

DESIGN OF
MULTIDIMENSIONAL PERFECT-RECONSTRUCTION FILTER BANKS
WITH COMPRESSION APPLICATIONS

by

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Submitted to the Institute for Graduate Studies in Science and Engineering in partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

BOĞAZIÇI UNIVERSITY

1996

ACKNOWLEDGEMENTS

I am grateful to Prof. Dr. Bülent Sankur, who, according to his philosophy of life, is always aimed to study, to go forward, to race against time, thus, to be helpful to us by accumulating in one person the knowledge of several others. Even though he judges sometimes necessary to take some measures in order to protect us, he is always fatherly close to us with all his heart and he usually sheds light on our problems.

Every time when we are tired of our research work or we cannot find out our way or on the contrary, when we are full of new ideas, Prof. Dr. Emin Anarım is a real friend who always supports, encourages, leads and guides us with his global approach and ability to good solutions.

I owe thanks to the talented friend Doç. Dr. Hakan Çağlar, who helped and supported me in every phase of my thesis.

I am thankful to our dear professor and Department chairman Prof. Dr. Yorgo Istefanopulos, who was a member of my Ph.D. jury and as I has stated in my MSc. thesis, who, with his endless patience, always approves and encourages us.

I am also thankful to my valuable professor Prof. Dr. Ahmet H. Kayran, who had advised my BSc. thesis at İ.T.Ü. enlightening my research work with his brilliant ideas.

I am thankful to Doç. Dr. Engin Tuncer, who came from Ankara to participate to my jury.

I am thankful to my very talented professor Prof. Dr. Erdal Panayırıcı, Doç. Dr. Aysın Ertüzün and Doç. Dr. Acar Savacı, who had the kindness of accepting to be on my jury and attending my presentation.

I am also thankful to our Dean Prof. Dr. Sabih Tansal and all of our professors for attending my presentation.

I want also thank to Şükrü Ergür.

I would thank to Prof. Dr. Khalid Sayood and my friends Yücel Yemez, Kerem Harmancı, Sinan Güntürk for their valuable discussions on the thesis.

I would also like to call on my dear friends, Selçuk Öğrenci, Dr. Mustafa A. Altınkaya, Bilgin Eşme, Kıvanç Nurdan, Şebnem Zorlu, Burak Özer, Aykut Hocanın,

Levent Fadiloğlu and all BUSIM crew saying that, we are together in this way, we are walking about, working, enduring, suffering, enjoying and bearing all difficulties together forever.

I want thank to all of our Department's staff for their valuable support.

I would like to thank with all my heart and express my profound gratitude to these wonderful people, constituting my family, who always put my life before their own, making every sacrifice for me and devoting all they possess to me.

DESIGN OF MULTIDIMENSIONAL PERFECT RECONSTRUCTION FILTER BANKS

ABSTRACT

In this thesis, design of 1, 2 and 3-dimensional filter banks, specially for compression applications is studied and related methods are proposed.

In 1-D, design issues related to statistically and waveform matched wavelet design, where one has an analyzing filter that statistically matches a process or that resembles a given waveform, are proposed. Parameters tradeoffs with proper bounds in statistically matched wavelets and energy compaction performance of two-band schemes vis-à-vis Karhunen Loeve transform is investigated. A novel matched wavelet design technique based on time domain approach has been proposed and compared with Frazier Kumar design technique in terms of flexibility and exactness.

A novel time domain method is proposed for the design of nonseparable, linear phase, two and then three dimensional perfect reconstruction filters where linear phase and nonseparability are guaranteed by the centrosymmetry of the filters, so that no additional constraint equations have to be solved. The proposed design method first finds for all sampling lattices, a suitable lowpass filter, while the remaining $M-1$ filters are obtained through use of signed shuffling operations on the coefficients of that filter. These filter structures for M -band design are solved then using an optimization procedure and applied to digital image and video signal. All filter requirements are imposed on the lowpass filter (nonseparability, linear phase, regularity, etc.).

This design method could be of interest for its following advantages : i) true 2 and 3-dimensional processing can be done by allowing 2 and 3-dimensional filtering, ii) increase in the number of free variables, when compared to separable filtering case, allows better design performances, iii) centrosymmetry is used as general symmetricity (the design can be applied to all other kind of symmetries) and linear phase is guaranteed even for 4

and 8-band cases in 2 and 3-dimensions respectively, iv) repeated coefficients mitigates the computational load and memory requirements brought in by the nonseparable structure.

The method is proposed for possible combinations of filter structures in nonseparable linear phase PRFBs design. The results show that the proposed design method can provide higher transform coding gains vis-à-vis Discrete Cosine Transform (DCT) and separable filtering.

ÇOK BOYUTLU TAM GERİÇATMALI SÜZGEÇ KÜMELERİNİN TASARIMI

ÖZET

Bu tezde; 1, 2 ve 3-boyutlu süzgeç kümelerinin tasarımı özellikle sıkıştırma uygulamalarına yönelik olmak üzere ele alınmış ve yöntemler önerilmiştir.

Bir boyutta verilen dalgabiçimini ya da işaret istatistiklerini gözönüne alarak tasarımı uyumlu dalgacık süzgeçleri ile elde edilebilecek sıkıştırma kazancının alt ve üst sınırları araştırılmış, bu sınırlar Karhunen Loeve Dönüşümü kazancı cinsinden ifade edilmiştir ve tasarımda kullanılan parametrelerin ödünleşimi incelenmiştir. Zaman domeninde yeni bir uyumlu dalgacık tasarım tekniği geliştirilerek sıklık domeninde tasarımı Frazier Kumar tekniği ile başarımlı karşılaştırmaları yapılmıştır.

İki ve üç boyutlu M -bantlı doğrusal-evreli, ayrışmaz, tam geriçatmalı dikgen süzgeç kümelerinin zaman domeninde tasarımları için bir yöntem geliştirilmiştir. Tasarım önce bir alçak-geçiren süzgeç bulmağa dayandırılmış olup diğer süzgeçler bu alçak-geçiren süzgecin katsayılarının yer ve işaret değiştirilmeleri ile elde edilmektedir. Kodlama kazancı gibi eniyileme istekleri ve dikgenlik, düzenlilik gibi süzgeç kümeleri için aranılan özelliklerin tümü alçak-geçiren süzgecin katsayıları üzerine yüklenebilmektedir.

Bu yöntemin kullanılmasıyla elde edilebilecek avantajlar şöyle sıralanabilir : i) gerçek iki ve üç-boyutlu yapılar kullanılmaktadır, ii) ayrıştır süzgeçlemeyle karşılaştırıldığında aynı boyutta daha fazla değişken içerdikleri için daha iyi tasarım imkanı sağlarlar, iii) iki ve üç boyut için sırasıyla dört ve sekiz bantta bile doğrusal evre elde edilebilir ve bu yöntemle elde edilen sonuçlar genel bakışlılıkla tasarımı yürütmektedirler, iv) işaretli karımlar kullanılarak değişken ve kısıt denklemi sayısında

büyük azalmalar elde edilmiştir. Katsayılar M bantta da yinelendiği için süzgeç yürütümü hızlı olabilmektedir. İki boyutlu ayrışmaz süzgeçler için işlem karmaşası ve bellek sorunları azaltılmış genel bir tasarım yöntemi geliştirilmiştir.

İşaretli karılmalara dayalı iki ve üç boyuttaki ayrışmaz, doğrusal evreli süzgeç kümeleri kullanılarak Ayrık Kosinüs Dönüşümüne ve ayrışmaz süzgeçlemeye göre dönüşüm kodlama kazancı cinsinden iyileşme elde edilmektedir.

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LIST OF SYMBOLS

$\mathbf{D}$	sampling lattice
$\det(\mathbf{D})$	Determinant of $\mathbf{D}$
$\text{LAT}(\mathbf{D}), \text{LAT}^{-1}(\mathbf{D})$	Lattice and reciprocal lattice of $\mathbf{D}$
$H(\mathbf{z})$	Transfer matrix
$\tilde{H}(\mathbf{z})$	Complex conjugate transpose of $H(\mathbf{z})$
J	Objective function
$\mathbf{n} \circ \mathbf{k} = (n_1 k_1, \dots, n_n k_n)$	Schur product of two vectors
$\mathbf{R}_{xx}$	Covariance matrix
p -D	p -Dimensional
$S_{xx}(e^{j\omega})$	Input power spectral density function
$x(\mathbf{n})$	Discrete time signal
$\hat{x}(\mathbf{n})$	Reconstructed signal
$x_a(\mathbf{Dn})$	Sampled version of a p -dimensional signal $x_a(\mathbf{t})$
$x_a(\mathbf{t})$	Analog time function
$X(\mathbf{z})$	Z transform of $x(\mathbf{n})$
$X(e^{j\omega}) \leftrightarrow x(\mathbf{n})$	FT and IFT of $x(\mathbf{n})$
$X_a(j\Omega) \leftrightarrow x_a(\mathbf{t})$	FT and IFT of $x_a(\mathbf{t})$
X_m	Modulated version of signal
$\text{Proj}_{V_m}(r)$	Successive projections onto spaces $\bar{V}_m$ of r
$V_m \subset L^2(\mathfrak{R})$	Embedded subspaces
y_{Dec}	Output of the decimator
y_{E}	Output of the expander
$\mathbf{z}^{\mathbf{D}}$	Raising $\mathbf{z}$ to a matrix power $\mathbf{D}$
$\uparrow, \downarrow$	Upsampling, downsampling processes
$\delta_{\mathbf{n}\mathbf{k}}$	equal to 1 if the vector $(\mathbf{n}-\mathbf{k})$ is a zero vector

ϕ		Scaling function
Ψ		Wavelet function
ρ		Correlation coefficient
σ_L^2		Low band variance

ABBREVIATIONS

AR.....	AutoRegressive model
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DWT	Discrete Wavelet Transform
Dt	Distortion
FB.....	Filter Bank
FIR	Finite Impulse Response
FK	Frazier Kumar design method
FPD	Fundamental Parallelepiped
G_{TC}	Gain of Transform Coding
H_{AC}	aliasing component matrix
HH.....	High-High component
HPF	HighPass Filter
HVS	Human Visual System
IDFT.....	Inverse Discrete Fourier Transform
IIR	Infinite Impulse Response
KLT.....	Karhunen Loeve Transform
LAT.....	Lattice
LL.....	Low-Low component
LO	Linearly Orthogonality
LP	Linear Phase
LPF.....	LowPass Filter
M.....	number of bands
MD	MultiDimensional
MR	MultiRate
MRA	MultiResolution Approach
MSE	Mean Square Error
NS	Nonseparability
PR	Perfect Reconstruction
PSNR	Peak Signal to Noise Ratio

QMF.....	Quadrature Mirror Filter
RO.....	Rotationally Orthogonality
SBC.....	SubBand Coding
SBD.....	SubBand Decomposition
SER.....	Signal to Error Ratio
UC.....	Unit Circle
WP.....	Wavelet Packet

1. INTRODUCTION

1.1. Statement of the Problem

Multiresolution techniques are most naturally applied to images which are successively approximated starting from a coarse version and going fine resolution version with notions such as resolution and scale being very intuitive for images. Multiresolution techniques have become increasingly popular in image analysis and processing. To mention most important ones, pyramids and quadtrees could be stated. These include methods for processing which are developed for segmentation, thresholding, smoothing and texture analysis. Multiresolution techniques have been used in computer vision for tasks such as object recognition and motion estimation as well as in image compression.

In signal coding, one aims to concentrate the signal energy into as few subspectra or subbands as possible for efficient transmission of information. The uneven distribution of signal energy has made signal decomposition an important practical problem. So the monitoring of signal energy components within the subbands is possible and rate-distortion theory shows that the uneven spectral nature of real world signals can provide the basis for source compression techniques. Compression methods are built on both redundancies in the data and the nonlinearities of human vision. They exploit correlation in space for still images and in both space and time for video signals [1]. The lossy algorithms generally exploit aspects of the human visual system [2].

Multirate [3] filter banks (MRFB) find applications in a wide variety of digital signal processing systems. The analysis section of a filter bank decomposes a signal into a set of subband components and the synthesis section reconstructs the input signal from the components. MRFB's application areas include speech and image coding systems [4-6], spectrum analyzers, signal scramblers, geophysics, sampling format conversions between various video standards, high definition television systems, medical applications and so on.

Filter banks that operate on multidimensional signals find application primarily in the subband coding of images and video data [7-11]. A fundamental difference between

one dimensional and multidimensional filter banks is the manner in which the subband components are formed. In one dimension, subsampling a sequence can be defined in one manner by a predescribed factor, while in more than one dimension the subsampling factor is replaced by a subsampling matrix, called subsampling lattice. So, the idea of “sampling” a multidimensional signal is fundamental and involves the mathematical concept of “lattices”. Some of the earliest contributions to multidimensional filter banks were due to Vetterli [12], Wackershruther [13] and Woods and O’Neil [14]. The use of lattices [15, 16], in sampling has also been extended to the discrete time decimation and interpolation [17]. They have also been used for the description and analysis of multidimensional filter banks [18-27].

There are three kinds of distortions in the subband coding system, they are amplitude distortion, phase distortion, and alias distortion respectively. The cancellation of all the three kinds of distortions in the filter banks leads to filter banks having the property of perfect reconstruction (PR). PRFB have been proposed as structures suitable for hierarchical subband coding and also for multiresolution signal decomposition, as might be used in image pyramid coding. The two principal reasons for their popularity are that PRFB is shown to be strongly related to the wavelet transform and the orthogonality property of PR enabled the use of a measure of error in coding applications, giving the possibility of controlling the error when lossy compression is desired.

Most of the previously reported work on subband coding of images and image sequences is based on separable filtering. One limitation of separable PRFB in image coding example is the emphasis given to two perpendicular directions, horizontal and vertical preferences, while neglecting other directional properties. Another shortcoming is the obligation of design only with separable downsampling schemes. Flexibility in use of different downsampling schemes and possibility to design linear phase filters are ways to obtain better performance. Nonseparable filters have more degrees of freedom [28]. They can have linear phase. Thus, nonseparability has to be considered in order to take advantage of properties of the signals which are inherently multidimensional. Moreover, separable filters constitute only a small special subset of possible multidimensional filter solutions.

This thesis focuses on the optimal design of nonseparable (NS), linear phase (LP) multidimensional perfect reconstruction filter banks with multidimensional data

compression applications. In Section 1.2. we start with the discussion of existing design of filter banks. In Section 1.3. we present the scope and an overview of the contents of this thesis and in section 1.4. we state the main contributions of this thesis.

1.2. Design Techniques of Filter Banks

Most efforts in the literature have been given for the image subband coding system with separable filters and regular sampling. All such methods of designing two-dimensional separable filters are based on one-dimensional quadrature mirror filters (QMF) [29], and the pioneer subband coding system which uses these separable filter banks was developed by [14]. However, Woods et al. [14] did not consider the perfect reconstruction problem of the filter banks. Smith et al. [30] were the first one to develop the perfect reconstruction filter banks. Vaidyanathan et al. [31, 32] have proposed a series of PRFB with various structures.

