ON A VOLUME ESTIMATE FOR SOME SUBSETS OF A GRASSMANNIAN

by

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ABSTRACT

ON A VOLUME ESTIMATE FOR SOME SUBSETS OF A GRASSMANNIAN

Measure theory gives us a technique for proving existence theorems. In simple terms this technique can be described as follows: start with a set and put a measure on it. Then find the exact measure of a specified set A. By showing the set of elements B in A that do not satisfy a certain condition P has measure strictly smaller than the measure of the ambient set A, one can deduce the existence of elements in A that satisfies the desired property P. This line of argument can only work when we have methods to find the measure of the set A and also estimate from above the measure of B. This thesis is about one such argument introduced by Ben-Artzi *et al.* and utilized later successfully by Foias and Olson to prove the existence of a Mane's projection whose inverse is Hölder continuous. The key estimate in those works was an inequality from integral geometry due to Santalo for measures of some subsets of the Grassmannian Space. The aim of this thesis is to expose and improve an alternate argument given by Friz and Robinson where they avoid working on the Grassmannian space and introduce the measures and estimates on a larger space while keeping track of the sizes of the coefficients.

ÖZET

GRASSMANNIAN'IN BAZI ALT KÜMELERİNİN HACİM KESTİRİMİ ÜZERİNE

Ölçü teorisi bize varlık teoremleri için bir ispat tekniği verir. Basitçe, bu teknik aşağıdaki gibi tarif edilebilir: Bir küme ile başlayıp ve üzerine bir ölçü koyalım. Sonra belirli bir A kümesinin kesin ölçüsünü bulalım. A kümesinin P koşulunu sağlamayan elemanlarının kümesi B'nin ölçüsünün onu çevreleyen A kümesinin ölçüsünden küçük olduğunu göstererek A kümesinin P koşulunu sağlayan elemanlarının var olduğu sonucunu çıkarabiliriz. Bu tip bir argüman ancak A kümesinin ölçüsünü bulmak ve B kümesinin ölçüsünü yukarıdan tahmin etmek için yöntemlerimiz olduğunda çalışır. Bu tez Ben-Artzi *et al.* tanıttığı ve daha sonra Foias ve Olson tarafından tersi Hölder sürekli olan bir Mane projeksiyonunun varlığını kanıtlamak için başarıyla kullanılan bir argüman hakkındadır. Bu eserlerdeki kilit kestirim Grassmannian'ın bazı alt kümeleri için Santalo sayesinde integral geometriden bir eşitsizliktir. Bu tezin amacı Friz ve Robinson tarafından verilen Grassmannian uzayında çalışmaktan kaçınan ve katsayıları boyutlarını takip ederken daha büyük bir uzayda ölçü ve kestirimler tanıtan alternatif bir argümanı incelemek ve geliştirmektir.

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LIST OF SYMBOLS

$B_\rho(x)$	Ball with center at x and radius ρ in $\mathbb{R}^N$
$B_\rho^S(x)$	Ball with center at x and radius ρ in space S
$C_c(G)$	Set of compactly supported continuous functions on G
$\{e_i\}_{i=1}^N$	Standard basis of $\mathbb{R}^N$
$d_F(X)$	Fractal dimension of X
$D_{N,k}$	$\{(x_1,\ldots,x_k) \in S_{N,k} \vert \{x_i\}_{i=1}^k$ linearly dependent$\}$
$D_{N,k}^l$	$\{(x_1,\ldots,x_k) \in S_{N,k} \vert \{x_i\}_{i=1}^l$ linearly dependent$\}$
G	A topological group
$G_{N,k}$	Space of k-dimensional subspaces of $\mathbb{R}^N$: Grassmannian
$G_{N,k}(B)$	$\{x \in G_{N,k} \mid x \cap B \neq \emptyset\}$
$GL(N)$	The set of all $N \times N$ invertible matrices
I_N	N by N identity matrix
$\mathcal{K}$	Set of compact subsets of G
L	Linear map
$\mathcal{L}(\mathbb{R}^N, \mathbb{R}^k)$	Set of linear maps from $\mathbb{R}^N$ to $\mathbb{R}^k$
$L_x^{G/H}$	Function from G/H to G/H such that $[L_x^{G/H}(\xi)](gH) = \xi(xgH)$
L_x^G	Function from G to G such that $[L_x^G(f)](g) = f(xg)$
$\mathbb{N}$	Set of natural numbers
$N_\epsilon(X)$	Minimum number of ϵ-balls necessary to cover X
$O(N)$	Set of $N \times N$ orthogonal matrices
P	Projection map
R	Function from $C_c(G)$ to $C_c(G/H)$ such that $$Rf(xH) = \int_H f(xh)d\mu(h)$$
$proj_S(A)$	Projection of the set A onto subspace S
q	Canonical quotient mapping from G to G/H

$\mathbb{R}^+$	Positive real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
S^n	n-dimensional unit sphere
$S_{N,k}$	$S^{N-1} \otimes S^{N-1} \otimes \cdots \otimes S^{N-1}$ (k times)
$S^g_{N,k}$	$S_{N,k} - D_{N,k}$
$S_{N,k}(B)$	$\{(x_1, \ldots, x_k) \in S_{N,k} \mid [x_1, \ldots, x_k] \cap B \neq \emptyset\}$
$\mathcal{U}$	Set of open subsets of G containing identity
$vol(A)$	Volume of the set A
$\mu_{N,k}$	Haar measure on $G_{N,k}$
$\nu_{N,k}$	The product measure on $S_{N,k}$
ν_{N-1}	The surface measure on S^{N-1}
ω_{N-1}	Surface area of unit ball in $\mathbb{R}^N$
$\Gamma(x)$	Gamma function
$\square$	End of proof
$\subset$	Subset
$\|\| \cdot \|\|$	Euclidean norm
$< \cdot, \cdot >$	Inner product in $\mathbb{R}^N$
$[x_1, \ldots, x_k]$	Space spanned by $x_1, \ldots, x_k$
$f \circ g$	Function f composed with g
G/H	Quotient set of G by H
$f\|_X$	Restriction of f on X
f^{-1}	Inverse function of f
$f^{-1}(A)$	Inverse image of the set A under f
$\mathring{A}$	Interior of the set A
$\bar{A}$	Closure of the set A
$A - B$	Set difference of set B from set A
$supp(f)$	Support of f

1. INTRODUCTION

Through linear maps $L : \mathbb{R}^N \to \mathbb{R}^k$, one can embed sets X in $\mathbb{R}^N$ into $\mathbb{R}^k$ where $N > k$. Moreover, the inverse mapping L^{-1} is a parametrisation of X. One can study some features of finite-dimensional sets by mapping them 'nicely' into a finite-dimensional Euclidean space [1].

Attractors that are emerged from (infinite-dimensional) dynamical systems are the motivating instances of such sets [2]. In [2], it is said that "In order to study the dynamics on these sets, one is obliged to project them onto a finite dimensional space."

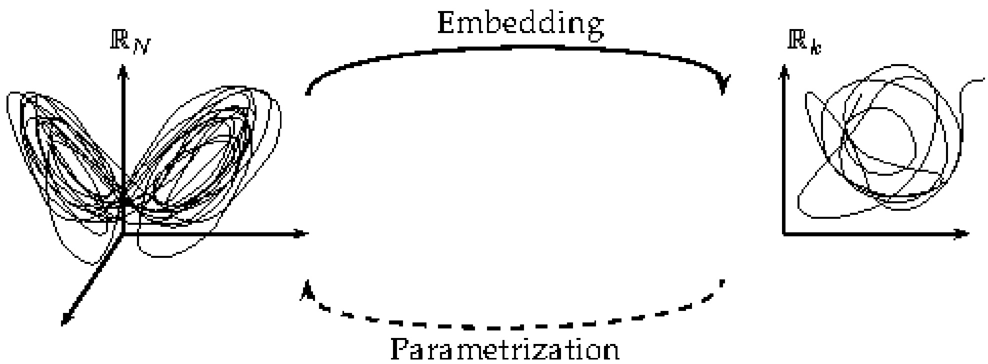

Figure 1.1. Embedding.

In this context, 'nicely' corresponds to having a Hölder continuous inverse on the image. That is, for some $\alpha \in (0, 1)$ and a constant $C > 0$,

$$||x - y|| \leq C||Lx - Ly||^{\alpha} \tag{1.1}$$

for all $x, y \in X$.

One way to prove existence of such linear maps is based on a technique in measure theory, called probabilistic method. In order to apply this technique, we must equip the space of linear maps, $\mathcal{L}(\mathbb{R}^N, \mathbb{R}^k)$, or some subset E of it, with a measure. Then, show that the subset of E that does not satisfy (1.1) for some x and y in X, has measure strictly smaller than the total measure of E. That will imply the existence of $L \in E$ satisfying (1.1).

In [1], E is chosen to be the set $\{(l_1^*, \ldots, l_k^*) \mid l_j \in B_1(0)\}$ where l_j^* be the linear map on $\mathbb{R}^N$ given by $l_j^*(x) =< l_j, x >$. E possesses a probability measure μ by identifying E with $(B_1(0))^k$. Observe that a linear map does not satisfy (1.1) if there exist x and y in X such that $||x - y|| > C||Lx - Ly||^\alpha$. Setting $z = x - y$, $||Lz|| < C^{-\alpha}||z||^\alpha$. So, the following theorem serves to bound the measure of linear maps that do not satisfy condition (1.1).

Theorem 1.1. *For any $\alpha \in \mathbb{R}^k$ and $x \in \mathbb{R}^N$,*

$$\mu\{L \in E \mid ||\alpha + Lx|| \leq \epsilon\} \leq cN^{k/2}\left(\frac{\epsilon}{||x||}\right)^k$$

where c is an absolute constant.

This inequality allows us to use probabilistic method in the proof of the theorem which asserts that 'almost' every linear map in E has a Hölder continuous inverse, cf. [1].

A similar argument appears in the article [4]. Foias and Olson are especially interested in projections.

Definition 1.2. *The fractal dimension of X, denoted by $d_F(X)$, is defined by the limit*

$$d_F(X) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(X)}{\log(1/\epsilon)}$$

where $N_\epsilon(X)$ is the minimum number of ϵ-balls necessary to cover X.

They prove the following theorem which generalizes the results of [2] and [3]:

Theorem 1.3. *Let H be a real Hilbert space and $X \subset H$ be such that $d_F(X) < m/2$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $||\tilde{P} - P|| < \delta$ and $\tilde{P}|_X$ has a Hölder continuous inverse. Here, the norm on the space of projections is the operator norm.*

In the proof of this theorem, Foias and Olson [4] make use of the one-to-one correspondence between orthogonal projections with their kernels, and utilize the space of subspaces and the measure on it.

Definition 1.4. *For any $N \in \mathbb{N}$ and $k < N$, the space of k-dimensional subspaces of $\mathbb{R}^N$ is called the* Grassmannian *and is denoted by $G_{N,k}$. Moreover, $G_{N,k}(B) = \{x \in G_{N,k} \mid x \cap B \neq \emptyset\}$ for any subset B of $\mathbb{R}^N$.*

This space can be endowed with an invariant measure $\mu_{N,k}$. Santalo gives both the construction of the measure $\mu_{N,k}$ and measures of some subsets of $G_{N,k}$ in [5], which relies on techniques from integral geometry. Consider a set X with finite fractal dimension. For any $\rho > 0$, there exist points $\{a_i\}_{i=1}^{N_\rho(X)}$ such that $X \subset \cup_{i=1}^{N_\rho(X)} B_\rho(a_i)$.

$$\begin{aligned} & \mu_{N,k}(\{\xi \in G_{N,k} \mid \exists i = 1 \ldots N_\rho(X) \xi \cap B_{2\rho}(a_i) \neq \emptyset\}) \\ = \; & \mu_{N,k}\left(\cup_{i=1}^{N_\rho(X)} G_{N,k}(B_{2\rho}(a_i))\right) \\ \leq \; & \sum_{i=1}^{N_\rho(X)} \mu_{N,k}(G_{N,k}(B_{2\rho}(a_i))) \end{aligned} \tag{1.2}$$

If this measure can be bounded for some $\rho > 0$, then there exists $\xi_0 \in G_{N,k}$ such that $\xi_0 \cap B_{2\rho}(a_i) = \emptyset$ for all $i = 1, \ldots N_\rho(X)$. So, the distanfce between ξ_0 and any $x \in X$ is greater than ρ since $X \subset \cup_{i=1}^{N_\rho(X)} B_\rho(a_i)$. As a result, the orthogonal projection P_0 whose kernel is ξ_0 satisfies $||P_0 x|| > \rho$ for all $x \in X$. Thus, P_0 also satisfies (1.1).

