pricing, scheduling and also admission control, i.e. when to accept or reject the customers. They show that the celebrated $C\mu$ rule need not be optimal for setting the priorities among the customers. This rule states that the customer types should be given priorities according to their $C\mu$ values, where C is the waiting cost per unit time and μ is the expected needed service time. They show that the optimal policy may insert an idleness to the server even if there is a customer in the system. This insertion of strategic delay deters time sensitive customers from purchasing the low-priority class. (Cui *et al.*, 2009) analyze a similar problem. They use an $M/M/1$ queueing system to model a profit maximizing firm that serves customers whose waiting cost per unit time and valuation of the service can take two distinct values, low or high. In their study, the server also decides on a *probabilistic* admission control, i.e. there is a probability of rejecting customers. They show that a probabilistic admission may force customers to reveal their true valuations and induces the delay-sensitive customers to choose high priorities and hence enables the server to receive more revenue from his customers.

(Aféche and Mendelson, 2004) propose a generalized delay cost structure and capture the interdependence between delay cost and value of the service of homogeneous customers. They show that the priority auction mechanisms perform better than setting a single price under the social optimization and revenue maximization. They also show that the structure of the delay cost has a *paramount* effect on the system behaviour.

(Gilland and Warsing, 2009) analyze a model where the customers have heterogeneous delay costs, picked up from a uniform distribution. The manufacturer offers a non-preemptive prioritization for a premium. They play a Stackelberg game where the manufacturer is the leader and decides for the premium. The customers are the followers. The total arrival rate is known and fixed. The customers decide the high priority arrival rate. They analyze the model according to the behaviour of the customers. *Co-ordinated customers* act to minimize the total costs per unit time across all customers. *Independent customers* decide based only on the economics of that job. They show that coordinated customers submit high-priority jobs at a rate which is below the rate which minimizes total delay cost. On the other hand, independent customers choose a submission rate for high priority jobs which minimize the total delay costs.

(Hassin, 2007) studies an $M/M/1$ system with a single customer type where the service rate, the value of the service and the waiting cost per unit time are random variables that can take two values and are known only to the service provider. He shows that the server is motivated to reveal the information except the case where the value of the service is random and waiting cost is small.

The priority pricing problem for the $M/M/1$ queueing system, both under social optimization or revenue maximization, is designed for service type systems in which the inventory decision is not incorporated. An extensive summary and classification of queueing systems can be found in (Hassin and Haviv, 2003). In this thesis, we study a server which can hold inventory. Hence our work is also related with inventory-production systems which are modeled as a queueing system.

A recent study by (Zhao and Lian, 2011) consider a two customer queueing-inventory system. Customers arrive at a service facility according to Poisson processes and their service times follow exponential distributions. Each service uses one item in the attached inventory supplied by an outside supplier with exponentially distributed lead times. Such a system is called a queueing-inventory system. An (r, Q) policy is used as the inventory replenishment policy. They show that if there is inventory on hand, it is optimal to serve the customer with the larger $\mu_i C_i$ index where μ_i is the service rate and C_i is the waiting cost. They also provide necessary and sufficient conditions for the stability of the system and compute steady state probabilities.

If the inventory of an item is low, it is reasonable to ration the on-hand inventory by rejecting demand from low priority (less valuable) customers in anticipation of future demand from the high priority (more valuable) customers. The stock rationing literature is also relevant to our problem in the sense that the manufacturer has to choose the priority levels by choosing the rationing levels. (Ha, 1997a,b, 2000) consider the stock rationing problem of a single item. (Ha, 1997a, 2000) study a make-to-stock production system with several demand classes where the unsatisfied demands are lost. He shows that there exists stock rationing levels of each priority class and it is optimal to start rejecting the demand of a low priority class in anticipation of future arrival of higher priority demands at or below this rationing level.

(Ha, 1997b) analyzes a system with two priority customer classes where the customers, that are not satisfied from the inventory, join the backorder queues. He shows that the rationing policy is characterized by a monotone switching curve which depends on the length of the low priority queue.

Inventory is not an explicit decision in many of the priority pricing problems with multiple classes of customers. We extend the priority-pricing literature by incorporating inventory decisions. We extend the inventory literature where the problem is modeled with queueing models by incorporating pricing and priority decisions with delay sensitive customers.

3. JOINT PRICING AND INVENTORY CONTROL UNDER REFERENCE EFFECTS

In this chapter, we analyze the periodic review joint inventory and pricing problem of a single item with stochastic demand. The demand is contingent on the price and the price history which is captured by the reference price. The randomness of the demand is introduced with an additive and a multiplicative random term. The objective is to find the optimal dynamic pricing and inventory policy which maximizes the total discounted profit.

We first give the description of the demand model with reference effects that we analyze throughout this chapter in Section 3.1. Then we formulate the problem using dynamic programming in Section 3.2. We analyze the case where there is only an additive random term in Section 3.3. Then we analyze the general case, i.e. the demand model where the randomness is due to an additive and a multiplicative random term in Section 3.4. We provide an extensive computational study in Section 3.5

3.1. Description of the Demand Model with Reference Effects

We start by introducing the structure of the demand model, customer types and reference effect models that are used in this study. In stochastic inventory and pricing literature, the randomness of demand is generally modeled by incorporating an additive and/or a multiplicative random term over the mean demand which is a function of the price (p). In our setting, mean demand is also affected by the reference price (r). Hence the demand is determined by:

$$D_t(p, r, \theta_t, \epsilon_t) = \hat{d}_t(p, r)\theta_t + \epsilon_t,$$

where $D_t(p, r, \theta_t, \epsilon_t)$ and $\hat{d}_t(p, r)$ are the demand and the mean demand functions of period t , respectively. Here θ_t and ϵ_t are independent random variables with $E[\theta_t] =$

1 and $E[\epsilon_t] = 0$ and they stand for the multiplicative and additive random terms, respectively. The variance of the demand is affected by the pricing decisions through the multiplicative term.

For the definition of reference effect, we follow (Popescu and Wu, 2007). *Reference effect* on demand is defined as the excess or lost demand due to the change of the reference price at a given price level and is captured by the following functional relationship:

$$\nu_t(p) + F_t(r - p, r) = \hat{d}_t(p, r), \quad (3.1)$$

where $\nu_t(p) = \hat{d}_t(p, p)$ is called the base demand and $F_t(r - p, r)$ is called the reference effect on demand. In a demand model with reference effects, customers decide to buy the product comparing the reference price r and the price p . Hence the larger the difference, the greater the impact. This is known as *reference dependence*. We have the following assumption for the mean demand:

Assumption 3.1. (i) $\hat{d}_t(p, r)$ is bounded, non-negative and continuous, strictly decreasing with p and increasing with r . The corresponding reference effect $F_t(\delta, r)$ is increasing with δ . (ii) The sum of the mean demand and the additive random term of distinct periods is non-negative, i.e. $\hat{d}_t(p, r) + \epsilon_{t'} \geq 0$ for any t and t' .

$F_t(r - p, r) = 0$ for $r = p$ by the definition of reference effect. Thus $F_t(r - p, r) \geq 0$ for $r > p$ and $F_t(r - p, r) \leq 0$ for $r < p$ since $F_t(\delta, r)$ is increasing with δ . Furthermore $F_t(r - p, r)$ is increasing with r and decreasing with p . All of the examples in (Popescu and Wu, 2007), (Gimpl-Heersink, 2008), (Zhang, 2010) and (Chen *et al.*, 2011) obey Assumption 3.1(i). Although Assumption 3.1(ii) seems restrictive, it reduces to nonnegativity of the demand when the random term is stationary over time, as in the case of an infinite horizon problem. It holds whenever the sum of the minimum of the mean demand and the minimum of the random term is non-negative. We will be explicit about the assumption requirements of the results throughout the study.

We assume that the price and the reference price are bounded with a lower and an upper bound. Let p^l and p^u denote these lower and the upper bounds, respectively. There is a linear ordering cost c_t and we assume that the price cannot be smaller than the production cost, i.e. $p^l \geq c_t$ for every t . Let $\mathbb{P}$ denote the set of the feasible price and the reference price pairs. Then $\mathbb{P} = \{(p, r) : p \in [p^l, p^u], r \in [p^l, p^u]\}$.

Customers may react differently to losses and gains. Consider the following mean demand function:

$$\hat{d}_t(p, r) = a_0 - a_1 p^n + a_2 \frac{(r-p)^+}{\kappa(r)} - a_3 \frac{(p-r)^+}{\kappa(r)}, \quad n > 1 \quad (3.2)$$

where $(x)^+$ denotes $\max(x, 0)$ with $a_0 \geq 0$, $a_1 \geq 0$, $a_2 \geq 0$, $\nu(p) = a_0 - a_1 p^n$ and $F_t(r-p, r) = a_2 \frac{(r-p)^+}{\kappa(r)} - a_3 \frac{(p-r)^+}{\kappa(r)}$ and $\kappa(r)$ is positive function of r . If the current price is larger than the reference price, the demand decreases by an amount of $a_3 \frac{(p-r)}{\kappa(r)}$. However if the price is below the reference price, then the demand increases by an amount of $a_2 \frac{(r-p)}{\kappa(r)}$. In general, defining $\chi_-(r)$ and $\chi_+(r)$ as the left and right derivatives of $F_t(\delta, r)$ at $\delta = 0$, let $\tilde{\chi}(r) = \frac{\chi_-(r)}{\chi_+(r)}$. The customers are loss-averse if $\tilde{\chi}(r) > 1$. In this example, $a_2 < a_3$ means loss-aversion. Customers are loss-neutral if $\tilde{\chi}(r) = 1$ ($a_2 = a_3$) and loss-seeking if $\tilde{\chi}(r) < 1$ ($a_2 > a_3$) (Köbberling and Wakker, 2005).

There are two models in the literature for modeling the reference effects on demand. In the ADRE model, the reference effect is increasing with the difference $r-p$ and has a zero effect when $p=r$. In general $R_{AD}(\delta, r) = g(\delta)$, with g increasing and $g(0) = 0$. The example in Equation 3.2 employs an ADRE model with $g(r-p) = a_2(r-p)^+ - a_3(p-r)^+$ whenever $\kappa(r) = 1$. A major drawback of the ADRE model is that the change in the demand is the same at every price level for the same value of difference.

RDRE model in which the reference effect is increasing with the relative difference $(r-p)/r$ is another model that is not subject to this drawback. In general $R_{RD}(\delta, r) = g(\delta/r)$, with g increasing and $g(0) = 0$. For example, the model in Equation 3.2 employs RDRE model for the reference effect when $\kappa(r) = r$. Now the reference effect

is not contingent on the absolute difference but on the relative difference of price and reference price.

Customers develop a common reference price using the price history of a product, hence with announcement of new price levels during successive periods, they update their reference price. A commonly used model for the evolution of the reference price is the exponential smoothing model (Mazumdar *et al.*, 2005):

$$r_{t+1} = \alpha r_t + (1 - \alpha)p_t. \quad (3.3)$$

α is the memory parameter and $0 \leq \alpha \leq 1$. The larger the memory parameter, the longer the memory. If α is high, then customers have a long memory and past price effect is larger. If α is small, then current price has a greater effect than the past on the reference price.

3.2. Dynamic Joint Inventory and Pricing Model

In this section we formulate the problem using dynamic programming. Consider a setting where a price setting firm faces stationary stochastic demand through successive periods and has to decide on the order quantity and the price. The demand in a period is dependent on the price and the reference price. The reference price mimics a price expectation, or a standard price, for that product and evolves with the exponential smoothing model given in Equation 3.3. The randomness in demand is due to the additive random and the multiplicative random term. Excess demand is backlogged. There are holding and backorder costs and a linear production cost. The order of events is as follows: At the start of the horizon, the firm observes (or predicts) the initial reference price level. At the beginning of every period the firm observes the initial inventory level (x) and the reference price (r). Then it decides on its inventory level (y) and a price (p) simultaneously. Ordering cost (c) is incurred per unit item. Then the demand is realized and unsatisfied demand is backlogged. The firm collects revenues (if any) and incurs holding and backorder costs according to the inventory level at the end of the period. If $H_t(\cdot)$ denotes the holding and backorder cost for

period t , the expected holding and backorder cost is given by:

$$\hat{G}_t(y, p, r) = E[H_t(y - \hat{d}_t(p, r)\theta_t - \epsilon_t)].$$

This problem can be formulated as a dynamic program and the Bellman equations for this multi-period problem can be written as follows:

$$\begin{aligned} \hat{J}_t(y, p, r) &= p\hat{d}_t(p, r) - c_t y - \hat{G}_t(y, p, r) \\ &\quad + \gamma E \left[\hat{V}_{t+1} \left(y - \hat{d}_t(p, r)\theta_t - \epsilon_t, \alpha r + (1 - \alpha)p \right) \right] \\ \hat{V}_t(x, r) &= \max_{\substack{y, p \\ (p, r) \in \mathbb{P}, y \geq x}} \left\{ \hat{J}_t(y, p, r) \right\} + c_t x \end{aligned} \tag{3.4}$$

where $\hat{V}_t$ is the profit-to-go function and $\hat{J}_t$ is the value function of the period t and $\gamma \in [0, 1]$ is the discount rate. The terminal value is given by $\hat{V}_{T+1}(x, r) = c_{T+1}x$. Let I^* denote the system that is subject to the problem given in Equation 3.4, i.e. the joint inventory and pricing problem with reference effects of a single item with stochastic demand where the randomness is due to an additive and a multiplicative random term. In the next section we study the demand models with an additive random term only and in Section 3.4 we analyze the demand models with both an additive and a multiplicative random term.

3.3. Demand Models with an Additive Random Term

In this section we study the problem in Equation 3.4 with an additive random term, i.e. $\theta \equiv 1$. We analyze the optimal pricing and inventory policy and characterize the optimal policy parameters. We make a variable transformation in order to facilitate the analysis of the problem. Let Δ_t denote the safety stock of period t , i.e. $\Delta_t = y - \hat{d}_t(p, r)$. The problem is formulated in such a way that, instead of deciding the order-up-to level, the firm decides on the quantity of the safety stock. For a similar transformation please see (Ceryan *et al.*, 2011). Note that, the existence of the multiplicative term does not allow us to model the problem using the safety stock variable

Δ . The problem can be reformulated as follows:

$$\begin{aligned}\phi_t(\Delta, p, r) &= (p - c_t)\hat{d}_t(p, r) - c_t\Delta - \hat{G}_t(\Delta) + \gamma E\left[\hat{V}_{t+1}(\Delta - \epsilon, \alpha r + (1 - \alpha)p)\right] \\ \hat{V}_t(x, r) &= \max_{\substack{\Delta, p \\ (p, r) \in \mathbb{P}, \hat{d}_t(p, r) + \Delta \geq x}} \left\{ \phi_t(\Delta, p, r) \right\} + c_t x\end{aligned}\quad (3.5)$$

where ϕ_t is the value function with the safety stock variable Δ and $\hat{G}_t(\Delta) = \hat{G}_t(\hat{d}_t(p, r) + \Delta, p, r)$. The terminal value is given by $\hat{V}_{T+1}(x, r) = c_{T+1}x$. Note that with the introduction of Δ , $\hat{G}_t$ turns out to be dependent only on Δ since one can write $H_t(y - \hat{d}(p, r) - \epsilon) = H_t(\Delta - \epsilon)$. The notation of $\hat{G}$, as $\hat{G}(y, p, r)$ or $\hat{G}(\Delta)$, will be clear from the context.

In order to have a zero terminal value, we define $\Phi_t(x, r) = \hat{V}_t(x, r) - c_t x$. The problem in Equation 3.5 can now be rewritten as:

$$\begin{aligned}\phi_t(\Delta, p, r) &= \Pi_t(p, r) - G_t^\gamma(\Delta) + \gamma E\left[\Phi_{t+1}(\Delta - \epsilon_t, \alpha r + (1 - \alpha)p)\right] \\ \Phi_t(x, r) &= \max_{\substack{\Delta, p \\ (p, r) \in \mathbb{P}, \hat{d}_t(p, r) + \Delta \geq x}} \left\{ \phi_t(\Delta, p, r) \right\}\end{aligned}\quad (3.6)$$

with terminal value $\Phi_{T+1} = 0$. Here $G_t^\gamma(\Delta) = \hat{G}_t(\Delta) + (c_t - \gamma c_{t+1})\Delta$ and $\Pi_t(p, r) = (p - c_t)\hat{d}_t(p, r)$. $G_t^\gamma(\Delta)$ and $\Pi_t(p, r)$ are called the *modified inventory related cost* and *modified revenue*, respectively. The single period profit function is now composed of two parts: the first one is the modified revenue function $\Pi_t(p, r)$ and is dependent on the price and the reference price. The second part $G_t^\gamma(\Delta)$ is the modified inventory related cost which depends on only the safety stock Δ .

3.3.1. Decomposition and Optimal Safety Stock

In this section, we show that the joint inventory and pricing problem can be decomposed into two subproblems if the starting inventory is low and safety stocks are non-decreasing over time. We follow the definition of (Chen *et al.*, 2011) and define a system where return of the products are allowed by relaxing the inventory constraint

$\hat{d}_t(p, r) + \Delta \geq x$. Let I_a^* denote the original system with the problem defined in Equation 3.6 and I^0 denote the system with the relaxed problem that is given in the following:

$$\begin{aligned}\phi_t^0(\Delta, p, r) &= \Pi_t(p, r) - G_t^\gamma(\Delta) + \gamma E\left[\Phi_{t+1}^0(\Delta - \epsilon_t, \alpha r + (1 - \alpha)p)\right] \\ \Phi_t^0(x, r) &= \max_{\substack{\Delta, p \\ (p, r) \in \mathbb{P}}} \left\{ \phi_t^0(\Delta, p, r) \right\}.\end{aligned}\tag{3.7}$$

Note that the constraint $\hat{d}_t(p, r) + \Delta \geq x$ is removed in the problem given in Equation 3.7. The dependence of the transformed profit-to-go function, Φ_t^0 , on the variable x is via the inventory constraint. Hence removing the inventory constraint enables us to write $\Phi_t^0(x, r)$ as $\Phi_t^0(r)$ (Note that this is not possible if $\hat{V}_t(x, r)$ is used in the recursion). With the elimination of x , the expectation on the last term of ϕ_t^0 can be removed. Therefore the problem in Equation 3.7 can be divided into two subproblems. First problem is a multi-period pricing problem with a deterministic demand with reference effects. Second problem is a single period inventory problem where optimal safety stock minimizes $G_t^\gamma(\Delta)$. Let Δ_t^0 denote the optimal safety stock which is given by $\Delta_t^0 = \arg \min G_t^\gamma(\Delta)$. For example, if the holding and backorder costs are linear and given by h and b per unit, respectively, then $\Delta_t^0 = F^{-1}\left(\frac{b - (c_t - \gamma c_{t+1})}{h + b}\right)$ where F is the distribution function of the random variable ϵ_t . The problem in Equation 3.7 further simplifies to:

$$\begin{aligned}\phi_t^0(p, r) &= \left[\Pi_t(p, r) + \gamma \Phi_{t+1}^0(\alpha r + (1 - \alpha)p) \right] - G_t^\gamma(\Delta_t^0) \\ \Phi_t^0(r) &= \max_{\substack{p \\ (p, r) \in \mathbb{P}}} \left\{ \phi_t^0(p, r) \right\}.\end{aligned}\tag{3.8}$$

Let $p_t^0(r)$ denote the optimal price, $d_t^0(r) = \hat{d}(p_t^0(r), r)$ denote the optimal mean demand, $\Delta_t^0(r)$ denote the optimal safety stock and $y_t^0(r) = d_t^0(r) + \Delta_t^0(r)$ denote the optimal inventory level after ordering (order-up-to level) for the I^0 system (equivalently for the problem in Equation 3.8). Note that $\Delta_t^0(r)$ is independent of the reference price of the current period and can be denoted by Δ_t^0 . Also let x_t denote the initial inventory level in period t . We show that if the initial inventory level x_1 is low and if

Δ_t^0 is non-decreasing, then the optimal decisions for I^0 are also optimal for I_a^* when the time horizon is finite.

Theorem 3.1. *Given Assumption 3.1, if (i) $d_1^0(r) + \Delta_1^0 \geq x_1$ and (ii) $\Delta_{t+1}^0 \geq \Delta_t^0$ for $t = 1, \dots, T$, then $(p_t^0(r), \Delta_t^0)$ for $t = 1, \dots, T$ are optimal for the system I_a^* in periods $t = 1, \dots, T$.*

Proof. See Appendix A.1. □

We now give examples of cases when Δ_t^0 is increasing over time. Assume that $H_t(x)$ is convex. Then $G_t^\gamma(\Delta)$ is convex and the value that minimizes $G_t^\gamma(\Delta)$ is found by solving the first order condition $\hat{G}'(\Delta) = \gamma c_{t+1} - c_t$, where $\hat{G}'$ is the first order derivative of $\hat{G}$. If the inventory related cost is stationary over time, i.e. $H_t = H$ and the production costs satisfy $\gamma(c_{t+1} - c_t) \geq c_t - c_{t-1}$, then $\Delta_{t+1}^0 \geq \Delta_t^0$ since $\hat{G}'(\Delta) = \gamma c_{t+1} - c_t$ and $\hat{G}'$ is increasing. Note that one can generalize the constraint $\gamma(c_{t+1} - c_t) \geq c_t - c_{t-1}$ to the following:

$$(c_{t+1+\tau} - c_{t+\tau}) \geq (c_t - c_{t-1})/\gamma^\tau \quad (3.9)$$

for $t = 1, \dots, T$ and $\tau = 0, \dots, T - t$. The constraint in Equation 3.9 is satisfied if production cost is constant throughout the horizon. If the production costs are decreasing with time, then the constraint is getting more relaxed over time but if the production costs are increasing with time, then the constraint gets tighter.

We now show that the result of Theorem 3.1 holds in the infinite horizon model as well. We assume that, all costs, revenue parameters and random terms distributions are stationary when the time horizon is infinite. We have $c_t = c$, $\hat{G}_t = \hat{G}$ and $d_t(\cdot) = d(\cdot)$ for all $t = 1, 2, \dots$, and all ϵ_t are independent and identically distributed as a random variable ϵ . Let $M = \max_{p, r \in \mathbb{P}} p \hat{d}(p, r)$. Since $\hat{d}(p, r)$ is bounded by Assumption 3.1(i) and $\mathbb{P}$ is bounded by definition, $M < \infty$.

Let $p^0(r)$ denote the optimal price and $d^0(r) = \hat{d}(p^0(r), r)$ denote the optimal

mean demand for the I^0 system when the planning horizon is infinite. Also let r_1 be the initial reference price. Since $G_t^\gamma = G^\gamma$, the optimal safety stock is constant throughout the horizon. Let Δ^0 denote this optimal safety stock which is given by $\Delta^0 = \arg \min G^\gamma(\Delta)$. Assumption 3.1(ii) reduces to nonnegativity of the demand, i.e. $D(p, r, \epsilon) \geq 0$ since $\epsilon_t = \epsilon$. We have the following theorem:

Theorem 3.2. *Given Assumption 3.1, if $d^0(r_1) + \Delta^0 \geq x_1$ then $p^0(r)$ is the optimal price and Δ^0 is the optimal safety stock for the system I_a^* .*

Proof. The proof is similar to that of Theorem 3.1. Hence it is omitted. $\square$

(Popescu and Wu, 2007) study pricing with reference effects when demand is deterministic, hence their results apply to our subproblem of pricing with reference effects. Next, we give their result for the loss-neutral customers without proof. Please refer to (Popescu and Wu, 2007) for more detailed analysis of the pricing with reference effects under deterministic demand problem. Let $\Pi(p, r) = B(p) + \Pi^R(p, r)$. Here $B(p) = (p - c)\nu(p)$ and $\Pi^R(p, r) = pF(r - p, r)$. Also let $\chi(r)$ denote the derivative of $F(\delta, r)$ with respect to δ at $\delta = 0$. The subscripts denote the partial derivatives with respect to the variable in the subscript.

Assumption 3.2. *(Popescu and Wu, 2007) (i) $B(p)$ is non-monotonic and concave in p . (ii) $\Pi_p(r, r) = B'(r) - (r - c)\chi(r)$ is strictly decreasing with r . (iii) $\Pi^R(p, r)$ is concave in p and supermodular in (p, r) . (d) $F(\delta, r)$ is twice differentiable in x and r .*

Theorem 3.3. *(Popescu and Wu, 2007) Given Assumptions 3.1 and 3.2, the problem in Equation 3.6 admits a unique steady state p^{0*} to which all optimal price paths converge. All corresponding reference price paths converge to p^{0*} monotonically. The steady state price p^{0*} is the unique solution of:*

$$\frac{B'(p)}{1 - \gamma} = \frac{p\chi(p)}{1 - \alpha\gamma} \quad (3.10)$$

Furthermore, p^{0*} is continuously decreasing with α , increasing in γ and satisfies $p^{M*} \leq p^{0*} \leq \bar{p}$ where p^{M*} is the steady state of a myopic firm, i.e. $\gamma = 0$.

The reference effect vanishes at the steady state since $r^{0*} = p^{0*}$. Let d^{0*} denote the steady state mean demand and y^{0*} denote the steady state order-up-to level. The optimal mean demand converges to a unique steady state which is given by $d^{0*} = \hat{d}(p^{0*}, p^{0*})$. Similarly $y^{0*} = d^{0*} + \Delta^0$ is the optimal steady state order-up-to level.

As a summary, we show that for *any* joint inventory and pricing problem with an additive random term, if the (i) initial inventory on hand is low, (ii) the single period safety stocks are non-decreasing and (iii) the sum of the mean demand and the random term is non-negative, then the problem can be decomposed into two subproblems: a multi period pricing problem under deterministic demand with reference effects and a single period inventory problem. These two problems are easier to solve than the original one. The single period problem has a closed form solution under linear holding and backorder costs. The second problem is a deterministic dynamic programming problem with a single state and a single action variable. Solution of this problem is much easier when compared with the original problem in Equation 3.7 which is a stochastic dynamic programming problem with two decision variables and two state variables. When the time horizon is infinite, non-decreasing safety stock condition becomes trivial and the second condition of Assumption 3.1 reduces to non-negativity of the realized demand in a period. Moreover, we show that the problem converges to a steady state and the solutions for the steady state can be characterized by Theorem 3.2 and Theorem 3.3.

For the remaining cases (when the initial inventory is high or optimal safety stocks are not monotonic) one may expect that the system does not place an order and waits until the on-hand inventory level drops below the optimal order-up-to level. In order to show this, one must show that an SDO policy is optimal.

3.3.2. Optimality of the Order-up-to Policy

A sufficient condition for the optimality of an SDO policy for the problem in Equation 3.6 is concavity of the value function ϕ_t . In this section we show that an SDO policy is optimal for concave demand models with concave modified revenue functions

when the time horizon is finite or infinite. The following additional assumptions are required to show the concavity of ϕ_t :

Assumption 3.3. (i) $H_t(x)$ is convex¹ (ii) $\lim_{|x| \rightarrow \infty} H_t(x) = \infty$ for all $t = 1, \dots, T$.

Assumption 3.4. (i) $\hat{d}_t(p, r)$ is concave on $\mathbb{P}$ for all $t = 1, \dots, T$. (ii) $\Pi_t(p, r)$ is concave on $\mathbb{P}$ for all $t = 1, \dots, T$.

We first give the results for the finite horizon case.

