REVISITING MASSLESS SCALAR TWO POINT FUNCTIONS ON dS_4

by

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Master period was like an oscillation between some painful and fruitful mood states. At the end of the story, it collapsed to this thesis that you are about to read.

ABSTRACT

REVISITING MASSLESS SCALAR TWO POINT FUNCTIONS ON dS_4

Scalar field theories in de Sitter space, as a special case of Quantum Field Theory in Curved spacetime, are currently popular research areas in physics. Scalar field theory is worth studying for several reasons among all the tensor field theories in de Sitter. For example, a well-established scalar field theory in de Sitter might pave the way for the construction of higher rank tensor fields in de Sitter [1], or massless scalar fields might explain the homogeneity of the Cosmic Wave Background with the inflation era of the universe [1, 2]. Hence, we mainly focused on massless scalar two-point functions because they exhibit different properties than massive ones; therefore, they require alternative approaches. From the representation theory perspective, in 4D de Sitter, among spin-0 particles $m = 0$ belongs to Discrete Series, which requires special attention different from Principle and Complementary Series [3]. While earlier works emphasize the lack of a unique vacuum and breaking of de Sitter symmetry as the main problems of massless two-point functions [4], these may be connected to an inadequate analysis of the boundary conditions. The recent ideas suggest more promising treatments for discrete series two-point functions [3, 5]. The boundary conditions cannot completely constrain the form of the solution because of the lack of a preferred vacuum, while it is well defined for the massive case as the Bunch-Davies vacuum [3]. Here we follow a Fourier Transformation procedure of two-point functions [4] to address the boundary conditions for the massless Wightman Function and its singularities. Finally, we are left with one free parameter that denotes an overall constant contribution which can be set to zero. The form of singularities, on the other hand, coincides with the Minkowskian type singularities in the incident point limit. Additionally, we summarized two-contemporary studies [3, 5] and discussed their results.

ÖZET

DE SİTTER'DE KÜTLESİZ İKİ NOKTA FONKSİYONLARININ TEKRAR ELE ALINMASI

Günümüzde, de Sitter uzayında skaler alanlar ve iki nokta fonksiyonları kozmolojideki en aktif çalışma alanlarından biridir. Daha yüksek dereceli tensör alanlar yerine skaler alanları çalışmanın arkasında önemli motivasyonlar yatmaktadır. Örneğin, de Sitter'de düzgün çalışan tutarlı bir skaler alan teorisi, de Sitter uzayında daha yüksek dereceli tensör alanlarının inşasının yolunu açabilir [1]. Öte yandan, gözlemsel olarak da özellike kütlesiz skaler alanlar, Kozmik Dalga Arkaplan Işınımını açıklamaya en iyi adaylardandır [1, 2]. Bu nedenlerle, tezimizde, esas olarak kütlesiz skaler alanlar ve onlara karşılık gelen iki nokta fonksiyonları üzerine odaklandık. Çünkü kütlesiz iki nokta fonksiyonları kütleli olanlardan farklı davranışlara sahiptirler; özel olarak ele alınmalı ve alternatif yaklaşımlar hesaba katılmalıdır. Temsil teorisi açısındanin spin-0 ve m=0 parçacıklar Discrete Seriye takabül ederler ve Principle ve Complementary serilerden ayrı özel olarak değerlendirilmelidir [3]. Bu konudaki öncü çalışmalar [4] daha çok biricik vakum durumunun eksikliğine ve de Sitter simetrisinin kırılmasına odaklanmıştır fakat bu problemler sınır koşullarının eksik anlaşılmasından kaynaklanıyor olabilir. Örneğin, bazı son çalışmalar da [3, 5] kütlesiz alanlarla ilgili daha kapsayıcı bir anlayış sunmaya çalışır. Tüm bunlardan ilham alarak, [4]'teki integrallerden de yararlanarak iki nokta fonksiyonlarının sınır koşullarını ve tekillik yapılarını anlamaya çalıştık. En nihayetinde, tekillik noktalarının beklentilerimizle uyuştuğunu fakat sınır koşullarının iki nokta fonksiyonundaki 4 serbest parametreden sadece 3 tanesini belirleyebildiğini gördük. Söz konusu serbest parametrenin makul koşullar altında sıfır seçilebileceğinden bahsettik. Ek olarak da literatürde ilgimizi çeken ve bu probleme farklı bakış açıları getiren iki farklı çalışmayı [3, 4] özetledik ve elde ettikleri sonuçları karşılaştırdık.

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$a_{\vec{k}} - a^{\dagger}_{\vec{k}}$	Annihilation and creation operators
$\mathcal{C}_i$	Special conformal transformation Killing vector
$\mathcal{D}$	Dilatation Killing vector
ds^2	Invariant length square
$f_{\vec{k}}(k\eta)$	Mode functions
${}_2F_1(a,b,c,z)$	Hypergeometric function
g	Determinant of the metric $g_{\mu\nu}$
$g_{\mu\nu}$	Components of metric
$g^{\mu\nu}$	Components of inverse metric
G_w	Wightman function
G_{wh}	Wightman function in Hadamard form
G_F	Feynman propagator
H	Hubble parameter
$H^{(1)}_{\nu}(z) - H^{(2)}_{\nu}(z)$	Hankel function of the first and second kind
l	Radius of 4D de Sitter hyper-surface
m^2	Mass parameter in Klein-Gordon equation
n_s	Spectral index
P_k	Power spectrum of CMB spectrum
$\mathcal{Q}$	1^{st}Casimir element
$R_{\mu\nu}$	Ricci tensor
$\mathbf{R}_{ij}$	Rotation Killing vector
s	Geodesic separation between two points
$\mathbf{T}_i$	Translation Killing vector
$\mathcal{W}$	2^{nd} Casimir element
$W(f,g)$	Wronskian of the two functions f and g
X^A	Coordinates of 5D Minkowski space
x^{μ}	Local coordinates of 4D de Sitter space
x^{μ}	de Sitter coordinates with $\mu = 0,1,2,3$

X^A	Minkowski coordinates with $A = 0, 1, 2, 3, 4$
$d\Omega_d$	Metric on d-sphere
Δ	Scale factor
Δ_k	Scale invariant power spectrum
η_{AB}	Metric components of 5D Minkowski space
Λ	Cosmological constant
ν	Order of Bessel and Hankel functions
Φ	Scalar fields in Minkowski space
$\phi_{\vec{k}}(\eta)$	Mode expansion fields
$\varphi(\eta, \vec{x})$	Scalar field in de sitter
$\varphi_{\vec{k}}(\eta)$	Scalar field in momentum space of de Sitter
$\mathcal{X}_{AB}$	Elements Of SO(1,4) group
$\Box_{dS}$	d'Alembert operator in $\mathbb{M}^5$ which respecting de Sitter surface

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
3D	Three Dimensional
CMB	Cosmic Microwave Background
FLRW	Friedmann–Lemaître–Robertson–Walker
QFT	Quantum Field Theory

1. INTRODUCTION

Cosmology as an attempt to explore the large structure of the universe is a well-established field of physics concerning the major discoveries such as expansion of the universe [6], cosmic microwave background [7], galactic and intergalactic structures [8], the formation of stars, etc. However, there are some unexpected patterns in the universe that surprise us so that they made us question on our theories of cosmology. For instance, if the initial phase of the universe is taken as radiation dominated there would exist causally independent regions so that they would not have opportunity to interact in their history, yet we observe a universe on very large scale to exhibit notable physical similarities such as temperature [2]. In other words, these regions would be space-like separated throughout whole their existence if the universe was initially radiation dominated. They would never interacted and exchanged energy such as heat. Inflation with quantum fluctuations, therefore, serves the one of the most promising solutions to these kind of anomalies in the history of the universe.

The phenomenon of large-scale homogeneity observed in the universe brings about a renowned challenge known as the horizon problem, for which inflation theory stands as the most promising solution [2]. Introducing an inflation phase to universe such that it expands rapidly can be represented by de Sitter metric. de Sitter space, therefore, as a special case of FLRW geometries, comes into the game to understand the possible results of the quantum fluctuations at the inflation phase which might explain the horizon problem [2, 9].

Although it is not yet clear what type of field(scalar, vector, tensor) could possibly maintained the inflation era, studying scalar fields as a simple toy model provides an understanding of the procedure that should be followed while constructing any ranked tensor field theory in de Sitter [1, 3]. In addition, scalar field theory in de Sitter might also explain the CMB spectrum which is briefly discussed in Section 3.

Quantum Field Theory in Curved spacetime is covered in two famous books [10, 11] for those looking for the generally used methods and more technical details. As an essential ingredient of Quantum Field Theory, two-point correlation functions in general and massless two point functions for scalar fields in particular are of main interest for this study. What makes them worth studying is that massless two point functions have particular difficulties and unresolved essential points such as IR(infrared) divergences that inevitably appear in the massless limit [3]. For a two-point function solution to be meaningful, the boundary conditions should be respected by two-point functions. For mode functions in dS, it is common to choose the Bunch-Davies vacuum in the early time limit [2] and there is no actual difficulty in deciding on solutions in momentum space. In position space, however, neither massless limit of the two-point function is well defined nor its early time limit can be matched to any reasonable boundary conditions owing to uncontrollable singularities.

On the way of understanding the boundary conditions, we first summarized the general properties of de Sitter geometry and its origin in Chapter 2. Those who are not familiar with de Sitter geometry are welcomed to recap Chapter 2. Appreciating the geometrical structure of de Sitter is essential to understand how a scalar field is affected by a fixed background geometry, here de Sitter. Accordingly, by ignoring the metric-field couplings, we offer a self-consistent and properly normalized real free scalar field theory in the Chapter 3.1.

Group theoretical aspects and symmetries of de Sitter space is discussed in section 2.2 and we present how eigenvalues of Casimir element categorizes different representations, namely principle, complementary, and discrete series (see Table 2.1). It can be seen from the Casimir element–Killing vector relations in Equation (2.17) that the symmetries of the space is directly related to symmetries of two-point functions since two-point functions solve the Klein-Gordon equation as in Equation (3.4).

Normally, in a homogeneous and isotropic space, two-point functions are expected to be only a function of the distance between two points [1]. Consequently, they are supposed to exhibit de Sitter invariant behavior. Although this expectation is properly satisfied by massive two-point functions, the situation becomes less clear for the massless two point functions.

One of the main problems is that two point function in position space(Wightman Function) for massive fields is well defined for heavy and light scalars but their massless limit are uncontrollably divergent [3]. On the other hand, Wightman Function(see Equation (3.3)) as a solution to following equation:

$$\Box_{dS} G_w(X,Y) - m^2 G_w(X,Y) = 0 \tag{1.1}$$

exists if the $m \to 0$ limit is taken in advance of solving the equation. Either way, understanding boundary conditions with standard methods for the massless Wightman function is ambiguous. For instance, for the massive case, imposing the condition that two-point function on de Sitter should behave as in Minkowski for incident point limit gives us a valuable idea about boundary conditions [12] which is not the case for the massless two-point function. Another valuable -but cumbersome- method to understand boundary conditions of massless Wightman Function is using the Fourier transform of the two-point function in momentum space, which is already set to respect the initial Bunch-Davies conditions.

Fourier transform method is discussed in Allen's paper [4] to investigate the symmetries of massless two-point functions. It is showed that de Sitter symmetries that is expected from a two-point function to obey is broken while conserving symmetries of some subgroups such as $O(4), E(3), O(1,3)$. Furthermore, these kind of anomalies in the massless case cause the famous vacuum problem in de Sitter called α-vacua. α stands as a label for each vacua and implies that there are infinitely many de Sitter vacuum [4,13,14]. Lack of unique vacuum is really crucial problem of de Sitter because observer dependent vacuum might cause a certain state of field can be measured as vacuum, one particle, two particle... states in an observer dependent way.

Considering these, we decided to pursue a Fourier Transformation procedure which Bruce Allen [4] uses for another reasons. He uses the Fourier Transformation of the two-point functions to show how de Sitter invariance is broken while preserving invariance under some subgroups of $O(1,4)$. Keeping or disclaiming of de Sitter invariance is and open discussion problem. Locality and dS Invariance of two-point functions have not been fully established and understand concepts yet [3]. Consequently, we try to understand the boundary conditions and singularity structures of the massless two point functions by taking the Fourier Transformation of the two-point function from momentum space to position space. This is because two-point functions in momentum space are already properly normalized and fixed according to the Bunch-Davies vacuum. Finally, the calculations converged to some extent but resulted in under-constrained equations that have free parameters that cannot be fixed by any physical input. Nevertheless, singularity structure of the incident point limit coincide with the Minkowskian type two-point singularities.

In addition to our naive attempts, we conclude by pointing out two more promising and contemporary solutions [3,5] to such uncontrollable divergences in the massless limit of the two-point function in the Chapter 5. Both study suggests some sort of modifications on the form of two-point function that solves the $\mathrm{m} \to 0$ limit problem to a certain extent.

2. COORDINATES AND EMBEDDING FORMALISM

In this section, we briefly present some widely used coordinate systems for de Sitter space. Each coordinate system has its own advantages and can be better suited for a specific purpose. For example, in conformal coordinates, metric is conformal to Minkowskian metric with a conformal factor, say $a^2(\eta)$, which makes equations simpler. For those experienced in coordinate systems for de Sitter are welcomed to skip this section keeping in mind that we used Conformal Planar Patch as a main framework in this study, where the metric takes the form:

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2) \tag{2.1}$$

with $\vec{x} \in \mathbb{R}^3$ and $\eta \in (-\infty, 0]$.

In short, Conformal Planar Patch will be referred as Conformal Coordinates from now on.

As a main guide, we will follow the [15] for the geometry of the de Sitter. de Sitter space is one of the vacuum($T_{\mu\nu} = 0$) solutions to Einstein's Equations in case of positive cosmological constant Λ. Einstein equations reduces to following form for vanishing $T_{\mu\nu}$

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{2.2}$$

The reasoning begins with the most general spherically symmetric metric with a Lorentzian signature, which reads [15]:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.3}$$

and looking for Einstein's equations to restrict its form. Before addressing Einstein's equation, one more simplification can be made of the static metric above. In order for static metric to conserve its Lorentzian signature, which is (-,+,+,+) in our convention, both $f(r)$ and $g(r)$ should be positive. Therefore, it is more convenient to reparameterize the metric in the following form:

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.4}$$

Calculating Ricci Tensor components and letting Einstein's equations do its job result in:

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.5}$$

which has spacial radius $l = \sqrt{\frac{3}{\Lambda}}$. In conclusion, de Sitter metric in static coordinates ends up as:

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right)dt^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.6}$$

Note that at $r = l$, there is a singularity implying that dt part vanishes and dr part diverges. Therefore, de Sitter space possesses an intrinsic horizon at $r = l$(see $r = l = \frac{1}{H}$ line on Figure 2.3).