The crucial inequality in [4] bounds the measure of some subsets is the following:

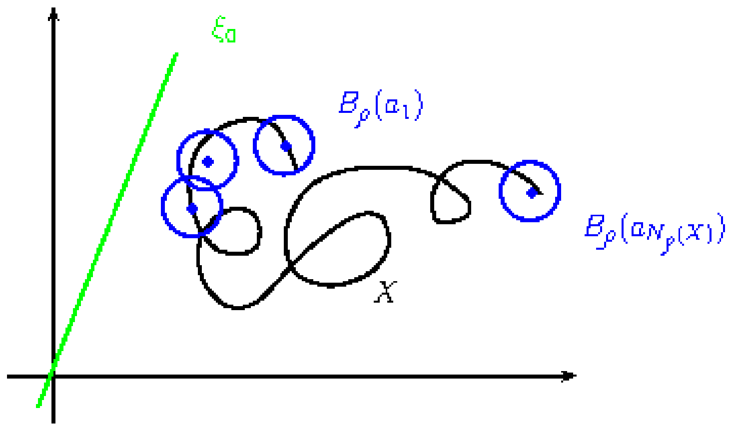

Figure 1.2. Projecting Finite Fractal Dimensional Set.

Theorem 1.5. *Let* $a \in \mathbb{R}^N$ *be nonzero and* $0 < \rho < ||a||$. *Then*

$$\mu_{N,k}(G_{N,k}(B_\rho(a))) \leq \mu_{N,k}(G_{N,k}) M_2(N,k) \left(\frac{\rho}{||a||}\right)^{N-k} \tag{1.3}$$

where $M_2(N,k) = \frac{\omega_{N-k-1}}{N-k} \left(\frac{N\pi}{2}\right)^{\frac{N-k+1}{2}} \left(\frac{1}{2}\right)^{N-k}$, $G_{N,k}(B) = \{x \in G_{N,k} \mid x \cap B \neq \emptyset\}$, *and* $\mu_{N,k}$ *is the invariant measure on* $G_{N,k}$.

On the other hand, Friz and Robinson [6] propose that one can work in a different space and obtain a similar result without the knowledge of integral geometry.

Definition 1.6. *Let* $S_{N,k}$ *be the product space* $S^{N-1} \otimes S^{N-1} \otimes \cdots \otimes S^{N-1}$ *(*k *times), which we refer as the alternative space in the rest of the paper, for any* $N \in \mathbb{N}$ *and* $k < N$. *Moreover, let* $\nu_{N,k}$ *be the product measure of surface measure* ν_{N-1} *on* S^{N-1}.

The measure $\nu_{N,k}$ on $S_{N,k}$ is more manageable than the measure $\mu_{N,k}$ on $G_{N,k}$, in the sense that the calculations with ν_{N-1} only require standard techniques from analysis. Using them, the following result is proven in [6].

Theorem 1.7. *Let $a \in \mathbb{R}^N$ be nonzero and $0 < \rho < ||a||/\pi$. Then,*

$$\nu_{N,k}(S_{N,k}(B_\rho(a))) \leq \nu_{N,k}(S_{N,k})K_2(N,k)\left(\frac{\rho}{||a||}\right)^{N-k} \tag{1.4}$$

where $K_2(N,k)$ is a constant such that $K_2(N,k) = 2\frac{\omega_{N-k-1}}{N-k}\left(\frac{N\pi}{2}\right)^{\frac{N-k+1}{2}}\left(\frac{1}{2}\right)^{N-k}$ and $S_{N,k}(B) = \{(x_1,\ldots,x_k) \in S_{N,k} \mid [x_1,\ldots,x_k] \cap B \neq \emptyset\}$.

In this thesis, we examine how Theorem 1.7 differs from Theorem 1.5 by investigating the paper [6]. In Chapter 2, we study the measure on the Grassmannian and prove Theorem 1.5. In Chapter 3, we introduce the alternative space and compare it to Grassmannian. Next, we prove Theorem 1.7 following the paper [6].

2. GRASSMANNIAN

As in Definition 1.4, Grassmannian $G_{N,k}$ consists of all k-dimensional subspaces of $\mathbb{R}^N$. Grassmannian is a space that can be endowed with a special measure called the Haar measure that we will see in Section 2.2. To this end, we study the structure of Grassmannian in next section.

2.1. Grassmannian as a Homogeneous Space

In order to define a Haar measure, we need a group structure. We begin with definitions taken from [7] and [8].

A topological group is both a group and a topological space for which the group operations are compatible with the topology, formally,

Definition 2.1. *A group G equipped with a topology is called a* topological group *provided that*

(i) The function $\phi : G \times G \to G$ defined by $\phi(x, y) = xy$ is continuous.
(ii) The function $\psi : G \to G$ defined by $\psi(x) = x^{-1}$ is continuous.

Let $GL(N)$ denote the set of invertible $N \times N$ matrices. $GL(N)$ is a group under matrix multiplication, and is called *the general linear group*. $GL(N)$ also possesses topological structure due to its one-to-one correspondence with $\mathbb{R}^{N^2}$. The topology of $GL(N)$ inherited from $\mathbb{R}^{N^2}$ is compatible with the group structure of $GL(N)$, i.e., $GL(N)$ is a topological group.

An *orthogonal matrix* is an $N \times N$ matrix whose columns form an orthonormal basis of $\mathbb{R}^N$. The set of all orthogonal $N \times N$ matrices, denoted by $O(N)$, under the matrix multiplication is called the *orthogonal group*.

$O(N)$ is both a subgroup and a subspace of $GL(N)$. $O(N)$ is also a topological group endowed with the induced topology. Moreover, one can show that $O(N)$ is closed and bounded, so is compact. (We can use Heine-Borel theorem since the topology is inherited from $\mathbb{R}^{N^2}$.)

Let G be a topological group and X be a topological space.

Definition 2.2. *A* transitive left action *of G on X is a continuous map $(g, x) \to g \cdot x$ from $G \times X$ to X such that*

(i) $x \to g \cdot x$ is a homeomorhism of X for all $g \in G$,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$,
(iii) $e \cdot x = x$ for all $x \in X$,
(iv) for every $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$,

where e is the unit element of G

A locally compact Hausdorff space X equipped with a transitive left action is called a transitive G-space*.*

If X is a transitive G-space and $x_0 \in X$, then the subgroup $H = \{h \in G \mid h \cdot x_0 = x_0\}$ is called the isotropy group *of x_0 and we call that x_0 is the* center of the isotropy*.*

Let G be a topological group and H be a subgroup of G. The set $\{gh \mid h \in H\}$ is called a left coset of H and denoted by gH, for any $g \in G$. The collection of all left cosets of H is called quotient set of G by H and denoted by G/H.

We can also talk about the topology on G/H induced by the topology of G. Consider the mapping q on G to G/H defined by $q(g) = gH$, which is called the canonical quotient map. The quotient topology G/H is defined in the following way: $U \subset G/H$ is open if and only if $q^{-1}(U)$ is open in G. By the way, the quotient topology is the strongest topology (i.e., containing the most open sets) on G/H that makes q

continuous.

Definition 2.3. *A* homogeneous space *is a transitive G-space that is isomorphic to a quotient space G/H, where G is a locally compact group.*

Now, we want to show that Grassmannian can be equipped with a topology in such a way that it becomes a homogeneous space. First, define

$$A \cdot \xi = \{Ax \mid x \in \xi\} \tag{2.1}$$

for all $A \in O(N)$ and $\xi \in G_{N,k}$.

Note that for any $A \in O(N)$ and $\xi \in G_{N,k}$, the set $A \cdot \xi$ also lies in $G_{N,k}$ since the matrix A is of full rank, and therefore preserves the dimension of ξ.

Proposition 2.4. *The map (2.1) satisfies properties (ii), (iii) and (iv) in Definition 2.2.*

Proof. (ii) Let A and B be orthogonal matrices and $\xi \in G_{N,k}$. We want to show that $(AB) \cdot \xi = A \cdot (B \cdot \xi)$. If $w \in (AB) \cdot \xi$, then there exists $v \in \xi$ such that $w = (AB)v = A(Bv)$. Hence, Bv is in $B \cdot \xi$, and so $w = A(Bv) \in A \cdot (B \cdot \xi)$. Conversely, if $w \in A \cdot (B \cdot \xi)$, then there exists $v \in B \cdot \xi$ such that $w = Av$. Since $v \in B \cdot \xi$, there exists $u \in \xi$ such that $v = Bu$. All in all, $w = Av = A(Bu) = (AB)u$, so $w \in (AB) \cdot \xi$.

(iii) $I_N \cdot \xi = \xi$ where I_N is the N by N identity matrix. If $v \in I_N \cdot \xi$, then there exists $u \in \xi$ such that $v = I_N u = u$. So, v is in ξ. Conversely, if u is in ξ, then $I_N u$ is in $I_N \cdot \xi$, so is u.

(iv) Let ξ and η be arbitrary elements of $G_{N,k}$. Then, we can find an orthonormal bases $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ for ξ and η, respectively. Complete both of them to orthonormal bases of $\mathbb{R}^N$: $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$. Let K and L be the matrices with columns $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$, respectively. So, $K^TK = KK^T = L^TL = LL^T = I$. Let $M = LK^T$. Then, $M^TM = (LK^T)^TLK^T = KL^TLK^T = KK^T = I$, hence, $M \in O(N)$. Moreover, $Mx_i = L(K^Tx_i) = Le_i = y_i$, for any $i = 1, 2, \ldots, k$. So, $M \cdot \xi$ is equal to η, since M sends basis elements of ξ to the basis elements of η. □

Proposition 2.5. *Let $e_1, \ldots, e_N$ denote the standard basis vectors in $\mathbb{R}^N$. Fix $\xi_0 \in G_{N,k}$ as the span of $e_1, \ldots, e_k$. Then, $\Xi = \{A \in O(N) \mid A \cdot \xi_0 = \xi_0\}$ is a subgroup of $O(N)$. Moreover, it is isomorphic to $O(k) \times O(N-k)$.*

Proof. First, we show that Ξ is a subgroup of $O(N)$. Let A and B be elements of Ξ. Then, $(AB) \cdot \xi_0 = A \cdot (B \cdot \xi_0) = A \cdot \xi_0 = \xi_0$ and $A^{-1} \cdot \xi_0 = A^{-1} \cdot (A \cdot \xi_0) = (A^{-1}A) \cdot \xi_0 = I_N \cdot \xi_0 = \xi_0$. So AB and A^{-1} belong to Ξ.

Let $\mathcal{Z}_{n,m}$ denote the n by m matrix consisting only zeros and M_i denote the i^{th} column of the matrix M. It is enough to show the that

$$\Xi = \left\{ \begin{bmatrix} A & \mathcal{Z}_{k,N-k} \\ \mathcal{Z}_{N-k,k} & D \end{bmatrix} \in O(N) \mid A \in O(k) \quad D \in O(N-k) \right\} \tag{2.2}$$

in order to prove that Ξ is isomorphic to $O(k) \times O(N-k)$.

First, take an element of M of the right-hand side of (2.2). Then, it is clear that $Me_i \in \xi_0$ for all i in $\{1, \ldots, k\}$. Thus, $M \cdot \xi_0 = \xi_0$, so $M \in \Xi$.

Conversely, take an element $M \in \Xi$. Then, M^T also lies in Ξ. In fact, $M^T = M^{-1}$ since $M \in O(N)$, and M^{-1} is in Ξ since $M \in \Xi$. We can write the matrix M in the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C and D are matrices of sizes $k \times k$, $k \times (N-k)$, $(N-k) \times k$ and $(N-k) \times (N-k)$, respectively. Now, let $i \in \{1, \ldots, k\}$.

$$\begin{bmatrix} A_i \\ C_i \end{bmatrix} = Me_i \in M\xi_0 = \xi_0,$$

$$\begin{bmatrix} (A^T)_i \\ (B^T)_i \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} e_i = M^T e_i \in M^T \xi_0 = \xi_0.$$

Thus, both B and C consist only of zeros. That is $B = \mathcal{Z}_{k,N-k}$ and $C = \mathcal{Z}_{N-k,k}$. Since M is in $O(N)$, we have

$$\begin{aligned} I_N &= MM^T \\ &= \begin{bmatrix} A & \mathcal{Z}_{k,N-k} \\ \mathcal{Z}_{N-k,k} & D \end{bmatrix} \begin{bmatrix} A^T & \mathcal{Z}_{k,N-k} \\ \mathcal{Z}_{N-k,k} & D^T \end{bmatrix} \\ &= \begin{bmatrix} AA^T & \mathcal{Z}_{k,N-k} \\ \mathcal{Z}_{N-k,k} & DD^T \end{bmatrix}. \end{aligned}$$

This implies $A \in O(k)$ and $D \in O(N-k)$. Therefore, M lies in the right-hand side of (2.2). □

Now, let ϕ be the function from $O(N)/\Xi$ to $G_{N,k}$, defined by $\phi(A\Xi) = A \cdot \xi_0$ for all $A \in O(N)$. ϕ establishes a one-to-one correspondence between $O(N)/\Xi$ and $G_{N,k}$ via Proposition 2.4. Therefore, we can equip $G_{N,k}$ by the topology τ given by $U \in \tau$ if and only if $\phi^{-1}(U)$ is open in the quotient topology of $O(N)/\Xi$.