Theorem 3.4. *Given Assumptions 3.1 (i), 3.3(i) and 3.4, the optimal ordering policy for the problem in Equation 3.6 is given as follows. Let (i) $(\Delta_t^*(r), p_t^*(r))$ be the maximizer of $\phi_t(\Delta, p, r)$ over $\mathbb{P}$ for a given r , (ii) $d_t^*(r) = \hat{d}_t(p_t^*(r), r)$ and (iii) $y_t^*(r) = d_t^*(r) + \Delta_t^*(r)$. If $x \leq y_t^*(r)$, then the firm orders $y_t^*(r) - x$ many units and charges the price $p_t^*(r)$. If $x > y_t^*(r)$, then the firm orders nothing and charges the price $p_t^*(x, r)$ which is the maximizer of $\phi_t(x - \hat{d}_t(p, r), p, r)$ over $\mathbb{P}$ for a given r .*

Proof. See Appendix A.2. □

We now extend this result to the infinite horizon case. In order to do this, we make a transformation of the single-period profit function. In particular, we subtract M from each of ϕ_t to have a non-positive single-period profit function. For a similar transformation, please refer to (Federgruen and Heching, 1999). Let Φ_t^* and ϕ_t^* be transformed profit-to-go and value functions, i.e:

$$\Phi_t^* = \Phi_t - M \frac{1 - \gamma^{t+1}}{1 - \gamma} \quad \phi_t^* = \phi_t - M \frac{1 - \gamma^{t+1}}{1 - \gamma}. \quad (3.11)$$

In order to facilitate the analysis, the periods are relabeled such that the index i shows

¹In this study, unless stated otherwise, by concavity (convexity) of a function we mean joint concavity (convexity) of that function in all of its variables.

number of periods to go, i.e. $i = T - t$. Then Φ_i^* and ϕ_i^* satisfy the following recursion:

$$\begin{aligned} \phi_i^*(\Delta, p, r) &= (p - c_i) \hat{d}_i(p, r) - \hat{G}_i^\gamma(\Delta) - M + \gamma E \left[\Phi_{i-1}^*(\Delta - \epsilon, \alpha r + (1 - \alpha)p) \right] \\ \Phi_i^*(x, r) &= \max_{\substack{\Delta, p \\ (p, r) \in \mathbb{P}, \hat{d}_i(p, r) + \Delta \geq x}} \left\{ \phi_i^*(\Delta, p, r) \right\} \end{aligned} \quad (3.12)$$

The following theorem shows the optimality of the SDO policy for the infinite horizon case:

Theorem 3.5. *Given Assumptions 3.1(i), 3.3 and 3.4; (i) The limits $\Phi^* = \lim_{i \rightarrow \infty} \Phi_i^*$, $\Phi = \lim_{i \rightarrow \infty} \Phi_i$, $\phi^* = \lim_{i \rightarrow \infty} \phi_i$ and $\phi = \lim_{i \rightarrow \infty} \phi_i$ all exist. (ii) Φ and ϕ satisfy the optimality equation in Equation 3.6 and Φ^* and ϕ^* satisfy the optimality equation in Equation 3.12. (iii) ϕ and Φ are concave. (iv) the optimal ordering policy for the problem in Equation 3.6 is given as follows. Let (i) $(\Delta^*(r), p^*(r))$ be the maximizer of $\phi(\Delta, p, r)$ over $\mathbb{P}$ for a given r , (ii) $d^*(r) = \hat{d}(p^*(r), r)$ and (iii) $y^*(r) = d^*(r) + \Delta^*(r)$. If $x \leq y^*(r)$, then the firm orders $y^*(r) - x$ many units and charges the price $p^*(r)$. If $x > y^*(r)$, then the firm orders nothing and charges the price $p^*(x, r)$ where $p^*(x, r)$ is the maximizer of $\phi(x - \hat{d}(p, r), p, r)$ over $\mathbb{P}$ for a given r .*

Proof. See Appendix A.3. □

Calculation of the optimal policy parameters $p^*(r)$ and $\Delta^*(r)$ is what we take up next. The following theorem shows that the optimal policy parameters of the system I_a^* are equal to the optimal parameters of the I^0 system. (Chen *et al.*, 2011) provide the same result for the linear demand models.

Theorem 3.6. *Given Assumptions 3.1, 3.3 and 3.4; $\Delta^*(r) = \Delta^0$ and $p^*(r) = p^0(r)$.*

Proof. See Appendix A.4. □

We now give examples where the concavity assumption holds. Consider the demand model in Equation 3.2 where customers are loss-neutral with a concave base

demand $\nu_t(p)$:

$$\hat{d}_t(p, r) = \nu_t(p) + a_t \frac{r - p}{\kappa(r)} \quad (3.13)$$

One can show that Assumption 3.4 holds if an RDRE model ($\kappa(r) = r$) is employed with $c_t = 0$. Assumption 3.4 also holds with an ADRE model ($\kappa(r) = 1$). To the best of our knowledge, this is the first study that shows the optimality of an SDO policy for a demand model with concave base demand with ADRE model. For details, please see Appendix A.5.

(Gimpl-Heersink, 2008), (Zhang, 2010) and (Chen *et al.*, 2011) study linear demand models. We show that an SDO policy is optimal for a demand model with concave revenue and with ADRE or RDRE when there is an additive random term. Our result holds when the planning horizon is finite or infinite. This is, to the best of our knowledge, not shown before in the literature. We also show that the optimal solution of the problem in Equation 3.6 is given by the optimal solution of the subproblems. (Chen *et al.*, 2011) provide the same result for linear demand models.

3.3.3. Characterization of the Optimal Policy Parameters

In this section, we show the order-up-to level for the relaxed problem in Equation 3.8 is increasing with the reference price when the planning horizon is infinite. When the planning horizon is finite, we show that the same result holds if on-hand inventory is low. First we give the following lemma:

Lemma 3.1. *Given Assumption 3.1(i), $\Phi_t(x, r)$ is decreasing with x and increasing with r .*

Proof. The proof follows from (Zhang, 2010). Since a larger x gives a smaller feasible region for the optimization, $\Phi_t(x, r)$ is decreasing with x . The proof of the second part is by induction. Let p^* be the optimal price for the states x, r . For a small $\delta > 0$, let

$r' = r + \delta$. One can find a $p' > p^*$ such that $d_t(p', r') = \hat{d}_t(p^*, r)$. Then:

$$\begin{aligned}\Phi_T(x, r') &\geq (p' - c_t)\hat{d}_t(p', r') - G_t^\gamma(\Delta) + \gamma\Phi_{T+1}\left(\Delta - \epsilon_T, \alpha r' + (1 - \alpha)p'\right) \\ &\geq (p^* - c_t)\hat{d}_t(p^*, r) - G_t^\gamma(\Delta) + \gamma\Phi_{T+1}\left(\Delta - \epsilon_T, \alpha r + (1 - \alpha)p^*\right) = \Phi_T(x, r)\end{aligned}$$

where the first inequality is by definition and the second one follows from the fact that $\Phi_0 = 0$. Hence $\Phi_1(x, r)$ is increasing with r . The proof for $\Phi_t(x, r)$ is similar and uses induction in the second inequality. $\square$

Now we transform the problem in such a way that the mean demand becomes the decision variable instead of the price. Every period starts with a given reference price r . The mean demand $\hat{d}_t(p, r)$ is assumed to be strictly decreasing with p by Assumption 3.1 (i). Hence for a given r , there is a one-to-one mapping between the mean demand and the price. Therefore price can be written as a function of the reference price and the mean demand as $\hat{p}_t(d, r)$ where $\hat{p}$ is the inverse function of the mean demand for a given r . By this mapping, choosing d corresponds to setting a price value through $\hat{p}_t(d, r)$ at the reference price r . Let $\mathbb{D}_t = \{(d, r) : \hat{p}_t(d, r) \in [p^l, p^u], r \in [p^l, p^u]\}$ be the set of d and r defined by the bounds. Since the set depends on the price function $\hat{p}_t(d, r)$, it can change throughout the periods. Note that this set is the image set of the function $\hat{d}_t(p, r)$ together with r . For every value of r , there is one to one correspondence between the variables d and p and between the sets $\mathbb{D}_t$ and $\mathbb{P}$. Then the problem can be written as:

$$\begin{aligned}\hat{\phi}_t^0(d, r) &= \left[\Pi_t(\hat{p}_t(d, r), r) + \gamma\Phi_{t+1}^0(\alpha r + (1 - \alpha)\hat{p}_t(d, r)) \right] - G_t^\gamma(\Delta_t^0) \\ \Phi_t^0(r) &= \max_{\substack{d \\ (d, r) \in \mathbb{D}_t}} \left\{ \hat{\phi}_t^0(d, r) \right\}\end{aligned}\tag{3.14}$$

with terminal value $\Phi_{T+1}^0(x, r) = 0$. For a similar transformation, see (Zhang, 2010) and (Chen *et al.*, 2011). Before stating the main result, we provide some properties of the price function.

Proposition 3.1. *Given Assumption 3.1(i), $\hat{p}_t(d, r)$ is strictly decreasing with d and*

increasing with r for $t = 1, \dots, T$.

Proof. For $\delta > 0$ and a pair $\bar{p}, \bar{r} \in \mathbb{P}$, define $d_1 = \hat{d}_t(\bar{p}, \bar{r})$ and $d_2 = \hat{d}_t(\bar{p} + \delta, \bar{r})$. Then $d_1 > d_2$ since demand is strictly decreasing with p (Assumption 3.1). Define $p_1 = \hat{p}_t(d_1, \bar{r}) = \bar{p}$ and $p_2 = \hat{p}_t(d_2, \bar{r}) = \bar{p} + \delta$ which implies that for a given $\bar{r}$, if $d_1 > d_2$, then $p_2 > p_1$ since $\delta > 0$.

Now define $d_3 = \hat{d}_t(\bar{p}, \bar{r} + \delta)$. Clearly $d_3 \geq d_1$ since demand is increasing with r (Assumption 3.1(i)). Let $\Delta = d_3 - d_1$. One can write $\bar{p} = \hat{p}_t(d_1, \bar{r})$ and also $\bar{p} = \hat{p}_t(d_3, \bar{r} + \delta) = \hat{p}_t(d_1 + \Delta, \bar{r} + \delta)$. Then $\hat{p}_t(d_1, \bar{r}) = \hat{p}_t(d_1 + \Delta, \bar{r} + \delta) \leq \hat{p}_t(d_1, \bar{r} + \delta)$ since $\hat{p}_t(d, r)$ is strictly decreasing with d . $\square$

We will show that $\hat{d}_t^0(r_t)$, hence $y_t^0(r)$ is increasing with r . In order to show this, we need supermodularity of the price function.

Assumption 3.5. $\hat{p}_t(d, r)$ is supermodular in (d, r) for $t = 1, \dots, T$.

Theorem 3.7 shows the result for the finite horizon problem.

Theorem 3.7. (i) Given Assumption 3.1 (i), $\mathbb{D}_t$ is a lattice. (ii) Given Assumptions 3.1 (i), 3.4 and 3.5, $\hat{\phi}_t^0(d, r)$ is supermodular and $y_t^0(r)$ is nondecreasing with r for $t = 1, \dots, T$.

Proof. See Appendix A.6. $\square$

Note that we do not need the concavity of the holding cost since Theorem 3.7 is a result for the I^0 system. Theorem 3.7 states that, for the finite horizon case, the optimal order-up-to level of the system I^0 is nondecreasing with the reference price. Hence optimal order-up-to level of the system I_a^* is nondecreasing with r whenever $y_t^*(r) \geq x$ since the optimal policy of the system I_a^* is to mimic the system I^0 when $y_t^0(r) \geq x$. Following statement is the corollary of Theorem 3.6 and Theorem 3.7. It

shows that $y^*(r)$ is nondecreasing with r even if $y_t^0(r) < x$ for the infinite horizon problem.

Corollary 3.1. *Given Assumptions 3.1, 3.3, 3.4 and 3.5, $y^*(r)$ is nondecreasing with r .*

Proof. One can show that $y^0(r)$ is nondecreasing with r in similar way to the one given in the proof of Theorem 3.7. Since $y^*(r) = y^0(r)$ from Theorem 3.6, $y^*(r)$ is nondecreasing with r . $\square$

(Chen *et al.*, 2011) prove the same result for linear demand models when the time horizon is finite or infinite. A technical condition on the demand function to have a supermodular price function is given in Appendix A.7.

3.3.4. Customer Behaviour

Until now, we did not impose any conditions on customer behaviour like loss-neutrality or loss-aversion except Theorem 3.3. In this section, we show that our results hold for loss-averse customers as well as loss-neutral customers. In particular we provide demand examples where customers are loss-averse and show that the results of the theorems hold under this customer type.

Theorem 3.1 and Theorem 3.2 need no further assumptions, hence they are valid for any demand function. (Popescu and Wu, 2007) extended the result of Theorem 3.3 to the case where the customers are loss-averse. For details, please refer to Theorem 4 of (Popescu and Wu, 2007).

Theorems 3.4 to 3.7 and Corollary 3.1 need Assumption 3.4 which states that the demand and the modified revenue functions are concave. Consider the demand model in Equation 3.2. The mean demand and the modified revenue of this model can be

written as follows:

$$\begin{aligned}\hat{d}_t(p, r) &= \min \left(\nu(p) + a_t^1 \frac{(r-p)}{\kappa(r)}, \nu_t(p) + a_t^2 \frac{(r-p)}{\kappa(r)} \right) \\ \Pi_t(p, r) &= \min \left((p - c_t) \left(\nu_t(p) + a_t^1 \frac{(r-p)}{\kappa(r)} \right), (p - c_t) \left(\nu_t(p) + a_t^2 \frac{(r-p)}{\kappa(r)} \right) \right)\end{aligned}$$

whenever the customers are loss-averse, i.e. $a_t^1 \leq a_t^2$ for $t = 1, \dots, T$. Hence if $\nu_t(p) + a_t^i \frac{r-p}{\kappa(r)}$ is concave for $i = 1, 2$, then the mean demand and the modified revenue are concave since minimum operator preserves concavity. In general, consider the following assumption from (Popescu and Wu, 2007):

Assumption 3.6. (*Popescu and Wu, 2007*) *The kinked reference effect is given by*

$$F_K(\delta, r) = 1_{\delta \geq 0} F_G(\delta, r) + 1_{\delta \leq 0} F_L(\delta, r) \quad (3.15)$$

where $F_L(\delta, r)$ and $F_G(\delta, r)$ are twice differentiable, increasing with δ with $F_G(\delta, r) - F_L(\delta, r)$ single crossing in δ , and satisfying $\chi_L(r) > \chi_G(r)$ for all r .

If the reference effect satisfies Assumption 3.6 and the mean demands $\nu(p) + F_G(r - p, r)$ and $\nu(p) + F_L(r - p, r)$ satisfy Assumption 3.4, then $\nu(p) + F_K(r - p, r)$ satisfies Assumption 3.4. Hence optimality of the SDO policy holds for loss-averse customers as well.

3.4. Demand Models with an Additive and a Multiplicative Random Term

In this section, we analyze demand models with an additive and a multiplicative random term. The existence of the multiplicative term does not allow us to model the problem using the safety stock variable Δ . In order to enable the analysis, we work on a larger action space: we re-define $\mathbb{P}$. The general assumption on the bounds of the price and the reference price is that they are bounded with a lower and an upper bound, i.e. $(p, r) \in [\underline{p}, \bar{p}]$ where $\underline{p}$ and $\bar{p}$ are the lower and upper bounds respectively (Popescu and Wu, 2007; Gimpl-Heersink, 2008; Zhang, 2010; Chen *et al.*, 2011). For models where the mean demand is dependent only on the price, setting an upper(lower) bound to the

price is equivalent to setting a lower(upper) bound on the mean demand. The general way is to set a lower bound on the price such that the demand is always non-negative. However for models where the mean demand is dependent also on the reference price, the mean demand can assume different values at the same price level. Hence it is more convenient to set a lower bound on the mean demand instead of imposing an upper bound on the price for such settings. Therefore, in this section, we assume that the price has a non-negative lower bound p^l and the mean demand has a non-trivial and non-negative lower bound d^l , i.e. $\hat{d}_t(p, r) \geq d^l$. By non-triviality we mean that the mean demand attains the lower bound at some p and r .

We assume that the reference price has an upper bound r^u since the price perception of the buyers of a product should be finite. The reference price is the price expectation of the customers, hence we also assume that it is bounded by the same lower bound with the price, that is $r \geq p^l$. Hence, from now on we re-define the feasible set of the price and the reference price as $\mathbb{P}_t = \{(p, r) : p \geq p^l, r \in [p^l, r^u], \hat{d}(p, r) \geq d^l\}$. Since the set depends on the demand function $\hat{d}(p, r)$, it depends on that particular period. This is also the case in finite horizon problems (Federgruen and Heching, 1999; Chen and Simchi-Levi, 2004a). The corresponding $\mathbb{D}_t$ can be written as $\mathbb{D}_t = \{(d, r) : \hat{p}_t(d, r) \geq p^l, r \in [p^l, r^u], d \geq d^l\}$.

In order to show the optimality of an order-up-to policy, a sufficient condition is to show the concavity of the value function $\hat{J}_t(\cdot)$ and the profit-to-go function $\hat{V}_t(\cdot)$. A sufficient condition for the concavity of $\hat{J}_t(\cdot)$, which is frequently used in inventory research, is to show the concavity of the revenue function and the convexity of the expected inventory cost function $\hat{G}_t(\cdot)$. The only way for $\hat{G}_t(\cdot)$ to be convex in (p, r) when there is convex inventory cost $H_t(\cdot)$ is to have a mean demand model which is linear in p and r . For a similar discussion please refer to (Chen and Simchi-Levi, 2004a). However the expected revenue term for this linear model is not concave. To deal with this challenge, (Chen *et al.*, 2011) develop a transformation technique and show the concavity of the transformed value function.

On the other hand, if the mean demand models with concave expected revenue

terms are considered, $\hat{G}_t(\cdot)$ turns out to be non-convex. One way to circumvent this challenge is to transform the problem by defining an inverse demand function and take the mean demand as the variable instead of the price. Then $\hat{G}_t(\cdot)$ becomes a linear function of y and the mean demand d which becomes the decision variable after the transformation. For models where the mean demand is a function of price only, the existence of such a function is clear since mean demand is strictly decreasing with the price and hence there is a one-to-one mapping between the mean demand and the price. (Chen and Simchi-Levi, 2004a) use such a transformation.

For demand models where the mean demand is also a function of the reference price, this transformation is also possible, however its existence and properties need more elaboration as it shown in Section 3.3.3. In subsequent sections, we use the inverse demand function and provide some of its properties. We then show that (i) the transformed action space is convex for concave demand models, (ii) the transformed value function and the profit-to-go function are concave if the transformed expected revenue function is concave and hence (iii) an SDO policy is optimal.

Proposition 3.1 in Section 3.3.3 and Proposition 3.2 below show the properties of the price function $\hat{p}_t(d, r)$ and its connection with the mean demand function $\hat{d}_t(p, r)$.

For our subsequent analysis, we need the concavity of the price function $\hat{p}_t(d, r)$ over the set $\mathbb{D}$. Define $p_t^u(r)$ as the price that makes the mean demand equal to its lower bound at a reference price r , i.e. $\hat{d}_t(p_t^u(r), r) = d^l$ and define p_t^u as the maximum of these prices, i.e. $p_t^u = \max_{r \in [p^l, r^u]} p_t^u(r)$. Since mean demand is increasing with r and strictly decreasing with p , $p_t^u(r)$ is increasing with r . Therefore $p_t^u = p_t^u(r^u)$. Note that $\hat{d}_t(p_t^u, r) \leq d^l$ for $r \leq r^u$, hence this upper bound on the price is not the same with the upper bounds in Section 3.3, (Gimpl-Heersink, 2008), (Zhang, 2010) and (Chen *et al.*, 2011).

Let $\hat{\mathbb{P}}_t = \{(p, r) : p \in [p^l, p_t^u], r \in [p^l, r^u]\}$. $\hat{\mathbb{P}}_t$ is a convex set and $\mathbb{P}_t \subseteq \hat{\mathbb{P}}_t$. We will show that $\hat{p}_t(d, r)$ is concave over the convex set $\mathbb{D}_t$ if $\hat{d}_t(p, r)$ is concave on $\hat{\mathbb{P}}_t$.

Proposition 3.2. *Given Assumption 3.1: $\mathbb{D}_t$ is a convex set and $\hat{p}_t(d, r)$ is concave on $\mathbb{D}_t$ if $\hat{d}_t(p, r)$ is concave on $\hat{\mathbb{P}}_t$.*

Proof. See Appendix A.8. □

Figure 3.1a shows the action space of the form $\mathbb{P} = \{(p, r) : p \in [p^l, p^u], r \in [p^l, p^u]\}$ with a contour on which the demand attains its lower bound. Figure 3.1b gives the corresponding $\mathbb{D}_t$ set which can be observed to be non-convex. Figure 3.2a shows the action space in the form $\mathbb{P}_t = \{(p, r) : p \geq p^l, r \in [p^l, r^u], \hat{d}(p, r) \geq d^l\}$ and its corresponding $\mathbb{D}_t$ is given in Figure 3.2b. We provide a necessary and sufficient condition for the set $\mathbb{D}_t$ to be convex and $\hat{p}_t(d, r)$ to be concave on $\mathbb{D}_t$ in Appendix A.9.

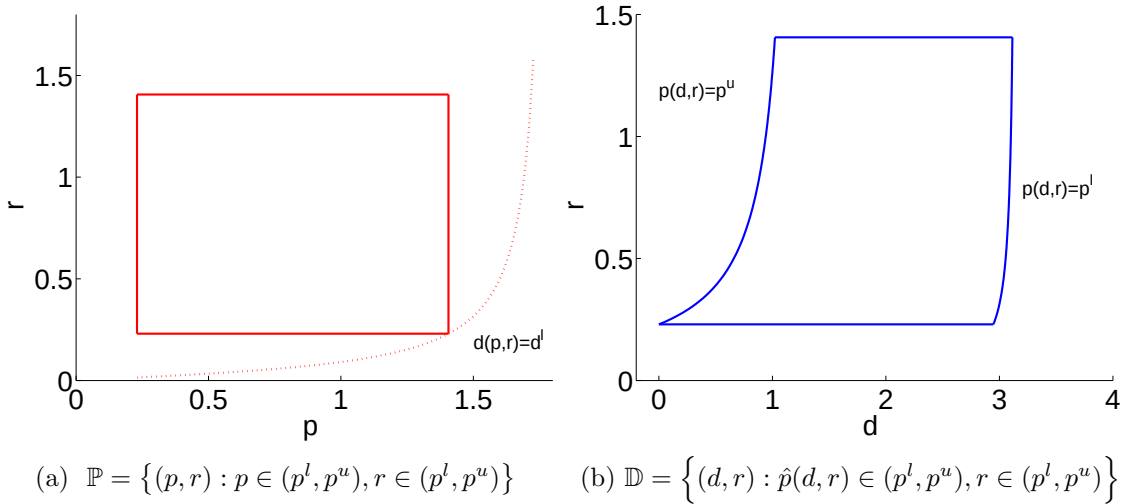


Figure 3.1. Original Action Space.

The expected revenue can equivalently be written as a function of the mean demand and the reference price as $\Pi_t(p, r) = \Pi_t^p(d, r) = \hat{p}_t(d, r)d$. $\Pi_t^p(d, r)$ is the expected revenue which is now a function of the demand and the reference price. For our subsequent analysis, we need that $\hat{d}_t(p, r)$ and $\Pi_t^p(d, r)$ are concave.

Assumption 3.7. $\Pi_t^p(d, r)$ is concave over $\mathbb{D}_t$.

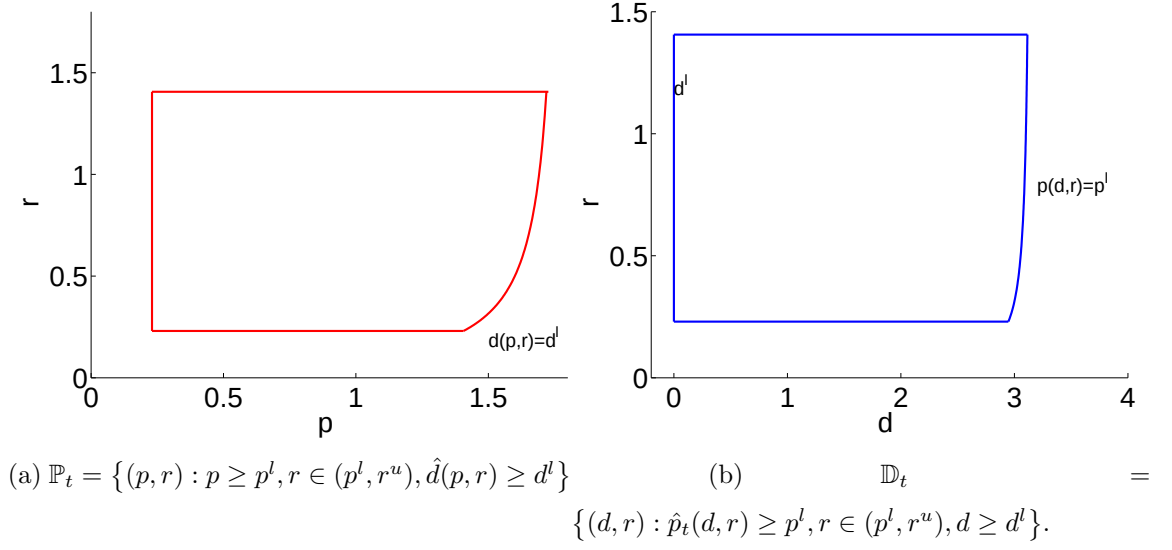


Figure 3.2. Transformed Action Space.

Now we give an example which satisfies Assumption 3.4(i) and Assumption 3.7. A demand model where customers are loss averse with RDRE is given as follows:

$$\hat{d}_t(p, r) = 3 - p^2 + \frac{(r - p)^+}{10r} - \frac{(r - p)^+}{5r}. \quad (3.16)$$

One can show that the price function $\hat{p}_t(d, r)$ and the revenue function $\Pi_t^p(d, r)$ of the expected demand model in Equation 3.16 is concave over

$$\mathbb{D}_t = \{(p, r) : \hat{p}_t(d, r), r \geq 0.23, d \geq 0.35\}$$

For details please see Appendix A.9. Note that this demand model is not linear in r and p and does not satisfy the concavity assumptions of (Gimpl-Heersink, 2008). Hence the results of (Gimpl-Heersink, 2008), (Zhang, 2010) and (Chen *et al.*, 2011) do not apply to this demand model.

3.4.1. Optimal Policy

We employ two transformations to rewrite the dynamic model given in Equation 3.4 in a slightly more manageable form. First we change our decision variables from

(y, p) to (y, d) . This is justified by the analysis in the previous sections. By this transformation, the expected holding and backorder cost $\hat{G}_t(y, p, r)$ can be now written as a function of only y and d (Equation 3.18) which enables us to establish the joint concavity of the transformed value function which is now a function of y , d and r and will be denoted as $J_t(y, d, r)$. Secondly, we transform the optimal value function $V_t(x, r) = \hat{V}_t(x, r) - c_t x$ which is standard in many inventory problems (Federgruen and Heching, 1999). With these transformations, the dynamic model in Equation 3.4 can equivalently be written as follows:

$$\begin{aligned}
 J_t(y, d, r) &= \hat{p}_t(d, r)d - \gamma c_{t+1}d + (\gamma c_{t+1} - c_t)y - G_t(y, d) \\
 &\quad + \gamma E[V_{t+1}(y - d\theta_t - \epsilon_t, \alpha r + (1 - \alpha)\hat{p}_t(d, r))] \\
 V_t(x, r) &= \max_{\substack{y, d \\ y \geq (d, r) \in \mathbb{D}_t}} \left\{ J_t(y, d, r) \right\}
 \end{aligned} \tag{3.17}$$

with terminal value $V_{T+1}(x, r) = 0$. $G_t(y, d)$ is the expected holding and backorder cost and is defined as:

$$G_t(y, d) = E[H_t(y - d\theta_t - \epsilon_t)]. \tag{3.18}$$

The following theorem states our main result for the finite horizon problem:

Theorem 3.8. *Given Assumptions 3.1(i), 3.3(i), 3.4 (i) and 3.7:*

- (i) $J_t(y, d, r)$ is concave,
- (ii) $V_t(x, r)$ is increasing with r and concave,
- (iii) Defining $(y_t^*(r), d_t^*(r))$ as the maximizer of $J_t(y, d, r)$ over $\mathbb{D}_t$ for a given r , the optimal policy for period $t = 1, \dots, T$ is given as follows: If $x \leq y_t^*(r)$, then the firm orders $y_t^*(r) - x$ many units and charges the price $p_t^*(r) = \hat{p}(d_t^*(r), r)$. If $x > y_t^*(r)$, then the firm orders nothing and charges the price $p_t^*(x, r) = \hat{p}(d_t^*(x, r), r)$ where $d_t^*(x, r)$ is the maximizer of $J_t(x, d, r)$ over $\mathbb{D}_t$ for a given r .