Other than static coordinates, flat slicing coordinates are also commonly used to cover de Sitter space. The coordinate transformation from static to flat slicing coordinates is given as follows [12]:

$$\begin{aligned} r &= \rho e^{\tau/l} \\ t &= \tau - \frac{l}{2} ln(\rho^2 e^{2\tau/l} - l^2), \end{aligned} \tag{2.7}$$

where the metric takes the form:

$$ds^2 = -d\tau^2 + e^{2\tau/l} d\vec{x}^2, \tag{2.8}$$

where $\vec{x} \in \mathbb{R}^3$ and $\tau \in (-\infty, +\infty)$. Expansion of the universe can be seen by looking at monotonically increasing function in front of the spatial part of the metric. Moreover, it can be related to general form of Friedmann–Lemaître–Robertson–Walker [16]:

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2 \tag{2.9}$$

with $a(t) = e^{t/l}$. Apparently, we abused the notation a bit by using t and τ interchangeably in the last equation but it is only about labeling the variables so there is no actual problem.

As a final step, we can go from flat slicing coordinates to conformal coordinates which appears as conformally Minkowskian metric [17]. Introducing the new parametrization of time η as follows:

$$\eta = -\frac{1}{H} e^{-Ht}, \tag{2.10}$$

where $\eta \in (-\infty, 0]$ and we replaced l with Hubble constant according to $l = 1/H$. Then in conformal coordinates: dS metric becomes:

$$ds^2 = \frac{1}{H^2\eta^2}(-dt^2 + d\vec{x}^2). \tag{2.11}$$

There are some benefits of working with conformally Minkowskian coordinates such as angles between vectors is independent of conformal factor. Therefore, space-like and time-like geodesics show up between $0 - 45°$, $45° - 90°$, respectively and null-geodesic travels with $45°$ in spacetime diagrams which is the same scenario as geodesics in Minkowskian Space. To see them pictorially, we can draw Penrose Diagrams of de Sitter space for different coordinate systems [17].

Global coordinates is a good starting point to draw Penrose Diagrams since they cover whole de Sitter Space. The metric in Global coordinates looks like:

$$ds^2 = -d\tau^2 + cosh^2(H\tau)d\Omega_3^2 \tag{2.12}$$

with the $d\Omega_3$ metric of 3-Sphere. Then, Penrose diagram in global coordinates shows up as in Figure 2.1.

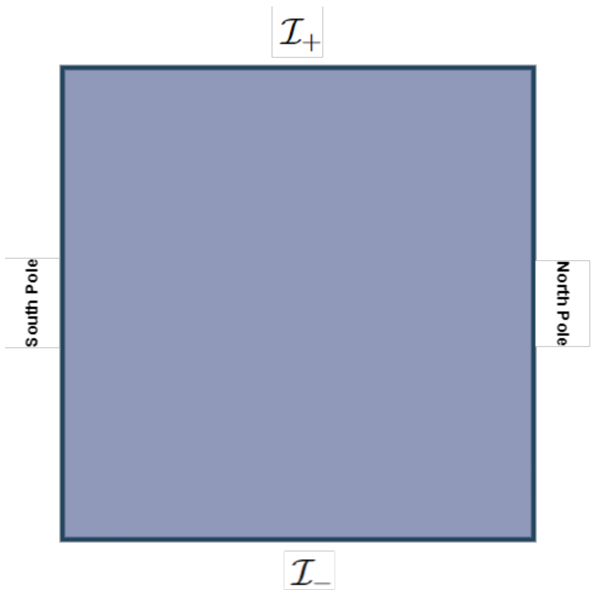

Figure 2.1. Penrose diagram of de Sitter in global coordinates.

Conformal Planar Coordinates, on the other hand, covers the one half of the de Sitter at a time(upper or lower half can be chosen by replacing $\tau \to -\tau$ in Equation (2.8)) which can be drawn as follows.

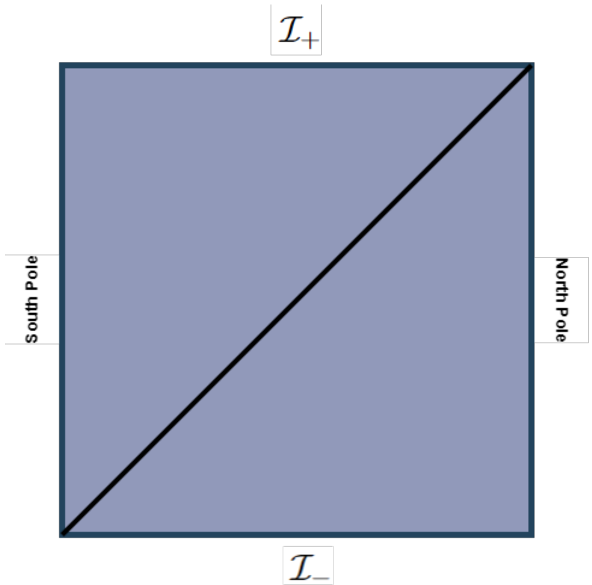

Figure 2.2. Penrose diagram of de Sitter in planar coordinates.

Finally, static coordinates introduced in Equation(2.6) can be considered physical observer's coordinates [17] with the horizon at $r = l$ and can be visualized(yellow region represents the static coordinates coverage) as a Penrose Diagram as in Figure 2.3.

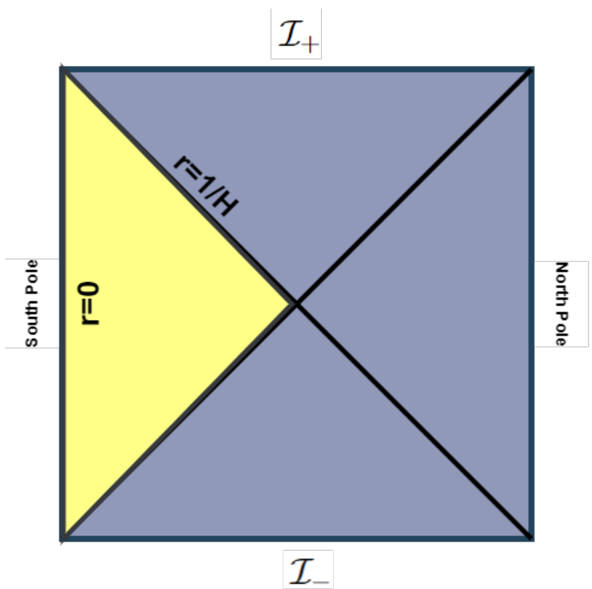

Figure 2.3. Penrose diagram of de Sitter in static coordinates.

Those who are looking for more deep understanding are welcomed to visit review of de Sitter Geometry in [17].

2.1. De Sitter as Embedding

An alternative way to understand de Sitter space is considering it a 4-dimensional hypersurface embedded in 5-dimensional(1,4) Minkowskian Space. We can create a hypersurface by imposing following constraints on the coordinates of the Minkowskian (embedding) space coordinates:

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = \frac{1}{H^2}, \tag{2.13}$$

where $X_A \in \mathbb{M}_{1,4}$ for $A = 0, 1, 2, 3, 4$.

Here, we only recite and skip the global coordinates since they are out of our scope for this study.

Whole de Sitter can be covered by Global Coordinates

$$\begin{aligned} X_0 &= \frac{1}{H} sinh(H\tau) \\ X_1 &= \frac{1}{H} cosh(H\tau) cos\theta_1 \\ X_2 &= \frac{1}{H} cosh(H\tau) sin\theta_1 cos\theta_2 \\ X_3 &= \frac{1}{H} cosh(H\tau) sin\theta_1 sin\theta_2 cos\theta_3 \\ X_4 &= \frac{1}{H} cosh(H\tau) sin\theta_1 sin\theta_2 sin\theta_3, \end{aligned} \tag{2.14}$$

where capital X's denote 5D Minkowski Coordinates. Consquently, metric takes the form:

$$ds^2 = -d\tau^2 + cosh^2(H\tau) d\Omega_3^2. \tag{2.15}$$

As we introduced in the earlier chapter, we use conformal coordinates throughout the thesis. Each coordinate patch captures the certain part of de Sitter space and serves different purpose. Conformal Patch, for example, covers half of the space which emerge from one single point in the past infinity. [12] It is important to access the relations between embedding coordinates (X^A) and de Sitter coordinates (x^μ) for further calculations so we list them here for the Conformal Coordinates:

$$\begin{aligned} \rightarrow X^0 &= \frac{1}{2H\eta}(\eta^2 - 1 - \vec{x}^2), \\ \rightarrow X^i &= -\frac{1}{H\eta}x^i, \\ \rightarrow X^4 &= \frac{1}{2H\eta}(\vec{x}^2 - 1 - \eta^2), \end{aligned} \tag{2.16}$$

where we used shorthand notation for $x^0 = \eta$ and $x^i = \vec{x}$. We can also invert the relations above and write inverse relations as: [18]

$$\begin{aligned} \rightarrow \eta &= -\frac{1}{H}(X^0 + X^4)^{-1}, \\ \rightarrow x^i &= \frac{X^i}{(X^0 + X^4)}. \end{aligned} \tag{2.17}$$

2.2. Symmetries and Killing Vectors of de Sitter

de Sitter space, as an embedded manifold to Minkowski Space ($M_{1,4}$), inherits its symmetries from $M_{1,4}$ so de Sitter space naturally respects the $SO(1,4)$ symmetry group. SO(1,4) lie algebra can be written in the standard basis $\mathcal{X}_{AB}$ with the following commutation relation [18]:

$$[\mathcal{X}_{AB}, \mathcal{X}_{CD}] = \eta_{BC}\mathcal{X}_{AD} - \eta_{AC}\mathcal{X}_{BD} + \eta_{AD}\mathcal{X}_{BC} - \eta_{BD}\mathcal{X}_{AC}, \tag{2.18}$$

where $\eta_{AB} = diag(-1, 1, 1, 1, 1)$ and

$$\mathcal{X}_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A} \tag{2.19}$$

for $A, B = 0, 1, 2, 3, 4$. Components of the $\mathcal{X}_{AB}$ can be calculated by using the coordinate relations listed in Equation (2.16) and Equation (2.17). One can check appendix for the detailed calculation of the components of $\mathcal{X}_{AB}$ in Appendix B.1,

As a result, certain components of the $\mathcal{X}_{AB}$ are composed of combination of the Killing vectors as follows:

$$\begin{aligned} \mathcal{X}_{0i} &= \frac{1}{2}\left[\mathcal{C}_i - \mathcal{T}_i\right], \\ \mathcal{X}_{4i} &= \frac{1}{2}\left[\mathcal{C}_i + \mathcal{T}_i\right], \\ \mathcal{X}_{ij} &= \mathcal{R}_{ij}, \\ \mathcal{X}_{04} &= \mathcal{D}, \end{aligned} \tag{2.20}$$

where $\mathcal{D}, \mathcal{T}_i, \mathcal{R}_{ij}, \mathcal{C}_i$ correspond to dilatations, translations, rotations and special conformal transformations killing vectors respectively. Moreover, inverse relations between $\mathcal{X}_{AB}$ and Killing vectors are also worth mentioning:

$$\begin{aligned} \mathcal{C}_i &= \mathcal{X}_{0i} + \mathcal{X}_{4i}, \\ \mathcal{T}_i &= \mathcal{X}_{4i} - \mathcal{X}_{0i}, \\ \mathcal{R}_{ij} &= \mathcal{X}_{ij}, \\ \mathcal{D} &= \mathcal{X}_{04}. \end{aligned} \tag{2.21}$$

Killing vectors as solution to equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ are listed below in conformal coordinates with unit parameters [18]:,

$$\begin{aligned} \mathcal{C}_i &= 2x_i\eta\frac{\partial}{\partial\eta} + \left[2x^j x_i + \left(\eta^2 - |\vec{x}|^2\right)\delta_i^j\right]\frac{\partial}{\partial x^j}, \\ \mathcal{T}_i &= \partial_i, \\ \mathcal{R}_{ij} &= x_i\frac{\partial}{x^j} - x_j\frac{\partial}{\partial x^i}, \\ \mathcal{D} &= -\eta\frac{\partial}{\partial\eta} - x^i\frac{\partial}{\partial x_i}. \end{aligned} \quad . \tag{2.22}$$

Another fundamental object, as usual, is Casimir element of the algebra. There are two Casimir elements which are [18]:

$$Q = -\frac{1}{2}\mathcal{X}_{AB}\mathcal{X}^{AB}$$
$$W = -w_A w^A, \tag{2.23}$$

where $w_A = -\frac{1}{8}\epsilon^{ABCDE}\mathcal{X}_{BC}\mathcal{X}_{DE}$. The first quadratic form Q can be used to label unitary representations as an eigenvalue problem. Second Casimir element W is trivial for spin 0 fields [18] so it brings nothing important. Consider Q operator with eigenvalues $m^2 \in \mathbb{C}$:

$$Q\psi(x) = m^2\psi(x). \tag{2.24}$$

with $\psi(x)$ is some arbitrary function of de Sitter coordinates. Surprisingly, Casimir element Q is related to d'Alembertian by $Q = H^{-2}\Box_{dS}$, see appendix B.2 for detailed demonstration of it. It is also conventional to study the eigenvalue problem with the redefinition of $m^2 = H^2\Delta(3-\Delta)$. Furthermore, for scalar fields, certain behaviours of m^2 (as well as Δ) correspond to different representations, namely, Principle Series, Complementary Series, Discrete Series and Exceptional Series(In 4D Exceptional and Discrete series are equal). The following table might be useful as a reference for reader [18].

Table 2.1. Representations and their labels for spinless fields in de Sitter.

s=0, D=4	Principle Series	Complementary Series	Discrete Series
Δ	$\frac{3}{2}+c$	$\frac{3}{2}+c$	0 or 3
m^2	$\frac{m^2}{H^2} > \frac{9}{4}$	$\frac{m^2}{H^2} < \frac{9}{4}$	$m^2 = 0$
c	$\pm i\nu$	$\pm\rho$	$\frac{3}{2}$

Note that c is generally called scaling weight with $c = \{\pm i\nu, \pm\rho\}$ and can be related to Δ as $\Delta = \frac{3}{2}+c$.

Mind that $\rho = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$ and $\nu = \sqrt{\frac{m^2}{H^2} - \frac{9}{4}}$. As a remark, the table 2.1 is specific to spinless and non-tachionic fields. For more general discussion see [18].

To summarize, range of the m^2 determines the representation. In addition, principle, complementary and discrete series are related to heavy, light, and massless fields as long as mass parameter m of the fields is concerned in table 2.1. A very short review of complementary and principle series can be found in [19] and relatively more comprehensive and detailed discussion of them exists in the [20]. Discrete series, however, is not a well-understood concept yet compared to principle and complementary cases due to some technical problems [4]; therefore, it requires alternative methods different from the massive cases.

3. TWO POINT FUNCTIONS

Two-point functions as special case of n-point correlation functions are extremely important quantities both in Quantum Field Theory and Cosmology. There are a variety of two point functions, each one serve special purposes and we listed some of them in Equation (3.3). For Quantum Field Theory in Minkowski Space, it is more-widely used that Green's function is written in the form of Feynman Propagator, which is a kind of time-ordered two point function(see (3.3)). Feynman Propagator in Minkowskian QFT , for example, is generally used to describe scattering processes in the presence of certain interactions.

While it is tried to apply the same procedure about interactions and scattering processes to cosmology, it is not a well-understood mechanism yet and it has its own technical details and difficulties which are beyond the scope of this study. One can still go through references [21–23] for further information on the S-Matrix formalism in de Sitter.