Hence, the map (2.1) is a transitive group action with the topology τ on $G_{N,k}$ and by Proposition 2.4. Moreover, Proposition 2.5 shows that $G_{N,k}$ is a homogeneous group isomorphic to $O(N)/(O(k) \times O(N-k))$ by abuse of notation.

2.2. Measure on Grassmannian

In this thesis, we will be working with a measure on the Grassmannian that is invariant under the action of orthogonal group. In this section, we prove the existence of such a measure. Moreover, we will see that this measure is unique up to a multiplicative constant.

Definition 2.6. *[9, p.206] Let X be a Hausdorff topological space. A* Borel measure *on X is a measure whose domain is the Borel σ-algebra on X. Suppose that $\mathcal{A}$ is a σ-algebra on X containing the Borel σ-algebra on X. A positive measure μ on $\mathcal{A}$ is* regular *if*

(i) each compact subset K of X satisfies $\mu(K) < \infty$,

(ii) each set A in $\mathcal{A}$ satisfies

$$\mu(A) = \inf\{\mu(U) \mid A \subset U \textit{ and } U \textit{ is open}\},$$

(iii) each open subset U of X satisfies

$$\mu(U) = \sup\{\mu(K) \mid K \subset U \textit{ and } K \textit{ is compact}\}.$$

A measure that satisfies condition (ii) and a measure that satisfies condition (iii) are often called outer regular *and* inner regular*, respectively.*

Definition 2.7. *Let G be a topological group. We say that a measure μ on G is a left (right) Haar measure if it is a nonzero regular Borel measure with the property that*

$$\mu(gA) = \mu(A) \quad (\mu(Ag) = \mu(A))$$

for all $g \in G$ and all measurable subsets A of G.

Lemma 2.8. *Let G be a topological group and μ be a left Haar measure on G. Then, the measure defined by $\tilde{\mu}(A) = \mu(A^{-1})$ is a right Haar measure on G.*

Proof. Since G is a topological group, the measure $\tilde{\mu}$ is also a regular Borel measure. For an arbitrary $g \in G$ and measurable set A, we have

$$\tilde{\mu}(Ag) = \mu((Ag)^{-1}) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \tilde{\mu}(A)$$

Therefore, $\tilde{\mu}$ is a right Haar measure on G. □

Theorem 2.9. *For all locally compact groups G, there exist a left and a right Haar measure on G.*

Proof. We follow the proof of Gleason in [10].

First of all, note that the set $\{g\mathring{V} | g \in G\}$ forms an open cover of G for any subset V with non empty interior. If we further take a compact set K, then this cover must possess a finite subcover. So, we can define

$$(K : V) = \min\{n \in \mathbb{N} \mid \exists g_1 \dots g_n, K \subset \cup_{k=1}^{n} g_k \mathring{V}\},$$

for non empty K and $(K : V) = 0$ for $K = \emptyset$. Informally, $(K : V)$ is the number of $\mathring{V}$ needed to cover K. Thus, we now have a way to determine how big a compact set once we fix an open set V.

Let $\mathcal{K}$ denote the collection of compact subsets of G and let $\mathcal{U}$ denote the collection of open subsets of G containing identity. For further use, we make the following normalization.

We can fix a compact subset K_0 of G with non empty interior since G is locally compact. Then, define

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)},$$

for all $K \in \mathcal{K}$ and all $U \in \mathcal{U}$.

Now, observe that both $(K : U)$ and $(K_0 : U)$ are nonnegative, so is $\mu_U(K)$ for any $K \in \mathcal{K}$ and $U \in \mathcal{U}$. Moreover, if $K \subset \cup_{k=1}^{m} g_k \mathring{K}_0$, and $K_0 \subset \cup_{k=1}^{n} h_k U$, then $K \subset \cup_{i=1}^{m} \left[\cup_{j=1}^{n} g_i h_j U\right]$. So, $(K : U) \leq (K : \mathring{K}_0)(K_0 : U)$. Hence, $\mu_U(K) \leq (K : \mathring{K}_0)$.

$[0, (K : \mathring{K}_0)]$ is compact with respect to the induced topology of $\mathbb{R}$, for each

$K \in \mathcal{K}$. Define the space

$$X = \prod_{K \in \mathcal{K}} [0, (K : \mathring{K}_0)]$$

whose topology is given by the product topology. By Tychonoff's theorem, X is compact. Moreover, μ_U is a point of the space X for each $U \in \mathcal{U}$ by previous observation. We will take an appropriate $\mu_0 \in X$ that will be a Haar measure on G defined for compact subsets of G.

Define $C(V) = \overline{\{\mu_U : U \in \mathcal{U}, U \subset V\}}$. Thus, $C(V)$ is a closed subset of X for each $V \in \mathcal{U}$. Moreover, for any finite collection $\{V_i\}_{i=1}^k \subset \mathcal{U}$, we have $\mu_{\cap_{i=1}^k V_i} \in \cap_{i=1}^k C(V_i)$. So, we get $\cap_{V \in \mathcal{U}} C(V) \neq \emptyset$, by compactness of X.

Choose $\mu_0 \in \cap_{V \in \mathcal{U}} C(V)$. This choice has two advantages. First, we can show monotonicity and subadditivity of μ_U for any open set $U \in \mathcal{U}$. These properties are carried over for any element of $C(V)$ by continuity. Secondly, we can show that if $U \in \mathcal{U}$ and $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$, then $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. For any $K_1 \cap K_2 = \emptyset$, we can find a set V such that $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$ for all $U \subset V^{-1}$. Thus, $\mu_0(K_1 \cup K_2) = \mu_0(K_1) + \mu_0(K_2)$. Moreover, μ_0 is non negative on $\mathcal{K}$ since $\mu_0 \in X = \prod_{K \in \mathcal{K}} [0, (K : \mathring{K}_0)]$. Also, $\mu_0(\emptyset) = 0$ since $(\emptyset : \mathring{K}_0) = 0$.

Up to now, we have constructed μ_0 for all $K \in \mathcal{K}$. We would like to extend μ_0 to all subsets of G. First, for all $U \subset G$ open, let $\mu_0(U) = \sup\{\mu_0(K) \mid K \subset U, K \in \mathcal{K}\}$. Next, for all $A \subset G$, let $\mu_0(A) = \inf\{\mu_0(U) \mid U \subset G \text{ open}, A \subset U\}$. The extension of μ_0 is an outer measure.

The restriction of μ_0 is additive on Borel σ-algebra. Let μ be the restriction of μ_0 to the Borel σ-algebra. Then, μ is a measure on G.

We finish the proof by showing that μ satisfies the desired properties of Haar measure:

(i) μ is a Borel measure, since the domain of definition of μ is the Borel σ-algebra.

(ii) μ is regular. Inner and outer regularity of μ follow directly from the way of extension. Finiteness on compact subsets is the result of $\mu_0(K) \leq (K : \mathring{K}_0)$, for $K \subset G$ compact subset.

(iii) μ is non zero. Consider the set K_0. For each $U \in \mathcal{U}$, $\mu_U(K_0) = \dfrac{(K_0 : U)}{(K_0 : U)} = 1$. So, $\mu_0(K_0) = 1$. Since K_0 is compact, $\mu(K_0) = \mu_0(K_0) = 1$.

(iv) μ is translation invariant. First, take an arbitrary $g \in G$ and $K \in \mathcal{K}$. Fix $V \in \mathcal{U}$. For all $U \in \mathcal{U}$ with $U \subset V$, $\mu_U(gK) = \mu_U(A)$ because of the fact that $\{x_1U, \ldots, x_kU\}$ covers K if and only if $\{gx_1U, \ldots, gx_kU\}$ covers gK. Thus, $(K : U) = (gK : U)$ and so, $\mu_0(gK) = \mu_0(K)$.

To sum up, we make use of locally compactness of G in order to obtain a measure μ_0 on compact subsets of G. We extended μ_0 by following and respecting inner and outer regularity conditions. Then, the measure μ induced by μ_0 became a left Haar measure by (i-iv). By Lemma 2.8, there exists also a right Haar measure on G. □

Until now, we have shown the existence of Haar measure. Next, we prove that this measure is unique up to a multiplicative constant.

We start by giving three lemmas that will be utilized in the proof of uniqueness.

Lemma 2.10. *Let μ be a left Haar measure on G. If $f \in C_c(G)$ and $f > 0$, then $\int_G f d\mu > 0$.*

Proof. Let μ be a left Haar measure on G. Take a function $f \in C_c(G)$ that is not identically zero. Then, there exists a set A such that $\mu(A) > 0$. By outer regularity, there exists an open set O such that $\mu(O) \geq \mu(A) > 0$. By inner regularity, there exists a compact set K such that $\mu(K) \geq \mu(O)/2 > 0$. Let $U = f^{-1}(\mathbb{R}^+)$. $U \neq \emptyset$ since f is not identically zero, and U is open since f is continuous. By compactness of K, there exist $g_1, g_2, \ldots, g_n$ in G such that $K \subset \cup_{k=1}^{n} g_k U$. Then,

$$\begin{aligned} 0 &< \mu(K) \leq \sum_{k=1}^{n} \mu(g_k U) = n\mu(U) = n\mu\left(\cup_{m\in\mathbb{N}}\{x \in G \mid f(x) \geq \frac{1}{m}\}\right) \\ &\leq n \sum_{m\in\mathbb{N}} \mu\left(\{x \in G \mid f(x) \geq \frac{1}{m}\}\right). \end{aligned}$$

So there is $m_0 \in \mathbb{N}$ such that $\{x \in G \mid f(x) \geq \frac{1}{m_0}\}$ has positive measure, call this set V. Then,

$$\int_G f d\mu \geq \int_V f d\mu \geq \frac{1}{m_0}\mu(V) > 0.$$

□

Lemma 2.11. *Let μ and ν be left and right Haar measures on G, respectively. Then, $\int_G f(xg)d\mu(g) = \int_G f(g)d\mu(g)$ and $\int_G f(gx)d\nu(g) = \int_G f(g)d\nu(g)$ for all integrable functions f on G and for all $x \in G$.*

Proof. We prove the lemma for left Haar measures, the proof for right Haar measures is similar. For any characteristic function χ_A where A is a measurable set,

$$\int_G \chi_A(xg)d\mu(g) = \int_G \chi_{x^{-1}A}(g)d\mu(g) = \mu(x^{-1}A) = \mu(A) = \int_G \chi_A(g)d\mu(g).$$

Thus, the lemma is valid for characteristic functions. The general case follows by approximating any function by simple functions. □

Lemma 2.12. *Let μ and μ' be two left Haar measures on G. If there is a constant $a > 0$ such that $\int_G f d\mu = a\int_G f d\mu'$ for all $f \in C_c(G)$, then $\mu = a\mu'$.*

Proof. Define the measure $\nu = a\mu'$. Then, for any $f \in C_c(G)$, $\phi(f) = \int_G f d\mu$ and $\psi(f) = \int_G f d\nu$. So,

$$\phi(f) = \int_G f d\mu = a\int_G f d\mu' = \int_G f d\nu = \psi(f).$$

By Riesz representation theorem [9, p. 209], $\mu = \nu = a\mu'$. □

We can now proceed to the uniqueness theorem of the Haar measure.

Theorem 2.13. *Let G be a locally compact group and let μ and μ' be two left Haar measures on G. Then, $\mu = a\mu'$ for some $a > 0$.*

Proof. Take a non-negative function $g \in C_c(G)$ which is not identically zero. By Lemma 2.10, $\int_G g(tx)d\mu'(t) \neq 0$ for any $x \in G$. Now, for any $f \in C_c(G)$, define the function h on $G \times G$ by

$$h(x,y) = \frac{f(x)g(yx)}{\int_G g(tx)d\mu'(t)}$$

h is compactly supported, since f and g are so. We will show that h is a continuous function. Since, both f and g are continuous, it is enough to show that the denominator in h, $I(x) = \int_G g(tx)d\mu'(t)$ is a continuous function. Let x_0 be an element of G. Let $\epsilon > 0$ be given. There exists an open neighborhood U of x_0 whose closure is compact, since G is locally compact. Moreover, g has a compact support, say K. Then, $K \times \overline{U}$ and $K\overline{U}^{-1}$ are compact.

By regularity of μ', $\mu'(K\overline{U}^{-1})$ is finite. So, there is $\delta > 0$ such that $\mu'(K\overline{U}^{-1}) < \frac{\epsilon}{\delta}$. Since $g \in C_c(G)$, there exists an open neighborhood V of identity such that
$|g(x) - g(y)| < \delta$ whenever $y \in xV$.

