

CONSTRUCTION OF SURFACE BUNDLES WITH NON-ZERO SIGNATURE

by

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ABSTRACT**CONSTRUCTION OF SURFACE BUNDLES WITH
NON-ZERO SIGNATURE**

In this thesis, we study a new method introduced by H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, and A. Stipsicz in their paper [1] to construct surface bundles over a surface of genus 9 with signature 4.

ÖZET

İMZASI SIFIR OLMAYAN YÜZEY DEMETLERİNİN İNŞASI

Bu tezde, imzası 4 ve tabanı 9 delikli yüzey olan yüzey demetlerinin H. Endo, M. Korkmaz, D. Kotschick, B. Özbağcı, ve A. Stipsicz tarafından [1] tanıtılan yeni bir yöntemle inşası çalışılmıştır.

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LIST OF SYMBOLS

b_n^\pm	The partial n^{th} Betti number
CP^n	The n -dimensional complex projective space
$\dim X$	The dimension of X
$H_2(X, G)$	The 2^{nd} homology group of X with the coefficient group G
$i(-, -)$	The geometric intersection number of closed curves
$PD(\alpha)$	Poincaré dual of α
\mathbb{Q}	The set of rationals
\mathcal{Q}_X	The intersection form of the 4-manifold X
S^n	The n -dimensional sphere
$SL(n, F)$	The special linear group of degree n over the field F
$Sp(2n, F)$	The symplectic group of degree $2n$ over the field F
t_α	The right-handed Dehn twist about the curve α
\mathbb{Z}	The set of integers
Γ_g^r	The mapping class group of a surface of genus g with r boundary components
$\chi(X)$	The Euler characteristic of X
$\pi(X)$	The fundamental group of X
Σ_g^r	A surface of genus g with r boundary components
$\sigma(X)$	The signature of X
∂X	The boundary of X
$[X]$	The fundamental class of X
$\#$	The connected sum
\otimes	Tensor product
$\tilde{\times}$	The twisted product
\cong	Diffeomorphism

1. INTRODUCTION

After the 4-sphere, the best known compact 4-manifolds are Cartesian products of surfaces. By allowing to twist the product structure we obtain a *fiber bundle*. We define a surface bundle over a surface as an oriented fiber bundle whose fibers are compact, oriented 2-manifolds and whose base is also a compact, oriented 2-manifold. The fundamental question of the topology of fiber bundles is how the topological invariants of the total space, the fiber space, and the base space are related. One of the most important topological invariant of a 4-manifold is its signature, that is described in Section 2.1. It is known that signature vanishes for the 4-sphere and the Cartesian products of surfaces. The first examples of surface bundles over surfaces with non-zero signature were constructed independently by Atiyah [2] and Kodaira [3]. These examples have base genus 129. After these examples, there have been many efforts to find out the smallest possible genera of surface bundles over a surface with nonvanishing signature. In the case of bundles whose fibers are sphere or torus, the signature vanishes. It also vanishes for all bundles with fiber genus 2 because $H_2(\Gamma_2, \mathbb{Q}) = 0$ where Γ_2 is the mapping class group of a surface of genus 2. We will introduce the mapping class group of a surface in Section 2.2. Therefore, only the signatures of surface bundles with fiber genus $h \geq 3$ are interesting as far as the signature is concerned.

In this work, we aim to study the method introduced by Endo, Kotschick, Korkmaz, Ozbagci, and Stipsicz that helps to construct examples of surface bundles over surfaces with non-zero signature. The main theorem in the paper [1] is the following:

Theorem 1.1. *For every $h \geq 3$ there is a surface bundle of genus h over the surface of genus 9 with signature 4.*

To construct a new smooth surface bundle over a surface Endo et al use *Lefschetz fibrations*, a wider class of 4-manifolds in which the product structure is allowed to have critical points of the simplest type, as the building blocks. They introduce an operation, called *the subtraction of Lefschetz fibrations*, that removes

neighborhoods of singular fibers and glues the remaining isomorphic boundaries via fiber-preserving diffeomorphisms. More information about Lefschetz fibrations and the details of the operation can be found in Section 2.3 and Section 3.1, respectively. It is known that every surface bundle or Lefschetz fibration is described by a *word* representing identity in the mapping class group of its base surface, and this word is expressed as a product of right handed *Dehn twists*, that are special elements, and in fact generators, of mapping class groups. We call this word the *monodromy factorization* of the surface bundle or the Lefschetz fibration. They describe various commutator relations in mapping class groups by using several well-known relations as *braid relation*, *lantern relation*, and *chain relation*. These relations in mapping class groups can be found in Section 2.2. Endo et al use these relations to construct several examples of Lefschetz fibrations, and by subtracting particular pairs of these Lefschetz fibrations they construct the surface bundles in the main theorem.

In the last section of this work, we show how to compute the signature of a surface bundle by using a formula introduced by Meyer [4]. This formula allows us to compute the signature by using the monodromy information of the surface bundle and Lefschetz fibration. The formula, alongside an example showing precise computation of the signature of a Lefschetz fibration that corresponds to a relation found in the paper using this formula, can be found in Section 3.3. Other signature computations are done in a similar way.

2. PRELIMINARY

2.1. Intersection form and the signature

Definition 2.1. *Given any closed oriented 4-manifold X , its intersection form is the symmetric 2-form defined as follows:*

$$\begin{aligned}\mathcal{Q}_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) &\rightarrow \mathbb{Z} \\ \mathcal{Q}_X(\alpha, \beta) &= PD(\alpha) \smile PD(\beta)[X]\end{aligned}$$

where $PD(\alpha)$ and $PD(\beta)$ represent Poincaré duals of α and β respectively, \smile is the cup product, and $[X]$ the fundamental class of X .

The intersection form basically governs the topology of 4-manifolds. It is proved by Whitehead that two simply-connected, closed, and oriented 4-manifolds are orientation preserving homotopy equivalent if and only if their intersection forms are isomorphic [5]. Later it is shown by Freedman that if two smooth simply-connected 4-manifolds have isomorphic intersection forms then they are homeomorphic [6]. The intersection form is bilinear, symmetric and unimodular. For defining \mathcal{Q}_X more geometrically, we will represent classes α and β from $H_2(X; \mathbb{Z})$ by embedded surfaces S_α and S_β , and then equivalently $\mathcal{Q}_X(\alpha, \beta)$ is defined as the intersection number of S_α and S_β counted with sign. For the connected sum of any given manifolds X and Y the intersection form of the sum $X \# Y$ is given as $\mathcal{Q}_{X \# Y} = \mathcal{Q}_X \otimes \mathcal{Q}_Y$. Also, $\mathcal{Q}_{\bar{X}} = -\mathcal{Q}_X$. Here is some basic examples of the intersection forms of well-known manifolds: We first calculate $\mathcal{Q}_{CP^2} = [+1]$ since $[CP^1]$ is a generator for $H_2(CP^2, \mathbb{Z})$, and any two projective line meet in a point. It follows that $\mathcal{Q}_{CP^2 \# \overline{CP^2}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The intersection form of the twisted product $S^2 \widetilde{\times} S^2$, that is the unique nontrivial sphere-bundle over S^2 , is

$$\mathcal{Q}_{S^2 \widetilde{\times} S^2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

It is known that $S^2 \widetilde{\times} S^2$ and $CP^2 \# \overline{CP^2}$ are diffeomorphic so their intersection forms must be isomorphic, this isomorphism is given by a simple change of basis.

An important invariant of an intersection form is its signature.