For example, [22] is a very comprehensive review on S-Matrix and presents the difficulty in preparing asymptotic states in de Sitter space because of the event horizon. Although they suggest possible solutions to observable problem by studying different types of geometries, it is claimed that S-Matrix itself cannot be a well defined observable in a physical de Sitter space. Instead it can be meta-observable that might give meaningful output only under certain conditions.

On the other hand, [21] emphasizes the famous problem of de Sitter space stems from the absence of time translation symmetry. Additionally, it is mentioned that a unique vacuum is not possessed by de Sitter Space while it is crucial to develop an S-matrix formalism in a space.

Without referring to an S-matrix, two-point functions in cosmology, play different roles that can be related to observations about the Cosmic Microwave Background by statistical arguments (see [2, 7, 9]).

Besides observational predictions, understanding the behaviour of two point functions of a scalar field in terms of boundary conditions might give important clues about the construction of higher rank tensor fields in de Sitter space. [1]. This being the case, we tried to piece some prominent approaches together to investigate or compare them and discussed some technical methods on the way of understanding the boundary conditions. In this manner, there are two main possibilities for us to examine two-point functions and its boundary conditions: through momentum space and position space two-point functions.

Two-point functions give idea about how much a field evaluated on two different points on space are correlated. Therefore, it might be argued that it is more meaningful to study two point functions in position space. Although it is right to some extend, in practice, some calculations happen to be much simpler in momentum space than position space or vice versa. Fortunately, introducing a proper framework will enable us to go back and forth between momentum and position space. Such a process can be very well done by Fourier Transformations as long as the integrals are "nice" enough to take them. In other words, Fourier transform of two point functions will give us freedom to use momentum and position space framework as long as we can take the integrals.

Especially in some cases, such as Cosmic Microwave Background observations, momentum space two-point functions can be addressed to spectrum of CMB data. How this reasoning goes is briefly presented in the following sections.

Here, a short review of propagators and Green's function in field theory will be presented for convenience in order for reader not to mix up between terms.

These are the formal definitions of the two-point function in position space assuming the fields are quantized.

$$G_w(x,y) = \langle 0|\varphi(\eta,\vec{x})\varphi(\eta',\vec{y})|0\rangle \rightarrow \text{(Wightman Function)} \tag{3.1}$$

$$G_{wh}(x,y) = \langle 0|\varphi(\eta,\vec{x})\varphi(\eta',\vec{y}) + \varphi(\eta',\vec{y})\varphi(\eta,\vec{x})|0\rangle \rightarrow \text{(Hadamard Form)} \tag{3.2}$$

$$G_F(x,y) = -i\langle 0|\Theta(\eta-\eta')G_w(x,y) + \Theta(\eta'-\eta)G_w(y,x)|0\rangle \rightarrow \text{(Feynman)}, \tag{3.3}$$

where $\Theta(\eta)$ is step function in time and deals with the causal order between two fields. From the relations above, it can be seen that how fundamental the Wightman Function is so that any propagator can be written in terms of it. It is also quite important to know which equation is satisfied by which two-point function in Equation (3.3) satisfies [24]:

$$\begin{aligned} \Box_{dS}G_w(X,Y) - m^2G_w(X,Y) &= 0 \\ \Box_{dS}G_{wh}(X,Y) - m^2G_{wh}(X,Y) &= 0 \\ \Box_{dS}G_F(X,Y) - m^2G_F(X,Y) &= \frac{1}{\sqrt{g}}\delta(X-Y). \end{aligned} \tag{3.4}$$

One of the motivations why we are interested in Wightman Function is that its Fourier transform can be related to power spectrum which possibly explains the power spectrum of the Cosmic Microwave Background. While quantization of the fields is left to the next section(see Section 3.1.1) one can take the following quantized field and expansion of the field in momentum space:

$$\varphi(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\varphi_k(\eta)e^{i\vec{k}.\vec{x}} \tag{3.5}$$

with

$$\varphi_k(\eta) = a_{\vec{k}}\phi_{\vec{k}}(\eta) + a^\dagger_{\vec{k}}\phi^*_{-\vec{k}}(\eta). \tag{3.6}$$

It also is essential to list the normalization relations which are chosen in Appendix A.1:

$$\begin{aligned}
[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] &= (2\pi)^3\delta^3(\vec{k}-\vec{k}') \\
\langle\vec{k}|\vec{k}'\rangle &= (2\pi)^3\delta^3(\vec{k}-\vec{k}'), \\
[\varphi(\eta,\vec{x}), \pi(\eta,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}).
\end{aligned} \tag{3.7}$$

Then using the definition of Wightman Function we can write:

$$\begin{aligned}
G_w(x,y) &= \langle 0|\varphi(\eta,\vec{x})\varphi(\eta,\vec{y})|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}[\phi_{\vec{k}}(\eta)\phi^*_{\vec{k}}(\eta)]e^{i\vec{k}.(\vec{x}-\vec{y})} \\
&= \int \frac{d^3k}{(2\pi)^3}P_k(\eta)e^{i\vec{k}.(\vec{x}-\vec{y})},
\end{aligned} \tag{3.8}$$

where we introduced a new function $P_k(\eta) = \phi_{\vec{k}}(\eta)\phi^*_{\vec{k}}(\eta)$ called power spectrum. This new quantity is closely related to power spectrum of Cosmic Microwave Background. To understand the relation, a few algebraic steps to be taken too for the equation above. We can rewrite the equation above by taking $\vec{x}$ along the k_z direction. By doing so, we can take angle integrals easily and the equation ends up with:

$$\begin{aligned}
G_w(x,y) &= \int \frac{dk}{2\pi^2}\frac{sin(kr)}{r}kP_k(\eta)e^{ikr} \\
&= \int \frac{d(lnk)}{2\pi^2}\frac{sin(kr)}{kr}k^3P_k(\eta)e^{ikr} \\
&= \int d(lnk)\frac{sin(kr)}{kr}\mathit{\Delta}_k(\eta)e^{ikr}.
\end{aligned} \tag{3.9}$$

Again, we introduced a new function called dimensionless power spectrum:

$$\mathit{\Delta}_k(\eta) = \frac{k^3}{2\pi^2}P_k(\eta). \tag{3.10}$$

Approximate behaviour of this function can be written as:

$$\mathit{\Delta}_k(\eta) \propto k^{n_s-1}, \tag{3.11}$$

where n_s is called spectral index and it is the physical observable from experiments [7]. Interestingly, the power spectrum of the Cosmic Microwave Background data almost agrees with a scale invariant spectrum Δ_k in k which corresponds to $n_s = 1$. In 2018, for example, spectral index was measured in Planck data [7] as follows:

$$n_s = 0.964 \pm 0.0042. \tag{3.12}$$

We will also see in section 3.1 that in the case of massless two point functions, power spectrum possess the scale invariant property, that is, $n_s = 1$.

All in all, one of the possible explanation for homogeneous Cosmic Microwave Background might come from studying a scalar quantum fields in de Sitter. This is one of the promising observational motivations behinds these kind of studies. For this reason, constructing a well-established Quantum Field Theory on de Sitter might shed light on some other aspects of the universe too. [2].

Among scalar field theories theories in de Sitter, for example, massless scalar field is of main interest for us since massless two point function of scalar fields hands us a k-independent power spectrum Δ_k. Therefore, investigating the properties of the massless scalar field will be our main concern yet we will briefly cover other two cases as well, namely, light and heavy massive scalar fields. Unfortunately, the boundary conditions of the two-point functions are relatively more appreciated for heavy and light scalar fields compared to massless one.

The subsequent chapter, therefore, is devoted to general properties of two point functions in position and momentum space but their connections through Fourier Transformation is left to the chapters 4 and 5.

3.1. Two Point Functions in Momentum Space via Canonical Quantization

Occasionally, the physical world is described in a much simpler way, or the equations take more elegant forms to solve in momentum space. Once fields are solved in momentum space, for example, any desired quantity which is some function of the fields can be transformed back to the position space by Fourier Transformation. For the scope of this study, we will be interested in two-point functions in both momentum and position space. Accordingly, how we can go back and forth between momentum space two-point functions to position space is a straightforward calculation if one is good at taking integrals.

Before move on two point functions, it is more appropriate to establish a self-consistent notation for scalar field theory in de Sitter. The following section, therefore, is devoted to mode functions with their normalization and quantization.

3.1.1. Mode Functions for Free Scalar Fields in de Sitter

As a crucial first step, we should quantize our fields and establish a consistent framework between position and momentum spaces. Examining a scalar field in de Sitter space by action principle would work properly. We can write a free scalar field in curved space-time with some mass parameter m as:

$$S = \int \sqrt{|g|} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right) d\eta d^3x, \tag{3.13}$$

where minus sign in front of the first term is conventional to make sure the kinetic term with time derivative appears as positive and g is the determinant of the metric $g_{\mu\nu}$ for:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{1}{\eta^2 H^2} \left(-d\eta^2 + d\vec{x}^2 \right), \end{aligned} \tag{3.14}$$

where $\vec{x} \in \mathbb{R}$ and $\eta \in (-\infty, 0]$. Assuming the metric $g_{\mu\nu}$ is just a background and does not couple with the fields, we can take the variation of action with respect to field φ.

$$\begin{aligned}\delta S[\varphi] &= \delta \int \sqrt{|g|}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right) d\eta d^3x \\ &= 2\int \sqrt{|g|}\left(-g^{\mu\nu}\partial_\mu\varphi\partial_\nu(\delta\varphi) - m^2\varphi\delta\varphi\right) \\ &= 2\int \left[\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu\varphi) - \sqrt{|g|}m^2\varphi\right]\delta\varphi = 0.\end{aligned} \tag{3.15}$$

Since this relation is hold for any $\delta\varphi$, we can write the equation of motion for scalar field in de Sitter as:

$$-\partial_\nu(\sqrt{|g|}g^{\mu\nu}\partial_\mu\varphi) + \sqrt{|g|}m^2\varphi = 0 \tag{3.16}$$

or in more implicit form:

$$\ddot{\varphi}(x,\eta) - \frac{2\dot{\varphi}(x,\eta)}{\eta} - \nabla^2\varphi(x,\eta) + \frac{m^2\varphi(x,\eta)}{H^2\eta^2} = 0, \tag{3.17}$$

where dots over φ's denote derivative with respect to conformal time η.

This is a partial differential equation and it is not that easy to solve it in general. Nevertheless, we can introduce a mode expansion of $\varphi(x,\eta)$' in momentum space with a promise of taking Fourier Transform of the fields whenever needed. We are looking for a real scalar fields so the general form can be written as follows:

$$\varphi(x,\eta) = \int \left[A_{\vec{k}}\phi_{\vec{k}}(\eta)e^{i\vec{k}.\vec{x}} + A^*_{\vec{k}}\phi^*_{\vec{k}}(\eta)e^{-i\vec{k}.\vec{x}}\right]\frac{d^3k}{(2\pi)^3}, \tag{3.18}$$

where $A_{\vec{k}}$'s are some complex number and will be responsible for normalization and quantization within few pages. $\phi_{\vec{k}}$'s, on the other hand, can be called as mode functions. Still, there can be made further simplification in the equation (3.18) before we address this solution.

First of all, one can easily check that imposing reality condition on the fields $\varphi(x,\eta)$ yields a condition on the $A_{\vec{k}}$'s as:

$$A^*_{\vec{k}} = A_{-\vec{k}}. \tag{3.19}$$

Secondly, making change of variable ($\vec{k} \to -\vec{k}$) in the second term in the equation (3.18) results in:

$$\varphi(\eta,\vec{x}) = \int \left[A_{\vec{k}}\phi_{\vec{k}}(\eta) + A^*_{-\vec{k}}\phi^*_{-\vec{k}}(\eta)\right] e^{i\vec{k}.\vec{x}} \frac{d^3k}{(2\pi)^3}. \tag{3.20}$$

Calling a new mode function $\varphi_{\vec{k}}(\eta) = A_{\vec{k}}\phi_{\vec{k}}(\eta) + A^*_{-\vec{k}}\phi^*_{-\vec{k}}(\eta)$ makes the relation even much simpler:

$$\varphi(\eta,\vec{x}) = \int \varphi_{\vec{k}}(\eta) e^{i\vec{k}.\vec{x}} \frac{d^3k}{(2\pi)^3}. \tag{3.21}$$

Finally, taking this expansion and inserting it into the action given in the equation (3.13) gives an action integral which is function of $\varphi_{\vec{k}}(\eta)$ this time. Again, minimizing the action by varying with respect to $\varphi_{\vec{k}}(\eta)$ hands us the following equation of motion for the mode functions:

$$\eta^2\ddot{\varphi}_{\vec{k}}(\eta) - 2\eta\dot{\varphi}_{\vec{k}}(\eta) + \varphi_{\vec{k}}(\eta)\left(k^2\eta^2 + \frac{m^2}{H^2}\right) = 0, \tag{3.22}$$

which is much simpler equation to solve compared to partial differential equation obtained in (3.17). One of the legitimate ways of solving this equation is to make it resemble to a well known differential equation whose solution is named after the famous mathematician who solved it. Here is the idea; we can make an intelligently guessed change of variable as $\varphi_{\vec{k}}(\eta) = \eta^{\frac{3}{2}}\chi_{\vec{k}}(\eta)$ transform the equation into :

$$\frac{d^2\chi_{\vec{k}}(u)}{du^2} + \frac{1}{u}\frac{d\chi_{\vec{k}}(u)}{du} + \chi_{\vec{k}}(u)\left(1 + \left(\frac{m^2}{H^2} - \frac{9}{4}\right)\frac{1}{u^2}\right) = 0, \tag{3.23}$$

which is in the form of Bessel's Equation:

$$\frac{d^2\chi_{\vec{k}}(u)}{du^2} + \frac{1}{u}\frac{d\chi_{\vec{k}}(u)}{du} + \chi_{\vec{k}}(u)\left(1 - \frac{\nu^2}{u^2}\right) = 0, \tag{3.24}$$

with $\nu^2 = \dfrac{9}{4} - \dfrac{m^2}{H^2}$ and $u = k\eta$. Then, the solution for $\chi_{\vec{k}}(u)$ is given in terms of Bessel Functions: $J_\nu(\eta)$ and $Y_\nu(\eta)$. Then, any linear combination of Bessel functions would work in the absence of a boundary condition. We pick Bunch-Davies vacuum [2] at the beginning of the universe so that mode functions match with that vacuum in the early time limit $\eta \to -\infty$ and luckily we have a nice linear combination of the Bessel Functions that successfully satisfies this limit behaviour, namely, Hankel Functions:

$$\begin{aligned} H_\nu^{(1)}(k\eta) &= J_\nu(k\eta) + iY_\nu(k\eta), \\ H_\nu^{(2)}(k\eta) &= J_\nu(k\eta) - iY_\nu(k\eta). \end{aligned} \tag{3.25}$$

where $H_\nu^1(k\eta)$ and $H_\nu^2(k\eta)$ are Hankel function of the first and second kind respectively. Up to this point, our scalar fields became determined up to normalization:

$$\varphi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\varphi_{\vec{k}}(\eta)e^{i\vec{k}.\vec{x}} \tag{3.26}$$

together with:

$$\begin{aligned} \varphi_{\vec{k}}(\eta) &= N a_{\vec{k}}\phi_{\vec{k}}(\eta) + N^* a^\dagger_{-\vec{k}}\phi^*_{-\vec{k}}(\eta) \\ &= N a_{\vec{k}} H_\nu^{(1)}(k\eta) + N^* a^\dagger_{-\vec{k}} H_\nu^{(2)}(k\eta), \end{aligned} \tag{3.27}$$

where N is normalization constant which will be determined by Klein-Gordon Normalization. Also note that, we quantized the fields by replacing $A_{\vec{k}}$ and $A^*_{\vec{k}}$ by creation and annihilation operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. Process of setting up a self-consistent normalized theory is presented in the appendix A.1 yet we recite them for convenience:

$$\begin{aligned} [a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] &= (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \\ \langle \vec{k} | \vec{k}' \rangle &= (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \\ [\varphi(\eta, \vec{x}), \pi(\eta, \vec{y})] &= i\delta^3(\vec{x} - \vec{y}). \end{aligned} \tag{3.28}$$

Klein-Gordon normalization is a condition that comes on mode functions $f_{\vec{k}}(\eta, \vec{x})$'s naturally once you try to construct a normalized theory:

$$\begin{aligned} \big(f_{\vec{k}}(\vec{x}, \eta), f_{\vec{k}'}(\vec{x}, \eta)\big)_{KG} &= -i \int d^3x \sqrt{|g|} g^{0\mu} \Big[f^*_{\vec{k}}(\vec{x}, \eta) \partial_\mu f_{\vec{k}'}(\vec{x}, \eta) - \\ &\quad f_{\vec{k}'}(\vec{x}, \eta) \partial_\mu f^*_{\vec{k}}(\vec{x}, \eta) \Big] \\ &= \delta_{kk'}, \end{aligned} \tag{3.29}$$

where $f_{\vec{k}}(\eta, \vec{x}) = N e^{i\vec{k}.\vec{x}} \phi_{\vec{k}}(\eta)$.