Set $W = U \cap x_0V$. Then, W is an open neighborhood of x_0. Let $x \in W$. Then, $tx \in tx_0V$, and so $|g(tx) - g(tx_0)| < \delta$ for any $t \in G$. Moreover, if $t \notin K\overline{U}^{-1}$, then $tx \notin K$ and $tx_0 \notin K$. Thus, $|g(tx) - g(tx_0)|$ vanishes outside of $K\overline{U}^{-1}$. Therefore,

$$|I(x) - I(x_0)| \leq \int_G |g(tx) - g(tx_0)|d\mu'(t) < \frac{\epsilon}{\delta}\delta = \epsilon$$

This shows that I is continuous, and so is h.

We can apply Fubini's theorem [9, p. 247] to h.

$$\begin{aligned}
\int_G \left[\int_G h(x,y)d\mu'(y)\right] d\mu(x) &= \int_G \left[\int_G h(x,y)d\mu(x)\right] d\mu'(y) && (2.3)\\
&= \int_G \left[\int_G h(y^{-1}x,y)d\mu(x)\right] d\mu'(y) && (2.4)\\
&= \int_G \left[\int_G h(y^{-1}x,y)d\mu'(y)\right] d\mu(x) && (2.5)\\
&= \int_G \left[\int_G h(y^{-1},xy)d\mu'(y)\right] d\mu(x). && (2.6)
\end{aligned}$$

Here, we used Fubini's theorem in (2.3) and (2.5), and Lemma 2.11 in (2.4) and (2.6). Using this equality, we obtain

$$\begin{aligned}
\int_G f(x)d\mu(x) &= \int_G f(x)\frac{\int_G g(yx)d\mu'(y)}{\int_G g(tx)d\mu'(t)}d\mu(x) = \int_G\int_G \frac{f(x)g(yx)}{\int_G g(tx)d\mu'(t)}d\mu'(y)d\mu(x)\\
&= \int_G \left[\int_G h(x,y)d\mu'(y)\right] d\mu(x) = \int_G \left[\int_G h(y^{-1},xy)d\mu'(y)\right] d\mu(x)\\
&= \int_G\int_G \frac{f(y^{-1})g(x)}{\int_G g(ty^{-1})d\mu'(t)}d\mu'(y)d\mu(x)\\
&= \int_G g(x)d\mu(x)\left[\int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})d\mu'(t)}d\mu'(y)\right]
\end{aligned}$$

Hence, the fraction

$$\frac{\int_G f(x)d\mu(x)}{\int_G g(x)d\mu(x)} = \frac{f(y^{-1})}{\int_G g(ty^{-1})d\mu'(t)}d\mu'(y)$$

is independent of μ. Thus, $\dfrac{\int_G f(x)d\mu(x)}{\int_G g(x)d\mu(x)} = \dfrac{\int_G f(x)d\mu'(x)}{\int_G g(x)d\mu'(x)}$ which implies $\int_G f(x)d\mu(x) = \left(\dfrac{\int_G g(x)d\mu(x)}{\int_G g(x)d\mu'(x)}\right)\int_G f(x)d\mu'(x)$.

By Lemma 2.12, $\mu = a\mu'$ where $a = \dfrac{\int_G g(x)d\mu(x)}{\int_G g(x)d\mu'(x)}$. □

Similar to Haar measure on locally compact groups, we can define invariant Radon measure on homogeneous spaces as follows:

Definition 2.14. *Let G be a locally compact group and X be a homogeneous space isomorphic to G/H. Then, a regular Borel measure μ is called* a left invariant Radon measure on X *if $\mu(gA) = \mu(A)$ for all $g \in G$ and all Borel subsets of X.*

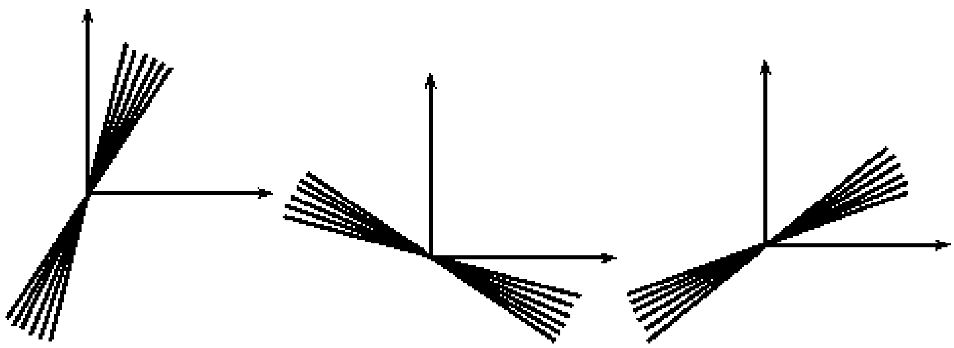

Figure 2.1. Invariance for Lines in $\mathbb{R}^2$.

Invariant Radon measure on homogeneous spaces may be seen as a generalization of Haar measure on locally compact groups when we consider the trivial action of group on itself.

Since $O(N)$ is compact, it is enough to show the existence of invariant Radon measure on homogeneous spaces when the underlying topological group is compact. However, the existence of invariant Radon measure on a homogeneous space can be shown under looser conditions. In fact, in [7], the existence of invariant Radon measure on a homogeneous space is characterised with a condition on so called modular function. We will follow the same proof in [7] but for compact groups.

Let G be a compact group and μ be a left Haar measure on G. Then, μ is a right Haar measure on G. In fact, we can define new measure μ_x by $\mu_x(A) = \mu(Ax)$, for any $x \in G$. μ_x is also a left Haar measure, since

$$\mu_x(yA) = \mu((yA)x) = \mu(y(Ax)) = \mu(Ax) = \mu_x(A)$$

for any $y \in G$ and measurable A. Thus, there exists a constant $\Delta(x)$ such that $\mu_x = \Delta(x)\mu$, by uniqueness. In particular, $\mu(G) = \mu(Gx) = \mu_x(G) = \Delta(x)\mu(G)$. But, $\mu(G) < \infty$, since G is compact and μ is regular. Thus, $\Delta(x) = 1$, for all $x \in G$. Therefore, $\mu(Ax) = \mu_x(A) = \Delta(x)\mu(A) = \mu(A)$. This shows that μ is a right Haar measure on G.

Moreover, the measure defined by $\tilde{\mu}(A) = \mu(A^{-1})$ is a left Haar measure and equals to μ. First, we show that $\tilde{\mu}$ is a right Haar measure:

$$\tilde{\mu}(Ag) = \mu((Ag)^{-1}) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \tilde{\mu}(A).$$

By previous argument, $\tilde{\mu}$ is also a left Haar measure on G. So, there exists a constant $c > 0$ such that $\tilde{\mu} = c\mu$, by uniqueness. $\mu(G) = \mu(G^{-1}) = \tilde{\mu}(G) = c\mu(G)$. But, $\mu(G) < \infty$, since G is compact and μ is regular. Hence, $c = 1$, so $\tilde{\mu} = \mu$. So $\int_G f(g)d\mu(g) = \int_G f(g^{-1})d\mu(g)$.

Theorem 2.15. *Let G be a compact group and X be a homogeneous space isomorphic to G/H. Then, there exists a left invariant Radon measure on X unique up to a multiplicative constant.*

Proof. First, we will show the existence of a left invariant Radon measure on G/H. Let μ be the left Haar measure on G, by Theorem 2.9. For any f in $C_c(G)$, define

$$Rf(xH) = \int_H f(xh)d\mu(h).$$

Rf is in $C_c(G/H)$ since f is in $C_c(G)$. Since μ is a left invariant measure, R is a well defined function from $C_c(G)$ into $C_c(G/H)$.

Now, define the functional ψ on $C_c(G/H)$ by

$$\psi(Rf) = \int_G f d\mu.$$

First, we show that ψ is well-defined. Since ψ is linear, it is enough to show that $Rf = 0$ implies $\int_G f d\mu = 0$. Let $f \in C_c(G)$ such that $Rf = 0$. Let E be a compact neighborhood of $supp(f)$. There exists a compact $K \subset G$ such that $q(K) = E$. Take a nonnegative function $g \in C_c(G)$ such that $g > 0$ on K and a continuous function ϕ on G/H supported in E such that $\phi = 1$ on $supp(f)$. Define

$$\Phi = \frac{\phi \circ q}{Rg \circ q} g$$

whenever denominator is nonzero, and $\Phi = 0$ when denominator vanishes. Then, the function $\Phi \in C_c(G)$ such that $R\Phi|_{supp(f)} = 1$,

$$\begin{aligned}
0 &= \int_G \Phi(x) Rf(xH) d\mu(x) = \int_G \Phi(x) \left[\int_H f(xh) d\mu(h) \right] d\mu(x) && (2.7) \\
&= \int_G \left[\int_H \Phi(x) f(xh^{-1}) d\mu(h) \right] d\mu(x) && (2.8) \\
&= \int_H \left[\int_G \Phi(x) f(xh^{-1}) d\mu(x) \right] d\mu(h) && (2.9) \\
&= \int_H \left[\int_G \Phi(xh) f(x) d\mu(x) \right] d\mu(h) && (2.10) \\
&= \int_G \left[\int_H \Phi(xh) f(x) d\mu(h) \right] d\mu(x) && (2.11) \\
&= \int_G f(x) R\Phi(xH) d\mu(x) = \int_G f(x) d\mu(x). && (2.12)
\end{aligned}$$

Here, we used Fubini's theorem in (2.9) and (2.11); Lemma 2.11 in (2.10). Thus, ψ is a well-defined positive linear functional on $C_c(G)$.

Let $L_x^{G/H}$ be the function from G/H to G/H such that $[L_x^{G/H}(\xi)](gH) = \xi(xgH)$ Similarly, let L_x^G be the function from G to G such that $[L_x^G(f)](g) = f(xg)$. Then,

$$[L_x^{G/H}(Rf)](yH) = Rf(xyH) = \int_H f(xyh) d\mu(h) = \int_H (L_x^G f)(yh) d\mu(h) = [R(L_x^G f)](yH),$$

for any $yH \in G/H$, and so,

$$\psi(L_x^{G/H}(Rf)) = \psi(R(L_x^G f)) = \int_G L_x^G f d\mu(x) = \int_G f d\mu(x) = \psi(Rf).$$

Therefore, the measure ν on G/H associated to ψ is a left invariant Radon measure on G/H by Riesz representation theorem.

Now, given any left invariant Radon measure ν on G/H, consider the function ψ on $C_c(G)$ defined by $\psi(f) = \int_{G/H} Rf d\nu$. ψ is a non-zero left invariant positive linear functional on $C_c(G)$, so there exists a constant $c > 0$ such that

$$\int_{G/H} Rf d\nu = c \int_G f d\mu, \tag{2.13}$$

by Theorem 2.13 and Riesz representation theorem. Therefore, ν can be characterised by the formula (2.13), so ν is unique up to a multiplicative constant. □

In Section 2.1, we have seen that the Grassmannian is a homogeneous space under the action of the orthogonal group. By Theorem 2.15, there exists an invariant Radon measure on $G_{N,k}$ unique up to a multiplicative constant. Thus, the measure of $G_{N,k}$ completely determines the invariant Radon measure on $G_{N,k}$.

However, there is not any consensus on the total measure of $G_{N,k}$. On one hand, Klain and Rota approach to Grassmannian in a combinatorial perspective in [11]. They construct the measure $G_{N,k}$ in such a way that total measure is $\binom{N}{k} \frac{\omega_N}{\omega_{N-k}\omega_k}$.

On the other hand, the construction of invariant Radon measure on $G_{N,k}$ in [5] relies on integral geometry and the manifold structure of Grassmannian. In the sequel, we will use the measure on $G_{N,k}$ given in [5], and denote it by $\mu_{N,k}$. In this construction,

$$\mu_{N,k}(G_{N,k}) = \frac{\omega_{N-1}\omega_{N-2}\ldots\omega_{N-k}}{\omega_{k-1}\omega_{k-2}\ldots\omega_0}.$$

Even though topological group structure of Grassmannian is sufficient for the existence of invariant Radon measure as we have shown earlier, Santalo takes advantage of the differentiable manifold structure in order to calculate measures on Grassmannian. As before, we need a group structure compatible with the manifold structure.

A *Lie group* is a topological group where the functions ϕ and ψ in Definition 2.1 are not only continuous but also differentiable.

$O(N)$ is an example of a Lie group. Manifold structure of $O(N)$ enables us to define 1-forms which may seen as vectors on the space. These 1-forms give rise to directions and lengths on these directions. For instance, dx and dy on $\mathbb{R}^2$ are independent 1-forms on $\mathbb{R}^2$. Moreover, it is possible to compound 1-forms in order to obtain higher dimensional forms by the operation called *wedge product*. If the dimension of a space is n, then the wedge product of n independent 1-forms gives a volume element on that space. For instance, $dxdy$ is a volume element on $\mathbb{R}^2$. Invariant 1-forms on a Lie group, according [5], can be calculated by $\Omega = g^{-1}dg$. Each element ω_{ij} of the matrix Ω corresponds to a left invariant 1-form on $O(N)$.