Definition 2.2. *The signature of X , denoted as $\sigma(X)$, is defined to be the signature of its intersection form \mathcal{Q}_X . It is obtained as follows: first diagonalize \mathcal{Q}_X as a matrix over \mathbb{R} , separate the resulting positive and negative eigenvalues, then subtract their counts; that is*

$$\sigma(X) = \dim H_+^2(X; \mathbb{R}) - \dim H_-^2(X; \mathbb{R})$$

where $H_{\pm}^2(X; \mathbb{R})$ are any maximal positive/negative-definite subspaces for \mathcal{Q}_X . We can set partial Betti numbers $b_2^{\pm}(X) = \dim H_{\pm}^2(X; \mathbb{R})$, and thus we can read $\sigma(X) = b_2^+(X) - b_2^-(X)$.

Theorem 2.3 (Novikov's Additivity Theorem.) *Let M and N be two 4-manifolds with non-empty boundaries. Assume that their boundary 3-manifolds ∂M and ∂N admit an orientation-reversing diffeomorphism $\partial M \cong \overline{\partial N}$. Then the closed manifold $M \cup_{\partial} N$, built by identifying the boundaries ∂M and ∂N , has signature*

$$\sigma(M \cup_{\partial} N) = \sigma(M) + \sigma(N)$$

The complete proof of this theorem can be found in the second chapter of R. Kirby's The topology of 4-manifolds [7].

2.2. Mapping Class Groups

In this section, we define the mapping class group of a surface and describe basic relations in the mapping class group. The basic source is B. Farb and D. Margalit's *A Primer on Mapping Class Groups*, Chapters 2 to 5 [8].

Definition 2.4. *Let Σ_g^r be an oriented surface of genus g with r boundary components. The mapping class group of Σ_g^r , denoted Γ_g^r , is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g^r that are identity on the boundary.*

When boundary components are not present, we drop r from our notation. Elements of the mapping class group are called mapping classes and they are applied right to left. A particular type of mapping classes are Dehn twists, introduced by Max Dehn in 1938 [9]. A right handed Dehn twist about a simple closed curve α , denoted t_α , is a diffeomorphism obtained by cutting the surface along α , twisting a neighborhood of one boundary component through an angle of 2π to the right, and then regluing. The inverse diffeomorphism t_α^{-1} is called a left handed Dehn twist.

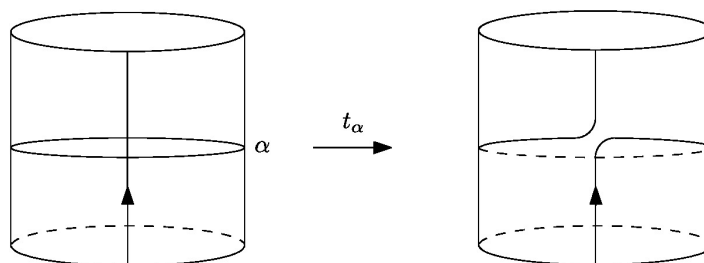


Figure 2.1. A view of a right handed Dehn twist

A Dehn twist about a simple closed curve α does not depend on the choice of the simple closed curve α within its isotopy class. Therefore, we will sometimes abuse notation slightly and write t_α instead of any Dehn twist about a simple closed curve that is in the same isotopy class as α . Dehn twists are nontrivial elements of the mapping class group if the corresponding simple closed curves are not isotopic to a point. It is

very well known that the mapping class group is generated by Dehn twists. In fact, in 1979 Humphries proved that twists about the $2g + 1$ curves suffice to generate Γ_g [10].

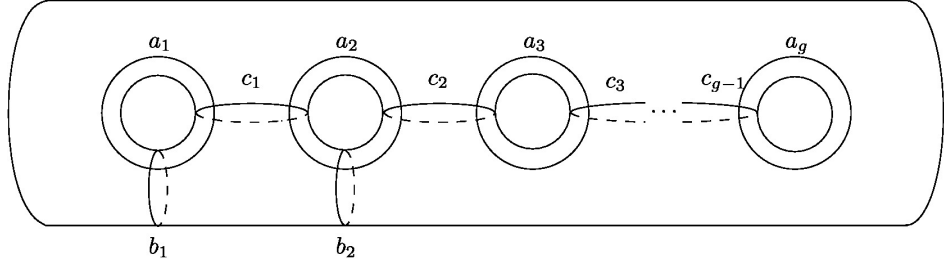


Figure 2.2. Humphries generators

The mapping class group for a torus, Γ_1 , is $\text{SL}(2, \mathbb{Z})$. Suppose that we call the $(1,0)$ -curve and the $(0,1)$ -curve in a torus a and b , respectively. The Dehn twists about a and b are represented by the matrices $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in $\text{SL}(2, \mathbb{Z})$, and they generate the group.

It is very effective to analyze Dehn twists via their actions on simple closed curves. Now we state some fundamental facts about and relations between Dehn twists that will be used repeatedly. Throughout the rest of the section a and b denote arbitrary isotopy classes of simple closed curves.

Proposition 2.5. $t_a = t_b \Leftrightarrow a = b$

Proposition 2.6. For any $f \in \Gamma_g$ and any isotopy class a of simple closed curves in Σ_g we have

$$t_{f(a)} = ft_a f^{-1}$$

Proposition 2.7. For any $f \in \Gamma_g$ and any isotopy class a of simple closed curves in Σ_g we have

$$f \text{ commutes with } t_a \Leftrightarrow f(a) = a$$

Proposition 2.7 follows directly from Proposition 2.5 and Proposition 2.6,

$$ft_a = t_af \Leftrightarrow ft_af^{-1} = t_a \Leftrightarrow t_{f(a)} = t_a \Leftrightarrow f(a) = a$$

Proposition 2.8. *For any nonseparating simple closed curves a and b there exists h in Γ_g which maps a to b*

Hence, it follows from Proposition 2.6 that for any nonseparating simple closed curves a and b in Σ_g , t_a and t_b are conjugate in Γ_g . Let $i(a, b)$ denote the *geometric intersection number* between two isotopy classes of a and b of simple closed curves in Γ_g that is defined to be the minimal number of intersection points between a representative curve in the class a and a representative curve in the class b :

$$i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}$$

Proposition 2.9. *Let a and b be arbitrary isotopy classes of essential simple closed curves in a surface and let k be an arbitrary integer. We have*

$$i(t_a^k(b), b) = |k|i(a, b)^2$$

We conclude that Dehn twists have infinite order as an important consequence of this proposition. The next fact follows from Proposition 2.7 and Proposition 2.9.

Proposition 2.10. *For any two isotopy classes a and b of simple closed curves in a surface Σ_g , we have*

$$i(a, b) = 0 \Leftrightarrow t_a(b) = b \Leftrightarrow t_at_b = t_bt_a$$

Proposition 2.11. *If a and b are isotopy classes of simple closed curves with $i(a,b) = 1$, then $t_a t_b(a) = b$.*

Proof. The computation is shown in Figure 2.3, where α is some representative of a and β is some representative of b .