Recently, attention has been drawn to the development of two-dimensional nonseparable filters and nonregular sampling. Simoncelli et al. developed the multiple dimensional non-separable filter banks by extending the one-dimensional QMF filters but his method does not provide the properties of perfect reconstruction. It has been demonstrated that a great number of multidimensional analysis/synthesis filters can be conveniently designed using a specific transformation applied on one dimensional prototype filters [33]. Examples to these specific transformations are given as McClellan, rotational transformations for the specific 2-D diamond shape and parallelogram shape subfilters [34, 35].

Kim et al. developed the diamond shape filter bank for the quincunx sampling pattern. Kim's method of designing the PRFB uses the properties of two-dimensional half-band filter. Bamberger et al. [35] uses a change of variables which is applied to the checkerboard-shaped response to effect the rotation, skewing, and scaling.

Considering an ideal 1-dimensional filter $G(\exp(jw))$ with magnitude response shown in Figure 1.1. Suppose we define a 2-dimensional transfer function $G(z_0, z_1)$ which has frequency response $G(\exp(j(w_0 + w_1)))$ shown in Figure 1.1b. A diamond of wrong size is obtained as in Figure 1.1.c if the product of 2-dimensional transfer function $G(z_0, z_1)$ and its flipped version with respect to the horizontal axis is considered. The desired diamond support is obtained in Figure 1.1.d from which fan filters in Figure 1.1.e, f, g can be obtained by π shifts.

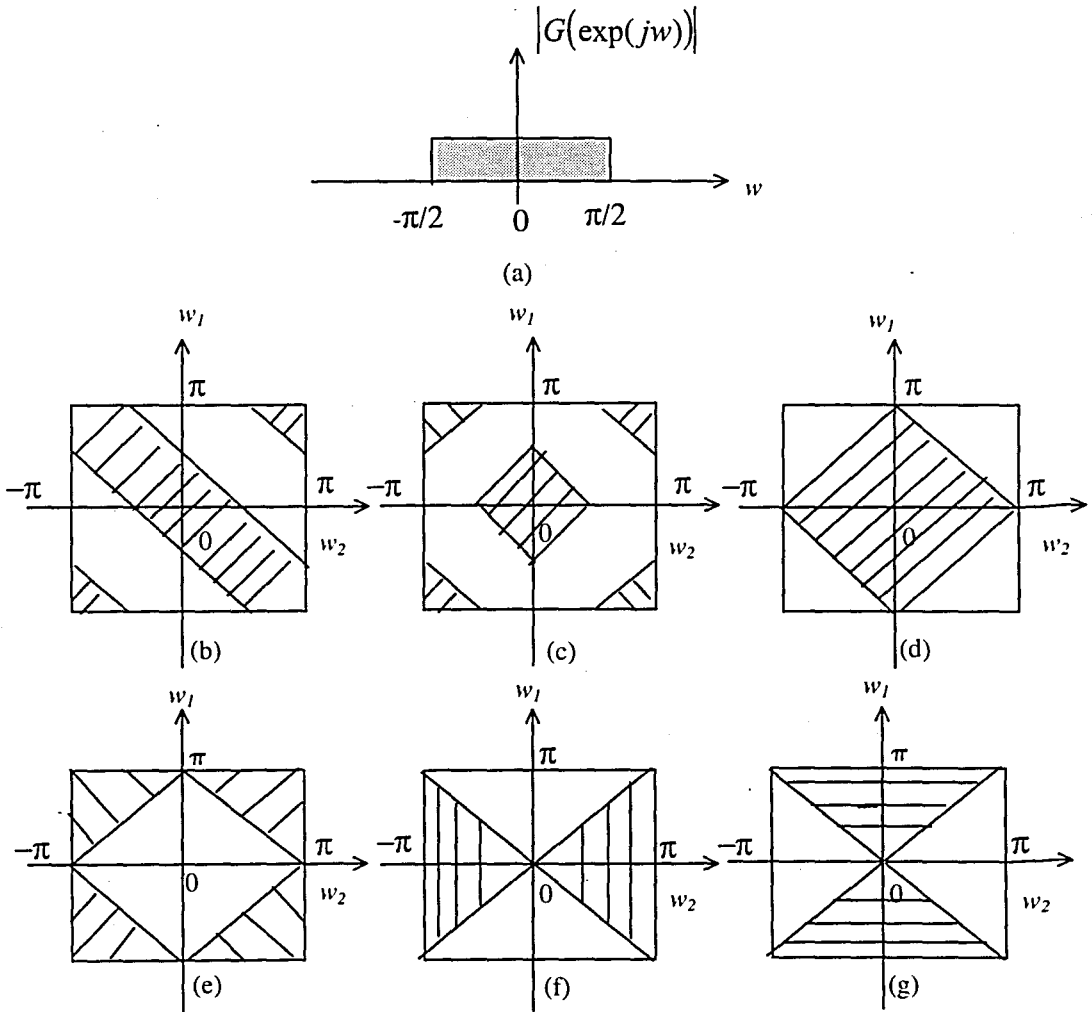


FIGURE 1.1. Nonseparable 2-dimensional filters designed from a 1-dimensional filter

The Fourier transform of a zero phase symmetric filter can be written as a function of $\cos(nw)$. Using Tchebycheff polynomials, one can replace $\cos(nw)$ by $T_n[\cos(w)]$ and the idea of the McClellan transformation is to replace $\cos(w)$ by a zero phase two dimensional filter. This results in an overall zero phase two dimensional filter.

Vetterli and Karlsson [26] proposed a method of designing the multidimensional perfect reconstruction filter banks for nonseparable sampling (quincunx and hexagonal). They classify the PR systems into two classes, one is paraunitary and the other one is nonparaunitary systems. A linear system with rational transfer function (or transfer matrix) $\mathbf{H}(z)$ is said to be paraunitary if it satisfies,

$$\tilde{\mathbf{H}}(z)\mathbf{H}(z) = d\mathbf{I} \quad (1.1)$$

for some $d>0$ and with $\tilde{\mathbf{H}}(z)$ being complex conjugate transpose of $\mathbf{H}(z)$.

Recently some results have emerged, setting up the theory of general, nonseparable multidimensional filter banks and so concentrating on “true” multidimensional case where “true” means allowing both nonseparable sampling and nonseparable filtering in design. Although true multidimensional approach suffers from higher computational complexity, use of nonseparable filters leads to more degrees of freedom in design and so to better filters, and nonseparable sampling opens a possibility of having schemes better adapted to the human visual system. PRFBs including nonseparability and linear phase are studied in [36-38] and all of these designs are carried in polyphase domain..While providing solutions for vertically and horizontally symmetric filters [39], they pose the question in their papers if their approach would cover all feasible filters. Almost all such design procedures for multidimensional PRFBs in the literature are based on polyphase domain analysis [38, 40-44]. The only exceptions are the 1-D designs based on time domain as in [45] where the perfect reconstruction conditions are formulated purely in the time domain.

1.3. Scope and Outline of this Thesis

Our approach differ from the existing multidimensional PRFBs in the literature in that they are purely time domain formulation which makes it possible to incorporate easily various performance desiderato. tries to tune the basis functions or bandpass filters of the filter banks with respect to a set of objective performance criteria. This leads to the concept of optimal filter banks which can also incorporate many design parameters simultaneously in filter bank design [46].

In Chapter 2, we follow a general analysis of multidimensional filter banks. In Chapter 3, we propose optimal design issues related statistically and waveform matched wavelet design while details are relegated to Appendices A-H.

We present in Chapter 4, a novel design method for 2-dimensional M -band nonseparable, linear phase, PRFB design. This method is a direct generalization of the work on 1-dimensional M -band PRFB time-domain design given in [47, 48]. The proposed time-domain design method, first finds for any sampling lattices (or dilation matrices) a suitable lowpass filter, while the remaining $M-1$ (for M -band) filters are obtained through use of signed shuffling operations on the coefficients of that filter. All filter bank requirements are imposed only on the lowpass filter. The method is computationally efficient since only the coefficients of the lowpass filter need to be stored and the shuffling structure can be shown to facilitate the VLSI implementation. The proposed method provides a complete search for all possible combinations of filter structures in 2-D M -band nonseparable linear phase PRFBs design. As an application to image coding, an adaptive tree structure using wavelet packets employing a framework that includes rate and distortion is implemented. The performance of the nonseparable filters proposed in this thesis are given for different distortion measures for this adaptive tree structure in Appendix G.

Some 3-dimensional PRFB design procedures are studied by Kovacevic et al. [37, 49, 50] where they extend their work in 2-dimensions [36]. The 2-D shuffling based time domain design approach presented in this thesis is extended also for three dimensional design of PRFBs which will be given in Chapter 5.

1.4. Main Contributions of the Thesis

A. Design issues related to statistically matched and waveform matched wavelet design are investigated:

A. ii. Parameter tradeoffs in statistically matched wavelets and energy compaction performance of two-band schemes vis-à-vis Karhunen Loeve transform is investigated

A. iii. Variations of the Frazier-Kumar (FK) matched wavelet filter design method are studied and analysis of this design method in terms of energy compaction criterion is given.

A. iv. A novel matched wavelet design technique based on time domain approach has been proposed and compared with the FK design technique

B. A time domain method is advanced for the design of nonseparable, linear phase, regular multidimensional (proper design methods for two dimensional and three dimensional signals respectively) perfect reconstruction filters.

B. i. Linear phase and nonseparability are guaranteed by the centrosymmetry of the filters, so that no additional constraint equations have to be solved. Symmetries other than centrosymmetry and lattices other than rectangular are also addressed.

B. ii. Compaction performance of the proposed PRFB is compared with respect to separable filtering and DCT (Discrete Cosine Transform) as tested on images.

B. iii. A permutation-elimination based proof of uniqueness of the time domain shuffle based multidimensional, nonseparable, linear phase design

2. ANALYSIS OF MULTIDIMENSIONAL MULTIRATE SYSTEMS

In this chapter, while dealing with the basic concepts of 1-dimensional multirate signal processing, we also review their extension to the general case of multidimensional signals (for p an integer number representing dimension, $1 \leq p < \infty$), with examples set from the two and three dimensional signals. An example of a 2-dimensional signal is an image, whose intensity is a function of two variables (horizontal and vertical coordinates). A popular example of a 3-dimensional signal is a video signal, “time” being the third dimension.

Transform domain expressions of multidimensional signals, their properties, and general structures are considered in Sections 2.1. and 2.2. Such preliminary information is essential to understand the multidimensional multirate filter banks. Different architectures for multidimensional multirate filter banks are revisited in Section 2.3.1 and 2.3.2. Other properties of filter banks such as, separability, linear phase and perfect reconstruction are explicated in Section 2.3.3 and in Section 2.3.4. In Section 2.3.5 we point out the underlying relations between wavelet transform, subband decomposition and multiresolution analysis.

This chapter provides necessary tools and concepts for the design of filter banks [32] used in the following chapters.

2.1. Multidimensional Signals

It is often more convenient to work with transformed versions of signals such as Fourier transform and Z-transform. These transforms will be considered in this section.

2.1.1. Fourier Transform and Z-Transform

Continuous time functions are denoted as $x_a(t)$ where the subscript “a” stands for “analog”. The 1-D Fourier Transform of $x_a(t)$, if it exists, is defined as,

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) \exp(-j\Omega t) dt \quad (2.1.a)$$

and the inverse transform relation is

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) \exp(j\Omega t) d\Omega \quad (2.1.b)$$

where the frequency variable Ω has the dimension of radians per second.

A p -dimensional signal $x_a(t_0 \dots t_{p-1})$ is a function of p real variables $t_0 \dots t_{p-1}$. Defining the column vector $\mathbf{t} = [t_0 \dots t_{p-1}]^T$ and the column vector of frequencies $\Omega = [\Omega_0 \ \Omega_1 \ \dots \ \Omega_{p-1}]^T$, the Fourier transform and its inverse of $x_a(\mathbf{t})$ are given, respectively by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(\mathbf{t}) \exp(-j\Omega^T \mathbf{t}) d\mathbf{t} \quad (2.2.a)$$

and

$$x_a(\mathbf{t}) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} X_a(j\Omega) \exp(j\Omega^T \mathbf{t}) d\Omega \quad (2.2.b)$$

Thus Ωt is replaced with the inner product $\Omega^T \mathbf{t}$ and the scale factor is $1/(2\pi)^p$ instead of $1/(2\pi)$ when passing from 1-dimensional to the general multidimensional expressions of continuous time Fourier transform. The region where $X_a(j\Omega)$ is allowed to be nonzero, is said to be the *support* of $X_a(j\Omega)$.

Fourier transform relations for discrete time 1-dimensional signals were defined as

$$X(\exp(jw)) = \sum_{n=-\infty}^{\infty} x(n) \exp(-jwn) dw \quad (2.3.a)$$

the inverse transform relation is given by

$$x(n) = 1/2\pi \int_0^{2\pi} X(\exp(jw)) \exp(jwn) dw \quad (2.3.b)$$

In the p -dimensional case, discrete time signals are denoted by $x(\mathbf{n})$ for the column integer vector $\mathbf{n} = [n_0 \ n_1 \ \dots \ n_{p-1}]^T$. With $\mathbf{w} = [\omega_0 \ \omega_1 \ \dots \ \omega_{p-1}]^T$, the Fourier transform, if it exists, and the inverse one are given respectively by

$$X(\mathbf{w}) = \sum_{\mathbf{n} \in \mathbf{N}} x(\mathbf{n}) \exp(-j\mathbf{w}^T \mathbf{n}) \quad (2.4.a)$$

and

$$x(\mathbf{n}) = \frac{1}{(2\pi)^p} \int_0^{2\pi} X(\mathbf{w}) \exp(j\mathbf{w}^T \mathbf{n}) d\mathbf{w} \quad (2.4.b)$$

Similarly, the Z-transform of $x(\mathbf{n})$ is denoted by

$$X(\mathbf{z}) = \sum_{\mathbf{n}} x(\mathbf{n}) \mathbf{z}^{-\mathbf{n}} \quad (2.5)$$

where $\mathbf{z} = [z_0 \ z_1 \ \dots \ z_{p-1}]^T$. If it converges for all $z_k = \exp(jwk)$, $0 \leq k \leq p-1$, it reduces to the Fourier transform $X(\mathbf{w})$.

2.1.2. Some Properties of the Fourier and Z-Transforms

Linearity

If $X_1(\mathbf{z})$ and $X_2(\mathbf{z})$ are the Z-transforms of $x_1(\mathbf{n})$ and $x_2(\mathbf{n})$, then the transform of $a_1 x_1(\mathbf{n}) + a_2 x_2(\mathbf{n})$ is given by $a_1 X_1(\mathbf{z}) + a_2 X_2(\mathbf{z})$ for arbitrary a_1 and a_2 .

Shifting a sequence

The Z-transform of the shifted sequence $x(\mathbf{n}-\mathbf{k})$ is given by $\mathbf{z}^{-\mathbf{k}} X(\mathbf{z})$ where $\mathbf{z}^{-\mathbf{k}} = z_1^{-k_1} z_2^{-k_2} \dots z_p^{-k_p}$.