Any element g of the orthogonal group satisfies $g^T g = I_N$. By exterior differentiation, we have the identity $\omega_{ij} + \omega_{ji} = 0$. Thus, any left invariant 1-form can be obtained by combinations of $\{\omega_{ij} \mid j < i\}$. Moreover, the number of elements of this set coincides with the dimension of the orthogonal group which is $\frac{N(N-1)}{2}$, by D.1. Therefore, the elements in the lower triangle excluding the diagonal of Ω form an independent set of left invariant 1-forms for $O(N)$.

In Section 2.1, we have seen that Grassmannian is isomorphic to the quotient space $O(N)/\Xi$. Moreover, Ξ consists of elements of $O(N)$ whose components with indices $\{(i,j) \mid k < i \leq N, 1 \leq j \leq k \text{ or } 1 \leq i \leq k, k < j \leq N\}$ vanish. Similarly, this group can be given by the conditions $\{\omega_{ij} = 0 \mid k+1 \leq i \leq N, 1 \leq j \leq k\}$ through Cartan's moving frames [5, p. 202]. Thus, we may think that Ξ is a subgroup of $O(N)$ deprived of the directions $\{\omega_{ij}\}_{k+1 \leq i \leq N, 1 \leq j \leq k}$ so that the quotient space $O(N)/\Xi$ varies exactly in these directions. That is, $\{\omega_{ij}\}_{k+1 \leq i \leq N, 1 \leq j \leq k}$ forms an independent set of left invariant 1-forms for Grassmannian. So, wedge product of these forms gives the volume element on $G_{N,k}$. Abusing the notation, we can write $dG_{N,k} = \prod_{k<i\leq N, 1\leq j\leq k} \omega_{ij}$.

When we consider $\omega_{2,1}, \ldots, \omega_{k,1}$ as a subset of $\{\omega_{ij} \mid i \leq k, j \leq k\}$, they are left invariant 1-forms on $G_{k,1}$. Similarly, $\{\omega_{ij} \mid k+1 \leq i \leq N, 2 \leq j \leq k\}$ forms left invariant 1-forms of $G_{N-1,k-1}$ viewed as the subset of $\{\omega_{ij} \mid 2 \leq i \leq N, 2 \leq j \leq N\}$. Then, we get

$$\begin{aligned} |dG_{N,k} dG_{k,1}| &= \left| \prod_{k+1\leq i\leq N, 1\leq j\leq k} \omega_{ij} \prod_{2\leq i\leq k} \omega_{i1} \right| \\ &= \left| \prod_{2\leq i\leq N} \omega_{i1} \prod_{k+1\leq i\leq N, 2\leq j\leq k} \omega_{ij} \right| \\ &= |dG_{N,1} dG_{N-1,k-1}| . \end{aligned}$$

If we use the equality inductively we obtain the relation,

$$|dG_{N,k} dG_{k,1} dG_{k-1,1} \ldots dG_{2,1}| = |dG_{N,1} dG_{N-1,1} \ldots dG_{N-k+1,1}|. \tag{2.14}$$

$G_{n,1}$ consists of lines in $\mathbb{R}^n$, we can replace $dG_{n,1}$ by dS^{n-1}.

Integrating both sides of the equation 2.14 results in

$$\mu_{N,k}(G_{N,k}) = \frac{\omega_{N-1}\omega_{N-2}\ldots\omega_{N-k}}{\omega_{k-1}\omega_{k-2}\ldots\omega_0}.$$

2.3. Volume of Some Subsets of $G_{N,k}$

Now, let us focus on the measure of some subsets of $G_{N,k}$. We want to find the measure of the subset of $G_{N,k}$ whose elements intersect a given spherical subset of $\mathbb{R}^N$. We denote the set $\{x \in G_{N,k} \mid x \cap B \neq \emptyset\}$ by $G_{N,k}(B)$ and the ball with center at x

and radius ρ by $B_\rho(x)$.

In [5], Santalo calculates

$$\begin{aligned}\mu_{N,k}(G_{N,k}(B_\rho(x))) &= \frac{\omega_{N-2}\omega_{N-3}\ldots\omega_{N-k-1}}{\omega_{k-2}\omega_{k-3}\ldots\omega_0}\int_0^{\arcsin(\rho/||x||)}\cos^{k-1}(s)\sin^{N-k-1}(s)ds \\ &= \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\int_0^{\arcsin(\rho/||x||)}\cos^{k-1}(s)\sin^{N-k-1}(s)ds,\end{aligned}$$

for any $x \in \mathbb{R}^N - \{0\}$ and $0 < \rho < ||x||$.

Using this equality, upper bound to $\mu_{N,k}(G_{N,k}(B_\rho(x)))$ is derived both in [6] and [4] as follows:

$$\mu_{N,k}(G_{N,k}(B_\rho(x))) = \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\int_0^{\arcsin(\rho/||x||)}\cos^{k-1}(s)\sin^{N-k-1}(s)ds \tag{2.15}$$

$$\begin{aligned}&\leq \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\int_0^{\arcsin(\rho/||x||)} s^{N-k-1}ds \\ &= \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\left.\frac{s^{N-k}}{N-k}\right|_{s=0}^{s=\arcsin(\rho/||x||)} \\ &= \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\arcsin^{N-k}(\rho/||x||) \\ &\leq \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\left(\frac{\rho}{||x||}\right)^{N-k} \\ &= \mu_{N,k}(G_{N,k})M_1(N,k)\left(\frac{\rho}{||x||}\right)^{N-k}\end{aligned} \tag{2.16}$$

where $M_1(N,k) = \frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}$. In the first inequality, we used $\cos(s) \leq 1$ and $\sin(s) \leq s$ for positive s. In the second one, we used the fact that $\arcsin(x) \leq \frac{\pi}{2}x$ for all $x \in [0,1]$.

So far, we have proven Theorem 1.5 with a smaller constant. In order to obtain Theorem 1.5, we plug the formula for the surface area of the unit ball in terms of gamma function (B.4), and obtain the following:

$$\begin{aligned}
M_1(N,k) &= \frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k} && (2.17)\\
&= \frac{\omega_{N-k-1}}{N-k}\frac{\frac{2\pi^{k/2}}{\Gamma(k/2)}}{\frac{2\pi^{N/2}}{\Gamma(N/2)}}\left(\frac{\pi}{2}\right)^{N-k} && (2.18)\\
&= \frac{\omega_{N-k-1}}{N-k}\left(\frac{1}{2}\right)^{N-k}\frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{k}{2})}\pi^{\frac{N-k}{2}} && (2.19)\\
&\leq \frac{\omega_{N-k-1}}{N-k}\left(\frac{N\pi}{2}\right)^{\frac{N-k+1}{2}}\left(\frac{1}{2}\right)^{N-k} && (2.20)\\
&= M_2(N,k) && (2.21)
\end{aligned}$$

where we used (A.3) in (2.20). So we have the inequality $M_1(N,k) \leq M_2(N,k)$. Replacing M_1 with M_2 in (2.16), we get

$$\mu_{N,k}(G_{N,k}(B_\rho(x))) \leq \mu_{N,k}(G_{N,k})M_2(N,k)\left(\frac{\rho}{||x||}\right)^{N-k}. \quad (2.22)$$

This inequality (2.3) is indeed the one in Theorem 1.5 that is used in the article [4] of Foias and Olson, hereby we proved the Theorem 1.5.

However, when $k \geq 2$, it is possible to prove Theorem 1.5 with an even smaller constant. Instead of replacing all consine terms with 1 in (2.15), we can preserve one cosine term in the integral and get the inequality:

$$\begin{aligned}
\mu_{N,k}(G_{N,k}(B_\rho(x))) &\leq \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\int_0^{\arcsin(\rho/||x||)}\cos(s)\sin^{N-k-1}(s)ds\\
&= \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{\omega_{N-1}}\left.\frac{\sin^{N-k}(s)}{N-k}\right|_{s=0}^{s=\arcsin(\rho/||x||)}\\
&= \mu_{N,k}(G_{N,k})\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\rho}{||x||}\right)^{N-k}\\
&= \mu_{N,k}(G_{N,k})M_1(N,k)\left(\frac{2}{\pi}\right)^{N-k}\left(\frac{\rho}{||x||}\right)^{N-k}.
\end{aligned}$$

Since $k < N$ and $\frac{2}{\pi} < 1$, the constant is smaller. In other words, we get rid of the exponential term $\left(\frac{\pi}{2}\right)^{N-k}$ in N. If we follow the steps (2.17)-(2.20), then

$$\mu_{N,k}(G_{N,k}(B_\rho(x))) \leq \mu_{N,k}(G_{N,k})\frac{\omega_{N-k-1}}{N-k}\left(\frac{N\pi}{2}\right)^{\frac{N-k+1}{2}}\left(\frac{1}{\pi}\right)^{N-k}\left(\frac{\rho}{||x||}\right)^{N-k}.$$

3. ALTERNATIVE MEASURE SPACE

We previously discussed the measure on $G_{N,k}$ and derived the inequality (2.3). However, both of them rely on techniques from integral geometry. Instead, Friz and Robinson [6] propose to work with another space $S_{N,k}$ which is similar to $G_{N,k}$. Any element of $G_{N,k}$ which is a k-dimensional subspace of $\mathbb{R}^N$ is spanned by k unit vectors in itself. Conversely, any k linearly independent set of unit vectors spans an element of $G_{N,k}$. So, one can represent any $A \in G_{N,k}$ by an element of $S_{N,k} = S^{N-1} \times S^{N-1} \times \cdots \times S^{N-1}$ (k times), which spans A. The main advantage in working with $S_{N,k}$ is that it is naturally endowed with an invariant Radon measure and it is easier to handle the measure of subsets using tools only from analysis.

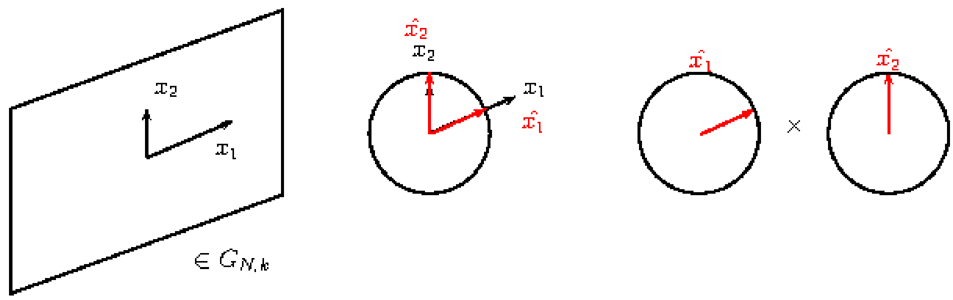

Figure 3.1. Motivation for Alternative Space.

First, we can define surface measure on S^{N-1} induced by the Lebesgue measure on R^N. On $S_{N,k}$, we define the measure $\nu_{N,k}$ as the product of surface measures. $\nu_{N,k}$ is an invariant Radon measure, since the surface measure is invariant under the action $O(N)$ on S^{N-1}.

3.1. Relation with Grassmannian

The advantageous part of working in $S_{N,k}$ is the product measure $\nu_{N,k}$ of the uniform surface measures which is more natural and manageable. Multivariable calculus suffices to calculate the measure of some subsets of $S_{N,k}$. Nevertheless, there are two apparent drawbacks: First, some elements of $S_{N,k}$ do not span a k-dimensional subspace of $\mathbb{R}^N$. Secondly, representation of any k-dimensional subspace of $\mathbb{R}^N$ is not unique. Thus, there is not a one-to-one correspondence between $S_{N,k}$ and $G_{N,k}$ and $S_{N,k}$ is even a 'bigger' set than $G_{N,k}$.

We overcome the first problem by ignoring the elements which are linearly dependent. In fact, the set of such elements has measure zero.