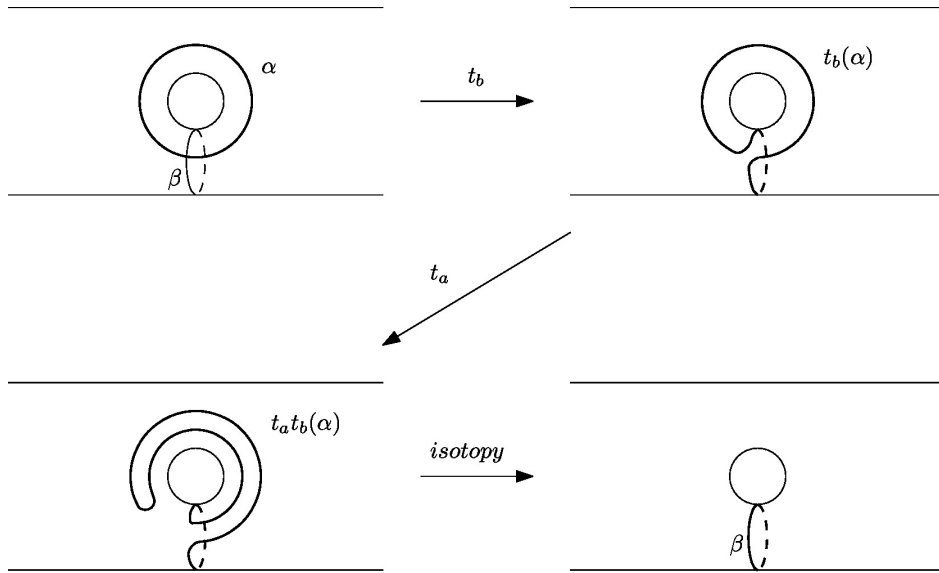


Figure 2.3. The proof of Proposition 2.11

□

It follows from Proposition 2.7 and Proposition 2.11 that we obtain a basic relation between Dehn twists in Γ_g called *Braid Relation*.

Proposition 2.12. (Braid Relation) *If a and b are isotopy classes of simple closed curves with $i(a,b) = 1$, then*

$$t_a t_b t_a = t_b t_a t_b$$

Proposition 2.13. (Lantern Relation) *Let S be a sphere with four boundary components b_1, b_2, b_3, b_4 . Suppose that S is embedded in Γ_g . Then there are three simple*

closed curves x, y, z , as illustrated in Figure 2.4, which satisfy the relation

$$t_x t_y t_z = t_{b_1} t_{b_2} t_{b_3} t_{b_4}$$

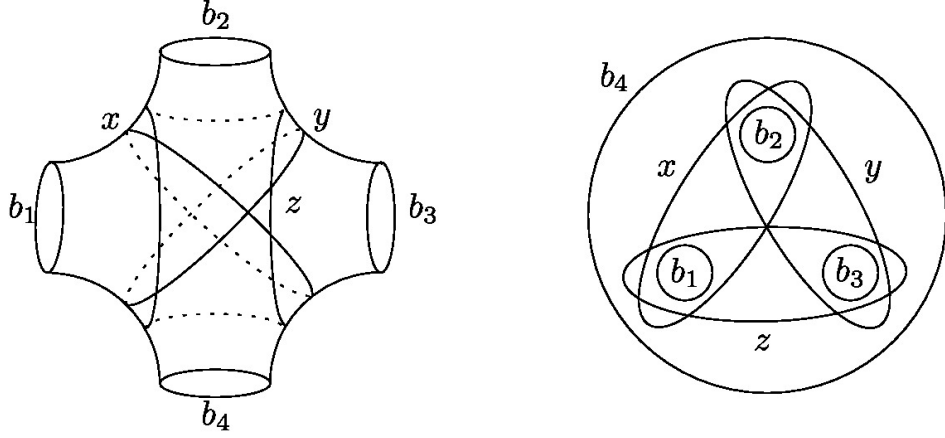


Figure 2.4. Two views of the lantern relation in Σ_0^4

In order to state the next result we need to set up some terminology. A sequence of isotopy classes c_1, \dots, c_k in a surface S is called a *chain* if $i(c_i, c_{i+1})=1$ for all i and $i(c_i, c_j)=0$ for $|i - j| > 1$.

Proposition 2.14. (Chain Relation) *Suppose $r \in 1, 2$ and c_1, \dots, c_k be a chain of curves in a surface Σ_g^r . Take representatives for the c_i that are in minimal position. Denote the isotopy classes of the boundary curves by d in the $r = 1$ case and by d_1 and d_2 in the $r = 2$ case.*

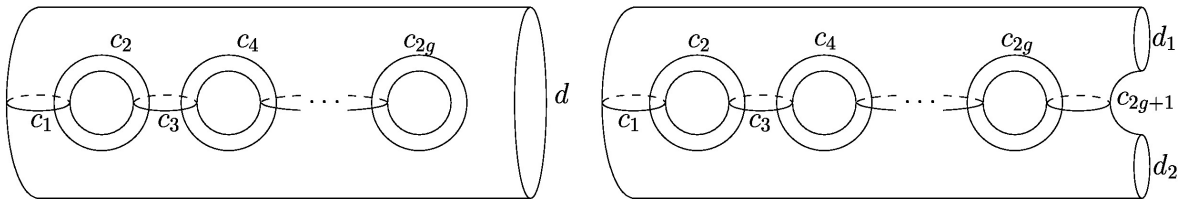


Figure 2.5. Two views of the chain relation in Σ_{2g}^1 and Σ_{2g}^2

Then the following relations hold in Γ_g^r :

$$\begin{aligned} (t_{c_1} \dots t_{c_{2g}})^{4g+2} &= t_d & r=1 \\ (t_{c_1} \dots t_{c_{2g+1}})^{2g+2} &= t_{d_1} t_{d_2} & r=2 \end{aligned}$$

2.3. Lefschetz fibrations

In this section we define Lefschetz fibrations which are the building blocks for our main result.

Definition 2.15. *Let X be a compact, connected, oriented, smooth 4-manifold, and Σ is a compact, connected, oriented surface. A smooth surjective map $f : X \rightarrow \Sigma$ is called a Lefschetz fibration if for each critical point $p \in X$ there are local complex coordinates (z_1, z_2) on X around p and z on Σ around $f(p)$ compatible with the orientations and such that $f(z_1, z_2) = z_1^2 + z_2^2$.*

By definition a Lefschetz fibration has at most finitely many critical points, and we may assume that f is injective on its critical set $C = \{p_1, \dots, p_k\}$. We assume that all regular fibers have the same genus so we identify the regular fiber with the closed oriented surface F of genus h . The *genus* of f is defined to be the genus of a regular fiber. Fibers of f passing through any critical point is called *singular* fibers. Let $\nu(f(C))$ denote an open neighborhood of the set of critical values $f(C)$, then we notice that the restriction of f to $f^{-1}(\Sigma - \nu(f(C)))$ is a smooth surface bundle over $\Sigma - \nu(f(C))$.

A singular fiber $f^{-1}(q_i)$ can be described by its *monodromy*, which is an element in Γ_h , the mapping class group of F . To determine this element, we consider the preimage of a loop c_i in $\Sigma - f(C)$ based at a regular value which has linking number $+1$ with q_i . This preimage is an F -bundle over S^1 which can be described by this element. Let t_i denote this element. It can be shown that t_i is a right-handed Dehn twist along a simple closed curve $v_i \subset F$ called the *vanishing cycle*. It can be shown that a singular fiber is obtained from the regular fiber by collapsing a corresponding vanishing cycle.

After fixing the generating system $\{a_1, b_1, \dots, a_h, b_h, c_1, \dots, c_k\}$ for $\pi_1(\Sigma - f(C))$ we get a map $\varphi : \pi_1(\Sigma - f(C)) \rightarrow \Gamma_h$ taking a_i to α_i , b_i to β_i , and c_i to t_i as described

above. It is easy to see that these elements satisfy the relation

$$\prod_{i=1}^g [\alpha_i, \beta_i] \cdot \prod_{j=1}^k t_j = 1 \text{ in } \Gamma_h,$$

and this is called the monodromy factorization of a Lefschetz fibration. Conversely, a factorization $\prod_{i=1}^g [\alpha_i, \beta_i] \cdot \prod_{j=1}^k t_j = 1$ in Γ_h gives rise to a genus h Lefschetz fibration over Σ_g . It follows immediately that surface bundles over surfaces are described by the monodromy factorizations that consist of only commutators.