Convolution Theorem

Given two multidimensional sequences $x(\mathbf{n})$ and $h(\mathbf{n})$, the convolution sum is defined as $y(\mathbf{n}) = \sum_{\mathbf{m} \in \mathbf{N}} h(\mathbf{m})x(\mathbf{n} - \mathbf{m}) = \sum_{\mathbf{m} \in \mathbf{N}} h(\mathbf{n})x(\mathbf{m} - \mathbf{n})$ and their Z-transform and Fourier transform are given respectively by $Y(\mathbf{z}) = H(\mathbf{z}) X(\mathbf{z})$ and $Y(\mathbf{w}) = H(\mathbf{w}) X(\mathbf{w})$.

Modulation and Frequency Shift

The Z-transform for multiplication of $x(\mathbf{n})$ with $\exp(\mathbf{a}^T \mathbf{n})$, where $\mathbf{a}$ is a column vector, is obtained from $X(\mathbf{z})$ by replacing for $0 \leq i \leq p-1$ each variable z_i with $z_i \exp(-a_i)$.

2.2. Sampling of Multidimensional Signals

Subsampling means that some of the samples of a discrete signal are discarded. Therefore fewer samples are used to represent the same signal, thus yielding a data reduction. Effectively this means that the sampling distance is increased. According to Nyquist criterion, if the sampling distance is increased, the bandwidth of the signal should be decreased. A decrease of the bandwidth affects the fine details of the signal, corresponding to the high frequency contents of the signal. Hence a data reduction is achieved at the expense of some distortion or information loss. The justification for the subsampling being the Nyquist rule we can reduce the number of samples at the output of the filter, then the range of frequencies at the output filter is less than the range of frequencies at the input to the filter. Subsampling can be thought as a method to compress data. As example, a digital image sequence consists of a spatial and temporal set of picture elements. In a subsampling system the information to be transmitted is reduced by transmitting only part of the pixels. At the receiver, the discarded pixels have to be recovered by using information which was transmitted. To facilitate a good synthesis, it is important that the transmitted pixels contain sufficient information.

The theory behind sampling of a multidimensional signal is fundamentally complicated because of the many different ways to choose the sampling geometry.

In this Section, we present a summary of relevant concepts from the theory of lattices. Each point on a uniform sampling grid (such as rectangular, hexagonal) is defined by an integer vector from the origin to the point. We will refer to the collection of all such integer vectors as $\mathbf{D}$, a lattice, i.e., a regular arrangement of points in a space. A lattice being a sampling grid, a sublattice is a subset of a lattice obtained by an integral linear mapping of a lattice, and cosets represent different ways to shift the sampling grid. A brief discussion of these ideas follows.

2.2.1. Sampling Lattices

An integral part of subband coding is the downsampling of the analyzed signal and the reciprocal upsampling before the synthesis. For one dimensional signals, uniform or equidistant, sampling structures are most common. In multiple dimensions the counterpart is a signal sampled on a uniform grid with one sample located at each vertex point of the grid. This grid is referred to as a lattice. Assuming a two dimensional signal defined on an arbitrary sampling lattice, a subsampling of this signal can be seen as a linear transformation from the input lattice to a subsampling lattice (i.e. the subsampling lattice is a coset of the input lattice). For a system with critical subsampling, where the number of samples is conserved, the number of subbands is equal to the subsampling factor.

The sampled version of a p -dimensional signal $x_a(t)$ is defined as

$$x(\mathbf{n}) = x_a(\mathbf{D}\mathbf{n}) \quad (2.6)$$

where $\mathbf{D}$ is a real $p \times p$ nonsingular matrix

$$\mathbf{D} = [\mathbf{d}_0 \mathbf{d}_1 \dots \mathbf{d}_{p-1}] \quad (2.7)$$

In the special case of 1-dimensional signals ($p=1$), $\mathbf{D}$ is equal to a scalar quantity T where T is the sample spacing. Let us give some basic definitions on the subject of lattices.

Lattice : A lattice is a discrete set of vectors (points, p -tuples) in Euclidean p -dimensional space $\mathbb{R}^p$ that forms a group under ordinary vector addition (i.e., given two points in a lattice, their Euclidean difference and sum are also in the lattice). The set of all sample points for a given lattice is the set $\mathbf{t} = \mathbf{D}\mathbf{n}$ that is the set of vectors and they are given as,

$$\sum_{k=0}^{p-1} n_k \mathbf{d}_k \quad (2.8)$$

In the thesis, $\mathbf{D}$ is chosen always as an integer matrix so that this is the set of all integer linear combinations of the columns $\mathbf{d}_0, \dots, \mathbf{d}_{p-1}$ of $\mathbf{D}$, which are linearly independent vectors. This set denoted $\text{LAT}(\mathbf{D})$ is called the lattice generated by the matrix $\mathbf{D}$. The matrix $\mathbf{D}$ is called the sampling matrix (also called dilation matrix), and is also said to be the basis which generates $\text{LAT}(\mathbf{D})$. So, $\text{LAT}(\mathbf{D})$ contains an infinite number of points

$$\text{LAT}(\mathbf{D}) = \{ \mathbf{t} \mid \mathbf{t} = n_1 \mathbf{d}_1 + \dots + n_{p-1} \mathbf{d}_{p-1} \} \quad (2.9)$$

$\text{LAT}(\mathbf{D}^{-T})$ is said to be the reciprocal lattice of $\text{LAT}(\mathbf{D})$.

Resampling Matrix: A resampling matrix is a nonsingular square integer matrix that maps $\mathbf{D}$ into itself.

Sublattice : A sublattice is a lattice which is contained within a larger lattice. We will only consider here sublattices of the integer lattice $\mathbf{D}$.

Cosets and Coset vectors : A coset is obtained by simply shifting a sublattice by an integer vector, $\mathbf{k}$. Any vector from the origin to a point on a coset is a coset vector. There are an infinite number of coset vectors. The distinct coset vectors associated with the matrix $\mathbf{D}$ are the $M = \det \mathbf{D}$ integer vectors that lie within the fundamental parallelepiped (FPD) of $\mathbf{D}$. The coset vectors are denoted by

$$\mathbf{k} = \text{FPD}(\mathbf{D}) = \{ \mathbf{D}\mathbf{x} \mid 0 \leq x_i < 1, i = 1, \dots, p \} \quad (2.10)$$

$\{ \mathbf{k}_0 \dots \mathbf{k}_{M-1} \}$ with $\mathbf{k}_0 = \mathbf{0}$. The distinct coset vectors provide a convenient way to enumerate a block of samples for resampling.

In Figure 2.1., lattice and its inverse generated by the matrix $\mathbf{D}$ with their proper FPDs are shown.

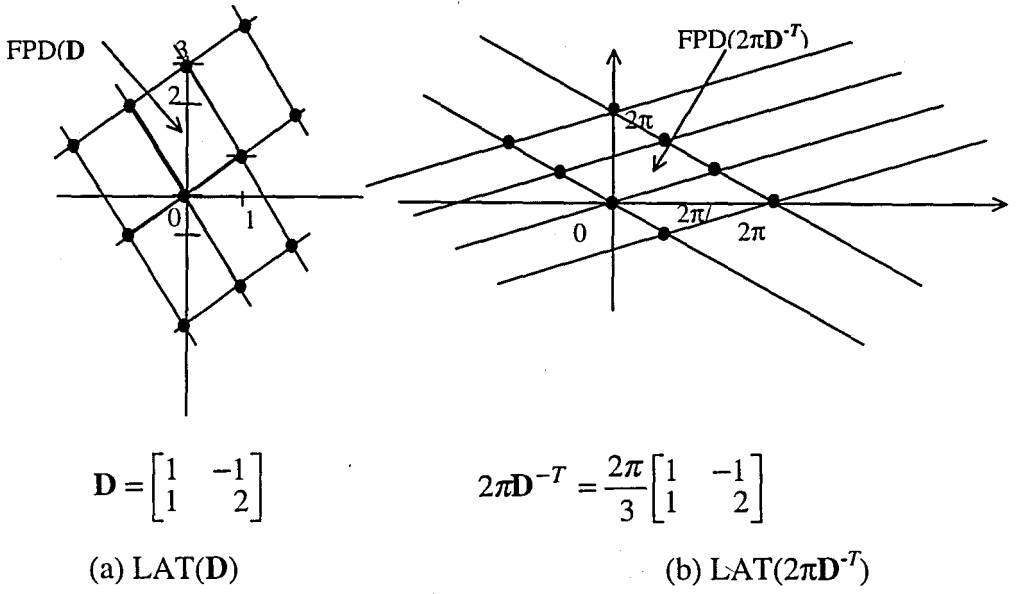


FIGURE 2.1. (a) Lattice generated by $\mathbf{D}$ and (b) scaled reciprocal lattice, generated by $2\pi\mathbf{D}^{-T}$

Sampling Density and Determinant of $\mathbf{D}$: The determinant of the sampling lattice, $\mathbf{D}$, plays a fundamental role in the sampling of a multidimensional signal. It is well known that the volume of the fundamental parallelepiped is equal to the determinant of $\mathbf{D}$. The term “volume” degenerates to “area” in the 2-dimensional case and “length” in the 1-dimensional case. A parallelepiped with one edge aligned to the horizontal axis (there is no loss of generality in such alignment since we can always rotate a parallelepiped without changing its area) and with a and b denoting the lengths of the sides, the area is given by $ab|\sin\theta|$ shown in Figure 2.2.

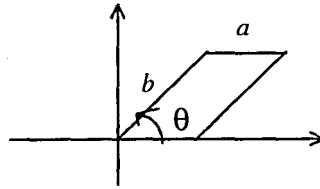


FIGURE 2.2. The area of the parallelogram being equal to $ab|\sin\theta|$

By definition of $\text{FPD}(\mathbf{D})$, only one point falls inside it, namely the origin. Similarly, there is one lattice point in each shifted version as seen in Figure 2.3. The number of sample points per unit volume is equal to the number of $\text{FPD}(\mathbf{D})$'s which can fit into the unit volume.

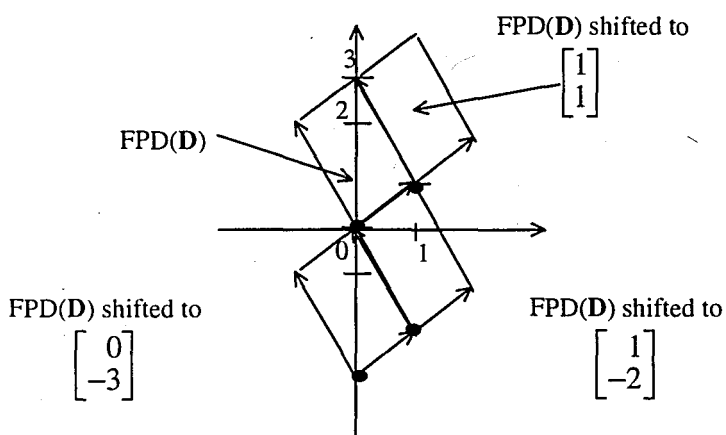


FIGURE 2.3. $\text{FPD}(\mathbf{D})$ and copies obtained by shifting it to various points on $\text{LAT}(\mathbf{D})$

$$\text{for } \mathbf{D} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

Polyphase Decomposition : For the defined subsampling pattern, the filters can be decomposed into polyphase components. The polyphase components together cover all the points on the input lattice, where each polyphase component is formed from the points of a coset of that lattice. The multidimensional multirate system being shift-variant one can use polyphase components obtained by the cosets of the corresponding sampling lattice as:

$$h(\mathbf{n}) = h(\mathbf{Dn} - \mathbf{k}) \quad (2.11)$$

where $\mathbf{k}$ are the cosets in the fundamental parallelepiped.

Similarly the polyphase components of the filter $H(\mathbf{z})$ can be expressed as

$$H(\mathbf{z}) = \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D})} \mathbf{z}^{-\mathbf{k}} H_{\mathbf{k}}(\mathbf{z}^{\mathbf{D}}) \quad (2.12)$$

where

$$H_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{Z}^p} h(\mathbf{D}\mathbf{n} + \mathbf{k}) \mathbf{z}^{-\mathbf{n}} \quad (2.13)$$

with $h(\mathbf{D}\mathbf{n} + \mathbf{k})$ denoting the polyphase components.

The system is shift variant due to downsampling. It is this shift variance that leads to aliased versions of the input signal at the output. The polyphase domain analysis where a signal is decomposed into its polyphase components is an important and useful tool in multirate processing [51, 52]. A signal can be recovered from its polyphase components simply by upsampling, appropriately shifting and summing them.

Time domain and polyphase domain analysis can be derived from each other as in Equation 2.11 and 2.12 and they are so equivalent. All of the design procedures in this thesis are given in time domain and we will not use this important concept of polyphase domain analysis but only compare the results obtained by using it in the literature.

2.2.1.1. Two Dimensional Sampling Lattices

The main difference when compared to the 1-dimensional treatment is that two dimensional sampling requires the use of lattices. A lattice, as defined in the previous section is the set of all vectors generated by $\mathbf{D}\mathbf{n}$, $\mathbf{n} \in \mathbb{Z}^p$ where the sampling lattice $\mathbf{D}$, is defined by

$$\mathbf{D} = \begin{bmatrix} d_{00} & d_{01} \\ 0 & d_{11} \end{bmatrix} \quad (2.14)$$

for $d_{00} > 1$, $d_{11} > 1$, $0 \leq d_{01} < d_{00}$ in the subsampling. A necessary requirement on the matrix $\mathbf{D}$ is that it represent a dilation in all dimensions (otherwise, the associated wavelet analysis would not change resolution in all dimensions). This is equivalent to requiring that all eigenvalues of $\mathbf{D}$ should have magnitudes greater than 1. The eigenvalues $\mathbf{e}$ of $\mathbf{D}$ are given as $\mathbf{e} = \{d_{00}, d_{11}\}$. So, ' d_{00} ' and ' d_{11} ' in $\mathbf{D}$ must be both greater than 1.