3.1.2. Klein-Gordon Normalization of the Fields

Recall that we expected for our field $\varphi(\eta, \vec{x})$ can be expanded as follows:

$$\varphi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[a_{\vec{k}} f_{\vec{k}}(\vec{x}, \eta) + a^{\dagger}_{\vec{k}} f^*_{\vec{k}}(\vec{x}, \eta) \right], \tag{3.30}$$

where

$$f_{\vec{k}}(\vec{x}, \eta) = N \phi_{\vec{k}}(\eta) e^{i\vec{k}.\vec{x}}. \tag{3.31}$$

Furthermore, Klein-Gordon Normalization reads:

$$\begin{aligned}\big(f_{\vec{k}}(\vec{x},\eta), f_{\vec{k'}}(\vec{x},\eta)\big) =& -i\int d^3x\sqrt{|g|}g^{0\mu}\Big[f^*_{\vec{k}}(\vec{x},\eta)\partial_\mu f_{\vec{k'}}(\vec{x},\eta) \\ & - f_{\vec{k'}}(\vec{x},\eta)\partial_\mu f^*_{\vec{k}}(\vec{x},\eta)\Big] \\ &= 1.\end{aligned} \tag{3.32}$$

We recite the metric and inverse metric component for convenience:

$$\begin{aligned}g^{\mu\nu} &= diag(-H^2\eta^2, H^2\eta^2, H^2\eta^2, H^2\eta^2) \\ g_{\mu\nu} &= diag(-\frac{1}{H^2\eta^2}, \frac{1}{H^2\eta^2}, \frac{1}{H^2\eta^2}, \frac{1}{H^2\eta^2}) \\ \sqrt{|g|} &= \frac{1}{\eta^4 H^4}.\end{aligned} \tag{3.33}$$

Firstly, observe relation in (3.32)that we have contribution only from the term with g^{00}, that is, $\mu = 0$ and only η derivative acts on $f_{\vec{k}}$ terms.

Secondly, the constraint on $\varphi_{\vec{k}}(\eta)$ for our fields $\varphi(\eta,\vec{x})$ to be a real field is :

$$\varphi_{-\vec{k}}(\eta) = \varphi^*_{\vec{k}}(\eta) \tag{3.34}$$

so we have:

$$\begin{aligned}\big(f_{\vec{k}}(\vec{x},\eta), f_{\vec{k'}}(\vec{x},\eta)\big) =& -i\int d^3x\sqrt{|g|}g^{00}\Big[\underbrace{f^*_{\vec{k}}(\vec{x},\eta)\partial_0 f_{\vec{k'}}(\vec{x},\eta)}_{\text{include } e^{i(\vec{k}'-\vec{k}).\vec{x}}} \\ & - \underbrace{f_{\vec{k'}}(\vec{x},\eta)\partial_0 f^*_{\vec{k}}(\vec{x},\eta)}_{\text{include } e^{i(\vec{k}'-\vec{k}).\vec{x}}}\Big].\end{aligned} \tag{3.35}$$

As given above, under-braced terms include $e^{i(\vec{k}'-\vec{k}).\vec{x}}$ and taking the integral over x results in $\vec{k'} \to \vec{k}$ according to following identity:

$$\int e^{i(\vec{k}'-\vec{k}).\vec{x}}d^3x = (2\pi)^3\delta(\vec{k'}-\vec{k}). \tag{3.36}$$

Then, it ends up with:

$$\left(f_{\vec{k}}(\vec{x},\eta), f_{\vec{k}'}(\vec{x},\eta)\right) = -i(2\pi)^3\sqrt{|g|}g^{00}\left[N^*\phi^*_{\vec{k}}(\eta)\partial_0(N\phi_{\vec{k}}) - N\phi_{\vec{k}}(\eta)\partial_0(N^*\phi^*_{\vec{k}})\right], \quad (3.37)$$

which is:

$$-i(2\pi)^3\sqrt{|g|}g^{00}|N|^2\left[\phi^*_{\vec{k}}\partial_0\phi_{\vec{k}} - \phi_{\vec{k}}\partial_0\phi^*_{\vec{k}}\right]. \quad (3.38)$$

Inserting: $\partial_0\phi_{\vec{k}}(\eta) = \frac{\partial}{\partial\eta}\left(\eta^{3/2}H^{(1)}_\nu(k\eta)\right) = \frac{3}{2}\eta^{1/2}H^{(1)}_\nu(k\eta) + \eta^{3/2}\frac{\partial}{\partial\eta}H^{(1)}_\nu(k\eta)$

and using the property of Hankel functions: $\left(H^{(1)}_\nu(k\eta)\right)^* = H^{(2)}_\nu(k\eta)$ results in:

$$\begin{aligned}\left(f_{\vec{k}}(\vec{x},\eta), f_{\vec{k}'}(\vec{x},\eta)\right) = &-i(2\pi)^3\sqrt{|g|}g^{00}|N_{\vec{k}}|^2\eta^3 k\Bigg(H^{(2)}_\nu(k\eta)\frac{\partial H^{(1)}_\nu(k\eta)}{\partial(k\eta)} \\ &- H^{(1)}_\nu(k\eta)\frac{\partial H^{(2)}_\nu(k\eta)}{\partial(k\eta)}\Bigg),\end{aligned} \quad (3.39)$$

where $W\left(H^{(2)}_\nu(k\eta), H^{(1)}_\nu(k\eta)\right)$ is the Wronskian of $H^{(1)}_\nu(k\eta)$ and $H^{(2)}_\nu(k\eta)$ which can be easily obtained from the DLMF [25] website:

$$W\left(H^{(2)}_\nu(k\eta), H^{(1)}_\nu(k\eta)\right) = -W\left(H^{(1)}_\nu(k\eta), H^{(2)}_\nu(k\eta)\right) = \frac{4i}{\pi k\eta}. \quad (3.40)$$

Combining all relations together results in:

$$N = \frac{H}{4\pi\sqrt{2}}. \quad (3.41)$$

Finally, our K-G normalized field becomes:

$$\varphi(\eta,\vec{x}) = \frac{H}{4\pi\sqrt{2}}\int\left(a_{\vec{k}}\phi_{\vec{k}}(\eta)e^{i\vec{k}.\vec{x}} + a^\dagger_{\vec{k}}\phi^*_{\vec{k}}(\eta)e^{-i\vec{k}.\vec{x}}\right)\frac{d^3k}{(2\pi)^3}. \quad (3.42)$$

It is worth noting that we give up the notation $N\phi_k(\eta)$ for normalized fields from now on. Instead we will use $\phi_{\vec{k}}(\eta)$ as normalized field by definition :

$$\phi_{\vec{k}} = \frac{H}{4\pi\sqrt{2}}\eta^{\frac{3}{2}}H_{\nu}^{(1)}(k\eta). \tag{3.43}$$

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3.2. Position Space(Wightman Function)-Massive Case

Wightman Function($G_w(X,Y)$), as presented in the Introduction section, is the solution to the following equation and can be considered as two-point function in position space

$$\Box_{dS}G_w(X,Y) - m^2G_w(X,Y) = 0. \tag{3.44}$$

We are working with de Sitter space as a four dimensional hypersurface embedded in five dimensional Minkowski space with same signature. Therefore, the $\Box_{dS}$ stands for five dimensional d'Alembertian operator projected on the dS_4 surface.

X and Y are Lorentzian coordinates of two points live on de Sitter hypersurface with the property that Y is fixed but X is free to move on the surface. Living in an homogeneous universe, motivates us to pick Wightman Function as a function of only distance between two points $r = X - Y$. Isotropy in the universe, on the other hand, lets Wightman Function to be angle independent, that is, function of only the magnitude of the separation vector between two points: $|r| = |X - Y|$. As a result of these settings, it is common and useful to work with the new variables $P(X,Y)$ and $z = \dfrac{P+1}{2}$.

Consider the scalar product of two vectors of $M_{1,4}$ which points the $X(x)$ and $Y(y)$ on the de Sitter surface:

$$\eta_{AB}X^A(x)Y^B(y) = |X||Y|cos\theta, \tag{3.45}$$

where small x and y stand for the corresponding de Sitter coordinates on the surface for X and Y respectively.

The most remarkable advantage of the variable P is that defining geodesic distance between two points becomes trivial in the following sense:

$$\begin{aligned} P(X,Y) &= H^2\eta_{AB}X^A(x)Y^B(y) \\ &= cos\theta, \end{aligned} \tag{3.46}$$

where θ is the angle separation between two points and $|X| = |Y| = 1/H$.

It is also clear that $P(X,Y)$ is de Sitter invariant so it can be a proper variable to describe the Wightman Function as $G_w(P)$.

This being the case, one can write the geodesic separation s in terms of new variables:

$$\begin{aligned} s &= R\theta \\ &= \frac{1}{H}\theta \\ &= \frac{1}{H}cos^{-1}(P). \end{aligned} \tag{3.47}$$

Now this is the right time to rewrite the equation (3.44) in terms of new variable P. The very first step is to transform d'Alembertian operator $\Box$ for the coordinates on de Sitter. We defined dS_4 as embedded in a 5-dimensional Minkowski Space in a way that defining an hypersurface equation:

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = R^2, \tag{3.48}$$

where $R = \dfrac{1}{H}$.

With embedding approach, d'Alembertian operator respecting de Sitter conformal coordinates (see Equations.(2.16) and (2.17)) takes the form [26]:

$$\Box_{dS} = \partial^2 - 3H^2(X.\partial) - H^2(X.\partial)(X.\partial). \tag{3.49}$$

Then, equation for massive Wightman Function (3.44) becomes:

$$\left[\partial^2 - 3(X.\partial) - (X.\partial)(X.\partial)\right] G_w(X,Y) - m^2 G_w(X,Y) = 0. \tag{3.50}$$

For the sake of simplicity, we took $H = 1$ in the last step. Now we let P variable be in the game by rewriting whole equation in terms of $P = \eta_{AB}X^AY^B$.

$$(1-P^2)\frac{\partial^2}{\partial P^2}G(P) - 4P\frac{\partial}{\partial P}G(P) - m^2G(P) = 0. \tag{3.51}$$

The last equation have an essential property one can realize at first glance. It has 3 regular singular points at ∞,1 and 0 so it can be turned into a Hyper-geometric Differential Equation by some change of variable. If we introduce a new variable z such that $z = \dfrac{P+1}{2}$, we end up with:

$$z(1-z)G''(z) + (2-4z)G'(z) - m^2G(z) = 0, \tag{3.52}$$

where each prime denotes a derivative with respect to z. This new form of the equation can be easily compared with the general form of the Hyper-geometric equation:

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0, \tag{3.53}$$

which has well defined solutions as follows:

$$\omega(z) = c_1\,{}_2F_1(a,b,c,z) + c_2\,{}_2F_1(a,b,c,1-z). \tag{3.54}$$

Not quite surprisingly, we obtained two solutions in terms of ${}_2F_1$ but with different arguments like z and $1 - z$. This is because the equation (3.53) is invariant under the transformation $z \to 1 - z$.

Comparing the equations (3.52) and (3.53) gives us following parameter relations:

- $c = 2$
- $a + b = 3$
- $ab = m^2$,

then, a and b can be solved algebraically:

- $c = 2$
- $a = \frac{3}{2} \pm \sqrt{\frac{9}{4} - m^2}$
- $b = \frac{3}{2} \mp \sqrt{\frac{9}{4} - m^2}$.

Making use of the property of Hyper-geometric function(being symmetric in the first and second argument), we can pick any $a - b$ couple as solutions. Thus, here is the most general solution to equation (3.52):

$$G(z) = c_1 \, {}_2F_1(\Delta_+, \Delta_-, 2, z) + c_2 \, {}_2F_1(\Delta_+, \Delta_-, 2, 1 - z), \tag{3.55}$$

where $\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - m^2}$.

To pick one of those solutions above, it is crucial to understand behaviours of the two solution around singular points in terms of our variables P and/or z. First of all, Hyper-geometric Function ${}_2F_1(a, b, c, z)$ is singular as $z \to 1$; or equivalently $P \to 1$. This is very special value for P as long as its definition is concerned. $P = 1$ value corresponds to $\theta = 0$ separation which indicates that x and y are incident points on the de Sitter surface. This logic is very fundamental part of decision making on the solutions in a way that taking incident point limit will make points close and close to each other so that they "feel" Minkowskian space. Then one can fairly expect the Green Function in Equation (3.55) behave as if Green function of Minkowksi space in the $z \to 1$ limit.

Consider two independent solution presented in (3.55), ${}_2F_1(\Delta_+, \Delta_-, 2, z)$ and ${}_2F_1(\Delta_+, \Delta_-, 2, 1-z)$, we expect de Sitter Wightman Function function to get singular similar to Minkowskian Wightman Function as $z \to 1$. From this perspective, only the first one satisfies this singularity condition since ${}_2F_1(\alpha, \beta, \gamma, x)$ becomes singular when the last argument is 1. We therefor picked c_2 to be zero since ${}_2F_1(\Delta_+, \Delta_-, 2, 1-z)$ becomes finite when $z = 1$.

