SOME MEAN VALUES RELATED TO DIRICHLET L-FUNCTIONS

by

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To Yıldız...

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ABSTRACT

SOME MEAN VALUES RELATED TO DIRICHLET L-FUNCTIONS

In this work, mean values of derivatives of the Riemann zeta-function and Dirichlet L-functions at the zeros of Dirichlet L-functions have been computed.

ÖZET

DIRICHLET L-FONKSİYONLARI İLE İLGİLİ BAZI ORTALAMA DEĞERLER

Bu çalışmada Riemann zeta-fonksiyonunun ve Dirichlet L-fonksiyonlarının türevlerinin, Dirichlet L-fonksiyonlarının sıfırlarındaki ortalama değerleri hesaplanmıştır.

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LIST OF SYMBOLS/ABBREVIATIONS

$\mathbb{Z}_q^\times$	The group of multiplicative units modulo q
$L(s,\chi)$	A Dirichlet L-function.
$\mathrm{li}(x)$	$=\int_2^x \frac{dt}{\log t}$; the logarithmic integral.
$N(T)$	The number of zeros $\rho=\sigma+i\gamma$ of $\zeta(s)$ with $0<\gamma\le T$.
$N(T,\chi)$	The number of zeros $\rho=\sigma+i\gamma$ of $L(s,\chi)$ with $\sigma>0$ and $0\le\gamma\le T$.
β	An exceptional zero of an L-function.
$\Gamma(s)$	The Gamma function.
γ_i	The i-th Stieltjes constant.
$\zeta(s)$	The Riemann zeta-function.
$\Lambda(n)$	The von-Mangoldt function.
$\pi(x)$	The number of primes not exceeding x.
τ	$=\|t\|+4$.
$\tau(\chi)$	The Gauss sum of χ.
$\phi(n)$	The number of a, $1\le a\le n$, for which $(a,n)=1$; known as Euler's totient function.
$\chi(n)$	A Dirichlet character.
$\chi(s,\chi)$	The factor in the funcional equation of $L(s,\chi)$.
$\psi(x)$	$=\sum_{n\le x}\Lambda(n)$.
$\Omega(n)$	The number of prime factors of n, counting multiplicity.
$f(x)=O(g(x))$	$\|f(x)\|\le Cg(x)$ where C is an absolute constant.
$f(x)=o(g(x))$	$\lim f(x)/g(x)=0$.
$f(x)\ll g(x)$	$f(x)=O(g(x))$.
$f(x)\asymp g(x)$	$cf(x)\le g(x)\le Cf(x)$ for some positive absolute constants c and C.
$f(x)\sim g(x)$	$\lim f(x)/g(x)=1$.
G.R.H.	Generalized Riemann Hypothesis

1. INTRODUCTION

The main objects of study in the present work are the zeta-function of Riemann and Dirichlet L-functions. Here we will give an outline of their history and some of their relevant properties, and try to convey a sense of their arithmetical significance. For a detailed account of the theory we refer the reader to Davenport's classical text [1].

1.1. Origins

Riemann's zeta function is initially defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for a complex variable $s = \sigma + it$, $\sigma > 1$. Euler considered, for real s, the identity

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1},$$

which is essentially an analytical statement of the unique factorization of integers. This is because expanding the factors in a geometric series, the right hand side equals $\prod_p \left(1 + p^{-s} + p^{-2s} \cdots\right)$, so when multiplied out, n^{-s} can be obtained by one and only one combination of powers of distinct primes. Taking logarithms in the product expression,

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-ms}, \tag{1.1}$$

and since $\zeta(s) \to \infty$ as $s \to 1$ from the right and the terms with $m \geq 2$ contribute a bounded amount, it follows that the the sum of reciprocals of primes diverges, entailing the infinitude of primes.

Dirichlet's aim was to prove that every arithmetic progression $kq + a$ with

$(a, q) = 1$ contains infinitely many primes. Following Euler, he set out to show that

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod q}} p^{-1}$$

diverges. For simplicity, we shall sketch his arguments for the case when q is prime. Now for any totally multiplicative function f (i.e. a function defined on integers which preserves multiplication) we have a product expression as in the case of $\zeta(s)$.

$$\sum_{n=1}^{\infty} f(s) n^{-s} = \prod_{p} \left(1 - f(p)p^{-s}\right)^{-1}.$$

But simply declaring a function to be the characteristic function of integers congruent to $a \pmod q$ does not produce a multiplicative function, so Dirichlet introduced the so called Dirichlet's characters which we shall denote by $\chi(n)$. They are totally multiplicative functions with period q. That they are periodic means that in fact they are defined on the group of integers modulo q, multiplicativity entails $\chi(1) = 1$, and since every integer has order $q - 1$ in the multiplicative group of units modulo q, we see that $\chi(n)$ must be a $(q-1)$-st root of unity for $(n, q) = 1$. We know that there is a primitive root modulo q when q is prime (in the language of algebra, $\mathbb{Z}_q^{\times}$ is cyclic and has a generator), so if we fix a primitive root g, defining $\chi(g)$ to be a $(q - 1)$-st root of unity completely determines $\chi(n)$ for $(n, q) = 1$, and we set $\chi(n) = 0$ for $(n, q) > 1$. Hence there are $\phi(q)$ characters to the modulus q. One of them takes the value 1 for all integers relatively prime to q and 0 otherwise, this is called the *principal character* and is distinguished by a naught in the subscript. Characters have the following imporant properties.

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod q, \\ 0 & \text{Otherwise}, \end{cases} \tag{1.2}$$

and

$$\frac{1}{\phi(q)} \sum_{n \pmod q} \chi(n) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{Otherwise}, \end{cases} \tag{1.3}$$

so they serve to select a residue class modulo q by virtue of

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod q, \\ 0 & \text{otherwise}. \end{cases} \tag{1.4}$$

Dirichlet defined L-functions in terms of the characters

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1},$$

and these have an Euler product expession as in the case of $\zeta(s)$. In particular, taking logarithms gives

$$\log L(s,\chi) = \sum_p \sum_{m=1}^{\infty} m^{-1}\chi(p^m)p^{-ms} \tag{1.5}$$

for $s > 1$, whence a suitable combination of them gives

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(a) \log L(s,\chi) &= \sum_p \sum_{\substack{m=1 \\ p^m \equiv a \pmod q}}^{\infty} m^{-1}p^{-ms} \\ &= \sum_{p \equiv a \pmod q} p^{-s} + O(1). \end{aligned} \tag{1.6}$$

Showing that the left hand side tends to infinity as $s \to 1$ from the right establishes the divergence of $\sum_{p \equiv a \pmod q} p^{-1}$.

Characters can be generalized with ease to composite moduli while retaining their crucial properties. To a general modulus q, however, an important distinction must be observed. A character $\chi(\text{mod } q)$, when restricted to the values of n with $(n,q) = 1$, may have a least period less than q, say q_1. Then necessarily $q_1 \mid q_2$, and there is a character χ_1 to the modulus q_1 such that for all $(n, q) = 1$, $\chi(n) = \chi_1(n)$. In this case we say χ_1 induces χ, and χ is called *imprimitive*, otherwise if q is the least period of χ restricted to those n with $(n, q) = 1$, we say that χ is *primitive*. We leave the principal character unclassified. Since the only character to the modulus 2 is the principal character, any mention of primitive characters presumes $q \geq 3$.

1.2. Relationships with the distribution of primes

We denote by $\pi(x)$ the number of primes not exceeding the real number x. The prime number theorem, in its simplest form, asserts that

$$\pi(x) \sim \frac{x}{\log x},$$

or in a more precise formulation

$$\pi(x) = \operatorname{li} x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

for some positive constant c, where

$$\operatorname{li} x = \int_2^x \frac{dt}{\log t} \tag{1.7}$$

is the logarithmic integral. That $\pi(x) \sim \operatorname{li} x$ was first conjectured by Gauss in 1849. Riemann, in his memoir of 1860, considered $\zeta(s)$ as a function of the complex variable s and showed that $\zeta(s)$, from its original half-plane of definition, admits an analytic continuation to the whole plane with the exception of a simple pole at $s = 1$ with residue 1. He also showed that $\zeta(s)$ satisfies the funcional equation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left[\frac{1}{2}(1-s)\right]\zeta(1-s). \tag{1.8}$$

From the functional equation one can infer the behaviour of $\zeta(s)$ for $\sigma < 0$ from that for $\sigma > 1$. By the absolute convergence of the Euler product for $\sigma > 1$, we see that $\zeta(s)$

does not vanish there, so the only zeros for $\sigma < 0$ are located at the poles of $\Gamma(\frac{1}{2}s)$, i.e. at $s = -2, -4, -6, \ldots$. These are called *trivial zeros.* The remaining part of the plane, where $0 \le \sigma \le 1$ is called the *critical strip.* He further stated without proof that $\zeta(s)$ has infinitely many zeros in the critical strip, (necessarily symmetrical with respect to the real axis by the reflection principle and to $\sigma = \frac{1}{2}$ by the functional equation) the number $N(T)$ of which that satisfy $0 \le t \le T$ is given by

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{1.9}$$

He also stated the so called Riemann-von Mangoldt formula

$$\sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log(1 - x^{-2}), \tag{1.10}$$

where $\Lambda(n)$ is the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}$$

In (1.10), the ρ are the zeros of $\zeta(s)$ in the critical strip, and it is understood that in the sum they are taken in conjugate pairs. The validity of (1.10) rests on the fact that $\Lambda(n)$ are the coefficients of the Dirichlet series for $-\frac{\zeta'}{\zeta}(s)$, and to sum them up to x, one may use the properties of the discontinuous integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = \frac{1}{2}, \\ 1 & \text{if } y > 1, \end{cases} \tag{1.12}$$

to obtain

$$\sum_{n \le x}{}' \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[-\frac{\zeta'}{\zeta}(s) \right] \frac{x^s}{s} ds, \tag{1.13}$$

where $\sum'$ denotes that the last term is to be halved for integral x. Of course there are questions of convergence, and the rigorous formulation of this method is known as Perron's Formula. In practice it is more convenient to start with an integral from $c-iT$ to $c+iT$, regard this as the right side of the rectangle extending to the left, and use the theory of residues. Letting $T \to \infty$ then renders plain the terms appearing in (1.10): the $\frac{1}{s}$ factor contributes $-\frac{\zeta'}{\zeta}(0)$, the nontrivial zeros contribute the sum over ρ, the pole at $s=1$ gives x, and the residues at trivial zeros are collected in the logarithm.

It is an elementary matter to pass to an estimate for $\pi(x)$ from (1.10), and this makes clear the significance of the zeros of the Riemann zeta-function; the smaller their real parts, the smaller the contribution of the sum over them. Thus, knowledge of larger zero-free regions for $\zeta(s)$ in general gives rise to sharper estimates for $\pi(x)$. With the classical zero-free region

$$\sigma \geq 1 - \frac{c}{\log(|t|+4)}, \tag{1.14}$$

which was first established by de la Vallée-Poussin in 1899, one obtains

$$\pi(x) = \operatorname{li} x + O\left(x \exp(-c\sqrt{\log x})\right), \tag{1.15}$$

and the truth of the only hitherto unproved statement from Riemann's memoir, the Riemann Hypothesis, to the effect that all the non-trivial zeros have $\sigma = \frac{1}{2}$, would imply

$$\pi(x) = \operatorname{li} x + O\left(x^{\frac{1}{2}} \log x\right). \tag{1.16}$$

All these have analogues for Dirichlet L-functions. We know that $L(s,\chi)$ admits a continuation to the whole complex plane as an entire function, and for primitive χ, it satisfies a functional equation whose form depends on whether $\chi(-1)$ is 1 or -1. The number $N(T,\chi)$ of zeros of $L(s,\chi)$ in the critical strip with non-negative imaginary parts not exceeding T is given by

$$N(T,\chi)=\frac{T}{2\pi}\log\frac{qT}{2\pi e}+O(\log qT). \tag{1.17}$$

However, there is one critical exception. The analogue of the method for deriving zero-free regions of $\zeta(s)$ fails in the case of L-functions of a real character. For a complex character χ to the modulus q, $L(s,\chi)$ is known to be non-zero in the region

$$\sigma\geq 1-\frac{c}{\log(q\,(|t|+4))}, \tag{1.18}$$

but in case χ is a real character, it has not been possible to rule out the possibility that a zero exists in this region. What is known is that there can be at most one, necessarily real, zero, for at most one real character modulo q. Such a zero β is called an *exceptional zero* and the corresponding character is called an *exceptional character.* This is summarized in Theorem 11.3 of [2], reproduced here since we shall have recourse to it often in the sequel. In the notation of [2], τ denotes $|t|+4$.

1.1 Theorem. *There is an absolute constant $c>0$ such that if χ is a Dirichlet character modulo q, then the region*

$$R_q=\{s:\sigma>1-c/\log q\tau\}$$

contains no zero of $L(s,\chi)$ unless χ is a quadratic character, in which case $L(s,\chi)$ has at most one, necessarily real, zero $\beta<1$ in R_q.

However, it has been shown that an exceptional zero β of an L-function to the modulus q, if it exists, has to satisfy the following upper bound [1, p.96,(12)].

$$\beta<1-\frac{c}{q^{\frac{1}{2}}(\log q)^2}. \tag{1.19}$$

The analogue of the Riemann Hypothesis, to the effect that all non-trivial zeros of an

L-function lie on $\sigma = \frac{1}{2}$, is known as the Generalized Riemann Hypothesis.

Using L-functions one can generalize the prime number theorem to arithmetic progressions. By means of Perron's formula, we can obtain estimates for $\psi(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n)$, and taking a suitable linear combination we may pass to

$$\sum_{\substack{n\le x \\ n\equiv a \,(\mathrm{mod}\ q)}} \Lambda(n) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi}(a)\psi(x,\chi). \tag{1.20}$$

However, because of the possible presence of an exceptional zero β, Perron's formula produces a term of the magnitude x^{β}, and for this to be of a lesser order than the main term, which turns out to be $x/\phi(q)$, one has to impose very severe restrictions on the size of q with respect to x. For instance, one form of the approximations to the number of primes not exceeding x in the progression $kq + a$ is

$$\frac{1}{\phi(q)} \operatorname{li} x + O\left(\exp\left(c\sqrt{\log x}\right)\right), \tag{1.21}$$

under the quite harsh restriction $q \le (\log x)^N$. Here c is a constant which depends only on N.

1.3. Results of the present work

There are various results concerning sums and mean values related to the zeta-function or L-functions. We have considered sums of the derivatives of the Riemann zeta-function at the non-trivial zeros of an L-function up to a prescribed height T, and the same for the derivatives of an L-function at the zeros of another (possibly same) L-function. We have tried to obtain as precise error terms as possible which are uniform in the moduli, and these are to be found in various forms in the main body of the work. Considering the moduli fixed, the asymptotic forms of our results read

$$\sum_{0\le\gamma_\chi\le T} \zeta^{(\mu)}(\rho_\chi) \sim \begin{cases} \frac{L'}{L}(1,\overline{\chi})\frac{T}{2\pi} + N(T,\chi) & \text{if } \mu = 0, \\ \left[\left(\frac{d}{ds}\right)^{\mu}\Big|_{s=1} \frac{L'}{L}(s,\overline{\chi})\right]\frac{T}{2\pi} & \text{otherwise,} \end{cases} \tag{1.22}$$

$$\sum_{0\le\gamma_{\chi_1}\le T} L^{(\mu)}(\rho_{\chi_1},\chi_1) \sim \frac{(-1)^{\mu+1}}{2\pi(\mu+1)} T\left(\log\frac{qT}{2\pi}\right)^{\mu+1}, \tag{1.23}$$

$$\sum_{0\le\gamma_{\chi_1}\le T} L(\rho_{\chi_1},\psi_1) \sim -\delta(q_1,q_2)\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)} L(1,\chi_1\overline{\psi}_1)\frac{T}{2\pi} + \frac{L'}{L}(1,\overline{\chi}_1\psi_1)\frac{T}{2\pi} + N(T,\chi_1), \tag{1.24}$$

and for $\mu \ge 1$

$$\sum_{0\le\gamma_{\chi_1}\le T} L^{(\mu)}(\rho_{\chi_1},\psi_1) \sim \begin{cases} -\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)} L(1,\chi_1\overline{\psi}_1)\frac{T}{2\pi}(\log T)^{\mu} & \text{if } q_1 \mid q_2, \\ (-1)^{\mu}\left[\left(\frac{d}{ds}\right)^{\mu}\Big|_{s=1}\frac{L'}{L}(s,\overline{\psi}_1\chi_1)\right]\frac{T}{2\pi} & \text{otherwise.} \end{cases} \tag{1.25}$$

Here ρ_χ are the non-trivial zeros of $L(s,\chi)$ with $\Im\rho_\chi = \gamma_\chi$, χ_1 and ψ_1 are primitive characters to the moduli q_1 and q_2 respectively, and $\delta(q_1,q_2)$ is the function which takes the value 1 if $q_1 \mid q_2$ and 0 otherwise. These results, along with expressions involving explicit error terms are derived in chapters 3, 4 and 5 respectively. The method we have employed in those derivations is essentially that of Gonek's in [3]. We made extensive use of Perron's formula to obtain approximations to sums of coefficients of various Dirichlet series; those computations are to be found in section 2.3.

The first thing to note about these results is that the greatest magnitude is achieved when summing the derivatives of an L-function over the zeros of itself. This may be regarded as a loose indication that not too many of the zeros can be multiple zeros. Comparing (1.24) with the case $\mu = 0$ of (1.22), we see that they agree if we regard $\zeta(s)$ as the L-function of a "character modulo 1". Considering (1.25), we note that the magnitude of an L-function at the given set of zeros of another L-function is mainly determined by the divisibility relations between their moduli. Hence, one might expect more correlated behaviour from L-functions of characters to moduli having greater common divisors than from those of characters to relatively prime moduli.

Our (1.24) and (1.25) are related to the following theorem of Fujii's, namely Theorem 13 of [4].

1.2 Theorem. *Let* χ *be a primitive character* $\chi \bmod q \geq 3$ *and* ψ *be a primitive character* $\bmod k \geq 1$. *We suppose that* $\chi \neq \psi$. *Then we have*

$$\lim_{T\to\infty} \frac{1}{\frac{T}{2\pi}} \sum_{0<\gamma(\psi)\leq T} \left(L\left(\frac{1}{2}+i\gamma(\psi),\chi\right) - 1\right)$$
$$= -\delta(k,q)L(1,\overline{\chi}\psi)\chi(-1)\tau(\chi)\frac{\tau(\overline{\psi}\chi_0)}{\phi(q)} + \frac{L'}{L}(1,\chi\overline{\psi}),$$

where χ_0 *is the principal character* $\bmod q$.

This is an immediate corollary of (1.24). With (1.25) we give estimates with the derivatives in the summand as well.

Counterparts of our results, i.e. the same sums over the zeros of $\zeta(s)$, may be found in Karabulut [5].

1.4. Some miscellaneous facts

For any character $\chi(n)$ to the modulus q, the Gaussian sum $\tau(\chi)$ is defined as

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m)e^{2\pi i m/q}. \tag{1.26}$$

If χ is a primitive character, the Gaussian sum satisfies $|\tau(\chi)| = q^{\frac{1}{2}}$.

The sum of $\chi(n)$ over any set of complete residues (mod q) is zero, so we have the obvious upper bound $\sum_{M\leq n\leq N}\chi(n) \leq q$. Pólya-Vinogradov inequality further states that

$$\sum_{M\leq n\leq N} \chi(n) \ll q^{\frac{1}{2}}\log q. \tag{1.27}$$

It will be apt to also reproduce here Theorem 11.4 from [2] which we used extensively.

1.3 Theorem. *Let* χ *be a non-principal character modulo* q*, let* c *be the constant in* Theorem 3*, and suppose that* $\sigma \geq 1 - c/(2\log q\tau)$*. If* $L(s,\chi)$ *has no exceptional zero, or if* β_1 *is an exceptional zero of* $L(s,\chi)$ *but* $|s-\beta_1| \geq 1/\log q$*, then*

$$\frac{L'}{L}(s,\chi) \ll \log q\tau, \tag{1.28}$$

$$|\log L(s,\chi)| \leq \log\log q\tau + O(1), \tag{1.29}$$

and

$$\frac{1}{L(s,\chi)} \ll \log q\tau. \tag{1.30}$$

Alternatively, if β_1 *is an exceptional zero of* $L(s,\chi)$ *and* $|s-\beta_1| \leq 1/\log q$*, then*

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q) \quad (s \neq \beta_1), \tag{1.31}$$

$$|\arg L(s,\chi)| \leq \log\log q + O(1) \quad (s \neq \beta_1), \tag{1.32}$$

and

$$|s-\beta_1| \ll |L(s,\chi)| \ll |s-\beta_1|(\log q)^2. \tag{1.33}$$

1.5. Remarks on notation and conventions

We have taken the liberty of using the Greek letter χ to denote the factor in the functional equation of both the Riemann zeta-function and L-functions, as well as a general Dirichlet character. There is no danger of ambiguity, as the object it denotes will be clear from the arguments it takes or the lack thereof. Also the letter τ denotes both Gaussian sums as $\tau(\chi)$, and the number $|t| + 4$ where $t = \Im s$, following the convention employed in [2]. The latter is to facilitate the expression of order relations which degenerate when $|t|$ is small.

As usual, we denote by ε a positive real number, which need not be the same at each instance. This is especially true in the case of order relations, that is, we will freely write expressions such as $T^\varepsilon \log T \ll T^\varepsilon$. The same goes for the letters c and C which denote unspecified constants that also need not always be the same. In the few instances where it was necessary to keep track of such constants, we have employed subscripts.

2. PRELIMINARIES

This chapter consists of some lemmata concerning L-functions and estimates of some sums and integrals related to them.

2.1. Estimates regarding $\chi(s,\chi)$

The functional equation for $L(s,\chi)$, where χ is a primitive character to the modulus q, is given by (see, for instance, Davenport [1], §9, equations (8) and (11))

$$\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma\left[\tfrac{1}{2}(s+\mathfrak{a})\right]L(s,\chi)=\frac{\tau(\chi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}}\left(\frac{\pi}{q}\right)^{-\frac{1}{2}(1-s+\mathfrak{a})}\Gamma\left[\tfrac{1}{2}(1-s+\mathfrak{a})\right]L(1-s,\overline{\chi}). \tag{2.1}$$

Here $\mathfrak{a}$ is defined as

$$\mathfrak{a}=\begin{cases}0 & \text{if } \chi(-1)=1,\\ 1 & \text{if } \chi(-1)=-1.\end{cases}$$

In analogy with the Riemann zeta-function, we write the functional equation in the unsymmetric form as

$$L(s,\chi)=\chi(s,\chi)L(1-s,\overline{\chi}), \tag{2.2}$$

where

$$\chi(s,\chi)=\begin{cases}\dfrac{\tau(\chi)}{q^{\frac{1}{2}}}\left(\dfrac{\pi}{q}\right)^{s-\frac{1}{2}}\dfrac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)} & \text{if } \chi(-1)=1,\\[2ex] \dfrac{\tau(\chi)}{iq^{\frac{1}{2}}}\left(\dfrac{\pi}{q}\right)^{s-\frac{1}{2}}\dfrac{\Gamma(1-\frac{1}{2}s)}{\Gamma\left[\frac{1}{2}(1+s)\right]} & \text{if } \chi(-1)=-1.\end{cases} \tag{2.3}$$

As an estimate of $\chi(s,\chi)$, we have the following

2.1 Lemma. *Let χ be a primitive character to the modulus $q \geq 3$. In any fixed half-strip $\alpha \leq \sigma \leq \beta$, $t \geq \delta > 0$, where δ is an arbitrarily small positive number, we have*

$$\chi(s,\chi) = \frac{\tau(\chi)e^{\frac{\pi i}{4}}}{q^{\frac{1}{2}}} \left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\Big(-it\log\Big(\frac{qt}{2\pi e}\Big)\Big)\Big\{1+O\Big(\frac{1}{t}\Big)\Big\}$$

and

$$\chi(1-s,\chi) = \frac{\tau(\chi)e^{\frac{-\pi i}{4}}}{q^{\frac{1}{2}}} \left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}} \exp\Big(it\log\Big(\frac{qt}{2\pi e}\Big)\Big)\Big\{1+O\Big(\frac{1}{t}\Big)\Big\}.$$

Proof. Stirling's formula asserts that [1, (5) of §10]:

$$\log\Gamma(s) = (s-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi + O\left(|s|^{-1}\right).$$

This gives, in any fixed half strip $\alpha \leq \sigma \leq \beta$, $t \geq \delta > 0$,

$$\begin{aligned}\log\Gamma(\sigma+it) &= (\sigma+it-\tfrac{1}{2})\log(\sigma+it) - \sigma - it + \tfrac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right)\\ &= (\sigma+it-\tfrac{1}{2})(\log it) + it\log\left(1+\frac{\sigma}{it}\right) - \sigma - it + \tfrac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right).\end{aligned}$$

For t large enough, say $t \geq t_0$, we have

$$it\log\left(1+\frac{\sigma}{it}\right) = -it\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{\sigma}{it}\right)^n\right) = \sigma + O\left(\frac{1}{t}\right),$$

so

$$\log\Gamma(\sigma+it) = (\sigma+it-\tfrac{1}{2})(\log it) - it + \tfrac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right). \tag{2.4}$$

But in the rectangle $\alpha \leq \sigma \leq \beta$, $\delta \leq t \leq t_0$, there hold

$$\log\Gamma(\sigma+it) - (\sigma+it-\tfrac{1}{2})(\log it) + it - \tfrac{1}{2}\log 2\pi \ll 1,$$

and $1 \ll 1/t$, so in fact (2.4) holds for $t \geq \delta$. Exponentiating (2.4), we get

$$\Gamma(\sigma+it) = t^{\sigma-\frac{1}{2}+it} e^{-\frac{\pi t}{2}-it+\frac{1}{2}i\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}\Big\{1+O\Big(\frac{1}{t}\Big)\Big\},$$

for $t \geq \delta$ uniformly in any fixed strip $\alpha \leq \sigma \leq \beta$. This also implies

$$\Gamma(\sigma-it) = t^{\sigma-\frac{1}{2}-it} e^{-\frac{\pi t}{2}+it-\frac{1}{2}i\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}\Big\{1+O\Big(\frac{1}{t}\Big)\Big\},$$

for $t \geq \delta$ there, by the reflection principle.