Lemma 3.1. $\nu_{N,k}(D_{N,k}) = 0$ *where*

$$D_{N,k} = \{(x_1, \dots, x_k) \in S_{N,k} | \{x_i\}_{i=1}^k \textit{ linearly dependent}\}.$$

Proof. Define the sets $D^l_{N,k}$, for $1 \leq l \leq k$, as follows

$$D^l_{N,k} = \{(x_1, \dots, x_k) \in S_{N,k} | \{x_i\}_{i=1}^l \text{ linearly dependent}\}.$$

Note that $D^1_{N,k} \subset D^2_{N,k} \subset \cdots \subset D^k_{N,k} = D$. We will prove that $\nu_{N,k}(D^l_{N,k}) = 0$ for each l by induction on l. For $l = 1$, $D^l_{N,k} = \{0\} \times S^{N-1} \times \cdots \times S^{N-1}$ and so $\nu_{N,k}(D^1_{N,k}) = 0$. Assume that $D^l_{N,k}$ has measure zero, for some $l < k$. Then,

$$\nu_{N,k}(D^{l+1}_{N,k}) = \nu_{N,k}(D^l_{N,k}) + \nu_{N,k}(D^{l+1}_{N,k} - D^l_{N,k}) = \nu_{N,k}(D^{l+1}_{N,k} - D^l_{N,k}). \tag{3.1}$$

$D^{l+1}_{N,k} - D^l_{N,k}$ consist of k-tuples whose first l vectors are linearly independent but first $l+1$ vectors are linearly dependent. Hence, if $(x_1, \dots, x_k) \in D^{l+1}_{N,k} - D^l_{N,k}$, then x_{l+1} is linear combination of $\{x_i\}_{i=1}^l$ i.e., $x_{l+1} \in [x_1, \dots, x_l] \cap S^{N-1} \simeq S^{l-1}$. Since $l < k$, $\nu_{N,k}(D^{l+1}_{N,k} - D^l_{N,k}) = 0$. By induction, $\nu_{N,k}(D_{N,k}) = 0$. □

We denote $S^g_{N,k} = S_{N,k} - D_{N,k}$, where g stands for generating. Lemma 3.1 provides $\nu_{N,k}(S^g_{N,k}) = \nu_{N,k}(S_{N,k})$. Even though every element of $S^g_{N,k}$ spans a k-dimensional subspace of $\mathbb{R}^N$ and so corresponds to an element of $G_{N,k}$, this correspondence is not unique. In fact, consider the relation $\sim$ on $S^g_{N,k}$ defined by $(x_1, \dots x_k) \sim (y_1, \dots y_k)$ if and only if $[x_1, \ldots x_k] = [y_1, \ldots y_k]$. This is an equivalence relation and the quotient set $S^g_{N,k}/\sim$ is in one-to-one correspondence with $G_{N,k}$. Moreover, there is a one-to-one correspondence between each equivalence class and $S^g_{k,k}$. Thus, we may say that $G_{N,k} \simeq S^g_{N,k}/S^g_{k,k}$ by abuse of notation.

3.2. Volume of Some Subsets of $S_{N,k}$

As in Section 2.3, we are interested in volumes of certain subsets of $S_{N,k}$. For any subset X of $\mathbb{R}^N$, $S_{N,k}(X)$ consists of elements $(x_1, \ldots, x_k) \in S_{N,k}$ such that the span of $(x_1, \ldots, x_k)$ intersects X. In particular, we bound the sets $S_{N,k}(X)$ where X is a ball in $\mathbb{R}^N$. The following lemma enables us to pull the center of any ball to S^{N-1}, so it suffices to deal with the balls whose center lies on S^{N-1}.

Lemma 3.2. *For any $\rho > 0$ and $a \in \mathbb{R}^N - \{0\}$ such that $||a|| < \rho$, we have $S_{N,k}(B_\rho(a)) = S_{N,k}(B_{\rho/||a||}(a/||a||)$.*

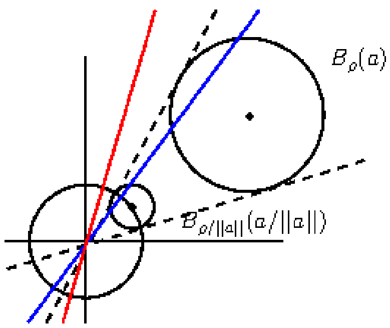

Figure 3.2. Normalization of Center.

Proof. The span of an element $(x_1, \ldots, x_k) \in S_{N,k}$ intersects $B_\rho(a)$ means that there is an element $y \in [x_1, \ldots, x_k]$ such that $||y - a|| < \rho$. Equivalently, $\frac{y}{||a||} \in [x_1, \ldots, x_k]$ and $\left|\left|\frac{y}{||a||} - \frac{a}{||a||}\right|\right| < \dfrac{\rho}{||a||}$. That is, the span of $(x_1, \ldots, x_k)$ intersects $S_{N,k}(B_{\rho/||a||}(a/||a||)$. □

Now, we introduce our main theorem. Theorem 1.7 used in [6] is a corollary of our this theorem. In fact, this theorem is essentially same as Theorem 1.7 with a different constant. Although we follow the proof of Friz and Robinson [6], we handled some inequalities more carefully and omitted some of them. As a result, the constant we obtained is smaller but less explicit. We begin by a lemma which is a special case of the following theorem.

Lemma 3.3. *Let $N \geq 2$, $\rho \in (0,1)$ and $a \in S^{N-1}$. Then, we will have*

$$\begin{aligned} \nu_{N,1}(S_{N,1}(B_\rho(a))) &= 2\omega_{N-2}\int_0^{\arcsin\rho} sin^{N-2}\theta d\theta \\ &\leq \nu_{N,1}(S_{N,1})\frac{2\omega_{N-2}}{(N-1)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-1}\rho^{N-1}. \end{aligned}$$

Proof. First, we consider the set $S_{N,1}(B_\rho(a))$. $S_{N,1}$ is essentially same as S^{N-1}. For any $x \in S^{N-1}$, let $\theta_x \in [0,\pi]$ be the angle between x and a. $x \in S_{N,1}(B_\rho(a))$ means the line spanned by x intersects the ball $B_\rho(a)$. This occurs if and only if the nearest point of that line to a lies in $B_\rho(a)$. Therefore,

$$\begin{aligned} x \in S_{N,1}(B_\rho(a)) &\Leftrightarrow ||a - proj_{[x]}a|| \leq \rho \\ &\Leftrightarrow ||a- <x,a> x||^2 \leq \rho^2 \\ &\Leftrightarrow 1- <x,a>^2 \leq \rho^2 \\ &\Leftrightarrow 1-\cos^2\theta_x \leq \rho^2 \\ &\Leftrightarrow \sin\theta_x \leq \rho. \end{aligned}$$

If we call $A = \{x \in S_{N,1} : \sin\theta_x \leq \rho\}$, then $S_{N,1}(B_\rho(a)) = A$. Hence,

$$\begin{aligned} \nu_{N,1}(S_{N,1}(B_\rho(a)) &= \int_A dS^{N-1} && (3.2) \\ &= \int_{S^{N-2}}\int_{\sin\theta_x\leq\rho} \sin^{N-2}\theta_x \quad d\theta dS^{N-2} && (3.3) \\ &\leq 2\omega_{N-2}\int_0^{\arcsin\rho} \theta_x^{N-2} \quad d\theta_x && (3.4) \end{aligned}$$

$$\begin{aligned}
\nu_{N,1}(S_{N,1}(B_\rho(a)) &\leq 2\omega_{N-2}\frac{(\arcsin\rho)^{N-1}}{N-1} && (3.5)\\
&\leq \frac{2\omega_{N-2}}{N-1}\left(\frac{\pi}{2}\rho\right)^{N-1} && (3.6)\\
&= \nu_{N,1}(S_{N,1})\frac{2\omega_{N-2}}{(N-1)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-1}\rho^{N-1}. && (3.7)
\end{aligned}$$

Equality (3.3) stems from C.1. In (3.4) and (3.6), we use inequalities $\sin x \leq x$ for $x \in [0, \pi/2]$ and $\arcsin y \leq \frac{\pi}{2}y$ for $y \in [0, 1]$, respectively. □

Theorem 3.4. *Let* $\alpha \in (0, 1]$*. If* $a \in \mathbb{R}^N - \{0\}$ *and* $\rho \in (0, \alpha||a||)$*, then there exists a constant* $K_0(N, k)$ *for every* $0 < k < N$ *such that*

$$\nu_{N,k}\left(S_{N,k}(B_\rho(a))\right) \leq \nu_{N,k}(S_{N,k})K_0(N, k)\left(\frac{\rho}{||a||}\right)^{N-k} \qquad (3.8)$$

where $K_0(N, k)$ *can be chosen as*

$$2\frac{\omega_{k-1}}{\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{\omega_{N-j-1}}{(N-j)\omega_{k-j}}. \qquad (3.9)$$

Proof. First, we fix $\alpha \in (0, 1]$. In order to simplify the calculations, we first make a reduction that $||a|| = 1$ and $\rho \in (0, \alpha)$. In fact, suppose that we proved the theorem under these assumptions. Then, for any $a \in \mathbb{R}^N - \{0\}$ and $\rho < \alpha||a||$, we have

$$\begin{aligned}
\nu_{N,k}(S_{N,k}(B_\rho(a)) &= \nu_{N,k}(S_{N,k}(B_{\rho/||a||}(a/||a||)) && (3.10)\\
&\leq \nu_{N,k}(S_{N,k})K_0(N, k)\left(\frac{\rho/||a||}{a/||a||}\right)^{N-k} && (3.11)\\
&= \nu_{N,k}(S_{N,k})K_0(N, k)\left(\frac{\rho}{||a||}\right)^{N-k}. && (3.12)
\end{aligned}$$

The equality (3.10) is due to the Lemma 3.2, and we use our supposition in (3.11).

Now, we will make an induction on N. Since, $0 < k < N$, we can take the basis step as $k = 1$ and $N = 2$. In fact, Lemma 3.3 is a special case of the theorem for $k = 1$ and $N \geq 2$.

Observe that

$$K_0(N,1) = 2\frac{\omega_0}{\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-1}\sum_{j=1}^{1}\left(\frac{\alpha\pi}{2}\right)^{1-j}\frac{\omega_{N-j-1}}{(N-j)\omega_{1-j}} \quad (3.13)$$

$$= \frac{2\omega_{N-2}}{(N-1)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-1}. \quad (3.14)$$

So the constant in Lemma 3.3 coincides with the constant in Theorem 3.4. This proves the basis step.

We proceed by induction. Let $N \geq 2$ be a positive integer. We assume that the theorem is valid for all integers less than N. In order to prove the theorem for N, we break $S_{N,k}(B_\rho(a))$ into two sets A and B as follows:

$$A = \{(x_1,\ldots,x_k) \in S_{N,k} \mid [x_1] \cap B_\rho(a) \neq \emptyset\},$$
$$B = \{(x_1,\ldots,x_k) \in S_{N,k} \mid [x_1,\ldots,x_k] \cap B_\rho(a) \neq \emptyset \text{ and } [x_1] \cap B_\rho(a) = \emptyset\}.$$

Since A and B are disjoint and their union is $S_{N,k}(B_\rho(a))$, we get

$$\nu_{N,k}(S_{N,k}(B_\rho(a))) = \nu_{N,k}(A) + \nu_{N,k}(B).$$

In the sequel, we will bound the measures of sets A and B using basis step and induction hypothesis, respectively. So, we will obtain an upper bound for $\nu_{N,k}(S_{N,k}(B_\rho(a)))$.

First we write the measure of A in integral form,

$$\nu_{N,k}(A) = \int_{x_1\in S^{N-1}:[x_1]\cap B_\rho(a)\neq\emptyset}\int_{x_2\in S^{N-1}}\cdots\int_{x_N\in S^{N-1}} dx_1dx_2\ldots dx_N \quad (3.15)$$

$$= (\omega_{N-1})^{k-1}\int_{x_1\in S^{N-1}:[x_1]\cap B_\rho(a)\neq\emptyset} dx_1 \quad (3.16)$$

$$\leq (\omega_{N-1})^{k}K_0(N,1)\rho^{N-1}. \quad (3.17)$$

Then, we can pass to (3.16) since there is no constraint on the variables $x_2\ldots x_N$. The inequality (3.17) is derived from Lemma 3.3.

Now, we deal with the measure of the set B. On the set B, there are two constraints: $[x_1] \cap B_\rho(a) = \emptyset$ and $[x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset$. In the latter one, we want to get rid of x_1 in order to separate these constraints from each other. To this end, we wil project elements in B on $[x_1]^\perp$. Denote $P = proj_{[x_1]^\perp}$. Now, define the following: $\theta_1 = \arccos < x_1, a >$, $\hat{a} = \frac{Pa}{\sin\theta_1}$, $\theta_j = \arccos < x_1, x_j >$ and $\hat{x}_j = \frac{Px_j}{\sin\theta_j}$ for $j = 2, \ldots k$. Once we fix x_1, there is one-to-one correspondence between x_j and the couple $(\hat{x}_j, \theta_j)$. Moreover, $\hat{x}_j$ can be viewed as an element of S^{N-2} in the space $[x_1]^\perp$.

We derive a necessary condition for $[x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset$ and $[x_1] \cap B_\rho(a) = \emptyset$ in terms of $\hat{x}_2 \ldots \hat{x}_k$.