Let us show some examples of dilation matrices to indicate how they can always be written in the form in Equation 2.14; for the hexagonal case, we can have two equivalent forms as given in Figure 2.4 with the corresponding sampling matrices:

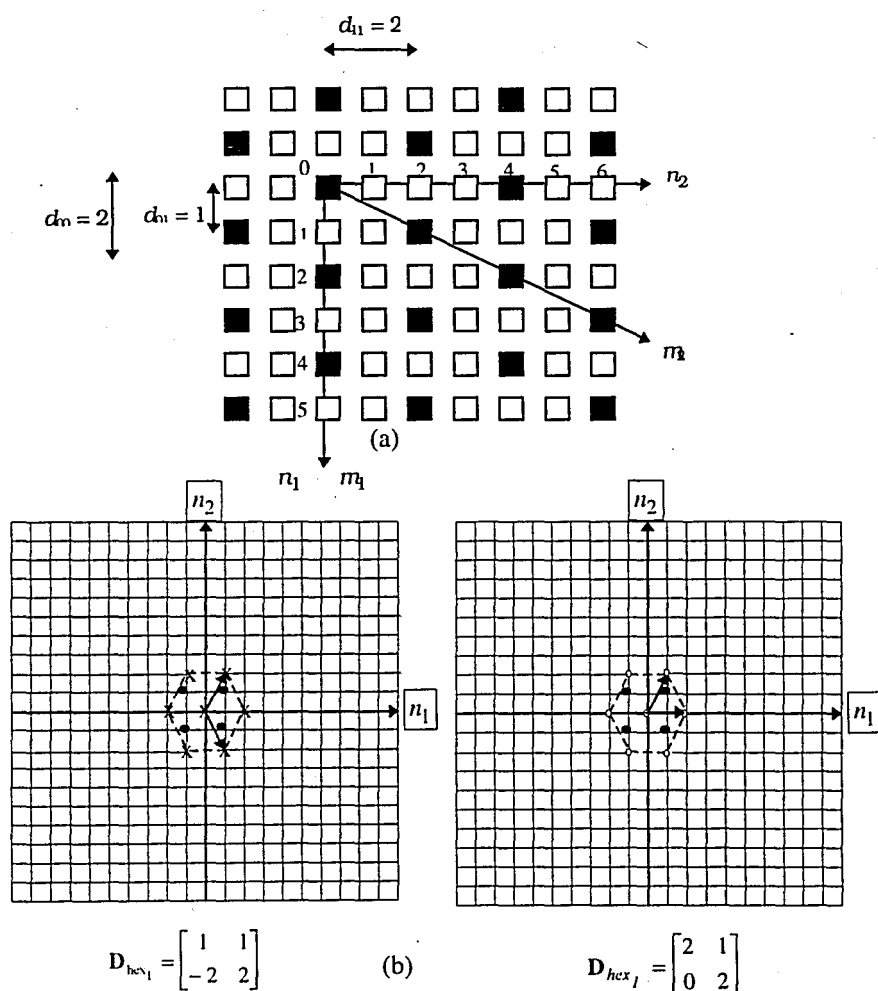


FIGURE 2.4. (a) Hexagonal sampling process is a representative for the sampling process for any dilation matrix, (b) Hexagonal lattice generated using two different $\mathbf{D}$ matrices : Bounded regions represent fundamental parallelepipeds (FPD) for each case.

For the quincunx case, we can also have two equivalent forms such as:

$$\mathbf{D}_{\text{qncx1}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{D}_{\text{qncx2}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Quincunx sampling lattice is not a valid dilation matrix since while dilating by 2 in one direction, it does not dilate at all in the other direction.

The number of subbands obtained by the subsampling operation is given by the determinant of $\mathbf{D}$ as $\det \mathbf{D} = d_{00} \cdot d_{11}$. The type of sampling is called “quadratic” when $d_{00} = d_{11}$ and $d_{01} = 0$ in the sampling lattice, “rectangular” when $d_{00} \neq d_{11}$ and $d_{01} = 0$, and for the general $d_{01} \neq 0$ case the sampling is called “non-rectangular” or only “polygonal”. In this “polygonal” case, for $d_{00} = d_{11}$ and $d_{01} \neq 0$ there are $(d_{00}-1)$ different sampling lattices, while for $d_{00} \neq d_{11}$ and $d_{01} \neq 0$ there are $(d_{00}-1) + (d_{11}-1)$ or $(d_{00} + d_{11} - 2)$ different sampling lattices, all providing in each case the same type of subsampling. The proposed shuffling based design method that will be explained later in Chapter 4, provides a complete search over the space for all possible structures of sampling lattices. Some numerical instances of the three types of sampling lattices are given below.

- Quadratic Sampling : $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, etc.
- Rectangular Sampling : $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, etc.
- Polygonal Sampling : $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, etc.

Examples for polygonal sampling where different lattices providing the same type of subsampling scheme can be given as,

- An example of $d_{00} = d_{11}$ and $d_{01} \neq 0$, with $(d_{00}-1)$ different sampling lattices is:
(for $\mathbf{D} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, we have $(d_{00} - 1) = (3 - 1) = 2$ different sampling lattices as $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$)
- An example of for $d_{00} \neq d_{11}$ and $d_{01} \neq 0$, with $(d_{00}-1) + (d_{11} - 1)$ different sampling lattices is:

(for $\mathbf{D} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, we have $(d_{00} - 1) + (d_{11} - 1) = (3 - 1) + (2 - 1) = 3$ different sampling lattices as $\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$)

In the thesis, designs for all types of two dimensional sampling cases are provided .

2.2.1.2. Three-Dimensional Sampling Lattices

Three dimensional sampling lattices find applications mainly in image sequence and video processing.

Digital video can be obtained by horizontally sampling analog video which has already been scanned in the vertical temporal direction or by applying an inherently 3-dimensional sampling structure to sample the time-varying image, as in the case of some solid state sensors. The sampling lattice prescribing the specific location of the samples, the shape of the sampling lattice can be adapted to the spectrum of the image.

Numerous sampling structures have been proposed for three dimensional sampling where the sampling lattice is a set of discrete points in the three dimensional space. The commonly used 3-dimensional sampling matrices in the literature are given in [17]. To give an example for 3-D sampling lattices, cubic sampling is given by the vectors $\mathbf{d}_0 = [2 \ 0 \ 0]$, $\mathbf{d}_1 = [0 \ 2 \ 0]$, $\mathbf{d}_2 = [0 \ 0 \ 2]$ in $\mathbf{D} = [\mathbf{d}_0 \ \mathbf{d}_1 \ \mathbf{d}_2]$. Cubical subsampling is redrawn in Figure 2.5 in a three-dimensional viewing.

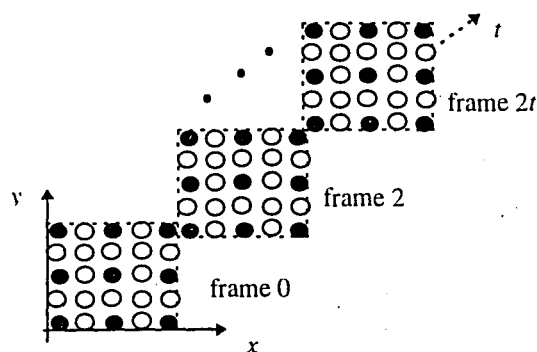
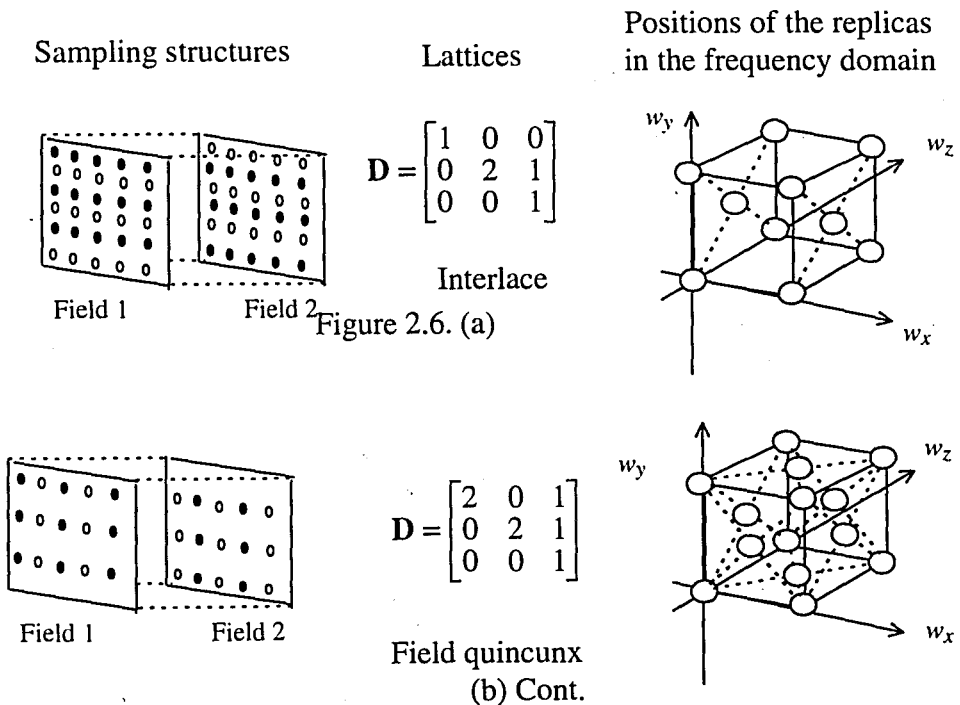


FIGURE 2.5. Cubical 8 band subsampling

If the sampling structure in Figure 2.5 is used to represent an image sequence, the sequence is called progressively scanned [53]. The spectral replicas are located on an orthogonal lattice in the frequency domain.

The scanning procedure commonly used in television is the interlace scan. The corresponding lattice is depicted in Figure 2.6a. Half of the number of rows of the frame are discarded, hereby reducing the vertical resolution. Effectively, this means that extra spectral replicas are introduced between the existing spectral replicas in the vertical direction. The pattern in which the rows are discarded is shifted from frame to frame. Therefore the extra replicas have an offset in the w_z direction. After subsampling, the individual images are called fields instead of frames. In practice, interlace sampling is used to increase the temporal resolution. While maintaining the same total number of samples as orthogonal (cubical) sampling, the field rate is doubled with respect to the frame rate in the case of orthogonal sampling, hereby doubling the temporal resolution. The field quincunx sampling structure is on Figure 2.6.b where half the pixels of a line are alternately discarded. The pattern in which the pixels are discarded is shifted from field to field. The consequence is that the resolution in the vertical and horizontal direction is same but the resolution in the diagonal direction is reduced. The quincunx structure can be recognized by combining the two fields in the temporal direction. The line quincunx sampling structure is illustrated in Figure 2.6.c where again half the pixels of a line are discarded but now the pattern is shifted from line to line within each field. The vertical and horizontal resolution is again at the expense of the diagonal reduction. The difference between the field quincunx lattice and the line quincunx lattice is that their temporal resolution is different.



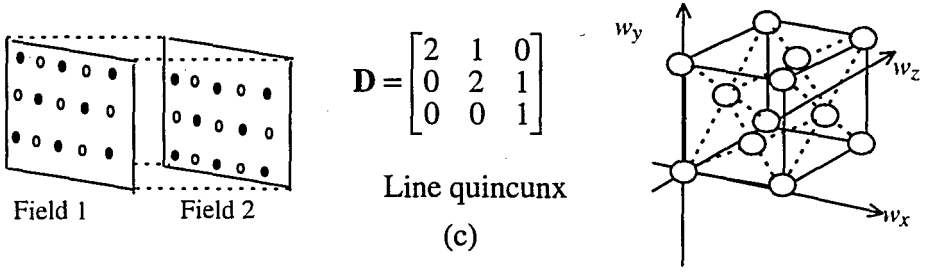


FIGURE 2.6. Sampling structures, lattice matrices and the positions of the replicas in the frequency domain are shown for a) orthogonal, b) interlace, c) field quincunx and d) line quincunx lattices [53].

In this thesis, only 8-band cubical (orthogonal) subsampling is used in the context of design of 3-dimensional filters as will be seen in Chapter 6 .

2.2.2 Aliasing Effects created by Sampling

Recall that in the 1-D case, the relationship between the continuous time Fourier transform of a signal and the discrete time Fourier transform is given by

$$X(\exp(jw)) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\frac{1}{T}(w - 2\pi k)\right) \quad (2.15)$$

In the p -dimensional case there is a similar effect. Let $x_a(\mathbf{t})$ be a p -dimensional signal with Fourier transform $X_a(j\Omega)$. Define the sequence $x(\mathbf{n}) = x_a(\mathbf{Dn})$. Then its Fourier transform is given by

$$X(\exp(j\mathbf{w})) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \mathbf{N}} X_a(j\mathbf{D}^{-T}(\mathbf{w} - 2\pi \mathbf{k})) \quad (2.16)$$

where $\mathbb{N}$ is the set of all p -component integers. With $\mathbf{D} = T$ (1 dimensional case), this reduces to Equation 2.15.

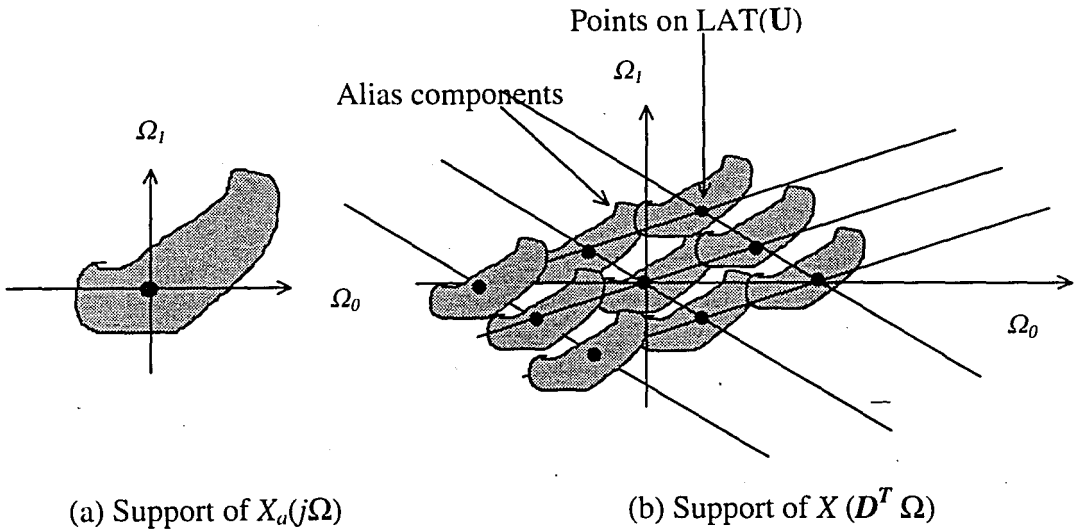
To illustrate the procedure [32] for constructing $X(\mathbf{w})$ from $X_a(j\Omega)$, we first make the change of variables

$$\Omega = \mathbf{D}^{-T} \mathbf{w} \quad (2.17)$$

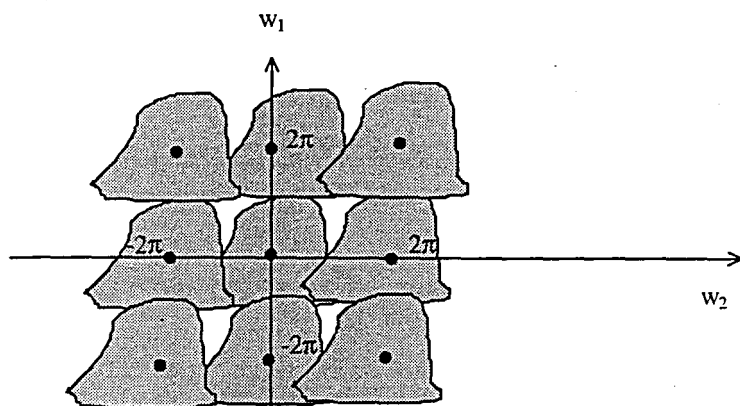
and with $\mathbf{U} \equiv 2\pi\mathbf{D}^{-T}$, the equation becomes

$$X(\mathbf{D}^T \Omega) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \mathbb{N}} X_a(j(\Omega - \mathbf{U}\mathbf{k})) \quad (2.18)$$

Considering a two dimensional signal bandlimited to the gray region shown in Figure 2.7 the infinite number of copies obtained by shifting its origin to the points on LAT($\mathbf{U}$) are shown in Figure 2.7.