Using these, we obtain

$$\begin{aligned}\frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)} &= \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\sigma-\frac{1}{2}it)}{\Gamma(\frac{1}{2}\sigma+\frac{1}{2}it)} \\ &= \frac{\left(\frac{t}{2}\right)^{-\frac{\sigma}{2}-\frac{it}{2}} e^{-\frac{\pi t}{4}+\frac{it}{2}+\frac{i\pi\sigma}{4}}}{\left(\frac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}+\frac{it}{2}} e^{-\frac{\pi t}{4}-\frac{it}{2}+\frac{i\pi\sigma}{4}-\frac{i\pi}{4}}}\left\{1+O\Big(\frac{1}{t}\Big)\right\} \\ &= \left(\frac{t}{2}\right)^{\frac{1}{2}-\sigma-it} e^{i(t+\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma\left[\frac{1}{2}(1+s)\right]} &= \frac{\Gamma(1-\frac{1}{2}\sigma-\frac{1}{2}it)}{\Gamma(\frac{1}{2}+\frac{1}{2}\sigma+\frac{1}{2}it)} \\ &= \frac{\left(\frac{t}{2}\right)^{\frac{1}{2}-\frac{\sigma}{2}-\frac{it}{2}} e^{-\frac{\pi t}{4}+\frac{it}{2}+\frac{i\pi\sigma}{4}-\frac{i\pi}{4}}}{\left(\frac{t}{2}\right)^{\frac{\sigma}{2}+\frac{it}{2}} e^{-\frac{\pi t}{4}-\frac{it}{2}+\frac{i\pi\sigma}{4}}}\left\{1+O\Big(\frac{1}{t}\Big)\right\} \\ &= \left(\frac{t}{2}\right)^{\frac{1}{2}-\sigma-it} e^{i(t-\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\}.\end{aligned}$$

Hence, if $\chi(-1)=1$,

$$\begin{aligned}\chi(s,\chi) &= \frac{\tau(\chi)}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i(t+\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\} \\ &= \frac{\tau(\chi)e^{\frac{\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma}\exp\left(-it\log\Big(\frac{qt}{2\pi e}\Big)\right)\left\{1+O\Big(\frac{1}{t}\Big)\right\}.\end{aligned}$$

Since writing $1-s$ for s in $\frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)}$ gives its reciprocal,

$$\begin{aligned}\chi(1-s,\chi) &= \frac{\tau(\chi)}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\sigma+it-\frac{1}{2}} e^{-i(t+\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\}\\ &= \frac{\tau(\chi)e^{\frac{-\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}} \exp\Big(it\log\Big(\frac{qt}{2\pi e}\Big)\Big)\left\{1+O\Big(\frac{1}{t}\Big)\right\}.\end{aligned}$$

Similarly when $\chi(-1)=-1$, we get

$$\begin{aligned}\chi(s,\chi) &= \frac{\tau(\chi)}{iq^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i(t-\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\}\\ &= \frac{\tau(\chi)e^{\frac{\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\Big(-it\log\Big(\frac{qt}{2\pi e}\Big)\Big)\left\{1+O\Big(\frac{1}{t}\Big)\right\},\end{aligned}$$

and

$$\begin{aligned}\chi(1-s,\chi) &= \frac{\tau(\chi)}{iq^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\sigma+it-\frac{1}{2}} e^{-i(t-\frac{\pi}{4})}\left\{1+O\Big(\frac{1}{t}\Big)\right\}\\ &= \frac{\tau(\chi)e^{\frac{-\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}} \exp\Big(it\log\Big(\frac{qt}{2\pi e}\Big)\Big)\left\{1+O\Big(\frac{1}{t}\Big)\right\}.\end{aligned}$$ □

Since $|\tau(\chi)| = q^{\frac{1}{2}}$ for a primitive character $\chi(\text{mod } q)$, we have as a corollary

$$|\chi(s,\chi)| \sim \left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-\sigma} \tag{2.5}$$

and

$$|\chi(1-s,\chi)| \sim \left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}}, \tag{2.6}$$

as $t\to\infty$.

The following is an estimate on the logarithmic derivative of $\chi(s,\chi)$.

2.2 Lemma. *For any fixed σ and $|t| > 1$, we have*

$$\frac{\chi'}{\chi}(s, \chi) = -\log \frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right).$$

Proof. First assume $\chi(-1) = 1$. Then

$$\chi(s, \chi) = \frac{\tau(\chi)}{q^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)}.$$

Taking logarithmic derivatives:

$$\frac{\chi'}{\chi}(s, \chi) = \log \frac{\pi}{q} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left[\tfrac{1}{2}(1-s)\right] - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}s\right).$$

Now we use $\frac{\Gamma'}{\Gamma}(s) = \log s + O(\frac{1}{|t|})$ [1, (6) of §10], which holds uniformly for $|s| \geq \delta$ and $|\arg s| < \pi - \delta$ for any fixed positive δ, and get

$$\begin{aligned}
\frac{\chi'}{\chi}(s, \chi) &= \log \frac{\pi}{q} - \frac{1}{2}\log\left[\frac{1}{2}(1-\sigma-it)\right] - \frac{1}{2}\log\left[\frac{1}{2}(\sigma+it)\right] + O\left(\frac{1}{|t|}\right) \\
&= \log \frac{2\pi}{q} - \frac{1}{2}\log\left[t^2\left(1 - \frac{\sigma^2 - \sigma + 2i\sigma t - it}{t^2}\right)\right] + O\left(\frac{1}{|t|}\right) \\
&= \log \frac{2\pi}{q} - \log|t| - \frac{1}{2}\log\left(1 - \frac{\sigma^2 - \sigma + 2i\sigma t - it}{t^2}\right) + O\left(\frac{1}{|t|}\right),
\end{aligned}$$

whence we obtain the assertion of the lemma. Similarly when $\chi(-1) = -1$,

$$\chi(s, \chi) = \frac{\tau(\chi)}{iq^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma\left[\frac{1}{2}(1+s)\right]}.$$

Therefore

$$\begin{aligned}
\frac{\chi'}{\chi}(s, \chi) &= \log \frac{\pi}{q} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1 - \frac{1}{2}s\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + \frac{1}{2}s\right) \\
&= \log \frac{\pi}{q} - \frac{1}{2}\log\left[\frac{1}{2}(2-s)\right] - \frac{1}{2}\log\left[\frac{1}{2}(1+s)\right] + O\left(\frac{1}{|t|}\right) \\
&= \log \frac{\pi}{q} - \frac{1}{2}\log\left[t^2\left(1 + O\left(\frac{1}{|t|}\right)\right)\right] + O\left(\frac{1}{|t|}\right),
\end{aligned}$$

so again we obtain the desired result. □

Next we have an estimate for the derivatives of $\chi(s,\chi)$.

2.3 Lemma. *For any fixed* σ, $\nu \geq 0$, *and* $|t| \geq 1$ *we have*

$$\chi^{(\nu)}(1-s,\chi) = \chi(1-s,\chi)\left(-\log\frac{q|t|}{2\pi}\right)^{\nu} + O\left(q^{\sigma-\frac{1}{2}}|t|^{\sigma-\frac{3}{2}}\left(\log q|t|\right)^{\nu-1}\right).$$

Proof. We use induction on ν. The case $\nu = 0$ is obvious. Assume the claim proved for $\nu = 0,\ldots,\mu-1$. We differentiate the identity

$$\chi'(1-s,\chi) = \chi(1-s,\chi)\cdot\frac{\chi'}{\chi}(1-s,\chi)$$

$\mu - 1$ times and obtain

$$\chi^{(\mu)}(1-s,\chi) = \sum_{\nu=0}^{\mu-1}\binom{\mu-1}{\nu}\chi^{(\nu)}(1-s,\chi)\left(\frac{\chi'}{\chi}\right)^{(\mu-\nu-1)}(1-s,\chi). \tag{2.7}$$

By Lemma 2.2 we know that

$$\frac{\chi'}{\chi}(1-s,\chi) = -\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right). \tag{2.8}$$

We have from (2.3)

$$\frac{\chi'}{\chi}(1-s,\chi) = \begin{cases} -\log\frac{\pi}{q} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1}{2}s) + \frac{1}{2}\frac{\Gamma'}{\Gamma}[\frac{1}{2}(1-s)] & \text{if } \chi(-1) = 1, \\ -\log\frac{\pi}{q} + \frac{1}{2}\frac{\Gamma'}{\Gamma}[\frac{1}{2}(1+s)] + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1-\frac{1}{2}s) & \text{if } \chi(-1) = 1. \end{cases} \tag{2.9}$$

We also have

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \sum_{n=0}^{\infty}\frac{1}{(s+n)^2},$$

which is obtained by logarithmic differentiation of the Weierstrass product for $\Gamma(s)$ [6, 12.16], and further termwise differentiation of this gives

$$\left(\frac{\Gamma'}{\Gamma}\right)^{(\nu)}(s) = \sum_{n=0}^{\infty} \frac{(-1)^{\nu+1}\nu!}{(s+n)^{\nu+1}} = O\left(\frac{1}{|t|}\right) \qquad (\nu \geq 1),$$

since

$$\begin{aligned}\Big|\sum_{n=0}^{\infty} \frac{(-1)^{\nu+1}\nu!}{(s+n)^{\nu+1}}\Big| &\leq \nu! \sum_{n=0}^{\infty} \frac{1}{|\sigma+n+it|^{\nu+1}} \\ &\leq \nu! \sum_{n=0}^{\infty} \frac{1}{(\sigma+n)^2+|t|^2} \\ &\leq \nu! \int_0^{\infty} \frac{du}{(\sigma+u)^2+|t|^2} \\ &= \nu! \int_{\sigma}^{\infty} \frac{du}{u^2+|t|^2} \\ &= \nu! \frac{1}{|t|} \arctan \frac{u}{|t|}\Big|_{\sigma}^{\infty} \ll \frac{1}{|t|}.\end{aligned}$$

Using this with (2.9), we have

$$\left(\frac{\chi'}{\chi}\right)^{(\nu)}(1-s,\chi) = O\left(\frac{1}{|t|}\right) \qquad (\nu \geq 1).$$

Also, by Lemma 2.1

$$\chi(1-s,\chi) = O\big((q|t|)^{\sigma-\frac{1}{2}}\big),$$

so combining these and the induction hypothesis in (2.7) yields the result. □

2.4 Lemma. *Let χ be a primitive character to the modulus q. For any $A > 0$, in the region $-A \leq \sigma \leq \frac{1}{2}$, we have*

$$L(s,\chi) \ll (q\tau)^{\frac{1}{2}-\sigma}\,|L(1-s,\overline{\chi})|.$$

Proof. This amounts to showing that $\chi(s,\chi) \ll (q\tau)^{\frac{1}{2}-\sigma}$. We make use of the formulae (see [2, Appendix C])

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$
$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s),$$

and Stirling's formula in the form

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} \left(1 + O\left(\frac{1}{|s|}\right)\right).$$

If $\chi(-1) = 1$, we have

$$|\chi(s,\chi)| = \left(\frac{\pi}{q}\right)^{\sigma-\frac{1}{2}} \left|\frac{\Gamma[\frac{1}{2}(1-s)]}{\Gamma(\frac{1}{2}s)}\right|.$$

Here

$$\frac{\Gamma[\frac{1}{2}(1-s)]}{\Gamma(\frac{1}{2}s)} = \frac{1}{\pi}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\sin\frac{\pi s}{2} = \pi^{-\frac{1}{2}}2^s\Gamma(1-s)\sin\frac{\pi s}{2},$$

and in the region under consideration, Stirling's formula gives

$$|\Gamma(1-s)| \asymp |(1-s)^{\frac{1}{2}-s}| = |1-s|^{\frac{1}{2}-\sigma}\exp\left(t\arg(1-s)\right).$$

But $\arg(1-s) = -\arctan t/(1-\sigma) = -\pi/2 + O(1/t)$ and $|1-s| \asymp \tau$, so $|\Gamma(1-s)| \asymp \tau^{\frac{1}{2}-\sigma}\exp(-\pi t/2)$. Also

$$\sin\frac{\pi s}{2} = \frac{e^{\frac{i\pi s}{2}} - e^{-\frac{i\pi s}{2}}}{2i} \ll \exp\left(\frac{\pi t}{2}\right).$$

These give the desired bound for $\chi(s,\chi)$. The case $\chi(-1) = -1$ is handled in the same fashion, since

$$\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma[\frac{1}{2}(1+s)]} = \frac{1}{\pi}\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\sin\frac{\pi(s+1)}{2} = \pi^{-\frac{1}{2}}2^s\Gamma(1-s)\sin\frac{\pi s}{2}. \qquad \square$$

2.2. Bounds for $L^{(\kappa)}(s,\chi)$

2.5 Lemma. *If χ is a primitive character to the modulus q, for any fixed $\varepsilon > 0$ and all $\kappa = 0, 1, 2, \ldots,$ we have*

$$L^{(\kappa)}(\sigma + it, \chi) \ll \begin{cases} (q\tau)^{\frac{1}{2}-\sigma+\varepsilon} & \text{if } \sigma \le 0, \\ (q\tau)^{\frac{1}{2}(1-\sigma)+\varepsilon} & \text{if } 0 \le \sigma \le 1, \\ (q\tau)^{\varepsilon} & \text{if } 1 \le \sigma. \end{cases}$$

Proof. Throughout the proof we are working with a prescribed ε, which can be arbitrarily small. Given any small but fixed $\delta > 0$, for $\sigma \ge \delta$, we have

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n \le A} \frac{\chi(n)}{n^s} + \sum_{n > A} \frac{\chi(n)}{n^s}, \tag{2.10}$$

where A is a number greater than 1 that may be freely specified. We bound the first series trivially as

$$\Big| \sum_{n \le A} \frac{\chi(n)}{n^s} \Big| \le \sum_{n \le A} \frac{1}{n^\sigma} \le \int_1^{A+1} \frac{dx}{x^\sigma}. \tag{2.11}$$

For the second, partial summation and Pólya-Vinogradov inequality give

$$\sum_{n > A} \frac{\chi(n)}{n^s} = s \int_A^{\infty} \Big(\sum_{A < n \le u} \chi(n) \Big) u^{-1-s} du \ll q^{\frac{1}{2}+\varepsilon} |s| A^{-\sigma}. \tag{2.12}$$

Therefore, we have

$$L(s,\chi) \ll \int_1^{A+1} \frac{dx}{x^\sigma} + q^{\frac{1}{2}+\varepsilon} |s| A^{-\sigma}, \quad (\sigma \ge \delta). \tag{2.13}$$

Now for any fixed $\delta_1 > 0$, in the half plane $\sigma \ge 1 + \delta_1$, we have $|L(s,\chi)| \le \zeta(\sigma) \le \zeta(1+\delta_1) \ll \frac{1}{\delta_1} \ll 1$. Also, in the strip $1 \le \sigma \le 1 + \delta_1$, we have $|s| \asymp \tau$, so there (2.13) with $A = q^{\frac{1}{2}}\tau$ gives

$$L(s,\chi) \ll (q\tau)^{\varepsilon}, \quad (\sigma \geq 1). \tag{2.14}$$

From this Lemma 2.4 immediately yields

$$L(s,\chi) \ll (q\tau)^{\frac{1}{2}-\sigma+\varepsilon}, \quad (\sigma \leq 0). \tag{2.15}$$

Also, using (2.13) in the bounded region $\frac{1}{2} \leq \sigma < 1$, $|t| \leq 1$ with $A = q^{\frac{1}{2}}$, we obtain

$$L(s,\chi) \ll q^{\frac{1}{2}(1-\sigma)+\varepsilon}, \tag{2.16}$$

since $|s| \asymp 1$ there. Using this with $\overline{\chi}$ in place of χ in Lemma 2.4, we obtain for $0 < \sigma \leq \frac{1}{2}$, $|t| \leq 1$,

$$L(s,\chi) \leq q^{\frac{1}{2}-\sigma} L(1-s,\overline{\chi}) \ll q^{\frac{1}{2}-\sigma} q^{\frac{1}{2}\sigma+\varepsilon} \ll q^{\frac{1}{2}(1-\sigma)+\varepsilon}, \tag{2.17}$$

so that (2.16) holds throughout the region $0 \leq \sigma \leq 1$, $|t| \leq 1$. Thus the case $\kappa = 0$ of the Lemma is established except for the region $0 < \sigma < 1$, $|t| > 1$.

Now we put $\sigma_1 = 0$ and $\sigma_2 = 1$, so that $L(\sigma_1 + it) = O\left((qt)^{\frac{1}{2}+\varepsilon}\right)$ and $L(\sigma_2 + it) = O\left((qt)^{\varepsilon}\right)$ for $t \geq 1$. We denote by $k(\sigma)$ the linear function satisfying $k(\sigma_1) = \frac{1}{2} + \varepsilon$ and $k(\sigma_2) = \varepsilon$, i.e. $k(s) = \frac{1}{2}(1-s) + \varepsilon$.

Put $\psi(s) = (-iqs)^{k(s)} = e^{k(s)\log(-iqs)}$. Then $\psi(s)$ is analytic for $\sigma_1 \leq \sigma \leq \sigma_2$, $t \geq 1$. If $k(s) = as + b$,

$$\begin{aligned} \Re\big(k(s)\log(-iqs)\big) &= \Re\Big(\big(k(\sigma) + iat\big)\big(\log(qt) + O\left(1/t\right)\big)\Big) \\ &= k(\sigma)\log(qt) + O(1), \end{aligned}$$

whence $|\psi(s)| = (qt)^{k(\sigma)} e^{O(1)}$.

So $\frac{L(s,\chi)}{\psi(s)}$ is analytic and bounded for $\sigma = \sigma_1$ and $\sigma = \sigma_2$ by (2.14) and (2.15), and also on the interval $[it, 1+it]$, by (2.16). Call the greater of these bounds M and put $\phi(s) = e^{\varepsilon i s}\frac{L(s,\chi)}{\psi(s)}$. Also, using (2.13) in the fixed half strip $\frac{1}{2} \le \sigma < 1$ with $A = 0$ with Lemma 2.4 shows that $L(s,\chi) = O\left((qt)^K\right)$ for some K as $t \to \infty$ uniformly in $0 \le \sigma \le 1$. So when T is sufficiently large, we have

$$|\phi(s)| = e^{-\varepsilon t}\left|\frac{L(s,\chi)}{\psi(s)}\right| \le M$$

on the boundary of the rectangle with vertices $\sigma_2 + i$, $\sigma_2 + iT$, $\sigma_1 + iT$ and $\sigma_1 + i$, whence on the whole rectangle, by the maximum modulus principle. So we have, in the region $\sigma_1 \le \sigma \le \sigma_2$, $t \ge 1$,

$$|L(s,\chi)| \le e^{\varepsilon t}(qt)^{k(\sigma)}e^{O(1)}M.$$

Letting $\varepsilon \to 0$ completes the proof for the case $\nu = 0$ since the same bound carries to the negative values of t by virtue of the identity $L(\overline{s},\chi) = \overline{L(s,\overline{\chi})}$.

We write

$$\nu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma \le 0, \\ \frac{1}{2}(1-\sigma) & \text{if } 0 \le \sigma \le 1, \\ 0 & \text{if } 1 \le \sigma, \end{cases} \tag{2.18}$$

so that our result reads $L(s,\chi) \ll (q\tau)^{\nu(\sigma)+\varepsilon}$. Now for any σ_0 in a given fixed strip, we apply this with $\varepsilon/2$ in place of ε and note that if $|\sigma - \sigma_0| \le \varepsilon/2$, we then have $L(\sigma + it) \ll (q\tau)^{\nu(\sigma_0)+\varepsilon}$. So Cauchy's Theorem gives

$$L'(\sigma_0 + it, \chi) = \frac{1}{2\pi i}\int_C \frac{L(\omega,\chi)}{(\omega - \sigma_0 - it)^2}d\omega \ll (q\tau)^{\nu(\sigma_0)+\varepsilon},$$

where C is the circle of radius $\varepsilon/2$ around $\sigma_0 + it$. After κ iterations of Cauchy's Theorem, we obtain the same bounds for $L^{(\kappa)}(s)$. This completes the proof. □

If χ is imprimitive and χ_1 is the character which induces χ, we have

$$L(s,\chi) = L(s,\chi_1)\prod_{p|q}[1-\chi_1(p)p^{-s}].$$

But

$$\Big|\prod_{p|q}[1-\chi_1(p)p^{-s}]\Big| \leq \prod_{p|q}[1+p^{-\sigma}],$$

so when $\sigma > 0$, $\prod_{p|q}[1-\chi_1(p)p^{-s}] \ll q^{\varepsilon}$, since there are $\omega(n)$ factors each ≤ 2 and $2^{\omega(q)} \ll q^{\varepsilon}$. When $\sigma < 0$, expanding the product gives $2^{\omega(q)}$ terms, the greatest of which is $\prod_{p|q} p^{-\sigma} \leq q^{-\sigma}$, so that in that case $\prod_{p|q}[1-\chi_1(p)p^{-s}] \ll q^{-\sigma+\varepsilon}$. With these factors the preceding lemma can be generalized as follows.

2.6 Lemma. *Let χ be a Dirichlet character to the modulus q, not necessarily primitive, but non-principal. For any fixed $\varepsilon > 0$ and all $\kappa = 0, 1, 2, \ldots,$ we have*

$$L^{(\kappa)}(\sigma+it,\chi) \ll \begin{cases} (q\tau)^{\frac{1}{2}-\sigma+\varepsilon}\, q^{-\sigma} & \text{if } \sigma \leq 0, \\ (q\tau)^{\frac{1}{2}(1-\sigma)+\varepsilon} & \text{if } 0 \leq \sigma \leq 1, \\ (q\tau)^{\varepsilon} & \text{if } 1 \leq \sigma. \end{cases}$$

2.3. Some sums of Dirichlet coefficients

Throughout this section it is understood that κ does not exceed some fixed μ, and that any term involving an exceptional zero exists only if there exists such a zero.

2.3.1. Coefficients of $\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)$

First we use Perron's formula to evaluate ${\sum}'_{n\leq x} A_n(\kappa,\chi)$, where $A_n(\kappa,\chi)$ are the coefficients of the Dirichlet series of $\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)$ for $\sigma > 1$ and ${\sum}'$ denotes that the last term is to be halved in case x is an integer. This sum will be needed in

Chapter 3. Using Perron's formula [1, §17,Lemma] we have, when $\sigma_0 > 1$,

$$\Big|\sum_{n\le x}{}' A_n(\kappa,\chi) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds\Big|$$
$$< \sum_{\substack{n=1\\ n\ne x}}^{\infty} A_n(\kappa,\chi)\left(\frac{x}{n}\right)^{\sigma_0}\min\left(1, U^{-1}\big|\log\tfrac{x}{n}\big|^{-1}\right) +' c_0\, U^{-1}A_x(\kappa,\chi), \quad (2.19)$$

where c_0 is some constant and the $+'$ term is present only when x is an integer. Now

$$|A_n(\kappa,\chi)| = \left|\sum_{d|n}\Lambda(d)\chi(d)\left(\log\frac{n}{d}\right)^{\kappa}\right| \ll \Omega(n)(\log n)^{\kappa+1} \ll (\log n)^{\kappa+2}, \qquad (2.20)$$

since $\Omega(n)$, the number of prime factors of n counted with multiplicity, is $\ll \log n$. Also

$$\left|\log\frac{x}{n}\right| = \left|\log\frac{n}{x}\right| = \left|\log\left(1-\frac{x-n}{x}\right)\right| \asymp \left|\frac{x-n}{x}\right|$$

uniformly for $-1 \le \frac{x-n}{x} \le \frac{1}{2}$. We choose $\sigma_0 = 1 + (\log x)^{-1}$, assume that

$$q \le U \le x, \qquad (2.21)$$

and bound the series on the right hand side of (2.19).