If $[x_1] \cap B_\rho(a) = \emptyset$, then

$$\begin{aligned} [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset \quad &\Rightarrow \quad P([x_1, \ldots, x_k] \cap B_\rho(a)) \neq \emptyset \\ &\Rightarrow \quad P([x_1, \ldots, x_k]) \cap P(B_\rho(a)) \neq \emptyset \\ &\Rightarrow \quad [\hat{x}_2 \ldots \hat{x}_k] \cap B_\rho^{[x_1]^\perp}(Pa) \neq \emptyset \\ &\Rightarrow \quad [\hat{x}_2 \ldots \hat{x}_k] \cap B_{\rho/\sin\theta_1}^{[x_1]^\perp}(\hat{a}) \neq \emptyset. \end{aligned}$$

For simplicity, we name the conditions as follows:

$$\begin{aligned} C_1 &: [x_1] \cap B_\rho(a) = \emptyset, \\ C_2 &: [\hat{x}_2, \ldots, \hat{x}_k] \cap B_{\rho/\sin\theta_1}^{[x_1]^\perp}(\hat{a}) \neq \emptyset. \end{aligned}$$

Thus, we have the inequality:

$$\nu_{N,k}(B) \leq \nu_{N,k}((x_1, \ldots, x_k) \in S_{N,k} \mid C_1 \text{ and } C_2). \tag{3.18}$$

Now, we can plunge into calculations:

$$\begin{aligned}
\nu_{N,k}(B) &\leq \int_{C_1,C_2} dS^{N-1}(x_1)\dots dS^{N-1}(x_k) && (3.19)\\
&= \int\limits_{\substack{x_1\in S^{N-1}\\ C_1}} \int\limits_{\substack{\hat{x}_2\dots\hat{x}_k\\ C_2}}\cdots\int \int\limits_{\theta_2\dots\theta_k}\cdots\int \sin^{N-2}\theta_2\dots\sin^{N-2}\theta_k && (3.20)\\
&\qquad d\theta_2\dots d\theta_k dS^{N-2}(\hat{x}_2)\dots dS^{N-2}(\hat{x}_k)dS^{N-1}(x_1)\\
&= \left(\frac{\omega_{N-1}}{\omega_{N-2}}\right)^{k-1} \int\limits_{\substack{x_1\in S^{N-1}\\ C_1}} \int\limits_{\substack{\hat{x}_2\dots\hat{x}_k\\ C_2}}\cdots\int && (3.21)\\
&\qquad dS^{N-2}(\hat{x}_2)\dots dS^{N-2}(\hat{x}_k)dS^{N-1}(x_1).
\end{aligned}$$

Here, we used (C.1) in (3.20) and (C.2) in (3.21). The inner integral in (3.21) is simply the measure of the set in $S_{N-1,k-1}$ with the constraint C_2, so that we can use the induction hypothesis on this integral:

$$\begin{aligned}
\nu_{N,k}(B) &\leq \left(\frac{\omega_{N-1}}{\omega_{N-2}}\right)^{k-1} && (3.22)\\
&\quad \cdot\int_{x_1} \nu_{N-1,k-1}(S_{N-1,k-1})K_0(N-1,k-1)\left(\frac{\rho}{\sin\theta_1}\right)^{N-k} dS^{N-1}\\
&= (\omega_{N-1})^{k-1} K_0(N-1,k-1)\rho^{N-k} && (3.23)\\
&\qquad \cdot\int_{\hat{x}_1}\int_{\theta_1}\left(\frac{1}{\sin\theta_1}\right)^{N-k} \sin(\theta_1)^{N-2} d\theta_1 dS^{N-2}\\
&= (\omega_{N-1})^{k-1} K_0(N-1,k-1)\rho^{N-k}\frac{\omega_{k-1}}{\omega_{k-2}}\int_{\hat{x}_1} dS^{N-2} && (3.24)\\
&= (\omega_{N-1})^{k-1} \frac{\omega_{N-2}\omega_{k-1}}{\omega_{k-2}} K_0(N-1,k-1)\rho^{N-k}. && (3.25)
\end{aligned}$$

Again, we use (C.1) and (C.2) in the steps (3.23) and (3.24), respectively. At last, we obtain the result (3.25) by reordering.

$$\begin{aligned}
\nu_{N,k} & (S_{N,k}(B_\rho(a))) \\
\leq & (\omega_{N-1})^k K_0(N,1)\rho^{N-1} \\
& + (\omega_{N-1})^{k-1} \frac{\omega_{N-2}\omega_{k-1}}{\omega_{k-2}} K_0(N-1,k-1)\rho^{N-k} \\
= & \rho^{N-k}\left[\rho^{k-1}\frac{2(\omega_{N-1})^{k-1}\omega_{N-2}}{N-1}\left(\frac{\pi}{2}\right)^{N-1}\right. \\
& + (\omega_{N-1})^{k-1}\frac{\omega_{N-2}\omega_{k-1}}{\omega_{k-2}} \\
& \left.\cdot \frac{2\omega_{k-2}}{\omega_{N-2}}\left(\frac{\pi}{2}\right)^{N-k}\sum_{j=1}^{k-1}\left(\frac{\alpha\pi}{2}\right)^{k-j-1}\frac{\omega_{N-j-2}}{(N-j-1)\omega_{k-1-j}}\right] \\
\leq & \rho^{N-k}\nu_{N,k}(S_{N,k})2\frac{\omega_{k-1}}{\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k} \\
& \cdot\left[\frac{\omega_{N-2}}{(N-1)\omega_{k-1}}\left(\frac{\alpha\pi}{2}\right)^{k-1} + \sum_{j=1}^{k-1}\left(\frac{\alpha\pi}{2}\right)^{k-j-1}\frac{\omega_{N-j-2}}{(N-j-1)\omega_{k-1-j}}\right] \\
= & \rho^{N-k}\nu_{N,k}(S_{N,k})2\frac{\omega_{k-1}}{\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\left[\sum_{j=0}^{k-1}\left(\frac{\alpha\pi}{2}\right)^{k-j-1}\frac{\omega_{N-j-2}}{(N-j-1)\omega_{k-1-j}}\right] \\
= & \nu_{N,k}(S_{N,k})K_0(N,k)\rho^{N-k}.
\end{aligned}$$

Here, we use (3.17) and (3.25). □

Hereby, we proved Theorem 3.4. Observe that the choice for $K_0(N,k)$ is not unique, and any upper bound to (3.9) can play the role of $K_0(N,k)$. In fact, Friz and Robinson [6] get rid of the summation and choose appropriate α. As a result, they obtain a more explicit form at the expense of having a larger constant. We can derive that constant as follows.

$$\begin{aligned}
K_0(N,k) &= 2\frac{\omega_{k-1}}{\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{\omega_{N-j-1}}{(N-j)\omega_{k-j}} \\
&= 2\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{(N-k)\omega_{N-j-1}}{(N-j)\omega_{N-k-1}\omega_{k-j}}.
\end{aligned}$$

We bound the summation:

$$\begin{aligned}
&\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{(N-k)\omega_{N-j-1}}{(N-j)\omega_{N-k-1}\omega_{k-j}}\\
=\;&\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{(N-k)2\frac{\pi^{(N-j)/2}}{\Gamma((N-j)/2)}}{(N-j)2\frac{\pi^{(N-k)/2}}{\Gamma((N-k)/2)}2\frac{\pi^{(k-j+1)/2}}{\Gamma((k-j+1)/2)}}\\
=\;&\frac{1}{2\sqrt{\pi}}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{\frac{N-k}{2}\Gamma(\frac{N-k}{2})\Gamma(\frac{k-j+1}{2})}{\frac{N-j}{2}\Gamma(\frac{N-j}{2})}\\
=\;&\frac{1}{2\sqrt{\pi}}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{\Gamma(\frac{N-k+2}{2})\Gamma(\frac{k-j+1}{2})}{\Gamma(\frac{N-j+2}{2})}\\
\leq\;&\frac{1}{2\sqrt{\pi}}\sum_{j=1}^{k}\left(\frac{\alpha\pi}{2}\right)^{k-j}\frac{\sqrt{\pi}}{k-j+1}
\end{aligned}$$

where last inequality stems from Theorem A.2

In [6], Friz and Robinson choose α as $1/\pi$. Then, the last row above is equal to $\frac{1}{2}\sum_{j=1}^{k}\left(\frac{1}{2}\right)^{k-j}\frac{1}{k-j+1}$ which is bounded above by 1.

Thus, $K_0(N,k)\leq 2\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k} := K_1(N,k)$.

As in Section 2.3, we can follow the steps (2.17)-(2.20) and obtain

$$K_1(N,k)\leq 2\frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}}\left(\frac{\pi}{2}\right)^{N-k}\leq 2\frac{\omega_{N-k-1}}{N-k}\left(\frac{1}{2}\right)^{N-k}\left(\frac{N\pi}{2}\right)^{\frac{N-k+1}{2}} = K_2(N,k).$$

This bound and Theorem 3.4 proves the Theorem 1.7.

Nevertheless, we could have chosen α as 1 in order to weaken the condition $\rho < \frac{||a||}{\pi}$

to $\rho < ||a||$. Then, we get

$$\frac{1}{2\sqrt{\pi}} \sum_{j=1}^{k} \left(\frac{\alpha\pi}{2}\right)^{k-j} \frac{\sqrt{\pi}}{k-j+1} = \frac{1}{2} \sum_{i=0}^{k-1} \left(\frac{\pi}{2}\right)^{i} \frac{1}{i+1} \tag{3.26}$$

$$\leq \left(\frac{\pi}{2}\right)^{k-1} \tag{3.27}$$

where the inequality (3.27) can be shown by induction for $k \geq 1$. Hereby,

$$K_0(N,k) \leq 2 \frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}} \left(\frac{\pi}{2}\right)^{N-1}$$

for $\alpha = 1$. Hence, Theorem 3.4 results in following corollary:

Corollary 3.5. *If* $a \in \mathbb{R}^N - \{0\}$ *and* $\rho \in (0, ||a||)$*, then*

$$\nu_{N,k}\left(S_{N,k}(B_\rho(a))\right) \leq \nu_{N,k}(S_{N,k}) 2 \frac{\omega_{k-1}\omega_{N-k-1}}{(N-k)\omega_{N-1}} \left(\frac{\pi}{2}\right)^{N-1} \left(\frac{\rho}{||a||}\right)^{N-k} \tag{3.28}$$

for every $0 < k < N$.

4. CONCLUSION

In [4], Foias and Olson prove the existence of a special kind of projection, called Mañé's projection. They showed that a finite fractal dimensional set X can be projected to a lower dimensional subspace in such a way that the projection's inverse is Hölder continuous on X, see Theorem 1.3. Their proof is based on the following facts: It is possible to cover the set X with $N_\rho(X)$ many balls with radius ρ, call them $B_\rho(a_i)$. If the kernel of a projection does not intersect any of the balls $B_{2\rho}(a_i)$, then the projection's inverse is Hölder continuous on X. So, it suffices to find a subspace that does not intersect any of the balls $B_{2\rho}(a_i)$. One way to show the existence of such a subspace is to use the probabilistic method. That is, one should bound the measure of the set of subspaces which intersect any of the balls $B_{2\rho}(a_i)$. By countable subadditivity, this measure is less than the sum of $N_\rho(X)$ terms:

$$\mu_{N,k}\left(\cup_{i=1}^{N_\rho(X)} G_{N,k}(B_{2\rho}(a_i))\right) \leq \sum_{i=1}^{N_\rho(X)} \mu_{N,k}(G_{N,k}(B_{2\rho}(a_i))).$$

Since X has finite fractal dimension, the number $N_\rho(X)$ can be controlled for sufficiently small ρ.

In this line of argument, it is crucial to bound the measure of the set containing the subspaces which intersect a given spherical subset of $\mathbb{R}^N$. In this thesis, we focused on such bounds and proved the following inequalities:

$$\begin{aligned} \mu_{N,k}(G_{N,k}(B_\rho(a))) &\leq \mu_{N,k}(G_{N,k})M_2(N,k)\left(\frac{\rho}{||a||}\right)^{N-k}; \\ \nu_{N,k}(S_{N,k}(B_\rho(a))) &\leq \nu_{N,k}(S_{N,k})K_2(N,k)\left(\frac{\rho}{||a||}\right)^{N-k}. \end{aligned}$$

Foias and Olson utilize the first inequality which is based on the measure on the Grassmannian, see page 28 and [4]. On the other hand, the second inequality and the underlying space, referred to as alternative space, are introduced by Friz and Robinson, see page 39 and [6].

Differences and similarities between these inequalities are discussed throughout this thesis. As a result, we observe that both of these inequalities are valid for sufficiently small ρ and the constants are independent of ρ. Hence, it is possible to replace the Grassmannian with the alternative space in the proof of Theorem 1.3.

On the other hand, these theorems differ from each other in two insignificant aspects when we are concerned with Theorem 1.3. First, the constant in $K_2(N,k)$ in Theorem 1.7 is as twice as the constant $M_2(N,k)$ in Theorem 1.5. Secondly, Theorem 1.5 is valid for all $\rho < ||a||$, whereas Theorem 1.7 requires that $\rho < \frac{||a||}{\pi}$. Finally note that when we handle calculations more carefully, we obtain smaller constants in both inequalities, see page 27 and Theorem 3.4.