(Continued)



(c) Support of $X(\mathbf{w})$

FIGURE 2.7 Effect of sampling of 2-dimensional signal in the frequency domain[32]

The shifted versions of $X_a(j\Omega)$ mentioned above are called **alias components**. If any of these overlaps with the unshifted version, we cannot recover $X_a(j\Omega)$ from $X(\mathbf{w})$ (similar to the aliasing effect in 1-dimensional) .

2.3. Multidimensional Multirate Filter Banks

The analysis section of a filter bank decomposes a signal into a set of subband components and the synthesis section reconstructs the input signal from the components. A fundamental difference between 1-dimensional and multidimensional filter banks is the manner in which the subband components are formed, i.e., the idea of lattices which is closely related to the concepts of multidimensional generalized sampling.

Multidimensional sampling is represented by a lattice which can be separable (i.e. diagonal) or nonseparable. While in 1-dimension, downsampling a signal by K is done by simply retaining every K^{th} sample, in two and higher dimensions the downsampling is represented by a nonsingular matrix $\mathbf{D}$ with integer entries as seen in previous sections.

The technique of subband coding is illustrated in Figure 2.8 where a general multidimensional M -channel system is shown. The input signal is passed through a bank of M bandpass filters which can be separable (i.e. tensor product of 1-D filters) or nonseparable, the latter regardless of the separability/nonseparability of the sampling lattice. Each resulting component is subsampled by $M = \det \mathbf{D}$ to its new multidimensional Nyquist frequency thus yielding the subband signals. These subband signals are encoded, transmitted and at the receiver side decoded. When reconstructing the original signal at the receiver side, all the subbands are upsampled to the sampling rate of the input and passed through the bandpass synthesis (separable/nonseparable) filters. When the sampling density and the number of channels are equal in order to preserve the same number of samples through the various stages of the system, this is called a critically sampled filter bank.

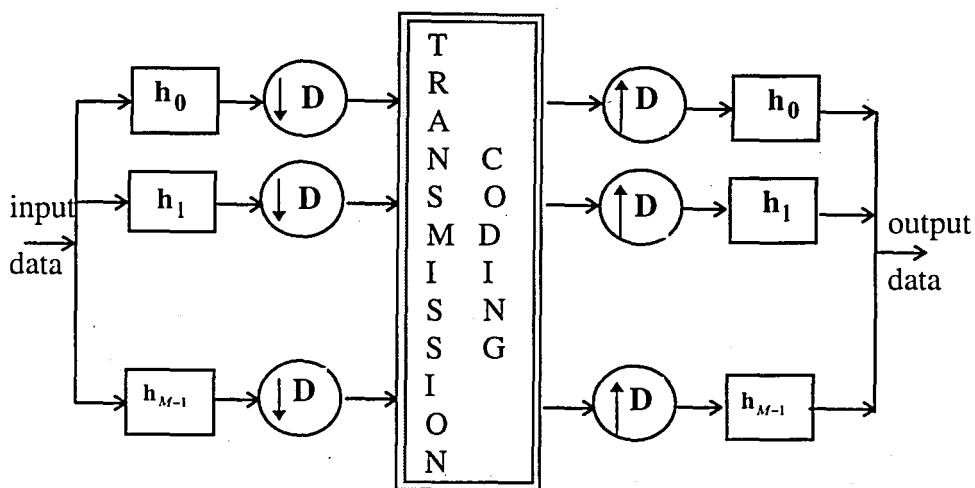


FIGURE 2.8. Analysis/synthesis p -dimensional filter bank

2.3.1. Multidimensional Filter Bank Structures

2.3.1.1. One-Dimensional Filter Banks

In the case of a two ($M=2$) channel filter bank, typically the input signal is split into a lowpass and a highpass subband. The analysis/synthesis parts are given accordingly in Figure 2.9. An easy way to construct multichannel filter banks is to cascade two-channel banks appropriately. One case can be seen in Figure 2.10.a where frequency analysis is obtained by simply iterating a two-channel division on the previous lowpass channel. This is called an octave-band filter bank since each successive highpass output contains an octave of the input bandwidth. The case when equal bandwidth channels are desired appears as in Figure 2.10.b. Another possibility is building an arbitrary tree structured filter bank, giving rise to wavelet packets as shown in Figure 2.10c.

appears as in Figure 2.10.b. Another possibility is building an arbitrary tree structured filter bank, giving rise to wavelet packets as shown in Figure 2.10c.

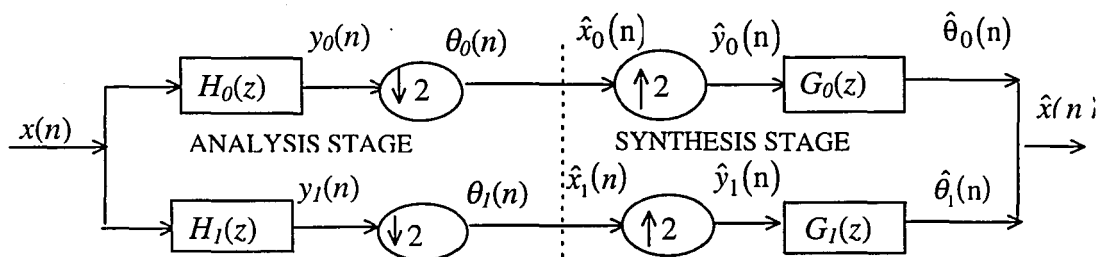
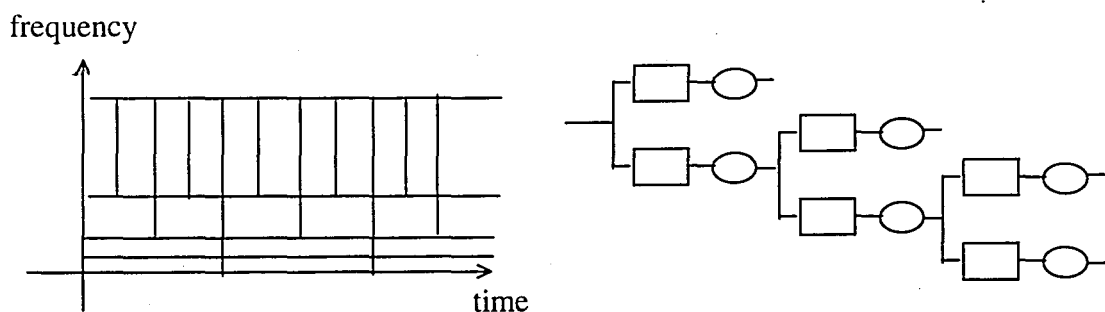
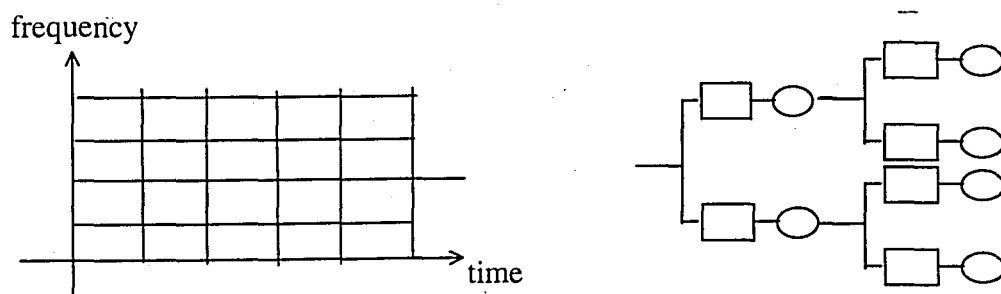


FIGURE 2.9. One dimensional two channel filter bank

It is interesting to consider the time-frequency analysis performed by various filter banks. This is shown schematically in Figure 2.10. for three particular cases of binary trees where one can note the tradeoffs in time and frequency resolutions.



(a) Octave band tree or wavelet series.



(b) Full tree Continued...

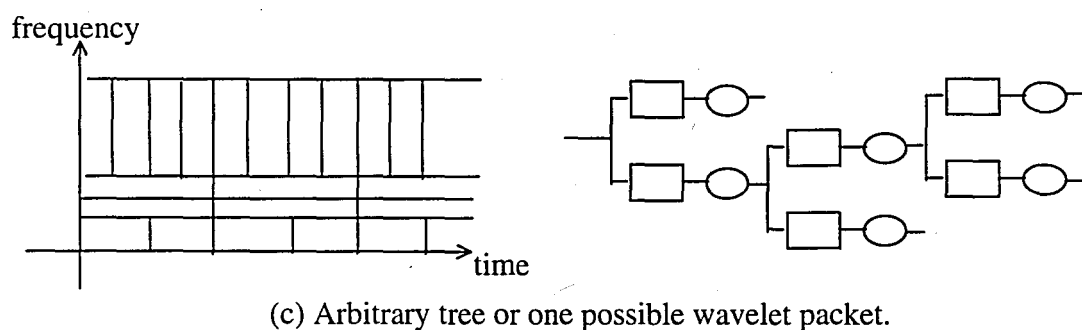


FIGURE 2.10. Time-frequency analysis achieved by different binary subband trees

Time and frequency resolutions can be modified as time evolves. This is called a dynamic time-frequency analysis and is shown in Figure 2.11.[44].

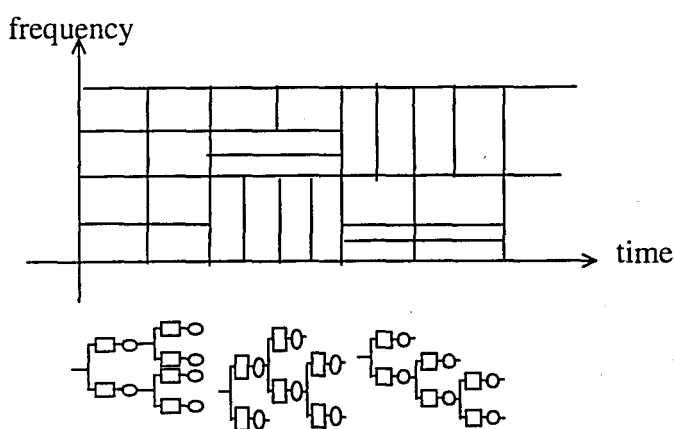


FIGURE 2.11. Dynamic time-frequency analysis achieved by concatenating the analyses from Figure 2.10.

This dynamic tree structure is used in compression applications. Starting with a collection of bases given by all binary subband coding trees of a given depth is called wavelet packets and from a full tree, the best basis algorithm uses dynamic programming to prune back to the best tree or best basis [39, 54].

2.3.1.2. Two-Dimensional Filter Banks

For 2-dimensional signals, e.g. images, there are two different tree structures that can be discerned, a separable and a non-separable tree. In the separable tree structure, first all the rows are split into subbands using a certain one-dimensional tree, after which all the columns are split into subbands, again using a one-dimensional tree. Of course, the order in which the rows and columns are split can be interchanged.

The non-separable splitting tree is constructed using building blocks that split an image into 4 subbands. In Figure 2.12. both types of splitting trees are illustrated by splitting the image into 9 subbands using the separable tree and 10 subbands using the nonseparable tree.

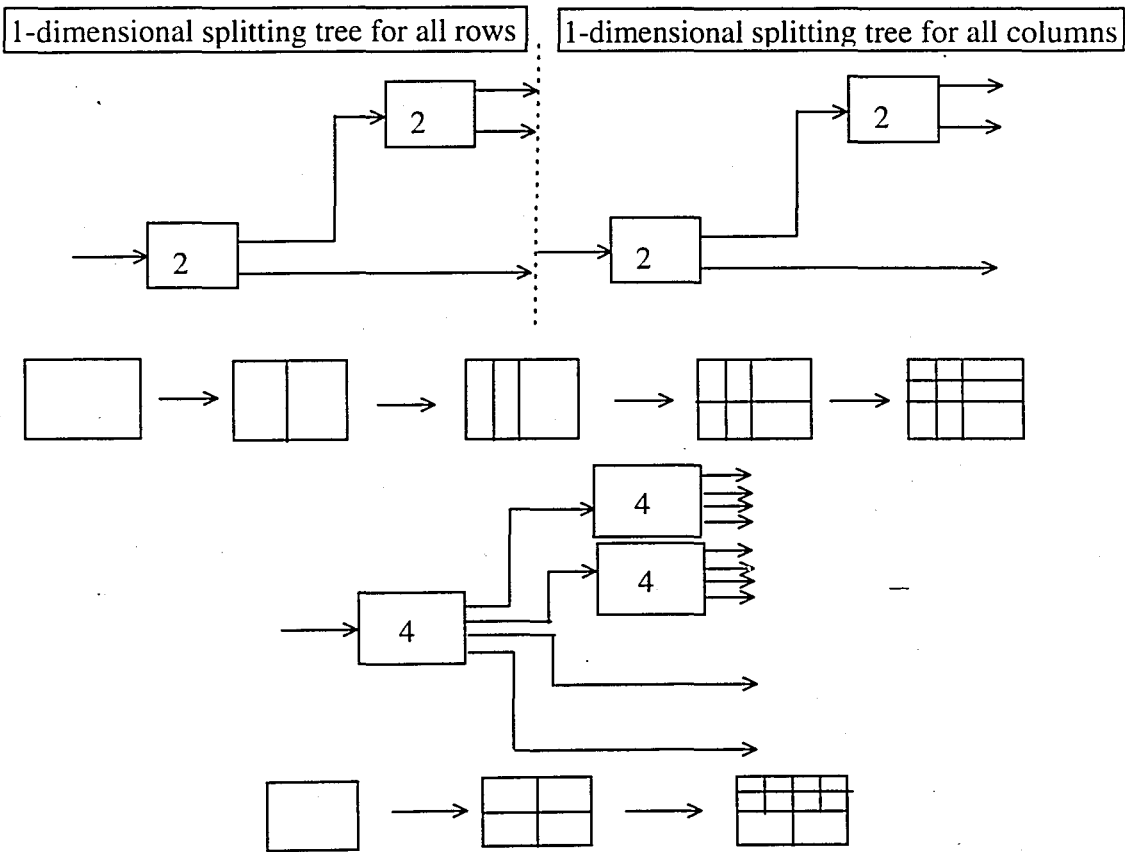


FIGURE 2.12. Tree structures to split an image into subbands : separable tree and nonseparable tree("2" means splitting into two one-dimensional subbands and "4" means splitting into four two-dimensional subbands).

The image frequency band division at each stage is shown at the bottom of each tree.

By using a separable decomposition into four bands and iterating it once, 16 subbands can be obtained as shown in Figure 2.13.a. Instead of uniformly decomposing the spectrum of the image, iteration of a separable four-band decomposition three times is shown in Figure 2.13.b.