With this argument we constraint the form of the solution to:

$$G_w(z) = c_1\, {}_2F_1(\Delta_+, \Delta_-, 2, z), \tag{3.56}$$

where c_1 is some constant. Furthermore, to determine the c_1 we can impose the condition that $G_w(z)$ should get singular with the same speed as Minkowskian Wightman function in the mentioned limit. For this reason, we expanded ${}_2F_1(\Delta_+, \Delta_-, 2, z)$ function around $z = 1$.

$$\lim_{z\to 1} {}_2F_1(\Delta_+, \Delta_-, 2, z) \approx \frac{1}{1-z}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)} + ln(z-1) + O(z-1) + O^2(z-1), \tag{3.57}$$

noting that we picked only the most dominant term in this limit then it becomes:

$$\lim_{z\to 1} {}_2F_1(\Delta_+, \Delta_-, 2, z) \approx \frac{1}{1-z}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)} \tag{3.58}$$

since $\frac{1}{1-z}$ gets infinity faster than $ln(1-z)$ in $z \to 1$ limit. We can make further simplifications in the equation above by using the relation that $z = \frac{P+1}{2}$ and $P =$

$cos\theta$.

$$\begin{aligned}\lim_{z\to 1} {}_2F_1(\Delta_+,\Delta_-,2,z) &\approx \frac{1}{(1-p)/2}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}\\ &\approx \frac{1}{(1-cos\theta)/2}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}\\ &\approx \frac{1}{sin^2(\frac{\theta}{2})}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}\\ &\approx \frac{1}{(\frac{\theta}{2})^2}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}\\ &\approx \frac{4}{\theta^2}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}.\end{aligned} \tag{3.59}$$

Also observe that in our embedding convention, we picked radius of the de Sitter $\frac{1}{H}=1$. This choice makes the angle separation θ and geodesic separation $s=\frac{1}{H}\theta$ the same. Therefore, incident point limit behaviour is generally written in terms of s in the following form [12]:

$$\lim_{z\to 1} {}_2F_1(\Delta_+,\Delta_-,2,z) \approx \frac{4}{s^2}\frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_-)}. \tag{3.60}$$

To compare last result with the two-point function of Minkowski space we have the following relation:

$$\langle 0|\Phi(x,t)\Phi(y,t)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}.(\vec{x}-\vec{y})}, \tag{3.61}$$

where, $\omega_k=\sqrt{|\vec{k}|^2+m^2}$ and Minkowskian field $\Phi(x,t)$ is:

$$\Phi(x,t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\omega_k}}(a_k e^{-i\vec{k}.\vec{x}} + a_k^\dagger e^{i\vec{k}.\vec{x}}). \tag{3.62}$$

Then the solution to integral in Equation (3.61) is given in terms of Modified Bessel Function $K_1(r)$ [27]:

$$\langle 0|\Phi(x,t)\Phi(y,t)|0\rangle = \frac{m}{4\pi r}K_1(mr), \tag{3.63}$$

again $r = |\vec{x} - \vec{y}|$. Expanding the Minkowskian Wightman solution around $r = 0$ gives:

$$\frac{m}{4\pi r} K_1(mr) \approx \frac{1}{4\pi r^2} + ... \ . \tag{3.64}$$

All in all, demanding the singularity of the de Sitter Wightman Function in Equation (3.60) behaves as the Minkowskian one in Equation (3.64) makes us to impose:

$$c_1 \frac{4}{s^2} \frac{1}{\Gamma(\Delta_+)\Gamma(\Delta_+)} \stackrel{!}{=} \frac{1}{4\pi^2 s^2}, \tag{3.65}$$

which fixes the constant c_1:

$$c_1 = \frac{\Gamma(\Delta_+), \Gamma(\Delta_-)}{16\pi^2}. \tag{3.66}$$

In the end, we obtained the de Sitter Wightman Function for massive scalar field that respects the Minkowskian singularity in the incident point limit:

$$G_w(z) = \frac{\Gamma(\Delta_+), \Gamma(\Delta_-)}{16\pi^2} {}_2F_1(\Delta_+, \Delta_-, 2, z). \tag{3.67}$$

Unfortunately, massless limit of the $G_w(z)$ diverges because of the first term in the following expansion [3]:

$$G_w(z) \approx \frac{1}{16\pi^2} \frac{6}{m^2} + \frac{1}{16\pi^2} \left(\frac{1}{z} - 2 \ln z \right) + \mathcal{O}\left(m^2\right). \tag{3.68}$$

We discussed two proposed solution to this divergence in the conclusion chapter. Briefly, they both suggest possible modifications to Wightman function to escape ill-defined $m \rightarrow$ limit. In [3], possible modifications are studied such as shift-invariant two-point functions. In [5], on the other hand,massless limit from tachionic side is examined to understand massless Wightman Function.

Apart from contemporary studies, we calculated Fourier transformation of the two-point function from momentum space to position space and compared their singularities by using the integrals in [4].

3.3. Wightman Function-Massless Case

As we showed in the previous section, massless limit of Wightman Function inevitably diverges. However, let's take the $m \to 0$ in the equation for Wightman function in Equation (3.51):

$$(1-P^2)\frac{\partial^2}{\partial P^2}G_w(P) - 4P\frac{\partial}{\partial P}G_w(P) = 0. \tag{3.69}$$

This equation has the following solution:

$$G_w(P) = a_1 + a_2\left(\frac{2P}{1-P^2} + ln\left[\frac{1+P}{1-P}\right]\right), \tag{3.70}$$

where a_1 and a_2 are constants that should hopefully be determined according to boundary conditions. Note that massless Wightman function in the last equation has singularities around $P = 1$ and $P = -1$ which correspond to incidence of points and incidence of point-antipodal point pairs respectively. [4]

Since we can not fix the coefficients by using the same approach as massive case, Fourier transform of the properly normalized Momentum-Space two-point function is taken to talk about boundary conditions in the following section.

4. COMPARING BOUNDARY CONDITIONS THROUGH FOURIER

Making a choice between the solutions in Equation (3.70) happened to be impossible by imposing Minkowskian type singularity in the incident point limit which worked perfectly for massive case. For this reason we tried to take Fourier transform of the massless two point function from momentum space to position space and hoped to decide on which solutions should be kept.

4.1. Fourier Transformation of Massless Two-Point Function

For the massless scalar field in momentum space, we can pick $\nu = \frac{3}{2}$ and the solution $\varphi_{\vec{k}}(k\eta)$ becomes:

$$\varphi_{\vec{k}}(k\eta) = \frac{H\eta^{\frac{3}{2}}}{4\pi\sqrt{2}} \left[a_{\vec{k}} H^{(1)}_{\frac{3}{2}}(k\eta) + a^{\dagger}_{-\vec{k}} H^{(2)}_{\frac{3}{2}}(k\eta) \right]. \tag{4.1}$$

Then two point function in position space can be expanded in terms of mode functions as follows:

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) > = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} < 0|\varphi_{\vec{k}}(\eta)\varphi_{\vec{k}'}(\eta')|0 > e^{i\vec{k}.\vec{x}} e^{i\vec{k}'.\vec{y}}. \tag{4.2}$$

Taking k' integral results in:

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= \int \frac{d^3k}{(2\pi)^3} \left[\frac{H^2\eta^{3/2}\eta'^{3/2}}{32\pi^2} H^{(1)}_{\frac{3}{2}}(k\eta) H^{(2)}_{\frac{3}{2}}(k\eta') \right] e^{i\vec{k}.(\vec{x}-\vec{y})}. \tag{4.3}$$

From the multiplication of the Hankel functions we have

$$H^{(1)}_{\frac{3}{2}}(k\eta) H^{(2)}_{\frac{3}{2}}(k\eta') = \frac{2e^{ik(\eta-\eta')} \left[1 - ik(\eta - \eta') + k^2\eta\eta'\right]}{\pi k^3 \eta^{3/2} \eta'^{3/2}}.$$

Let $\eta - \eta' = \tau$ and $\vec{x} - \vec{y} = \vec{r}$, then we have:

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= \int \frac{d^3k}{(2\pi)^3} \left[\frac{H^2}{16\pi^2} \frac{[1 - ik\tau + k^2\eta\eta']}{\pi k^3} \right] e^{i\vec{k}.\vec{r}} e^{ik\tau}. \tag{4.4}$$

Let's expand the measure and take angle integral of k,

$$= \int \frac{k^2 dk sin\theta_k d\theta_k d\phi_k}{(2\pi)^3} \left[\frac{H^2}{16\pi^3} \frac{[1 - ik\tau + k^2\eta\eta']}{k^3} \right] e^{ikrcos\theta_k} e^{ik\tau}. \tag{4.5}$$

Define $\cos\theta_k = u$, then $-sin\theta_k d\theta_k = du$ and u runs from 1 to -1.

Note also that, integral along $d\phi_k = 2\pi$.

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= - \int \frac{dkdu}{(2\pi)^2} \left[\frac{H^2}{16\pi^3} \frac{[1 - ik\tau + k^2\eta\eta']}{k} \right] e^{ikru} e^{ik\tau}. \tag{4.6}$$

Then u integral gives,

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= \int \frac{dk}{(2\pi)^2} \left[\frac{H^2}{16\pi^3} \frac{[1 - ik\tau + k^2\eta\eta']}{k} \right] e^{ik\tau} \frac{e^{ikr} - e^{-ikr}}{ikr}. \tag{4.7}$$

And finally:

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= \int \frac{dk}{(2\pi)^2} \left[\frac{H^2}{8\pi^3 r} \frac{[1 - ik\tau + k^2\eta\eta']}{k^2} \right] e^{ik\tau} sin(kr). \tag{4.8}$$

Taking all constants in front of the integral gives:

$$< \varphi(\eta, \vec{x}), \varphi(\eta', \vec{y}) >= \frac{H^2}{32\pi^5 r} \int dk \left[\frac{[1 - ik\tau + k^2\eta\eta']}{k^2} \right] e^{ik\tau} sin(kr), \tag{4.9}$$

replacing $e^{ik\tau} = cos(k\tau) + i\sin(k\tau)$ gives sum of the two follofing integrals.

$$I = \frac{H^2}{32\pi^5 r}[I_1 + I_2], \tag{4.10}$$

where

$$I_1 = \int dk\, cos(k\tau) sin(kr) \left[\frac{[1 - ik\tau + k^2 \eta\eta']}{k^2} \right] \tag{4.11}$$

and

$$I_2 = \int dk\, isin(k\tau) sin(kr) \left[\frac{[1 - ik\tau + k^2 \eta\eta']}{k^2} \right]. \tag{4.12}$$

Let's consider the first integral I_1. It can also be written in terms of 3 integrals:

$$I_1 = \underbrace{\int \frac{cos(k\tau)sin(kr)}{k^2} dk}_{C_2(\tau,r)} - i\tau \underbrace{\int \frac{cos(k\tau)sin(kr)}{k} dk}_{C_1(\tau,r)} + \eta\eta' \underbrace{\int \frac{cos(k\tau)sin(kr)}{k^0} dk}_{C_0(\tau,r)}. \tag{4.13}$$

Then,

$$I_1 = C_2(\tau, r) - i\tau C_1(\tau, r) + \eta\eta' C_0(\tau, r). \tag{4.14}$$

Similarly, we can rewrite integral I_2 as:

$$I_2 = i \underbrace{\int \frac{sin(k\tau)sin(kr)}{k^2}}_{S_2(\tau,r)} + \tau \underbrace{\int \frac{sin(k\tau)sin(kr)}{k}}_{S_1(\tau,r)} + i\eta\eta' \underbrace{\int \frac{sin(k\tau)sin(kr)}{k^0} dk}_{S_0(\tau,r)}. \tag{4.15}$$

Then we have:

$$I_2 = iS_2(\tau, r) + \tau S_1(\tau, r) + i\eta\eta' S_0(\tau, r). \tag{4.16}$$

Where S_n and C_n integral are calculated according to the procedure in Allen(1985) [4] in Appendix B.3 which can be listed as:

$$S_n = \int_0^\infty \frac{sin(k\tau)sin(kr)}{k^n}dk$$
$$C_n = \int_0^\infty \frac{cos(k\tau)sin(kr)}{k^n}dk \tag{4.17}$$

with

$$\begin{aligned}
S_2(\tau, r) &= \frac{\pi}{2}\left[(r+\tau)\Theta(r+\tau) + (r-\tau)\Theta(\tau - r) - r\right] \\
S_1(\tau, r) &= \frac{1}{2}ln\left[\frac{r+\tau}{r-\tau}\right] \\
S_0(\tau, r) &= \frac{\pi}{2}\left[\delta(\tau - r) - \delta(\tau + r)\right] \\
C_2(\tau, r) &= -\frac{r}{2}ln[r^2-\tau^2] - \frac{\tau}{2}ln\left[\frac{r+\tau}{r-\tau}\right] + \psi(r) \\
C_1(\tau, r) &= \frac{\pi}{2}[\Theta(r+\tau) - \Theta(\tau - r)] \\
C_0(\tau, r) &= \frac{r}{r^2-\tau^2}.
\end{aligned} \tag{4.18}$$

In the light of these results, integrals I_1 and I_2 becomes:

$$\begin{aligned}
I_1 =& -\frac{r}{2}ln[r^2-\tau^2] - \frac{\tau}{2}ln\left[\frac{r+\tau}{r-\tau}\right] + \psi(r) \\
& - i\tau\frac{\pi}{2}\left[\Theta(r+\tau) - \Theta(\tau - r)\right] + \eta\eta'\frac{r}{r^2-\tau^2}.
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
I_2 =& \frac{i\pi}{2}\left[(r+\tau)\Theta(r+\tau) + (r-\tau)\Theta(\tau - r) - r\right] + \frac{\tau}{2}ln\left[\frac{r+\tau}{r-\tau}\right] \\
& + i\eta\eta'\frac{\pi}{2}\left[\delta(\tau - r) - \delta(\tau + r)\right].
\end{aligned} \tag{4.20}$$

Then the Fourier transform of the mode functions in equation 14 happens to be:

$$
\begin{aligned}
<\varphi(\eta,\vec{x}),\varphi(\eta',\vec{y})>=&\frac{H^2}{32\pi^5 r}(I_1+I_2)\\
=&\frac{H^2}{32\pi^5}\Bigg(-\frac{1}{2}ln[r^2-\tau^2]+\frac{\psi(r)}{r}\\
&-i\tau\frac{\pi}{2r}\left[\Theta(r+\tau)-\Theta(\tau-r)\right]\\
&\qquad+\eta\eta'\frac{1}{r^2-\tau^2}\\
&\qquad+\frac{i\pi}{2r}\left[(r+\tau)\Theta(r+\tau)+(r-\tau)\Theta(\tau-r)-r\right]\\
&\qquad+i\eta\eta'\frac{\pi}{2r}\left[\delta(\tau-r)-\delta(\tau+r)\right]\Bigg).
\end{aligned}
\tag{4.21}
$$

The last expression becomes simplified for the following two cases:

1-Equal time two-point function($\tau=0, r>0$)

It is worth noting that we did not proved that the equal time limit and Fourier integral commute. This is a huge assumption that should be considered carefully but legitimacy of this approach is left to further studies. (Special thanks to Tonguc Rador for his valuable advices and warnings about this issue.)

$$
<\varphi(\eta,\vec{x}),\varphi(\eta,\vec{y})>=\frac{H^2}{32\pi^5}\left(-\frac{1}{2}ln[r^2]+\frac{\eta^2}{r^2}+\frac{\psi(r)}{r}\right), \tag{4.22}
$$

where $\psi(r)=b_1r+b_2$ with constant coefficients b_1 and b_2.

$$
<\varphi(\eta,\vec{x}),\varphi(\eta,\vec{y})>=\frac{H^2}{32\pi^5}\left(-\frac{1}{2}ln[r^2]+\frac{\eta^2}{r^2}+b_1+\frac{b_2}{r}\right). \tag{4.23}
$$

2- Spacelike separated$(r > \tau, |\tau| \neq r\ )$

$$< \varphi(\eta,\vec{x}),\varphi(\eta',\vec{y}) >= \frac{H^2}{32\pi^5}\Bigg(-\frac{1}{2}ln[r^2-\tau^2]+\frac{\psi(r)}{r}-i\tau\frac{\pi}{2r} + \eta\eta'\frac{1}{r^2-\tau^2}+\frac{i\pi}{2r}\tau\Bigg). \tag{4.24}$$

$$< \varphi(\eta,\vec{x}),\varphi(\eta',\vec{y}) >= \frac{H^2}{32\pi^5}\Bigg(-\frac{1}{2}ln[r^2-\tau^2]+\eta\eta'\frac{1}{r^2-\tau^2}+b_1+\frac{b_2}{r}\Bigg). \tag{4.25}$$

Let us recite the Wightman function of massless field in terms of $P(X(x),Y(y)) = \eta_{AB}X^A(x)Y^B(y)$

$$G_w(P) = c_1 + c_2\left[\frac{2P}{1-P^2} + ln\left[\frac{1+P}{1-P}\right]\right]. \tag{4.26}$$

Determination of the coefficient according to boundary conditions is left to the following Section.