For the terms with $n \le \frac{x}{2}$ or $n \ge 2x$, $|\log\frac{x}{n}|$ has a positive lower bound, so these terms contribute to the sum in the right-hand side of (2.19)

$$\ll \frac{x}{U}\sum_{n=1}^{\infty}\frac{|A_n(\kappa,\chi)|}{n^{\sigma_0}} \ll \frac{x}{U}\left|\frac{\zeta'}{\zeta}(\sigma_0)\zeta^{(\kappa)}(\sigma_0)\right| \ll \frac{x}{U}(\log x)^{\kappa+2}, \qquad (2.22)$$

since $\frac{\zeta'}{\zeta}(\sigma_0) \asymp \frac{1}{\sigma_0-1}$ and $\zeta^{(\kappa)}(\sigma_0) \asymp \frac{1}{(\sigma_0-1)^{\kappa+1}}$ by consideration of their Laurent expansions around $s = 1$.

Now we deal with the terms $\frac{x}{2} < n < 2x$. When n is closest to x, we take

$\min(1, U^{-1}|\log \frac{x}{n}|^{-1})$ to be 1, otherwise we use $U^{-1}|\log \frac{x}{n}|^{-1} \ll \frac{x}{U|x-n|}$. Then since $\left(\frac{x}{n}\right)^{\sigma_0} \ll 1$ in this range, the contribution of the terms under consideration is

$$\ll \frac{x}{U}(\log x)^{\kappa+2}\left(1+\sum_{n\le x}\frac{1}{n}\right) \ll \frac{x}{U}(\log x)^{\kappa+3}.$$

Combining these and noting that if x is an integer $A_x(\kappa,\chi) \ll (\log x)^{\kappa+2}$, we get

$$\left|\sum_{n\le x}{}' A_n(\kappa,\chi) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds\right| \ll \frac{x}{U}(\log x)^{\kappa+3}, \tag{2.23}$$

subject to $q \le U \le x$, with $\sigma_0 = 1 + (\log x)^{-1}$.

Now let C be the rectangle with vertices $\sigma_0 - iU$, $\sigma_0 + iU$, $\sigma_1 + iU$ and $\sigma_1 - iU$, where σ_1 is to be specified.

Case I. *There is no exceptional zero.* We take $\sigma_1 = 1 - \frac{c_1}{5\log qU}$, where c_1 is the constant in Theorem 11.3 of [2], so that the only pole of the integrand is that of $\zeta^{(\kappa)}(s)$ at $s = 1$ and the estimate $\frac{L'}{L}(s,\chi) \ll \log qU$ holds in C by (1.28). Then we have, by the theory of residues

$$\frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds = R_1(x,\kappa),$$

where $R_1(x,\kappa)$ is the residue of the integrand at $s = 1$.

Now, using $\frac{L'}{L}(s,\chi) \ll \log qU$ and $\zeta^{(\kappa)}(s) \ll U^{\frac{1}{2}(1-\sigma_1)+\varepsilon}$ [3, (20)],

$$\int_{\sigma_0+iU}^{\sigma_1+iU}\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds \ll \log qU \cdot U^{\frac{c_1}{10\log qU}+\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1) \ll \frac{x}{U^{1-\varepsilon}}.$$

Analogously we have

$$\int_{\sigma_1-iU}^{\sigma_0-iU}\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

Also, since $\zeta^{(\kappa)}(\sigma_1+it) \ll (\log qU)^{\kappa+1}$ for $|t|<1$ and $\zeta^{(\kappa)}(\sigma_1+it) \ll U^{\frac{c_1}{10\log qU}+\varepsilon} \ll U^{\varepsilon}$ for $|t|\geq 1$, we may use the bound $\zeta^{(\kappa)}(s) \ll U^{\varepsilon}$ for all t. We have

$$\begin{aligned}\int_{\sigma_1+iU}^{\sigma_1-iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds &\ll \log qU \cdot x^{\sigma_1}\int_{-U}^{U}\left|\frac{\zeta^{(\kappa)}(\sigma_1+it)}{\sigma_1+it}\right|dt \\ &\ll x\exp\left(\frac{-c_1\log x}{5\log qU}\right)U^{\varepsilon}\int_{-U}^{U}\frac{dt}{1+|t|} \\ &\ll xU^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right).\end{aligned}$$

Noting also that dropping the condition on the last term changes the sum an amount $\ll (\log x)^{\kappa+2}$, we obtain

$$\sum_{n\leq x} A_n(\kappa,\chi) = R_1(x,\kappa)+O\left(x\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U}+U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right). \quad (2.24)$$

Case II. *There is an exceptional zero β, and it satisfies $\beta \geq 1-c_1/(4\log qU)$.* In this case we take $\sigma_1 = 1-c_1/(3\log qU)$, so the integrand has another pole at β with residue $-\frac{x^{\beta}}{\beta}\zeta^{(\kappa)}(\beta)$. As before

$$\begin{aligned}\int_{\sigma_0+iU}^{\sigma_1+iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds &\ll \log qU\cdot U^{\frac{c}{6\log qU}+\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1) \\ &\ll \frac{x}{U^{1-\varepsilon}}\end{aligned}$$

and

$$\int_{\sigma_1-iU}^{\sigma_0-iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

For the vertical segment, this time the appeal is to (1.31) which gives $\frac{L'}{L}(\sigma_1+it) \ll$

$\log qU$ for small $|t|$ also in this case. Then the estimates proceed as before to yield

$$\begin{aligned}\sum_{n\le x} A_n(\kappa,\chi) &= R_1(x,\kappa) - \frac{x^\beta}{\beta}\zeta^{(\kappa)}(\beta) \\ &\quad + O\left(x\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^\varepsilon \exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.25}$$

Case III. *There is an exceptional zero β, but it satisfies $\beta < 1 - c_1/(4\log qU)$.* Then we proceed exactly as in Case I and obtain (2.24). But in this case we have $x^\beta\zeta^{(\kappa)}(\beta) \ll x\exp\left(\frac{-c_1\log x}{10\log U}\right)(\log qU)^{\kappa+2} \ll xU^\varepsilon\exp\left(\frac{-c_1\log x}{10\log U}\right)$, so we may pass to (2.25) within error. Hence (2.25) holds unconditionally if we agree that the term involving β is present only when the exceptional zero exists.

To compute $R_1(x,\kappa)$, we consider the series expansions of the factors of the integrand.

$$\begin{aligned} x^s &= x\sum_{j=0}^\infty \frac{(\log x)^j(s-1)^j}{j!}, \\ \frac{1}{s} &= \sum_{k=0}^\infty (-1)^k(s-1)^k, \\ \frac{L'}{L}(s,\chi) &= \sum_{i=0}^\infty \left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]\frac{(s-1)^i}{i!}, \\ \zeta^{(\kappa)}(s) &= \frac{(-1)^\kappa\kappa!}{(s-1)^{\kappa+1}} + \sum_{n=0}^\infty \frac{(-1)^{n+\kappa}}{n!}\gamma_{n+\kappa}(s-1)^n, \end{aligned}$$

where γ_i is the i-th Stieltjes constant. From this it is evident that R_1 is $(-1)^\kappa\kappa!$ times the coefficient of $(s-1)^\kappa$ in the expansion of the product $\frac{L'}{L}(s,\chi)\frac{x^s}{s}$, so

$$R_1(x,\kappa) = \sum_{i+j+k=\kappa}(-1)^{\kappa+k}\frac{\kappa!}{i!j!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x(\log x)^j.$$

We consider the sum $\sum_{n\le x}(\log n)^{\mu-\kappa}A_n(\kappa,\chi)$. Applying partial summation to (2.25)

we have

$$\sum_{n\le x}(\log n)^{\mu-\kappa}A_n(\kappa,\chi) = R_1(x,\kappa)(\log x)^{\mu-\kappa} - \frac{x^\beta}{\beta}\zeta^{(\kappa)}(\beta)(\log x)^{\mu-\kappa}$$
$$- (\mu-\kappa)\int_1^x \Big(\sum_{n\le t} A_n(\kappa,\chi)\Big)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \tag{2.26}$$
$$+ O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^\varepsilon \exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).$$

We want to replace the sum in the integrand with our approximation in (2.25) but it is valid only for $t \ge U$, so we split the integral at U.

$$\int_1^x \Big(\sum_{n\le t} A_n(\kappa,\chi)\Big)\frac{(\log t)^{\mu-\kappa-1}}{t}dt = \left(\int_1^U + \int_U^x\right)\Big(\sum_{n\le t} A_n(\kappa,\chi)\Big)\frac{(\log t)^{\mu-\kappa-1}}{t}dt. \tag{2.27}$$

Now (2.25) may be used for the second integral. We are interested in the error introduced in replacing the sum in the first integral. We bound the series trivially: since $A_n(\kappa,\chi) \ll (\log n)^{\kappa+2}$,

$$\int_1^U \left(\sum_{n\le t} A_n(\kappa,\chi)\right)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \ll U(\log U)^{\mu+1}.$$

This is the error made in omitting the first integral in (2.27). It is to be replaced by

$$\int_1^U R_1(t,\kappa)\frac{(\log t)^{\mu-\kappa-1}}{t} - \frac{1}{\beta}\zeta^{(\kappa)}(\beta)\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta}}$$
$$+ O\left((\log t)^{\mu-\kappa-1}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log t)^{\kappa+3}}{U} + U^\varepsilon \exp\left(\frac{-c_1\log t}{10\log U}\right)\right)\right)dt. \tag{2.28}$$

At this point it is appropriate to investigate the magnitude of $\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]$. If $L(s,\chi)$ has no exceptional zero, we have $\frac{L'}{L}(s,\chi) \ll \log q$ in the region $|1-s| \le \frac{c_1}{2\log 5q}$ by Theorem 1.3, otherwise if there is an exceptional zero β, we know that it satisfies $\beta < 1 - \frac{c}{q^{\frac12}(\log q)^2}$ for some constant c [1, §14,(12)], so in the region $|1-s| \le \frac{c_2}{2q^{\frac12}(\log q)^2}$ with c_2 small enough, we have $\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta} + O(\log q)$, where β is the exceptional zero by Theorem 1.3. But in this region $|s-\beta| \gg \frac{1}{q^{\frac12}(\log q)^2}$, so we have $\frac{L'}{L}(s,\chi) \ll q^{\frac12}(\log q)^2$.

Now we may derive a bound for higher derivatives of $\frac{L'}{L}(s,\chi)$ at $s=1$ using Cauchy's theorem as follows. First assume that there exists an exceptional zero. Let i be a given positive integer less than or equal to κ. For those s in the disk $|1-s| \le \frac{c_2(i-1)}{2(i+1)q^{\frac{1}{2}}(\log q)^2}$, we have

$$\left(\frac{d}{ds}\right)\frac{L'}{L}(s,\chi) = \frac{1}{2\pi i}\int_C \frac{(L'/L)(s,\chi)}{(\omega-s)^2}d\omega \ll q^{\frac{3}{2}}(\log q)^6.$$

Here C is the circle around s with radius $\frac{c_2}{2iq^{\frac{1}{2}}(\log q)^2}$. Proceeding in this manner, accumulating a factor of $q(\log q)^4$ at each step, we find

$$\left(\frac{d}{ds}\right)^i \frac{L'}{L}(s,\chi) \ll q^{i+\frac{1}{2}}(\log q)^{4i+2}.$$

Otherwise if there's no exceptional zero, we use disks with radii $\frac{c_1}{2(i+1)\log 5q}$, so accumulating factors of order $\log q$ and get

$$\left(\frac{d}{ds}\right)^i \frac{L'}{L}(s,\chi) \ll (\log q)^{i+1}.$$

Henceforth we denote by $E(i,\chi)$ the quantity $q^{i+\frac{1}{2}}(\log q)^{4i+2}$ if χ is exceptional and $(\log q)^{i+1}$ otherwise. From this it is clear that

$$R_1(t,\kappa) \ll \sum_{i=0}^{\kappa} t(\log t)^{\kappa-i}E(i,\chi),$$

whence

$$\begin{aligned}\int_1^U R_1(t,\kappa)\frac{(\log t)^{\mu-\kappa-1}}{t}dt &\ll \sum_{i=0}^{\kappa} U(\log U)^{\mu-i-1}E(i,\chi)\\ &\ll U(\log U)^{\mu-1}q^{\kappa+\frac{1}{2}}(\log q)^{4\kappa+2} \ll U^{\kappa+2}.\end{aligned}$$

Also

$$\int_1^U \zeta^{(\kappa)}(\beta)\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta}}dt \ll \zeta^{(\kappa)}(\beta)U^{\beta}(\log U)^{\mu-\kappa-1}$$
$$\ll U^{\beta}(\log U)^{\mu-\kappa-1}q^{\frac{1}{2}(\kappa+1)}(\log q)^{2\kappa+2} \ll U^{\kappa+1}$$

since $\zeta^{(\kappa)}(\beta) \ll q^{\frac{1}{2}(\kappa+1)}(\log q)^{2\kappa+2}$, and

$$\int_1^U (\log t)^{\mu-\kappa-1}\exp\left(\frac{-c_1\log t}{10\log U}\right)dt \ll U(\log U)^{\mu-\kappa-1}.$$

Thus (2.28) is

$$\ll U^{\kappa+2}.$$

Hence we may replace the integrand in (2.26) with the right hand side of (2.25).

$$\begin{aligned}\sum_{n\le x}(\log n)^{\mu-\kappa}A_n(\kappa,\chi) &= R_1(x,\kappa)(\log x)^{\mu-\kappa} - \frac{x^{\beta}}{\beta}\zeta^{(\kappa)}(\beta)(\log x)^{\mu-\kappa} \\ &- (\mu-\kappa)\int_1^x R_1(t,\kappa)\frac{(\log t)^{\mu-\kappa-1}}{t} - \frac{t^{\beta}}{\beta}\zeta^{(\kappa)}(\beta)\frac{(\log t)^{\mu-\kappa-1}}{t} \\ &+ O\left((\log t)^{\mu-\kappa-1}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log t)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_1\log t}{10\log U}\right)\right)\right)dt \\ &+ O\left(U^{\kappa+2}\right) \\ &+ O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.29}$$

Now repeated partial integration shows

$$\int_1^x (\log t)^{\nu}dt = x\sum_{j=0}^{\nu}\frac{\nu!}{(\nu-j)!}(-1)^j(\log x)^{\nu-j} + (-1)^{\nu+1}\nu!, \tag{2.30}$$

so we have

$$
\begin{aligned}
&(\mu-\kappa)\int_1^x R_1(t,\kappa)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \\
&\quad= \sum_{i+j+k=\kappa}(-1)^{\kappa+k}\frac{\kappa!}{i!j!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right](\mu-\kappa)\int_1^x(\log t)^{\mu-1-\kappa+j}dt \\
&\quad= \sum_{i+j+k=\kappa}\sum_{l=0}^{\mu-1-\kappa+j}(-1)^{\kappa+k+l}\frac{\kappa!}{i!j!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right] \\
&\qquad\cdot\left\{\frac{(\mu-\kappa)(\mu-1-\kappa+j)!}{(\mu-1-\kappa+j-l)!}x(\log x)^{\mu-1-\kappa+j-l}+O(1)\right\} \\
&\quad= \sum_{i+j+k=\kappa}\sum_{l=0}^{\mu-1-\kappa+j}(-1)^{\kappa+k+l}\frac{\kappa!}{i!j!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right] \\
&\qquad\cdot\frac{(\mu-\kappa)(\mu-1-\kappa+j)!}{(\mu-1-\kappa+j-l)!}x(\log x)^{\mu-1-\kappa+j-l}+O\left(E(\kappa,\chi)\right).
\end{aligned}
$$

Similarly as in (2.30),

$$
\int_1^x\frac{(\log t)^\nu}{t^{1-\beta}}dt=\sum_{j=0}^{\nu}\frac{(-1)^j\nu!}{(\nu-j)!}\frac{x^\beta}{\beta^{j+1}}(\log x)^{\nu-j}+\frac{(-1)^{\nu+1}\nu!}{\beta^{\nu+1}}, \tag{2.31}
$$

so

$$
\begin{aligned}
&(\mu-\kappa)\int_1^x\frac{t^\beta}{\beta}\zeta^{(\kappa)}(\beta)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \\
&\qquad= \sum_{j=0}^{\mu-\kappa-1}\frac{(\mu-\kappa)!}{(\mu-\kappa-1-j)!}\frac{(-1)^j\zeta^{(\kappa)}(\beta)}{\beta^{j+2}}x^\beta(\log x)^{\mu-\kappa-1-j} \\
&\qquad\quad+O(U^{\kappa+2}).
\end{aligned}
$$

As regards the integral of the error term, we note that

$$
\begin{aligned}
\int_1^x(\log t)^{\mu-\kappa-1}\exp\left(\frac{-c_1\log x}{10\log U}\right)dt &\le \int_0^{\log x}e^{u\left(1-\frac{c_3}{10\log U}\right)}u^{\mu-\kappa-1}du \\
&\le x(\log x)^{\mu-\kappa}\exp\left(\frac{-c_3\log x}{10\log U}\right),
\end{aligned} \tag{2.32}
$$

where we've put $c_3=\min(c_1,10)$, say, to ensure that the integrand is majorized by substituting the end point. So (2.29) becomes

$$\begin{aligned}\sum_{n\le x}(\log n)^{\mu-\kappa}A_n(\kappa,\chi) &= R_1(x,\kappa)(\log x)^{\mu-\kappa}-\frac{x^\beta}{\beta}\zeta^{(\kappa)}(\beta)(\log x)^{\mu-\kappa}\\ &-\sum_{i+j+k=\kappa}\sum_{l=0}^{\mu-1-\kappa+j}(-1)^{\kappa+k+l}\frac{\kappa!}{i!j!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]\\ &\cdot\frac{(\mu-\kappa)(\mu-1-\kappa+j)!}{(\mu-1-\kappa+j-l)!}x(\log x)^{\mu-1-\kappa+j-l}\\ &+\sum_{j=0}^{\mu-\kappa-1}\frac{(\mu-\kappa)!}{(\mu-\kappa-1-j)!}\frac{(-1)^j\zeta^{(\kappa)}(\beta)}{\beta^{j+2}}x^\beta(\log x)^{\mu-\kappa-1-j}\\ &+O\left(U^{\kappa+2}\right)\\ &+O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U}+U^\varepsilon\exp\left(\frac{-c_3\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.33}$$

We multiply this by $\binom{\mu}{\kappa}$ and sum over κ, for it is this sum that we shall need in the sequel, and by the nature of the cancellations to come the error terms are most effectively bounded if this summation is done before specifying U. The first term yields

$$\begin{aligned}&\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}R_1(x,\kappa)(\log x)^{\mu-\kappa}\\ &\quad=\sum_{\kappa=0}^{\mu}\sum_{i+j+k=\kappa}\frac{\mu!}{\kappa!(\mu-\kappa)!}\frac{\kappa!}{i!j!}(-1)^{\kappa+k}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x(\log x)^{\mu-\kappa+j}\\ &\quad=\sum_{n=0}^{\mu}\sum_{i=0}^{n}(-1)^{n-i}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x(\log x)^{\mu-n}\frac{\mu!}{i!(\mu-n)!}\\ &\qquad\cdot\sum_{\kappa=n}^{\mu}\frac{(\mu-n)!}{(\kappa-n)!(\mu-\kappa)!}(-1)^\kappa\\ &\quad=\sum_{n=0}^{\mu}\sum_{i=0}^{n}(-1)^{i}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x(\log x)^{\mu-n}\frac{\mu!}{i!(\mu-n)!}\\ &\qquad\cdot\sum_{\kappa=0}^{\mu-n}\frac{(\mu-n)!}{\kappa!(\mu-n-\kappa)!}(-1)^\kappa,\end{aligned}$$

where in rearranging the terms we've put $n=\kappa-j=i+k$. The inner sum vanishes unless $n = \mu$, in which case it is 1, so we get

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} R_1(x,\kappa)(\log x)^{\mu-\kappa} = \left(\sum_{i=0}^{\mu} (-1)^i \frac{\mu!}{i!} \left[\left(\frac{d}{ds}\right)^i \Big|_{s=1} \frac{L'}{L}(s,\chi) \right] \right) x.$$

Similarly,

$$\begin{aligned} & - \sum_{\kappa=0}^{\mu-1} \sum_{i+j+k=\kappa} \sum_{l=0}^{\mu-1-\kappa+j} \binom{\mu}{\kappa} (-1)^{\kappa+k+l} \frac{\kappa!}{i!j!} \left[\left(\frac{d}{ds}\right)^i \Big|_{s=1} \frac{L'}{L}(s,\chi) \right] \\ & \cdot \frac{(\mu-\kappa)(\mu-1-\kappa+j)!}{(\mu-1-\kappa+j-l)!} x (\log x)^{\mu-1-\kappa+j-l} \\ = & - \sum_{n=0}^{\mu-1} \sum_{i=0}^{n} \sum_{l=0}^{n-i} (-1)^{i+l} \left[\left(\frac{d}{ds}\right)^i \Big|_{s=1} \frac{L'}{L}(s,\chi) \right] x (\log x)^{\mu-1-n} \\ & \cdot \frac{\mu!}{i!(\mu-1-n)!} \sum_{\kappa=n-l}^{\mu-1} \frac{(\mu-1-n+l)!}{(\mu-1-\kappa)!(\kappa-n+l)!} (-1)^{\kappa-n+l}. \end{aligned}$$

This time we've put $\kappa - j + l = i + k + l = n$. Here we have summed up to $\mu - 1$, since for $\kappa = \mu$ the sum is zero because of the presence of the factor $(\mu - \kappa)$. Again, the innermost sum vanishes unless $n - l = \mu - 1$, which necessitates $n = \mu - 1$ and $l = 0$, whereupon we obtain

$$- \left(\sum_{i=0}^{\mu-1} (-1)^i \frac{\mu!}{i!} \left[\left(\frac{d}{ds}\right)^i \Big|_{s=1} \frac{L'}{L}(s,\chi) \right] \right) x.$$

We combine the foregoing computations with (2.33) and obtain

$$\begin{aligned} & \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{n \le x} (\log n)^{\mu-\kappa} A_n(\kappa,\chi) \\ & \quad = (-1)^{\mu} \left[\left(\frac{d}{ds}\right)^{\mu} \Big|_{s=1} \frac{L'}{L}(s,\chi) \right] x \\ & \quad - \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{j=0}^{\mu-\kappa} \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{(-1)^j \zeta^{(\kappa)}(\beta)}{\beta^{j+2}} x^{\beta} (\log x)^{\mu-\kappa-j} \\ & \quad + O\left(U^{\mu+2}\right) \\ & \quad + O\left(x(\log x)^{\mu} \left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^3}{U} + U^{\varepsilon} \exp\left(\frac{-c_3 \log x}{10 \log U} \right) \right) \right). \end{aligned}$$

We finally note that if we put, say, $U = \exp\left(\sqrt{c_3 \log x}\right)$ with ε sufficiently small, the error terms are $\ll x\exp(-C\sqrt{\log x})$ for some positive C. Thus we have obtained

2.7 Lemma. *Let $A_n(\kappa,\chi)$ be the coefficients of the Dirichlet series for $\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)$. There is a constant c such that for $q \le \exp\left(c\sqrt{\log x}\right)$ we have*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{n\le x}(\log n)^{\mu-\kappa}A_n(\kappa,\chi) = (-1)^{\mu}\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x$$
$$-\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{j=0}^{\mu-\kappa}\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{(-1)^j\zeta^{(\kappa)}(\beta)}{\beta^{j+2}}x^{\beta}(\log x)^{\mu-\kappa-j}$$
$$+O\left(x\exp(-C\sqrt{\log x})\right)$$

for some positive constant C. Here β is the possible exceptional zero of $L(s,\chi)$ and the terms involving it are present only when it exists.