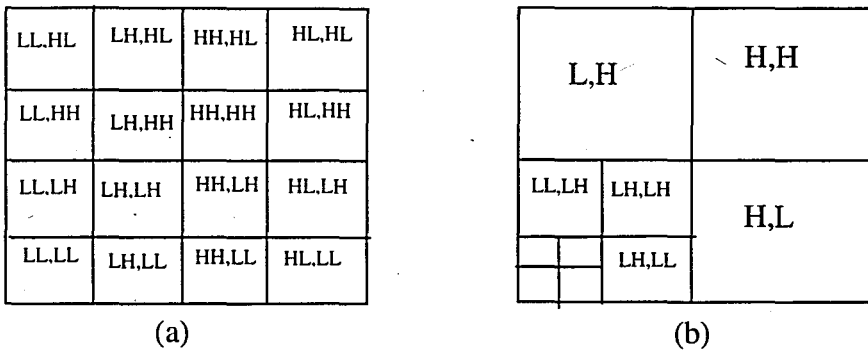


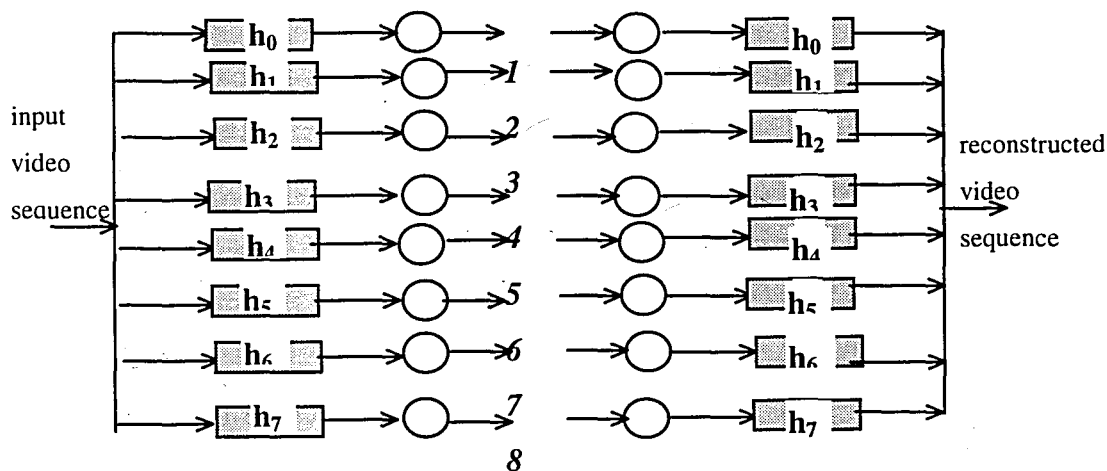
FIGURE 2.13. The spectral decomposition of the channels shown with the first two letters correspond to horizontal filtering and the last two to vertical filtering a) Uniform subband decomposition of an image into 16 subbands, b) Octave-band or wavelet decomposition of an image into unequal subbands

Starting with a collection of bases given by all binary subband coding trees of a given depth and using again dynamic programming to prune back to the best tree or best basis can also be used for 2-dimensional applications.

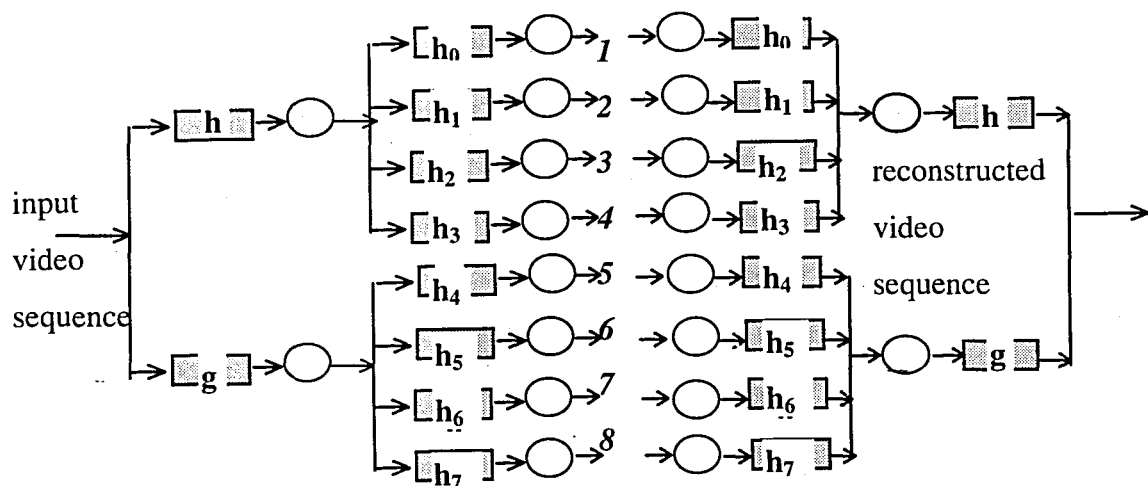
2.3.1.3 Three -Dimensional Filter Banks

Three dimensional signals are decomposed into filtered and subsampled signals in the 3-dimensional subband coding. The first component in subband decomposition shown in Figure 2.14.a corresponds to lowpass component in both spatial and time dimension.

Similarly, other branches indicates highpass components in different directions in spatial dimension or time dimension. Alternatively, subband decomposition can be iterated along the low-low branch.



(a) Three dimensional nonseparable filter bank structure



(b) One dimensional filtering in the time and two dimensional nonseparable filtering in the spatial coordinates

FIGURE 2.14. 3-D nonseparable filter structures for 8-band cubic sampling

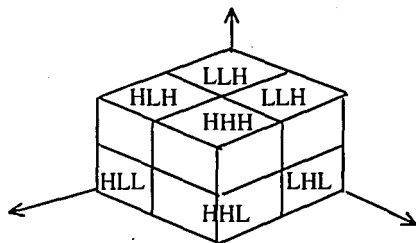


FIGURE 2.15. 3-D data decomposed into 8 subbands using three dimensional FBs

As illustrated in Figure 2.15. , the 3-dimensional data is decomposed into eight subbands by using three-dimensional filter banks. The highpass subbands have seven orientations denoted by LHL, LLH, LHH, HLL, HHL, HLH and HHH (L indicates the lowpass direction and H indicates the highpass one)

2.3.2 Multirate Fundamentals

2.3.2.1. Basic Multirate Operations

The most basic operators in multirate digital signal processing are decimators (called also as downsampler, subsampler, compressor) and expanders (called also upsampler).

In the 1-dimensional case, only those samples of $x(n)$ which occur at time equal to multiples of K are retained by the decimator, i.e.,

$$y_{Dec}(n) = x(Kn) \quad (2.19)$$

where K is an integer.

The second operator is that of expander which for an input $x(n)$ produces an output sequence as

$$y_E(n) = \begin{cases} x(n/L) & \text{if } n \text{ is an integer multiple of } L \\ 0 & \text{otherwise} \end{cases} \quad (2.20)$$

Transform domain relations for decimators and expanders can be given respectively, as,

$$Y_E(z) = \sum_{n=-\infty}^{\infty} y_E(n) z^{-n} = \sum_{n=\text{multiple of } L} y_E(n) z^{-n} = \sum_k y_E(kL) z^{-kL} = X(z^L) \quad (2.21)$$

$$Y_E(\exp(jw)) = X(\exp(jwL)) \quad (2.22)$$

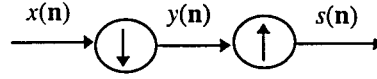
The multiple copies of the compressed spectrum are called images so that expander creates an imaging effect.

$$Y_{Dec}(\exp(jw)) = \frac{1}{K} \sum_{k=0}^{K-1} X(\exp(j(w - 2\pi k) / K)) \quad (2.23)$$

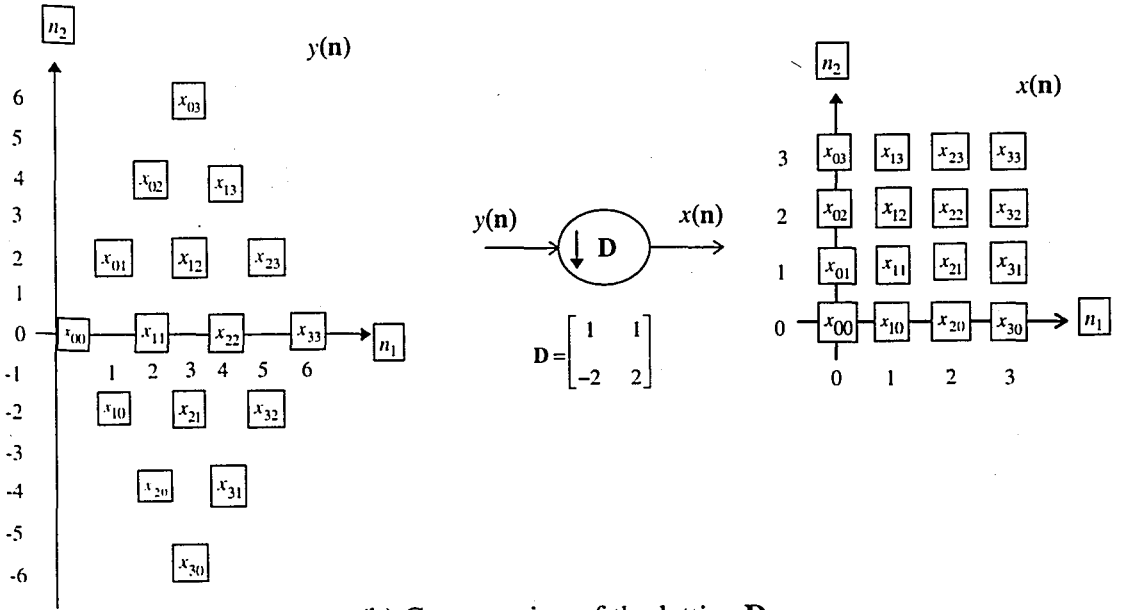
$$Y_{Dec}(z) = \frac{1}{K} \sum_{k=0}^{K-1} X(z^{1/K} W_K^k) \quad \text{with} \quad W_K = \exp(-j2\pi / K) \quad (2.24)$$

The stretched versions $X(\exp(jw/K))$ can in general overlap with its shifted replicas resulting in aliasing. It is clear that aliasing can be avoided if $x(n)$ is a lowpass signal bandlimited to the region $|w| < \pi / K$. The decimator and expander are linear but time varying systems.

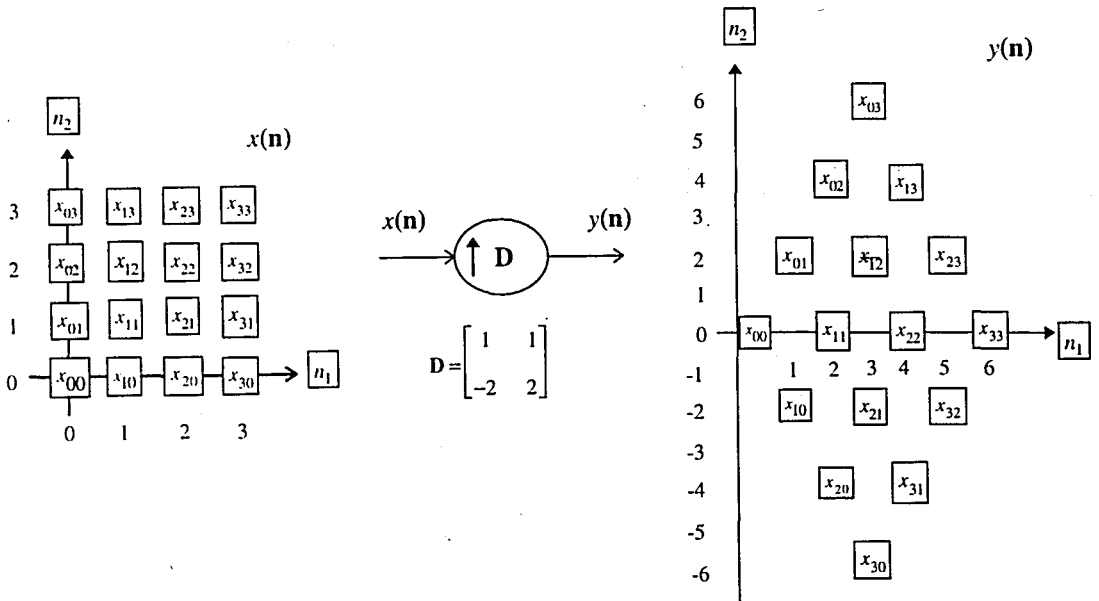
To examine the operations involving sampling used in 2-D filter banks, let us consider time, Fourier and Z-domain expressions for outputs of a downsampler and then a subsampler with $\mathbf{D}$ sampling lattice as given in Figure 2.16a.



(a) Upsampling and downsampling for any sampling lattice $\mathbf{D}$,



(b) Compression of the lattice $\mathbf{D}$,



(a) Expansion of the lattice by $\mathbf{D}$,

FIGURE 2.16. Expander/decimator process implemented for hexagonal sampling lattice example

Given the p -dimensional sequence $x(\mathbf{n})$, the $\mathbf{D}$ -fold decimated version is

$$y_{Dec}(\mathbf{n}) = x(\mathbf{Dn}) \quad (2.25)$$

the decimated version $y(\mathbf{n})$ contains only those samples of $x(\mathbf{n})$ which fall at the points belonging to $\text{LAT}(\mathbf{D})$. This is demonstrated for some examples of $\mathbf{D}$ in Figure 2.17 here the black dots represent the lattice points, white circles integers which are sample points discarded by the decimator, black dots and white circles falling inside $\text{FPD}(\mathbf{D})$ are given by surrounding outer circles.