Recall that Γ_1 is isomorphic to $\text{SL}(2, \mathbb{Z})$, the isomorphism sends t_a and t_b to the matrices $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ respectively, then the matrix $C = BA = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ has order 6 in $\text{SL}(2, \mathbb{Z})$. Hence, we obtain the relation $(t_a t_b)^{6n} = 1$ for each $n \geq 1$. It is also possible to obtain the same result from the chain relation. So, each relation gives rise to a genus-1 Lefschetz fibrations over S^2 with $12n$ critical points. In particular, the resulting fibration $\pi_n : X_n \rightarrow S^2$ is isomorphic to a generic elliptic fibration $\pi : E(n) \rightarrow S^2$.

3. PROOF OF THE MAIN THEOREM

3.1. Subtraction of Lefschetz Fibrations

In this section we explain an operation, called the subtraction of Lefschetz fibrations, introduced by H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, A. Stipsicz to construct surface bundles over surfaces from Lefschetz fibrations [1].

A singular fiber $f^{-1}(q_i)$ is *nonseparating* and of *type 0* if the corresponding vanishing cycle v_i has nonzero homology class in $H_1(F; \mathbb{Z})$, and it is *separating* and of *type* $j \in \{1, \dots, \lfloor \frac{h}{2} \rfloor\}$ if the corresponding vanishing cycle has trivial homology class and separates the surface of genus h into two components with genera j and $h - j$. Two singular fibers of the same type have fiber- and orientation-preservingly diffeomorphic tubular neighborhoods since from the classification of surfaces for two simple closed curves of the same type there exists a diffeomorphism of the ambient surface mapping one into the other.

Definition 3.1. *The vector $\mu_{comb}(X) = (\mu_0, \dots, \mu_{\lfloor \frac{h}{2} \rfloor}) \in \mathbb{Z}^{\lfloor \frac{h}{2} \rfloor + 1}$ associated to the Lefschetz fibration $f : X \rightarrow \Sigma_g$ is constructed by taking μ_j to be the number of singular fibers of type j ($j = 0, \dots, \lfloor \frac{h}{2} \rfloor$). We say that two fibrations $f_i : X_i \rightarrow \Sigma_{g_i}$ ($i = 1, 2$) are combinatorially equivalent if $\mu_{comb}(X_1) = \mu_{comb}(X_2)$.*

Now we introduce a way to construct a surface bundle by taking the difference of two combinatorially equivalent Lefschetz fibrations. Suppose that X_1 and X_2 are combinatorially equivalent as in Definition 3.1, with critical values $\{q_i^1\}_{i=1}^s$ and $\{q_i^2\}_{i=1}^s$ that are ordered in a way that singular fibers with coinciding lower index have the same type. Fix an orientation- and fiber-preserving diffeomorphism ϕ_i between the boundaries of tubular neighborhoods of fibers with lower index i ($i = 1, \dots, s$), let ϕ denote the union of these diffeomorphisms. Now glue $X_1 - (\cup_{i=1}^s \nu(f_1^{-1}(q_i^1)))$ to $\overline{X_2 - (\cup_{i=1}^s \nu(f_2^{-1}(q_i^2)))}$ using ϕ . finally we obtain the resulting manifold $Y = X_1 - X_2$ which inherits a natural orientation since the map ϕ becomes orientation-reversing after reversing the orienta-

tion on X_2 . and is an oriented smooth surface bundle over a compact surface Σ which we denote by $B_1 - B_2$.

Lemma 3.2. *If $X_1 \rightarrow \Sigma_{g_1}$ and $X_2 \rightarrow \Sigma_{g_2}$ are combinatorially equivalent Lefschetz fibrations with k singular fibers, then $Y = X_1 - X_2$ is a smooth surface bundle with signature $\sigma(Y) = \sigma(X_1) - \sigma(X_2)$ over the surface $\Sigma = \Sigma_{g_1} - \Sigma_{g_2}$ with Euler characteristic $\chi(\Sigma) = \chi(\Sigma_{g_1}) + \chi(\Sigma_{g_2}) - 2s$.*

Proof. By construction, Y is an oriented smooth surface bundle over a surface Σ . Since the base Σ is formed by deleting a ball around each critical value inside Σ_{g_1} and Σ_{g_2} such that the corresponding singular fibers are of the same type and gluing their boundary, the resulting surface is a surface of genus $g_1 + g_2 + s - 1$, and hence, with Euler characteristic $\chi(\Sigma) = \chi(\Sigma_{g_1}) + \chi(\Sigma_{g_2}) - 2s$. The claim about the signature follows from Novikov additivity. \square

Another variation of this construction goes as follows:

Proposition 3.3. *Suppose $f_1 : X_1 \rightarrow \Sigma_{g_1}$ and $f_2 : X_2 \rightarrow \Sigma_{g_2}$ are combinatorially equivalent Lefschetz fibrations with s singular fibers. We group the critical values $\{q_i^1\}_{i=1}^s$ and $\{q_i^2\}_{i=1}^s$ according to their types and choose disjoint disks $D_j^k \subset \Sigma_{g_k}$ ($k = 1, 2$ and $j = 1, \dots, m$) such that each disk contains only one type. Then*

$$Y = (X_1 - \left(\bigcup_{j=1}^m f_1^{-1}(\text{int}D_j^1)\right)) \cup \overline{(X_2 - \left(\bigcup_{j=1}^m f_2^{-1}(\text{int}D_j^2)\right))}$$

is a surface bundle obtained by subtracting X_2 from X_1 along the disks D_j^k . The signature $\sigma(Y)$ is again $\sigma(X_1) - \sigma(X_2)$ while the Euler characteristic of the base is $\chi(\Sigma_{g_1}) + \chi(\Sigma_{g_2}) - 2m$.

Proof. These disks $D_j^k \subset \Sigma_{g_k}$ ($k = 1, 2$ and $j = 1, \dots, m$) can be paired up in a way that $X_1 - f_1^{-1}(\text{int}D_j^1)$ and $X_2 - f_2^{-1}(\text{int}D_j^2)$ have diffeomorphic boundaries. The rest is an immediate consequence of the subtraction of Lefschetz fibrations operation. \square

3.2. Commutators in mapping class groups

Lemma 3.4. *Let a, b, c and d be four simple closed curves on Σ such that a is disjoint from b , c is disjoint from d , and the complements of $a \cup b$ and $c \cup d$ in Σ are connected. Then $t_a t_b^{-1} t_c t_d^{-1}$ is a commutator.*

Proof. By the classification of surfaces, there exists a diffeomorphism g of Σ such that $g(a)=d$ and $g(b)=c$. Then

$$t_a t_b^{-1} t_c t_d^{-1} = t_a t_b^{-1} t_{g(b)} t_{g(a)}^{-1} = t_a t_b^{-1} g t_b t_a^{-1} g^{-1} = [t_a t_b^{-1}, g]$$

□

Proposition 3.5. *Let $h \geq 3$ and let a be a simple closed curve on Σ_h . In the mapping class group $\Gamma_{h,s}^r$ of Σ_h*

- (i) t_a can be written as a product of two commutators,
- (ii) t_a^2 can be written as a product of two commutators,
- (iii) if a is nonseparating, then t_a^4 can be written as a product of three commutators.