2.3.2. Coefficients of $\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)$

Another estimate which we shall need in Chapters 4 and 5 is the following

2.8 Lemma. *There is a positive constant c such that if χ_1 and χ_2 are non-principal characters to arbitrary moduli q_1 and q_2 respectively, with $q_1, q_2 \le \exp\left(c\sqrt{\log x}\right)$ and $B_n(\kappa,\chi_1,\chi_2)$ is the n-th coefficient in the Dirichlet series for $\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)$, there holds*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{n\le x}(\log n)^{\mu-\kappa}B_n(\kappa,\chi_1,\chi_2)$$
$$= -\sum_{\kappa=0}^{\mu}\sum_{j=0}^{\mu-\kappa}L^{(\kappa)}(\beta_1,\chi_2)(-1)^j\frac{\mu!}{\kappa!(\mu-\kappa-j)!}\frac{x^{\beta_1}}{\beta_1^{j+1}}(\log x)^{\mu-\kappa-j}$$
$$+O\left(x\exp\left(-C\sqrt{\log x}\right)\right),$$

where C is some positive constant. Here β_1 is a possible exceptional zero of $L(s,\chi_1)$ and it is understood that the sum involving it is present only when such a zero exists.

Proof. Let $B_n(\kappa,\chi_1,\chi_2)$ be the coefficients of the Dirichlet series for $\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)$

for $\sigma > 1$. We put $q = \max(q_1, q_2)$. As in (2.19) and (2.20), we have

$$\Big|\sum_{n\le x}{}' B_n(\kappa,\chi_1,\chi_2) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds\Big|$$
$$< \sum_{\substack{n=1\\ n\neq x}}^{\infty} B_n(\kappa,\chi_1,\chi_2)\left(\frac{x}{n}\right)^{\sigma_0}\min\left(1, U^{-1}\big|\log\tfrac{x}{n}\big|^{-1}\right) +' c_0\, U^{-1}B_x(\kappa,\chi_1,\chi_2) \quad (2.34)$$

for $\sigma_0 > 1$, and

$$|B_n(\kappa,\chi_1,\chi_2)| = \left|\sum_{d|n}\chi_1(d)\Lambda(d)\chi_2\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa}\right| \ll \Omega(n)(\log n)^{\kappa+1} \ll (\log n)^{\kappa+2}.$$

We again choose $\sigma_0 = 1 + (\log x)^{-1}$, assume that $q_1, q_2 \le U \le x$, and proceed as before. The contribution of the terms with $n \le \frac{x}{2}$ or $n \ge 2x$ to the sum on the right-hand side of (2.34) is

$$\ll \frac{x}{U}\sum_{n=1}^{\infty}\frac{|B_n(\kappa,\chi_1,\chi_2)|}{n^{\sigma_0}} \ll \frac{x}{U}\left|\frac{\zeta'}{\zeta}(\sigma_0)\zeta^{(\kappa)}(\sigma_0)\right| \ll \frac{x}{U}(\log x)^{\kappa+2},$$

as in (2.22). To deal with the terms $\frac{x}{2} < n < 2x$, again when n is closest to x we take $\min\left(1, U^{-1}\big|\log\frac{x}{n}\big|^{-1}\right)$ to be 1, and use $U^{-1}|\log\frac{x}{n}|^{-1} \ll \frac{x}{U|x-n|}$ otherwise. So the terms under consideration contribute

$$\ll (\log x)^{\kappa+2}\frac{x}{U}\left(1 + \sum_{n\le x}\frac{1}{n}\right) \ll \frac{x}{U}(\log x)^{\kappa+3}.$$

Combining these, we get

$$\Big|\sum_{n\le x}{}' B_n(\kappa,\chi_1,\chi_2) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds\Big| \ll x\frac{(\log x)^{\kappa+3}}{U},$$

when $U \le x$, with $\sigma_0 = 1 + (\log x)^{-1}$.

Now let C be the rectangle with vertices $\sigma_0 - iU$, $\sigma_0 + iU$, $\sigma_1 + iU$ and $\sigma_1 - iU$.

Case I. $L(s,\chi_1)$ *has no exceptional zero.* We take $\sigma_1 = 1 - \frac{c_1}{5\log qU}$, where c_1 is the constant such that $L(s,\chi_1) \neq 0$ for $\sigma > 1 - \frac{c_1}{\log q_1 U}$. So again the integrand is analytic on and inside C and we have the bound $\frac{L'}{L}(s,\chi_1) \ll \log q_1 U$. Then we have, by the theory of residues

$$\frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds = 0,$$

Now, using Lemma 2.6 for $L^{(\kappa)}(s,\chi_2)$,

$$\begin{aligned}\int_{\sigma_0+iU}^{\sigma_1+iU} \frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds &\ll \log q_1U \cdot (q_2U)^{\frac{c_1}{10\log qU}+\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1)\\ &\ll \frac{x}{U^{1-\varepsilon}}.\end{aligned}$$

Analogously we have

$$\int_{\sigma_1-iU}^{\sigma_0-iU} \frac{L'}{L}(s,\chi)L^{(\kappa)}(s,\chi)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

Also, since as above $\frac{L'}{L}(\sigma_1+it,\chi_1)L^{(\kappa)}(\sigma_1+it,\chi_2) \ll (q_2U)^{\varepsilon}$, we have

$$\begin{aligned}\int_{\sigma_1+iU}^{\sigma_1-iU} \frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds &\ll U^{\varepsilon}\cdot x^{\sigma_1}\int_{-U}^{U}\left|\frac{1}{\sigma_1+it}\right|dt\\ &\ll xU^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right).\end{aligned}$$

Combining our estimates, and dropping the condition on the last term as before, we obtain

$$\sum_{n\le x} B_n(\kappa,\chi_1,\chi_2) = O\left(x\left(\frac{(\log x)^{\kappa+3}}{U} + \frac{1}{U^{1-\varepsilon}} + U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right). \qquad (2.35)$$

Case II. *There is an exceptional zero β_1, and it satisfies $\beta_1 \ge 1 - c_1/(4\log qU)$.* In this case we take $\sigma_1 = 1 - c_1/(3\log qU)$, so the integrand has another pole at β_1

with residue $-\frac{x^{\beta_1}}{\beta_1}L^{(\kappa)}(\beta_1,\chi_2)$. As before

$$\int_{\sigma_0+iU}^{\sigma_1+iU}\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}},$$

and

$$\int_{\sigma_1-iU}^{\sigma_0-iU}\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_2)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

For the vertical segment, when $|s-\beta|\geq 1/\log q$, we use (1.28) as before and for small $|t|$ this time we use (1.31) which gives $\frac{L'}{L}(\sigma_1+it,\chi_1)\ll \log qU$ also in this case. Then the estimates proceed exactly as before to yield

$$\begin{aligned}\sum_{n\leq x}B_n(\kappa,\chi_1,\chi_2) = &-\frac{x^{\beta_1}}{\beta_1}L^{(\kappa)}(\beta_1,\chi_2)\\ &+O\left(x\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.36}$$

Case III. *There is an exceptional zero β_1, but it satisfies $\beta_1<1-c_1/(4\log qU)$.* In this case we proceed exactly as in Case 1 and obtain (2.35). Since

$$L^{(\kappa)}(\beta_1,\chi_2)\ll q_2^{\frac{1}{2}(1-\beta_1)+\varepsilon}\ll q_2^{\frac{1}{2}} \tag{2.37}$$

by Lemma 2.6, we have $x^{\beta_1}L^{(\kappa)}(\beta_1,\chi_2)\ll xq_2^{\frac{1}{2}}\exp\left(\frac{-c_1\log x}{10\log U}\right)$. Hence, majorizing the last error terms in the previous bounds by a factor of $q_2^{\frac{1}{2}}$, we may unconditionally use

$$\begin{aligned}\sum_{n\leq x}B_n(\kappa,\chi_1,\chi_2) = &-\frac{x^{\beta_1}}{\beta_1}L^{(\kappa)}(\beta_1,\chi_2)\\ &+O\left(x\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.38}$$

Now we sum by parts and pass to

$$\sum_{n\le x}(\log n)^{\mu-\kappa}B_n(\kappa,\chi_1,\chi_2) = -\frac{x^{\beta_1}}{\beta_1}(\log x)^{\mu-\kappa}L^{(\kappa)}(\beta_1,\chi_2)$$
$$-(\mu-\kappa)\int_1^x\Big(\sum_{n\le t}B_n(\kappa,\chi_1,\chi_2)\Big)\frac{(\log t)^{\mu-\kappa-1}}{t}dt$$
$$+O\left(x(\log x)^{\mu-\kappa}\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).$$

As in the previous lemma, with a view to replacing the sum in the integrand with (2.38) we observe that

$$\int_1^U\Big(\sum_{n\le t}B_n(\kappa,\chi_1,\chi_2)\Big)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \ll U(\log U)^{\mu+1},$$

and that

$$L^{(\kappa)}(\beta_1,\chi_2)\int_1^U\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta_1}}dt \ll q_2^{\frac{1}{2}}U^{\beta_1}(\log U)^{\mu-\kappa-1},$$

using (2.37) for $L^{(\kappa)}(\beta_1,\chi_2)$. Also

$$\int_1^U(\log t)^{\mu-\kappa-1}\left(\frac{(\log t)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_1\log t}{10\log U}\right)\right)dt \ll q_2U^{1+\varepsilon}(\log U)^{\mu+2}.$$

These upper bounds, when lumped together in an error term of $O(U^2)$, justify substituting our estimate for the sum in the integral.

$$\sum_{n\le x}(\log n)^{\mu-\kappa}B_n(\kappa,\chi_1,\chi_2) = -\frac{x^{\beta_1}}{\beta_1}(\log x)^{\mu-\kappa}L^{(\kappa)}(\beta_1,\chi_2)$$
$$+\frac{L^{(\kappa)}(\beta_1,\chi_2)}{\beta_1}(\mu-\kappa)\int_1^x\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta_1}}dt$$
$$+O\left(\int_1^x(\log t)^{\mu-\kappa-1}\left(\frac{(\log t)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_1\log t}{10\log U}\right)\right)dt\right)$$
$$+O\left(U^2\right)+O\left(x(\log x)^{\mu-\kappa}\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).$$

The integral in the error is handled as in (2.32), and contributes

$$\ll x(\log x)^{\mu-\kappa}\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_3\log x}{10\log U}\right)\right).$$

We also note that

$$\begin{aligned}
&-\frac{x^{\beta_1}}{\beta_1}(\log x)^{\mu-\kappa}L^{(\kappa)}(\beta_1,\chi_2)+\frac{L^{(\kappa)}(\beta_1,\chi_2)}{\beta_1}(\mu-\kappa)\int_1^x\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta_1}}dt\\
&=-\frac{x^{\beta_1}}{\beta_1}(\log x)^{\mu-\kappa}L^{(\kappa)}(\beta_1,\chi_2)\\
&\quad+\frac{L^{(\kappa)}(\beta_1,\chi_2)}{\beta_1}(\mu-\kappa)\left(\sum_{j=0}^{\mu-\kappa-1}(-1)^j\frac{(\mu-\kappa-1)!}{(\mu-\kappa-1-j)!}\frac{x^{\beta_1}}{\beta_1^{j+1}}(\log x)^{\mu-\kappa-1-j}+O(1)\right)\\
&=-L^{(\kappa)}(\beta_1,\chi_2)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{x^{\beta_1}}{\beta_1^{j+1}}(\log x)^{\mu-\kappa-j}+O\big(q_2^{\frac{1}{2}}\big),
\end{aligned}$$

whence

$$\begin{aligned}
\sum_{n\le x}(\log n)^{\mu-\kappa}B_n(\kappa,\chi_1,\chi_2)&=-L^{(\kappa)}(\beta_1,\chi_2)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{x^{\beta_1}}{\beta_1^{j+1}}(\log x)^{\mu-\kappa-j}\\
&\quad+O\left(U^2\right)\quad+O\left(x(\log x)^{\mu-\kappa}\left(\frac{(\log x)^{\kappa+3}}{U}+\frac{1}{U^{1-\varepsilon}}+q_2^{\frac{1}{2}}U^{\varepsilon}\exp\left(\frac{-c_2\log x}{10\log U}\right)\right)\right).
\end{aligned}$$

We multiply this with $\binom{\mu}{\kappa}$ and sum over κ, and note that putting $U=\exp\left(\frac{1}{10}\sqrt{c_2\log x}\right)$ and imposing $q_2\le U$ make the error terms $\ll x\exp\left(-C\sqrt{\log x}\right)$ for some positive C, and obtain our result. □

2.3.3. Coefficients of $\frac{L'}{L}(s,\chi)L^{(\kappa)}(s,\psi_0)$

Now we consider the the sum ${\sum}'_{n\le x}C_n(\kappa,\chi,q_2)$ of the coefficients of the Dirichlet series of $\frac{L'}{L}(s,\chi)L^{(\kappa)}(s,\psi_0)$, where χ is a non-principal character to the modulus q_1 and ψ_0 is the principal character to the modulus q_2. This sum will be needed in Chapter 5. We have

$$L(s,\psi_0)=\zeta(s)\prod_{p|q_2}(1-p^{-s}),$$

so that

$$L^{(\kappa)}(s,\psi_0) = \sum_{\nu=0}^{\kappa} \binom{\kappa}{\nu} \zeta^{(\kappa-\nu)}(s) \left(\prod_{p|q_2} (1-p^{-s}) \right)^{(\nu)}.$$

Therefore we may write

$$\sum_{n\le x}{}' C_n(\kappa,\chi,q_2) = \sum_{\nu=0}^{\kappa} \binom{\kappa}{\nu} \sum_{n\le x}{}' C_n^{(\nu)}(\kappa,\chi,q_2),$$

where $C_n^{(\nu)}(\kappa,\chi,q_2)$ is the n-th coefficient in the Dirichlet series for

$$\frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s) \left(\prod_{p|q_2} (1-p^{-s}) \right)^{(\nu)}.$$

We compute $\sum_{n\le x}' C_n^{(\nu)}(\kappa,\chi,q_2)$ as in the preceding lemmata. Since

$$\left(\prod_{p|q_2} (1-p^{-s}) \right)^{(\nu)} = \sum_{d|q_2} \frac{\mu(d)\,(-\log d)^{\nu}}{d^s},$$

we have

$$\begin{aligned} |C_n^{(\nu)}(\kappa,\chi,q_2)| &= \Big| \sum_{\substack{klm=n \\ m|q_2}} \Lambda(k)\chi(k)(\log l)^{\kappa-\nu}\mu(m)(\log m)^{\nu} \Big| \\ &\ll \Omega(n)(\log n)^{\kappa-\nu+1} d(q_2)(\log q_2)^{\nu} \\ &\ll (\log n)^{\kappa-\nu+2} q_2^{\varepsilon}, \end{aligned}$$

since $d(q_2) \ll q_2^{\varepsilon}$. Again,

$$\left|\sum_{n\le x}{}' C_n^{(\nu)}(\kappa,\chi,q_2) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\left(\prod_{p|q_2}(1-p^{-s})\right)^{(\nu)}\frac{x^s}{s}ds\right|$$
$$< \sum_{\substack{n=1\\ n\neq x}}^{\infty} C_n^{(\nu)}(\kappa,\chi,q_2)\left(\frac{x}{n}\right)^{\sigma_0}\min\left(1, U^{-1}\left|\log\tfrac{x}{n}\right|^{-1}\right) +' c_0\, U^{-1}C_x^{(\nu)}(\kappa,\chi,q_2), \quad (2.39)$$

with $\sigma_0 = 1 + (\log x)^{-1}$. We assume that $q_1, q_2 \le U \le x$.

The terms with $n \le \frac{x}{2}$ or $n \ge 2x$ contribute

$$\ll \frac{x}{U}\sum_{n=1}^{\infty}\frac{|C_n^{(\nu)}(\kappa,\chi,q_2)|}{n^{\sigma_0}} \ll \frac{x}{U}\left|\frac{\zeta'}{\zeta}(\sigma_0)\zeta^{(\kappa-\nu)}(\sigma_0)\sum_{d|q_2}\frac{|\mu(d)|(\log d)^{\nu}}{d^{\sigma_0}}\right|$$
$$\ll \frac{x}{U}(\log x)^{\kappa-\nu+2}q_2^{\varepsilon}. \quad (2.40)$$

For the terms $\frac{x}{2} < n < 2x$, as usual when n is closest to x, we take $\min(1, U^{-1}|\log\frac{x}{n}|^{-1})$ to be 1 and otherwise use $U^{-1}|\log\frac{x}{n}|^{-1} \ll \frac{x}{U|x-n|}$. Then the contribution of these terms is

$$\ll \frac{x}{U}(\log x)^{\kappa-\nu+2}d(q_2)(\log q_2)^{\nu}\left(1+\sum_{n\le x}\frac{1}{n}\right) \ll \frac{x}{U}(\log x)^{\kappa-\nu+3}q_2^{\varepsilon}.$$

Combining these, we get

$$\left|\sum_{n\le x}{}' C_n^{(\nu)}(\kappa,\chi,q_2) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds\right|$$
$$\ll \frac{x}{U}(\log x)^{\kappa-\nu+3}q_2^{\varepsilon}, \quad (2.41)$$

subject to $U \le x$, with $\sigma_0 = 1 + (\log x)^{-1}$.

Now let C be the rectangle with vertices $\sigma_0 - iU$, $\sigma_0 + iU$, $\sigma_1 + iU$ and $\sigma_1 - iU$.

Case I. *There is no exceptional zero.* As usual we take $\sigma_1 = 1 - \frac{c_1}{5\log q_1 U}$, so that the only pole of the integrand is that of $\zeta^{(\kappa-\nu)}(s)$ at $s=1$ and the estimate (1.28) holds. Then we have, by the theory of residues

$$\frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds = R_3(\kappa,\nu),$$

where $R_3(\kappa,\nu)$ is the residue of the integrand at $s=1$.

As has become habitual, using $\frac{L'}{L}(s,\chi) \ll \log q_1 U$ and $\zeta^{(\kappa-\nu)}(s) \ll U^{\frac{c_1}{10\log q_1 U}+\varepsilon} \ll U^\varepsilon$,

$$\begin{aligned}\int_{\sigma_0+iU}^{\sigma_1+iU} &\frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds \\ &\ll \log q_1 U\cdot U^{\frac{c_1}{10\log q_1 U}+\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1)d(q_2)(\log q_2)^\nu \\ &\ll \frac{x}{U^{1-\varepsilon}}.\end{aligned}$$

Analogously we have

$$\int_{\sigma_1-iU}^{\sigma_0-iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

Since we have $\zeta^{(\kappa-\nu)}(\sigma_1+it) \ll (\log q_1 U)^{\kappa-\nu+1} \ll U^\varepsilon$ for $|t|<1$ also, we may use that bound for the whole vertical contour.

$$\begin{aligned}\int_{\sigma_1+iU}^{\sigma_1-iU} &\frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds \\ &\ll \log q_1 U\cdot x^{\sigma_1}q_2^\varepsilon\int_{-U}^{U}\left|\frac{\zeta^{(\kappa-\nu)}(\sigma_1+it)}{\sigma_1+it}\right|dt \\ &\ll x(\log q_1 U)q_2^\varepsilon U^\varepsilon \exp\left(\frac{-c_1\log x}{5\log q_1 U}\right)\int_{-U}^{U}\frac{dt}{1+|t|} \\ &\ll xU^\varepsilon\exp\left(\frac{-c_1\log x}{10\log U}\right).\end{aligned}$$

Combining our estimates, and ridding the sum of the constraint of halving the last term for integral x as usual, we obtain

$$\sum_{n\le x} C_n^{(\nu)} = R_3(\kappa,\nu) + O\left(x\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa-\nu+3}}{U^{1-\varepsilon}} + U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right). \quad (2.42)$$

Case II. *There is an exceptional zero β_1, and it satisfies $\beta_1 \ge 1 - c_1/(4\log q_1 U)$.* In this case we take $\sigma_1 = 1 - c_1/(3\log q_1 U)$, so the integrand has another pole at β_1 with residue $-\frac{x^{\beta_1}}{\beta_1}\zeta^{(\kappa-\nu)}(\beta_1)\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}$. As before

$$\int_{\sigma_0+iU}^{\sigma_1+iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds \ll \log q_1 U \cdot U^{\frac{c_1}{6\log q_1 U}+\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1)q_2^{\varepsilon}$$
$$\ll \frac{x}{U^{1-\varepsilon}}$$

and

$$\int_{\sigma_1-iU}^{\sigma_0-iU} \frac{L'}{L}(s,\chi)\zeta^{(\kappa-\nu)}(s)\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)}\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

For the vertical segment, (1.31) gives $\frac{L'}{L}(\sigma_1+it) \ll \log q_1 U$ for small $|t|$ in this case also. Then the estimates proceed as before to yield

$$\begin{aligned}\sum_{n\le x} C_n^{(\nu)}(\kappa,\chi,q_2) = R_3(\kappa,j) - \frac{x^{\beta_1}}{\beta_1}\zeta^{(\kappa-\nu)}(\beta_1)\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)} \\ + O\left(x\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa-\nu+3}}{U^{1-\varepsilon}} + U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).\end{aligned} \quad (2.43)$$

Case III. *There is an exceptional zero β_1, but it satisfies $\beta_1 < 1 - c_1/(4\log q_1 U)$.* This case proceeds as in Case I and we obtain (2.42). To pass to (2.43) we note that

$$x^{\beta_1}\zeta^{(\kappa-\nu)}(\beta_1) \ll xU^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right).$$

Hence (2.43) holds unconditionally.