Proof. See Appendix A.11. □

We now extend the result of Theorem 3.8 to the case where the planning horizon is infinite. We assume that, in addition to the stationary assumptions cited in Section 3.3.2, all θ_t are independent and identically distributed as a random variable θ . Since the construction of the auxiliary functions (ϕ_t^*, Φ_t^*) for the proof of the infinite planning horizon can be done in a similar way as we did in Section 3.3.2 for Theorem 3.5, we give the results without proof. Let the periods be relabeled such that the index i shows the number of periods to go, i.e. $i = T - t$.

Theorem 3.9. *Given Assumptions 3.1(i), 3.3, 3.4(i) and 3.7;*

- i. The limits $V = \lim_{i \rightarrow \infty} V_i$ and $J = \lim_{i \rightarrow \infty} J_i$ all exist.*
- ii. V and J satisfy the optimality equation in Equation 3.17.*
- iii. $J(y, d, r)$ is concave,*
- iv. $V(x, r)$ is increasing with r and concave,*
- v. Defining $(y^*(r), d^*(r))$ as the maximizer of $J(y, d, r)$ over $\mathbb{D}$ for a given r , the optimal policy is given as follows: If $x \leq y^*(r)$, then the firm orders $y^*(r) - x$ many units and charges the price $p^*(r) = \hat{p}_t(d^*(r), r)$. If $x > y^*(r)$, then the firm orders nothing and charges the price $p^*(x, r) = \hat{p}_t(d^*(x, r), r)$ where $d^*(x, r)$ is the maximizer of $J(x, d, r)$ over $\mathbb{D}$ for a given r .*

As a summary, we define a larger and a more convenient action space by imposing a lower bound on the mean demand instead of setting an upper bound on the price. Then we make a transformation in such a way that the mean demand becomes the new decision variable instead of the price which makes the expected inventory cost $G_t(y, d)$ convex in (y, d) . We then show that (i) the new action space preserves its convexity if demand is concave and (ii) the value function $J_t(y, d, r)$ is concave if the expected revenue function after transformation is concave. Using (i) and (ii) we show the concavity of the profit-to-go function $V_t(x, r)$. This enables us to prove Theorem 3.8 which states that a state dependent order-up-to policy is optimal for the joint inventory and pricing problem with reference effects for concave demand models and concave expected revenue functions after the transformation. Although (Gimpl-Heersink, 2008) shows a similar result, she requires an additional assumption

in addition to concavity of the revenue function: she assumes that the expected holding and backorder cost $\hat{G}_t(y, p, r)$ is convex in (y, p, r) . This condition implies the linearity of the expected demand in its variables p and r . (Chen and Simchi-Levi, 2004a) provide a similar discussion for inventory cost of a mean demand model depending only on price. However the revenue function of a linear model is not concave. Hence there is no demand model that can satisfy the set of assumptions provided by (Gimpl-Heersink, 2008). (Chen *et al.*, 2011) show the optimality of an SDO policy for this linear demand model (ADRE and loss-neutral customers). (Zhang, 2010) show that the same policy is optimal for piecewise linear demand models which includes loss-averse customers. To the best of our knowledge, ours is the first study that shows the optimality of a state dependent order-up-to policy for *concave* demand models and *concave* transformed expected revenue functions and the randomness is due to multiplicative and/or additive random terms. We show that these mean demand models include models with a RDRE and *loss-averse* customers. Since we do not impose any differentiability assumption, the results in Theorem 3.8 and Theorem 3.9 are valid for demand models that are not differentiable as well.

3.5. Computational Study

In this section, the aim is to analyze the effects of parameters and behaviour of optimal decisions which cannot be derived analytically. Throughout the section, we use the following stationary demand model with loss neutral customers:

$$\hat{d}(p, r) = 3 - p^2 + \frac{r - p}{5r}. \quad (3.19)$$

We have $\mathbb{D} = \{(d, r) : \hat{p}(d, r) \geq 0.25, r \in [0.25, 1.73], d \geq 0.35\}$. All of the assumptions of Theorem 3.8 are satisfied, hence a state dependent order-up-to policy is optimal. We employ both additive and multiplicative random terms. The additive random term (ϵ) is a truncated normal random variable whereas the multiplicative random term (θ) is distributed uniformly. The lower bounds for the random terms and the mean demand is chosen such that the realized demand is non-negative, i.e. $D(p, r, \theta, \epsilon) \geq 0$. In

order to see the effect of the standard deviation, runs with low and high variances are conducted. Table 3.1 gives the standard deviation and the boundaries of the random terms. r^u is the price that makes the base demand zero, i.e. $3 - (r^u)^2 = 0$. This upper

Table 3.1. Random Terms.

		Low	High
Variances	Additive	$\sigma_a^L = 0.1310$	$\sigma_a^H = 0.9172$
	Multiplicative	$\sigma_m^L = 0.0578$	$\sigma_m^H = 0.173$
Boundaries	Additive()	$[-0.24, 0.24]$	$[-0.24, 0.24]$
	Multiplicative	$[0.9, 1.1]$	$[0.7, 1.3]$

bound can be justified with the following: If a period starts with $r_t = r^u$, then $p_t \leq r^u$ in order to keep the mean demand non-negative. Let d^u be the maximum value that the mean demand can attain. Then $d^u = \hat{d}(p^l, r^u) = 3.1$. Variance of the additive term is chosen such that $\sigma_a^L = (0.075)(d^l + d^u)/2$ and $\sigma_a^H = (7)(\sigma_a^L)$; in order to mimic the coefficient of variation.

Our first aim is to answer questions about steady state, optimal policy through and within periods and the end of horizon effect. Second aim is to observe the marginal and joint effects of variances of multiplicative and additive random terms, the parameter α and initial reference price levels (r_0) over the optimal decisions and total expected profit. The third aim is to observe the behaviour of a myopic firm and evaluating the value of dynamic programming, and the value of reference effect information. For the last two goals, two firms are compared via simulation.

Holding and backorder costs are taken as a multiple of the production cost and values of the cost parameters are given in Table 3.2. Section 3.5.1 provides an illustrative example and Section 3.5.2 shows the effect of α , r_0 and random terms over the optimal decisions and total expected profit. Sections 3.5.3 and 3.5.4 investigate the value of dynamic programming and value of reference effects, respectively. The lower and the higher levels of the parameters are denoted with a superscript, i.e. c^L and c^H stand for the low and high levels of the production cost, respectively. We use MATLAB

Table 3.2. Cost Parameters.

		Low	High
Production Cost	(c)	0.425	0.620
Holding Cost	(h)	0.25c	0.50c
Backorder Cost	(b)	1.50c	3.00c

R2010a in our computations.

3.5.1. An Illustrative Example

As an illustrative example, consider a firm facing the demand model in Equation 3.19 with parameters σ_a^L , σ_m^L , c^H , h^H and b^H over 60 periods. Figure 3.3 shows that the system has a steady state.

As shown in Figure 3.3a, the firm starts with a high price and adopts a skimming strategy when initial reference price is high ($r_0 = 1.6$) and adopts a penetration strategy if the system starts with a low reference price ($r_0 = 0.85$). During the initial periods, the firm tries to balance the current and future profits. Decreasing the price increases the current revenue but decreases future revenues because of the reference effect. However importance of the ‘future’ decreases towards the end of horizon, since the whole process terminates at the end of 60 periods. Hence the firm decreases the price towards the end of the horizon. The effect is the same for both of the systems starting with different initial reference price levels. (Fibich *et al.*, 2003) show that there is a similar end of horizon effect for linear models in a continuous time framework.

The system uses the benefit of the reference effects and starts with a high price and high mean demand when the initial reference price is high (Figure 3.3b). Staying constantly at the high price removes the reference effect, thus price is decreased gradually and hence the system makes use of the excess reference effect demand. Optimal order-up-to level just follows the optimal mean demand pattern as shown in Figure 3.3c.

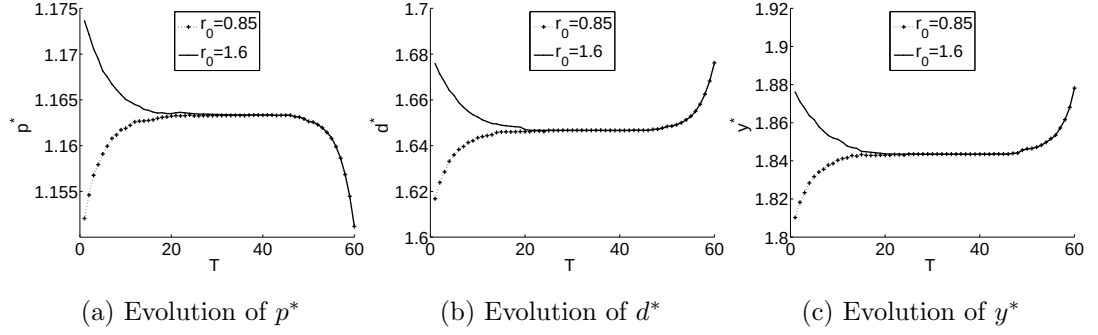


Figure 3.3. Evolution of Optimal Decisions with r_0 .

When r_0 is low, which means starting with a low mean demand, the firm starts with a low price in order not to decrease the mean demand too much and gradually increases the price and the mean demand utilizing the reference effect. Both systems become equivalent at the steady state, thus the price decrease at the end of horizon increases mean demand in both systems.

Figure 3.4 shows the optimal policy as a function of on hand inventory when $r = 1$. The firm tries to deplete excess on hand inventory. Hence optimal price decreases and optimal mean demand increases with on-hand inventory level.

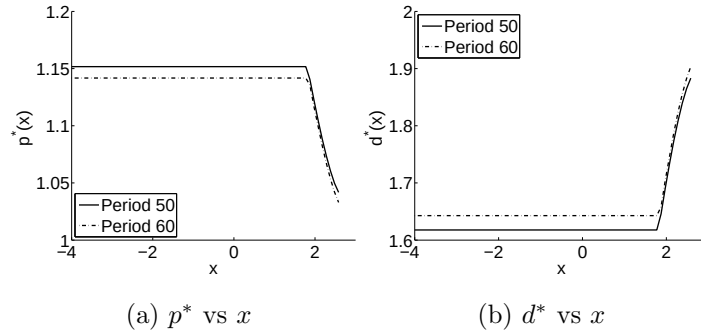


Figure 3.4. p^* and d^* with On Hand Inventory at Different Periods.

Figure 3.5 shows the optimal policy as a function of current reference price when there is no initial on hand inventory, i.e. $x = 0$. Since at a given price level mean demand increases with r , this gives the firm an incentive to increase the price (Figure 3.5a). However this increase does not reduce the mean demand to its original value (Figure 3.5b). In order to sustain the optimal inventory level, optimal order-up-to level also increases (Figure 3.5c). (Chen *et al.*, 2011) also show that $y^*(r)$ increases with r

in a linear demand model with an additive random term.

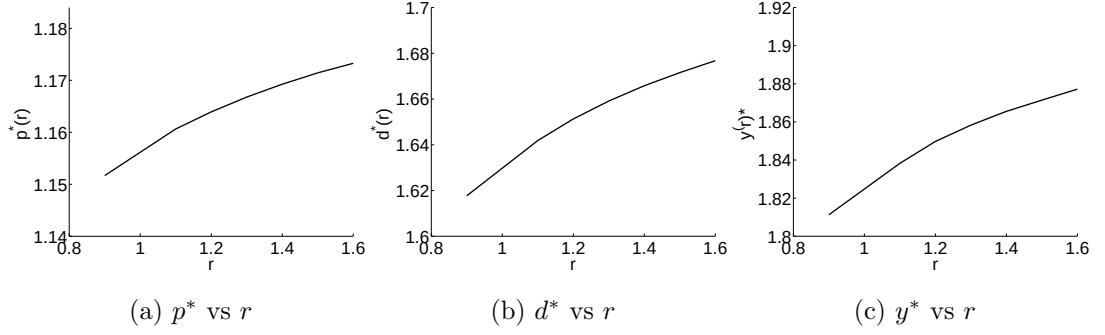


Figure 3.5. Optimal Policy vs Reference Price at Different Periods.

3.5.2. Effect of α , r_0 and Random Terms

During the experiments, it is observed that the system reaches a steady state after a while and hence the reference effect vanishes. This is also the case in (Popescu and Wu, 2007), (Zhang, 2010) and (Chen *et al.*, 2011). However as the product life cycles are getting shorter, firms introduce new substitutes of the product changing the reference price expectation sooner. Therefore the reference effect analysis with short horizons has greater importance than the problems with a long horizon or an infinite horizon. For further analysis the length of horizon is set to 10.

In order to observe the effects of r_0 and α on the optimal policy and the total expected profit, the parameter set in Table 3.3 is employed.

Table 3.3. Experiment Values for α and r_0 .

Parameter	Parameter Set
α	0, 0.2, 0.5, 0.8, 1
r_0	0.85, 1.00, 1.15, 1.30, 1.45, 1.60

All of the experiments in Figures 3.6-10 show results with high variances and low costs with a horizon length of 10. Whenever there are changes in the base parameters,

we will be explicit throughout the remainder of the text.

Figure 3.6a shows the total expected profit with r_0 and α values. There are two important observations. First, starting with low r_0 causes a lower reference price throughout the horizon and hence a lower mean demand. Thus for all α values, total expected profit increases with r_0 . Second, as α decreases, reference price depends more on the current price (Equation 3.3), reference price is updated fast and dependence on the past decreases. So if α is large, importance of r_0 increases. Therefore:

- The increase in the total expected profit decreases as α decreases.
- When compared to lower α values, higher values result in lower total expected profits at low r_0 values but in higher profits at high r_0 values. Thus high dependence on a ‘good’ history increases the profit while high dependence on a ‘bad’ history decreases the profit.

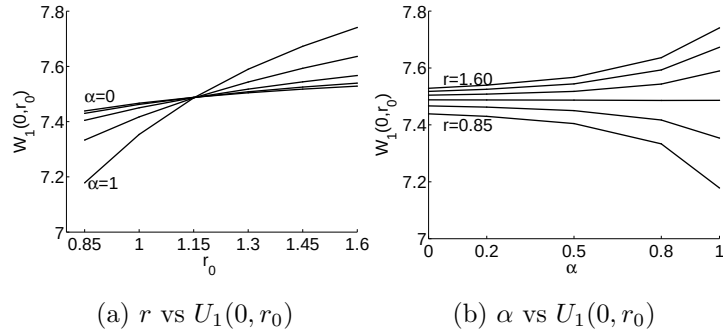


Figure 3.6. Total Expected Profit with r_0 and α .

Figure 3.7c shows the optimal price for the first period with α for different r_0 values. For a fixed α value, the initial optimal price increases with initial reference price. The change in the price level when $\alpha \in [0, 0.8]$ is moderate when it is compared with the change when α changes from 0.8 to 1. This is due to the fact that the system is able to change r when $\alpha \in [0, 0.8]$ but this is not possible when $\alpha = 1$. Same effect can also be seen in the optimal mean demand and order-up-to levels which is shown in Figures 3.7b and 3.7c.

Figure 3.8 shows the evolution of the optimal values with different α and r_0

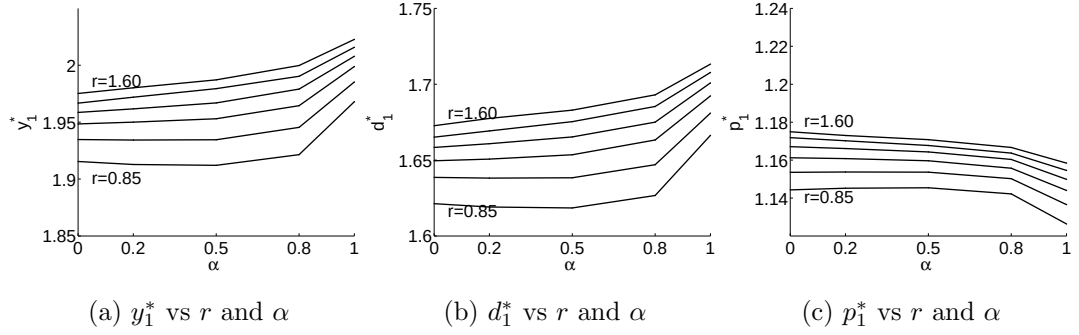


Figure 3.7. Optimal Policy of First Period with α and r_0 .

levels. The steady state values depend on α , but not on the initial reference price levels (Popescu and Wu, 2007).

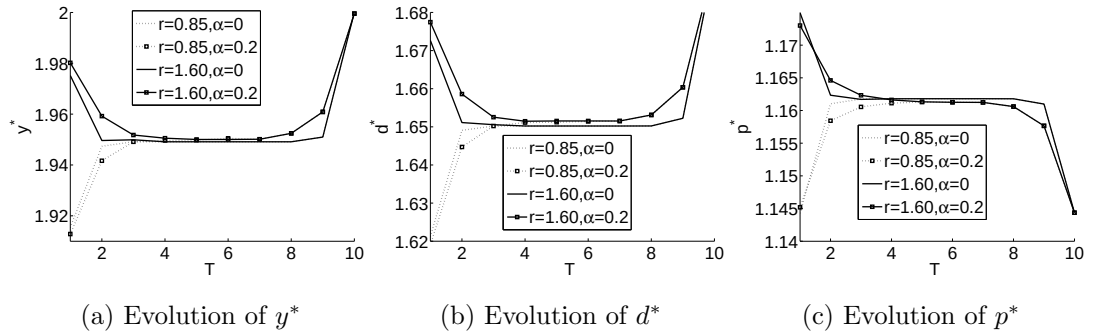


Figure 3.8. Evolution of Optimal Decisions with α and r_0 .

Figure 3.9 shows the effect of variances over optimal policies. As it can be seen from Figure 3.9a, optimal mean demand and optimal order-up-to levels change when the variance of the multiplicative random term changes while the variance of additive random term is kept constant. When the variance is high, optimal mean demand quantity is decreased while the optimal order quantity is increased. This is due to the fact that a high mean demand results in a high demand variance through the multiplicative random term. Thus the firm reduces its mean demand to decrease this variance. However decreasing the mean demand reduces the revenue, thus the decrease in demand is limited. As a result of this trade-off, the mean demand is reduced to lower the inventory cost and the order-up-to level is increased to compensate increased variance. Figure 3.9c shows the optimal price in the same setting.

Figure 3.9b shows the case when the variance of the additive random term changes

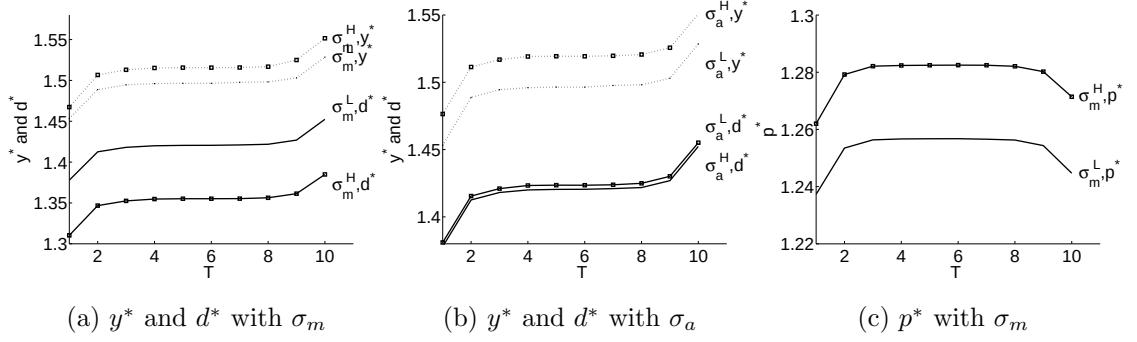


Figure 3.9. Effect of Variances Over Optimal Decisions.

while the variance of multiplicative random term is kept constant. The change in variance of the additive term results in a change only on the optimal order-up-to level and does not affect the optimal mean demand. (Zhang, 2010) shows a similar result with linear demand models.

3.5.3. Investigating the Value of Dynamic Programming

This section and Section 3.5.4 investigate the value of dynamic programming and value of reference effects, respectively. In order to show these, we make a Monte Carlo simulation for two firms who face the same environment in order to calculate their total expected profits. The following are common for these two firms:

- (i) The demand is affected by price and reference price and is given in Equation 3.19.
- (ii) There are additive and/or multiplicative random terms whose properties are in Table 3.1.
- (iii) The customers have a common reference price which has an initial value r_0 and it is updated with current price and α as in Equation 3.3 during successive periods.

Firm 1, has the correct information related to three issues above and gives optimal decisions. Firm 2, acts suboptimally due to either incorrect modeling of demand or an incorrect estimation for α or r_0 value. At the start of every period, both firms observe their on-hand inventory levels and their estimate for reference price levels. Then they optimize their profit function and decide on an order quantity and a price.

Then demand is realized, costs are incurred and they update their estimated reference price. Since the demand has additive and/or multiplicative random terms, it is not possible to correct the demand model and r or α values for Firm 2. Hence, although the parameters of the underlying model evolves with correct parameters, Firm 2 cannot observe this and gives decisions according to the demand model and parameters that it estimates incorrectly. Firm 1 on the other hand, has correct information and acts optimally. In this section we show the lower and the higher levels of the parameters with a superscript, i.e. c^L stands for the low level of the production cost.

This section shows the value of dynamic programming by comparing it to a myopic solution. We simulate two firms who face the same environment. Both of the firms observe the reference price but one of them, Firm 1, acts optimally whereas Firm 2 gives myopic decisions. Firm 2 decides after checking the single period profit function only, ignoring the future periods. 100 simulations are conducted for every parameter set combination given in Tables 3.1, 3.2 and 3.3 with length of horizon being also a parameter ($T \in \{5, 10\}$). Profit improvement in favor of Firm 1 as a percentage between Firm 1 and Firm 2 is reported for different values of the parameters. For example, Table 3.4 shows that when T is low ($T = 5$), all of the runs yield an average percent improvement of 10.07% in favor of Firm 1.

The improvement in the profits has an average value of 9.12% and can be as high as 38.59% which is in a 5-period problem with c^H , h^L , b^L , σ_a^L , σ_m^L and with $\alpha = 0.8$, $r_0 = 1.3$. Table 3.4 shows the change in profits for the parameters at their low and high values. As the length of horizon increases, the profit improvement decreases. The increase in profit when production cost is high, is more than double of that of the low production cost, thus acting optimally becomes more important if the profit margin is low. The improvement is high when both holding and backorder costs are low but the effect of the backorder cost is higher than the holding cost.

Table 3.5 shows that for any given value of σ_m (σ_a), the profit increases when σ_a (σ_m) increases. The maximum improvement in profits is observed when both of the variances are high. Hence the value of dynamic programming increases when

Table 3.4. Average Profit Improvement(%) with T , c , h and b .

Value	T	c	h	b
Low	10.07	5.32	9.57	11.92
High	8.18	12.93	8.68	6.32

randomness is high. Although there is no significant effect of α in the improvement

Table 3.5. Average Profit Improvement(%) with Variances of Random Terms.

		σ_m		
		0	Low	High
σ_a	0	-	4.11	14.32
	Low	4.65	5.88	14.76
	High	6.87	7.26	15.14

of profits (9.13% on the average), r_0 levels seem to increase the improvement (Table 3.6). However this effect is relatively small when compared with the effects in Tables 3.4 and 3.5.

Table 3.6. Average Profit Improvement(%) with r_0 .

r_0	0.85	1.00	1.15	1.30	1.45	1.6
	9.34	9.21	9.11	9.11	9.01	8.99

Another important issue is the comparison of the myopic and optimal decisions. Subscripts (M) showing the decisions of the myopic firm, Figure 3.10 shows the evolution of optimal decisions at different α values in a 10-period problem with $r_0 = 0.85$. It can be seen that the myopic firm undershoots the optimal order-up-to levels during the horizon. The increase in the last period is due to the salvage value at the end of the horizon. Although one may expect the optimal mean demand to be also undershot, it is not the case (Figure 3.10b). The price is undershot by the myopic firm (Figure 3.10c) which is also justified by (Popescu and Wu, 2007) and (Zhang, 2010). As a

combined effect, the myopic firm does not suffer from the revenue but it suffers from the inventory costs.

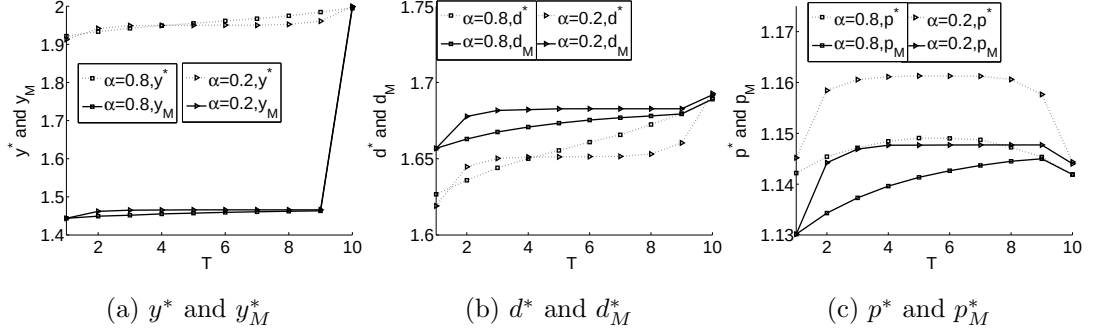


Figure 3.10. Myopic and Optimal Decisions.

3.5.4. Investigating the Value of Reference Effect

In this section, the aim is to evaluate the value of reference effects. In order to see this, two firms facing the same environment are simulated. One of the firms, Firm 1, is acting optimally. The other firm, Firm 2, acts sub-optimally by either ignoring or incorrectly predicting the reference effect on demand.

3.5.4.1. Value of Modeling with Reference Effects. Assume that Firm 2 anticipates the demand model of the product by assessing the demand information during the steady state where $r = p$. This demand information can be collected from either similar products or previous versions of the same product which are on sale for a considerably long time. Since the reference effects vanish at the steady state, Firm 2 models the demand as a function of price only. Therefore the mean demand model that it is trying to optimize is:

$$\hat{d}(p, r) = 3 - p^2. \quad (3.20)$$

However the real demand evolves with reference effects, that is the underlying real demand model is the one given in Equation 3.19. Even if the profit of Firm 1 turns out to be larger than Firm 2, we check whether this is statistically significant or not. The profit improvements are tested according to the following hypothesis with a significance

level of 0.05:

$$H_0 : \Theta_1 = \Theta_2$$

$$H_1 : \Theta_1 > \Theta_2$$

where Θ_1 and Θ_2 are the mean of total profit of Firm 1 and Firm 2, respectively. 100 simulations are performed for the same setting described in Section 3.5.3. The percentage of the significance over all runs are reported. The main goal is to observe the value of modeling the demand correctly with reference effects.

Average improvement in profits over all simulation runs for all parameter sets turns out to be 0.21%. The improvement in profits can be as high as 5.49% which turns out to be the setting with 10-periods, $\alpha = 1$ and $r_0 = 0.85$ when there is a multiplicative random term with low variance and a constant additive random term ($\epsilon = 1$) with high production cost and inventory related costs. Note that $\alpha = 1$ corresponds to the case when Firm 1 always observes $r_0 = 0.85$ as the reference price, whereas Firm 2 does not observe this. We also note that, even though $\alpha = 1$ may seem to be an unfair comparison for Firm 1 and Firm 2, there is still a significant difference (2.38%) for $\alpha = 0.8$ as shown in Table 3.7.