Proof. (i)

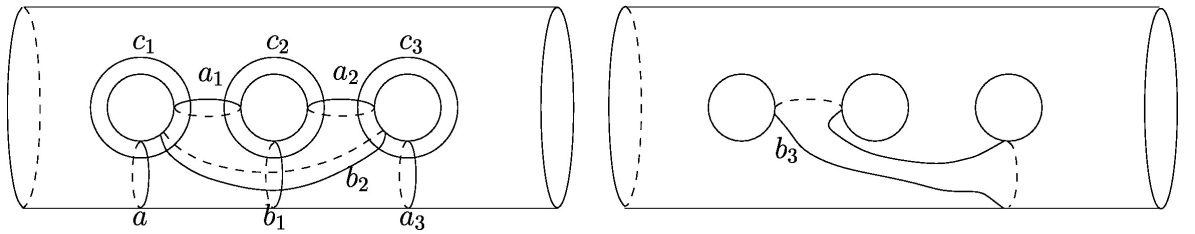


Figure 3.1. If a is nonseparating

The vanishing cycle v is topologically equivalent to the curve a when the genus 3 subsurface is chosen suitably. We first assume the case where a is nonseparating. We

consider curves $a, a_1, a_2, a_3, b_1, b_2, b_3$, and c_1, c_2, c_3 on a genus 3 subsurface of Σ_h as in Figure 3.1

Now we will show how to write a Dehn twist as a product of two commutators and how to make explicit choices for these commutators.

Then we obtain the lantern relation between Dehn twists about $a, a_1, a_2, a_3, b_1, b_2, b_3$ as

$$t_{b_3}^{-1} t_{a_3} t_{b_2}^{-1} t_{a_2} t_{b_1}^{-1} t_{a_1} t_a = 1$$

By commuting t_{a_2} and $t_{b_2}^{-1}$ and inserting $t_{c_1} t_{b_2} t_{b_2}^{-1} t_{c_1}^{-1}$ we obtain

$$t_{b_3}^{-1} t_{a_3} t_{a_2} t_{c_1} t_{b_2} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_2}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

the curves b_2 and c_1 intersect once so the Dehn twists about these curves satisfy the braid relation so we get

$$t_{b_3}^{-1} t_{a_3} t_{a_2} t_{c_1} t_{b_2} t_{c_1}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

by commuting t_{a_2} with $t_{c_1}, t_{b_2}, t_{c_1}^{-1}$ and inserting $t_{c_2} t_{a_2} t_{a_2}^{-1} t_{c_2}^{-1}$ the relation becomes

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_1}^{-1} t_{a_2} t_{c_2} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

t_{a_2} and t_{c_2} satisfy the braid relation

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_1}^{-1} t_{c_2} t_{a_2} t_{c_2}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

by inserting $t_{b_1} t_{c_2} t_{c_2}^{-1} t_{b_1}^{-1}$

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_1}^{-1} t_{c_2} t_{a_2} t_{c_2} t_{b_1} t_{c_2}^{-1} t_{b_1}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

now t_{b_1} and t_{c_2} satisfy the braid relation

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_1}^{-1} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{b_1} t_{c_2}^{-1} t_{b_1}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

by commuting $t_{c_1}^{-1}$ with $t_{c_2}, t_{a_2}, t_{b_1}$ and inserting $t_{a_1} t_{c_1} t_{c_1}^{-1} t_{a_1}^{-1}$,

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{b_1} t_{a_1} t_{c_1} t_{c_1}^{-1} t_{a_1}^{-1} t_{c_1}^{-1} t_{c_2}^{-1} t_{b_1}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

$t_{c_1}^{-1}$ and $t_{a_1}^{-1}$ satisfy the braid relation

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{b_1} t_{a_1} t_{c_1} t_{a_1}^{-1} t_{c_1}^{-1} t_{a_1}^{-1} t_{c_2}^{-1} t_{b_1}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

by commuting t_{b_1} with $t_{a_1}, t_{c_1}, t_{a_1}^{-1}$

$$t_{b_3}^{-1} t_{a_3} t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{a_1} t_{c_1} t_{a_1}^{-1} t_{b_1} t_{c_1}^{-1} t_{a_1}^{-1} t_{c_2}^{-1} t_{b_1}^{-1} t_{a_2}^{-1} t_{c_2}^{-1} t_{b_2}^{-1} t_{c_1}^{-1} t_{b_1}^{-1} t_{a_1} t_a = 1$$

now defining $\phi_1 = t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{a_1} t_{c_1}$ we obtain that

$$t_{b_3}^{-1} t_{a_3} [\phi_1, t_{a_1}^{-1} t_{b_1}] t_a = 1$$

Similarly we can write $t_{b_3}^{-1} t_{a_3}$ as $[t_{a_3}, \phi_2]$ where $\phi_2 = t_{c_3} t_{b_3} t_{a_3} t_{c_3}$. Finally we obtain that

$$[t_{a_3}, \phi_2] [\phi_1, t_{a_1}^{-1} t_{b_1}] t_a = 1$$

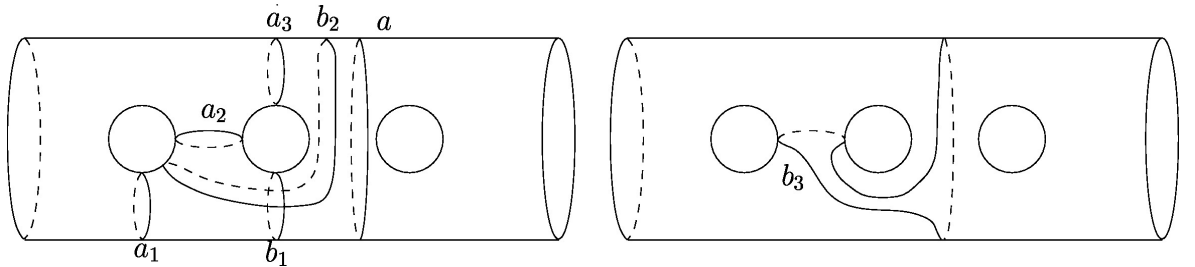


Figure 3.2. If a is separating

For the case a is separating, we consider the curves $a, a_1, a_2, a_3, b_1, b_2, b_3,$ and c_1, c_2, c_3 as in Figure 3.2. We again have the following lantern relation

$$t_{a_3} t_{b_3}^{-1} t_{a_2} t_{b_2}^{-1} t_{a_1} t_{b_1}^{-1} t_a = 1$$

We set the diffeomorphisms $\phi_1 = t_{c_2} t_{a_2} t_{c_1} t_{b_2} t_{a_1} t_{c_1} t_{b_1} t_{c_2}$, and $\phi_2 = t_{c_2} t_{a_3} t_{b_3} t_{c_2}$ such that we obtain that $\phi_1(a_1) = b_2$, $\phi_1(b_1) = a_2$, and $\phi_2(a_3) = b_3$. Applying Lemma 3.4 to this relation proves that t_a is a product of two commutators. This completes the proof of (i).

(ii) On a genus 3 subsurface with the specified curves as described in Figure 3.3,