Now since

$$\Big(\prod_{p|q_2}(1-p^{-s})\Big)^{(\nu)} = \sum_{d|q_2}\mu(d)(-\log d)^{\nu} d^{-s} = \sum_{r=0}^{\infty}\frac{1}{r!}\Big(\sum_{d|q_2}\frac{\mu(d)(-\log d)^{\nu+r}}{d}\Big)(s-1)^r,$$

we have

$$R_3(\kappa,\nu) = \sum_{i+j+k+r=\kappa-\nu}\frac{(-1)^{\kappa-\nu+k}(\kappa-\nu)!\, x(\log x)^j}{i!\; j!\; r!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right] S(\nu+r),$$

where $S(\nu+r)$ denotes $\sum_{d|q_2}\mu(d)(-\log d)^{\nu+r}d^{-1}$. Combining sums of $C_n^{(\nu)}(\kappa,\chi,q_2)$, we obtain

$$\begin{aligned}
\sum_{n\le x} C_n(\kappa,\chi,q_2) &= \sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\sum_{n\le x} C_n^{(\nu)}(\kappa,\chi,q_2)\\
&= \sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{\kappa!}{i!j!r!\nu!}x(\log x)^j\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right] S(\nu+r)\\
&\quad -\sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\frac{x^{\beta_1}}{\beta_1}\zeta^{(\kappa-\nu)}(\beta_1)\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}\\
&\quad + O\left(x\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U^{1-\varepsilon}}+U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).
\end{aligned} \tag{2.44}$$

Summing by parts we have

$$\begin{aligned}
&\sum_{n\le x}(\log n)^{\mu-\kappa}C_n(\kappa,\chi,q_2)\\
&= \sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{\kappa!}{i!j!r!\nu!}x(\log x)^{\mu-\kappa+j}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right] S(\nu+r)\\
&\quad -\sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1}x^{\beta_1}(\log x)^{\mu-\kappa}\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}\\
&\quad -(\mu-\kappa)\int_1^x\left(\sum_{n\le t}C_n(\kappa,\chi,q_2)\right)\frac{(\log t)^{\mu-\kappa-1}}{t}dt\\
&\quad + O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U^{1-\varepsilon}}+U^{\varepsilon}\exp\left(\frac{-c_1\log x}{10\log U}\right)\right)\right).
\end{aligned}$$

As before we want to replace the integrand with the right hand side of (2.44). We have

$$\int_1^U \left(\sum_{n\le t} C_n(\kappa,\chi,q_2)\right)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \ll U(\log U)^{\mu+1}q_2^{\varepsilon} \ll U^{1+\varepsilon}, \tag{2.45}$$

since $C_n(\kappa,\chi,q_2)=\sum\binom{\kappa}{\nu}C_n^{(\nu)}(\kappa,\chi,q_2)(n)\ll(\log n)^{\kappa+2}q_2^{\varepsilon}$. Also,

$$\begin{aligned}\int_1^U \sum_{i+j+k+r+\nu=\kappa}&(-1)^{\kappa-\nu+k}\frac{\kappa!}{i!j!r!\nu!}(\log t)^{\mu-\kappa-1+j}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\nu+r)dt\\ &\ll q_2^{\varepsilon}\sum_{j\le\kappa}E(i,\chi)U(\log U)^{\mu-\kappa-1+j}\ll q_2^{\varepsilon}q_1^{\kappa+1}(\log q_1)^{4\kappa+2}U(\log U)^{\mu}\ll U^{\kappa+3}\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}\int_1^U\sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\frac{1}{\beta_1}\zeta^{(\kappa-\nu)}(\beta_1)&\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta_1}}dt\\ &\ll q_2^{\varepsilon}q_1^{\frac{1}{2}(\kappa+1)}(\log q_1)^{2\kappa+2}U^{\beta_1}(\log U)^{\mu-\kappa-1}\ll U^{\kappa+2}.\end{aligned}$$

The integral up to U of the error term in (2.44) is handled as in (2.32), so we get

$$\begin{aligned}&\sum_{n\le x}(\log n)^{\mu-\kappa}C_n(\kappa,\chi,q_2)\\ &=\sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{\kappa!}{i!j!r!\nu!}x(\log x)^{\mu-\kappa+j}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\nu+r)\\ &\quad-\sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1}x^{\beta_1}(\log x)^{\mu-\kappa}\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}\\ &\quad-\sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{(\mu-\kappa)\kappa!}{i!j!r!\nu!}\left[\left(\frac{d}{ds}\right)^i\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\nu+r)\int_1^x(\log t)^{\mu-\kappa-1+j}dt\\ &\quad+(\mu-\kappa)\sum_{\nu=0}^{\kappa}\binom{\kappa}{\nu}\frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1}\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}\int_1^x\frac{(\log t)^{\mu-\kappa-1}}{t^{1-\beta_1}}dt\\ &\quad+O(U^{\kappa+3})+O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U^{1-\varepsilon}}+U^{\varepsilon}\exp\left(\frac{-c_3\log x}{10\log U}\right)\right)\right).\end{aligned} \tag{2.47}$$

Next, we multiply this equation by $\binom{\mu}{\kappa}$, and sum over κ. The first term then becomes

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{\kappa!}{i!j!r!\nu!}x(\log x)^{\mu-\kappa+j}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\nu+r)$$
$$=\sum_{\substack{m=0\\ i+k+r+\nu=m}}^{\mu}(-1)^{m+\nu+k}\frac{\mu!}{i!r!\nu!(\mu-m)!}x(\log x)^{\mu-m}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\nu+r)$$
$$\cdot\sum_{\kappa=m}^{\mu}\frac{(\mu-m)!}{(\mu-\kappa)!\,(\kappa-m)!}(-1)^{\kappa-m}$$
$$=(-1)^{\mu}x\sum_{\substack{n=0\\ i+k=n}}^{\mu}(-1)^k\frac{\mu!}{i!(\mu-n)!}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]S(\mu-n)\sum_{\nu=0}^{\mu-n}\frac{(\mu-n)!}{\nu!(\mu-n-\nu)!}(-1)^{\nu}$$
$$=S(0)\left(\sum_{i=0}^{\mu}(-1)^i\frac{\mu!}{i!}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]\right)x.$$

Here in the second equality we have observed that the innermost sum is zero unless $m=\mu$, and similarly in the last equality unless $n=\mu$. The third term, using (2.30) for the integral, gives

$$-\sum_{\kappa=0}^{\mu-1}\binom{\mu}{\kappa}\sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{(\mu-\kappa)\kappa!}{i!j!r!\nu!}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]$$
$$\cdot S(\nu+r)\int_1^x(\log t)^{\mu-\kappa-1+j}dt$$
$$=-\sum_{\kappa=0}^{\mu-1}\binom{\mu}{\kappa}\sum_{i+j+k+r+\nu=\kappa}(-1)^{\kappa-\nu+k}\frac{(\mu-\kappa)\kappa!}{i!j!r!\nu!}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]$$
$$\cdot S(\nu+r)\left\{\sum_{l=0}^{\mu-1-\kappa+j}(-1)^l\frac{(\mu-1-\kappa+j)!}{(\mu-1-\kappa+j-l)!}x(\log x)^{\mu-1-\kappa+j-l}+O(1)\right\}$$
$$=-\sum_{m=0}^{\mu-1}x(\log x)^{\mu-1-m}\sum_{i+k+r+\nu+l=m}(-1)^{\nu+k+m}\frac{\mu!}{i!r!\nu!(\mu-1-m)!}\left[\left(\frac{d}{ds}\right)^i\Big|_{s=1}\frac{L'}{L}(s,\chi)\right]$$
$$\cdot S(\nu+r)\sum_{\kappa=m-l}^{\mu-1}\frac{(\mu-1-m+l)!}{(\mu-1-\kappa)!(\kappa+l-m)!}(-1)^{\kappa+l-m}+O\left(q_2^{\varepsilon}E(\mu-1,\chi)\right).$$

We have summed over κ up to $\mu-1$ since when $\kappa=\mu$, the summands vanish by the presence of the factor $(\mu-\kappa)$. The sum over κ vanishes unless $m-l=\mu-1$, which implies $m = \mu - 1$ and $l = 0$. So the sixfold sum collapses to

$$
\begin{aligned}
&(-1)^\mu x \sum_{i+k+r+\nu=\mu-1} (-1)^{\nu+k} \frac{\mu!}{i!r!\nu!} \left[\left(\frac{d}{ds} \right)^i \Bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] S(\nu+r) \\
&= (-1)^\mu x \sum_{i+k=0}^{\mu-1} (-1)^k \frac{\mu!}{i!(\mu-1-i-k)!} \left[\left(\frac{d}{ds} \right)^i \Bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] S(\mu-1-i-k) \\
&\cdot \sum_{\nu=0}^{\mu-1-i-k} \frac{(\mu-1-i-k)!}{\nu!(\mu-1-i-k-\nu)!} (-1)^\nu \\
&= (-1)^\mu x \sum_{i=0}^{\mu-1} (-1)^{\mu-1-i} \frac{\mu!}{i!} \left[\left(\frac{d}{ds} \right)^i \Bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] S(0) \\
&= -S(0) \left(\sum_{i=0}^{\mu-1} (-1)^i \frac{\mu!}{i!} \left[\left(\frac{d}{ds} \right)^i \Bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] \right) x.
\end{aligned}
$$

We note also that

$$
\begin{aligned}
&-x^{\beta_1} (\log x)^{\mu-\kappa} + (\mu-\kappa) \int_1^x \frac{(\log t)^{\mu-1-\kappa}}{t^{1-\beta_1}} dt \\
&= -x^{\beta_1} (\log x)^{\mu-\kappa} + (\mu-\kappa) \sum_{j=0}^{\mu-1-\kappa} (-1)^j \frac{(\mu-1-\kappa)!}{(\mu-1-\kappa-j)!} \frac{x^{\beta_1}}{\beta_1^{j+1}} (\log x)^{\mu-1-\kappa-j} + O(1) \\
&= -x^{\beta_1} (\log x)^{\mu-\kappa} - \sum_{j=1}^{\mu-\kappa} (-1)^j \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{x^{\beta_1}}{\beta_1^{j}} (\log x)^{\mu-\kappa-j} + O(1) \\
&= - \sum_{j=0}^{\mu-\kappa} (-1)^j \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{x^{\beta_1}}{\beta_1^{j}} (\log x)^{\mu-\kappa-j} + O(1),
\end{aligned}
$$

and so (2.47), when summed over κ gives

$$
\begin{aligned}
&\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{n \le x} (\log n)^{\mu-\kappa} C_n(\kappa,\chi,q_2) = S(0)(-1)^\mu \left[\left(\frac{d}{ds} \right)^\mu \Bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] x \\
&- \sum_{\kappa=0}^{\mu} \sum_{\nu=0}^{\kappa} \sum_{j=0}^{\mu-\kappa} \binom{\mu}{\kappa} \binom{\kappa}{\nu} \Big(\prod_{p|q_2} (1-p^{-\beta_1}) \Big)^{(\nu)} (-1)^j \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1^{j+1}} x^{\beta_1} (\log x)^{\mu-\kappa-j} \\
&+ O(U^{\mu+3}) + O\left(x (\log x)^\mu \left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^3}{U^{1-\varepsilon}} + U^\varepsilon \exp\left(\frac{-c_3 \log x}{10 \log U} \right) \right) \right).
\end{aligned}
$$

To conclude, we note that $S(0) = \phi(q_2)/q_2$ and that when we put $U = \exp\left(\frac{1}{10}\sqrt{c_3 \log x}\right)$ the error terms are $\ll x \exp\left(-C\sqrt{\log x}\right)$ for some positive C, to obtain

2.9 Lemma. *Let χ be a non-principal character to the modulus q_1 and ψ_0 be the principle character to the modulus q_2, and let $C_n(\kappa, \chi, q_2)$ be the n-th coefficient of the Dirichlet series for $\frac{L'}{L}(s,\chi)L^{(\kappa)}(s,\psi_0)$. There is a positive constant c such that for $q_1, q_2 \le \exp\left(c\sqrt{\log x}\right)$, we have*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{n\le x}(\log n)^{\mu-\kappa}C_n(\kappa,\chi,q_2) = (-1)^{\mu}\frac{\phi(q_2)}{q_2}\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\chi)\right]x$$
$$-\sum_{\kappa=0}^{\mu}\sum_{\nu=0}^{\kappa}\sum_{j=0}^{\mu-\kappa}\binom{\kappa}{\nu}\Big(\prod_{p|q_2}(1-p^{-\beta_1})\Big)^{(\nu)}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1^{j+1}}x^{\beta_1}(\log x)^{\mu-\kappa-j}$$
$$+O\left(x\exp\left(-C\sqrt{\log x}\right)\right)$$

for some positive constant C. Here β_1 is a possible exceptional zero of $L(s,\chi)$, and the sums involving it are present only when it exists.

2.3.4. Coefficients of $\frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)$

We conclude this section with the evaluation of $\sum_{n\le x}' D_n(\kappa,\psi)$, where $D_n(\kappa,\psi)$ are the coefficients of the Dirichlet series of $\frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)$. Here ω_0 is the principal character to the modulus q_1 and ψ is any nonprincipal character to the modulus q_2. We shall have use for this sum in Chapter 5. As usual,

$$\left|\sum_{n\le x}{}' D_n(\kappa,\psi) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s)\frac{x^s}{s}ds\right|$$
$$<\sum_{\substack{n=1\\ n\ne x}}^{\infty}D_n(\kappa,\psi)\left(\frac{x}{n}\right)^{\sigma_0}\min\left(1,U^{-1}\left|\log\tfrac{x}{n}\right|^{-1}\right)+'c_0\,U^{-1}D_x(\kappa,\psi). \quad (2.48)$$

Now

$$|D_n(\kappa,\psi)| = \left|\sum_{d|n}\Lambda(d)\omega_0(d)\psi\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa}\right| \ll \Omega(n)(\log n)^{\kappa+1} \ll (\log n)^{\kappa+2}. \quad (2.49)$$

The estimates proceed in the fashion which has become routine by now: we choose $\sigma_0 = 1+(\log x)^{-1}$, assume that $q_1, q_2 \le U \le x$ and bound the series on the right hand side of (2.48). The terms with $n \le \frac{x}{2}$ or $n \ge 2x$ contribute

$$\ll \frac{x}{U}\sum_{n=1}^{\infty}\frac{|D_n(\kappa,\psi)|}{n^{\sigma_0}} \ll \frac{x}{U}\left|\frac{\zeta'}{\zeta}(\sigma_0)\zeta^{(\kappa)}(\sigma_0)\right| \ll \frac{x}{U}(\log x)^{\kappa+2}, \tag{2.50}$$

and the terms $\frac{x}{2} < n < 2x$ contribute

$$\ll \frac{x}{U}(\log x)^{\kappa+2}\left(1+\sum_{n\le x}\frac{1}{n}\right) \ll \frac{x}{U}(\log x)^{\kappa+3}.$$

Combining these we get

$$\left|\sum_{n\le x}{}' D_n(\kappa,\psi) - \frac{1}{2\pi i}\int_{\sigma_0-iU}^{\sigma_0+iU}\frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)\frac{x^s}{s}ds\right| \ll \frac{x}{U}(\log x)^3, \tag{2.51}$$

subject to $U \le x$, with $\sigma_0 = 1+(\log x)^{-1}$.

Now let C be the rectangle with vertices $\sigma_0 - iU$, $\sigma_0 + iU$, $\sigma_1 + iU$ and $\sigma_1 - iU$. We note that $L(s,\omega_0) = \zeta(s)\prod_{p|q_1}(1-p^{-s})$, and the product on the right does not vanish for, say, $\sigma > \frac{1}{2}$. So a zero free region for $\zeta(s)$ may serve as a zero free region for the L-function of the principal character. Accordingly we take $\sigma_1 = 1 - \frac{c_4}{\log U}$, so that $L(s,\omega_0)$ is free of zeros in C. Hence the only pole of the integrand is that of $\frac{L'}{L}(s,\omega_0)$ at $s=1$ and we have, by the theory of residues

$$\frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)\frac{x^s}{s}ds = -L^{(\kappa)}(1,\psi)x.$$

We have on our contour

$$\frac{L'}{L}(s,\omega_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q_1}\frac{\log p}{p^s-1} = \frac{\zeta'}{\zeta}(s) + O(\log q_1),$$

since $\log p/(p^s-1) \ll 1$ there and the number of primes dividing q_1 is $\ll \log q_1$. Also by [2, (6.6)] we have $\frac{\zeta'}{\zeta}(s) \ll \log U$ for $|t| \ge 1$ and and for $|t| < 1$, the series expansion gives $\frac{\zeta'}{\zeta}(s) \ll \log U$, and we may use $\frac{L'}{L}(s,\omega_0) \ll \log q_1 U$ unconditionally. As before we have $L^{(\kappa)}(s,\psi) \ll (q_2U)^{\varepsilon}$ by Lemma 2.6, so these combine to give

$$\int_{\sigma_0+iU}^{\sigma_1+iU} \frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)\frac{x^s}{s}ds \ll \log q_1U \cdot U^{\varepsilon}\frac{x}{U}(\sigma_0-\sigma_1) \\ \ll \frac{x}{U^{1-\varepsilon}},$$

and similarly

$$\int_{\sigma_1-iU}^{\sigma_0-iU} \frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)\frac{x^s}{s}ds \ll \frac{x}{U^{1-\varepsilon}}.$$

The same bounds are valid also for the vertical segment, and we have

$$\int_{\sigma_1+iU}^{\sigma_1-iU} \frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)\frac{x^s}{s}ds \ll x^{\sigma_1}U^{\varepsilon}\int_{-U}^{U}\left|\frac{1}{\sigma_1+it}\right|dt \\ \ll xU^{\varepsilon}\exp\left(\frac{-c_4\log x}{\log U}\right).$$

Combining our estimates and removing the condition on the sum, we obtain

$$\sum_{n\le x} D_n(\kappa,\psi) = -L^{(\kappa)}(1,\psi)x \\ + O\left(x\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_4\log x}{\log U}\right)\right)\right). \quad (2.52)$$

Now summing by parts gives

$$\sum_{n\le x}(\log n)^{\mu-\kappa}D_n(\kappa,\psi) = -L^{(\kappa)}(1,\psi)x(\log x)^{\mu-\kappa} \\ -(\mu-\kappa)\int_1^x\left(\sum_{n\le t}D_n(\kappa,\psi)\right)\frac{(\log t)^{\mu-\kappa-1}}{t}dt \\ + O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_4\log x}{\log U}\right)\right)\right).$$

As usual we intend to replace the sum in the integrand with our approximate expression.

By (2.49) we have

$$\int_1^U \left(\sum_{n\le t} D_n(\kappa,\psi)\right) \frac{(\log t)^{\mu-\kappa-1}}{t} dt \ll U(\log U)^{\mu+1}.$$

By Lemma 2.6 we know that $L^{(\kappa)}(1,\psi) \ll q_2^{\varepsilon}$, whence

$$\int_1^U L^{(\kappa)}(1,\psi)(\log t)^{\mu-\kappa-1} dt \ll q_2^{\varepsilon} U(\log U)^{\mu-\kappa-1} \ll U^2.$$

The integral of the error term can be bounded trivially.

$$\int_1^U (\log t)^{\mu-\kappa-1}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log t)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_4 \log t}{\log U}\right)\right) dt \\ \ll U^{1+\varepsilon}(\log U)^{\mu+2} \ll U^2.$$

Thus we may write

$$\sum_{n\le x} (\log n)^{\mu-\kappa} D_n(\kappa,\psi) = -L^{(\kappa)}(1,\psi)x(\log x)^{\mu-\kappa} \\ + (\mu-\kappa)L^{(\kappa)}(1,\psi)\int_1^x (\log t)^{\mu-\kappa-1} dt \\ + O\left(\int_1^x (\log t)^{\mu-\kappa-1}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log t)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_4\log t}{\log U}\right)\right) dt\right) \\ + O\left(U^{2+\varepsilon}\right) + O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_4 \log x}{\log U}\right)\right)\right).$$

Proceeding as in (2.32) the integral in the error is

$$\ll x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}} + \frac{(\log x)^{\kappa+3}}{U} + U^{\varepsilon}\exp\left(\frac{-c_5\log x}{\log U}\right)\right).$$

Where $c_5 = \min(c_4, 1)$, say. We expand $\int(\log t)^{\mu-\kappa-1}$ in a series as before.

$$(\mu-\kappa)\int_1^x (\log t)^{\mu-\kappa-1} dt = (\mu-\kappa)\sum_{j=0}^{\mu-\kappa-1}(-1)^j \frac{(\mu-\kappa-1)!}{(\mu-\kappa-1-j)!} x(\log x)^{\mu-\kappa-1-j} + O(1) \\ = -\sum_{j=1}^{\mu-\kappa}(-1)^j \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} x(\log x)^{\mu-\kappa-j} + O(1).$$

This holds for all $\kappa \le \mu$ if we convene that vacuous sums are zero. So we have

$$\sum_{n\le x}(\log n)^{\mu-\kappa}D_n(\kappa,\psi) = -L^{(\kappa)}(1,\psi)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}x(\log x)^{\mu-\kappa-j}$$
$$+O\left(U^{2+\varepsilon}\right)+O\left(x(\log x)^{\mu-\kappa}\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{\kappa+3}}{U}+U^{\varepsilon}\exp\left(\frac{-c_5\log x}{\log U}\right)\right)\right).$$

Multiplying this by $\binom{\mu}{\kappa}$ and summing over κ, we obtain

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{n\le x}(\log n)^{\mu-\kappa}D_n(\kappa,\psi)$$
$$=-\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}L^{(\kappa)}(1,\psi)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}x(\log x)^{\mu-\kappa-j}$$
$$+O\left(U^{2+\varepsilon}\right)$$
$$+O\left(x(\log x)^{\mu}\left(\frac{1}{U^{1-\varepsilon}}+\frac{(\log x)^{3}}{U}+U^{\varepsilon}\exp\left(\frac{-c_5\log x}{\log U}\right)\right)\right).$$

Finally, we note that if we put $U=\exp\left(\sqrt{c_2\log x}\right)$, then all the error terms are $\ll x\exp\left(-C\sqrt{\log x}\right)$ for some positive C, and we obtain

2.10 Lemma. *Let ω_0 be the primitive character to the modulus q_1 and ψ be any non-principal character to the modulus q_2, and let $D_n(\kappa,\psi)$ be the n-th coefficient of the Dirichlet series for $\frac{L'}{L}(s,\omega_0)L^{(\kappa)}(s,\psi)$. There is a positive constant c such that if $q_1,q_2\le \exp\left(c\sqrt{\log x}\right)$, we have*

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{n\le x}(\log n)^{\mu-\kappa}D_n(\kappa,\psi)$$
$$=-\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}L^{(\kappa)}(1,\psi)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}x(\log x)^{\mu-\kappa-j}$$
$$+O\left(x\exp\left(-C\sqrt{\log x}\right)\right)$$

for some positive constant C.

2.4. Estimates regarding some integrals involving $\chi(s,\chi)$

This section contains analogues of Gonek's lemmata 2-5 from [3].

First we record an immediate corollary of Lemma 2 of [3] in the following

2.11 Lemma. *For large A and $A \le r/q \le B \le 2A$*

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt = q^{-\frac{1}{2}}(2\pi)^{1-a}r^a e^{-ir/q+\pi i/4} + q^{a-\frac{1}{2}}E(r/q,A,B),$$

where a is fixed and where

$$E(r/q,A,B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r/q|+A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r/q|-B^{\frac{1}{2}}}\right). \quad (2.53)$$

For $r/q \le A$ or $r/q > B$,

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt = q^{a-\frac{1}{2}}E(r/q,A,B).$$

Proof. We have

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt = q^{a-\frac{1}{2}}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt,$$

and using Lemma 2 of Gonek [3] with r/q in place of r gives the desired result. □

2.12 Lemma. *For $m = 0,1,\ldots$, A large, and $A < r/q \le B \le 2A$,*

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt = q^{-\frac{1}{2}}(2\pi)^{1-a}r^a e^{-ir/q+\pi i/4}\left(\log\frac{r}{2\pi}\right)^m$$
$$+ q^{a-\frac{1}{2}}E(r/q,A,B)\left(\log qA\right)^m,$$

while for $r \le A$ or $r > B$,

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt = q^{a-\frac{1}{2}}E(r/q,A,B)\left(\log qA\right)^m,$$

where $E(r/q,A,B)$ is (2.53).