The average difference in 5-period problems is 0.24% while that of 10-period problems is 0.19%. When the horizon is long, steady state starts to dominate the whole horizon, thus the difference between the profits of the two firms decreases.

When all of the costs and variances are high, Table 3.7 shows the improvements in profits in a 5-period problem for different α values. It can be seen that when α gets large, the improvement increases. When reference price depends heavily on the past, Firm 1 gets a greater profit than Firm 2. When α is small, the reference price adapts itself to the current price quickly, thus the change in profits is small.

Table 3.8 shows the profit improvement with different r_0 levels with high costs

Table 3.7. Average Profit Improvement(%) with α .

α	0	0.2	0.5	0.8	1
	1.21	1.35	1.55	2.38	5.49

and variances when $\alpha = 0.8$. Optimal prices turn out to be around 1.25 which explains the difference vanishing at moderate values of r_0 ($r_0 = 1.30$). Improvement increases at both extreme values of r_0 . Since the mean demand model in Equation 3.19 has a RDRE, the same difference between price and reference price results in a lower reference effect when r_0 is high. Hence the change is larger at the lowest end. It can be seen

Table 3.8. Average Profit Improvement(%) with r_0 .

r_0	0.85	1.00	1.15	1.30	1.45	1.60
	2.38	1.06	0.11	0.02	0.43	0.77

that the loss in the profits when reference price is ignored is smaller than the loss when a myopic policy is employed. A similar argument can be found in (Fibich *et al.*, 2003) where they show that a constant price strategy may yield a very close profit to that of the optimal one.

Another important issue is how significant are the profit changes. Table 3.9 shows that the significance decreases with increasing variance. For example, when σ_m is low and additive random term is constant, the ratio of runs where profit improvement is statistically significant is 77.3%. The decrease is higher with multiplicative random term compared to the additive random term. This is due to the fact that choosing the mean demand (via pricing) does not affect the variance of demand when there is only additive random term but the variance of demand changes when there is multiplicative random term.

Table 3.10 shows percent difference in significance with α and T . For both T values, when α gets larger, significance increases. An important observation is, when α is small, significance in 5-period problems is larger than 10-period problems. This

Table 3.9. Significance(%) for Variances of Random Terms.

		σ_m		
		0	Low	High
σ_a	0	-	77.3	40.2
	Low	66.9	56.7	33.1
	High	65.6	55.8	32.9

relation changes as α gets larger. When α is small, reference price is updated fast, thus the difference between the two firms vanishes quickly. Only initial periods have significant difference and the weight of a single period in a 5-period problem is greater than a 10-period problem. When α gets larger, then the reference price is dependent more on the past which results in a slower admittance of steady state. Thus two firms have a significance difference not only at initial periods, but in later periods as well. So 10-period problems have greater significance percentages when α gets larger. Significance with r_0 levels is given in Table 3.11. The significance level is high at both

Table 3.10. Significance(%) for T and α .

		α				
		0	0.2	0.5	0.8	1
T	5	37.5	39.1	52.6	66.4	77.3
	10	32.3	34.1	43.8	63.5	89.1

extremes with the lowest value being the largest due to the same facts explained with Table 3.8.

Table 3.11. Significance(%) for r_0 .

r_0	0.85	1.00	1.15	1.30	1.45	1.60
	82.7	60.2	58.9	56.7	42.0	34.8

Costs have different effects on significance. The decrease in profit margin reduces

the significance (Table 3.12). Figure 3.11 shows the cumulative significance as a function of inventory related costs. On the x axis, the experiment results are ordered in increasing order with respect to their percentage of total inventory related costs to the revenue. Corresponding y axis values are the ratio of number of significant runs up to that x point, to all significant runs. It is observed that the increase in significance reduces when ratio of inventory cost with respect to the revenue increases.

Table 3.12. Significance(%) for Production Cost.

c	Low	High
	59.6	47.6

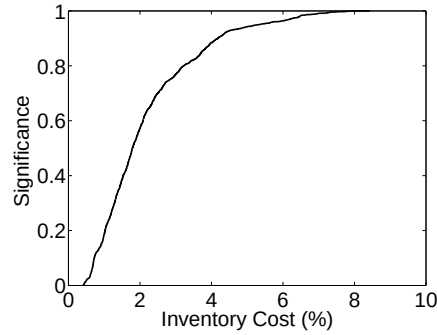


Figure 3.11. Significance with Inventory Cost Share on Profit.

3.5.4.2. Value of Correctly Assessing Reference Effects. Assessing the correct reference price update mechanism is important for demand models with reference effects. In this section we analyze the value of knowing the correct α or r_0 .

Firm 1 acts optimally with the correct demand model and correctly estimated parameters. Firm 2, on the other hand, estimates α or r_0 incorrectly. The evolution of the reference price continues with the correct α or r_0 values. Since the demand has additive and/or multiplicative random terms, it is not possible for Firm 2 to correct the parameter α or r_0 . We provide two examples.

Consider a case where the correct α is $\alpha^C = 0$ so that the reference price heavily

depends on the current price. However Firm 2 incorrectly assumes that the reference effect is effected heavily by the past and sets $\alpha^I = 0.8$. If all of the costs are high with parameters σ_m^L , $\epsilon = 1$ and $r_0 = 0.85$, then the profit improvement turns out to be 1.54% and 0.83% for a 5 and 10-period problem, respectively.

Consider the same setting with high costs and only multiplicative random term with parameters σ_m^L , $\epsilon = 1$ where $\alpha = 0.8$ and $r_0 = 0.85$. Again Firm 1 acts optimally with the correct demand model and the reference price update mechanism. It knows the true value of the initial reference price level and sets $r_0^C = 0.85$. The second firm anticipates that the initial reference price is a bit high and sets $r_0^I = 1.15$. The improvement in profit turns out to be 2% in the 5-period problem and 1.1% in the 10-period problem. As expected, the improvement is large if the prediction error is high. For the same setting the improvement is 7.33% if the correct r_0 level is $r_0^C = 0.85$ but the firm predicts it to be $r_0^I = 1.6$.

The results due to the length of horizon follows from the fact that the reference effect vanishes at the steady state. Hence if the horizon gets longer then the difference in the profits is getting smaller.

3.6. Extensions

In this section, we provide four extensions. In the first one we show that our result in Theorem 3.8 holds for a more general reference price update mechanism. The next two extensions are for cases where α is a random variable and follows a discrete time Markov chain, respectively. The last extension is for the capacitated problem where the order quantity is limited.

3.6.1. General Evolution Models

The result in Theorem 3.8 is still valid when the evolution function of the reference price obeys the following assumption. Define $\mathbb{P}_r = \left\{ r : (p, r) \in \mathbb{P} \right\}$.

Assumption 3.8. *The reference price of a period can be written as a function of the reference price and price of the previous period, $r_{t+1} = f(r_t, p_t)$ where $f : \mathbb{P} \rightarrow \mathbb{P}_r$ is a concave function. Moreover $f(r_t, p_t)$ is increasing with r_t and p_t .*

Note that the exponential smoothing model in Equation 3.3 satisfies this assumption.

3.6.2. α as a Random Variable

Consider the case where α is a random variable with $\alpha \in [0, 1]$. Then Theorem 3.8 still holds and a state dependent order-up-to policy is optimal. We omit its proof since it is similar to that of Theorem 3.8.

Theorem 3.10. *Given Assumptions 3.1(i), 3.4 (i) and 3.7, if α is a random variable with $\alpha \in [0, 1]$, then the policy in Theorem 3.8 is still optimal.*

3.6.3. α with Markovian Structure

Consider the case where α evolves according to a Markov Chain with a probability transition matrix $\mathbb{Q}$. Suppose α assumes m possible values that are in $[0, 1]$. Let $\mathbb{Q}_{ij}$ be the transition probability that if α in a period is equal to α_i then in the next period it becomes α_j . Then using the same transformations in Section 3.4.1, the dynamic recursion of the stated problem can be written as:

$$\begin{aligned} J_t(y, d, r, \alpha_i) &= \hat{p}_t(d, r)d - \gamma c_{t+1}d + (\gamma c_{t+1} - c_t)y - G_t(y, d) \\ &\quad + \gamma \sum_{j=1}^m \mathbb{Q}_{ij} E \left[U_{t+1} \left(y - d\theta - \epsilon, \alpha_i r + (1 - \alpha_i)\hat{p}(d, r), \alpha_j \right) \right] \\ V_t(x, r, \alpha_i) &= \max_{\substack{y, d \\ y \geq (d, r) \in \mathbb{D}}} \left\{ J(y, d, r, \alpha_i) \right\} \end{aligned} \quad (3.21)$$

with terminal value $V_{T+1}(x, r, \alpha_i) = 0$. The optimal policy for the problem in Equation 3.21 is an SDO policy where the state includes α as well as r .

Theorem 3.11. *Given Assumptions 3.1(i), 3.7 (i) and 3.7, if α evolves according to a*

Markov Chain with a probability transition matrix $\mathbb{Q}$, then optimal policy is as follows:

- i. $J_t(y, d, r, \alpha)$ is concave in (y, d, r) for a fixed α ,
- ii. $V_t(x, r, \alpha)$ is increasing with r and concave in (x, r) for a fixed α ,
- iii. Defining $(y_t^*(r, \alpha), d_t^*(r, \alpha))$ as the maximizer of $J_t(y, d, r, \alpha)$ over $\hat{D}$ for a given r and α , the optimal policy for period $t = 1, \dots, T$ is given as follows: If $x \leq y_t^*(r, \alpha)$, then the firm orders $y_t^*(r, \alpha) - x$ many units and charges the price $p_t^*(r, \alpha) = \hat{p}_t(d_t^*(r, \alpha), r)$. If $x > y_t^*(r, \alpha)$, then the firm orders nothing and charges the price $p_t^*(x, r, \alpha) = \hat{p}_t(d_t^*(x, r, \alpha), r)$ where $d_t^*(x, r, \alpha)$ is the maximizer of $J_t(x, d, r, \alpha)$ over $\mathbb{D}_t$ for a given r and α .

Proof. See Appendix A.12. □

3.6.4. Capacitated Problem

If order quantities are limited in every period, then concavity proof in Theorem 3.8 is still valid. Hence a modified state dependent order-up-to policy is optimal. Such a policy is the same with a state dependent order-up-to policy except the case where the order-quantity exceeds capacity. In such a situation the whole capacity is used for ordering and price is chosen accordingly. Same policy is also optimal for the extensions in Section 3.6.1, 3.6.2 and 3.6.3.

4. JOINT INVENTORY AND PRICING DECISIONS WITH DELAY SENSITIVE CUSTOMERS

In this section, we study a continuous review joint inventory and pricing problem of a capacity constrained firm which we call ‘the manufacturer’. The customers of the manufacturer incur a waiting cost during their waiting time for the product and they are not homogeneous in terms of their sensitivities for waiting. Some customers are more sensitive for the delays and incur larger waiting costs. The manufacturer holds inventory and charges different prices for the product, i.e. high prices in return for high priorities and less waiting times. Our aim is to maximize the steady state profit of the manufacturer by analyzing its joint inventory, priority and pricing policy. We give the model definition and formulation in Section 4.1. We provide solutions and analysis in Section 4.2. An approximation is provided in Section 4.3. Single type customer case is analyzed in Section 4.4 and we give some numerical illustrations in Section 4.5.

4.1. Definiton of the Model

In this section, we provide the problem definiton. Then we formulate the problem using an $M/M/1$ queueing system with non-preemptive priorities and finally give some preliminary results for the expected sojourn times of the customers.

4.1.1. Problem Definition

We consider a manufacturer who produces a single product. He holds inventory of the product and follows an *order-up-to* policy. In this policy, the manufacturer sets a target level for the inventory position and keeps the inventory position at this target level. As soon as a product is sold, he gives a production order to replenish that product and increase the inventory level to the order-up-to level. In our model, the production times are exponentially distributed with rate μ . The orders which are in the production process or waiting to be produced are called *outstanding orders*.

There are M types of customers and the interarrival time of type- i customer is exponentially distributed with rate λ_i . The customers incur a cost when they wait in the system. Each customer has two attributes. The first attribute is the sensitivity of the customers for waiting and called *the waiting cost*. Second attribute is the willingness to pay for this product and is called the *reservation price*. The customers of same type have the same reservation prices and waiting costs. The customer does not buy the product if the sum of the waiting cost and price of the product exceeds the reservation price. Otherwise he does.

The manufacturer assigns non-preemptive priorities to the customers. This is practically the same with the non-preemptive priority in a classical non-preemptive priority $M/M/1$ queueing system. In such a system, an arriving high-priority customer gets ahead of the low-priority customers who are in the queue. The customer who is in service cannot be preempted by any other customer even if the new customer has a higher priority over the one in the service. Let $\hat{I}^*$ denote the original system that we study and let $\tilde{I}^0$ denote an $M/M/1$ queueing system with M non-preemptive priority classes.

If there is inventory on hand upon the arrival of a customer, then the manufacturer sells the product to the customer irrespective of his group type. The customer buys the product and immediately leaves the system. If there is no product in the inventory, then the customer has to wait. Let N denote the number of outstanding orders and let y denote the order-up-to level. Note that inventory on-hand is given by $(y - N)^+$ and number of customers in the system is given by $(N - y)^+$. In order to analyze how sales take place under the non-preemptive priority, consider the following cases faced by the arriving customer. Without loss of generality, assume that type- i customers have a non-preemptive priority over types $i + 1, i + 2, \dots, M$.

Case 1 $y > N$: There is inventory on hand, therefore any arriving customer buys the product and leaves the system without waiting.

Case 2 $y = N$: In this case there is no inventory on hand and no customer in

the system. The arriving customer is *tagged* with the next product to be produced. The customer leaves the system as soon as the product becomes available *even if* a customer with higher priority arrives before its production is complete.

Case 3 $y < N$: There is no inventory and there is a tagged customer in the system. The type- i customer arriving in this case stay behind customers of type $i, i - 1, \dots, 1$ in the queue but precedes customers of type $i + 1, i + 2, \dots, M$. It is explanatory to mention the following instance. Let the tagged customer be a type- $i + 1$ customer and all other customers waiting in the queue be customers of type $i + 1, i + 2, \dots, M$. The type- i customer arriving in this case gets ahead of the queue but not ahead of the tagged customer. Being tagged for a customer in the $\hat{I}^*$ system is analogous to being served in the $\tilde{I}^0$ queueing system. Hence from now on, ‘the time in service’ indicates the time elapsed from the beginning of being tagged to the departure with the product. We use ‘customer in service’ and ‘tagged customer’ interchangeably.

There are three decisions for the manufacturer. First one is to choose a priority rule among customers. With this rule, the manufacturer can reduce waiting time of high priority customers. Second, the manufacturer chooses an order-up-to level. High order-up-to levels result in high holding costs for the manufacturer but low waiting times (and costs) for the customers. Third, the manufacturer chooses price in order to maximize its revenue.

The manufacturer has full knowledge about the arriving customers and charges prices according to the customer types. The customer either chooses to enter the system and wait for the product or to immediately leave the system without buying the product. Since the production process and the interarrival times are exponentially distributed, this problem can be formulated by an $M/M/1$ queueing model with non-preemptive priorities where the manufacturer is the server and the customers are the arrivals. On the long run, this queueing system ends up in a steady state and the manufacturer’s aim is to maximize its expected steady state profit rate. In Section 4.2.4, we relax the full information assumption and analyze the asymmetric information case where the manufacturer knows the attributes of the customer types but does not

know explicitly the type of an arriving customer.

4.1.2. Problem Formulation

In this section, we formulate the problem and show that the expected sojourn times can be written as a fraction of expected sojourn times of customers in the $\tilde{I}^0$ queueing system. Let p_i denote the price charged to type- i customer. Define $\underline{p} = (p_1, \dots, p_M)$ and $\underline{\lambda} = (\lambda_1, \dots, \lambda_M)$ as the vectors of prices and arrival rates. Also let $\mathbf{U}$ denote the set of all permutations of the set $\{1, 2, \dots, M\}$ and $u \in \mathbf{U}$ denote the priority rule among the customers. For example if there are three customer types and $u = \{2, 3, 1\}$, then type-2 customer has the highest priority and type-1 customer has the lowest priority. Finally $W_i(\underline{\lambda}, u)$ and $E[W_i(\underline{\lambda}, u)]$ denote the sojourn time and the expected sojourn time, respectively. Note that the waiting times are functions of the arrival rates and the priority rule. There are two cost components for the customers, the price and cost of waiting for the product. A customer buys the product if the cost incurred by the customer is smaller than his willingness to pay (reservation price). Let R_i and C_i denote the reservation price and the waiting cost per unit time of type- i customer, respectively. μ , R_i , and C_i are known by all of the customers and the manufacturer. We assume there is large potential demand from each customer type and the individual customers are infinitesimal, so the decision of a single customer has a negligible effect on the system statistics. We assume that the queue is unobservable, i.e. the customers do not observe the queue prior to their actions. The manufacturer announces the order-up-to level y , the priority rule u and the price $\underline{p}$. Then the customers join the system according to a Nash equilibrium rate. By our assumption of large potential demand, at equilibrium the marginal user will be indifferent between buying or not buying the product. Hence the relationship between the price and the arrival rate is given by the following:

$$R_i = p_i + C_i E[W_i(\underline{\lambda}, u)] \quad i = 1, 2, \dots, M. \quad (4.1)$$

The systems in (Mendelson and Whang, 1990), (Katta and Sethuraman, 2005), (Aféche, 2004) and (Hassin, 2007) satisfy the equilibrium condition in Equation 4.1. In addition to the arrival rates, the priority of a customer also effects the waiting time of the customer in the system. Hence W_i and $E[W_i]$ are functions of $\underline{\lambda}$ and u .

Define $\lambda = \sum_{i=1}^M \lambda_i$, $\rho_i = \lambda_i/\mu$ and $\rho = \lambda/\mu$. Note that for the system to have a steady state, $\rho < 1$ must hold. Regardless of the number of customers in the system, every arriving customer initiates a production process. Hence the number of outstanding orders has the same distribution with the number of people in an $M/M/1$ system and its probability distribution is geometric which is given by:

$$Pr(N = k) = \rho^k(1 - \rho) \quad k = 0, 1, 2 \dots$$

Let W_i^q denote the waiting time of a type- i customer in the queue. Also let $\tilde{W}_i^q$ and $\tilde{W}_i$ denote the waiting time and the sojourn time of a customer in the $\tilde{I}^0$ system, respectively. Following proposition shows that the expected waiting and sojourn times of customers can be written as a fraction of the corresponding times in an $\tilde{I}^0$ queueing system.

Proposition 4.1. *Expected waiting times and sojourn times of type- i customer can be written as:*

$$\begin{aligned} E[W_i^q] &= \rho^y E[\tilde{W}_i^q] \\ E[W_i] &= \rho^y E[\tilde{W}_i]. \end{aligned} \tag{4.2}$$

Proof. See Appendix B.1. □

The intuition is that a type- i customer does not wait in the queue and leaves immediately if there are $y - 1$ or less outstanding orders. Otherwise its expected waiting time is exactly the same as the expected waiting time of a type- i customer

in the $\tilde{I}^0$ queueing system. Since $P(N \geq y) = \rho^y$, Equation 4.2 follows. A similar argument is valid for the expected sojourn times.

The manufacturer's problem is to maximize its steady state profit rate, which consists of the revenue and inventory holding cost. The manufacturer does not incur any backorder cost. This is due to the following fact. Assume that the manufacturer have a large amount of products in its inventory. This enables him to set the price approximately equal to the reservation price, i.e. $p_i \approx R_i$ for type- i customer since the customers does not wait at all and hence does not incur any waiting cost. As the order-up-to level decreases, waiting time and waiting cost increases which reduces the price that can be charged to the customers. Hence the price of the product can be at most R_i and decreases with the waiting time. Therefore the manufacturer already incurs a cost due to the delays of the customers. Although an explicit backorder cost can still be employed, it does not give any extra flavor to the solution. Then the manufacturer's problem is given by the following:

$$\begin{aligned} \max_{\underline{p}, y, u} \quad & \sum_{i=1}^M p_i \lambda_i - hE[(y - N)^+]. \\ & p_i = R_i - C_i E[W_i(\underline{\lambda}, u)] \quad i = 1, 2, \dots, M \\ & \underline{p} \geq 0, y \geq 0; u \in \mathbf{U} \end{aligned} \quad (4.3)$$

where h is the inventory holding cost per unit time. Using Equations 4.1 and 4.2, the manufacturer's problem in Equation 4.3 can be rewritten as a function of $\underline{\lambda}$, y and r :

$$\max_{y \geq 0, \underline{\lambda} \geq 0; u \in \mathbf{U}} \left\{ \sum_{i=1}^M \lambda_i R_i - \rho^y \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda}, u)] - hE[(y - N)^+] \right\}. \quad (4.4)$$

4.2. Solution Approaches, Solutions and Analysis

In this section, we give the optimal solution for priority scheduling, provide an optimal order-up-to level when arrival rates are given and show that some customer types can be dominated, i.e. they do not exist in the optimal solution. Finally, we

show that the optimal prices for the problem in Equation 4.4 are still optimal when the manufacturer has asymmetric information, i.e. knows the attributes of the customers but does not know the type of an arriving customer.

4.2.1. Priority Rule Decision

In this section, we give the optimal priority rule. The celebrated $C\mu$ rule (Cox and Smith, 1961) states that, for given incoming rates λ_i , total expected waiting time in the $\tilde{I}^0$ queueing system is minimized if the priorities are given in decreasing order of $C_i\mu_i$. Therefore if $C_1\mu_1 > C_2\mu_2 > \dots C_M\mu_M$, then the total expected waiting time is minimized when type-1 job has the highest priority, type-2 has the second highest and so on. When service rates are equal, this rule reduces to assigning priorities according to waiting costs only. We show that $C\mu$ rule is the optimal priority scheduling rule for the problem in Equation 4.4.

Proposition 4.2. *For the optimization problem in Equation 4.4, the $C\mu$ rule gives the optimal priority schedule.*

Proof. Since the distribution of N does not depend on the scheduling rule u , the value of the holding cost ($hE[(y - N)^+]$) is independent of the variable u . Similarly, $\sum_{i=1}^M \lambda_i R_i$ is also independent of u , therefore the scheduling rule only changes the value of the total waiting costs. For a given $\underline{\lambda}$, $C\mu$ rule is known to be optimal for minimizing the total waiting costs in a system. Let u^0 be the schedule obtained by the $C\mu$ rule and $\underline{\lambda}^* = (\lambda_1^*, \dots, \lambda_M^*)$, y^* and u^* be the optimal decisions which maximize the problem in Equation 4.4. Then the following can be written:

$$\max_{\underline{\lambda} \geq 0, y \geq 0, u \in \mathbf{U}} \left\{ \sum_{i=1}^M \lambda_i R_i - \rho^y \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda}, u)] - hE[(y - N)^+] \right\}$$

$$\begin{aligned}
&= \sum_{i=1}^M \lambda_i^* R_i - \rho^{y^*} \sum_{i=1}^M C_i \lambda_i^* E[\tilde{W}_i(\underline{\lambda}^*, u^*)] - hE[(y^* - N)^+] \\
&\leq \sum_{i=1}^M \lambda_i^* R_i - \rho^{y^*} \sum_{i=1}^M C_i \lambda_i^* E[\tilde{W}_i(\underline{\lambda}^*, u^0)] - hE[(y^* - N)^+] \\
&\leq \max_{\lambda \geq 0, y \geq 0} \left\{ \sum_{i=1}^M \lambda_i R_i - \rho^y \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda}, u^0)] - hE[(y - N)^+] \right\}
\end{aligned}$$

where the first inequality follows from the definition of u^0 and the second inequality is by definition. $\square$

(Mendelson and Whang, 1990) show that $C\mu$ rule is optimal for a similar model where there is no inventory. Without loss of generality, let the types of customers be numbered such that $C_1 > C_2 > \dots > C_M$. Then $C\mu$ rule states that, type-1 customer having the highest priority, type-2 having the second highest and so on, is the optimal priority scheme.

4.2.2. Inventory Decision

In this section, we give the optimal order-up-to level when arrival rates are given. Since $C\mu$ rule is the optimal priority rule, we suppress u_0 from $E[\tilde{W}_i(\underline{\lambda}, u^0)]$, and the problem in Equation 4.4 reduces to the following:

$$\max_{y \geq 0, \underline{\lambda} \geq 0} \left\{ \sum_{i=1}^M \lambda_i R_i - \rho^y \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})] - hE[(y - N)^+] \right\}. \quad (4.5)$$

Let $\Pi(y, \underline{\lambda})$ denote the maximand in Equation 4.5:

$$\Pi(y, \underline{\lambda}) = \sum_{i=1}^M \lambda_i R_i - \rho^y \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})] - hE[(y - N)^+].$$

We will show that the second difference of the $\Pi(y, \underline{\lambda})$ over y is negative, hence $\Pi(y, \underline{\lambda})$ is concave in y . Then for a given $\underline{\lambda}$, optimal y can be found from first order differences

(FOD) which are the discrete versions of the first order conditions (FOC) for the continuous domain. Let $y(\underline{\lambda})$ denote the optimal order-up-to level for a given $\underline{\lambda}$ and define v such that $v = (1 - \rho)/\rho$. Since N has a geometric distribution, its expected value is $\rho/(1 - \rho)$ and $v^{-1} = E[N]$. Also let $H(\underline{\lambda})$ denote the expected waiting cost rate incurred by all types of customers in the $\tilde{I}^0$ queueing system. Then

$$H(\underline{\lambda}) = \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})]$$

Note that $H(\underline{\lambda}) \geq 0$ since $\tilde{W}_i \geq 0$ for $i = 1, \dots, M$. Let $\mathbb{N}$ denote the set of natural numbers.

Proposition 4.3. $\Pi(y, \lambda)$ is concave in y and $y(\underline{\lambda})$ is given by the following:

$$y(\underline{\lambda}) = \min \left\{ y : y \in \mathbb{N}; \rho^{y+1} \leq \frac{h}{h + vH(\underline{\lambda})} \right\}. \quad (4.6)$$

Proof. Define the first difference of $\Pi(y, \underline{\lambda})$ for y as $\Delta\Pi(y) = \Pi(y + 1, \underline{\lambda}) - \Pi(y, \underline{\lambda})$ which is given by:

$$\Delta\Pi(y) = -h(1 - \rho^{y+1}) + \rho^y(1 - \rho)H(\underline{\lambda}) \quad (4.7)$$

since $P(N \leq y) = 1 - \rho^{y+1}$. Let $\Delta^2\Pi(y) = \Delta\Pi(y + 1) - \Delta\Pi(y)$ be the second difference. Then $\Delta^2\Pi(y) = -h\rho^{y+1}(1 - \rho) - \rho^y(1 - \rho)^2H(\underline{\lambda})$. So $\Delta^2\Pi(y)$ is negative and $\Pi(y, \underline{\lambda})$ is concave in y . Then $y(\underline{\lambda})$ is the minimum nonnegative integer that has a zero or negative $\Delta\Pi(y)$ and it is given by Equation 4.6. $\square$

There are several observations to be inferred from Equation 4.6. Before getting into the observations, note that $vH(\underline{\lambda})$ increases with the waiting cost. First, for a given rate vector, the optimal order-up-to level decreases with the holding cost and increases with the waiting costs. Second, the optimal order-up-to level for a given arrival rate vector depends on only the unit waiting costs C_i and the holding cost h , but not on the reservation prices R_i . This is due to the fact once arrival rates are

given, the characteristics of the system like the expected number of customer is known and this is sufficient to calculate the order-up-to level. Another observation is that, rearranging the inequality in Equation 4.6, $y(\underline{\lambda})$ can also be found from the following inequality:

$$y(\underline{\lambda}) = \min \left\{ y : y \in \mathbb{N}; P(y \geq N) \geq \frac{vH(\underline{\lambda})}{h + vH(\underline{\lambda})} \right\} \quad (4.8)$$

which is similar to the critical fractile solution in a newsvendor problem where $vH(\underline{\lambda})$ is the equivalent underage cost. Recall that in the newsvendor problem, the underage cost is incurred when there is no product available. Finally, the term $vH(\underline{\lambda})$ can be written as:

$$vH(\underline{\lambda}) = \frac{1}{E[N]} \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i] = \frac{\sum_{i=1}^M C_i E[\tilde{L}_i]}{E[N]}$$

where $\tilde{L}_i$ is the number of type- i customers in the $\tilde{I}^0$ queueing system. Hence $vH(\underline{\lambda})$ is some sort of a weighted waiting cost where the weights are the number of type- i customers in the system.