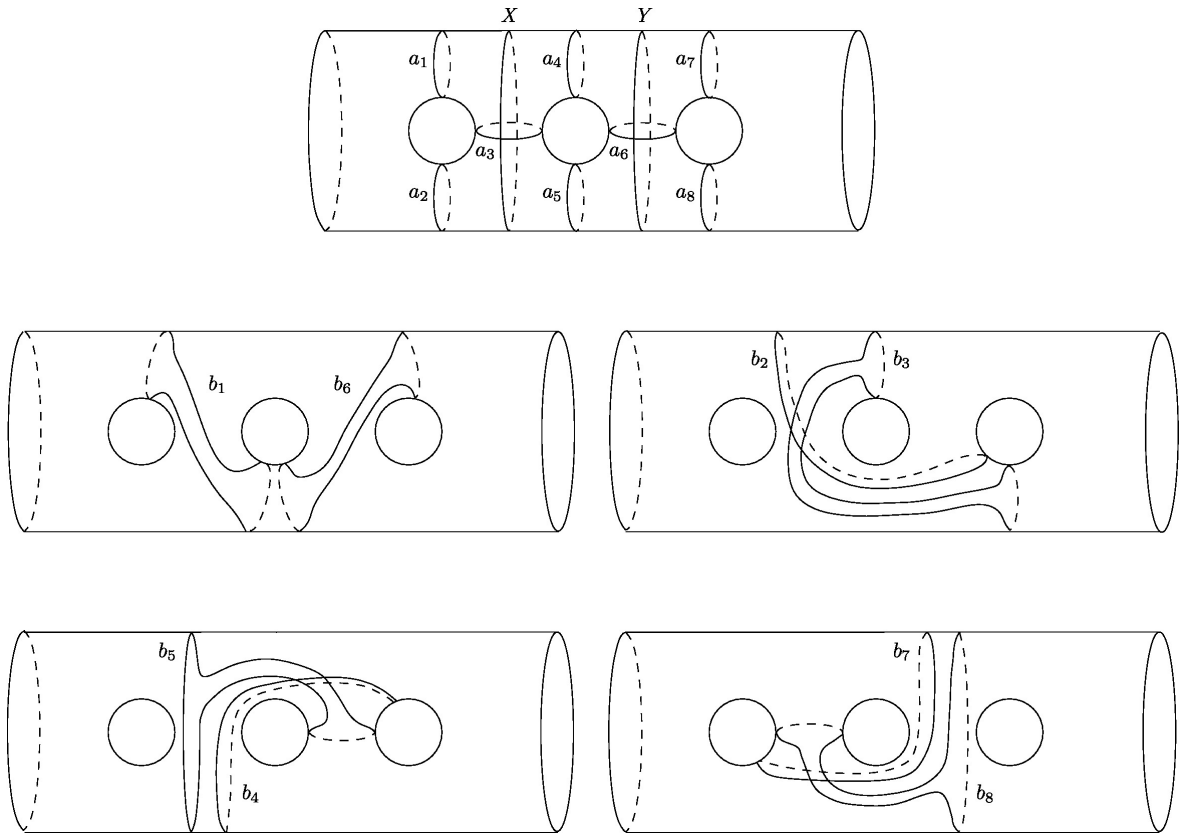


Figure 3.3. Curves on a genus 3 subsurface

the following three lantern relations hold,

$$t_{a_1}t_{a_2}t_{a_4}t_{a_5} = t_x t_{a_3} t_{b_1} \quad (3.1)$$

$$t_x t_{a_4} t_{a_6} t_{a_8} = t_{a_5} t_{b_2} t_{b_3} \quad (3.2)$$

and

$$t_x t_{a_5} t_{a_6} t_{a_7} = t_{a_4} t_{b_4} t_{b_5} \quad (3.3)$$

Multiplying both sides of (3.1) by $t_{b_2}t_{b_3}$, using (3.2) and cancelling t_x we obtain

$$t_{a_4}^2 t_{a_1} t_{a_2} t_{a_6} t_{a_8} = t_{a_3} t_{b_1} t_{b_2} t_{b_3} \quad (3.4)$$

or, equivalently,

$$t_{a_4}^2 = t_{a_3} t_{a_6}^{-1} t_{b_1} t_{a_8}^{-1} t_{b_2} t_{a_1}^{-1} t_{b_3} t_{a_2}^{-1} \quad (3.5)$$

Similarly, (3.2) and (3.3) yield the following equation

$$t_x^2 = t_{b_2} t_{a_6}^{-1} t_{b_3} t_{a_7}^{-1} t_{b_4} t_{a_6}^{-1} t_{b_5} t_{a_8}^{-1} \quad (3.6)$$

We conclude by Lemma 3.4 that $t_{a_4}^2$ and t_x^2 can be written as a product of two commutators. Given any simple closed curve a , if it is nonseparating we know that it is topologically equivalent to a_4 , and if it is separating the genus 3 subsurface can be chosen in a way that a is topologically equivalent to x . We complete the proof of (ii) since a conjugate of a commutator is again a commutator.

(iii) Similarly two more embeddings in the Figure 3.3 of the lantern give the relations

$$t_{a_4} t_{a_5} t_{a_7} t_{a_8} = t_{b_6} t_{a_6} t_y \quad (3.7)$$

and

$$t_y t_{a_2} t_{a_3} t_{a_4} = t_{a_5} t_{b_7} t_{b_8} \quad (3.8)$$

Multiplying (3.7) by $t_{b_7} t_{b_8}$ from the left and using (3.8) we obtain

$$t_{a_4}^2 t_{a_2} t_{a_3} t_{a_7} t_{a_8} = t_{b_7} t_{b_8} t_{b_6} t_{a_6} \quad (3.9)$$

By combining (3.4) and (3.9) and cancelling t_{a_3} and t_{a_6} , we obtain

$$t_{a_4}^4 t_{a_1} t_{a_2}^2 t_{a_7} t_{a_8}^2 = t_{b_1} t_{b_2} t_{b_3} t_{b_7} t_{b_8} t_{b_6} \quad (3.10)$$

The relation obtained in eq. 3.10 is equivalent to

$$t_{a_4}^4 = t_{b_1} t_{a_1}^{-1} t_{b_2} t_{a_2}^{-1} t_{b_3} t_{a_2}^{-1} t_{b_7} t_{a_7}^{-1} t_{b_8} t_{a_8}^{-1} t_{b_6} t_{a_8}^{-1} \quad (3.11)$$

Since a_1 is disjoint from b_1 , a_2 is disjoint from b_2 , a_2 is disjoint from b_3 , a_7 is disjoint from b_7 , a_8 is disjoint from b_8 , a_8 is disjoint from b_6 , and the complements of $a_1 \cup b_1$, $a_2 \cup b_2$, $a_2 \cup b_3$, $a_7 \cup b_7$, $a_7 \cup b_8$, and of $a_8 \cup b_6$ are all connected, Lemma 3.4 implies that $t_{a_4}^4$ is a product of three commutators. \square

Proposition 3.6. *Let $h \geq 2$ and let a and b are two simple closed curves intersecting each other transversely at one point on Σ_h . Then $t_a^4 t_b^4$ is a product of three commutators.*

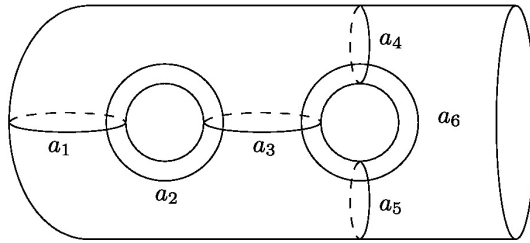


Figure 3.4. The chain relation in Σ_2^1

Proof. Suppose that the genus 2 subsurface as in Figure 3.4 embedded in Σ_h in a way that a_4 and a_5 are nonseparating on Σ_h .