5. LAST COMMENTS AND TWO NOVEL APPROACHES

Taking the Fourier transformation result into consideration of two-point function of equal times gave us the following result in Equation (4.23) (with H=1):

$$< \varphi(\eta,\vec{x}),\varphi(\eta,\vec{y}) >= \frac{1}{32\pi^5}\left(-\frac{1}{2}ln[r^2]+\eta^2\frac{1}{r^2}+b_1+\frac{b_2}{r}\right) \tag{5.1}$$

which is expected to match with the Wightman Function in the Equation (3.70) :

$$G_w(P) = a_1 + a_2\left(\frac{2P}{1-P^2}+ln\left[\frac{1+P}{1-P}\right]\right). \tag{5.2}$$

In general, P has the following form for $\tau = \eta - \eta' \neq 0$:

$$\begin{aligned} P(x,y) &= 1 + \frac{(\eta-\eta')^2-(\vec{x}-\vec{y})^2}{2\eta\eta'} \\ &= 1 + \frac{\tau^2 - r^2}{2\eta\eta'}, \end{aligned} \tag{5.3}$$

where it reduces to following form for equal times, that is $\tau = 0$,

$$P(x,y) = 1 - \frac{r^2}{2\eta^2}. \tag{5.4}$$

It can be seen at first glance that we can set $a_1 = \frac{1}{32\pi^5}b_1$ since only constants in the both expressions in Equation (5.1) and (5.2) are a_1 and b_1. Then, we are investigating their behaviour around incident point singularities which implies that $\eta \to 0$, $r \to 0$, $P \to 1$:

$$< \varphi(\eta,\vec{x}),\varphi(\eta,\vec{y}\to\vec{x}) >\approx \frac{1}{32\pi^5}\left(-\frac{1}{2}ln[r^2]+\frac{\eta^2}{r^2}+\frac{b_2}{r}\right), \tag{5.5}$$

where we ignored the constant b_1 term since it is small compared to ∞.

Before taking incident point limit ($r \to 0$) of Wightman Function in Equation (3.70), we can replace the variable P with r, η according to the definition of P in Equation (5.4).

$$\begin{aligned} G_w(r,\eta) &= a_1 + a_2 \left(\frac{2 - \frac{r^2}{\eta^2}}{\frac{r^2}{2\eta^2}(2 - \frac{r^2}{2\eta^2})} + ln \left[\frac{2 - \frac{r^2}{2\eta^2}}{\frac{r^2}{2\eta^2}} \right] \right) \\ &= a_1 + a_2 \left(\frac{\frac{4\eta^2}{r^2} - 2}{2 - \frac{r^2}{2\eta^2}} + ln \left[\frac{4\eta^2}{r^2} - 1 \right] \right). \end{aligned} \tag{5.6}$$

Then, taking incident point limit of $G_w(r,\eta)$ results in the following behaviour:

$$\begin{aligned} G_w(r \to 0, \eta) &\approx a_2 \left(\frac{\frac{4\eta^2}{r^2}}{2} + ln \left[\frac{4\eta^2}{r^2} \right] \right) \\ &\approx a_2 \frac{2\eta^2}{r^2} + a_2 ln \left[\frac{\eta^2}{r^2} \right]. \end{aligned} \tag{5.7}$$

Note that in the last step we applied the following trick:

$$\begin{aligned} \lim_{u\to\infty} ln(cu) &= \lim_{u\to\infty} (lnc + lnu) \\ &\approx lnu. \end{aligned} \tag{5.8}$$

From this perspective, we can expect:

$$2a_2 \left(\frac{\eta^2}{r^2} + \frac{1}{2} ln \left[\frac{\eta^2}{r^2} \right] \right) \stackrel{!}{=} \frac{1}{32\pi^5} \left(-\frac{1}{2} ln[r^2] + \frac{\eta^2}{r^2} + \frac{b_2}{r} \right). \tag{5.9}$$

First of all, since there is no $\frac{1}{r}$ singularity on the left hand side, b_2 can be chosen zero. Secondly, comparing the first and second terms in the both hand side, $\frac{\eta^2}{r^2}$ terms are dominant relative to $ln\left(\frac{1}{r^2}\right)$ terms in the $r \to 0$ limit. Then, we can compare only the following terms:

$$2a_2 \frac{\eta^2}{r^2} \stackrel{!}{=} \frac{1}{32\pi^5} \frac{\eta^2}{r^2}, \tag{5.10}$$

which implies:

$$a_2 = \frac{1}{64\pi^5} \tag{5.11}$$

and

$$a_1 = \frac{b_1}{32\pi^5}. \tag{5.12}$$

All in all, we tried to match Wightman Function of massless scalar field and Fourier transform of momentum space massless two-point functions in the incident point limit. The motivation behind this attempt can be justified as follows. Bunch-Davies Vacuum as a boundary condition for momentum-space two-point functions is set up so that mode functions agree with the flat space vacuum in the early time limit. This being the case, we tried to construct a Wightman Function that coincides with the flat space behaviour of Fourier transform of momentum space two-point functions. Finally, we end up with 3 equations while having 4 unknowns. a_1 and b_1 happened to be linearly dependent so once a_1 is fixed b_1 is corollary fixed or vice versa.

Here are the all final results coming from Fourier Transformation:

$$a_1 = \frac{b_1}{32\pi^5} \qquad a_2 = \frac{1}{64\pi^5} \qquad b_2 = 0. \tag{5.13}$$

As a result of the Fourier procedure, it seems more convenient to keep singular term (second term in the following equation) in the massless Wightman function in Equation (5.2). On the other hand, there is an arbitrariness in the choice of a_1 so that it might be taken zero for certain purposes.

$$G_w(P) = a_1 + \frac{1}{64\pi^5}\left(\frac{2P}{1-P^2} + ln\left[\frac{1+P}{1-P}\right]\right). \tag{5.14}$$

5.1. Possible Modifications To Massless Wightman Function in dS

In literature, two different approaches draw our attention that suggest two possible solution to such anomalies in the massless limit [3, 5]. The first one puts being finite for a correlation function as crucial condition in order for it to be meaningful physical observable. Therefore, suggests two possible form for correlation functions. In the second paper, they approaches the massless Wightman function problem from a tachionic perspective. In the end, they both suggest a new form for two-point functions that makes it finite in the $m \to 0$ limit.

5.1.1. Shift Invariant States and Modified Correlation Functions

In [3], it is claimed that such divergences in a theory is a sign of unphysical implication that should be got rid off. The suggestion is to work with physical two-point functions which do not have singularities in the massless limit. Two-family of well-behaving two-point functions can be defined as "two-point functions of differences" and "two-point functions of derivatives" of the fields.

Here are the example suggested two-point functions in [3]:

$$\bar{G}(x, y, u, v) = |\langle 0|[\varphi(x) - \varphi(y)][\varphi(u) - \varphi(v)]|0\rangle, \tag{5.15}$$

which has the following finite behaviour in $m \to 0$ limit

$$\begin{aligned} \lim_{m\to 0} \bar{G}(x, y, u, v) = &\lim_{m\to 0} [G_w(x, u) - G_w(y, u) - G_w(x, v) + G_w(y, v)] \\ =&\frac{1}{16\pi^2} \left\{ \left[\frac{1}{z(x, u)} - 2 \ln z(x, u) \right] - \left[\frac{1}{z(y, u)} - 2 \ln z(y, u) \right] \right. \\ &\left. - \left[\frac{1}{z(x, v)} - 2 \ln z(x, v) \right] + \left[\frac{1}{z(y, v)} - 2 \ln z(y, v) \right] \right\} \end{aligned} \tag{5.16}$$

with $z(x, u) = \dfrac{P(x, u) + 1}{2}$.

Not that by defining differences of two-point functions as fundamental objects we got rid of the first term in the following expansion in $m \to 0$ limit:

$$G_w(z) \approx \frac{1}{16\pi^2}\frac{6}{m^2} + \frac{1}{16\pi^2}\left(\frac{1}{z} - 2\ln z\right) + \mathcal{O}\left(m^2\right). \tag{5.17}$$

Thus, the terms that contains positive power of m disappeared in the limit.

Second suggestion on the form of the two-point functions in [3] is the following one:

$$G_\nabla(x,x') = \langle 0|\nabla_\mu\varphi(\vec{x},\eta)\nabla_{\mu'}\varphi(\vec{x}',\eta)|0\rangle. \tag{5.18}$$

Since ∇_μ acts only on x and $\nabla_{\mu'}$ acts only on x' coordinates, derivatives commutes with the field that is independent of derivative coordinates.

$$\begin{aligned} G_\nabla(x,x') &= \nabla_\mu\nabla_{\mu'}\langle 0|\varphi(\vec{x},\eta)\varphi(\vec{x}',\eta)|0\rangle \\ &= \nabla_\mu\nabla_{\mu'}G_w(x,x'). \end{aligned} \tag{5.19}$$

It is also clear that two-point function of derivative of the fields is well defined due to vanishing of the problematic mass dependent constant term in the Equation(5.17).

Motivation behind these sort of redefinition of the two-point function comes from what is said "physical condition" in [3]. In the massless field action, only derivative of the fields exist:

$$S = \int d^4x\sqrt{g}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi), \tag{5.20}$$

that is, action is invariant under the shift transformations: $\varphi \to \varphi + \varphi_0$ with φ_0 is constant. It is remarked that this shift-invariant property of the fields should also be possessed by correlation functions.

All in all, the dilemma between unique dS vacuum and de Sitter invariant correlation functions for massless correlation functions happens to be solved by restricting the form of correlation functions to correlation of differences and/or two point functions of the derivatives [3].

5.1.2. Tachionic Scalar Fields and Massless Limit

Here is another alternative solution to divergences in the massless limit of Wightman Function [5]. In this study, mass factor $m^2 = -\lambda(\lambda+d-1)$ is defined with respect to a family of parameters λ for d-dimensional de Sitter space, which is embedded in $d+1$ Minkowskian space. This convention corresponds to 4D de Sitter embedded in 5D Minkowski Space for our purpose. We rewrite their equation by choosing $d = 4$ [5]:

$$m^2 = -\lambda(\lambda + 3), \tag{5.21}$$

where, λ's are real parameter for tachionic and massless cases. Consequently, $\lambda = 0$ brings about the massless case. The equation for massive Wightman Function G_λ turns into [5]:

$$\Box_{dS}\, w_\lambda(X, Y) - \lambda(\lambda + 3)w_\lambda(X, Y) = 0, \tag{5.22}$$

which produce the following solution [5]:

$$w_\lambda(P) = \Gamma(-\lambda)G_\lambda(P). \tag{5.23}$$

Where $G_\lambda(P)$ is [5]:

$$G_\lambda = \frac{\Gamma(\lambda + 3)}{16\pi^2}\,{}_2F_1(-\lambda, \lambda + 3; 2, \frac{1 - P}{2}). \tag{5.24}$$

For a scalar field theory in de Sitter to be meaningful, they suggest following conditions on Wightman Function that one can expect to hold: [5]

(i) Invariance Under SO(1,4)

(ii) Locality :commuting $w_\lambda(X,Y)$ for space-like separated X and Y, which states that:

$$c_\lambda(X,Y) = w_\lambda(X,Y) - w_\lambda(Y,X) = 0.$$

(iii) Positive definiteness: to guarantee quantum-mechanical interpretation

It is showed that tachionic fields do not satisfy the requirements above [5]. However, there are a special family of solution produced by non-negative integer λ's, say $\lambda = n$. In this case the solution can be written in terms of Gegenbauer polynomials of degree n C_n as follows [5]:

$$ {}_2F_1(-n, n+3; \frac{1-P}{2}) = \frac{\Gamma(n+1)\Gamma(3)}{\Gamma(n+3)} C_n^{\frac{3}{2}}(P). \tag{5.25}$$

Again, since the massless limit diverges, a modified two-point function $\hat{w}_\lambda(X,Y)$ is proposed as follows [5]:

$$\begin{aligned} \hat{w}_\lambda(X,Y) &= \hat{w}_\lambda(P) \\ &= \lim_{\lambda \to n} \Gamma(-\lambda)[G_\lambda(P) - G_n(P)] \\ &= \frac{(-1)^{n+1}}{n!} \frac{\partial}{\partial \lambda} G_\lambda(P)|_{\lambda=n}. \end{aligned} \tag{5.26}$$

What is done here resembles the intention in [3] study in a way that defining two-point functions of derivatives to get rid of the constant problematic term $\propto \frac{1}{m^2}$ in the expansion in Equation (5.17). However, they differ in choosing derivatives with respect to different variables. It is suggested in [3] that two point functions can be chosen as two point function of covariant derivative of the fields [3]:

$$G_\nabla(x, x') = \nabla_\mu \nabla_{\mu'} \langle 0|\varphi(\vec{x}, \eta)\varphi(\vec{x}', \eta)|0\rangle, \tag{5.27}$$

which causes $\frac{1}{m^2}$ singularity term to vanish. In [5] on the other hand $\lambda \to n$ limit brings about a derivative operator on Wightman Function of the type $\frac{d}{d\lambda}$ as in Equation (5.26), which result in vanishing $\frac{1}{m^2}$ term again.