4.2.3. Pricing Decision

In this section, we find the prices that are given by the first order conditions. Moreover we show that a customer type can dominate less profitable customer types and the manufacturer sets high prices for the dominated customers to prevent them from buying the product. Before getting into the results, we give the following definitions and identities from (Heyman and Sobel, 2004). $E[\tilde{W}_i^q(\underline{\lambda})]$ and $E[\tilde{W}_i(\underline{\lambda})]$ can be written as follows:

$$\begin{aligned} E[\tilde{W}_i^q(\underline{\lambda})] &= \frac{\alpha_M}{\bar{\alpha}_{i-1}\bar{\alpha}_i} \\ E[\tilde{W}_i(\underline{\lambda})] &= E[\tilde{W}_i^q(\underline{\lambda})] + \frac{1}{\mu} = \frac{\alpha_M}{\bar{\alpha}_{i-1}\bar{\alpha}_i} + \frac{1}{\mu} \end{aligned} \quad (4.9)$$

where $\alpha_0 = 0$, $\alpha_i = \sum_{k=1}^i \lambda_k$, $\bar{\alpha}_i = \mu - \alpha_i$ for all $i = 1, \dots, R$. Since $E[W_i(\underline{\lambda})] = \rho^y E[\tilde{W}_i(\underline{\lambda})]$ and α_M is constant, $E[W_1] < E[W_2] < \dots < E[W_R]$.

Let $\underline{\lambda}(y)$ denote the solution found from the FOC of the function $\Pi(y, \lambda)$ for a given y . Also let $p_i(y)$ denote the corresponding prices for type- i customer. They are given by the following lemma.

Proposition 4.4. *Prices for type- i customer $p_i(y)$ are given by the following:*

$$p_i(y) = \frac{\partial \rho^y}{\partial \lambda_i} \sum_{k=1}^M C_k \lambda_k E[\tilde{W}_k(\underline{\lambda})] + h \frac{\partial E[(y - N)^+]}{\partial \lambda_i} + \rho^y \left\{ \sum_{k=1}^M \frac{\lambda_k^* C_k}{\bar{\alpha}_k \bar{\alpha}_{k-1}} + \sum_{k=i}^M \frac{\lambda_k^* C_k E[\tilde{W}_k^q(\underline{\lambda})] + \lambda_{k+1}^* C_{k+1} E[\tilde{W}_{k+1}^q(\underline{\lambda})]}{\bar{\alpha}_k} \right\} \quad (4.10)$$

Proof. See Appendix B.2. □

Since the derivative of ρ and the expected inventory is same for all λ_i , first three terms are the same for all $p_i(y)$. Hence the prices are increasing, i.e. $p_1(y) > p_2(y) > \dots > p_R(y)$. So the customers can increase their priority by paying a higher price.

The following lemma proves that changing the order of priority among identical customers does not affect the overall system.

Lemma 4.1. *Consider any two consecutive customer types, say type- i and type- $i + 1$ in the $\tilde{I}^0$ system. The total expected number of type- i and type- $i + 1$ customers in the system depends on the sum of the arrival rates $(\lambda_i + \lambda_{i+1})$ but not on their individual values or the priority rules between two customer types.*

Proof. Without loss of generality, consider customer type-1 and type-2. In the current priority setting, the expected number of customer type-1 and type-2 in the system is

given by the following:

$$E[\tilde{L}_1(\underline{\lambda}, u^0)] + E[\tilde{L}_2(\underline{\lambda}, u^0)] = \lambda_1 \left[\frac{\lambda}{(\mu)(\mu - \lambda_1)} + \frac{1}{\mu} \right] + \lambda_2 \left[\frac{\lambda}{(\mu - \lambda_1)(\mu - \lambda_1 - \lambda_2)} + \frac{1}{\mu} \right]$$

which reduces to the following:

$$E[\tilde{L}_1] + E[\tilde{L}_2] = \frac{\lambda(\lambda_1 + \lambda_2)}{\mu(\mu - \lambda_1 - \lambda_2)} + \frac{\lambda_1 + \lambda_2}{\mu} \quad (4.11)$$

Let $\hat{r}$ be the priority rule such that the order of the priority level of customer type-1 and type-2 is exchanged, i.e. $\hat{r} = \{2, 1, 3, \dots, M\}$. The total expected waiting number of customers under $\hat{r}$ is given by the following:

$$E[\tilde{L}_1(\underline{\lambda}, \hat{r})] + E[\tilde{L}_2(\underline{\lambda}, \hat{r})] = \lambda_1 \left[\frac{\lambda}{(\mu - \lambda_2)(\mu - \lambda_1 - \lambda_2)} + \frac{1}{\mu} \right] + \lambda_2 \left[\frac{\lambda}{(\mu)(\mu - \lambda_2)} + \frac{1}{\mu} \right]$$

which reduces to the same value in Equation 4.11. $\square$

Note that the expected sojourn time and expected number of customer of other customer types in the system, i.e. customer types $3, \dots, M$, do not change by the definitions in Equation 4.9. Let p_i^* and λ_i^* denote the optimal price and optimal arrival rate for customer type- i for the problem in Equation 4.5, respectively. Proposition 4.5 shows how a customer type is dominated by a more profitable type.

Proposition 4.5. *Given that all customer types are different, if $R_i \geq R_j$ and $C_i < C_j$, then type- i dominates type- j in the optimal solution of the problem in Equation 4.5. The optimal rate for the dominated customer is $\lambda_j^* = 0$ which is achieved by setting $p_j^* = R_j$.*

Proof. Without loss of generality assume that $M \geq 2$, $R_1 < R_2$ and $C_1 \geq C_2$. We will show that type-2 customer dominates type-1 customer and hence $\lambda_1^* = 0$. The problem

in Equation 4.5 can be rewritten as follows:

$$\max_{y \geq 0, \underline{\lambda} \geq 0} \left\{ \begin{array}{l} (\lambda_1 + \lambda_2)R_2 + \sum_{i=3}^M \lambda_i R_i - h(y - v^{-1}(1 - \rho^y)) \\ -\rho^y \left[C_2 \left(\lambda_1 E[\tilde{W}_1(\underline{\lambda})] + \lambda_2 E[\tilde{W}_2(\underline{\lambda})] \right) + \sum_{i=3}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})] \right] \\ -\lambda_1(R_2 - R_1) - \rho^y(C_1 - C_2)\lambda_1 E[\tilde{W}_1(\underline{\lambda})] \end{array} \right\}. \quad (4.12)$$

The problem in Equation 4.5 is the same with the problem in Equation 4.12. Hence we have to show that at optimality $\lambda_1 = 0$ for the problem in Equation 4.12. Consider the problem which employs only the first two lines of Equation 4.12 as the objective function:

$$\max_{y \geq 0, \underline{\lambda} \geq 0} \left\{ \begin{array}{l} (\lambda_1 + \lambda_2)R_2 + \sum_{i=3}^M \lambda_i R_i - h(y - v^{-1}(1 - \rho^y)) \\ -\rho^y \left[C_2 \left(\lambda_1 E[\tilde{W}_1(\underline{\lambda})] + \lambda_2 E[\tilde{W}_2(\underline{\lambda})] \right) + \sum_{i=3}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})] \right] \end{array} \right\}. \quad (4.13)$$

Let $\underline{\lambda}^- = (\lambda_1^-, \dots, \lambda_M^-)$ be the optimal solution vector for problem in Equation 4.13. Type-1 and type-2 customers are equivalent in the problem in Equation 4.13. Hence any solution vector that satisfies $\lambda_1 + \lambda_2 = \lambda_1^- + \lambda_2^-$ and $\lambda_i = \lambda_i^-$ for $i = 3, \dots, M$ is also optimal for the problem in Equation 4.13. This is due to Lemma 4.1 and the fact that the value of $(y - v^{-1}(1 - \rho^y))$ depends on only *total* arrival rate λ . Let $\Pi^-(\lambda, y)$ be the maximand in Equation 4.13. Since $(R_2 - R_1)$, $(C_1 - C_2)$ and $E[\tilde{W}_1(\underline{\lambda})]$ are non-negative, $\Pi^-(\underline{\lambda}, y) \geq \Pi(\underline{\lambda}, y)$ for any given $\underline{\lambda}$ and y . Then $\underline{\lambda}' = (0, \lambda_1^- + \lambda_2^-, \lambda_3^-, \dots, \lambda_M^-)$ is also optimal for the problem in Equation 4.12 since $\underline{\lambda}'$ is optimal for the problem in Equation 4.13 and $\Pi^-(\underline{\lambda}', y) \geq \Pi(\underline{\lambda}', y)$. $\square$

4.2.4. Asymmetric Information: Incentive Compatibility

In this section, we show that solutions found in previous sections are also optimal when the manufacturer knows the attributes of the customer types but does not explicitly know the type of an arriving customer. So far it is assumed that the manufacturer has full information: it knows the types of every arriving customer and charges the customer with the price of his own type. However this is not always the case. The

manufacturer has full information about the attributes of the customer types but may not know the exact type of an arriving customer. What the manufacturer can do is to announce a menu including the prices and the inventory level, and let the customers choose a price. By choosing a price from the menu, the customer *reveals* its own type.

In such a setting, the prices that are optimal for the problem in Equation 4.5 stays optimal as long as a type- i customer *chooses* p_i . However if a type- i customer has an *incentive* to choose another price, say p_j , then the decisions are not optimal any more. Therefore the problem for the manufacturer is to announce an *incentive compatible* price menu. In an incentive compatible price menu, a type i customer chooses p_i among the price menu to maximize his own utility. In other words, an incentive compatible price menu satisfies the following:

$$p_i + C_i E[W_i] < p_j + C_i E[W_j] \quad \forall j \neq i \quad (4.14)$$

for every type- i customer.

Next lemma is the key point in the following theorem. It shows that $p_i(y) \forall i = 1, \dots, M$ are incentive compatible.

Lemma 4.2. $p(y) = (p_1(y), \dots, p_M(y))$ is incentive compatible. That is

$$p_i(y) + C_i E[W_i] < p_k(y) + C_i E[W_k] \quad \text{for any } k \in \{1, \dots, R\} - \{i\}$$

Proof. See Appendix B.3. □

We conclude that the prices at the optimal order-up-to level are incentive compatible as well. This is given by the following theorem:

Theorem 4.1. $p(y^*)$ is incentive compatible.

Proof. Lemma 4.2 shows that for any given y , prices are incentive compatible. Hence they are incentive compatible for the optimal inventory level y^* as well. $\square$

As a summary, first we study the problem in Equation 4.4 under the assumption the manufacturer has full information about the types of arriving customers. We show that $C\mu$ rule is optimal as a priority rule. We give the optimal order-up-to level when arrival rates are given. We also show that a customer type can be dominated by a more profitable customer type and a dominated customer type does not appear in the solution. Then we show that the optimal solutions found in the full information case stay optimal if the optimal rates found from the first order conditions are feasible.

4.3. Continuous Approximation

Since N and y are discrete, it is hard to carry on analytical study over the profit function when searching for optimal y . In this section, we make a continuous approximation for the distribution of N . Then we show that $C\mu$ rule is optimal and provide optimal order-up-to level for given arrival rates.

The geometric distribution of N can be approximated with an exponential distribution with rate $1/E[N]$. Using the definition of v , the rate is given by v . Although this approximation can change the quantitative values, the qualitative structure of the problem still stays intact. Table 4.1 shows the the original probabilities versus approximated probabilities.

Using the exponential distribution, $P(N \geq y)$ can be approximated by:

$$P(N \geq y) = \rho^y \approx \int_y^\infty v e^{-vx} dx = e^{-vy}. \quad (4.15)$$

and $E[(y - N)^+]$ can be approximated by:

$$E[(y - N)^+] \approx \int_0^y (y - x) v e^{-vx} dx = y - v^{-1} (1 - e^{-vy}) \quad (4.16)$$

Table 4.1. Approximation Values.

		P(X≤2)	P(X≤5)	P(X≤10)
p=0.2	Geo	0,992	1,000	1,000
	Exp	1,000	1,000	1,000
p=0.5	Geo	0,875	0,984	1,000
	Exp	0,865	0,993	1,000
p=0.8	Geo	0,488	0,738	0,914
	Exp	0,393	0,713	0,918

Using Equations 4.15 and 4.16, the approximated problem of the manufacturer in Equation 4.4 can be written as follows:

$$\max_{y \geq 0, \underline{\lambda} \geq 0, u \in \mathbf{U}} \left\{ \sum_{i=1}^M \lambda_i R_i - e^{-vy} \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda}, u)] - h \left(y - \frac{1}{v} (1 - e^{-vy}) \right) \right\}. \quad (4.17)$$

We have the following proposition for the optimality of $C\mu$ rule for the priority scheduling of the problem in Equation 4.17.

Proposition 4.6. *For the optimization problem in Equation 4.17, the $C\mu$ rule gives the optimal priority schedule.*

Proof. Since $\sum_{i=1}^M \lambda_i R_i$ and $-h(y - v^{-1}(1 - e^{-vy}))$ does not depend on the priority schedule rule, the proof is similar to that of the Proposition 4.2. $\square$

Substituting $u = u^0$ in $E[\tilde{W}_i(\underline{\lambda}, u)]$, the problem in Equation 4.17 reduces to following:

$$\max_{y \geq 0, \underline{\lambda} \geq 0} \left\{ \sum_{i=1}^M \lambda_i R_i - e^{-vy} H(\underline{\lambda}) - h \left(y - C^{-1} (1 - e^{-vy}) \right) \right\}. \quad (4.18)$$

Let $\pi(y, \lambda)$ denote the maximand in Equation 4.18. Since $\partial \pi / \partial y = v e^{-vy} H(\underline{\lambda}) -$

$h + he^{-vy}$ and $\partial^2\pi/\partial y^2 = -v^2e^{-vy}H(\underline{\lambda}) - hve^{-vy} < 0$, the profit function of the manufacturer is concave in y . Therefore for a given $\underline{\lambda}$, FOC is necessary and sufficient for finding optimal order-up-to level for the problem in Equation 4.18. Let $y^a(\underline{\lambda})$ denote the optimal order-up-to level given $\underline{\lambda}$. Then $y^a(\underline{\lambda})$ satisfies the equation $e^{-vy} = \frac{h}{h+vH(\underline{\lambda})}$ which can also be derived by substituting ρ^y with e^{-vy} in Equation 4.6. Rearranging the equation, the optimal order-up-to level for a given arrival rate vector is given by:

$$y^a(\underline{\lambda}) = v^{-1} \log \left(\frac{h + vH(\underline{\lambda})}{h} \right). \quad (4.19)$$

Following proposition gives an analogous result to Proposition 4.5 for the approximated problem. Since the proof is identical, we give the proposition without proof. Let $\underline{\lambda}^a = (\lambda_1^a, \dots, \lambda_M^a)$ and $\underline{p}^a = (p_1^a, \dots, p_M^a)$ denote the optimal arrival rate vector and corresponding optimal price vector for the problem in Equation 4.18.

Proposition 4.7. *Given that all customer types are different, if $R_i \geq R_j$ and $C_i \leq C_j$, then type- i dominates type- j in the optimal solution of the problem in Equation 4.18. The optimal rate for the dominated customer is $\lambda_j^a = 0$ which is achieved by setting $p_j^a = R_j$.*

We give two corollaries for Proposition 4.7 in Section 4.4 which provides explicit solutions for the cases where the customer types have equal waiting costs or equal reservation prices.

Let $\underline{\lambda}^f(y)$ denote the solution found from the FOC of the function $\pi(y, \lambda)$ for a given y . Also let $p_i^f(y)$ denote the corresponding prices for type- i customer. Next theorem shows a similar result to Theorem 4.1:

Theorem 4.2. *$p^f(y^a)$ is incentive compatible.*

The advantage of the approximation is that $y(\underline{\lambda})$ can be substituted into the profit function in Equation 4.18. Then the problem can be written as a function of $\underline{\lambda}$ only:

$$\max_{\underline{\lambda} \geq 0} \left\{ \sum_{i=1}^M \lambda_i R_i - hv^{-1} \log \left(\frac{h + vH(\underline{\lambda})}{h} \right) \right\}. \quad (4.20)$$

Hence the problem reduces to choose the arrival rates. Although we cannot show the concavity or the quasi-concavity of the maximand in Equation 4.20, several numerical analysis show that it is at least quasi-concave. In Section 4.4 we analyze the case where there is a single customer type for which the continuous approximation enables to explicitly solve the optimal parameters and to make sensitivity analysis. This analysis provides valuable insights for the general structure of the multi-customer case as well. Before getting into single type customer case, we analyze the case where holding inventory is expensive.

4.3.1. If Inventory Holding is Expensive

When the holding cost is high with respect to the waiting costs, $vH(\underline{\lambda})/h \approx 0$. The Taylor series expansion of degree one for the logarithm function around the point x_0 is given by the following:

$$\log(x) = \log(x_0) + (x - x_0)\frac{1}{x_0} + O(x^2). \quad (4.21)$$

The expansion of $\log(x)$ around $x_0 = 1$ is given by $\log(x) \approx x - 1$. Using this approximation $\log(1 + vH(\underline{\lambda})/h) \approx vH(\underline{\lambda})/h$ can be written. Therefore, if the holding cost is high, the optimal order-up-to level for a given $\underline{\lambda}$ is approximately given by $y^*(\underline{\lambda}) \approx H(\underline{\lambda})/h$. which can be rewritten as $hy^*(\underline{\lambda}) \approx H(\underline{\lambda})$. The left hand side ($hy^*(\underline{\lambda})$) is the holding cost rate. The right hand side ($H(\underline{\lambda})$) is the expected waiting cost rate incurred by all types of customers in the $\tilde{I}^0$ queueing system, i.e. when there is no inventory. Therefore, when holding inventory is expensive, the manufacturer chooses an order-up-to level which equates the rate of the holding cost to the rate of the total expected waiting cost in a system without inventory.

4.4. Single Customer Type

In this section, we analyze the problem when there is a single customer type. We explicitly calculate the optimal solutions and make sensitivity analysis on the optimal

decisions with respect to the holding and waiting costs for the approximated problem. Assume that there is only a single customer type, i.e. $R_i = R$ and $C_i = C$ for $i = 1, \dots, M$. Then $vH(\underline{\lambda}) = c$ and the optimal order-up-to level in Equation 4.19 is given by $y^a(\rho) = v^{-1} \log(1 + C/h)$. Similar to Equation 4.8, optimal order-up-to level can be found from $P(y \leq N) = \frac{C}{h+C}$. With this solution, the analogy with the newsvendor problem is more obvious: The waiting cost per unit time (C) turns out to be the equivalent underage cost of the critical fractile solution of the newsvendor problem. Substituting $vH(\underline{\lambda}) = C$ in Equation 4.20, the problem reduces to the following:

$$\max_{\rho \in [0,1)} \left\{ R\mu\rho - h \frac{\rho}{1-\rho} \log \left(1 + \frac{C}{h} \right) \right\}. \quad (4.22)$$

Hence the decision of finding arrival rates reduces to finding ρ , or equivalently finding a single rate λ . Let $\hat{\pi}(\rho)$ denote the maximand in Equation 4.22. Define $\hat{\rho}$ as the value which satisfy the FOC, i.e. $\hat{\pi}'(\hat{\rho}) = 0$. Also define $K = h \log(1 + C/h)$ as the cost related coefficient. Note that $K > 0$. Then $\hat{\rho}$ is given by $\hat{\rho} = 1 - \sqrt{K/(R\mu)}$. The second derivative of $\hat{\pi}(\rho)$ is given by $\hat{\pi}''(\rho) = -\frac{2K}{(1-\rho)^3}$ which is negative, hence $\hat{\pi}$ is concave. So $\hat{\rho}$ is a global maximizer for $\hat{\pi}$. Let ρ^a denote the optimal ρ for the problem in Equation 4.22. Since ρ^a must be non-negative and $\hat{\pi}$ is concave, ρ^a is given by the following:

$$\rho^a = \begin{cases} 1 - \sqrt{\frac{K}{R\mu}} & \text{if } R\mu \geq K \\ 0 & \text{o/w} \end{cases}$$

Since the solution is trivial when $R\mu < K$, from now on we assume that $R\mu \geq K$.

The optimal values of the arrival rate ($\mu\rho^a$), the waiting time ($E[W^a]$), the waiting cost ($CE[W^a]$), the price (p^a) the order-up-to level (y^a) and the profit ($\hat{\pi}^a$) are found by substituting ρ^a into corresponding formulations. Our aim is to conduct sensitivity analysis of the problem with respect to cost parameters h and C . Hence we give the optimal values of the variables, the total expected waiting time, and the total expected waiting cost and their partial derivatives in Table 4.2. Their derivation is given in Appendix B.4. The subscripts denote the partial derivatives with respect to h or C .

We need K_h and K_C for the analysis of the derivatives of the variables.

We first analyze the changes of the optimal values with respect to the holding cost h . In order to show that $K_h > 0$, let $u = \frac{h+C}{h}$ and rewrite K_h as a function of u :

$$K_h(u) = \log(u) + \frac{1}{u} - 1 \quad (4.23)$$

Since h and C are positive, $u > 1$. Hence we analyze the sign of $K_h(u)$ for $u > 1$. The derivative of $K_h(u)$ is given by $K'_h(u) = 1/u - 1/u^2$. $K_h(u)$ is increasing for $u > 1$ since $K'_h(u) > 0$ for $u > 1$. Then $K_h(u) > 0$ for $u > 1$ since $K_h(1) = 0$ and $K_h(u)$ is increasing for $u > 1$.

When holding cost increases, the manufacturer starts to hold less inventory ($y_h^a < 0$). Low order-up-to levels increase the expected waiting cost, hence the manufacturer reduces the arrival rate to compensate high expected waiting cost ($\rho^a < 0$). The expected waiting time and cost increases with the arrival rate and decreases with the order-up-to level, both of which decrease with the holding cost ($y_h^a < 0$ and $\rho_h^a < 0$). Hence the expected waiting time and the expected waiting cost are exposed to two opposite effects when h increases. It turns out that the effect of the arrival rate is more dominant and the optimal expected waiting cost and time are decreasing with the holding cost ($(E[W^a])_h < 0$ and $(cE[W^a])_h < 0$). The manufacturer tries to compensate its loss due to the increase in holding cost from its customers by increasing the optimal price ($p_h^a > 0$).

The effects of the waiting cost can differ from the effects of the holding cost. The manufacturer reduces the arrival rate in order to decrease the expected waiting cost when C increases ($\rho_C^a < 0$) as in the case of the holding cost. Although $y_h^a < 0$, the optimal order-up-to level does not have a monotonic behaviour with respect to C . Let C^t denote the threshold value of C that satisfies $R\mu = 4K$ for a given h . Also let $C^{\max}$ denote the value that satisfies $R\mu = K$, i.e. the maximum value of C for which the system is profitable for a given h . Then they are given as $C^t = h(\exp(R\mu/(4h)) - 1)$

Table 4.2. Parametric Analysis.

Variable	Value	$\partial/\partial h$	$\partial/\partial C$
K	$h \log(1 + C/h)$	$\log\left(\frac{h+C}{h}\right) + \frac{h}{h+C} - 1$	$\frac{h}{h+C}$
ρ^a	$1 - \sqrt{K/(R\mu)}$	$-K_h/(2\sqrt{R\mu K})$	$-K_C/(2\sqrt{R\mu K})$
y^a	$(\sqrt{R\mu K} - K)/h$	$-\left(\frac{Ch}{2K(h+C)} + 0.5\right)(\sqrt{R\mu K} - K)/h^2 - K_h/2h$	$\frac{K_C}{h} \left(\frac{\sqrt{R\mu} - 2\sqrt{K}}{2K}\right)$
$E[W^a]$	$\sqrt{R/(\mu K)}$	$-(K_h\sqrt{R})/(2\sqrt{\mu K^3})$	$-(K_C\sqrt{R})/(2\sqrt{\mu K^3})$
$CE[W^a]$	$C\sqrt{R/(\mu K)}$	$-C(K_h\sqrt{R})/(2\sqrt{\mu K^3})$	$\frac{h\sqrt{R}}{2\sqrt{\mu K^3}}(2K_h + \frac{C}{h+C})$
p^a	$R - c\sqrt{R/(\mu K)}$	$C(K_h\sqrt{R})/(2\sqrt{\mu K^3})$	$-\frac{h\sqrt{R}}{2\sqrt{\mu K^3}}(2K_h + \frac{C}{h+C})$

and $C^{\max} = h(\exp(R\mu/(h)) - 1)$. It turns out that $y_C^a \geq 0$ if $C \leq C^t$ and $y_C^a < 0$ if $C^t < C \leq C^{\max}$. This effect stems from the following fact. When the waiting cost rate is low, the customers are profitable and the manufacturer increase the optimal order-up-to level in order to reduce the expected waiting time. The cost of increasing y^a is compensated by the revenue generated from the profitable customers. However after the threshold C^t , the customers are not that profitable. Therefore the manufacturer reduces y^a in order to reduce its costs and maintain its profitability.

When $C < C^t$, y^a is increasing and ρ^a is decreasing. Hence it is obvious that the expected waiting time decreases with C ($E[W^a]_C < 0$). Similar to the case of h , there is an opposite effect of C when $C \geq C^t$. It turns out that the decrease in ρ^a is more dominant than the decrease in y^a . Hence the optimal expected waiting time is decreasing with C ($(E[W^a])_C < 0$). The optimal expected waiting cost is given by the product of C and $E[W^a]$. Although $E[W^a]$ is decreasing with C , their product $CE[W^a]$ turns out to be increasing with C ($\{CE[W^a]\}_C > 0$). Although high holding cost is compensated from the customers, it is not the case for high waiting costs. If C increases, then the customers are willing to allocate less budget for the price of the product. Hence the manufacturer reduces the price to keep the customers in the system.

We give two corollaries for Proposition 4.7. First one states that if the customers have equal waiting costs, then in the optimal solution there exists only the customer type with the highest reservation price. Recall that $(y^a, \underline{\lambda}^a)$ is the optimal solution to the problem in Equation 4.18 and $\underline{p}^a$ denote the corresponding optimal price vector.

Corollary 4.1. *Assume that all of the unit waiting costs are identical, i.e. $C_i = C$ and let customer type- k has the greatest reservation price, i.e. $R_k = \max_{i=1,\dots,M} \{R_i\}$. Then the optimal solution for the problem in Equation 4.18 is given as follows: $y^a = (\sqrt{R_k \mu K} - K)/h$, $\lambda_k^a = \mu - \sqrt{\mu K / R_k}$ and $p_j^a = R_j$ (and hence $\lambda_j^a = 0$) for $j \neq k$ where $K = h \log(1 + C/h)$.*

The second corollary states that if the reservation prices are equal, then only the

customer type with the lowest unit waiting cost exists in the optimal solution.