By the chain relation we have $t_{a_4}t_{a_5} = (t_{a_1}t_{a_2}t_{a_3})^4$. Then, we obtain

$$\begin{aligned}
t_{a_4}t_{a_5} &= (t_{a_1}t_{a_2}t_{a_3}t_{a_1}t_{a_2}t_{a_3})(t_{a_1}t_{a_2}t_{a_3}t_{a_1}t_{a_2}t_{a_3}) \\
&= (t_{a_1}t_{a_2}t_{a_1}t_{a_3}t_{a_2}t_{a_3})(t_{a_1}t_{a_2}t_{a_1}t_{a_3}t_{a_2}t_{a_3}) \\
&= (t_{a_2}t_{a_1}t_{a_2}t_{a_2}t_{a_3}t_{a_2})(t_{a_2}t_{a_1}t_{a_2}t_{a_2}t_{a_3}t_{a_2}) \\
&= t_{a_2}t_{a_2}t_{a_3}t_{a_2}t_{a_2}t_{a_1}t_{a_2}t_{a_2}t_{a_3}t_{a_2}t_{a_2}t_{a_1} \\
&= (t_{a_2}t_{a_2}t_{a_3}t_{a_2}^{-1}t_{a_2}^{-1})t_{a_2}t_{a_2}t_{a_2}t_{a_2}(t_{a_1}t_{a_2}t_{a_2}t_{a_2}t_{a_2}t_{a_1}^{-1})t_{a_1}t_{a_1}(t_{a_1}^{-1}t_{a_2}^{-1}t_{a_2}^{-1}t_{a_3}t_{a_2}t_{a_2}t_{a_1})
\end{aligned}$$

We can assume that $a = a_2$ and $b = t_{a_1}(a_2)$ since a intersects b transversely at one point and any two such pairs are topologically equivalent. By setting $v = t_{a_2}^2(a_3)$ and $w = t_{a_1}^{-1}t_{a_2}^{-2}(a_3)$, we have the equality

$$t_a^4 t_b^4 t_{a_1}^2 = (t_{a_4} t_v^{-1} t_{a_5} t_w^{-1}) \quad (3.12)$$

Now, $t_{a_4} t_v^{-1} t_{a_5} t_w^{-1}$ is a commutator by Lemma 3.4 and $t_{a_1}^2$ is a product of two commutators from the previous proposition. Thus, we conclude that $t_a^4 t_b^4$ is a product of three commutators. \square

3.3. Signature Computations

In this section we introduce the Meyer's theorem [4] to calculate the signature and compute the signature by using a relation found Proposition 3.5.

Theorem 3.7. *Let $f : X \rightarrow \Sigma_{g,s}^r$ be an oriented surface bundle with monodromy representation $\rho : \pi_1(\Sigma_{g,s}^r) \rightarrow \Gamma_h$. Fix a standart representation of $\pi_1(\Sigma_{g,s}^r)$ as follows:*

$$\pi_1(\Sigma_{g,s}^r) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^r c_j = 1 \rangle$$

and let $\tau_h : Sp(2h, \mathbb{Z}) \times Sp(2h, \mathbb{Z}) \rightarrow \mathbb{Z}$ be the signature cocyle. Then the signature of X is given by the formula

$$\sigma(X) = \sum_{i=1}^g \tau_h(\kappa_i, \beta_i) - \sum_{i=2}^g \tau_h(\kappa_1 \dots \kappa_{i-1}, \kappa_i) - \sum_{j=1}^{r-1} \tau_h(\kappa_1 \dots \kappa_g \gamma_1 \dots \gamma_{j-1}, \gamma_j)$$

where $\alpha_i = \chi(a_i)$, $\beta_i = \chi(b_i)$, $\gamma_i = \chi(c_i)$ and $\kappa_i = [\alpha_i, \beta_i]$.

(χ is the composition $\phi \circ \rho : \pi_1(\Sigma_{g,s}^r) \rightarrow Sp(2h, \mathbb{Z})$)

For a pair of symplectic matrices $A, B \in Sp(2h, \mathbb{Z})$, the vector space $V_{A,B}$ is defined by:

$$V_{A,B} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid (A^{-1} - I_{2h})x + (B - I_{2h})y = 0\}$$

the symplectic bilinear form $\langle \cdot, \cdot \rangle_{A,B} : V_{A,B} \times V_{A,B} \rightarrow \mathbb{R}$ defined by

$$\langle (x, y), (x', y') \rangle_{A,B} := \langle x + y, (A^{-1} - I_{2h})x' \rangle = (x + y)^T \cdot J \cdot (A^{-1} - I_{2h})x'$$

where $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Meyer's signature cocyle $\tau_h : Sp(2h, \mathbb{Z}) \times Sp(2h, \mathbb{Z}) \rightarrow \mathbb{Z}$ is defined by

$$\tau_h(A, B) := \sigma(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}).$$

Here is an example of this formula:

Recall that we previously obtain the relation $[t_{a_3}, \phi_2] \cdot [\phi_1, t_{a_1}^{-1} t_{b_1}] t_a = 1$ where $\phi_1 = t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{a_1} t_{c_1}$ and $\phi_2 = t_{c_3} t_{b_3} t_{a_3} t_{c_3}$.

We choose a symplectic basis as follows:

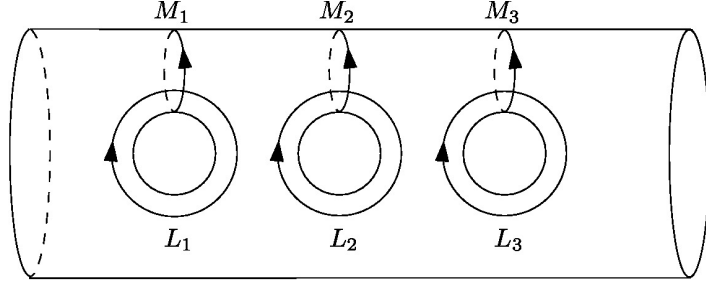


Figure 3.5. Symplectic basis $\{L_1, L_2, L_3, M_1, M_2, M_3\}$

The corresponding symplectic matrix of a diffeomorphism which acts trivially on the complement of the genus 3 subsurface has the shape $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \in Sp(2h, \mathbb{Z})$ with $A \in Sp(6, \mathbb{Z})$. Therefore, we denote this matrix by only A . According to Meyer's formula:

$$\sigma(X) = \sum_{i=1}^2 \tau_3(\kappa_i, \beta_i) - \tau_3(\kappa_1, \kappa_2) = \tau_3(\kappa_1, \beta_1) + \tau_3(\kappa_2, \beta_2) - \tau_h(\kappa_1, \kappa_2)$$

In the following we will show how to evaluate Meyer's signature cocycle $\tau_3(\kappa_1, \beta_1)$. We can simply compute κ_1 and β_1 .

$$\kappa_1 = [\alpha_1, \beta_1] = [\chi(a_1), \chi(b_1)] = [\phi(t_{a_3}), \phi(\phi_2)] = \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline 1 & -1 & 1 & 1 & & \\ -1 & 1 & -1 & & 1 & \\ 1 & -1 & & & & 1 \end{array} \right] \text{ and}$$

$$\beta_1 = \phi(\chi(b_i)) = \phi(\phi_2) = \left[\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ -1 & 1 & -1 & & \\ \hline -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & & 1 \\ -1 & 1 & -2 & & -1 \end{array} \right]$$

$$V_{\kappa_1, \beta_1} = \{(x, y) \in \mathbb{R}^6 \times \mathbb{R}^6 | (\kappa_1^{-1} - I)x + (\beta_1 - I)y = 0\}$$

$$\left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \\ \hline -1 & 1 & -1 & & & \\ 1 & -1 & 1 & & & \\ -1 & 1 & & & & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \left[\begin{array}{ccc|c} -1 & 1 & -2 & \\ \hline -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -2 & -2 \end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = 0$$

Solving this equation we obtain that

$$\begin{aligned} y_2 &= x_1 - x_2 + 2x_3 + y_1, \\ y_3 &= \frac{x_1}{2} - \frac{x_2}{2} + x_3 \end{aligned}$$

$$\langle (x, y), (x', y') \rangle_{\kappa_1, \beta_1} = \langle x + y, (\kappa_1^{-1} - I)x' \rangle$$

$$\left\langle \begin{bmatrix} x_1 + y_1 \\ x_1 + 2x_3 + y_1 \\ \frac{1}{2}x_1 - \frac{x_2}{2} + x_3 \\ x_4 + y_4 \\ x_5 + y_5 \\ x_6 + y_6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -x'_1 + x'_2 - x'_3 \\ x'_1 - x'_2 + x'_3 \\ -x'_1 + x'_2 \end{bmatrix} \right\rangle =$$

$$\begin{bmatrix} x_1 + y_1 \\ x_1 + 2x_3 + y_1 \\ \frac{1}{2}x_1 - \frac{x_2}{2} + x_3 \\ x_4 + y_4 \\ x_5 + y_5 \\ x_6 + y_6 \end{bmatrix}^T \begin{bmatrix} & & 1 \\ & & 1 \\ & & 1 \\ -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -x'_1 + x'_2 - x'_3 \\ x'_1 - x'_2 + x'_3 \\ -x'_1 + x'_2 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

$$\tau_3(\kappa_1, \beta_1) = \sigma \left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) = \sigma \left(\begin{bmatrix} -\frac{1}{2} & & \\ & 0 & \\ & & 2 \end{bmatrix} \right) = 0.$$

Similarly we compute that $\tau_3(\kappa_2, \beta_2) = -1$ and $\tau_3(\kappa_1, \kappa_2) = 0$. Finally we obtain the result that $\sigma(X) = -1$.