Having new two-point functions $\hat{w}_\lambda(P)$'s brings about a modification on the Klein-Gordon Equation as well [5]:

$$\Box\hat{w}_\lambda(P) - n(n+3)\hat{w}_\lambda(P) = \frac{(-1)^{n+1}(2n+3)\Gamma(\frac{3}{2})}{4\pi^{\frac{5}{2}}} C_n^{\frac{3}{2}}(P) \tag{5.28}$$

and for the special case $m \to 0$,that is $n = 0$, it reduces to [5]:

$$\Box\hat{w}_\lambda(P) = \frac{-\Gamma(\frac{5}{2})}{2\pi^{\frac{5}{2}}}. \tag{5.29}$$

The solution of the last equation gives(for d is non-negative even integers) [5]:

$$\hat{w}_\lambda(P) = \frac{1}{16\pi^2 z} - \frac{1}{8\pi^2} ln(z) - \frac{4-2\gamma}{16\pi^2} \tag{5.30}$$

which has well-defined finite $m \to 0$ limit as in the case of Equation (5.16).

To summarize, in this study [5], massless limit of the two-point function is intended to be taken from tachionic side. Still, it happened to be ambiguous to take $m \to 0$ unless introducing a redefinition on the Wightman Function. For the more detailed discussions on possible physical interpretations of tachionic fields in de Sitter see [5].

6. CONCLUSION

To sum up, among three irreducible representations of the scalar field in de Sitter Space(see Table 2.1), Discrete series is challenging to examine by conventional methods which are fairly valid for both principle and complementary series [19]. Obtaining a de Sitter invariant vacuum state and/or fixing the Wightman Function according to boundary conditions are main problems of the massless case.

Massive Wightman function in de Sitter was successfully fixed by taking incident point limit and forcing it to have the same singularity as its Minkowski counterpart(see Section 3.2). Taking its massless limit, however, resulted in $\frac{1}{m^2}$ type singularity which cannot be controlled by any boundary condition. On the other hand, solving massless Klein-Gordon equation in Equation (3.69), gave us a solution but its physical meaning became uncertain. At that point, standard coefficient fixing by looking at incident point limit(see Equation (3.67)) did not work for the massless Wightman function. For that reason, we hoped that the Fourier transformation of massless two-point functions from momentum space to position space might shed light on something new and meaningful. Finally, the expected singularity structure is successfully observed while one of the coefficients stayed unfixed by boundary conditions at the end of the calculations. Remaining of such undetermined coefficient in the Wightman function can be interpreted as arbitrariness in the choice of the constant. Let us recite the final solution for convenience:

$$G_w(P) = a_1 + \frac{1}{64\pi^5}\left(\frac{2P}{1-P^2} + ln\left[\frac{1+P}{1-P}\right]\right). \tag{6.1}$$

While the singular term is fixed according to Bunch-Davies boundary condition, the a_1 in the last equation happened to be free parameter at the end of the calculations. Therefore, It seems more reasonable to keep singular term to maintain Minkowski type singularity in the incident point limit($P \to 1$). However, one can choose a_1 to be zero as long as it is justified. Furthermore, a_1 might also be fixed if we can find an extra constraint or boundary condition on the massless Wightman function.

Besides our naive attempts, we gathered some important points and suggestions about the problem in question from two recent studies [3, 5]. What is common about these studies is that they both suggest some sort of redefinition of the two-point function so that they satisfy certain requirements to be physically meaningful.

These kind of attempts -such as modifications on two-point functions- might make sense in some physical framework one day in the future. This is somehow probable since in Quantum Cosmology, it is not completely figured out which object has which physical/observable interpretation [28].

All, in all, massless two-point functions and its boundary conditions continue to stay a mystery and open question.

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APPENDIX A: NOTATIONS

A.1. Normalization of the fields: arbitrariness on $(2\pi)^k$'s.

There is an arbitrariness on choosing power of π in the momentum decomposition integral. This arbitrariness can be clarified by looking at the relation between canonic quantization and creation and annihilation commutation relations.

We choose commutation relation of field and conjugate momentum to have no π power as the same as the classical Poisson bracket.

$$[\varphi(\vec{x},\eta),\pi(\vec{y},\eta)] \stackrel{!}{=} i\delta(\vec{x}-\vec{y}) \tag{A.1}$$

Consider decomposition of any field $h(\vec{x},\eta)$ can be written as follows [10]:

$$h(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^n}\left[f_{\vec{k}}(\vec{x},\eta)\underbrace{(f_{\vec{k}},h(\vec{y},\eta))}_{I} - f^*_{\vec{k}}(\vec{x},\eta)\underbrace{(f^*_{\vec{k}},h(\vec{y},\eta))}_{II}\right] \tag{A.2}$$

$$I \to -i\int \sqrt{|g|}g^{00}(f^*_k(\vec{y},\eta)\dot{h}(\vec{y},\eta) - h(\vec{y},\eta)\dot{f}^*_k(\vec{y},\eta))d^3y$$

$$II \to -i\int \sqrt{|g|}g^{00}(f_k(\vec{y},\eta)\dot{h}(\vec{y},\eta) - h(\vec{y},\eta)\dot{f}_k(\vec{y},\eta))d^3y$$

Inserting these into the equation 73 gives:

$$\begin{aligned}&\frac{-i}{\eta^2H^2}\int \frac{d^3y}{(2\pi)^n}d^3k[f_k(\vec{x},\eta)\dot{f}^*_k(\vec{y},\eta) - f^*_k(\vec{x},\eta)\dot{f}_k(\vec{y},\eta)]h(\vec{y},\eta)\\ &= \int h(\vec{y},\eta)\delta(\vec{y}-\vec{x})d^3y.\end{aligned} \tag{A.3}$$

Then, naturally:

$$\int \frac{d^3k}{\eta^2 H^2}[f_k(\vec{x},\eta)\dot{f}_k^*(\vec{y},\eta) - f_k^*(\vec{x},\eta)\dot{f}_k(\vec{y},\eta)] = i\delta(\vec{y}-\vec{x})(2\pi)^n \tag{A.4}$$

Now, consider the form of canonic quantization that we desire:

$$[\varphi(\vec{x},\eta)\pi(\vec{y},\eta)] \overset{!}{=} i\delta(\vec{x}-\vec{y})$$

with $\pi(\vec{y},\eta) = \dfrac{\dot{\varphi}(\vec{y},\eta)}{\eta^2 H^2}$

$$\varphi(\eta,\vec{x}) = \int \left[a f_{\vec{k}}(\vec{x},\eta) + a^\dagger f_{\vec{k}}^*(\vec{x},\eta)\right] \frac{d^3k}{(2\pi)^m}$$

$$\pi(\vec{y},\eta) = \frac{1}{H^2\eta^2}\int \left[a f_{\vec{k}}(\vec{y},\eta) + a^\dagger f_{\vec{k}}^*(\vec{y},\eta)\right] \frac{d^3k}{(2\pi)^m}$$

Then we get:

$$\begin{aligned}[\varphi(\vec{x},\eta)\pi(\vec{y},\eta)] = \int \frac{d^3k d^3k'}{(2\pi)^{2m}} \Bigg[& f_k(\vec{x},\eta)\dot{f}_{k'}(\vec{y},\eta)\underbrace{(a_k a_{k'} - a_{k'} a_k)}_{0} \\ & + f_k(\vec{x},\eta)\dot{f}_{k'}\underbrace{(a_k a_{k'}^\dagger - a_{k'}^\dagger a_k)}_{\delta_{kk'}(2\pi)^p} \\ & + f_k^*(\vec{x},\eta)\dot{f}_{k'}(\vec{y},\eta)\underbrace{(a_{k'}^\dagger a_k - a_k a_{k'}^\dagger)}_{-\delta_{kk'}(2\pi)^p} \\ & + f_k^*(\vec{x},\eta)\dot{f}_{k'}^*(\vec{y},\eta)\underbrace{(a_k^\dagger a_{k'}^\dagger - a_{k'}^\dagger a_k^\dagger)}_{0}\Bigg]\end{aligned} \tag{A.5}$$

$$[\varphi(\vec{x},\eta)\pi(\vec{y},\eta)] = \frac{(2\pi)^p}{(2\pi)^{2m}} \underbrace{\int \frac{1}{H^2\eta^2} d^3k [f_k(\vec{x},\eta)\dot{f}^*_{k'}(y,\vec{\eta}) - f^*_k(\vec{x},\eta)\dot{f}_{k'}(\vec{y},\eta)]}_{i\delta(\vec{y}-\vec{x})(2\pi)^n} \tag{A.6}$$

Finally,

$$[\varphi(\vec{x},\eta)\pi(\vec{y},\eta)] = (2\pi)^{p+n-2m} i\delta(\vec{y}-\vec{x}) \stackrel{!}{=} i\delta(\vec{y}-\vec{x}) \tag{A.7}$$

The condition for m, n, p is:

$$p + n - 2m = 0 \tag{A.8}$$

This is the so called arbitrariness on the power of π that pops up when the theory is built up. Any p, n, m combination will work self-consistently if we coose them according to relation above. Finally, we made the choice $n = m = p = 3$.

With this specific choice we have the following quantization and normalization conditions:

$$\begin{aligned} [a_{\vec{k}}, a^\dagger_{\vec{k}'}] &= (2\pi)^3 \delta^3(\vec{k}-\vec{k}'), \\ \langle \vec{k}|\vec{k}'\rangle &= (2\pi)^3 \delta^3(\vec{k}-\vec{k}'), \\ [\varphi(\eta,\vec{x}), \pi(\eta,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{A.9}$$

APPENDIX B: SOME CUMBERSOME CALCULATIONS

B.1. COMPONENTS OF $\mathcal{X}_{AB}$ AND KILLING VECTORS OF DE SITTER

Let's calculate the components of the Casimir operator in terms of dS_4 Coordinates and rewrite the Casimir components by using Killing vectors of dS_4. Besides having known dictionary between embedding and de Sitter coordinates in Equation (2.16) and Equation (2.17), we need to derive how $\dfrac{\partial}{\partial X^A}$ is written in terms of de Sitter coordinates as well. Let me recite the general formula for Casimir element for convenience:

$$\mathcal{X}_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \tag{B.1}$$

Let's investigate how derivative operators are related. We have the following transformation rule between derivatives:

$$\frac{\partial}{\partial X^A} = \frac{\partial x^\mu}{\partial X^A} \frac{\partial}{\partial x^\mu} \tag{B.2}$$

Where A=(0,1,2,3,4), and μ = (0,1,2,3).

$$\begin{aligned} \rightarrow \frac{\partial}{\partial X^0} &= \frac{\partial x^\mu}{\partial X^0} \frac{\partial}{\partial x^\mu} \\ &= \frac{\partial x^0}{\partial X^0} \frac{\partial}{\partial x^0} + \frac{\partial x^i}{\partial X^0} \frac{\partial}{\partial x^i} \\ &= -H\eta^2 \frac{\partial}{\partial x^0} - H\eta x^i \frac{\partial}{\partial x^i} \end{aligned} \tag{B.3}$$

$$\begin{aligned}\rightarrow \frac{\partial}{\partial X^i} &= \frac{\partial x^\mu}{\partial X^i}\frac{\partial}{\partial x^\mu} \\ &= \frac{\partial x^j}{\partial X^i}\frac{\partial}{\partial x^j} \\ &= -H\eta\frac{\partial}{\partial x^i}\end{aligned} \tag{B.4}$$

$$\begin{aligned}\rightarrow \frac{\partial}{\partial X^4} &= \frac{\partial x^\mu}{\partial X^4}\frac{\partial}{\partial x^\mu} \\ &= -H\eta^2\frac{\partial}{\partial x^0} - H\eta x^i\frac{\partial}{\partial x^i}\end{aligned} \tag{B.5}$$

Now we have all derivative and coordinate transformations so we can begin with $\mathcal{X}_{0i}$:

$$\begin{aligned}\rightarrow \mathcal{X}_{0i} &= X_0\frac{\partial}{\partial X^i} - X_i\frac{\partial}{\partial X^0} \\ &= \frac{1}{2H\eta}(\vec{x}^2 + 1 - \eta^2)(-H\eta)\frac{\partial}{\partial x^i} - \frac{1}{H\eta}x^i\left(H\eta^2\frac{\partial}{\partial x^0} + H\eta x^j\frac{\partial}{\partial x^j}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial x^i} + \frac{1}{2}\left[(\eta^2 - \vec{x}^2)\partial_i - 2x_j x^j\partial_j + 2x_i\eta\partial_0\right] \\ &= -\frac{1}{2}\underbrace{\frac{\partial}{\partial x^i}}_{\mathcal{T}_i} + \frac{1}{2}\underbrace{\left[((\eta^2 - \vec{x}^2)\delta_i^j - 2x_j x^j)\partial_j + 2x_i\eta\partial_0\right]}_{\mathcal{C}_i}\end{aligned} \tag{B.6}$$

Then we get:

$$\mathcal{X}_{0i} = \frac{1}{2}\left[\mathcal{C}_i - \mathcal{T}_i\right], \tag{B.7}$$

where $\mathcal{C}_i$ is special conformal transformations and $\mathcal{T}_i$ is translations Killing vector. Take $\mathcal{X}_{ij}$ for this time,

$$\begin{aligned}\rightarrow \mathcal{X}_{ij} &= X_i\frac{\partial}{\partial X^j} - X_j\frac{\partial}{\partial X^i} \\ &= \frac{x^i}{H\eta}(H\eta)\frac{\partial}{\partial x^j} - \frac{x^j}{H\eta}(H\eta)\frac{\partial}{\partial x^i} \\ &= x^i\partial_j - x^j\partial_i \\ &= \mathcal{R}_{ij},\end{aligned} \tag{B.8}$$

where $\mathcal{R}_{ij}$ is rotation Killing vector.