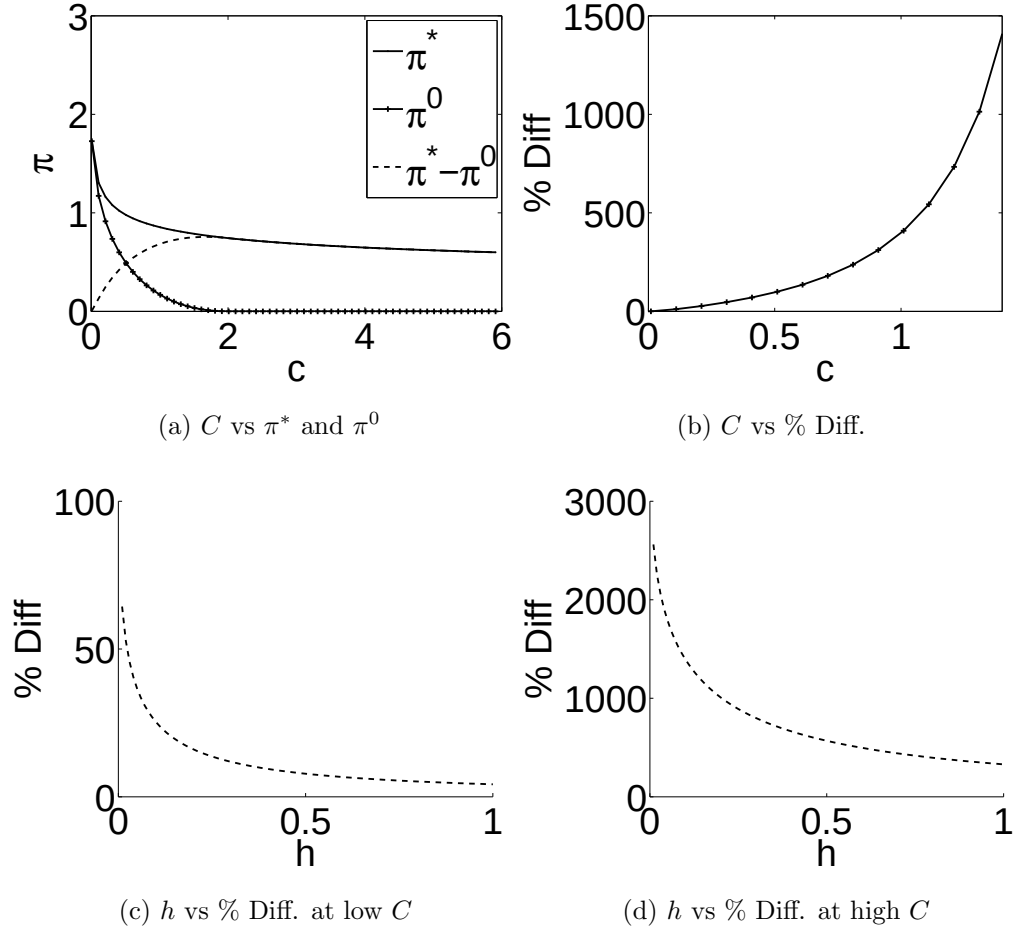
Corollary 4.2. *Assume that all of the reservations prices are identical, i.e. $R_i = R$. Then the optimal solution for the problem in Equation 4.18 is given as follows: $y^a = (\sqrt{R\mu K} - K)/h$, $\lambda_i^a = \mu - \sqrt{\mu K/R}$ and $p_j^a = R_j$ (and hence $\lambda_j^a = 0$) for $j \neq M$ where $K = h \log(1 + C_M/h)$.*

4.4.1. Benefit of Holding Inventory

In this section, we show the benefit of holding inventory by comparing our system $\hat{I}^*$ with the system without any inventory, $\tilde{I}^0$. The latter system is also studied in (Hassin, 2007). Let $\hat{\pi}^*$ and $\hat{\pi}^0$ denote the optimal profits of the systems $\hat{I}^*$ and $\tilde{I}^0$, respectively. Substituting ρ^a into Equation 4.22, the optimal profit of $\hat{I}^*$ is given by $\hat{\pi}^* = (\sqrt{R\mu} - \sqrt{K})^2$. From (Hassin, 2007), $\hat{\pi}^0 = (\sqrt{R\mu} - \sqrt{C})^2$ whenever $R\mu \geq C$. Since $\hat{I}^*$ system is maximized over a larger feasible space ($y \geq 0$) than the $\tilde{I}^0$ system ($y = 0$), we have $\hat{\pi}^* \geq \hat{\pi}^0$. Explicitly, the percent difference in the profits is given by the following:

$$\frac{(\sqrt{R\mu} - \sqrt{K})^2 - (\sqrt{R\mu} - \sqrt{C})^2}{(\sqrt{R\mu} - \sqrt{C})^2} \quad (4.24)$$

Several observations can be inferred from this analysis. First, regardless of the level of the holding cost, the ability to hold inventory is always preferable since $\hat{\pi}^* \geq \hat{\pi}^0$. Second, since $x \geq \log(x+1)$ for $x \geq 0$, one can write $C/h \geq \log(1 + C/h)$ and hence $C > K$. So the $\hat{I}^*$ system maintains its profitability and can continue its operation even if $R\mu \leq C$. Third, the difference in the profits increases when the waiting cost increases (Figure 4.1a). The plots are from a setting where $R = 2$, $\mu = 1$, $h = 0.1$ and $C^{\max} = 2$. The change in the percent difference convexly increases in C and is huge when C is large (Figure 4.1b). Although the change in the holding cost decreases with h , the percent difference depends heavily on the value of the waiting cost. Figure 4.1c shows the change in h when $C = 0.1C^{\max}$ and 4.1d shows the change in h when $C = 0.7C^{\max}$.

Figure 4.1. C vs π^* and π^0 .

One interesting observation is made in the case of high holding cost. When there is a single customer type, $K \approx C$ when holding cost is high since $vH(\underline{\lambda}) = C$. Therefore the $\hat{I}^*$ system reduces to the $\tilde{I}^0$ system. The optimal values of the variables are given by substituting C instead of K to the values given in Table 4.2. All of the approximated values turn out to be equal to the values given in (Hassin, 2007).

4.4.2. The Exact Problem

We now analyze the original problem with single customer type. Substituting the expected on hand inventory $E[(y - N)^+] = y - v^{-1}(1 - \rho^y)$ into the profit function in

Equation 4.5, the single type customer problem can be written as follows:

$$\max_{y \geq 0, \rho \geq 0} \left\{ R\mu\rho - C\rho^y \frac{\rho}{1-\rho} - h \left(y - \frac{\rho}{1-\rho}(1-\rho^y) \right) \right\} \quad (4.25)$$

Hence the problem is now to choose an optimal (y, ρ) pair. This pair satisfies the inequality given in Equation 4.6 which reduces to the following when there is single customer type:

$$y(\rho) = \min \left\{ y | y \in \mathbb{N}; \rho^{y+1} \leq \frac{h}{C+h} \right\}.$$

We give two corollaries for Proposition 4.5 which present similar results with Corollaries 4.1 and 4.2. Recall that $(y^*, \underline{\lambda}^*)$ is the optimal solution to the problem in Equation 4.5.

Corollary 4.3. *Assume that all of the unit waiting costs are identical, i.e. $C_i = C$ and let customer type- k has the greatest reservation price, i.e. $R_k = \max_{i=1, \dots, M} \{R_i\}$. Let (y^k, ρ^k) be the optimal solution to the problem in Equation 4.25 using $R = R_k$. Then the optimal solution for the problem in Equation 4.5 is given as follows: $y^* = y^k$, $\lambda_k^* = \rho^k \mu$, $p_k^* = R_k - CE[W_k(\underline{\lambda}_k^*)]$ and $p_j^* = R_j$ (and hence $\lambda_j^* = 0$) for $j \neq k$.*

Corollary 4.4. *Assume that all of the reservations prices are identical, i.e. $R_i = R$. Let (y^M, ρ^M) be the optimal solution to the problem in Equation 4.25 using $C = C_M$. Then the optimal solution for the problem in Equation 4.5 is given as follows: $y^* = y^M$, $\lambda_M^* = \rho^M \mu$ and $p_M^* = R - C_M E[W_M(\underline{\lambda}_M^*)]$ $p_j^* = R_j$ (and hence $\lambda_j^* = 0$) for $j \neq M$.*

4.5. Numerical Illustrations

In this section, we investigate two types of customers and show the effect of inventory on customers and on the manufacturer. Consider a setting where there are two types of customers with $R_1 = 2$, $R_2 = 1.9$, $C_1 = 0.3$ and $C_2 = 0.1$. Holding cost rate is $h = 0.02$ and production rate is $\mu = 1$. Optimal order-up-to level is $y^* = 13$. Figure 4.2 and 4.3 show plots of values of the variables at optimality with increasing

y . We use exact profit function to make the plots. The observations can be outlined as follows.

First, since type1 customers have high unit waiting cost, they do not enter the system until waiting time reduces to a reasonable amount. As soon as type-1 customers appear in the solution at $y = 7$, their incoming rate exceeds that of the type-2 customers after $y = 10$. As expected, total incoming rate is concave and increasing. Figure 4.2a shows the change of optimal arrival rates with inventory level y .

Second, although the aim for holding inventory is to decrease the waiting cost of the customers, the expected waiting cost of type-2 customers and total expected waiting cost increase with y after some threshold. Figure 4.2b shows the expected waiting costs of a customer who enters the system. This is due to the fact that the system starts to incur a large inventory holding cost as y increases. In order to decrease this cost, the system gradually increases the overall incoming rate as observed in Figure 4.2a. Since type-1 customers have a higher priority, they make use of the large y and have shorter waiting times. However type-2 customers are prone to longer waiting times due to priority rules. The expected number of waiting customers in the system ($\lambda E[W]$) is given in Figure 4.2c. Expected number of type-2 customers is zero until $y = 7$ since $\lambda_2 = 0$.

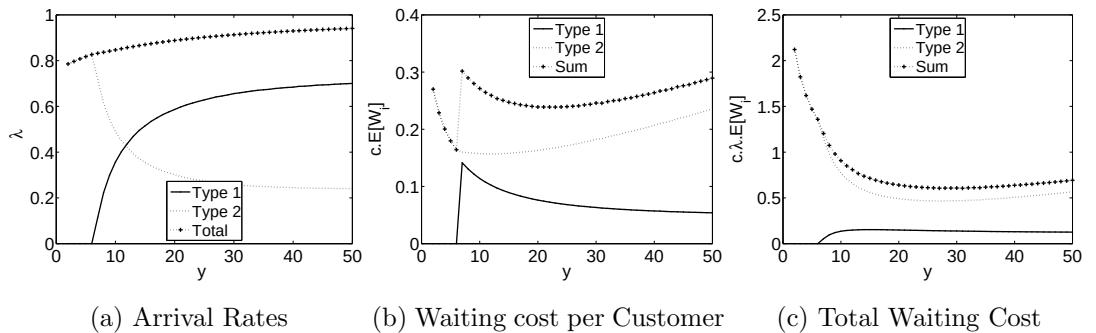


Figure 4.2. Waiting Costs.

Third, since waiting cost of type-1 customer decreases with y , the manufacturer makes use of this and makes more profit by increasing the price of type-1 customers.

However since waiting cost of type-2 customers increase, their price decreases. Figure 4.3a shows the optimal prices with y . The plot of optimal revenue in Figure 4.3b resembles the optimal incoming rates.

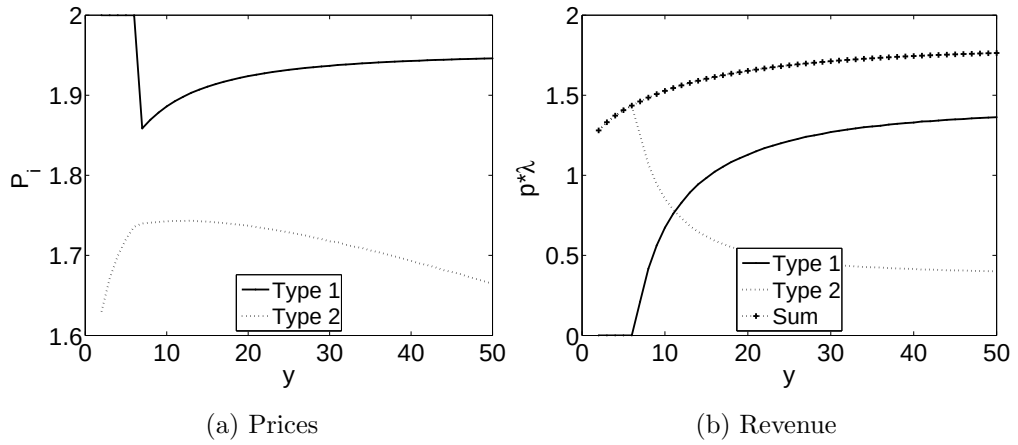


Figure 4.3. Optimal Values of λ , p and λp .

5. CONCLUSION

We analyze the multi period joint pricing and inventory problem of a single item with stochastic demand in the periodic and the continuous review settings. In the period review setting, the demand is modelled to be dependent on the current price and the price history which is captured by the reference price. The randomness is due an additive and a multiplicative random term. The planning horizon can be finite or infinite. In the continuous review problem, we analyze a capacity constrained firm who serves several types of delay sensitive customers. Our aim is to find optimal joint inventory and pricing policy.

In Chapter 3, we analyze the periodic review problem. We first study the case where the randomness is due to an additive random term only. We formulate the problem using the safety stock as a decision variable. We show that the problem can be decomposed into two subproblems if (i) the initial inventory is low, (ii) optimal safety stock is non-decreasing over time and (iii) sum of the mean demand and the random term at distinct periods is non-negative. The first subproblem is a single period inventory problem and the second problem is a deterministic multi-period pricing problem with reference effects. For the finite horizon we provide conditions which satisfy (ii) and (iii). When the time horizon is infinite, condition (ii) becomes trivial since safety stocks are the same throughout the horizon and (iii) reduces to non-negativity of the realized demand. Using the results from the literature, we show that reference price converges to a steady state when planning horizon is infinite and the solutions of the steady state can be characterized.

By formulating the problem with safety stock, we also circumvent the problem of non-convexity of the holding cost and show that an SDO is optimal for concave demand models with concave modified revenue functions. We show that demand models with RDRE or ADRE and with concave base demand satisfy the concavity assumptions. The optimal policy parameters are shown to be equivalent to the optimal solutions of the subproblems which are easier to calculate.

Finally, we transform the problem in such a way that the mean demand becomes the new decision variable and the price is determined through a price function which is dependent on the reference price and the mean demand. We show that when the horizon is infinite, the optimal order-up-to level is increasing with the reference price if the price function is supermodular. The same result holds for the finite horizon case, as long as the optimal order-up-to level is greater than the on-hand inventory level.

Then we analyze the demand models with an additive and also a multiplicative random term. We deal with the concavity problems of the original formulation by transforming the problem into a domain where the mean demand is the decision variable and by defining a larger action space which preserves convexity after the transformation. We show that a SDO is optimal for concave mean demand models and concave transformed expected revenue functions. Our result covers demand models with RDRE where the customers are loss-averse or loss-neutral.

We make several important observations from a computational study, most of which agrees with the theoretical findings over different but similar problems in previous studies. A skimming and penetration strategy depending on the initial reference price is observed to be valid. Excess inventory is depleted by a price discount but a firm can benefit from larger reference prices with larger prices and inventory levels. A firm goes to a price discount to the end of horizon since reference effect loses its importance. The steady state solutions are regardless of the initial reference price but they depend on the parameters of the reference price evolution function. If there is an additive random term in the demand function, the firm can deal with the variance of the demand by only changing the inventory level. However if the multiplicative random term exists in the demand model, the firm must change both the price and the inventory level.

When compared to a myopic firm, dynamic programming yields a substantial increase in the profit. It turns out that dynamic programming has greater importance when (i) time horizon is short, (ii) profit margin is low, (iii) inventory related costs are low and (iv) the variance is high. If the firm is myopic, then it undershoots both the optimal order-up-to levels and prices while overshooting the optimal mean demand

and suffers from inventory related costs. If a firm cannot recognize the existence of the reference effect, then it loses relatively high profits when (i) initial reference price is low, (ii) reference price depends heavily on the past and (iii) the planning horizon is short. It turns out that estimating the initial reference price correctly is more crucial than estimating the correct evolution parameter because the loss due to incorrect estimation of initial reference price is higher than that of the evolution parameter.

In Chapter 4, we study joint inventory and pricing problem in the continuous review setting. We analyze a capacity-constrained manufacturer whose production process is modeled with an $M/M/1$ queueing system with non-preemptive priorities. There are different customer types with respect to their willingness to pay and sensitivity to wait. The manufacturer holds inventory and assigns non-preemptive priorities to the customers in order to minimize total waiting cost and sets priorities and prices in order to maximize revenue. The manufacturer's objective is to maximize its steady state profits. There are three decisions for the manufacturer: the prices, the inventory level and the priority rule.

We first analyze the case where the manufacturer knows the type of an arriving customer. The celebrated $C\mu$ rule is shown to be optimal for the priority rule. Given the prices, we provide the solution for optimal order-up-to level which is similar to the critical fractile solution of the newsvendor model where the equivalent underage cost is given by the waiting cost incurred by the customers. We also provide solutions for the special cases where the reservation prices or waiting costs are equal. In particular we show that a customer type is dominated by another type whose waiting cost (reservation price) is equal but reservation price (waiting cost) is high (low). These solutions are still optimal if the prices are incentive compatible for the case where the manufacturer does not know the type of the arriving customer. We show that the prices found from the first order conditions are incentive compatible.

We make a continuous approximation for the number of outstanding orders. We provide optimal order-up-to level for the approximated problem and show that the optimal prices of the approximate problem are incentive compatible. We also provide

numerical illustrations for two customer types where we analyze the effect of holding inventory. We show the change in the optimal values at different levels of inventory. We make the following observations. The type-1 customer does not enter the system until the expected waiting time reduces to a reasonable amount. Second, when the order-up-to level gets too large, the system increases the arrival rates by reducing the price in order to reduce the holding cost. Since type-1 customers have a higher priority, they make use of the large order-up-to level and have shorter waiting times. However type-2 customers are prone to longer waiting times due to priority rules. The manufacturer increases the price of the type-1 customers since their waiting time decreases and there is more room left for the price of the product. However the price for the type-2 customer decreases since its expected waiting cost increases.

Using the continuous approximation, we provide explicit and simple formulations for the optimal values of the variables when there is single customer type. We also provide parametric analysis with respect to the holding cost and the waiting cost. Several observations are inferred from these analyses. One major observation is that, the optimal order-up-to level increases with the waiting cost when the waiting cost is low but starts to decrease after the threshold C_t . Another interesting observation is that, although the expected waiting time is decreasing with the waiting cost, the expected waiting cost increases with the waiting cost. Lastly, the expected waiting cost increases with the incoming rate and increases with the order-to-level, both of which decrease with the holding cost. Hence when the holding cost decreases, the expected waiting cost is exposed to two opposite effects. It turns out that the effect of decrease in the arrival rate dominates the effect of the decrease in the order-up-to level.

One of the main future directions is to change the reference price mechanism such that it includes other parameters. For example reference price in a perfect market can be modeled to be effected also with the competitor's price. Another direction in this setting is a hard but a promising way: to analyse the case where unsatisfied demand is lost. For the continuous review problem, a manufacturer selling different types of products which has different production times and sharing the same capacity is a promising future direction.

APPENDIX A: PROOFS OF THE RESULTS IN CHAPTER 3

A.1. Proof of Theorem 3.1

Fix a period t' . We show that given Assumption 3.1, if $\Delta_{t+1}^0 \geq \Delta_t^0$ for $t = t', \dots, T$ (ii) and if $(p_{t'}^0(r_{t'}), \Delta_{t'}^0)$ is feasible, i.e. $d_{t'}^0(r_{t'}) + \Delta_{t'}^0 \geq x_{t'}$, then

- (i) I_a^* can mimic the decisions of I^0 in periods $t', \dots, T$.
- (ii) $\Phi_{t'}^*(x, r) = \Phi_{t'}^0(r)$ and $(p_t^0(r_t), \Delta_t^0)$ are indeed optimal for I_a^* for $t = t', \dots, T$.

In order to show the feasibility, we need to show that $x_{t'+1} \leq d_{t'+1}^0(r_{t'+1}) + \Delta_{t'+1}^0$. Since $x_{t'+1} = d_{t'}^0(r_{t'}) + \Delta_{t'}^0 - (d_{t'}^0(r_{t'}) - \epsilon_{t'})$, we need to show following:

$$\Delta_{t'}^0 - \epsilon_{t'} \leq d_{t'+1}^0(r_{t'+1}) + \Delta_{t'+1}^0. \quad (\text{A.1})$$

$d_{t'+1}^0(r_{t'+1}) + \epsilon_{t'} \geq 0$ from Assumption 3.1(ii) and $\Delta_{t'}^0 \leq \Delta_{t'+1}^0$ from the hypothesis of the theorem. Hence Equation A.1 holds and this completes the proof of (i).

For (ii), observe that the profit of the I_a^* in last period cannot be greater than that of I^0 since I_a^* has the same profit function with I^0 but has a smaller feasible region. Hence $\Phi_t^0(r) \geq \Phi_t(x, r)$ for $\forall x$ can be shown inductively for $t = 1, \dots, T$. If the system I_a^* can mimic the decisions of I^0 starting from period t' , then $\Phi_t(x, r) = \Phi_t^0(r)$ for $t = t', \dots, T$. Therefore the decisions of $(p_t^0(r), \Delta_t^0)$ are optimal for I_a^* as well. Since we prove this for arbitrary t' , the proof is complete. $\square$

A.2. Proof of Theorem 3.4

We first show that the value function $\phi_T(\Delta, p, r)$ and the optimal value function $\Phi_T(x, r)$ are concave. Then we show by induction that $\phi_t(\Delta, p, r)$ and $\Phi_t(x, r)$ are

concave for every t .

The revenue function $\Pi_t(p, r)$ is concave by Assumption 3.4 and expected inventory related cost is convex by Assumption 3.3(i). Since $\Phi_0 = 0$, $\phi_1(\Delta, p, r)$ is concave over $(p, r) \in \mathbb{P}$.

The set $\{(\Delta, p, r, x) : \hat{d}_T(p, r) + \Delta - x \geq 0\}$ is a convex set since upper level set of a concave function is a convex set. Therefore the maximization over $\phi_T(\Delta, p, r)$ is done over a convex set. The concavity of $\Phi_T(x, r)$ follows from Theorem A.3 in (Porteus, 2002). For the induction step, we assume that Φ_{t+1} is concave and show that ϕ_t and Φ_T are concave. Since $\Pi_t(p, r)$ is concave by Assumption 3.4 and G_t^γ is convex by Assumption 3.3, in order to show the concavity of ϕ_t , we need to show that $\Phi_{t+1}(\Delta - \epsilon, \alpha r + (1 - \alpha)p)$ is concave in (Δ, p, r) . For arbitrary $\Delta_1, \Delta_2, (p_1, r_1) \in \mathbb{P}$ and $(p_2, r_2) \in \mathbb{P}$, following can be written:

$$\begin{aligned} & \Phi_{t+1}\left(\frac{\Delta_1 + \Delta_2}{2} - \epsilon, \alpha \frac{r_1 + r_2}{2} + (1 - \alpha) \frac{p_1 + p_2}{2}\right) \\ &= \Phi_{t+1}\left(\frac{(\Delta_1 + \epsilon) + (\Delta_2 + \epsilon)}{2}, \frac{(\alpha r_1 + (1 - \alpha)p_1) + (\alpha r_2 + (1 - \alpha)p_2)}{2}\right) \\ &\geq \frac{\Phi_{t+1}(\Delta_1 + \epsilon, \alpha r_1 + (1 - \alpha)p_1) + \Phi_{t+1}(\Delta_2 + \epsilon, \alpha r_2 + (1 - \alpha)p_2)}{2} \end{aligned}$$

where the inequality follows from the concavity of $\Phi_{t+1}(x, r)$. This completes the proof for the concavity of $\phi_t(\Delta, p, r)$ over $\mathbb{P}$. The concavity of $\Phi_T(x, r)$ follows from Theorem A.3 in (Porteus, 2002).

We now prove that an SDO policy is optimal in period t . Given r , let

$$(\Delta_t^*(r), p_t^*(r)) = \arg \max_{p \in \mathbb{P}} \phi_t(\Delta, p, r).$$

If $\Delta_t^*(r) + d_t^*(r) \geq x$, then $(\Delta_t^*(r), p_t^*(r))$ are the optimal decisions for the period and the optimal order-up-to level is given by $y_t^*(r) = \Delta_t^*(r) + d_t^*(r)$.

If the initial inventories are high, i.e. $\Delta_t^*(r) + d_t^*(r) < x$, we argue that the firm

does not give any order. We prove it by showing that the firm can make a better profit by not ordering. Assume that the firm decides on a price $p'_t(r)$ and a safety stock $\Delta'_t(r)$ which are feasible and this decision results in a positive order replenishment. Then we have $\Delta'_t(r) + d'_t(r) > x$ where $d'_t(r) = \hat{d}(p'_t(r), r)$ and $(p'_t(r), r) \in \mathbb{P}$. One can pick a pair $(\bar{\Delta}_t(r), \bar{p}_t(r))$ such that

$$\begin{pmatrix} \bar{\Delta}_t(r) \\ \bar{p}_t(r) \end{pmatrix} = \psi \begin{pmatrix} \Delta_t^*(r) \\ p_t^*(r) \end{pmatrix} + (1 - \psi) \begin{pmatrix} \Delta'_t(r) \\ p'_t(r) \end{pmatrix} \quad (\text{A.2})$$

and $\bar{\Delta}_t(r) + \bar{d}_t(r) = x$ where $\bar{d}_t(r) = \hat{d}(\bar{p}, r)$ and $\psi \in (0, 1)$. This is possible since the pairs $(p_t^*(r), \Delta_t^*(r))$ and $(p'_t(r), \Delta'_t(r))$ are in disjoint spaces and the line passing through those points crosses the boundary which is defined by the set $\{(\Delta, p) : \Delta + \hat{d}(p, r) = x\}$. Note that since $\{p_t^*(r), p'_t(r)\} \in [p^l, p^u]$ and $\bar{p}_t(r)$ is a convex combination of these two, $\bar{p}_t(r) \in [p^l, p^u]$. Hence the decision $(\bar{\Delta}_t(r), \bar{p}_t(r))$ is a feasible one. Now assume that the profit generated by the decision $(p'_t(r), \Delta'_t(r))$ is greater than that of the decision $(\bar{\Delta}_t(r), \bar{p}_t(r))$. Then one can write the following:

$$\phi_t(\bar{\Delta}_t(r), \bar{p}_t(r), r) \leq \psi \phi_t(\Delta_t^*(r), p_t^*(r), r) + (1 - \psi) \phi_t(\Delta'_t(r), p'_t(r), r) \quad (\text{A.3})$$

since $(p_t^*(r), \Delta_t^*(r))$ is the optimal solution for ϕ_t . However this contradicts the concavity of the function ϕ_t . Therefore it is optimal not to order if $\Delta_t^*(r) + d_t^*(r) < x$ and hence $\Delta = x - \hat{d}(p, r)$. The optimal price is the value that maximizes $\phi_t(x - \hat{d}(p, r), p, r)$ for a given r . $\square$

A.3. Proof of Theorem 3.5

(i) The transformed model has a non-positive single-period profit function, i.e.:

$$(p\hat{d}(p, r) - M) - c\hat{d}(p, r) - \hat{G}^\gamma(\Delta) \leq 0$$

since $\hat{G}^\gamma(\Delta) = \hat{G}(\Delta) + c(1 - \gamma)\Delta \geq 0$ by definition. Hence $\Phi_i^* \leq 0$ for all $i = 1, 2, \dots$. Using Proposition 9.17 in (Bertsekas and Shreve, 1996), it suffices to show that for all

$i = 1, 2, \dots$, and all β , the sets:

$$U_i(x, r, \beta) = \left\{ (\Delta, p) : \hat{d}(p, r) + \Delta \geq x, (p, r) \in \mathbb{P} \text{ and } \phi_i^*(\Delta, p, r) \geq -\beta \right\}$$

are compact. From (i) $\Phi_i^* \leq 0$ and (ii) the definition of M , $p\hat{d}(p, r) + \gamma E \left[\Phi_{i-1}^*(\Delta - \epsilon, \alpha r + (1 - \alpha)p) \right] \leq M$ can be written. Subtracting $c\hat{d}(p, r) + \hat{G}^\gamma(\Delta)$ from both sides and rearranging yields the following:

$$\phi_i^*(\Delta, p, r) \leq -c\hat{d}(p, r) - \hat{G}^\gamma(\Delta). \quad (\text{A.4})$$

From Assumption 3.3, $\lim_{|\Delta| \rightarrow \infty} (c\hat{d}(p, r) + \hat{G}^\gamma(\Delta)) = \infty$ on $(p, r) \in \mathbb{P}$. Moreover $\hat{G}^\gamma(\Delta)$ is convex. Hence for a given β , there exists Δ_l^β and Δ_u^β such that $\Delta_l^\beta \leq \Delta \leq \Delta_u^\beta$ implies $c\hat{d}(p, r) + \hat{G}^\gamma(\Delta) \leq \beta$. Define a new set $U^* = \left\{ (\Delta, p) : (p, r) \in \mathbb{P}, \Delta_l^\beta \leq \Delta \leq \Delta_u^\beta \right\}$. Note that U^* is a bounded set. From Equation A.4, $U_i(x, r, \beta) \subseteq U^*$ and hence $U_i(x, r, \beta)$ is also bounded. $\hat{d}(p, r)$ is continuous, and hence ϕ_i and Φ_i are continuous. Therefore $U_i(x, r, \beta)$ is closed since levels sets of continuous functions are closed (Rudin, 1976). Hence $U_i(x, r, \beta)$ is compact for all $i = 1, 2, \dots$

(ii) Φ^* and ϕ^* satisfy the optimality equations in Equation 3.12 from Proposition 9.8 in (Bertsekas and Shreve, 1996). ϕ_t and Φ_t satisfy the optimality equation in the original model (Equation 3.6) since $\phi_t = \phi_t^* + M \frac{1-\gamma^{t+1}}{1-\gamma}$ and $\Phi_t = \Phi_t^* + M \frac{1-\gamma^{t+1}}{1-\gamma}$.