3.4. Main Results

Meyer's formula allows us to compute the signatures of surface bundles that are the complements of singular fibers of Lefschetz fibration. Therefore, the following fact will be useful.

Proposition 3.8. *The signature of a fibered neighborhood of nonseparating, respectively separating, singular fiber in a Lefschetz fibration is equal to 0, respectively to -1.*

Now we can use the relations found in Section 3.2 to construct Lefschetz fibrations.

Proposition 3.9. *There is a Lefschetz fibration $X \rightarrow \Sigma_2$ with a unique singular fiber and with signature -1, whether the vanishing cycle is separating or not.*

Proof. We show that a Dehn twist, whether it is separating or not, can be written as a product of two commutators in the first part of Proposition 3.5. Hence, the word obtained in the first part of Proposition 3.5 representing 1 in Γ_h , where $h \geq 2$, gives rise to a Lefschetz fibration of genus h over a surface Σ_2 . For the vanishing cycle is nonseparating, the signature of the complement of the corresponding singular fiber is calculated to be -1 in the previous section. Similarly, evaluating the signature cocycle, the complement of the separating singular fiber is calculated to be 0. Finally, Proposition 3.8 and Novikov additivity complete the proof. \square

Similarly, the relations found in Proposition 3.5 and 3.6 give rise to Lefschetz fibrations described as follows:

Proposition 3.10. *There is a Lefschetz fibration $X \rightarrow \Sigma_2$ with two singular fibers whose monodromies are Dehn twists with the same nonseparating vanishing cycle and signature equal to -2.*

Proof. Along with the curves in Figure 3.3 we also consider the curves d and e as in the following.

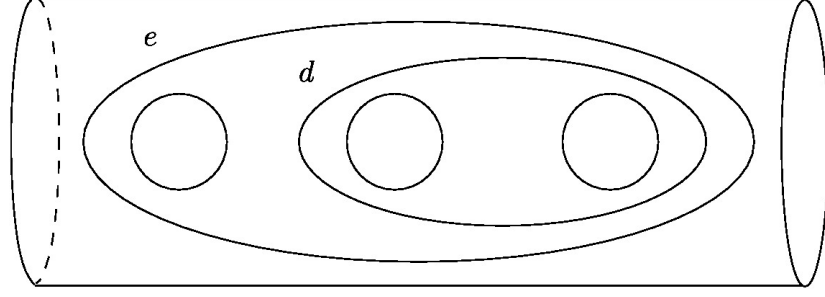


Figure 3.6. The curves e and d on genus 3 subsurface

We find the relation $t_{a_4}^2 = t_{a_3} t_{a_6}^{-1} t_{b_1} t_{a_8}^{-1} t_{b_2} t_{a_1}^{-1} t_{b_3} t_{a_2}^{-1}$ in 2nd part of Proposition 3.5.

By setting the diffeomorphisms

$$\phi_1 = t_e t_{b_2} t_{a_1} t_e t_{c_2} t_{a_6} t_{b_3} t_{c_2}$$

and

$$\phi_2 = t_{c_1} t_{a_3} t_{a_2} t_{c_1} t_d t_{a_8} t_{b_1} t_d \text{ we write the relation as } [t_{b_3}^{-1} t_{a_1}, \phi_1] [t_{b_1}^{-1} t_{a_2}, \phi_2] t_{a_4}^2 = 1.$$

Now this word gives rise to a Lefschetz fibration over Σ_2 with two singular fibers whose monodromies are Dehn twists with the same nonseparating vanishing cycle. By evaluating the Meyer's signature cocycle the signature of the complement of the singular fibers are calculated to be -2. Finally, Proposition 3.8 and Novikov additivity complete the proof. \square

Proposition 3.11. *There is a Lefschetz fibration $X \rightarrow \Sigma_3$ with four singular fibers whose monodromies are Dehn twists with the same nonseparating vanishing cycle and signature equal to -4.*

Proof. The relation found in 3rd part of Proposition 3.5 is written as

$$[t_{b_6}^{-1} t_{a_2}, \phi_1] [t_{b_7}^{-1} t_{a_8}, \phi_2] [t_{b_2}^{-1} t_{a_1}, \phi_3] t_{a_4}^4 = 1$$

where $\phi_1 = t_{c_1} t_{b_8} t_{a_2} t_{c_1} t_{c_3} t_{a_8} t_{b_6} t_{c_3}$ $\phi_2 = t_{c_3} t_{b_3} t_{a_8} t_{c_3} t_{c_1} t_{a_2} t_{b_7} t_{c_1}$ and,

$$\phi_3 = t_{c_1} t_{b_1} t_{a_1} t_{c_1} t_{c_3} t_{a_7} t_{b_2} t_{c_3}.$$

\square

Proposition 3.12. *Let a and b be two nonseparating simple closed curves which intersect transversely and precisely at one point. There is a Lefschetz fibration $X \rightarrow \Sigma_3$ with signature -4 which has eight singular fibers, four of which have monodromy a Dehn twist along a and four of which have monodromy a Dehn twist along b .*

Proof. The relation found in Proposition 3.6 can be written as

$$[t_{a_1}^{-1}t_{a_2}^{-2}t_{a_3}t_{a_2}^2t_{a_1}t_{a_5}^{-1}, \phi]t_{a_2}^4(t_{a_1}t_{a_2}t_{a_1})^4t_{a_1}^2 = 1$$

with $\phi = t_{a_2}^3t_{a_3}t_{a_6}t_{a_4}t_{a_5}t_{a_6}t_{a_3}t_{a_2}t_{a_6}t_{a_5}t_{a_3}t_{a_6}t_{a_2}^2t_{a_1}$. This relation gives rise to a Lefschetz fibration $X' \rightarrow \Sigma_2$ with 10 singular fibers, and the signature of X' is computed to be -6 . Subtracting the fibration obtained in Proposition 3.10 from X' completes the proof. \square

Finally, we can state and prove the main theorem.

Theorem 3.13. *For every $h \geq 3$ there is a surface bundle of genus h over the surface of genus 9 with signature 4.*

Proof. Consider the Lefschetz fibrations $X_1 \rightarrow \Sigma_3$, and $X_2 \rightarrow \Sigma_3$ obtained in Proposition 3.12, and Proposition 3.11, respectively. We choose two disjoint disks in the base surface of X_1 such that each contains four singular values whose corresponding vanishing cycles coincide, and one disk in the base surface of X_2 which contains all singular values. Subtracting two copies of X_2 from X_1 along these disks we obtain a surface bundle $Y \rightarrow \Sigma_9$ of fiber genus h with $\sigma(Y) = \sigma(X_1) - 2\sigma(X_2) = -4 - 2(-4) = 4$.

\square

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