APPENDIX A: THE GAMMA FUNCTION

The gamma function is one of the most frequently used function in this thesis. We will give some properties and inequalities regarding the gamma function. The gamma function is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for all $x \in \mathbb{R}$.

If $x \neq 0$, then $x\Gamma(x) = \int_0^\infty x t^{x-1} e^{-t} dt = t^x e^{-x}|_{t=0}^\infty + \int_0^\infty x t^x e^{-t} dt = \Gamma(x+1)$ by integration by parts. Moreover, when we use the formula (B.4) for $N = 2$ and $N = 3$, we get $\Gamma(1) = 1$ and $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$. If we apply $x\Gamma(x) = \Gamma(x+1)$ repeatedly, then we obtain

$$\Gamma(n+1) = n! \tag{A.1}$$

and

$$\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi} \tag{A.2}$$

for all $n = 0, 1, 2, 3 \ldots$.

Considering parity of k and N, (A.1) and (A.2) leads to following inequality

Theorem A.1. *For all integers k and N such that $1 \leq k < N$,*

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}. \tag{A.3}$$

Proof. If $k = 1$, then $\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}} \leq \frac{\left(\frac{N}{2}\right)^{\frac{N}{2}}}{\sqrt{\pi}} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}$. For $k \neq 1$, we have four cases:

Case 1: N and k are even.

There exist a and b such that $N = 2a + 2$ and $k = 2b + 2$.

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma(a+1)}{\Gamma(b+1)} = \frac{a!}{b!} \leq a^{a-b} = \left(\frac{N-2}{2}\right)^{\frac{N-k}{2}} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}.$$

Case 2: N is even and k is odd.

There exist a and b such that $N = 2a + 2$ and $k = 2b + 1$.

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma(a+1)}{\Gamma\left(\frac{2b+1}{2}\right)} = \frac{a!}{\left(\frac{2b-1}{2}\right)\left(\frac{2b-3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}} \leq a^{a-b+1}\sqrt{\pi} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}.$$

Case 3: N is odd and k is even.

There exist a and b such that $N = 2a + 1$ and $k = 2b + 2$.

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma\left(\frac{2a+1}{2}\right)}{\Gamma(b+1)} = \frac{\left(\frac{2a-1}{2}\right)\left(\frac{2a-3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{b!} \leq a^{a-b}\sqrt{\pi} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}.$$

Case 4: N and k are odd.

There exist a and b such that $N = 2a + 1$ and $k = 2b + 1$.

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma\left(\frac{2a+1}{2}\right)}{\Gamma\left(\frac{2b+1}{2}\right)} = \frac{\left(\frac{2a-1}{2}\right)\left(\frac{2a-3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{\left(\frac{2b-1}{2}\right)\left(\frac{2b-3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}} \leq a^{a-b} \leq \sqrt{\pi}\left(\frac{N}{2}\right)^{\frac{N-k+1}{2}}.$$

□

Another useful result for the gamma function is due to a convexity property. Let

λ be between zero and one. Then, we have

$$\begin{aligned}
\Gamma(\lambda x + (1-\lambda)y) &= \int_0^\infty t^{\lambda x + (1-\lambda)y - 1} e^{-t} dt \\
&= \int_0^\infty (t^{\lambda(x-1)} e^{-\lambda t})(t^{(1-\lambda)(y-1)} e^{-(1-\lambda)t}) dt \\
&\leq \left(\int_0^\infty t^{\lambda(x-1)} e^{-\lambda t} \right)^\lambda \left(\int_0^\infty t^{\lambda(y-1)} e^{-\lambda t} \right)^{1-\lambda} \\
&= (\Gamma(x))^\lambda (\Gamma(y))^{1-\lambda}
\end{aligned}$$

where we used Hölder's inequality. Taking logarithm of both sides results in the inequality $(\log \Gamma)(\lambda x + (1-\lambda)y) \leq \lambda (\log \Gamma)(x) + (1-\lambda)(\log \Gamma)(y)$ for all $\lambda \in (0,1)$. That is the composite function $\log \circ \Gamma$ is convex.

Theorem A.2. $\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-\frac{1}{2})} \leq \frac{\sqrt{\pi}}{2a}$ *for all* $a \in [1/2, \infty)$ *and* b *in* $[3/2, \infty)$.

Proof. Fix a. Consider the function $F(b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-\frac{1}{2})}$. Then, $F(3/2) = \frac{\Gamma(a)\Gamma(\frac{3}{2})}{\Gamma(a+1)} = \frac{\sqrt{\pi}}{2a}$. It is enough to show that F is decreasing, equivalently $\log \circ F$ is decreasing. If we take the derivative of $\log \circ F$, then we obtain $(\log \circ F)'(b) = (\log \circ \Gamma)'(b) - (\log \circ \Gamma)'(a+b-\frac{1}{2})$. Since $\log \circ \Gamma$ is convex, $(\log \circ \Gamma)'(b) \leq (\log \circ \Gamma)'(a+b-\frac{1}{2})$ so that $(\log \circ F)' \leq 0$. Thus, $\log \circ F$ is decreasing. □

APPENDIX B: THE SURFACE AREA OF UNIT SPHERE

We have frequently used the formula $\omega_{N-1} = \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$. Now, we derive this by calculating the following integral in two different ways:

$$I = \int_{\mathbb{R}^N} e^{-||x||^2} dx.$$

First, we manipulate the integral and obtain ω_{N-1} factor out of it:

$$\begin{aligned} I &= \int_{\mathbb{R}^N} \int_{\infty}^{||x||} \frac{d}{dt}(e^{-t^2}) dt dx \\ &= \int_{\mathbb{R}^N} \int_{||x||}^{\infty} 2te^{-t^2} dt dx \\ &= \int_0^{\infty} \int_{||x|| \leq t} 2te^{-t^2} dx dt. \end{aligned}$$

Here, we used the fundamental theorem of calculus and then the change of order of integration in $\mathbb{R}^N$.

$$\begin{aligned} I &= \int_0^{\infty} 2te^{-t^2} vol(B_t(0)) dt \\ &= \int_0^{\infty} 2t^{N+1} e^{-t^2} vol(B_1(0)) dt \\ &= \frac{\omega_{N-1}}{N} \int_0^{\infty} 2t^{N+1} e^{-t^2} dt. \end{aligned} \tag{B.1}$$

The last step (B.1) stems from the fact that $Nvol(B_1(0)) = \omega_{N-1}$. In the following lines, we first use the change of variables, and then $\Gamma(n+1) = n\Gamma(n)$

$$\begin{aligned} I &= \frac{\omega_{N-1}}{N}\int_0^\infty u^{N/2}e^{-u}du \\ &= \frac{\omega_{N-1}}{N}\Gamma\left(\frac{N}{2}+1\right) \\ &= \frac{\omega_{N-1}}{N}\frac{N}{2}\Gamma\left(\frac{N}{2}\right) \\ &= \frac{\omega_{N-1}}{2}\Gamma\left(\frac{N}{2}\right). \end{aligned} \tag{B.2}$$

Next, we calculate I in a different manner :

$$\begin{aligned} I &= \int_{\mathbb{R}^N} e^{-\sum_{i=1}^N |x_i|^2} dx_1 \ldots dx_N \\ &= \int_{\mathbb{R}^N} \prod_{i=1}^N e^{-|x_i|^2} dx_1 \ldots dx_N \\ &= \prod_{i=1}^N \left(\int_{\mathbb{R}} e^{-|x_i|^2} dx_i\right) \\ &= \left(\int_{\mathbb{R}} e^{-t^2} dt\right)^N \\ &= \left(2\int_0^\infty e^{-t^2} dt\right)^N \\ &= \pi^{N/2}. \end{aligned} \tag{B.3}$$

Equating (B.2) and (B.3), we get,

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{B.4}$$

APPENDIX C: SPHERICAL COORDINATES

One of the tools that facilitate the calculations of measure on S^{N-1} is spherical coordinates. Let u be an element of S^{N-1}. Then, u can be represented by $N-1$ angles between u and coordinate axis: $\phi \in [0, 2\pi)$ and $\theta_1 \ldots \theta_{N-2} \in [0, \pi]$. If we denote its i^{th} component in Cartesian coordinates by x_i, then

$$\begin{aligned} x_1 &= \cos\phi \prod_{k=1}^{N-2} \sin\theta_k; \\ x_2 &= \sin\phi \prod_{k=1}^{N-2} \sin\theta_k; \\ x_i &= \cos\theta_{i-2} \prod_{k=i-1}^{N-2} \sin\theta_k \qquad 3 \le i \le N-1; \\ x_N &= \cos\theta_{N-2}. \end{aligned}$$

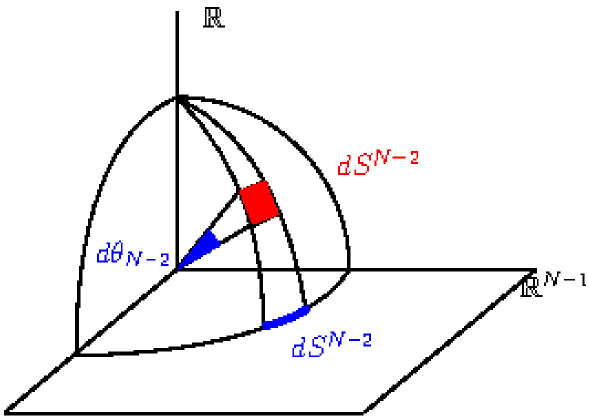

Figure C.1. Surface Area of the Unit Sphere in $\mathbb{R}^N$.

The surface element can be calculated by induction on N. For the basis step, take $N = 2$, i.e., consider a unit circle on $\mathbb{R}^2$. The only variable needed to determine the point on S^1 is the angle $\phi \in [0, 2\pi)$ given by the equations $\frac{x_2}{x_1} = \tan\phi$ if $x_1 \neq 0$ and else $\phi = \pi - \frac{x_2\pi}{2}$. Then, the surface element on S^1 is elementary $dS^1 = d\phi$. In order to pass higher dimensions, we project the surface element on the space spanned by the first $N-1$ coordinates. The intersection of this space and the S^{N-1} is just an $N-2$ dimensional sphere on which the surface element dS^{N-2} is assumed to be known. Then the base volume of the surface element dS^{N-1} is equal to $\sin^{N-2}\theta_{N-2} dS^{N-2}$ since $\dim(S^{N-2}) = N-2$. Therefore,

$$dS^{N-1} = \sin^{N-2}\theta_{N-2} dS^{N-2} d\theta_{N-2}. \tag{C.1}$$

By induction, we get $dS^{N-1} = \left(\prod_{k=1}^{N-2} \sin^k \theta_k\right) d\theta_{N-2} \dots d\theta_1 d\phi$. Moreover, we obtain a useful formula,

$$\omega_{N-1} = \omega_{N-2} \int_{\theta \in [0,\pi]} \sin^{N-2}\theta d\theta, \tag{C.2}$$

by integrating both sides of (C.1).

APPENDIX D: DIMENSION ANALYSIS

Proposition D.1. *$O(N)$ has dimension $\frac{N(N-1)}{2}$.*

Proof. We consider the columns of an orthonormal matrix one by one. Then, the only constraint for the first column is that its norm should be equal to one, the second column should have norm one and also be orthogonal to first column, generally, the kth column should be orthogonal to former $k-1$ columns and have norm one. Thus, $O(N)$ posesses $\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$ constraints.

$$\dim(O(N)) = N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}. \tag{D.1}$$

□

Proposition D.2. *$S_{N,k}$ has dimension $k(N-1)$.*

Proof. By definition of $S_{N,k}$:

$$\begin{aligned} \dim(S_{N,k}) &= \dim\left(\bigotimes_{j=1}^{k} S^{N-1}\right) \\ &= \sum_{j=1}^{k} \dim(S^{N-1}) \\ &= \sum_{j=1}^{k} N-1 \\ &= k(N-1). \end{aligned}$$

□

Proposition D.3. *$G_{N,k}$ has dimension $k(N-k)$.*

Proof. Following the formula $G_{N,k} \simeq O(N)/(O(k) \times O(N-k))$, and using (D.1), we have

$$
\begin{aligned}
\dim(G_{N,k}) &= \dim(O(N)/(O(k)\times O(N-k))) \\
&= \frac{N(N-1)}{2} - \frac{k(k-1)}{2} - \frac{N-k(N-k-1)}{2} \\
&= \frac{N^2 - N - k^2 + k - N^2 + Nk + N + kN - k^2 - k}{2} \\
&= kN - k^2 \\
&= k(N-k).
\end{aligned}
$$

This establishes the proof. On the other hand, the discussion in Section 3.1 yields

$$
\begin{aligned}
\dim(G_{N,k}) &= \dim(S_{N,k}) - \dim(S_{k,k}) \\
&= k(N-1) - k(k-1) \\
&= k(N-k).
\end{aligned}
$$

□

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