(iii) Concavity of ϕ and Φ follows from Lemma 8.4 in (Heyman and Sobel, 2004).

(iv) Follows from (c) and Theorem 3.4. □

A.4. Proof of Theorem 3.6

Proof. Let $S = \{(x, r) | r \in [p^l, p^u]\}$ denote the state space and let Define

$$S_1 = \{(x, r) \in S : d^0(r) + \Delta^0 \geq x\}$$

and $S_2 = S \setminus S_1$. For the I_a^* system, we will show the following:

- (i) If $(x, r) \in S_1$ and the system I_a^* chooses $(\Delta^0, p^0(r))$, then the new state (x', r') is in S_1 .
- (ii) If $(x, r) \in S_1$, then $\Phi(x, r) = \Phi^0(r)$ and $(p^0(r), \Delta^0)$ is also optimal for the system I_a^* as well.
- (iii) If $(x, r) \in S_2$, then it is optimal not to order for the system I_a^* .

(i) and (ii) are proven similar to Theorem 3.1. For (iii), we show that $(\Delta^0, p^0(r))$ is also a maximizer of the function ϕ over $\mathbb{P}$. In order to do this, we compare the value functions ϕ and ϕ^0 which are given in Equations 3.6 and 3.7, respectively. Note that the only difference between ϕ and ϕ^0 is the terms $E[\Phi(\cdot)]$ and $E[\Phi^0(\cdot)]$. Consider the following inequality:

$$\phi(\Delta, p, r) \leq \phi^0(\Delta, p, r) \leq \phi^0(\Delta^0(r), p^0(r), r) = \phi(\Delta^0(r), p^0(r), r)$$

The first inequality follows from the fact that $\Phi(x, r) \leq \Phi^0(r)$ for all (x, r) as shown in the proof of Theorem 3.1. The second inequality is from the definition of $(\Delta^0(r), p^0(r))$. For the equality, if the decision of the system I_a^* is $(\Delta^0(r), p^0(r))$ in the current period, then the state in the next period remains in S_1 . Once the state is in S_1 , it remains in S_1 and $\Phi_t(x, r) = \Phi_t^0(r)$ from (ii). Therefore $(\Delta^0(r), p^0(r))$ is a maximizer of ϕ over $\mathbb{P}$. Therefore $p^*(r) = p^0(r)$ and $\Delta^*(r) = \Delta^0$ by the definition of $p^*(r)$ and $\Delta^*(r)$. Also $y^*(r) = y^0(r)$ and $d^*(r) = d^0(r)$. $\square$

A.5. Concavity Examples

Consider the demand model in Equation 3.13 and assume that $B(p)$ is concave. The model in Equation 3.13 is concave which is concave if and only if $a_t \leq -2prB_t''(p)$ where $B_t''(p)$ is the second derivative of the base demand function. For example if $B_t(p) = a_t^0 - a_t^1 p^k$, then the demand model is concave if $p^l \geq \left(\frac{a_t^0}{2a_t^1 k(k-1)}\right)^{1/k}$. The modified revenue function is concave for all p, r if $c_t = 0$. Now consider the demand

model

$$\hat{d}_t(p, r) = B_t(p) + a_t(r - p). \quad (\text{A.5})$$

This demand model is concave since $B_t(p)$ is concave. Consider the following transformation which is given by (Chen *et al.*, 2011). Let $\Phi_t^\omega(x, r) = \Phi_t(x, r) - \omega_t r^2$ be the transformed profit-to-go function for the problem in Equation 3.6 where $\omega_t \geq 0$ is a positive scalar. We show that by choosing ω_t carefully, the value function ϕ_t can be shown to be concave. The recursion in Equation 3.6 can be written as follows:

$$\begin{aligned} \phi_t(\Delta, p, r) &= (p - c_t)B_t(p) + (p - c_t)(a_t(r - p)) - \omega_t r^2 \\ &\quad + \gamma \omega_{t+1}(\alpha r + (1 - \alpha)p)^2 - G_t^\gamma(\Delta) + \gamma E\left[\Phi_{t+1}^\omega(\Delta - \epsilon, \alpha r + (1 - \alpha)p)\right] \\ \Phi_t^\omega(x, r) &= \max_{\substack{\Delta, p \\ (p, r) \in \mathbb{P}, \hat{d}_t(p, r) + \Delta \geq x}} \left\{ \phi_t(\Delta, p, r) \right\} - \omega_t r^2 \end{aligned}$$

The term $(p - c_t)B_t(p)$ is concave since $p^l \geq c_t$ by the modelling assumptions. G_t^γ is concave from Assumption 3.3(i). For the concavity of the remaining part of the single period profit function, one must check the hessian matrix which is given in the following:

$$\begin{bmatrix} 2(\gamma(1 - \alpha)^2 \omega_{t+1} - a_t) & a_t + 2\gamma\alpha(1 - \alpha)\omega_{t+1} \\ a_t + 2\gamma\alpha(1 - \alpha)\omega_{t+1} & 2(\gamma\omega_{t+1}\alpha^2 - \omega_t) \end{bmatrix}$$

The hessian matrix is negative semi definite if and only if the following conditions hold:

$$\frac{a_t}{\gamma(1 - \alpha)^2} \geq \omega_{t+1} \quad (\text{A.6})$$

$$\omega_t \geq \frac{n_t^2/4 + a_t\alpha\gamma\omega_{t+1}}{a_t - \gamma(1 - \alpha)^2\gamma_{t+1}} \quad (\text{A.7})$$

Let construct ω_t using Equation A.7 as an equality, i.e. $\omega_{T+1} = 0$, $\omega_T = n_T/4$, ... If these ω_t satisfy Equation A.6, then the hessian matrix is negative semi-definite.

In order to find a sufficient condition over the parameters, let $\omega_{t+1} = k_t \frac{a_t}{\gamma(1 - \alpha)^2}$.

Then the hessian matrix is negative semi-definite if $0 \leq k_t < 1$ and

$$\frac{a_{t-1}}{\gamma a_t} \geq \frac{(1-\alpha)^2 + \alpha k_t}{4(1-k_t)k_{t-1}} \quad (\text{A.8})$$

Hence if there exists k_t which satisfy $0 \leq k_t < 1$ and Equation A.8, then the single period profit function is concave in (Δ, p, r) . For example, if $\frac{a_{t-1}}{a_t} \geq \gamma$, then $k_t = 0.5$ satisfies the required conditions.

A.6. Proof of Theorem 3.7

Without loss of generalite, we supress the time indices. Let $\{(d_1, r_1), (d_2, r_2)\} \in \mathbb{D}$ and $p_{ij} = \hat{p}(d_i, r_j)$.

(i) The case $d_1 \geq d_2$ and $r_1 \geq r_2$ is trivial. Let $d_1 \leq d_2$ and $r_1 \geq r_2$. Then we need to show $p^l \leq p_{12} \leq p^u$ and $p^l \leq p_{21} \leq p^u$. Then $p^l \leq p_{22} \leq p_{12} \leq p_{11} \leq p^u$ can be written since the price function is strictly decreasing with d and increasing with r (Proposition 3.1). Similarly it can be shown that $p^l \leq p_{21} \leq p^u$ which completes our proof of part (i).

(ii) We show that every term of $\hat{\phi}_t^0$ is supermodular. Let $d_2 \geq d_1$ and $r_2 \geq r_1$. We first show that the revenue term is supermodular. Following can be written:

$$d_2(p_{22} - p_{21}) \geq d_1(p_{22} - p_{21}) \geq d_1(p_{12} - p_{11}) \quad (\text{A.9})$$

where the first inequality follows from the fact that $\hat{p}$ is increasing with r (Proposition 3.1) and $d_2 \geq d_1$ and the second inequality follows from supermodularity of the price function (Assumption 3.5). This completes the proof of supermodularity of $d\hat{p}(d, r)$.

We now show that $\Phi_{t+1}^0(\alpha r + (1-\alpha)\hat{p}(d, r))$ is supermodular in d and r . Without loss of generality, let $r_2 \geq r_1$ and $d_2 \geq d_1$. Let $p_{12} = p_{11} + \delta_p/(1-\alpha)$. Since $\hat{p}$ is increasing with r , we have $\delta_p \geq 0$. From supermodularity of the price function, we have $p_{21} - p_{11} \leq p_{22} - p_{12}$. Let $p_{11} + p_{22} = p_{12} + p_{21} + \delta_s$. Note that $\delta_s \geq 0$. Rearranging

yields $p_{12} = p_{11} + (p_{22} - p_{21} - \delta_s)$ and hence $p_{22} - p_{21} - \delta_s = \delta_p/(1 - \alpha)$. Moreover let $r_2 = r_1 + \delta_r/\alpha$. Finally, it can be shown in a similar way to the proof of Theorem 3.4 that ϕ_{t+1}^0 is concave. Using these definitions and properties, following can be written:

$$\begin{aligned} & \Phi_{t+1}^0(\alpha r_2 + (1 - \alpha)p_{12}) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{11}) \\ = & \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{11} + \delta_r + \delta_p) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{11}) \\ \leq & \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21} + \delta_r + \delta_p) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21}) \end{aligned} \quad (\text{A.10})$$

where the second inequality follows from $p_{21} \leq p_{11}$ and from concavity of Φ_{t+1}^0 . Substituting δ_r and δ_s :

$$\begin{aligned} & \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21} + \alpha(r_2 - r_1) + (1 - \alpha)(p_{22} - p_{21} - \delta_s)) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21}) \\ = & \Phi_{t+1}^0(\alpha r_2 + (1 - \alpha)p_{22} - (1 - \alpha)\delta_s) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21}) \\ \leq & \Phi_{t+1}^0(\alpha r_2 + (1 - \alpha)p_{22}) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21}) \end{aligned} \quad (\text{A.11})$$

can be written. The inequality follows from the fact that $\Phi_{t+1}^0(r)$ is increasing with r (Lemma 3.1). From Equations A.10 and A.11, following can be written:

$$\begin{aligned} & \Phi_{t+1}^0(\alpha r_2 + (1 - \alpha)p_{12}) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{11}) \\ \leq & \Phi_{t+1}^0(\alpha r_2 + (1 - \alpha)p_{22}) - \Phi_{t+1}^0(\alpha r_1 + (1 - \alpha)p_{21}) \end{aligned}$$

which is the definition of supermodularity.

Since $\mathbb{D}$ is a lattice and $\hat{\phi}_t^0(d, r)$ is supermodular, $d_t^0(r)$ is increasing with r . Hence $y_t^0(r) = d_t^0(r) + \Delta_t^0$ is increasing with r .

A.7. Supermodularity of $\hat{p}(d, r)$

Without loss of generality, we suppress the time indices.

Proposition A.1. *Assume that the price function is twice differentiable. Then price function is supermodular if and only if $\hat{d}_r \hat{d}_{pp} \leq \hat{d}_p \hat{d}_{pr}$.*

Proof. Subscripts denoting the partial derivatives, $\hat{p}_d = \frac{1}{\hat{d}_p}$ and $\frac{\partial d}{\partial r} = \frac{\partial \hat{d}(p,r)}{\partial r}$ can be written. The last term reduces to $\hat{p}_r = -\frac{\hat{d}_r}{\hat{d}_p}$. Also:

$$\hat{p}_{dr} = \frac{\partial \hat{p}_d}{\partial r} = -\left(\frac{1}{\hat{d}_p^2}\right) \left(\hat{d}_{pp}\hat{p}_r + \hat{d}_{pr}\right)$$

and hence $\hat{p}_{dr} = \frac{\hat{d}_r\hat{d}_{pp}-\hat{d}_p\hat{d}_{pr}}{\hat{d}_p^3}$ can be written. Since $\hat{d}_p^3 \leq 0$ from Assumption 3.1, $\hat{d}_r\hat{d}_{pp} \leq \hat{d}_p\hat{d}_{pr}$ is necessary and sufficient for $\hat{p}(d, r)$ to be non-negative and hence $\hat{p}(d, r)$ to be supermodular. $\square$

Note that $\hat{d}_{pp} \leq 0$ for concave demand models and $\hat{d}_r \geq 0$ and $\hat{d}_p \leq 0$ from Assumption 3.1(ii). Then if $\hat{d}_{pr} \leq 0$, particularly if the mean demand has ADRE then the price function is supermodular. If the mean demand is supermodular in (p, r) , then one must check explicitly the condition $\hat{p}_{dr} \geq 0$.

A.8. Proof of Proposition 3.2

Assume that $\hat{d}$ is concave on $\hat{\mathbb{P}}$. We first show that the corresponding $\mathbb{D}$ is a convex set. Without loss of generality, we suppress the time indices. Let $\{(d_1, r_1), (d_2, r_2) \in \mathbb{D}\}$. We need $(\frac{d_1+d_2}{2}, \frac{r_1+r_2}{2}) \in \mathbb{D}$. Let $p_1 = \hat{p}(d_1, r_1)$ and $p_2 = \hat{p}(d_2, r_2)$. Note that $\{(p_1, r_1), (p_2, r_2)\} \in \hat{\mathbb{P}}$ follows from construction of $\hat{p}$. From concavity of $\hat{d}(p, r)$, $\hat{d}(\frac{p_1+p_2}{2}, \frac{r_1+r_2}{2}) \geq \frac{d_1+d_2}{2}$. Fix $r = \frac{r_1+r_2}{2}$. Note that, there exists a finite p^u since the mean demand is concave and strictly decreasing with p (Assumption 3.4 (i)). Since $\hat{d}$ is strictly decreasing and continuous in p for a given r , there exists a price, say p_{12} , such that $p_{12} \geq \frac{p_1+p_2}{2}$ and $p_{12} \in [p^l, p^u]$ which satisfies $\hat{d}(p_{12}, \frac{r_1+r_2}{2}) = \frac{d_1+d_2}{2}$ and equivalently $p_{12} = \hat{p}(\frac{d_1+d_2}{2}, \frac{r_1+r_2}{2})$. Since $p_{12} \geq p^l$ and $\frac{r_1+r_2}{2} \geq p^l$, $(p_{12}, \frac{r_1+r_2}{2}) \in \hat{\mathbb{P}}$. Hence $\mathbb{D}$ is a convex set. For concavity of $\hat{p}(d, r)$, we need $p_{12} \geq \frac{p_1+p_2}{2}$ which is already shown. $\square$

A.9. Example for a Concave Expected Demand and Revenue Function

Without loss of generality, we suppress the time indices.

A.9.1. Example for a Concave Demand Function

Mean demand can equivalently be written as minimum of two loss neutral mean demand models: $\hat{d}(p, r) = \min \left(\hat{d}(p, r, a_2), \hat{d}(p, r, a_3) \right)$ where

$$\hat{d}(p, r, a_i) = a_0 - a_1 p^n + a_i \frac{r - p}{r} \quad i = 2, 3. \quad (\text{A.12})$$

Hence $\hat{d}(p, r)$ is concave if both of the mean demand models are concave since minimum preserves concavity. First consider $\hat{d}(p, r, a_2)$. Subscripts denoting partial derivatives with respect to the variable in the subscript, $B_p(p) = -a_1 n p^{n-1}$, $B_{pp}(p) = -a_1 n(n-1)p^{n-2}$, $F_p(r-p, r) = -a_2/r$, $F_r(r-p, r) = -\frac{a_2 p}{r^2}$ and $F_{pr}(r-p, r) = \frac{a_2}{r^2}$ are the derivatives of base demand $B(p)$ and reference effect demand $F(r-p, r)$. Other derivatives, namely $F_{pp}(r-p, r)$, $B_r(p)$, $B_{pr}(p)$ and $B_{rr}(p)$, are zero. Then the derivatives of the mean demand function are given as:

$$\hat{d}_p(p, r, a_2) = B_p(p) + F_p(r-p, r) = -a_1 n p^{n-1} - a_2/r \quad (\text{A.13})$$

$$\hat{d}_{pp}(p, r, a_2) = B_{pp}(p) = -a_1 n(n-1)p^{n-2}$$

$$\hat{d}_r(p, r, a_2) = F_r(r-p, r) = -\frac{a_2 p}{r^2} \quad (\text{A.14})$$

$$\hat{d}_{rr}(p, r, a_2) = F_{rr}(r-p, r) = -\frac{2a_2 p}{r^3}$$

$$\hat{d}_{pr}(p, r, a_2) = F_{pr}(r-p, r) = \frac{a_2}{r^2},$$

Then the mean demand function is concave in p and concave in r when $n \geq 1$. For joint concavity the determinant of the hessian (ignoring the terms in paranthesis):

$$|H| = \hat{d}_{pp}^2 \hat{d}_{rr}^2 - \left(\hat{d}_{pr} \right)^2 = \frac{a_2}{r^4} (2a_1 n(n-1)p^{n-1}r - a_2),$$

must be non-negative. Since a_2 and r defined to be non-negative, this condition reduces to non-negativity of the term $(2a_1 n(n-1)p^{n-1}r - a_2)$ which is increasing with p and

r . Since the lower bounds of p and r are assumed to be same,

$$p \geq \left(\frac{a_2}{2a_1n(n-1)} \right)^{1/n}, \quad (\text{A.15})$$

is sufficient for joint concavity of $\hat{d}(p, r, a_2)$. Same argument is valid for $\hat{d}(p, r, a_3)$, hence:

$$p \geq \left(\frac{a_3}{2a_1n(n-1)} \right)^{1/n}, \quad (\text{A.16})$$

is sufficient for joint concavity for $\hat{d}(p, r, a_3)$. If customers are loss-averse or loss-neutral, then $a_3 \geq a_2$. Thus the inequality in Equation A.16 is sufficient for concavity of both mean demand functions. $\square$

A.9.2. Example for a Concave Revenue Function

Mean revenue can be written as the minimum of two revenue functions of loss neutral demand models:

$$p\hat{d}(p, r) = p \min \left(\hat{d}(p, r, a_2), \hat{d}(p, r, a_3) \right) = \min \left(p\hat{d}(p, r, a_2), p\hat{d}(p, r, a_3) \right) \quad (\text{A.17})$$

where $\hat{d}(p, r, a_i)$ for $i = 2, 3$ are defined in the followings:

$$\hat{d}(p, r, a_i) = a_0 - a_1p^n + a_i \frac{r-p}{r} \quad i = 2, 3.$$

Defining $\hat{p}(d, r, a_i)$ for $i = 2, 3$ as the inverse functions, following can be written:

$$\min \left(p\hat{d}(p, r, a_2), p\hat{d}(p, r, a_3) \right) = \min \left(d\hat{p}(d, r, a_2), d\hat{p}(d, r, a_3) \right) = d\hat{p}(d, r)$$

Both revenue functions in the second minimum operator can be shown to be concave over the set

$$\mathbb{D} = \{(d, r) : \hat{p}(d, r), r \geq 0.23, d \geq 0.35\}.$$

Hence the revenue function $d\hat{p}(d, r)$ is concave since minimum operator preserves concavity.

A.10. On the Convexity of $\mathbb{D}$

Here we provide a necessary and sufficient condition for the set $\mathbb{D}$ to be convex and $\hat{p}(d, r)$ to be concave. Without loss of generality, we suppress the time indices.

Proposition A.2. *Given Assumption 3.1(i), $\mathbb{P}$ is a convex set and $\hat{d}(p, r)$ is concave on $\mathbb{P}$ if and only if $\mathbb{D}$ is a convex set and $\hat{p}(d, r)$ is concave on $\mathbb{D}$.*

Proof. The first part is similar to the proof in Section A.8. The only difference is that $(\frac{p_1+p_2}{2}, \frac{r_1+r_2}{2}) \in \mathbb{P}$ follows from convexity of $\mathbb{P}$.

The proof of other direction is similar. Now assume that $\mathbb{D}$ is a convex set and $\hat{p}$ is concave on $\mathbb{D}$. Let the definitions of d_1, d_2, p_1, p_2 and p_{12} be the same and let $d_{12} = \hat{d}(\frac{p_1+p_2}{2}, \frac{r_1+r_2}{2})$. We need to show $d_{12} \geq \frac{d_1+d_2}{2}$. $(\frac{d_1+d_2}{2}, \frac{r_1+r_2}{2}) \in \mathbb{D}$ from convexity of $\mathbb{D}$ and $p_{12} \geq \frac{p_1+p_2}{2}$ from concavity of $\hat{p}$. Since $\hat{d}$ is decreasing with p for a given r (Proposition 3.1), $d_{12} \geq \hat{d}(p_{12}, \frac{r_1+r_2}{2})$ can be written. Since $\hat{d}(p_{12}, \frac{r_1+r_2}{2}) = \frac{d_1+d_2}{2}$ by construction, this completes our proof. $\square$

A.11. Proof of Theorem 3.8

The proof is by induction. We first show the concavity of $J_T(y, d, r)$. Then the optimality of an order-up-to policy for the last period follows. In order to proceed to the induction step, $V_T(x, r)$ is shown to be concave and also increasing with r .

Before going into the proof, first define the following: Let $(d_i, r_i) \in \mathbb{D}_T$ for $i = 1, 2$ and $p_i^t = \hat{p}_t(d_i, r_i)$. Also let $p_{12}^t = \hat{p}_t\left(\frac{d_1+d_2}{2}, \frac{r_1+r_2}{2}\right)$ and let $d_{12}^t = \hat{d}_t\left(\frac{p_1^t+p_2^t}{2}, \frac{r_1+r_2}{2}\right)$.

The first term in the last period's value function, the mean revenue, is assumed to be concave by Assumption 3.7. The second term and the third term are linear on the decision variables. For the expected holding and backorder cost, following shows the convexity of $H_T(y - d\theta_T - \epsilon_T)$ in y and d :

$$\begin{aligned} H_T\left(\frac{y_1 + y_2}{2} - \frac{d_1 + d_2}{2}\theta_T - \epsilon_T\right) &= H_T\left(\frac{y_1 - d_1\theta_T - \epsilon_T}{2} + \frac{y_2 - d_2\theta_T - \epsilon_T}{2}\right) \\ &\leq H_T\left(\frac{y_1 - d_1\theta_T - \epsilon_T}{2}\right) + H_T\left(\frac{y_2 - d_2\theta_T - \epsilon_T}{2}\right), \end{aligned}$$

where the inequality is written from the convexity of $H_T(\cdot)$. Concavity of $-G_T(y, d)$ follows. Thus $J_T(y, d, r)$ is concave. Since $\mathbb{D}_T$ is a convex set (Proposition 3.2), concavity of $V_T(x, r)$ follows directly from Theorem A.3 in (Porteus, 2002). Then there exists a maximizer $(y_T^*(r), d_T^*(r))$ for the function $J_T(y, d, r)$ for a given r over $\mathbb{D}_T$. If there are multiple optima, then define the optimal pair as the lexicographically smallest. If the inventory level at the beginning of the period is smaller than $y_T^*(r)$, i.e. $x \leq y_T^*(r)$, then the pair $(y_T^*(r), d_T^*(r))$ is feasible, hence optimum. Thus the firm orders $y_T^*(r) - x$ many units and charges the price $p_T^*(r) = \hat{p}_T(d_T^*(r), r)$. If the inventory level at the beginning is larger than $y_T^*(r)$, i.e. $x > y_T^*(r)$, then from the concavity of $J_T(y, d, r)$, it is optimal to order nothing and apply a state dependent price $p_T^*(x, r) = \hat{p}_T(d_T^*(x, r), r)$ where $d_T^*(x, r)$ is the maximizer of $J_T(x, d, r)$ for a given r over $\mathbb{D}_T$. Hence the optimal policy of the period is a state-dependent order-up-to policy.

In order to show that $V_T(x, r)$ is increasing with r , one needs to show two cases:

- (i) $x \leq y_T^*(r)$: The optimal solution is $(y_T^*(r), d_T^*(r))$. Without changing the optimal pair, if r is increased by an amount of $\delta > 0$, then the production cost, expected holding and backorder cost remain constant but the revenue term increases since $\hat{p}(d, r)$ is increasing with r . Since the terminal value $V_{T+1}(x, r)$ is defined to be zero, the optimal solution for the new state, namely $r + \delta$ will be larger than or

equal to the current solution, that is:

$$V_T(x, r) = J_T(y_T^*(r), d_T^*(r), r) \leq J_T(y_T^*(r), d_T^*(r), r + \delta) \leq V_T(x, r + \delta).$$

(ii) $x > y_T^*(r)$: The optimal solution is $(x, d_T^*(x, r))$. If r is increased by $\delta > 0$ while the solution remains the same, the argument in case 1 remains valid, hence:

$$V_T(x, r) = J_T(x, d_T^*(x, r), r) \leq J_T(x, d_T^*(x, r), r + \delta) \leq V_T(x, r + \delta).$$

In the induction step, one must show the concavity of $V_t(x, r)$ and $J_t(y, d, r)$ given that $V_{t+1}(x, r)$ and $J_{t+1}(y, d, r)$ are concave and $V_{t+1}(x, r)$ is increasing with r . The value function of period t is:

$$\begin{aligned} J_t(y, d, r) &= \hat{p}_t(d, r)d - \gamma c_{t+1}d + (\gamma c_{t+1} - c_t)y - G_t(y, d) \\ &\quad + \gamma E[V_{t+1}(y - d\theta_t - \epsilon_t), \alpha r + (1 - \alpha)\hat{p}_t(d, r)]. \end{aligned}$$

All the terms, but the last one, are shown to be concave. Consider the last term:

$$\gamma E[V_{t+1}(y - d\theta_t - \epsilon_t, \alpha r + (1 - \alpha)\hat{p}_t(d, r))]. \quad (\text{A.18})$$

From concavity of $\hat{p}_t(d, r)$ (Assumption 3.4 (i) and Proposition 3.2), the following can be written:

$$\alpha \frac{r_1 + r_2}{2} + (1 - \alpha)p_{12}^t \geq \frac{\alpha r_1 + (1 - \alpha)p_1^t}{2} + \frac{\alpha r_2 + (1 - \alpha)p_2^t}{2} \quad (\text{A.19})$$

Define $x_i^t = y_i - d_i\theta_t - \epsilon_t$, $f_i^t = \frac{\alpha r_i + (1 - \alpha)p_i^t}{2}$ for $i = 1, 2$ and $f_{12}^t = \alpha \frac{r_1 + r_2}{2} + (1 - \alpha)p_{12}^t$.