Proof. Call the integral under consideration I_m and assume $m > 0$. Then,

$$\begin{aligned} I_m &= \int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt \\ &= \int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{m-1}\left(\log\frac{r}{2\pi}+\log\frac{qt}{r}\right) dt \\ &= \left(\log\frac{r}{2\pi}\right)I_{m-1} + \int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{m-1}\left(\log\frac{qt}{r}\right). \end{aligned}$$

Integrating the last integral, call it J, by parts yields

$$\begin{aligned} i\left(\frac{2\pi}{q}\right)^{a-\frac{1}{2}} J &= \exp\left(it\log\frac{qt}{re}\right)t^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{m-1}\Bigg|_A^B \\ &\quad + \int_A^B \exp\left(it\log\frac{qt}{re}\right)t^{a-\frac{3}{2}} \\ &\quad \cdot\left((a-\tfrac{1}{2})\left(\log\frac{qt}{2\pi}\right)^{m-1} + (m-1)\left(\log\frac{qt}{2\pi}\right)^{m-2}\right)dt \\ &\ll A^{a-\frac{1}{2}}\left(\log qA\right)^{m-1} \ll E(r/q,A,B)\left(\log qA\right)^m. \end{aligned}$$

Since Lemma 2.11 provides the assertion for $m = 0$, the result follows by induction. □

2.13 Lemma. *Let $E(r/q,A,B)$ be as in (2.53), where A is large and $A < B \le 2A$. Let $a > 1$. If $\{b_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $b_n \ll n^{\varepsilon}$ for any $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty}\frac{b_n}{n^a}E(2\pi n/q,A,B) \ll A^{a-\frac{1}{2}}.$$

Proof. Choose ε so that $0 < \varepsilon < a - 1$. By (2.53)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{b_n}{n^a} E(2\pi n/q, A, B) &\ll \sum_{n=1}^{\infty} n^{-a+\varepsilon} E(2\pi n/q, A, B)\\ &\ll A^{a-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-a+\varepsilon} + A^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}\left(|A - 2\pi n/q| + A^{\frac{1}{2}}\right)}\\ &+ B^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}\left(|B - 2\pi n/q| + B^{\frac{1}{2}}\right)}.\end{aligned}$$

Now in the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{a-\varepsilon}\left(|C - 2\pi n/q| + C^{\frac{1}{2}}\right)},$$

There are $O(qC^{\frac{1}{2}})$ terms with $|C - 2\pi n/q| < C^{\frac{1}{2}}$ and since for these terms $n^{-a+\varepsilon} \ll (qC)^{-1}$ and $\left(|C - 2\pi n/q| + C^{\frac{1}{2}}\right)^{-1} \ll C^{-\frac{1}{2}}$, the contribution of these terms is $\ll C^{-1}$. For the terms with $2\pi n/q < C/2$ and $2\pi n/q > 2C$, we have $\left(|C - 2\pi n/q| + C^{\frac{1}{2}}\right)^{-1} \ll C^{-1}$, so these terms also contribute $\ll C^{-1}$. Now

$$\begin{aligned}\sum_{\frac{qC}{4\pi} \le n \le \frac{q(C - C^{\frac{1}{2}})}{2\pi}} \frac{1}{n^{a-\varepsilon}\left(|C - 2\pi n/q| + C^{\frac{1}{2}}\right)} &\ll \frac{1}{(qC)^{a-\varepsilon}} \sum_{\frac{qC}{4\pi} \le n \le \frac{q(C - C^{\frac{1}{2}})}{2\pi}} \frac{1}{(C - 2\pi n/q)}\\ &\ll \frac{1}{(qC)^{a-\varepsilon}} \int_{\frac{qC}{4\pi}}^{\frac{q(C - C^{\frac{1}{2}})}{2\pi}} \frac{dx}{C - 2\pi x/q}\\ &\ll \frac{q}{(qC)^{a-\varepsilon}} \int_{C^{\frac{1}{2}}}^{C/2} \frac{du}{u} \ll C^{-1}\end{aligned}$$

A similar estimate holds for the terms with $\frac{q(C - C^{\frac{1}{2}})}{2\pi} \le n \le \frac{qC}{\pi}$, and this completes the proof. ∎

2.14 Lemma. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that for any $\varepsilon > 0$, $b_n \ll n^{\varepsilon}$. Let $a > 1$ and let m be a non-negative integer. Then for $1 \le T_1 < 2$, T*

sufficiently large,

$$\frac{1}{2\pi}\int_{T_1}^{T}\left(\sum_{n=1}^{\infty}b_n n^{-a-it}\right)\chi(1-a-it,\chi)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$=\frac{\tau(\chi)}{q}\sum_{1\le n\le\frac{qT}{2\pi}}b_n e^{-2\pi i n/q}\left(\log n\right)^m+O\left(\left(qT\right)^{a-\frac{1}{2}}\left(\log qT\right)^m\right)+O\left(q^{2a-1}\left(\log q\right)^m\right).$$

Proof. By Lemma 2.1, we have

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}b_n n^{-a-it}\right)\chi(1-a-it,\chi)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$=\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}b_n n^{-a-it}\right)\frac{\tau(\chi)e^{\frac{-\pi i}{4}}}{q^{\frac{1}{2}}}\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\left(\frac{qt}{2\pi e}\right)\right)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$+O\left(\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}|b_n|n^{-a}\right)q^{a-\frac{1}{2}}t^{a-\frac{3}{2}}\left(\log qt\right)^m dt\right).$$

Since $b_n \ll n^{\varepsilon}$, $\sum_{n=1}^{\infty}|b_n|n^{-a} \ll 1$ for $a>1$. The error term is therefore

$$\ll (qT)^{a-\frac{1}{2}}\left(\log qT\right)^m.$$

To treat the main term, we write it as

$$\frac{\tau(\chi)}{q^{\frac{1}{2}}}\sum_{n=1}^{\infty}b_n n^{-a}e^{-\frac{\pi i}{4}}\left(\frac{1}{2\pi}\int_{T/2}^{T}\exp\left(it\log\frac{qt}{2\pi ne}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt\right), \qquad (2.54)$$

the interchange of summation and integration being justified by absolute convergence. The integral is estimable by Lemma 2.12 with $A=T/2$, $B=T$ and $r=2\pi n$, and has the value $q^{-\frac{1}{2}}(2\pi)n^a e^{-2\pi in/q+\pi i/4}(\log n)^m + q^{a-\frac{1}{2}}E(2\pi n/q,T/2,T)\left(\log\frac{qT}{2}\right)^m$ when $qT/4\pi < n \le qT/2\pi$ and $q^{a-\frac{1}{2}}E(2\pi n/q,T/2,T)\left(\log\frac{qT}{2}\right)^m$ otherwise. So (2.54) equals

$$\frac{\tau(\chi)}{q}\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}}b_n e^{-2\pi in/q}\left(\log n\right)^m+q^{a-\frac{1}{2}}\left(\log\frac{qT}{2}\right)^m\sum_{n=1}^{\infty}b_n n^{-a}E(2\pi n/q,T/2,T)$$

for large T. Using Lemma 2.13 for the second sum, this equals

$$\frac{\tau(\chi)}{q}\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} b_n e^{-2\pi i n/q}(\log n)^m + O\left((qT)^{a-\frac{1}{2}}(\log qT)^m\right).$$

Hence we have obtained

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it,\chi)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$= \frac{\tau(\chi)}{q}\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} b_n e^{-2\pi i n/q}(\log n)^m + O\left((qT)^{a-\frac{1}{2}}(\log qT)^m\right)$$

for $T > T_0$, say. Now let l be the unique integer such that $T_0 \le \frac{T}{2^l} < 2T_0$. Adding together the above estimate for for the ranges $\left[\frac{T}{2^j}, \frac{T}{2^{j-1}}\right]$, $(j = 1, 2, \ldots, l)$, we find

$$\frac{1}{2\pi}\int_{T/2^l}^{T}\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it,\chi)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$= \frac{\tau(\chi)}{q}\sum_{\frac{qT}{2^{l+1}\pi}<n\le\frac{qT}{2\pi}} b_n e^{-2\pi i n/q}(\log n)^m + O\left((qT)^{a-\frac{1}{2}}(\log qT)^m\right).$$

The proof is completed upon noting that

$$\frac{1}{2\pi}\int_{T_1}^{T/2^l}\left(\sum_{n=1}^{\infty} b_n n^{-a-it}\right)\chi(1-a-it,\chi)\left(\log\frac{qt}{2\pi}\right)^m dt \ll q^{a-\frac{1}{2}}(\log q)^m$$

and that

$$\sum_{T_1\le n\le\frac{qT}{2^l}} b_n e^{-2\pi i n/q}(\log n)^m \ll q^{2a-1}(\log q)^m,$$

using $b_n \ll n^{\varepsilon}$ with $\varepsilon = 2a-2$. $\square$

3. THE SUM $\sum \zeta^{(\mu)}(\rho_\chi)$

Our aim in this chapter is to estimate the sum

$$\sum_{0 \le \gamma_\chi \le T} \zeta^{(\mu)}(\rho_\chi),$$

where χ is a primitive character to the modulus $q \le T$ and the ρ_χ are nontrivial zeros of $L(s,\chi)$ with $\Im \rho_\chi = \gamma_\chi$.

We construct a sequence $\mathcal{T}$ of numbers T_n with $n \le T_n < n+1$ such that $|\gamma_\chi - T_n| \gg \frac{1}{\log(q(T_n+2))}$ (see Davenport [1, p.116]) and consider

$$I = \frac{1}{2\pi i} \int_R \frac{L'}{L}(s,\chi)\zeta^{(\mu)}(s)ds,$$

where R is the rectangle depicted in the following diagram:

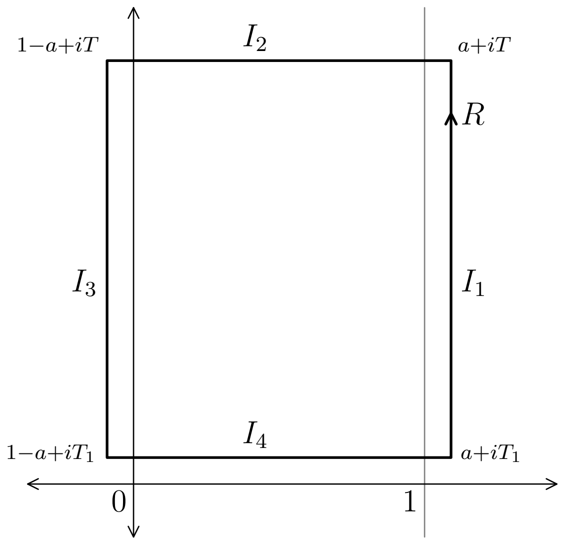

Figure 3.1.

Here a is arbitrary but fixed with $1 < a < 2$, and $T \in \mathcal{T}$. This restriction on T is to be removed later. We split I into four parts as in the figure, so that

$$I = I_1 + I_2 + I_3 + I_4. \tag{3.1}$$

For I_4, recall [1, §16, (4)] that

$$\frac{L'}{L}(s,\chi) = \sum_{|T_1-\gamma_\chi|<1} \frac{1}{s-\rho_\chi} + O\left(\log q(T_1+2)\right).$$

Since the number of zeros satisfying $|T_1 - \gamma_\chi| < 1$ is $\ll \log q$ [1, §16,(1)], and each satisfies $\frac{1}{|\gamma_\chi - s|} \ll \log q$ by our choice of T_1, we deduce

$$I_4 \ll (\log q)^2. \tag{3.2}$$

Similarly, with our choice of T, we have $\frac{L'}{L}(s,\chi) \ll (\log qT)^2$ uniformly for $-1 \le \sigma \le 2$. This together with $\zeta^{(\kappa)}(s) \ll T^{a-\frac{1}{2}+\varepsilon}$ on $[1-a+iT, a+iT]$ [3, (20)] implies

$$I_2 \ll T^{a-\frac{1}{2}+\varepsilon}. \tag{3.3}$$

As for I_1, we have

$$\begin{aligned}
|I_1| &= \left|\frac{1}{2\pi}\int_{T_1}^{T} \frac{L'}{L}(a+it,\chi)\zeta^{(\mu)}(a+it)dt\right| \\
&= \left|\frac{1}{2\pi}\int_{T_1}^{T} \left(\sum_{n=2}^{\infty} n^{-a-it}\sum_{d|n}\Lambda(d)\chi(d)\left(\log\frac{n}{d}\right)^{\mu}\right)dt\right| \\
&\le \frac{1}{2\pi}\sum_{n=2}^{\infty}\left(\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\mu} n^{-a}\left|\int_{T_1}^{T} n^{-it}dt\right|\right),
\end{aligned} \tag{3.4}$$

the interchange of summation and integration being justified by uniform convergence. But

$$\int_{T_1}^{T} n^{-it}dt = \int_{T_1}^{T} e^{(-i\log n)t}dt = \frac{e^{(-i\log n)t}}{-i\log n}\Bigg|_{T_1}^{T} \ll 1, \tag{3.5}$$

wherefore

$$I_1 \ll \left|\frac{\zeta'}{\zeta}(a)\zeta^{(\mu)}(a)\right| \ll 1. \tag{3.6}$$

We now consider I_3.

We have

$$I_3 = \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+iT_1} \frac{L'}{L}(s,\chi)\zeta^{(\mu)}(s)ds = \frac{1}{2\pi i}\int_{a-iT_1}^{a-iT} \frac{L'}{L}(1-s,\chi)\zeta^{(\mu)}(1-s)ds,$$

so

$$\overline{I}_3 = \frac{i}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi)}\zeta^{(\mu)}(1-s)ds.$$

Taking μ-fold derivative of both sides in $\zeta(1-s) = \chi(1-s)\zeta(s)$ we get

$$\zeta^{(\mu)}(1-s) = \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa}(-1)^{\kappa}\zeta^{(\kappa)}(s)\chi^{(\mu-\kappa)}(1-s)$$

and write

$$\overline{I}_3 = \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa}(-1)^{\kappa} I_{3\kappa}, \tag{3.7}$$

where

$$I_{3\kappa} = \frac{i}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi)}\zeta^{(\kappa)}(s)\chi^{(\mu-\kappa)}(1-s)ds. \tag{3.8}$$

By Lemma 6 of Gonek [3], we know that

$$\chi^{(\mu-\kappa)}(1-s) = \chi(1-s)\left(-\log\frac{|t|}{2\pi}\right)^{\mu-\kappa} + O\left(|t|^{\sigma-3/2}(\log|t|)^{\mu-\kappa-1}\right) \tag{3.9}$$

for $|t| \geq 1$. Differentiating logarithmically $L(1-s,\chi) = \chi(1-s,\chi)L(s,\overline{\chi})$ and using Lemma 2.2 we get

$$\frac{L'}{L}(1-s,\chi) = -\frac{L'}{L}(s,\overline{\chi}) - \log\frac{|t|}{2\pi} - \log q + O\left(\frac{1}{|t|}\right) \tag{3.10}$$

for σ fixed. Using this with $\overline{s}$ in place of s and taking conjugates, we obtain

$$\overline{\frac{L'}{L}(1-\overline{s},\chi)} = -\frac{L'}{L}(s,\chi) - \log\frac{|t|}{2\pi} - \log q + O\left(\frac{1}{|t|}\right). \tag{3.11}$$

Here we used $\overline{\frac{L'}{L}(s,\chi)} = \frac{L'}{L}(\overline{s},\overline{\chi})$, which is readily seen upon taking conjugates in the series expression. Note that this also entails

$$\overline{\frac{L'}{L}(1-\overline{s},\chi)} \ll \log qT \tag{3.12}$$

for $\sigma = a$ and $1 \leq |t| \leq T$, since $\frac{L'}{L}(s,\chi) \ll |\frac{\zeta'}{\zeta}(a)| \ll 1$ there. Also,

$$\zeta^{(\kappa)}(s) \ll 1 \tag{3.13}$$

when $\sigma = a$. We combine (3.9), (3.12) and (3.13) in (3.8) and obtain

$$\begin{aligned} I_{3\kappa} &= \frac{i}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi)}\zeta^{(\kappa)}(s)\chi^{(\mu-\kappa)}(1-s)ds \\ &= \frac{i(-1)^{\mu-\kappa}}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi)}\zeta^{(\kappa)}(s)\chi(1-s)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa} ds \\ &\quad + O\left(\int_{a+iT_1}^{a+iT}\left|\frac{L'}{L}(1-\overline{s},\chi)\zeta^{(\kappa)}(s)\right| t^{\sigma-3/2}(\log t)^{\mu-\kappa-1}ds\right) \\ &= \frac{i(-1)^{\mu-\kappa}}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi)}\zeta^{(\kappa)}(s)\chi(1-s)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa} ds + O\left(T^{a-\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

Also using (3.11) for $\overline{\frac{L'}{L}(1-\overline{s},\chi)}$ gives

$$\begin{aligned}
I_{3\kappa} &= \frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T}\frac{L'}{L}(a+it,\chi)\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa}dt \\
&+\frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T}\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa+1}dt \\
&+\frac{(-1)^{\mu-\kappa}(\log q)}{2\pi}\int_{T_1}^{T}\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa}dt \\
&+O\left(\int_{T_1}^{T}\left|\zeta^{(\kappa)}(a+it)\chi(1-a-it)\right|\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa}\frac{1}{t}dt\right) \\
&+O\left(T^{a-\frac{1}{2}+\varepsilon}\right).
\end{aligned}$$

The last integral is $\ll T^{a-\frac{1}{2}+\varepsilon}$, since $\zeta^{(\kappa)}(a+it)\ll 1$ and $\chi(1-a-it)\ll t^{a-\frac{1}{2}}$ [3, (13)]. So we may write

$$I_{3\kappa}=(-1)^{\mu-\kappa}\left(I_{3\kappa 1}+I_{3\kappa 2}+I_{3\kappa 3}\right)+O\left(T^{a-\frac{1}{2}+\varepsilon}\right),$$

whence (3.7) becomes

$$\overline{I}_3=(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 1}+(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2}+(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 3}+O\left(T^{a-\frac{1}{2}+\varepsilon}\right). \tag{3.14}$$

We first deal with $I_{3\kappa 2}$.

$$\begin{aligned}
I_{3\kappa 2} &= \frac{1}{2\pi}\int_{T_1}^{T}\zeta^{(\kappa)}(a+it)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa+1}dt \\
&= \frac{1}{2\pi}\int_{T_1}^{T}\left(\sum_{n=1}^{\infty}\frac{(-1)^{\kappa}(\log n)^{\kappa}}{n^{a+it}}\right)\chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^{\mu-\kappa+1}dt.
\end{aligned}$$

Clearly $(-1)^{\kappa}(\log n)^{\kappa}\ll n^{\varepsilon}$ for any $\varepsilon>0$, so Lemma 5 of Gonek [3] applies to give

$$I_{3\kappa 2}=(-1)^{\kappa}\sum_{n\leq T/2\pi}(\log n)^{\mu+1}+O\left(T^{a-\frac{1}{2}}(\log T)^{\mu-\kappa+1}\right),$$

so that

$$(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2}=(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^{\kappa}\sum_{n\le T/2\pi}(\log n)^{\mu+1}+O\left(T^{a-\frac{1}{2}}(\log T)^{\mu+1}\right)$$
$$=\begin{cases}O\left(T^{a-\frac{1}{2}+\varepsilon}\right) & \text{if } \mu\ge 1,\\ \frac{T}{2\pi}\log\frac{T}{2\pi e}+O(T^{a-\frac{1}{2}+\varepsilon}) & \text{if } \mu=0.\end{cases}\tag{3.15}$$

$I_{3\kappa 3}$ is analogous to $I_{3\kappa 2}$ with $\mu-1$ in place of μ and a factor of $\log q$, so we have

$$(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 3}=\begin{cases}O\left(T^{a-\frac{1}{2}+\varepsilon}\right) & \text{if } \mu\ge 1,\\ \frac{T}{2\pi}\log q+O\left(T^{a-\frac{1}{2}+\varepsilon}\right) & \text{if } \mu=0.\end{cases}\tag{3.16}$$

In order to evaluate $I_{3\kappa 1}$, we write

$$\frac{L'}{L}(s,\chi)\zeta^{(\kappa)}(s)=\sum_{n=1}^{\infty}\frac{A_n(\kappa,\chi)}{n^s}\quad(\sigma>1),$$

and assume $q\le\exp\left(c\sqrt{\log\frac{T}{2\pi}}\right)$ where c is the constant in Lemma 2.7. We have

$$|A_n(\kappa,\chi)|\ll n^{\varepsilon}$$

for any $\varepsilon>0$ by (2.20), so Lemma 5 of [3] gives,

$$I_{3\kappa 1}=\sum_{1\le n\le\frac{T}{2\pi}}(\log n)^{\mu-\kappa}A_n(\kappa,\chi)+O\left(T^{a-\frac{1}{2}}(\log T)^{\mu-\kappa}\right),$$

wherefore

$$
\begin{aligned}
(-1)^\mu \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} I_{3\kappa 1} &= (-1)^\mu \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{1\le n\le \frac{T}{2\pi}} (\log x)^{\mu-\kappa} A_n(\kappa,\chi) + O\left(T^{a-\frac{1}{2}} (\log T)^{\mu}\right) \\
&= \left[\left(\frac{d}{ds}\right)^{\mu} \bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] \frac{T}{2\pi} \\
&\quad - \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{j=0}^{\mu-\kappa} \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{(-1)^j \zeta^{(\kappa)}(\beta)}{\beta^{j+2}} \left(\frac{T}{2\pi}\right)^{\beta} \left(\log \frac{T}{2\pi}\right)^{\mu-\kappa-j} \\
&\quad + O\left(\frac{T}{2\pi} \exp\left(-C\sqrt{\log \frac{T}{2\pi}}\right)\right) + O\left(T^{a-\frac{1}{2}} (\log T)^{\mu-\kappa}\right),
\end{aligned}
\tag{3.17}
$$

for some positive C, by Lemma 2.7. Using (3.15), (3.16) and (3.17) in (3.14), we have

$$
\begin{aligned}
\overline{I}_3 &= \left[\left(\frac{d}{ds}\right)^{\mu} \bigg|_{s=1} \frac{L'}{L}(s,\chi) \right] \frac{T}{2\pi} +' \frac{T}{2\pi} \log \frac{qT}{2\pi e} \\
&\quad - \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{j=0}^{\mu-\kappa} \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{(-1)^j \zeta^{(\kappa)}(\beta)}{\beta^{j+2}} \left(\frac{T}{2\pi}\right)^{\beta} \left(\log \frac{T}{2\pi}\right)^{\mu-\kappa-j} \\
&\quad + O\left(\frac{T}{2\pi} \exp\left(-C\sqrt{\log \frac{T}{2\pi}}\right)\right) + O\left(T^{a-\frac{1}{2}} (\log T)^{\mu-\kappa}\right),
\end{aligned}
$$

where the $+'$ term is present if $\mu = 0$. We combine this with (3.2), (3.3) and (3.6) in (3.1) and obtain

$$
\begin{aligned}
I &= \left[\left(\frac{d}{ds}\right)^{\mu} \bigg|_{s=1} \frac{L'}{L}(s,\overline{\chi}) \right] \frac{T}{2\pi} +' \frac{T}{2\pi} \log \frac{qT}{2\pi e} \\
&\quad - \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \sum_{j=0}^{\mu-\kappa} \frac{(\mu-\kappa)!}{(\mu-\kappa-j)!} \frac{(-1)^j \zeta^{(\kappa)}(\beta)}{\beta^{j+2}} \left(\frac{T}{2\pi}\right)^{\beta} \left(\log \frac{T}{2\pi}\right)^{\mu-\kappa-j} \\
&\quad + O\left(\frac{T}{2\pi} \exp\left(-C\sqrt{\log \frac{T}{2\pi}}\right)\right) + O\left(T^{a-\frac{1}{2}} (\log T)^{\mu-\kappa}\right).
\end{aligned}
$$

To complete our estimates, we put $a = \frac{5}{4}$ and note that replacing T_1 by 0 introduces $O(\log q)$ terms each $\ll q^{\frac{1}{2}\mu+1+\varepsilon}$, and removing the restriction $T \in \mathcal{T}$ introduces $O(\log qT)$ terms each $\ll T^{\frac{3}{4}+\varepsilon}$ by (20) of [3]. We also note that $\frac{T}{2\pi} \log \frac{qT}{2\pi e} =$

$N(T,\chi)+O(\log qT)$. So we have proved our

3.1 Theorem. *Let χ be a primitive character to the modulus q, and let the ρ_χ be the zeros of $L(s,\chi)$ in the critical strip with $\Im\rho_\chi=\gamma_\chi$. There is a constant c such that for $q\le\exp\left(c\sqrt{\log\frac{T}{2\pi}}\right)$, we have*

$$\sum_{0\le\gamma_\chi\le T}\zeta^{(\mu)}(\rho_\chi)=\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\overline{\chi})\right]\frac{T}{2\pi}$$
$$-\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{j=0}^{\mu-\kappa}\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{(-1)^j\zeta^{(\kappa)}(\beta)}{\beta^{j+2}}\left(\frac{T}{2\pi}\right)^{\beta}\left(\log\frac{T}{2\pi}\right)^{\mu-\kappa-j}$$
$$+O\left(T\exp\left(-C\sqrt{\log\frac{T}{2\pi}}\right)\right)$$

when $\mu\ge1$, and

$$\sum_{0\le\gamma_\chi\le T}\zeta(\rho_\chi)=\frac{L'}{L}(1,\overline{\chi})\frac{T}{2\pi}+N(T,\chi)-\frac{\zeta(\beta)}{\beta^2}\left(\frac{T}{2\pi}\right)^{\beta}$$
$$+O\left(T\exp\left(-C\sqrt{\log\frac{T}{2\pi}}\right)\right)$$

for some positive constant C.

A much harsher restriction on q gives us a neater form of the approximation.

3.2 Corollary. *Let χ be a primitive character to the modulus q, and let the ρ_χ be the zeros of $L(s,\chi)$ in the critical strip with $\Im\rho_\chi=\gamma_\chi$. For those q satisfying $q(\log q)^4\le\log T$, we have*

$$\sum_{0\le\gamma_\chi\le T}\zeta^{(\mu)}(\rho_\chi)=\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\overline{\chi})\right]\frac{T}{2\pi}$$
$$+O\left(T\exp\left(-C\sqrt{\log\frac{T}{2\pi}}\right)\right)$$

when $\mu \geq 1$, *and*

$$\sum_{0 \leq \gamma_\chi \leq T} \zeta(\rho_\chi) = \frac{L'}{L}(1, \overline{\chi})\frac{T}{2\pi} + N(T,\chi) + O\left(T \exp\left(-C\sqrt{\log \frac{T}{2\pi}}\right)\right)$$

for some positive constant C.

Finally we state the asymptotic result in the case of fixed moduli.

3.3 Corollary. *Let* χ *be a primitive character of the fixed modulus* q, *and let the* ρ_χ *be the zeros of* $L(s,\chi)$ *in the critical strip with* $\Im \rho_\chi = \gamma_\chi$. *We have*

$$\sum_{0 \leq \gamma_\chi \leq T} \zeta^{(\mu)}(\rho_\chi) \sim \left[\left(\frac{d}{ds}\right)^\mu \bigg|_{s=1} \frac{L'}{L}(s, \overline{\chi})\right] \frac{T}{2\pi}$$

when $\mu \geq 1$, *and*

$$\sum_{0 \leq \gamma_\chi \leq T} \zeta(\rho_\chi) \sim \frac{L'}{L}(1, \overline{\chi})\frac{T}{2\pi} + N(T,\chi)$$

as $T \to \infty$.

We remark that the explicit terms in Theorem 3.1 and Corollary 3.2 are not bona fide main terms, since we have no lower bound for $\left[\left(\frac{d}{ds}\right)^\mu |_{s=1} \frac{L'}{L}(s, \overline{\chi})\right]$ to ensure that they are of dominant magnitude.