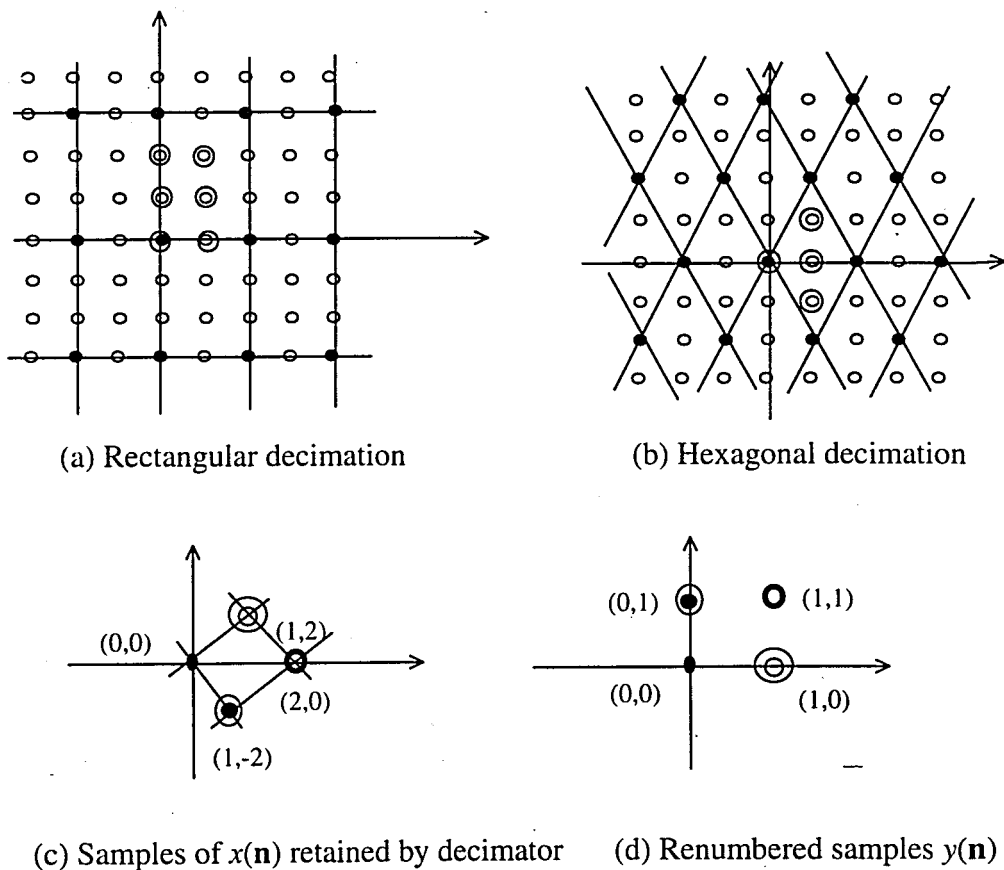


FIGURE 2.17. Two dimensional decimation for a)rectangular and b)hexagonal sampling lattices , c) and d) how hexagonal decimator renumbers the samples

A multidimensional expander is defined similarly,

$$y_E(\mathbf{n}) = \begin{cases} x(\mathbf{D}^{-1}\mathbf{n}) & \mathbf{n} \in \text{LAT}(\mathbf{D}) \\ 0 & \text{otherwise} \end{cases} \quad (2.26)$$

The expanded output is zero unless $\mathbf{n}$ belongs to this lattice. The output of the $\mathbf{D}$ -fold expander is zero at the points indicated by white circles and nonzero at other places. As transform domain expressions of the expander, we have its Fourier transform given by,

$$Y(\mathbf{w}) = X(\mathbf{D}^T \mathbf{w}) \quad (2.27)$$

and its Z-transform as,

$$Y(\mathbf{z}) = X(\mathbf{z}^{\mathbf{D}}) \quad (2.28)$$

When coming to the decimator, we have for its Fourier transform,

$$Y(\mathbf{w}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} X(\mathbf{D}^{-T}(\mathbf{w} - 2\pi\mathbf{k})) \quad (2.29)$$

and its Z transform

$$Y(\mathbf{z}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} X(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}^{\mathbf{D}^{-1}}) \quad (2.30)$$

with $\mathbf{W}_{\mathbf{D}}(\mathbf{w}) = (\exp(-j\langle w_1, d_1 \rangle), \dots, \exp(-j\langle w_n, d_n \rangle))$ and $\mathbf{n} \circ \mathbf{k} = (n_1 k_1, \dots, n_n k_n)$.

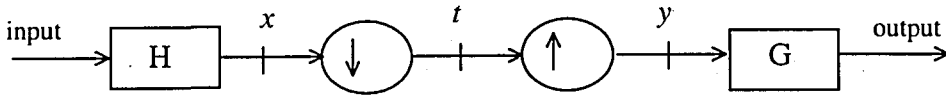


FIGURE 2.18. One branch of filter bank structure

Combining the Equations 2.27, 2.28, 2.29 and 2.30, expression for the output of a downsampler followed by an upsampler as shown in Figure 2.18 (replacing by zeros all samples that are not on the sublattice)

$$t(\mathbf{n}) = x(\mathbf{D}\mathbf{n}) \quad (2.31a)$$

$$y(\mathbf{n}) = \begin{cases} t(\mathbf{n}) & \mathbf{n} \in \text{LAT}(\mathbf{D}) \\ 0 & \text{otherwise} \end{cases} \quad (2.31b)$$

$$T(\mathbf{w}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} X(\mathbf{D}^{-T}(\mathbf{w} - 2\pi\mathbf{k})) \quad (2.32a)$$

$$Y(\mathbf{w}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} T(\mathbf{w} - 2\pi\mathbf{D}^{-T}\mathbf{k}) \quad (2.32b)$$

$$T(\mathbf{z}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} X(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}^{\mathbf{D}^{-1}}) \quad (2.33a)$$

$$Y(\mathbf{z}) = \frac{1}{\det \mathbf{D}} \sum_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} T(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}) \quad (2.33b)$$

so that in Equation 2.32, 2.33 the signals x and t and their modulated version ("m" will denote modulated) are given by

$$T_m(\mathbf{w}) = \left\{ T(\mathbf{w} - 2\pi\mathbf{D}^{-T}\mathbf{k}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad (2.34a)$$

$$T_m(\mathbf{z}) = \left\{ T(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad (2.34b)$$

$$X_m(\mathbf{w}) = \left\{ X(\mathbf{w} - 2\pi\mathbf{D}^{-T}\mathbf{k}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad (2.34c)$$

$$X_m(\mathbf{z}) = \left\{ X(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad (2.34d)$$

Thus the output of the system given by Equation 2.32 and 2.33 after upsampling and filtering in the synthesis bank, can be written as,

$$Y(\mathbf{w}) = \frac{1}{\det(\mathbf{D})} (G_0(\mathbf{w}) \dots G_{\det \mathbf{D}-1}(\mathbf{w})) \cdot H_m(\mathbf{w}) \cdot x_m(\mathbf{w}) \quad (2.36a)$$

$$Y(\mathbf{z}) = \frac{1}{\det(\mathbf{D})} (G_0(\mathbf{z}) \dots G_{\det \mathbf{D}-1}(\mathbf{z})) \cdot H_m(\mathbf{z}) \cdot x_m(\mathbf{z}) \quad (2.36b)$$

with

$$H_m(\mathbf{w}) = \left\{ H_i(\mathbf{w} - 2\pi \mathbf{D}^{-T} \mathbf{k}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad i \in \{0, \dots, \det \mathbf{D} - 1\} \quad (2.37a)$$

$$H_m(\mathbf{z}) = \left\{ H_i(\mathbf{W}_{\mathbf{D}^{-1}}(2\pi \mathbf{k}) \circ \mathbf{z}) \right\}_{\mathbf{k} \in \text{FPD}(\mathbf{D}^T)} \quad i \in \{0, \dots, \det \mathbf{D} - 1\} \quad (2.37b)$$

These expressions will be used in the next section for deriving equivalent time and frequency domain orthogonality equations in designing PRFBs.

2.3.2.2. Some Multirate Identities

A pair of useful identities gives conditions when shift-invariant filters can be passed across sampling rate changes. Any filter in the subsampled domain can be represented in the upsampled domain by simply upsampling its impulse response. Similarly, a filter with the Z-transform $H(\mathbf{z})$ placed in front of upsampling by $\mathbf{D}$, can be moved past the sampling rate change and represented as $H(\mathbf{z}^{\mathbf{D}})$. Similarly, a filter with the Z-transform $H(\mathbf{z})$ placed in front of upsampling by $\mathbf{D}$, can be moved past the sampling rate change and represented as $H(\mathbf{z}^{\mathbf{D}})$.

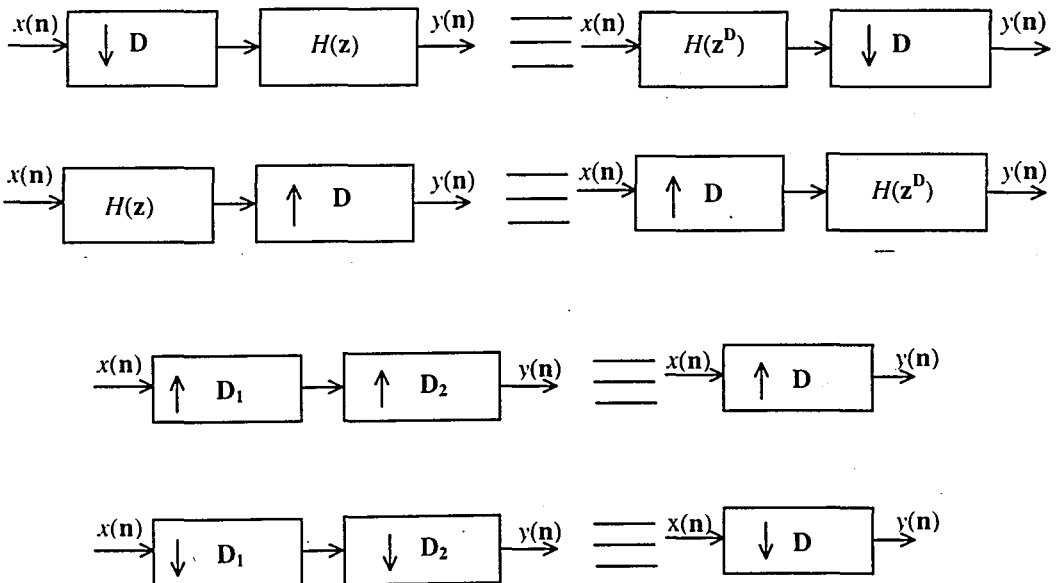


FIGURE 2.19. Useful multidimensional multirate identities

Applying these multirate identities to the tree structure leads to an equivalent parallel structure subband decomposition. Each branch in the parallel structure subband decomposition, gives us the relation between the original signal and the coefficients of the wavelet decomposition at a given level. For example, using the fact that sampling by $\mathbf{D}$ followed by filtering by $H(\mathbf{w})$ is equivalent to filtering $H(\mathbf{D}^T \mathbf{w})$ followed by sampling by $\mathbf{D}$ as shown in Figure 2.19. The last branch of the tree shows that the low resolution version of the original signal, is obtained by a filtering operation with $H^{(i)}(\mathbf{z})$ followed by a decimation by $\mathbf{D}^i$, where the filter $H^{(i)}(\mathbf{z})$ is given by[36]:

$$H^{(i)}(\mathbf{w}) = \prod_{l=0}^{i-1} H_0\left((\mathbf{D}^T)^l \mathbf{w}\right) \quad i=1,2,\dots \quad (2.38)$$

where $H_0(\mathbf{z})$ is the Z-domain expression of the lowpass filter.

Let $H^{(\infty)}(\mathbf{z})$ be the limit of the filter $H^{(i)}(\mathbf{z})$, as $i \rightarrow +\infty$. The iterated two-band filter bank on the lowpass branch is shown in Figure 2.20.

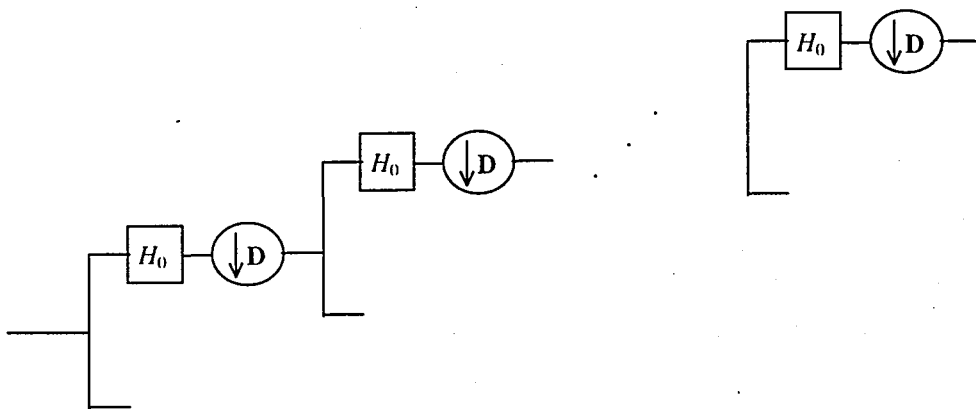


FIGURE 2.20. Iterated two-band filter bank on the lowpass branch

The aim is to construct a continuous-time function corresponding to $h^{(i)}(\mathbf{n})$, the latter being the impulse response of the iterated filter $H^{(i)}(\mathbf{w})$. Consequently, an iterated function defined as,

$$f^{(i)}(\mathbf{t}) = (\det \mathbf{D})^{i/2} \cdot h^{(i)}(\mathbf{n}) \quad (2.39)$$

is constant over regions of hypervolume $1/\det \mathbf{D}^i$ and normalization by $\det \mathbf{D}^{i/2}$ is chosen so as to make the norms of $h^{(i)}(\mathbf{n})$ and $f^{(i)}(\mathbf{t})$ equal to 1.

The equivalent filter after i steps in terms of the equivalent filter after $(i-1)$ steps as

$$h^{(i)}(\mathbf{n}) = \sum_{\mathbf{k}} h_0(\mathbf{k}) h^{(i-1)}(\mathbf{n} - \mathbf{D}^{i-1} \mathbf{k}) \quad (2.40)$$

Using Equation 2.39, the Equation 2.40 can be expressed in terms of iterated functions as

$$h^{(i)}(\mathbf{n}) = (\det \mathbf{D})^{-i/2} \cdot f^{(i)}(\mathbf{t}) \quad (2.41a)$$

$$h^{(i-1)}(\mathbf{n} - \mathbf{D}^{i-1} \mathbf{k}) = (\det \mathbf{D})^{-i/2} \cdot f^{(i-1)}(\mathbf{D} \mathbf{t} - \mathbf{k}) \quad (2.41b)$$

Substituting Equations and into yields

$$f^{(i)}(\mathbf{t}) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_0(\mathbf{k}) \phi(\mathbf{D} \mathbf{t} - \mathbf{k}) \quad (2.42)$$

Similarly for other branches the iterated filter expressions are given as

$$f_j^{(i)}(\mathbf{t}) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_j(\mathbf{k}) \phi(\mathbf{D} \mathbf{t} - \mathbf{k}) \text{ for } j = 1, 2, \dots, \det \mathbf{D} - 1 \quad (2.43)$$

The expression in Equations 2.42 and 2.43 will be used both in regularity and derivation of scaling function when constructing wavelet bases in Section 2.3.3. and Section 2.3.5.

2.3.3. Multidimensional Digital Filters

A multidimensional digital filter is an LTI system characterized by a transfer function

$$H(\mathbf{z}) = \sum_{\mathbf{n}} h(\mathbf{n}) \mathbf{z}^{-\mathbf{n}} \quad (2.44)$$

where $h(\mathbf{n})$ is the impulse response of the filter. The frequency response $H(\mathbf{w})$ of the filter is given by,

$$H(\mathbf{w}) = \sum_{\mathbf{n} \in N} h(\mathbf{n}) \exp(-j \mathbf{w}^T \mathbf{n}) \quad (2.45)$$

which is obtained by evaluating Equation 2.44 at $z_k = \exp(jw_k)$, $0 \leq k \leq p-1$. The input/output relation of the system is given by the convolution, $y(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbf{N}} x(\mathbf{k})h(\mathbf{n}-\mathbf{k})$ or

in terms of Z-transform $Y(\mathbf{z}) = H(\mathbf{z})X(\mathbf{z})$.