Then:

$$V_{t+1}\left(\frac{x_1^t + x_2^t}{2}, f_{12}^t\right) \geq V_{t+1}\left(\frac{x_1^t + x_2^t}{2}, \frac{f_1^t + f_2^t}{2}\right) \geq \frac{V_{t+1}(x_1^t, f_1^t) + V_{t+1}(x_2^t, f_2^t)}{2}$$

where the first inequality follows from Equation A.19 and $V_{t+1}(x, r)$ being increasing with r , and the second inequality follows from the concavity of $V_{t+1}(x, r)$. This completes the proof for the joint concavity of last term of $J_t(y, d, r)$, hence $J_t(y, d, r)$ itself. $\mathbb{D}_?t$ is convex, hence concavity of $V_t(x, r)$ follows directly from Theorem A.3 in (Porteus, 2002). Then the optimal policy in period t is a state-dependent order-up-to policy which is defined in Theorem 3.8.

$V_t(x, r)$ increasing with r can be shown with an additional argument to the case of $V_T(x, r)$. The terms in the value function are shown to be increasing with r except the last term (Equation A.18). From Proposition 3.1, $\hat{p}_t(\cdot)$ increases with r , hence $\alpha r + (1 - \alpha)\hat{p}_t(d, r)$ increases with r . Then the arguments for $V_T(x, r)$ still remain valid since $V_{t+1}(x, r)$ is increasing with r by the induction hypothesis. $\square$

A.12. Proof of Theorem 3.11

The dynamic recursion of the problem before transformations is given as:

$$\begin{aligned}\hat{J}_t(y, p, r, \alpha_i) &= p\hat{d}_t(p, r) - c_t y - \hat{G}_t(y, p, r) \\ &\quad + \gamma \sum_{j=1}^m \mathbb{Q}_{ij} E \left[\hat{V}_{t+1}(y - \hat{d}_t(p, r)\theta_t - \epsilon_t, \alpha_i r + (1 - \alpha_i)p, \alpha_j) \right] \\ \hat{V}_t(x, r, \alpha_i) &= \max_{\substack{y, p \\ y \geq (p, r) \in \mathbb{P}}} \left\{ \hat{J}_t(y, p, r, \alpha_i) \right\} + c_t x\end{aligned}$$

with terminal value $\hat{V}_{T+1}(x, r, \alpha_i) = c_{T+1}x$. Define $V_t(x, r, \alpha_i) = \hat{V}_t(x, r, \alpha_i) - c_t x$. Then $V_{T+1}(x, r, \alpha_i) = 0$ and

$$V_t(x, r, \alpha_i) = \max_{\substack{y, p \\ y \geq (p, r) \in \mathbb{P}}} \left\{ \hat{J}_t(y, p, r, \alpha_i) \right\}$$

Lastly $\hat{J}_t(y, p, r, \alpha_i)$ can be written as:

$$\begin{aligned}\hat{J}_t(y, p, r, \alpha_i) &= p\hat{d}_t(p, r) - c_t y - \hat{G}_t(y, p, r) \\ &\quad + \gamma \sum_{j=1}^m \mathbb{Q}_{ij} \left(E \left[V_{t+1}(y - \hat{d}_t(p, r)\theta_t - \epsilon_t, \alpha_j r + (1 - \alpha_i)p, \alpha_j) \right] + c_{t+1}y - c_{t+1}\hat{d}_t(p, r) \right)\end{aligned}$$

which finally reduces to the following:

$$\begin{aligned}\hat{J}_t(y, p, r, \alpha_i) &= (p - \gamma c_{t+1})\hat{d}_t(p, r) + (\gamma c_{t+1} - c_t)y - \hat{G}_t(y, p, r) \\ &\quad + \gamma \sum_{j=1}^m \mathbb{Q}_{ij} E \left[V_{t+1}(y - \hat{d}_t(p, r)\theta_t - \epsilon_t, \alpha_i r + (1 - \alpha_i)p, \alpha_j) \right]\end{aligned}$$

With the second transformation, the decision variables are changed from (y, p, r) to (y, d, r) and this results in the recursion given in Equation 3.21.

The proof of the optimal policy is by induction. We first show the concavity of $J_T(y, d, r, \alpha_i)$ on (y, d, r) for a given α_i . Then the optimality of an order-up-to policy for last period follows. In order to proceed the induction step, $V_T(x, r, \alpha_i)$ is shown to be concave on x and r and increasing with r for a given α_i .

Similar to the arguments given in the proof of Theorem 3.8, one can show that for the last period T , for a given α_i , (i) $J_T(y, d, r, \alpha_i)$ is concave on (y, d, r) , (ii) $V_T(x, r, \alpha_i)$ is concave on (x, r) and (iii) $V_T(x, r, \alpha_i)$ is increasing with r .

Then there exists a maximizer $(y_T^*(r, \alpha_i), d_T^*(r, \alpha_i))$ for $J_T(y, d, r, \alpha_i)$ for a given r and α_i over $\mathbb{D}$. If there are multiple optima, then one can define the optimal pair as the lexicographically smallest. If the inventory level at the beginning of the period is smaller than $y_T^*(r, \alpha_i)$, i.e. $x \leq y_T^*(r)$, then the pair $(y_T^*(r, \alpha_i), d_T^*(r, \alpha_i))$ is feasible, hence optimum. Thus the firm orders $y_T^*(r, \alpha_i) - x$ many units and charges the price $p_T^*(r, \alpha_i) = \hat{p}_T(d_T^*(r, \alpha_i), r)$. If the inventory level at the beginning is larger than $y_T^*(r)$, i.e. $x > y_T^*(r)$, then from concavity of $J_T(y, d, r, \alpha_i)$, it is optimal to order nothing and apply a state dependent price $p_T^*(x, r, \alpha_i) = \hat{p}_T(d_T^*(x, r, \alpha_i), r)$ where $d_T^*(x, r, \alpha_i)$ is the maximizer of $J_T(x, d, r, \alpha_i)$ for a given r and α_i over $\mathbb{D}$. Hence the optimal policy of the period is a state-dependent order-up-to policy given by Theorem 3.11.

In the induction step, given that $V_{t+1}(x, r, \alpha_i)$ is concave in (x, r) , $J_{t+1}(y, d, r, \alpha_i)$ is concave in (y, d, r) and $V_{t+1}(x, r, \alpha_i)$ is increasing with r for a given α_i , one must show that (i) $J_t(y, d, r, \alpha_i)$ is concave in (y, d, r) , (ii) $V_t(x, r, \alpha_i)$ is concave in (x, r)

and (iii) $V_t(x, r, \alpha_i)$ is increasing with r for a given α_i for all $i = 1, \dots, m$. The value function of period t is:

$$\begin{aligned} J_t(y, d, r, \alpha_i) &= \hat{p}_t(d, r)d - \gamma c_{t+1}d + (\gamma c_{t+1} - c_t)y - G_t(y, d) \\ &\quad + \gamma \sum_{j=1}^m \mathbb{Q}_{ij} E[V_{t+1}(y - d\theta_t - \epsilon_t, \alpha_j r + (1 - \alpha_j)\hat{p}_t(d, r), \alpha_j)] \end{aligned}$$

From previous analysis it is known that all the terms, but the last one, are concave. Consider the last term in the value function $J_t(\cdot)$:

$$\gamma \sum_{j=1}^m \mathbb{Q}_{ij} E \left[V_{t+1}(y - d\theta_t - \epsilon_t, \alpha_j r + (1 - \alpha_j)\hat{p}_t(d, r), \alpha_j) \right]. \quad (\text{A.20})$$

Concavity of $V_{t+1}(y - d\theta_t - \epsilon_t, \alpha_j r + (1 - \alpha_j)\hat{p}_t(d, r), \alpha_j)$ can be shown by similar arguments given in the proof of Theorem 3.8. Since expectation preserves concavity and sum of concave functions is concave, the last term of $J_t(y, d, r, \alpha_i)$ is concave, hence $J_t(y, d, r, \alpha_i)$ is concave in (y, d, r) for a given α_i . Concavity of $V_t(\cdot)$ follows directly from Theorem A.3 in (Porteus, 2002). Then the optimal policy in period t is a state-dependent order-up-to policy as given in Theorem 3.11.

$V_t(x, r, \alpha_i)$ increasing with r can be shown by a similar argument given in the proof of Theorem 3.8.

APPENDIX B: PROOFS OF THE RESULTS IN CHAPTER 4

B.1. Proof of Proposition 4.1

The proof is by induction. We first show that the equations hold for type-1 customers. Type-1 customers have the highest priority over the other customers and are not affected by the customers who are waiting in the queue. Let Q_i denote the number of type- i customers in queue, S_j denote the service time of j^{th} customer and S_t denote the service time of the customer which is tagged. Then the waiting time of a type-1 customer can be written as:

$$W_1^q = \sum_{j=1}^{Q_1} S_j + (S_t)(1_{N \geq y+1}),$$

where $1_{N \geq y+1}$ is the indicator function of the event $\{N \geq y+1\}$ taking on the value 1 if $N \geq y+1$ and 0 otherwise. Since service times are independent and exponentially distributed, $E[W_1^q]$ can be written as:

$$\begin{aligned} E[W_1^q] &= E[Q_1] \frac{1}{\mu} + \frac{1}{\mu} P(N \geq y+1) \\ &= (E[W_1^q] \lambda_1 + \rho^{y+1}) \frac{1}{\mu} \end{aligned}$$

since from the Little's law we have $E[Q_1] = E[W_1^q] \lambda$. This equation finally reduces to:

$$E[W_1^q] = \rho^y \frac{\lambda}{\mu(\mu - \lambda_1)}.$$

For the induction step, we assume that it holds for type- i customers and show that the induction holds for type- $(i+1)$ customers. In general, since type- $(i+1)$ customers do not differentiate between higher priority customers in terms of waiting,

the higher priority customers can be merged in a single customer type, for calculation of time in queue of a type- $(i + 1)$ customer.

An arriving type- $(i + 1)$ customer waits for the high priority customers currently in the queue and for the high priority customers who arrive during the service of the currently waiting high priority customers. If there is a customer who is tagged, then he also waits for the high priority customers who arrive until the tagged customer leaves the system. Hence a type- $i + 1$ customer on the average waits for a busy-period for (i) each of type-1, ..., type- $i + 1$ customers that are in the queue and also (ii) for the tagged customer, if any. So the waiting time of type- $(i + 1)$ customer can be written as:

$$E[W_{i+1}^q] = E \left[1_{N \geq y+1} + \sum_{j=1}^{i+1} Q_j \right] \frac{1}{\mu - \sum_{j=1}^i \lambda_j}$$

where $\frac{1}{\mu - \sum_{j=1}^i \lambda_j}$ is the busy time for the customers who have higher priority than type- $i + 1$ customers. By Little's law $E[Q_i^q] = \lambda_i E[W_i^q]$ and rearranging the terms we get:

$$E[W_{i+1}^q] \left(\mu - \sum_{j=1}^{i+1} \lambda_j \right) = \rho^{y+1} + \sum_{j=1}^i E[W_j^q] \lambda_j$$

Substituting the induction assumption $E[W_j^q] = \rho^y E[\tilde{W}_j^q]$ into above equation, we can write:

$$E[W_{i+1}^q] = \rho^y \left(\frac{\rho + \sum_{j=1}^i E[\tilde{W}_j^q] \lambda_j}{\mu - \sum_{j=1}^{i+1} \lambda_j} \right) \quad (\text{B.1})$$

For the $\tilde{I}^0$ queueing system, expected waiting time of a type- $(i + 1)$ customer is given by:

$$E[\tilde{W}_{i+1}^q] = E \left[1_{N \geq 1} + \sum_{j=1}^{i+1} \tilde{Q}_j \right] \frac{1}{\mu - \sum_{j=1}^i \lambda_j}.$$

Rearranging the terms we get:

$$E[\tilde{W}_{i+1}^q] = \frac{\rho + \sum_{j=1}^i E[\tilde{W}_j^q] \lambda_j}{\mu - \sum_{j=1}^{i+1} \lambda_j}.$$

Substituting this into Equation B.1, we have $E[W_i^q] = \rho^y E[\tilde{W}_{i+1}^q]$. This completes the proof of Equation 4.2.

A customer waits for the production process (or waits for a service) if there are y or more outstanding orders in the system. Hence the expected time of a type- i customer can be calculated as:

$$\begin{aligned} E[W_i] &= E[W_i^q] + \sum_{k=y}^{\infty} \frac{1}{\mu} \rho^k (1 - \rho) \\ &= \rho^y \left(E[\tilde{W}_i^q] + \frac{1}{\mu} \right) \frac{\rho}{\mu - \lambda_1} + \rho^y \frac{1}{\mu} \end{aligned}$$

which reduces to $E[W_i] = \rho^y E[\tilde{W}_i]$ and this completes our proof.

B.2. Proof of Proposition 4.4

The FOCs of $\Pi(y, \lambda)$ are given by the following:

$$R_i = \frac{\partial \rho^y}{\partial \lambda_i} H(\underline{\lambda}) + \rho^y \frac{\partial H(\underline{\lambda})}{\partial \lambda_i} + h \frac{\partial E[(y - N)^+]}{\partial \lambda_i}. \quad (\text{B.2})$$

We find $\frac{\partial H(\underline{\lambda})}{\partial \lambda_i}$ and substitute it into Equation B.2. Since $H(\underline{\lambda}) = \sum_{i=1}^M C_i \lambda_i E[\tilde{W}_i(\underline{\lambda})]$, we first find derivatives of $\lambda_i E[\tilde{W}_i(\underline{\lambda})]$:

$$\begin{aligned} \frac{\partial \lambda_i E[\tilde{W}_i(\underline{\lambda})]}{\partial \lambda_i} &= E[\tilde{W}_i(\underline{\lambda})] + \lambda_i \frac{\partial E[\tilde{W}_i(\underline{\lambda})]}{\partial \lambda_i} \\ \frac{\partial \lambda_j E[\tilde{W}_j(\underline{\lambda})]}{\partial \lambda_i} &= \lambda_j \frac{\partial E[\tilde{W}_j(\underline{\lambda})]}{\partial \lambda_i} \quad j \neq i. \end{aligned}$$

Hence we can write:

$$\frac{\partial H(\underline{\lambda})}{\partial \lambda_i} = \sum_{k \neq i}^M C_k \lambda_k \frac{\partial E[\tilde{W}_k]}{\partial \lambda_i} + C_i \left[E[\tilde{W}_i(\underline{\lambda})] + \lambda_i \frac{\partial E[\tilde{W}_i(\underline{\lambda})]}{\partial \lambda_i} \right]$$

which reduces to

$$\frac{\partial H(\underline{\lambda})}{\partial \lambda_i} = \sum_{k=1}^M C_k \lambda_k \frac{\partial E[\tilde{W}_k]}{\partial \lambda_i} + C_i E[\tilde{W}_i(\underline{\lambda})].$$

Substituting this equation into Equation B.2, we get:

$$R_i = \frac{\partial \rho^y}{\partial \lambda_i} H(\underline{\lambda}) + \rho^y \left\{ \sum_{k=1}^M C_k \lambda_k \frac{\partial E[\tilde{W}_k]}{\partial \lambda_i} + C_i E[\tilde{W}_i(\underline{\lambda})] \right\} + h \frac{\partial E[(y - N)^+]}{\partial \lambda_i}.$$

Substituting the steady state equalities $R_i = p_i + C_i E[W_i(\underline{\lambda})]$ into the equation above, we have following:

$$p_i(y) = \frac{\partial \rho^y}{\partial \lambda_i} H(\underline{\lambda}) + \rho^y \sum_{k=1}^M C_k \lambda_k \frac{\partial E[\tilde{W}_k(\underline{\lambda})]}{\partial \lambda_i} + h \frac{\partial E[(y - N)^+]}{\partial \lambda_i}. \quad (\text{B.3})$$

Using Equation 4.9, one can find the derivatives of the expected waiting times $E[\tilde{W}_i(\underline{\lambda})]$:

$$\frac{\partial E[\tilde{W}_k(\underline{\lambda})]}{\partial \lambda_i} = \frac{1}{\bar{\alpha}_{k-1} \bar{\alpha}_k} + \frac{\alpha_M}{\bar{\alpha}_{k-1}} \frac{1_{i \leq k}}{\bar{\alpha}_k^2} + \frac{\alpha_M}{\bar{\alpha}_k} \frac{1_{i \leq k-1}}{\bar{\alpha}_{k-1}^2}$$

which simplifies into

$$\frac{\partial E[\tilde{W}_k(\underline{\lambda})]}{\partial \lambda_i} = \frac{1}{\bar{\alpha}_{k-1} \bar{\alpha}_k} + \frac{E[\tilde{W}_k^q(\underline{\lambda})] 1_{i \leq k}}{\bar{\alpha}_k} + \frac{E[\tilde{W}_k^q(\underline{\lambda})] 1_{i \leq k-1}}{\bar{\alpha}_{k-1}}$$

Substituting the above equation into Equation B.3 results in Equation 4.10.

B.3. Proof of Lemma 4.2

The proof goes parallel to Theorem 2 in (Mendelson and Whang, 1990). It proceeds in two stages.

- (i) *Local Optimality*: For a type- i customer, the expected cost of being assigned to class $i - 1$ or $i + 1$ is greater than that of being assigned to its own class i .
- (ii) *Transitivity*: Define TC_i^j as the expected total cost when a type- i customer placed himself as a type- j customer. Then, for $k > j > i$, for a type- i customer being better off in type- i instead of type- j and a type- j customer being better off in type- j instead of type- k implies that type- i customer is better off in type- i instead of being type- k . That is

$$(TC_i^i < TC_i^j) \& (TC_j^j < TC_j^k) \Rightarrow TC_i^i < TC_i^k$$

Local optimality and repeated application of transitivity will prove the theorem. We will drop y from p_i^* for convenience.

Proof of local optimality: If a job type- i customer is assigned to class $(i + 1)$, by degrading its priority he gains following amount over the prices:

$$\begin{aligned} p_i^* - p_{i+1}^* &= \frac{\partial \rho^y}{\partial \lambda_i} \sum_{k=1}^M C_k \lambda_k E[\tilde{W}_k(\underline{\lambda})] + h \frac{\partial E[(y - N)^+]}{\partial \lambda_i} \\ &\quad + \rho^y \left\{ \sum_{k=1}^M \frac{\lambda_k^* C_k}{\bar{\alpha}_k \bar{\alpha}_{k-1}} + \sum_{k=i}^M \frac{\lambda_k^* C_k E[\tilde{W}_k^q(\underline{\lambda})] + \lambda_{k+1}^* C_{k+1} E[\tilde{W}_{k+1}^q(\underline{\lambda})]}{\bar{\alpha}_k} \right\} \\ &\quad - \frac{\partial \rho^y}{\partial \lambda_{i+1}} \sum_{k=1}^M C_k E[\tilde{L}_k(\underline{\lambda})] - h \frac{\partial E[(y - N)^+]}{\partial \lambda_{i+1}} \\ &\quad - \rho^y \left\{ \sum_{k=1}^M \frac{\lambda_k^* C_k}{\bar{\alpha}_k \bar{\alpha}_{k-1}} + \sum_{k=i+1}^M \frac{\lambda_k^* C_k E[\tilde{W}_k^q(\underline{\lambda})] + \lambda_{k+1}^* C_{k+1} E[\tilde{W}_{k+1}^q(\underline{\lambda})]}{\bar{\alpha}_k} \right\} \end{aligned}$$

Since the derivatives of $E[(y - N)^+]$ and ρ^y are the same for all λ_i , the above equation can be simplified as follows:

$$p_i^* - p_{i+1}^* = \rho^y \frac{\lambda_i^* C_i E[\tilde{W}_i^q(\lambda)] + \lambda_{i+1}^* C_{i+1} E[\tilde{W}_{i+1}^q(\lambda)]}{\bar{\alpha}_i}.$$

Same customer will lose a delay cost of:

$$\begin{aligned} C_i(E[W_{i+1}] - E[W_i]) &= C_i \rho^y (E[\tilde{W}_{i+1}(\lambda)] - E[\tilde{W}_i(\lambda)]) \\ &= C_i \rho^y (E[\tilde{W}_{i+1}^q(\lambda)] - E[\tilde{W}_i^q(\lambda)]) \end{aligned}$$

We need to show that the cost is more than the gain. Since $C_{i+1} < C_i$, it is sufficient to show the following:

$$\rho^y \frac{\lambda_i^* C_i E[\tilde{W}_i^q(\lambda)] + \lambda_{i+1}^* C_i E[\tilde{W}_{i+1}^q(\lambda)]}{\bar{\alpha}_i} \leq C_i \rho^y (E[\tilde{W}_{i+1}^q(\lambda)] - E[\tilde{W}_i^q(\lambda)]). \quad (\text{B.4})$$

Rearranging and substituting $E[\tilde{W}_i^q(\lambda)]$ and $E[\tilde{W}_{i+1}^q(\lambda)]$, the above expression reduces to:

$$\frac{\lambda_i^* \frac{\alpha_M}{\bar{\alpha}_{i-1} \alpha_i} + \lambda_{i+1}^* \frac{\alpha_M}{\bar{\alpha}_{i+1} \bar{\alpha}_i}}{\bar{\alpha}_i} \leq \frac{\alpha_M}{\bar{\alpha}_{i+1} \bar{\alpha}_i} - \frac{\alpha_M}{\bar{\alpha}_i \bar{\alpha}_{i-1}},$$

which is shown to hold in (Mendelson and Whang, 1990)

Same argument is valid for type- i customer choosing type- $(i-1)$. It is sufficient to show that:

$$\frac{\lambda_{i-1}^* E[\tilde{W}_{i-1}^q(\lambda)] + \lambda_i^* E[\tilde{W}_i^q(\lambda)]}{\bar{\alpha}_i} \geq E[\tilde{W}_i^q(\lambda)] - E[\tilde{W}_{i-1}^q(\lambda)]. \quad (\text{B.5})$$

Substituting λ_{i-1}^* , λ_i^* , $E[\tilde{W}_i^q(\underline{\lambda})]$ and $E[\tilde{W}_{i-1}^q(\underline{\lambda})]$ we can write the left hand side of the above equation as follows:

$$\begin{aligned} & \frac{(\bar{\alpha}_{i-2} - \bar{\alpha}_{i-1})\frac{\alpha_M}{\bar{\alpha}_{i-1}\bar{\alpha}_{i-2}} + (\bar{\alpha}_{i-1} - \bar{\alpha}_i)\frac{\alpha_M}{\bar{\alpha}_i\bar{\alpha}_{i-1}}}{\bar{\alpha}_{i-1}} \\ &= \frac{\alpha_M}{\bar{\alpha}_{i-1}} \left[\frac{1}{\bar{\alpha}_i} - \frac{1}{\bar{\alpha}_{i-2}} \right] \end{aligned}$$

Similarly, right hand side of Equation B.5 reduces to same value when $E[\tilde{W}_i^q(\underline{\lambda})]$ and $E[\tilde{W}_{i-1}^q(\underline{\lambda})]$ are substituted. Hence Equation B.5 holds as an equality.

The transitivity proof conditions of (Mendelson and Whang, 1990) depend on the facts that $C_i < C_j$ and $W_i > W_k$ for $i > j > k$. Hence we refer the reader to (Mendelson and Whang, 1990).

B.4. The Derivatives for Single Customer Type

Derivatives of K :

$$\begin{aligned} \frac{\partial K}{\partial h} &= \log(1 + C/h) + h \frac{-C/h^2}{(h+C)/h} = \log\left(\frac{h+C}{C}\right) - \frac{C}{h+C} \\ \frac{\partial K}{\partial C} &= h \frac{1/h}{(h+C)/h} = \frac{h}{h+C} \end{aligned}$$

Derivatives of ρ^* :

$$\frac{\partial \rho^*}{\partial h} = -\frac{1}{2} \frac{K_h}{\sqrt{R\mu K}} \quad \frac{\partial \rho^*}{\partial C} = -\frac{1}{2} \frac{K_C}{\sqrt{R\mu K}}$$

Derivatives of y^* :

$$\begin{aligned} y^* &= \frac{1 - \sqrt{\frac{K}{R\mu}}}{1 - 1 + \sqrt{\frac{K}{R\mu}}} K/h = \frac{\sqrt{R\mu} - \sqrt{K}}{\sqrt{K}} K/h \\ &= \frac{\sqrt{R\mu K} - K}{h} \end{aligned}$$

$$\begin{aligned}
\frac{\partial y^*}{\partial h} &= \left[\left(\frac{\sqrt{R\mu}}{2\sqrt{K}} K_h - K_h \right) h - (\sqrt{R\mu K} - K) \right] / h^2 \\
&= \left[\frac{K_h}{2K} (\sqrt{R\mu K} - 2K) h - (\sqrt{R\mu K} - K) \right] / h^2 \\
&= \left[\frac{K_h}{2K} (\sqrt{R\mu K} - K) h - hK \frac{K_h}{2K} - (\sqrt{R\mu K} - K) \right] / h^2 \\
&= \left[(\sqrt{R\mu K} - K) \left(\frac{K_h h}{2K} - 1 \right) - \frac{hK_h}{2} \right] / h^2
\end{aligned}$$

Substitute $K_h = \frac{K}{h} - \frac{C}{h+C}$:

$$\begin{aligned}
\frac{\partial y^*}{\partial h} &= \left[(\sqrt{R\mu K} - K) \left(\frac{\left(\frac{K}{h} - \frac{C}{h+C} \right) h}{2K} - 1 \right) - \frac{hK_h}{2} \right] / h^2 \\
&= \left[(\sqrt{R\mu K} - K) \left(-\frac{ch}{2K(h+C)} - \frac{1}{2} \right) - \frac{hK_h}{2} \right] / h^2
\end{aligned}$$

$$\frac{\partial y^*}{\partial C} = \left(\frac{\sqrt{R\mu}}{2K} K_C - K_C \right) / h = \frac{K_C}{h} \left(\frac{\sqrt{R\mu K}}{2K} - 1 \right)$$

Derivatives of $CE[W^*]$:

$$CE[W^*] = \frac{1}{\mu - \lambda^*} = C \frac{1}{\mu(1 - \rho^*)} = C \sqrt{\frac{R}{\mu K}}$$

$$\frac{\partial (CE[W^*])}{\partial h} = -\frac{CK_h \sqrt{R}}{2\sqrt{\mu K^3}}$$

$$\begin{aligned}
\frac{\partial (CE[W^*])}{\partial C} &= C \frac{\partial (E[W^*])}{\partial C} + E[W^*] = C \frac{-K_C \sqrt{R}}{2\sqrt{\mu K^3}} + \sqrt{\frac{R}{\mu K}} \\
&= \sqrt{\frac{R}{4\mu K^3}} (2K - CK_C)
\end{aligned}$$

substitute K and K_C

$$\begin{aligned}
\frac{\partial (CE[W^*])}{\partial C} &= \sqrt{\frac{R}{4\mu K^3}} \left(2h \log(1 + C/h) - C \frac{h}{h+C} \right) \\
&= h \sqrt{\frac{R}{4\mu K^3}} \left(2 \log(1 + C/h) - \frac{2C}{h+C} + \frac{C}{h+C} \right) \\
&= h \sqrt{\frac{R}{4\mu K^3}} \left(2K_h + \frac{C}{h+C} \right)
\end{aligned}$$

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