Then, we get:

$$\mathcal{X}_{ij} = \mathcal{R}_{ij} \tag{B.9}$$

Take $\mathcal{X}_{4i}$:

$$\begin{aligned}
\rightarrow \mathcal{X}_{4i} &= X_4 \frac{\partial}{\partial X^i} - X_i \frac{\partial}{\partial X^4} \\
&= \frac{1}{2H\eta}(\vec{x}^2 - 1 - \eta^2)(-H\eta)\frac{\partial}{\partial x^i} - \frac{1}{H\eta} x^i \left(H\eta^2 \frac{\partial}{\partial x^4} + H\eta x^j \frac{\partial}{\partial x^j} \right) \\
&= \frac{1}{2}\frac{\partial}{\partial x^i} + \frac{1}{2}\left[(\eta^2 - \vec{x}^2)\partial_i - 2x_j x^j \partial_j + 2x_i \eta \partial_4\right] \\
&= \frac{1}{2}\underbrace{\frac{\partial}{\partial x^i}}_{\mathcal{T}_i} + \frac{1}{2}\underbrace{\left[((\eta^2 - \vec{x}^2)\delta_i^j - 2x_j x^j)\partial_j + 2x_i \eta \partial_4\right]}_{\mathcal{C}_i}
\end{aligned} \tag{B.10}$$

so we get:

$$\mathcal{X}_{4i} = \frac{1}{2}[\mathcal{C}_i + \mathcal{T}_i]. \tag{B.11}$$

Last but not least : $\mathcal{X}_{04}$

$$\begin{aligned}
\rightarrow \mathcal{X}_{4i} &= X_0 \frac{\partial}{\partial X^4} - X_4 \frac{\partial}{\partial X^0} \\
&= \frac{1}{2H\eta}(\eta^2 - 1 - \vec{x}^2)(-H\eta^2 \frac{\partial}{\partial x^0} - H\eta x^i \frac{\partial}{\partial x^i}) \\
&- \frac{1}{2H\eta}(\vec{x}^2 - 1 - \eta^2)(-H\eta^2 \frac{\partial}{\partial x^0} - H\eta x^i \frac{\partial}{\partial x^i}) \\
&= \eta \partial_0 + x^i \partial_i \\
&= \mathcal{D}
\end{aligned} \tag{B.12}$$

with $\mathcal{D}$ is dilatation Killing Vector.

$$\mathcal{X}_{04} = \mathcal{D} \tag{B.13}$$

As as result we can lis whole compnents together for convenience:

- $\mathcal{X}_{0i} = \frac{1}{2}\left[\mathcal{C}_i - \mathcal{T}_i\right]$
- $\mathcal{X}_{4i} = \frac{1}{2}\left[\mathcal{C}_i + \mathcal{T}_i\right]$
- $\mathcal{X}_{ij} = \mathcal{R}_{ij}$
- $\mathcal{X}_{04} = \mathcal{D}$

together with the inverse relations:

- $\mathcal{C}_i = \mathcal{X}_{0i} + \mathcal{X}_{4i}$
- $\mathcal{T}_i = \mathcal{X}_{4i} - \mathcal{X}_{0i}$
- $\mathcal{R}_{ij} = \mathcal{X}_{ij}$
- $\mathcal{D} = \mathcal{X}_{04}$,

minding that we have following dS Killing vectors for unit parameters [20]:

- SCT :$\mathcal{C}_i = 2x_i\eta\frac{\partial}{\partial\eta} + \left[2x^jx_i + \left(\eta^2 - |\vec{x}|^2\right)\delta_i^j\right]\frac{\partial}{\partial x^j}$
- Dilatations: $\mathcal{D} = \eta\frac{\partial}{\partial\eta}x^i\frac{\partial}{\partial_i}$
- Translations: $\mathcal{T}_i = \partial_i$
- Rotations: $\mathcal{R}_{ij} = x_i\frac{\partial}{x^j} - x_j\frac{\partial}{\partial x^i}$

B.2. CASIMIR ELEMENT AND D'ALEMBERTIAN RELATION

Casimir operator is defined as: $Q = -\frac{1}{2}\mathcal{X}^{AB}\mathcal{X}_{AB}$ Where $\mathcal{X}$'s are :

$$\mathcal{X}_{AB} = X_A\partial_B - X_B\partial_A \tag{B.14}$$

$$Q = -\frac{1}{2}\mathcal{X}^{AB}\mathcal{X}_{AB} = -\frac{1}{2}\eta^{AC}\eta^{BD}[X_A\partial_B - X_B\partial_A][X_C\partial_D - X_D\partial_C]$$

$$= \frac{1}{2}\mathcal{X}^{AB}\mathcal{X}_{AB} = -\frac{1}{2}\eta^{AC}\eta^{BD}[X_A\eta_{BC}\partial_D + X_AX_C\partial_B\partial_D - X_A\eta_{BD}\partial_C - X_AX_D\partial_B\partial_C$$

$$-X_B\eta_{AC}\partial_D - X_BX_C\partial_A\partial_D + X_B\eta_{AD}\partial_C + X_BX_D\partial_A\partial_C]$$

$$= -\frac{1}{2}\left[X.\partial + X^2\partial^2 - 5X.\partial - X^CX^B\partial_B\partial_C - 5X.\partial - X^DX^A\partial_A\partial_D + X.\partial + X^2\partial^2\right]$$

$$Q_0^{(1)} = \frac{1}{2}\mathcal{X}^{AB}\mathcal{X}_{AB} = 3X.\partial + (X.\partial)(X.\partial) - H^{-2}\partial^2$$

Then we figured out:

$$Q_0^{(1)} = -H^{-2}\Box_{dS} \tag{B.15}$$

Where $\Box_{dS} = \partial^2 - 3H^2(X.\partial) - H^2(X.\partial)(X.\partial)$ [26].

B.3. INTEGRALS IN SECTION 4.1

For the massless two point function there are strict types of integrals which are expressed by the following general form:

$$\begin{aligned} S_n &= \int_0^\infty \frac{sin(k\tau)sin(kr)}{k^n}dk \\ C_n &= \int_0^\infty \frac{cos(k\tau)sin(kr)}{k^n}dk \end{aligned} \tag{B.16}$$

and we have to evaluate $S_0(\tau,r), S_1(\tau,r), S_2(\tau,r), C_0(\tau,r), C_1(\tau,r), C_2(\tau,r)$ to take Fourier transformation.

To deal with the integrals, it is important to observe the following property about the relation between the certain integrals with respect to their order parameter n.

For example, taking τ derivative of the C_n integral gives:

$$\begin{aligned} \frac{\partial}{\partial\tau}C_n &= \frac{\partial}{\partial\tau}\int_0^\infty \frac{cos(k\tau)sin(kr)}{k^n}dk \\ &= -\int_0^\infty \frac{sin(k\tau)sin(kr)}{k^{n-1}}dk \\ &= -S_{n-1} \end{aligned} \tag{B.17}$$

Similarly taking τ derivative of the S_n integral gives

$$\begin{aligned} \frac{\partial}{\partial\tau}S_n &= \frac{\partial}{\partial\tau}\int_0^\infty \frac{sin(k\tau)sin(kr)}{k^n}dk \\ &= \int_0^\infty \frac{cos(k\tau)sin(kr)}{k^{n-1}}dk \\ &= C_{n-1}. \end{aligned} \tag{B.18}$$

Therefore, instead of taking all integrals we can take only S_2 and S_1 and the rest can be deduced from the following relations:

$$\begin{aligned} S_n &= -\frac{\partial}{\partial\tau}C_{n+1} \\ C_n &= \frac{\partial}{\partial\tau}S_{n+1} \end{aligned} \tag{B.19}$$

We happened to be lucky since taking derivative is generally much easier than taking an integral. By the aid of the derived relation in equation (B.19), only a certain subset of the integrals can be calculated and the rest can be obtained by derivative relations.

As seen from the equations (4.13) and (4.15), only the integrals $C_0, C_1, C_2, S_0, S_1, S_2$ are of interest. However, only S_2 and S_1 need to be evaluated.

Let's start from the most complicated one: S_2:

$$S_2 = \int_0^\infty \frac{sin(rk)sin(\tau k)}{k^2}dk \tag{B.20}$$

First important observation is that integrand is symmetric in $k \to -k$ so we can turn it into:

$$S_2 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{sin(rk)sin(\tau k)}{k^2} dk \tag{B.21}$$

Rewriting *sin* as exponentials gives:

$$\begin{aligned} S_2 &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{k^2}\frac{e^{ikr} - e^{-ikr}}{2i}\frac{e^{ik\tau} - e^{-ik\tau}}{2i} dk \\ &= -\frac{1}{8}\int_{-\infty}^{\infty} \frac{1}{k^2}\left[e^{ik(r+\tau)} - e^{ik(r-\tau)} - e^{ik(\tau-r)} + e^{-ik(r+\tau)}\right] dk \end{aligned} \tag{B.22}$$

Which can be separated into four different integrals:

$$\begin{aligned} S_2 = -\frac{1}{8}\Bigg[\int_{-\infty}^{\infty} \frac{e^{ik(r+\tau)}}{k^2} - \int_{-\infty}^{\infty} \frac{e^{ik(r-\tau)}}{k^2} dk \\ - \int_{-\infty}^{\infty} \frac{e^{ik(\tau-r)}}{k^2} dk + \int_{-\infty}^{\infty} \frac{e^{-ik(r+\tau)}}{k^2} dk \Bigg] \end{aligned} \tag{B.23}$$

Instead of taking each integral we can take a general form of the integrals above as:

$$I(\alpha) = \int_{-\infty}^{\infty} \frac{e^{ik\alpha}}{k^2} dk \tag{B.24}$$

This integral can be taken in the complex plane as long as the sign of the α is carefully taken into consideration. A very first step is to determination the choice of the contour so that it can be either a semi circle from below or above the real axis according to the sign of the α. One can show that, for $\alpha > 0$ contour should be a semi-circle above the real axis, and for $\alpha < 0$ contour should be a semi circle below the real axis. Secondly, since the pole is on the real axis at $k = 0$, one should lift or decent the real axis by a change of variable in k like $k \to k \pm i\epsilon$. Again, the sign in front of the ϵ is to be determined according to contour. For contours below the real axis we should take $k \to k + i\epsilon$ and vice versa.

In the light of these, let's first consider the case where $\alpha > 0$:

Then, we can descent the real line by introducing $k \to k - i\epsilon$ as long as we promise to take $\epsilon \to 0$ limit at the end.

Then the integral $I(\alpha)$ becomes:

$$\begin{aligned} \lim_{\epsilon \to 0} I(\alpha) &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{i(k-i\epsilon)\alpha}}{(k-i\epsilon)^2} dk \\ &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{ik\alpha} e^{\epsilon\alpha}}{(k-i\epsilon)^2} dk \end{aligned} \tag{B.25}$$

Then we have a second order pole at $k = i\epsilon$. Thus, residue of the integrand can be calculated as:

$$\begin{aligned} Rez[k = i\epsilon] &= \lim_{k \to i\epsilon} \frac{\partial}{\partial k} \left[(k-i\epsilon)^2 \frac{e^{ik\alpha} e^{\epsilon\alpha}}{(k-i\epsilon)^2} \right] \\ &= \lim_{k \to i\epsilon} [i\alpha e^{-\epsilon\alpha} e^{\epsilon\alpha}] \\ &= i\alpha \end{aligned} \tag{B.26}$$

For $\alpha < 0$, on the other hand, we can lift the real line by $i\epsilon$. In this case, however, to obtain the desired integral from $-\infty$ to ∞), we should take the contour integral in clock wise direction, which is the opposite convention compared to standard counter clock wise contour integration. This, naturally, brings an extra minus sign in front of the residue calculation.

All in all, for $\alpha < 0$, we have:

$$\begin{aligned} \lim_{\epsilon \to 0} I(\alpha) &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{i(k+i\epsilon)\alpha}}{(k+i\epsilon)^2} dk \\ &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{ik\alpha} e^{-\epsilon\alpha}}{(k+i\epsilon)^2} dk \end{aligned} \tag{B.27}$$

Again to calculate the residue at $k = -i\epsilon$,

$$\begin{aligned} Rez[k = -i\epsilon] &= -\lim_{k \to -i\epsilon} \frac{\partial}{\partial k} \left[(k + i\epsilon)^2 \frac{e^{ik\alpha} e^{-\epsilon\alpha}}{(k + i\epsilon)^2} \right] \\ &= -\lim_{k \to -i\epsilon} [i\alpha e^{-\epsilon\alpha} e^{\epsilon\alpha}] \\ &= -i\alpha \end{aligned} \tag{B.28}$$

Then the result of the integral can be categorized as:

$$I(\alpha) = \left\{ \begin{array}{ll} (2\pi i) i\alpha & \text{if } \alpha > 0 \\ (2\pi i) - i\alpha & \text{if } \alpha < 0 \end{array} \right\} = 2\pi \left[-\alpha\Theta(\alpha) + \alpha(\Theta(-\alpha)) \right] \tag{B.29}$$

We used the property of Heavy-side Step Function which gives 1 when its argument is greater than zero and otherwise it vanishes. Now, we are able to take our integrals in S_2 which can be rewritten as:

$$\begin{aligned} S_2 &= -\frac{1}{8} \left[I(r + \tau) - I(r - \tau) - I(\tau - r) + I(-r - \tau) \right] \\ &= \frac{\pi}{4} \Big[(r + \tau)\Theta(r + \tau) - (r + \tau)\Theta(-r - \tau) \\ &- (r - \tau)\Theta(r - \tau) + (r - \tau)\Theta(-r + \tau) \\ &- (\tau - r)\Theta(\tau - r) + (\tau - r)\Theta(r - \tau) + \\ &(-r - \tau)\Theta(-r - \tau) + (r + \tau)\Theta(r + \tau) \Big] \\ &= \frac{\pi}{2} \Big[(r + \tau)\Theta(r + \tau) - (r + \tau)\Theta(-r - \tau) \\ &- (r - \tau)\Theta(r - \tau) + (r - \tau)\Theta(\tau - r) \Big] \end{aligned} \tag{B.30}$$

Furthermore, we can use the famous property of the Heavy-side Step Function to simplify the equation to the end:

$$\Theta(x) = 1 - \Theta(-x) \tag{B.31}$$

$$\begin{aligned} S_2 &= \frac{\pi}{2}\left[(r+\tau)(\Theta(r+\tau) - \Theta(-r-\tau)) + (r-\tau)(-\Theta(r-\tau) + \Theta(\tau-r))\right] \\ &= \frac{\pi}{2}\left[(r+\tau)(2\Theta(r+\tau) - 1) + (r-\tau)(-1 + 2\Theta(\tau-r))\right] \\ &= \frac{\pi}{2}\left[2(r+\tau)\Theta(r+\tau) - 2r + 2(r-\tau)\Theta(\tau-r)\right] \\ &= \pi\left[(r+\tau)\Theta(r+\tau) + (r-\tau)\Theta(\tau-r) - 2r\right] \end{aligned} \tag{B.32}$$

Then we end up with:

$$S_2(r,\tau) = \pi\left[(r+\tau)\Theta(r+\tau) + (r-\tau)\Theta(\tau-r) - 2r\right] \tag{B.33}$$

Now, as promised, one can calculate $C_1(r,\tau)$ as follows by using the relations mentioned in (B.19):

$$\begin{aligned} C_1(r,\tau) &= \frac{\partial S_2}{\partial \tau} \\ &= \pi\left[\Theta(r+\tau) + (r+\tau)\delta(r+\tau) - \Theta(\tau-r) + (r-\tau)\delta(\tau-r)\right] \\ &= \pi\left[\Theta(r+\tau) - \Theta(\tau-r)\right] \end{aligned} \tag{B.34}$$

$$C_1(r,\tau) = \pi\left[\Theta(r+\tau) - \Theta(\tau-r)\right] \tag{B.35}$$

Similarly, S_0 can be derived from C_1:

$$\begin{aligned} S_0(r,\tau) &= -\frac{\partial C_1(r,\tau)}{\partial \tau} \\ &= \pi\left[\delta(\tau-r) - \delta(r+\tau)\right] \end{aligned} \tag{B.36}$$

$$S_0(r,\tau) = \pi\left[\delta(\tau - r) - \delta(r + \tau)\right] \tag{B.37}$$

On the other hand, result for $S_1(r,\tau)$ is taken from the famous table of integrals book [29]:

$$S_1(r,\tau) = \frac{1}{4} ln \left[\frac{r+\tau}{r-\tau}\right]^2 \tag{B.38}$$

Then, $C_0(r,\tau)$ can be calculated as:

$$\begin{aligned} C_0(r,\tau) &= \frac{\partial S_1(r,\tau)}{\partial \tau} \\ &= \frac{r}{r^2 - \tau^2} \end{aligned} \tag{B.39}$$

$$C_0 = \frac{r}{r^2 - \tau^2} \tag{B.40}$$