4. THE SUM $\sum L^{(\mu)}(\rho_{\chi_1}, \chi_1)$

Now we estimate

$$\sum_{0 \le \gamma_{\chi_1} \le T} L^{(\mu)}(\rho_{\chi_1}, \chi_1).$$

Here χ_1 is a primitive character to the modulus q, the ρ_{χ_1} are the nontrivial zeros of $L(s, \chi_1)$ where $\Im \rho_{\chi_1} = \gamma_{\chi_1}$. It is assumed that $\mu \ge 1$, since the case $\mu = 0$ is trivial with the result 0. We impose the restriction $q \le T$ throughout. As in the previous chapter, we consider a sequence $\mathcal{T}$ of numbers T_n such that $n \le T_n < n+1$ and $|\gamma_{\chi_1} - T_n| \gg \frac{1}{\log(q(T_n+2))}$, and consider

$$I = \int_R \frac{L'}{L}(s, \chi_1) L^{(\mu)}(s, \chi_1) ds$$

over the same contour as before, shown in Figure 3.1. Again,

$$I = I_1 + I_2 + I_3 + I_4. \tag{4.1}$$

As before, we have $\frac{L'}{L}(s, \chi_1) \ll (\log q)^2$ on $[1 - a + iT_1, a + iT_1]$. Also, Lemma 2.5 furnishes the bound $L^{(\mu)}(s, \chi_1) \ll q^{a - \frac{1}{2} + \varepsilon}$ for any $\varepsilon > 0$ there. So we have

$$I_4 \ll q^{a - \frac{1}{2} + \varepsilon}. \tag{4.2}$$

For I_2, as in the previous chapter we have $\frac{L'}{L}(s, \chi_1) \ll (\log qT)^2$. With an appeal to Lemma 2.5 this time instead of (20) of [3], we obtain

$$I_2 \ll (qT)^{a - \frac{1}{2} + \varepsilon}. \tag{4.3}$$

Since for $\sigma = a$ we have $\frac{L'}{L}(s,\chi_1)L^{(\mu)}(s,\chi_1) \ll |\frac{\zeta'}{\zeta}(a)\zeta^{(\mu)}(a)| \ll 1$,

$$I_1 \ll T. \tag{4.4}$$

Now we estimate

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+iT_1} \frac{L'}{L}(s,\chi_1)L^{(\mu)}(s,\chi_1)ds \\ &= \frac{1}{2\pi i}\int_{a-iT_1}^{a-iT} \frac{L'}{L}(1-s,\chi_1)L^{(\mu)}(1-s,\chi_1)ds. \end{aligned}$$

Taking complex conjugates we have

$$\overline{I}_3 = \frac{i}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)L^{(\mu)}(1-\overline{s},\chi_1)}ds,$$

and we note that $\overline{L^{(\mu)}(1-\overline{s},\chi_1)} = L^{(\mu)}(1-s,\overline{\chi}_1)$. Taking μ-fold derivative of $L(1-s,\overline{\chi}_1) = \chi(1-s,\overline{\chi}_1)L(s,\chi_1)$, we get

$$L^{(\mu)}(1-s,\overline{\chi}_1) = \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^\kappa L^{(\kappa)}(s,\chi_1)\chi^{(\mu-\kappa)}(1-s,\overline{\chi}_1). \tag{4.5}$$

Using these we may write

$$\overline{I}_3 = \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^\kappa I_{3\kappa}, \tag{4.6}$$

where

$$I_{3\kappa} = \frac{i}{2\pi}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)}L^{(\kappa)}(s,\chi_1)\chi^{(\mu-\kappa)}(1-s,\overline{\chi}_1)ds.$$

Now we use

$$\chi^{(\mu-\kappa)}(1-s,\overline{\chi}_1) = \chi(1-s,\overline{\chi}_1)\left(-\log\frac{q|t|}{2\pi}\right)^{\mu-\kappa} + O\left(q^{\sigma-\frac{1}{2}}|t|^{\sigma-\frac{3}{2}}(\log q|t|)^{\mu-\kappa-1}\right),$$

which is Lemma 2.3, $\frac{L'}{L}(1-\overline{s},\chi_1) \ll \log qT$ for $\sigma = a$, $T_1 \le t \le T$ which is (3.12) and $L^{(\kappa)}(s,\chi_1) \ll 1$ for $\sigma = a$ and obtain

$$\begin{aligned} I_{3\kappa} &= \frac{-1}{2\pi}\int_{T_1}^{T} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)} L^{(\kappa)}(s,\chi_1)\chi(1-s,\overline{\chi}_1)\left(-\log\frac{qt}{2\pi}\right)^{\mu-\kappa} dt \\ &+ \frac{-1}{2\pi}\int_{T_1}^{T} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)} L^{(\kappa)}(s,\chi_1)\cdot O\left(q^{\sigma-\frac{1}{2}}t^{\sigma-\frac{3}{2}}(\log qt)^{\mu-\kappa-1}\right) dt \\ &= \frac{-1}{2\pi}\int_{T_1}^{T} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)} L^{(\kappa)}(s,\chi_1)\chi(1-s,\overline{\chi}_1)\left(-\log\frac{qt}{2\pi}\right)^{\mu-\kappa} dt \\ &+ O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

Also using (3.10) for $\overline{\frac{L'}{L}(1-\overline{s},\chi_1)}$ we get

$$\begin{aligned} I_{3\kappa} &= \frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T} \frac{L'}{L}(a+it,\chi_1)L^{(\kappa)}(a+it,\chi_1)\chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa} dt \\ &+ \frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T} L^{(\kappa)}(a+it,\chi_1)\chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa+1} dt \\ &+ O\left(\int_{T_1}^{T}\left|L^{(\kappa)}(a+it,\chi_1)\chi(1-a-it,\overline{\chi}_1)\right|\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa}\frac{1}{t}dt\right) \\ &+ O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

Lemma 2.5 and (2.5) allow us to put the first error term into the second so that we may write

$$I_{3\kappa} = (-1)^{\mu-\kappa}\big(I_{3\kappa 1} + I_{3\kappa 2}\big) + O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right), \tag{4.7}$$

whence

$$\overline{I}_3 = (-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa} I_{3\kappa 1} + (-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa} I_{3\kappa 2} + O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right).$$

Now $I_{3\kappa 1}$ and $I_{3\kappa 2}$ are amenable to Lemma 2.14. So since

$$\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\chi_1) = (-1)^{\kappa+1}\sum_{n=1}^{\infty} n^{-s}\chi_1(n)\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa},$$

we have

$$\begin{aligned}
I_{3\kappa 1} &= \frac{1}{2\pi}\int_{T_1}^{T} \frac{L'}{L}(a+it,\chi_1)L^{(\kappa)}(a+it,\chi_1)\chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa} dt \\
&= \frac{(-1)^{\kappa+1}}{2\pi}\int_{T_1}^{T}\left(\sum_{n=1}^{\infty} n^{-a-it}\chi_1(n)\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa}\right) \\
&\quad \cdot \chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa} dt \qquad (4.8)\\
&= \frac{(-1)^{\kappa+1}\tau(\overline{\chi}_1)}{q}\sum_{1\le n\le \frac{qT}{2\pi}} \chi_1(n)e^{-2\pi in/q}(\log n)^{\mu-\kappa}\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa} \\
&\quad + O\left((qT)^{a-\frac{1}{2}}(\log qT)^{\mu-\kappa}\right) + O\left(q^{2a-1}(\log q)^{\mu-\kappa}\right).
\end{aligned}$$

Since $q \le T$, we may include the last error term in the preceding one. Also

$$\begin{aligned}
I_{3\kappa 2} &= \frac{1}{2\pi}\int_{T_1}^{T} L^{(\kappa)}(a+it,\chi_1)\chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa+1} dt \\
&= \frac{(-1)^{\kappa}}{2\pi}\int_{a+iT_1}^{a+iT}\left(\sum_{n=1}^{\infty}\frac{\chi_1(n)(\log n)^{\kappa}}{n^{a+it}}\right)\chi(1-a-it,\overline{\chi}_1)\left(\log\frac{qt}{2\pi}\right)^{\mu-\kappa+1} dt \\
&= \frac{(-1)^{\kappa}\tau(\overline{\chi}_1)}{q}\sum_{1\le n\le \frac{qT}{2\pi}} \chi_1(n)e^{-2\pi in/q}(\log n)^{\mu+1} \\
&\quad + O\left((qT)^{a-\frac{1}{2}}(\log qT)^{\mu-\kappa+1}\right) + O\left(q^{2a-1}(\log q)^{\mu-\kappa+1}\right) \\
&= \frac{(-1)^{\kappa}\tau(\overline{\chi}_1)}{q}\sum_{1\le n\le \frac{qT}{2\pi}} \chi_1(n)e^{-2\pi in/q}(\log n)^{\mu+1} + O\left((qT)^{a-\frac{1}{2}}(\log qT)^{\mu-\kappa+1}\right).
\end{aligned}$$

From this we see that, since $\mu \ge 1$,

$$\begin{aligned}
(-1)^{\mu}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa} I_{3\kappa 2} &= (-1)^{\mu}\frac{\tau(\overline{\chi}_1)}{q}\sum_{1\le n\le \frac{qT}{2\pi}} \chi_1(n)e^{-2\pi in/q}(\log n)^{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^{\kappa} \\
&\quad + O\left((qT)^{a-\frac{1}{2}}(\log qT)^{\mu-\kappa+1}\right) \ll (qT)^{a-\frac{1}{2}+\varepsilon}.
\end{aligned}$$

It remains to compute $\sum\binom{\mu}{\kappa} I_{3\kappa 1}$. Recall that

$$e^{-2\pi in/q} = \frac{1}{\phi(q)}\sum_{\chi \,(\mathrm{mod}\ q)} \tau(\overline{\chi})\chi(-n) \qquad (4.9)$$

when $(n,q)=1$ [1, p.146]. Using this we have

$$\begin{aligned}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 1} &= \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\kappa+1}\tau(\overline{\chi}_1)}{q}\sum_{1\le n\le \frac{qT}{2\pi}}\chi_1(n)e^{-2\pi i n/q}\,(\log n)^{\mu-\kappa} \\ &\quad\cdot\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa}+O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right) \\ &= \frac{\tau(\overline{\chi}_1)}{q\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\ne\overline{\chi}_1}}\tau(\overline{\chi})\chi(-1)\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{1\le n\le\frac{qT}{2\pi}}(\log n)^{\mu-\kappa}B_n(\kappa,\chi\chi_1,\chi\chi_1) \\ &\quad+\frac{\tau(\overline{\chi}_1)\tau(\chi_1)\overline{\chi}_1(-1)}{q\phi(q)}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^{\kappa+1} \\ &\quad\cdot\sum_{1\le n\le\frac{qT}{2\pi}}(\log n)^{\mu-\kappa}\chi_0(n)\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa}+O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right),\end{aligned} \tag{4.10}$$

where $B_n(\kappa,\chi\chi_1,\chi\chi_1)$ are as in Lemma 2.8, which shows that the sum over χ on the right hand side is

$$\begin{aligned}&-\frac{\tau(\overline{\chi}_1)\tau(\chi_E\chi_1)\chi_E\overline{\chi}_1(-1)}{q\phi(q)} \\ &\cdot\sum_{\kappa=0}^{\mu}\sum_{j=0}^{\mu-\kappa}L^{(\kappa)}(\beta,\chi_E)(-1)^j\frac{\mu!}{\kappa!(\mu-\kappa-j)!}\frac{1}{\beta^{j+1}}\left(\frac{qT}{2\pi}\right)^{\beta}\left(\log\frac{qT}{2\pi}\right)^{\mu-\kappa-j} \\ &+O\left(\frac{qT}{2\pi}\exp\left(-C\sqrt{\log\frac{qT}{2\pi}}\right)\right),\end{aligned} \tag{4.11}$$

where χ_E is the possibly existent exceptional character. This is because for each χ in the outer sum, of which there are $\phi(q)-1$, the sum over κ contributes

$$\ll\frac{1}{\phi(q)}\frac{qT}{2\pi}\exp\left(-C\sqrt{\log\frac{qT}{2\pi}}\right)$$

by the lemma, and the sum corresponding to the possible exceptional character χ_E occurs when $\chi=\chi_E\overline{\chi}_1$, giving us the explicit terms. Here we of course restricted q to the range

$$q \le \exp\left(c\sqrt{\log T}\right), \tag{4.12}$$

required by the lemma.

Now there is a constant c_1 such that if an exceptional zero β exists, it satisfies $\beta > 1 - \frac{c_1}{\log q}$, by the definition of an exceptional zero. Then we have $n^{-\beta} = e^{-\beta \log n} \le e^{c_1 \kappa} n^{-1}$ for $n \le q^\kappa$. So,

$$\left|\sum_{n=1}^{q^\kappa} \chi\chi_1(n)(\log n)^\kappa n^{-\beta}\right| \le e^{c_1\kappa} \sum_{n=1}^{q^\kappa} (\log n)^\kappa n^{-1} \ll (\log q)^{\kappa+1},$$

and since $(\log n)^\kappa n^{-\beta}$ decreases for $n > q^\kappa$, by partial summation

$$\left|\sum_{n=q^\kappa+1}^{\infty} \chi\chi_1(n)(\log n)^\kappa n^{-\beta}\right| \le (\log q)^\kappa q^{-\beta} \max_N \left|\sum_{n=q^\kappa+1}^{q^\kappa+N} \chi\chi_1(n)\right| \ll (\log q)^\kappa,$$

so that $L^{(\kappa)}(\beta, \chi) \ll (\log q)^{\kappa+1}$. This together with $q/\phi(q) \ll \log\log q$ [2, Theorem 2.9] shows that (4.11) is

$$\ll T(\log qT)^\mu (\log q)(\log\log q) + qT \exp\left(-C\sqrt{\log \frac{qT}{2\pi}}\right),$$

wherefore (4.10) may be written as

$$\begin{aligned}\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} I_{3\kappa 1} = {} & \frac{\tau(\overline{\chi}_1)\tau(\chi_1)\overline{\chi}_1(-1)}{q\phi(q)} \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} (-1)^{\kappa+1} \\ & \cdot \sum_{1 \le n \le \frac{qT}{2\pi}} (\log n)^{\mu-\kappa} \chi_0(n) \sum_{d|n} \Lambda(d) \left(\log \frac{n}{d}\right)^\kappa + O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right) \\ & + O\left(T(\log qT)^\mu (\log q)(\log\log q)\right) + O\left(qT \exp\left(-C\sqrt{\log \frac{qT}{2\pi}}\right)\right).\end{aligned} \tag{4.13}$$

Now we compute

$$\sum_{1\le n\le x}\chi_0(n)\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa}=\sum_{d\le x}\chi_0(d)\Lambda(d)\sum_{k\le\frac{x}{d}}\chi_0(k)\left(\log k\right)^{\kappa}.$$

First assume $\kappa\ge 1$. By the Eratosthenes-Legendre sieve [2, Theorem 3.1], we know that

$$\sum_{k\le x}\chi_0(k)=\frac{\phi(q)}{q}x+O\left(q^{\varepsilon}\right). \tag{4.14}$$

Then partial summation gives

$$\begin{aligned}\sum_{k\le x}\chi_0(k)\left(\log k\right)^{\kappa}&=\left(\log x\right)^{\kappa}\sum_{k\le x}\chi_0(k)-\kappa\int_1^x\left(\sum_{k\le t}\chi_0(k)\right)\frac{(\log t)^{\kappa-1}}{t}dt\\&=\frac{\phi(q)}{q}x\left(\log x\right)^{\kappa}+O\left(\frac{\phi(q)}{q}x\left(\log x\right)^{\kappa-1}\right)+O\left(q^{\varepsilon}\left(\log x\right)^{\kappa}\right).\end{aligned} \tag{4.15}$$

Hence,

$$\begin{aligned}\sum_{d\le x}\chi_0(d)\Lambda(d)\sum_{k\le\frac{x}{d}}\chi_0(k)\left(\log k\right)^{\kappa}=&\frac{\phi(q)x}{q}\sum_{d\le x}\frac{\Lambda(d)\chi_0(d)}{d}\left(\log\frac{x}{d}\right)^{\kappa}\\&+O\left(\frac{\phi(q)x}{q}\sum_{d\le x}\frac{\Lambda(d)\chi_0(d)}{d}\left(\log\frac{x}{d}\right)^{\kappa-1}\right)\\&+O\left(q^{\varepsilon}\sum_{d\le x}\Lambda(d)\chi_0(d)\left(\log\frac{x}{d}\right)^{\kappa}\right).\end{aligned} \tag{4.16}$$

We first show

$$\sum_{d\le x}\frac{\Lambda(d)\chi_0(d)}{d}\left(\log\frac{x}{d}\right)^{\kappa}=\frac{(\log x)^{\kappa+1}}{\kappa+1}+O((\log x)^{\kappa}(\log q)). \tag{4.17}$$

We have $\psi(x,\chi_0) = x + E(x)$ where $E(x) \ll x \exp\left(-c\sqrt{\log x}\right) + (\log q)(\log x)$ [1, p.121], so

$$\sum_{d\le x} \frac{\Lambda(d)\chi_0(d)}{d}\left(\log\frac{x}{d}\right)^\kappa = \int_1^x \frac{\left(\log\frac{x}{u}\right)^\kappa}{u}\, d\psi(u,\chi_0)$$
$$= \int_1^x \frac{\left(\log\frac{x}{u}\right)^\kappa}{u}\, du + \int_1^x \frac{\left(\log\frac{x}{u}\right)^\kappa}{u}\, dE(u).$$

The first integral is plainly $(\log x)^{\kappa+1}/(\kappa+1)$. We integrate the second by parts and obtain,

$$\begin{aligned} &- E(1)(\log x)^\kappa + \int_1^x \frac{E(u)}{u^2}\left(\kappa\left(\log\frac{x}{u}\right)^{\kappa-1} + \left(\log\frac{x}{u}\right)^\kappa\right) du \\ &\ll (\log x)^\kappa\left(1 + \int_1^x \frac{\exp(-c\sqrt{\log u})}{u} du + (\log q)\int_1^x \frac{\log u}{u^2} du\right) \\ &\ll (\log x)^\kappa (\log q). \end{aligned}$$

This shows (4.17). The last error term in (4.16) is plainly

$$q^\varepsilon \sum_{d\le x} \Lambda(d)\chi_0(d)\left(\log\frac{x}{d}\right)^\kappa \ll q^\varepsilon x (\log x)^\kappa \ll q^\varepsilon \frac{\phi(q)}{q} x (\log x)^\kappa, \tag{4.18}$$

by virtue of the fact that $\frac{q}{\phi(q)} \ll q^\varepsilon$ [2, Theorem 2.9]. Combining (4.17) and (4.18) in (4.16), we obtain

$$\begin{aligned} \sum_{1\le n\le x} \chi_0(n) \sum_{d|n} \Lambda(d)\left(\log\frac{n}{d}\right)^\kappa &= \frac{\phi(q)}{(\kappa+1)\,q} x\,(\log x)^{\kappa+1} \\ &\quad + O\left(q^\varepsilon \frac{\phi(q)}{q} x\,(\log x)^\kappa\right). \end{aligned} \tag{4.19}$$

We note that when $\kappa = 0$,

$$\sum_{n\le x} \chi_0(n) \sum_{d|n} \Lambda(d) = \sum_{n\le x} \chi_0(n) \log n = \frac{\phi(q)}{q} x \log x + O\left(\frac{\phi(q)}{q} x\right) + O\left(q^\varepsilon \log x\right),$$

and this agrees with (4.19), so (4.19) holds generally for all $\kappa \geq 0$. Now by partial summation,

$$\sum_{n\leq x}(\log n)^{\mu-\kappa}\chi_0(n)\sum_{d|n}\Lambda(d)\left(\log\frac{n}{d}\right)^{\kappa} = \frac{\phi(q)}{(\kappa+1)\,q}x\,(\log x)^{\mu+1} + O\left(q^{\varepsilon}\frac{\phi(q)}{q}x\,(\log x)^{\mu}\right).$$

We use this with $x = qT/2\pi$ in (4.13) with q as in (4.12).

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 1} = -\frac{\tau(\overline{\chi}_1)\tau(\chi_1)\overline{\chi}_1(-1)}{2\pi q}T\left(\log\frac{qT}{2\pi}\right)^{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\kappa}}{\kappa+1} + O\left(q^{\varepsilon}T(\log T)^{\mu}\right) + O\left(T(\log qT)^{\mu}(\log q)(\log\log q)\right) + O\left((qT)^{a-\frac{1}{2}+\varepsilon}\right) + O\left(qT\exp\left(-C\sqrt{\log\frac{qT}{2\pi}}\right)\right).$$

If we further restrict q so that $q \leq (\log T)^{N}$ for some N, we may write

$$\overline{I}_3 = (-1)^{\mu+1}\frac{\tau(\overline{\chi}_1)\tau(\chi_1)\overline{\chi}_1(-1)}{2\pi q}T\left(\log\frac{qT}{2\pi}\right)^{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\kappa}}{\kappa+1} + O\left(T(\log T)^{\mu+\varepsilon}\right) + O\left(T^{a-\frac{1}{2}+\varepsilon}\right).$$

We finally note that

$$\begin{aligned}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\kappa}}{\kappa+1} &= \sum_{\kappa=0}^{\mu}\frac{\mu!}{(\kappa+1)!(\mu-\kappa)!}(-1)^{\kappa} = \frac{1}{\mu+1}\sum_{\kappa=0}^{\mu}\frac{(\mu+1)!}{(\kappa+1)!(\mu-\kappa)!}(-1)^{\kappa}\\ &= \frac{-1}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa+1}(-1)^{\kappa+1} = \frac{-1}{\mu+1}\sum_{\kappa=1}^{\mu+1}\binom{\mu+1}{\kappa}(-1)^{\kappa}\\ &= \frac{1}{\mu+1},\end{aligned}$$

and that

$$\begin{aligned}\tau(\overline{\chi}_1)\overline{\chi}_1(-1) &= \overline{\chi}_1(-1)\sum_{m=1}^{q}\overline{\chi}_1(m)e^{2\pi i m/q} \\ &= \sum_{m=1}^{q}\overline{\chi}_1(-m)\overline{e^{2\pi i(-m)/q}} \\ &= \overline{\sum_{m=1}^{q}\chi_1(m)e^{2\pi i m/q}} \\ &= \overline{\tau(\chi_1)},\end{aligned}$$

whereupon

$$I_3 = \frac{(-1)^{\mu+1}}{(\mu+1)}\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{\mu+1} + O\left(T(\log T)^{\mu+\varepsilon}\right) + O\left(T^{a-\frac{1}{2}+\varepsilon}\right), \tag{4.20}$$

since $|\tau(\chi_1)| = q^{\frac{1}{2}}$. We set $a = \frac{5}{4}$, and combine (4.4), (4.3) and (4.2) with (4.20) in (4.1). Noting that replacing T_1 with 0 introduces $O(\log q)$ terms each $\ll q^{\frac{3}{4}+\varepsilon}$ by Lemma 2.5 yields our

4.1 Theorem. *Let χ_1 be a primitive character to the modulus q and $\mu \geq 1$. If $q \leq (\log T)^N$ for some N, we have*

$$\sum_{0\leq\gamma_{\chi_1}\leq T} L^{(\mu)}(\rho_{\chi_1},\chi_1) = \frac{(-1)^{\mu+1}}{2\pi(\mu+1)}T\left(\log\frac{qT}{2\pi}\right)^{\mu+1} + O\left(T(\log T)^{\mu+\varepsilon}\right),$$

for any $\varepsilon > 0$.

The asymptotic version of this is

4.2 Corollary. *Let χ_1 be a primitive character of the fixed modulus q. Then we have*

$$\sum_{0\leq\gamma_{\chi_1}\leq T} L^{(\mu)}(\rho_{\chi_1},\chi_1) \sim \frac{(-1)^{\mu+1}}{2\pi(\mu+1)}T\left(\log\frac{qT}{2\pi}\right)^{\mu+1},$$

as $T \to \infty$.

5. THE SUM $\sum L^{(\mu)}(\rho_{\chi_1}, \psi_1)$

In this chapter we will be dealing with characters to (generally) distinct moduli, say q_1 and q_2. As a consequence, characters to the modulus $q = [q_1, q_2]$ will arise. We shall denote by the Greek letters χ, ψ and ω characters to the moduli q_1, q_2 and q respectively. A naught in the subscript will denote that the character is principal, as usual. We compute

$$\sum_{0 \le \gamma_{\chi_1} \le T} L^{(\mu)}(\rho_{\chi_1}, \psi_1),$$

where χ_1 and ψ_1 are primitive, $[q_1, q_2] \le T$. Again, $\mathcal{T}$ is a sequence of numbers T_n such that $n \le T_n < n + 1$ and $|\gamma_{\chi_1} - T_n| \gg \frac{1}{\log(q(T_n+2))}$. Our integral this time is

$$I = \int_R \frac{L'}{L}(s, \chi_1) L^{(\mu)}(s, \psi_1) ds$$

over the same rectangle R as in Figure 3.1. We split the integral as before.

$$I = I_1 + I_2 + I_3 + I_4. \tag{5.1}$$

Here we note that the estimate for I_1 of Chapter 3 will hold verbatim with $\frac{L'}{L}(s, \chi)\zeta^{(\kappa)}(s)$ replaced with $\frac{L'}{L}(s, \chi_1) L^{(\kappa)}(s, \chi_2)$, and likewise the estimates for I_2 and I_4 of Chapter 4 when $L^{(\mu)}(s, \chi_1)$ is replaced with $L^{(\mu)}(s, \psi_1)$, except that relevant bounds coming from $\frac{L'}{L}(s, \chi_1)$ and $L^{(\mu)}(s, \psi_1)$ will be with q_1 and q_2 respectively in place of q. But writing $q = [q_1, q_2]$ in place of q_1 and q_2 majorizes the bounds, so we may write

$$I_1 \ll 1, \tag{5.2}$$

$$I_2 \ll (qT)^{a - \frac{1}{2} + \varepsilon}, \tag{5.3}$$

$$I_4 \ll q^{a - \frac{1}{2} + \varepsilon}. \tag{5.4}$$

In the sequel we will do this majorization of q_i by q without forewarning.

We proceed with I_3 as before:

$$I_3 = \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+iT_1} \frac{L'}{L}(s,\chi_1)L^{(\mu)}(s,\psi_1)ds$$
$$= \frac{1}{2\pi i}\int_{a-iT_1}^{a-iT} \frac{L'}{L}(1-s,\chi_1)L^{(\mu)}(1-s,\psi_1)ds.$$

Taking conjugates,

$$\overline{I}_3 = \frac{1}{2\pi i}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)L^{(\mu)}(1-\overline{s},\psi_1)}ds. \tag{5.5}$$

Dealing with $\overline{L^{(\mu)}(1-\overline{s},\psi_1)}$ as in (4.5), we write

$$\overline{I}_3 = \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^{\kappa} I_{3\kappa}, \tag{5.6}$$

where

$$I_{3\kappa} = \frac{1}{2\pi i}\int_{a+iT_1}^{a+iT} \overline{\frac{L'}{L}(1-\overline{s},\psi_1)}L^{(\kappa)}(s,\psi_1)\chi^{(\mu-\kappa)}(1-s,\overline{\psi}_1)ds.$$

Again we use

$$\chi^{(\mu-\kappa)}(1-s,\overline{\psi}_1) = \chi(1-s,\overline{\psi}_1)\left(-\log\frac{q_2|t|}{2\pi}\right)^{\mu-\kappa} + O\left(q_2^{\sigma-\frac{1}{2}}|t|^{\sigma-\frac{3}{2}}(\log q_2|t|)^{\mu-\kappa-1}\right),$$

which is Lemma 2.3, $\frac{L'}{L}(1-\overline{s},\chi_1) \ll \log q_1 T$ for $\sigma = a$, $T_1 \le t \le T$ which is (3.12) and $L^{(\kappa)}(s,\chi_1) \ll 1$ for $\sigma = a$ and obtain

$$I_{3\kappa} = \frac{-1}{2\pi}\int_{T_1}^{T} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)}L^{(\kappa)}(s,\psi_1)\chi(1-s,\overline{\psi}_1)\left(-\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa}dt$$
$$+ O\left(\frac{-1}{2\pi}\int_{T_1}^{T}\left|\overline{\frac{L'}{L}(1-\overline{s},\chi_1)}L^{(\kappa)}(s,\psi_1)\right| \cdot q_2^{a-\frac{1}{2}}t^{a-\frac{3}{2}}(\log q_2 t)^{\mu-\kappa-1}dt\right)$$
$$= \frac{-1}{2\pi}\int_{T_1}^{T} \overline{\frac{L'}{L}(1-\overline{s},\chi_1)}L^{(\kappa)}(s,\psi_1)\chi(1-s,\overline{\psi}_1)\left(-\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa}dt$$
$$+ O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right).$$

Also using (3.10) for $\overline{\frac{L'}{L}(1-\overline{s},\chi_1)}$ and noting that $\log\frac{q_1 t}{2\pi} = \log\frac{q_2 t}{2\pi} + \log\frac{q_1}{q_2}$, we get

$$\begin{aligned} I_{3\kappa} = & \frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T}\frac{L'}{L}(a+it,\chi_1)L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt \\ & + \frac{(-1)^{\mu-\kappa}}{2\pi}\int_{T_1}^{T} L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa+1} dt \\ & + \frac{(-1)^{\mu-\kappa}}{2\pi}\left(\log\frac{q_1}{q_2}\right)\int_{T_1}^{T} L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt \\ & + \frac{(-1)^{\mu-\kappa+1}}{2\pi}\int_{T_1}^{T} L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} O\left(\frac{1}{t}\right) dt \\ & + O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

Lemma 2.5 and (2.5) allow us to put the last integral into the error term so that we may write

$$I_{3\kappa} = (-1)^{\mu-\kappa}\big(I_{3\kappa 1} + I_{3\kappa 2} + I_{3\kappa 3}\big) + O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}(\log q_1 T)\right). \tag{5.7}$$

Therefore, (5.5) becomes

$$\overline{I}_3 = (-1)^{\mu}\left(\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 1} + \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2} + \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 3}\right) + O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right). \tag{5.8}$$

Now $I_{3\kappa 1}$, $I_{3\kappa 2}$ and $I_{3\kappa 3}$ are amenable to Lemma 2.14. So since

$$\frac{L'}{L}(s,\chi_1)L^{(\kappa)}(s,\psi_1) = (-1)^{\kappa+1}\sum_{n=1}^{\infty} n^{-s}\sum_{d|n}\chi_1(d)\Lambda(d)\psi_1\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa},$$

we have

$$I_{3\kappa 1} = \frac{1}{2\pi}\int_{T_1}^{T} \frac{L'}{L}(a+it,\chi_1)L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt$$
$$= \frac{(-1)^{\kappa+1}}{2\pi}\int_{T_1}^{T}\left(\sum_{n=1}^{\infty} n^{-s}\sum_{d|n}\chi_1(d)\Lambda(d)\psi_1\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa}\right)$$
$$\cdot\,\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt$$
$$= \frac{(-1)^{\kappa+1}\tau(\overline{\psi}_1)}{q_2}\sum_{1\le n\le\frac{q_2 T}{2\pi}} e^{-2\pi i n/q_2}\left(\log n\right)^{\mu-\kappa}\sum_{d|n}\chi_1(d)\Lambda(d)\psi_1\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa}$$
$$+\,O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right)+O\left(q_2^{2a-1}\left(\log q_2\right)^{\mu-\kappa}\right).$$

Since $q \le T$, we may put the last error term into the preceding one. Also

$$I_{3\kappa 2} = \frac{1}{2\pi}\int_{T_1}^{T} L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa+1} dt$$
$$= \frac{(-1)^{\kappa}}{2\pi}\int_{T_1}^{T}\left(\sum_{n=1}^{\infty}\frac{\psi_1(n)\left(\log n\right)^{\kappa}}{n^{a+it}}\right)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa+1} dt$$
$$= \frac{(-1)^{\kappa}\tau(\overline{\psi}_1)}{q_2}\sum_{1\le n\le\frac{q_2 T}{2\pi}}\psi_1(n)e^{-2\pi i n/q_2}\left(\log n\right)^{\mu+1}$$
$$+\,O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right),$$

and

$$I_{3\kappa 3} = \frac{1}{2\pi}\left(\log\frac{q_1}{q_2}\right)\int_{T_1}^{T} L^{(\kappa)}(a+it,\psi_1)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt$$
$$= \frac{(-1)^{\kappa}}{2\pi}\left(\log\frac{q_1}{q_2}\right)\int_{T_1}^{T}\left(\sum_{n=1}^{\infty}\frac{\psi_1(n)\left(\log n\right)^{\kappa}}{n^{a+it}}\right)\chi(1-a-it,\overline{\psi}_1)\left(\log\frac{q_2 t}{2\pi}\right)^{\mu-\kappa} dt$$
$$= \frac{(-1)^{\kappa}\tau(\overline{\psi}_1)}{q_2}\left(\log\frac{q_1}{q_2}\right)\sum_{1\le n\le\frac{q_2 T}{2\pi}}\psi_1(n)e^{-2\pi i n/q_2}\left(\log n\right)^{\mu}$$
$$+\,O\left((q_2 T)^{a-\frac{1}{2}+\varepsilon}\right),$$

Clearly, for $\mu \geq 1$

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2}=\frac{\tau(\overline{\psi}_1)}{q_2}\sum_{1\leq n\leq \frac{q_2T}{2\pi}}\psi_1(n)e^{-2\pi i n/q_2}\left(\log n\right)^{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(-1)^{\kappa}$$
$$+O\left(\left(q_2T\right)^{a-\frac{1}{2}}\left(\log q_2T\right)^{\mu+1}\right)$$
$$\ll \left(q_2T\right)^{a-\frac{1}{2}+\varepsilon},$$

and likewise for $\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 3}$. If $\mu = 0$, using (4.9) we get

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2}=\frac{\tau(\overline{\psi}_1)}{q_2}\sum_{1\leq n\leq \frac{q_2T}{2\pi}}\psi_1(n)e^{-2\pi i n/q_2}\left(\log n\right)$$
$$=\frac{\tau(\overline{\psi}_1)}{q_2\phi(q_2)}\sum_{\psi \,(\mathrm{mod}\, q_2)}\psi(-1)\tau(\overline{\psi})\sum_{n\leq \frac{q_2T}{2\pi}}\psi_1\psi(n)(\log n)+O\left(\left(q_2T\right)^{a-\frac{1}{2}+\varepsilon}\right).$$

When $\psi \neq \overline{\psi}_1$ the inner sum is

$$\ll q_2^{\frac{1}{2}}(\log q_2)(\log q_2T),$$

by Pólya-Vinogradov inequality and partial summation. So the total contribution of all such characters is also $\ll q_2^{\frac{1}{2}}(\log q_2)(\log q_2T)$. The inner sum corresponding to $\psi = \overline{\psi}_1$ is, using the Eratosthenes-Legendre sieve as in (4.15),

$$\sum_{n\leq \frac{q_2T}{2\pi}}\psi_0(n)(\log n)=\frac{\phi(q_2)T}{2\pi}\log\frac{q_2T}{2\pi}-\int_1^{\frac{q_2T}{2\pi}}\left(\sum_{n\leq t}\psi_0(n)\right)\frac{dt}{t}$$
$$=\frac{\phi(q_2)T}{2\pi}\log\frac{q_2T}{2\pi}-\frac{\phi(q_2)T}{2\pi}+O(q_2^{\varepsilon}\log T).$$

Then, since $\tau(\psi)\tau(\overline{\psi}_1)\overline{\psi}_1(-1) = q_2$, the whole sum is

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 2}=\frac{T}{2\pi}\log\frac{q_2T}{2\pi}-\frac{T}{2\pi}+O\left(\left(q_2T\right)^{a-\frac{1}{2}+\varepsilon}\right),\quad (\mu=0).$$

Similarly, for the sum corresponding to $I_{3\kappa 3}$, we have

$$\frac{\tau(\overline{\psi}_1)}{q_2\phi(q_2)}\left(\log\frac{q_1}{q_2}\right)\sum_{\psi\,(\mathrm{mod}\ q_2)}\psi(-1)\tau(\overline{\psi})\sum_{n\le\frac{q_2T}{2\pi}}\psi_1\psi(n).$$

Using the Pólya-Vinogradov inequality for the inner sums when $\psi\neq\overline{\psi}_1$, these contribute

$$\ll\left(\log\frac{q_1}{q_2}\right)q_2^{\frac{1}{2}}(\log q_2).$$

Using (4.14), the inner sum for $\psi=\overline{\psi}_1$ is

$$\sum_{n\le\frac{q_2T}{2\pi}}\psi_0(n)=\phi(q_2)\frac{T}{2\pi}+O(q_2^{\varepsilon}),$$

so the whole sum is

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 3}=\left(\log\frac{q_1}{q_2}\right)\frac{T}{2\pi}+O\left(\left(\log\frac{q_1}{q_2}\right)q_2^{\frac{1}{2}}(\log q_2)\right),\quad(\mu=0).$$

Therefore

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(I_{3\kappa 2}+I_{3\kappa 2})=\begin{cases}O\left((q_2T)^{a-\frac{1}{2}+\varepsilon}\right) & \text{if } \mu\ge 1,\\ \frac{T}{2\pi}\log\frac{q_1T}{2\pi}-\frac{T}{2\pi}+O\left((q_2T)^{a-\frac{1}{2}+\varepsilon}\right) & \text{if } \mu=0.\end{cases}\tag{5.9}$$

We remark that $\frac{T}{2\pi}\log\frac{q_1T}{2\pi}-\frac{T}{2\pi}=N(T,\chi_1)+O(\log q_1T)$.

At this point some inquiries into the relationship between χ_1 and ψ are in order. We note that $\chi_1\psi$ is a character to the modulus q and that every character has a unique factorization into characters to distinct prime power moduli, i.e. to the factorization of the modulus into prime powers corresponds a factorization of the character [2, Corollary 4.6]. We are interested in the conditions for $\chi_1\psi$ to equal ω_0. For a prime factor p of q, we must have, in view of the above mentioned factorization, $\chi_{1p}\psi_p=\omega_{0p}$, where a subscript p denotes the factor of the character to the modulus p^α, such that p^α is the highest power of p dividing the corresponding modulus. Given primitive χ_1, such a ψ is definable if and only if p divides q_1 to a power less than or equal to the power to which it divides q_2. We see that $\chi_1\psi = \omega_0$ for some ψ if and only if $q_1 \mid q_2$ and χ_1 induces $\overline{\psi}$.

We have, by (4.9),

$$\begin{aligned}
\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa} I_{3\kappa 1} &= \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\kappa+1}\tau(\overline{\psi}_1)}{q_2}\sum_{1\le n\le \frac{q_2T}{2\pi}} e^{-2\pi i n/q_2}(\log n)^{\mu-\kappa} \\
&\quad\cdot \sum_{d|n}\chi_1(d)\Lambda(d)\psi_1\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa} \\
&\quad + O\left((q_2T)^{a-\frac{1}{2}}(\log q_2T)^{\mu}\right) + O\left(q_2^{2a-1}(\log q_2)^{\mu}\right) \\
&= \frac{\tau(\overline{\psi}_1)}{q_2\phi(q_2)}\sum_{\psi}\tau(\overline{\psi})\psi(-1) \\
&\quad\cdot \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{1\le n\le \frac{q_2T}{2\pi}}(\log n)^{\mu-\kappa}\sum_{d|n}(-1)^{\kappa+1}\psi\chi_1(d)\Lambda(d)\psi\psi_1\left(\frac{n}{d}\right)\left(\log\frac{n}{d}\right)^{\kappa} \\
&\quad + O\left((q_2T)^{a-\frac{1}{2}+\varepsilon}\right).
\end{aligned} \tag{5.10}$$

Now as ψ runs over characters modulo q_2 in the outer sum, the sums over κ are of forms estimable by our lemmata. If $q_1 \nmid q_2$, $\chi_1\psi$ is always non-principal modulo q. So for all characters modulo q_2 but $\overline{\psi}_1$, the sum over κ is given by Lemma 2.8. Their total contribution is

$$\begin{aligned}
&-\sum_{\substack{\psi_{e_i}\chi_1=\omega_E \\ \psi_{e_i}\ne\overline{\psi}_1}}\frac{\tau(\overline{\psi}_1)\tau(\psi_{e_i})\psi_\beta(-1)}{q_2\phi(q_2)}L^{(\kappa)}(\beta_1,\psi_{e_i}\psi_1) \\
&\cdot\sum_{j=0}^{\mu-\kappa}\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{(-1)^j}{\beta_1^{j+1}}\left(\frac{q_2T}{2\pi}\right)^{\beta_1}\left(\log\frac{q_2T}{2\pi}\right)^{\mu-\kappa-j} \\
&+O\left(q_2T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right),
\end{aligned} \tag{5.11}$$

where ω_E is the possible exceptional character modulo q, ψ_{e_i} are characters modulo q_2 such that $\chi_1\psi_{e_i} = \omega_E$, and β_1 is the corresponding exceptional zero. If we require that $q(\log q)^4 \le (\log T)$, then $T^{\beta_1} \ll T\exp\left(-C\sqrt{\log T}\right)$. Also, by the discussion following (4.12), we have $L^{(\kappa)}(\beta_1, \psi_{\beta_i}\psi_1) \ll (\log q_2)^{\kappa+1}$. Therefore (5.11) is

$$\ll T\exp\left(-C\sqrt{\log\frac{q_2 T}{2\pi}}\right).$$

The contribution of $\psi = \overline{\psi}_1$ is given by Lemma 2.9, i.e.

$$\begin{aligned}
&(-1)^{\mu}\frac{\tau(\overline{\psi}_1)\tau(\psi_1)\overline{\psi}_1(-1)}{q_2\phi(q_2)}\frac{\phi(q_2)}{q_2}\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\overline{\psi}_1\chi_1)\right]\frac{q_2T}{2\pi}\\
&-\frac{\tau(\overline{\psi}_1)\tau(\psi_1)\overline{\psi}_1(-1)}{q_2\phi(q_2)}\sum_{\kappa=0}^{\mu}\sum_{\nu=0}^{\kappa}\sum_{j=0}^{\mu-\kappa}\binom{\kappa}{\nu}\\
&\cdot\left(\prod_{p|q_2}(1-p^{-\beta_1})\right)^{(\nu)}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{\zeta^{(\kappa-\nu)}(\beta_1)}{\beta_1^{j+1}}\left(\frac{q_2T}{2\pi}\right)^{\beta_1}\left(\log\frac{q_2T}{2\pi}\right)^{\mu-\kappa-j}\\
&+O\left(q_2T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right).
\end{aligned} \tag{5.12}$$

Under the restriction $q(\log q)^4 \le (\log T)$, the second term may again be put into the error. So that when $q_1 \nmid q_2$, also using $\tau(\overline{\psi}_1)\tau(\psi_1)\overline{\psi}_1(-1) = q_2$, we have

$$\begin{aligned}
\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}I_{3\kappa 1} = {}&(-1)^{\mu}\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1}\frac{L'}{L}(s,\overline{\psi}_1\chi_1)\right]\frac{T}{2\pi}\\
&+O\left(T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right)+O\left((q_2T)^{a-\frac{1}{2}+\varepsilon}\right).
\end{aligned} \tag{5.13}$$

Now if $q_1 \mid q_2$, we know that $\overline{\psi}_1\chi_1 \ne \psi_0$ because for this it is necessary that ψ_1 is induced by $\overline{\chi}_1$, but our ψ_1 is primitive. When ψ is induced by $\overline{\chi}_1$ in the outer sum, $\psi\chi_1 = \psi_0$, and the sum over κ is estimable by Lemma 2.10, and the contribution of the said character is

$$
\begin{aligned}
&-\frac{\tau(\overline{\psi}_1)\tau(\chi_1\psi_0)\overline{\chi}_1\psi_0(-1)}{q_2\phi(q_2)} \\
&\cdot\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}L^{(\kappa)}(1,\overline{\chi}_1\psi_1)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{q_2T}{2\pi}\left(\log\frac{q_2T}{2\pi}\right)^{\mu-\kappa-j} \qquad (5.14)\\
&+O\left(q_2T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right).
\end{aligned}
$$

The sums which arose in the case $q_1 \nmid q_2$ are present in this case also. As always, we put $a = 5/4$, combine our estimates in (5.6), and make use of the identity $\tau(\chi)\chi_1(-1) = \overline{\tau(\overline{\chi}_1)}$ to obtain our

5.1 Theorem. *Let χ_1 and ψ_1 be primitive characters to the moduli q_1 and q_2 respectively, and assume that $q = [q_1, q_2]$ satisfies $q(\log q)^4 \le \log T$. Then we have*

$$
\begin{aligned}
\sum_{0\le\gamma_{\chi_1}\le T} L^{(\mu)}(\rho_{\chi_1},\psi_1) &= (-1)^\mu\left[\left(\frac{d}{ds}\right)^\mu\bigg|_{s=1}\frac{L'}{L}(s,\overline{\chi}_1\psi_1)\right]\frac{T}{2\pi}\\
&-\delta(q_1,q_2)\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)}\\
&\cdot\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}L^{(\kappa)}(1,\overline{\chi}_1\psi_1)\sum_{j=0}^{\mu-\kappa}(-1)^j\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{T}{2\pi}\left(\log\frac{q_2T}{2\pi}\right)^{\mu-\kappa-j}\\
&+' N(T,\chi)+O\left(T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right).
\end{aligned}
$$

where $\delta(q_1, q_2)$ is 1 or 0 according as $q_1 \mid q_2$ or not and the $+'$ term is present only when $\mu = 0$.

To simplify the expressions, when $\mu > 0$ we may single out the term with the highest power of the logarithm in (5.14), and collect the rest in an error term of $O\left(T(\log T)^{\mu-1+\varepsilon}\right)$.

5.2 Corollary. *Let χ_1 and ψ_1 be primitive characters to the moduli q_1 and q_2 respectively, and assume that $q = [q_1, q_2]$ satisfies $q(\log q)^4 \le \log T$. With $\mu \ge 1$, if $q_1 \nmid q_2$,* *we have*

$$\sum_{0\le\gamma_{\chi_1}\le T} L^{(\mu)}(\rho_{\chi_1},\psi_1) = (-1)^\mu \left[\left(\frac{d}{ds}\right)^\mu \Big|_{s=1} \frac{L'}{L}(s,\overline{\chi}_1\psi_1)\right]\frac{T}{2\pi} + O\left(T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right).$$

Otherwise if $q_1 \mid q_2$*,*

$$\sum_{0\le\gamma_{\chi_1}\le T} L^{(\mu)}(\rho_{\chi_1},\psi_1) = -\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)}L(1,\chi_1\overline{\psi}_1)\frac{T}{2\pi}\left(\log\frac{q_2T}{2\pi}\right)^\mu + O\left(T(\log T)^{\mu-1+\varepsilon}\right),$$

for any $\varepsilon > 0$.

When $\mu = 0$, our result reads

5.3 Corollary. *Let* χ_1 *and* ψ_1 *be primitive characters to the moduli* q_1 *and* q_2 *respectively, and assume that* $q = [q_1, q_2]$ *satisfies* $q(\log q)^4 \le \log T$*. Then we have*

$$\sum_{0\le\gamma_{\chi_1}\le T} L(\rho_{\chi_1},\psi_1) = -\delta(q_1,q_2)\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)}L(1,\chi_1\overline{\psi}_1)\frac{T}{2\pi} + \frac{L'}{L}(1,\overline{\chi}_1\psi_1)\frac{T}{2\pi} + N(T,\chi_1) + O\left(T\exp\left(-C\sqrt{\log\frac{q_2T}{2\pi}}\right)\right),$$

where $\delta(q_1,q_2)$ *is* 1 *or* 0 *according as* $q_1 \mid q_2$ *or not.*

The asymptotic expressions of our results follow.

5.4 Corollary. *Let* χ_1 *and* ψ_1 *be primitive characters to the fixed moduli* q_1 *and* q_2 *respectively. Then we have*

$$\sum_{0\le\gamma_{\chi_1}\le T} L(\rho_{\chi_1},\psi_1) \sim -\delta(q_1,q_2)\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)}L(1,\chi_1\overline{\psi}_1)\frac{T}{2\pi} + \frac{L'}{L}(1,\overline{\chi}_1\psi_1)\frac{T}{2\pi} + N(T,\chi_1),$$

and for $\mu \geq 1$,

$$\sum_{0 \leq \gamma_{\chi_1} \leq T} L^{(\mu)}(\rho_{\chi_1}, \psi_1) \sim \begin{cases} -\frac{\tau(\psi_1)\psi_1(-1)\tau(\overline{\chi}_1\psi_0)}{\phi(q_2)} L(1, \chi_1\overline{\psi}_1)\frac{T}{2\pi}(\log T)^\mu & \textit{if } q_1 \mid q_2, \\ (-1)^\mu \left[\left(\frac{d}{ds}\right)^\mu \Big|_{s=1} \frac{L'}{L}(s, \overline{\psi}_1\chi_1)\right]\frac{T}{2\pi} & \textit{otherwise.} \end{cases}$$

6. CONCLUSION

The derivations of the results announced in the introduction have thus been completed. As indicated earlier, the results are loosely suggestive of the correlative behaviour of the Riemann zeta-function and Dirichlet L-functions. In particular, if we consider moduli q_1 and q_2 with $q_1 \nmid q_2$ and compare the mean values of $L^{(\mu)}(s, \chi)$, with primitive χ to the modulus q_1 at its own zeros and at those of another L-function of a primite character modulo q_2, we see that their orders differ by a factor of $(\log T)^{\mu+1}$; this suggests that the zeros of these functions do not coincide very often.

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