In the sequel, properties of multidimensional filters germane to filter bank design are detailed.

Separability

A multidimensional (p -dimensional) filter is said to be separable if it satisfies the following relationship, respectively; in time, Z and Fourier domains:

$$h(\mathbf{n}) = h_1(n_1) \circ \dots \circ h_p(n_p) \quad (2.46 \text{ a})$$

$$H(\mathbf{z}) = H_1(z_1) \dots H_p(z_p) \quad (2.46 \text{ b})$$

$$H(\mathbf{w}) = H_1(\exp(jw_0)) \dots H_p(\exp(jw_p)) \quad (2.46 \text{ c})$$

In other words, the transfer function is a product of p one dimensional transfer functions. Separability implies, in particular, that the plot of $H(\mathbf{w})$ with respect to any one frequency variable is unchanged (except for a scale factor) if the remaining frequency variables are kept fixed.

Most of the previously reported work on subband coding of images is based on separable filtering. Several authors have used the 1-dimensional wavelets in the analysis and compression of images by processing the rows and columns of an image separately. When multiresolutional analysis is extended to 2-dimensions in this manner, a bias is introduced for the horizontal and vertical directions. Design of nonseparable 2-D filter has the following advantages:

- Directional properties other than horizontal and vertical can be obtained as pointed out by Woods and O'Neal [14].

- Nonseparable multiresolution filtering allows secondly nonrectangular subsampling patterns offering a way to possibly obtain better performance.
- Nonseparable filters have more degrees of freedom and hence better designs of appropriate optimization schemes are used. The nonseparable filters can have better frequency characteristics and more design freedom: in fact for an $N \times N$ filter, while nonseparable case can have up to N^2 free variables, the separable counterpart has at most $2N$ free variables. Similarly, for an $N \times N \times N$ filter, while nonseparable case can have up to N^3 free variables the separable counterpart has at most $3N$ free variables. The disadvantage of the increased degrees of freedom is that it is computationally more complex than its separable counterpart. This drawback is overcome by using a design method which will be proposed in this thesis, where only the coefficients of the lowpass filter are calculated and the coefficients for the remaining $M-1$ filters are obtained through the use of signed shuffling operations.
- Nonseparable filters can have linear phase, while, when 2-dimensional filter is obtained with a tensor product of two 1-dimensional filters, linear phase is not achievable. So, 4-band nonseparable orthogonal PR filters can have linear phase while 4-band separable PR filters in 2-dimensions (tensor product of two 1-dimensional filters) cannot have linear phase. Similarly, 8-band nonseparable orthogonal PR filters in 3 dimensions can have linear phase while 8-band separable PR filters (tensor product of two 1-D filters) cannot have linear phase.
- Nonseparable filters offer the possibility of having true multidimensional processing. Thus, they can exploit image properties that cannot be so easily exploited by separable ones.

Linear Phase

A multidimensional filter is said to have linear phase if

$$H(\mathbf{w}) = c \exp(-j\mathbf{k}^T \mathbf{w}) H_R(\mathbf{w}) \quad (2.47a)$$

where $\mathbf{k}$ is a real constant vector, and c is a (possibly complex) constant and $H_R(\mathbf{w})$ is an p -variable real function. Linear phase systems do not introduce phase distortion and therefore phase distortion and therefore linear phase is important in image processing unlike in speech applications where some amount of phase nonlinearity is acceptable. The human eye is very sensitive to the phase information in images. If the filters have nonlinear phase then the resulting distortion, e.g. contour deterioration, in low bit rate could be larger as compared to a linear phase situation.

In 1-D linear phase implies symmetry of the impulse response. In the 2-D and higher dimensional cases the impulse response array must be centrosymmetric. The number of free variables, for the $N \times N$ sized centrosymmetric filters, is $N^2/2$. The number of free variables, for $N \times N \times N$ sized centrosymmetric filters, is $N^3/4$. Notice that while centrosymmetry is the symmetry condition allowing for the largest number of free variables, there are other types of symmetries that reduces the number of variables to a smallest number (See Chapter 4).

Thus, for real coefficient cases, $\mathbf{D}$ being the dilation matrix,

$$h(n_1 \dots n_p) = \pm h(\det(\mathbf{D}) - 1 - n_1, \dots, \det(\mathbf{D}) - 1 - n_p) \quad (2.48a)$$

$$H(z_1 \dots z_p) = \pm z_1^{-(\det(\mathbf{D})-1)} \dots z_p^{-(\det(\mathbf{D})-1)} H(z_1^{-1} \dots z_p^{-1}) \quad (2.48b)$$

A zero phase filter, on the other hand, is for real coefficient case, and thus:

$$h(\mathbf{n}) = h(-\mathbf{n}) \quad (2.49)$$

Regularity

Several researchers have examined the problem of designing wavelets with regularity conditions. Compactly supported orthogonal one-dimensional wavelets with high regularity have been constructed by Daubechies [56]. These solutions are related to conjugate-quadrature filter solutions in a two-channel exact reconstruction filter bank described by Smith and Barnwell [27]. However, the two families of filters have opposite features. Daubechies filters are maximally regular but are not selective, whereas conjugate quadrature (CQ) filters are very selective but not regular. A large family of orthogonal wavelet filters with balanced regularity, selectivity and phase is designed in [57]. It is observed in [57] that the constraint of regularity is in contradiction with the requirement of good selectivity, and that trading regularity for selectivity affects the compression performances. The main novelty of wavelets compared to traditional subband coding is the additional requirement of regularity on the filters [58]. The regularity of the mother wavelet is important and is closely related with the processed signal [59].

Current trend in wavelet design tend towards maximum regularity, which attempts to place the maximum number of Z-plane zeros of the finite impulse response (FIR) filter at $z = -1$. This can be shown to be, in the limit, for 1-dimension a box-car filter which has a very sharp cut-off producing large amounts of ringing at higher compression ratios. The convergence can be guaranteed if the lowpass filter $H_0(z)$ has a root at $z = -1$. This condition is not very restrictive, since the majority of lowpass filters in filter banks have a zero response at half of the Nyquist frequency. For instance, if the lowpass filter is even and symmetrical, this condition is fulfilled. So that an orthogonal filter with a certain number of zeros at the aliasing frequency π is called regular if its iteration tends to a continuous function. The importance of this property for coding is potentially twofold when the decomposition is iterated. First, the presence of many zeroes at the aliasing frequency can improve the coding gain and second, compression artifacts might be less objectionable. To investigate the first effect, Rioul [57] compared the compression gain for filters of varying regularity used in a wavelet coder. As results he found that some regularity is desired and higher regularity improves compression further. As for the compression artifacts, let assume that a single quantization error in the lowpass channel.

This will add an error to the reconstructed signal depending only on the iterated lowpass filter. If the iterated filter is smooth, this will be less noticeable than if it is a highly irregular function.

In Daubechies construction the key concept is that of an iterated filter bank. The equivalent low branch after i steps of filtering and sampling by 2 will contain the following filter and sampling by 2^i using the fact that sampling by 2 followed by filtering by $H_0(z)$ is equivalent to filtering by $H_0(z^2)$ followed by 2

$$H^{(i)}(z) = \prod_{k=0}^{i-1} H_0(z^{2^k}) \quad (2.50)$$

Daubechies then constructs a piecewise continuous function from this discrete iterated filter as follows

$$f^{(i)}(t) = 2^{\frac{i}{2}} \cdot h^{(i)}(n) \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i} \quad (2.51)$$

The function is constant over intervals of length $\frac{1}{2^i}$, and normalization by $2^{\frac{i}{2}}$ is chosen, so that to make the norms equal to 1. The question now is under which conditions this function has a limit in L^2 , and when it does, whether this limit is continuous or not. Thus a filter having the above properties is termed regular. Daubechies in [27] gives a sufficient condition for regularity; if the lowpass filter has a certain amount of zeroes at the aliasing frequency π then the function in will converge to a continuous function $\phi(t)$, thus this construction would lead to the scaling function $\phi(t)$.

Results show that as the degree of regularity is increased better results are obtained in terms of human visual system requirements [60]. A certain degree of regularity is necessary to avoid fractal like behavior of the coefficients, because they may cause annoying artifacts and their entropy will increase. In multiple dimensions the basic ideas for constructing wavelet bases from iterated filter banks are the same, the only difference being that instead of dealing with dilation factors we deal with a dilation matrix. The equivalent low branch after i steps of filtering and sampling by $\mathbf{D}$ will contain the following filter and sampling by $\mathbf{D}^i$ as in Equation 2.42 in Section 2.3.2.2.

$$H^{(i)}(\mathbf{w}) = \prod_{k=0}^{i-1} H_0\left((\mathbf{D}^T)^i \mathbf{w}\right)$$

To construct a continuous function from this discrete iterated filter, Equation 2.39 was used in [36],

$$f^{(i)}(\mathbf{t}) = (\det \mathbf{D})^{i/2} \cdot h^{(i)}(\mathbf{n})$$

Kovacevic showed that the iterated filter can be made regular (i.e. the iterated function continuous) by putting a zero of a sufficiently high order at the points in frequency domain where the repeated spectra occur, that is, at the points belonging to the reciprocal lattice of $\mathbf{D}$.

Another important criteria is the number Y of zero moments of the wavelet ψ , i.e. its degree of oscillation. This corresponds to

$$\int_{\mathfrak{R}} x^n \psi(x) dx = 0 \quad \forall n = 0, 1, \dots, Y-1 \quad (2.52)$$

As ψ is a wavelet, Y is greater or equal to 1 and $\int_{-\infty}^{\infty} \psi(x) dx = 0$ is satisfied. In fact, this number of zero moments is related to the regularity of the wavelet. Practically, a wavelet which possess a number Y of zero moments imply to annulate all wavelet coefficients of polynomial signal of degree less than Y . Thus if f is a polynomial signal of degree less than N over the support $\psi_{m,n}$,

$$c_{m,n} = \langle f, \psi_{m,n} \rangle = 0 \quad (2.53)$$

This result is very important in image coding since it offers the possibility of obtaining high compression rates (great number of wavelet coefficients are zero or negligible).

In the thesis, zeroes are imposed such that frequency response of lowpass filter vanishes at the Nyquist frequencies by incorporating if possible the additional constraint equations formed on the coefficients of the filters in the optimization procedure.

2.3.4 Perfect Reconstruction Conditions

Perfect reconstruction filter banks (PRFBs) have been a popular multiresolution signal decomposition technique [38, 61-66]. The two principal reasons for their popularity are that the PRFB is shown to be strongly related with the wavelet transform (this relationship will be explained in Section 2.3.5.) and the orthogonality property of PR enables the use of a measure of error in coding applications[67], giving the possibility of controlling the error when lossy compression is desired.

In this section we derive the requirements and properties of perfect reconstruction in subband systems. Since perfect reconstruction is desired, only FIR filters are used in the subband system. The perfect reconstruction property is derived for 1-dimensional two-channel PR-QMF banks and then extended to higher dimensions.

One Dimensional Two-Channel PR-QMF Bank

The two-channel $M = 2$ filter bank provides the starting point for the study of subband coding systems. The two-channel filter bank is shown in the Figure 2.9. where $H_0(z)$ and $H_1(z)$ are analysis filters splitting the spectrum into two equal bands. Then according to the Nyquist theorem $m_0(n)$ and $m_1(n)$ are each downsampled by 2 to provide the subband signals $v_0(n)$ and $v_1(n)$ at the outputs of the analysis stage. The received $v_0(n)$ and $v_1(n)$ signals are each up-sampled by 2 to give the zero-interlaced signals $y_0(n)$ and $y_1(n)$ which are in turn processed by the interpolation filters $G_0(z)$ and $G_1(z)$ in the synthesis stage to yield the reconstructed signal.

Tracing the signals through the first branch in Figure 2.9 gives us as the outputs of the decimation and interpolation filters,

$$Y_0(z) = H_0(z)X(z) \quad (2.54)$$

$$\hat{\theta}_0(z) = G_0(z)\hat{Y}_0(z) \quad (2.55)$$

In frequency domain, $X(w)$ being the Fourier transform of the input signal $x(n)$ and $H_0(w)$, of the lowpass filter, the resulting signals in the frequency domain are, with the outputs of the decimation and interpolation filters, while the down sampler and the upsampler impose respectively;

$$X_0(z) = \frac{1}{2} [Y_0(z^{1/2}) + Y_0(-z^{1/2})] \quad (2.56)$$

$$X_0(w) = \frac{1}{2} \left[H_0\left(\frac{w}{2}\right) X\left(\frac{w}{2}\right) + H_0\left(\frac{w}{2} + \pi\right) X\left(\frac{w}{2} + \pi\right) \right]$$

$$\hat{Y}_0(z) = X_0(z^2) \quad (2.57)$$

$$\hat{Y}_0(w) = \hat{X}_0(2w)$$

Combining Equations 2.54, 2.55, 2.56 and 2.57 gives

$$\hat{\theta}_0(z) = \frac{1}{2} G_0(z) [H_0(z) X(z) + H_0(-z) X(-z)] \quad (2.58)$$

Similarly, tracing the signals through the second branch in Figure 2.9. gives us,

$$\hat{\theta}_1(z) = \frac{1}{2} G_1(z) [H_1(z) X(z) + H_1(-z) X(-z)] \quad (2.59)$$

Since the interpolation stage introduced images of the original subband spectrum, this is followed by bandpass filtering. In order to put each subband at its original position in the signal spectrum the signal $\hat{Y}_0(w)$ is lowpass filtered and $\hat{Y}_1(w)$ is highpass filtered. Finally, the filter results are added to obtain the reconstructed signal

$$\begin{aligned} \hat{X}(w) = & \frac{1}{2} [G_0(w)H_0(w) + G_1(w)H_1(w)]X(w) + \\ & \frac{1}{2} [G_0(w)H_0(w+\pi) + G_1(w)H_1(w+\pi)]X(w+\pi) \end{aligned} \quad (2.60a)$$

The z transform of the reconstructed signal and thus the input/output relationship can be written as,

$$\begin{aligned} \hat{X}(z) = & \frac{1}{2} [H_0(z)G_0(z) + H_1(z)G_1(z)]X(z) + \frac{1}{2} [H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z) \\ = & T(z)X(z) + A(z)X(-z) \end{aligned} \quad (2.60b)$$

which is a linear combination of the linear time invariant (LTI) system transfer function $T(z)$ and the aliasing term $A(z)$. For aliasing cancellation $A(z)$ should be zero and for perfect reconstruction $T(z)$ must be a pure delay. So,

$$(1) A(z) = 0, \text{ for all } z$$

$$(2) T(z) = cz^{-\Delta}, c \text{ being a constant}$$

In matrix form the PR conditions can be written as

$$\frac{1}{2} \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-\Delta} \\ 0 \end{bmatrix} \quad (2.61a)$$

$$\frac{1}{2} H_{AC}(z) \mathfrak{S}(z) = \begin{bmatrix} T(z) \\ 0 \end{bmatrix} \quad (2.61b)$$

Then,

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{1}{H_0(z)H_1(-z) - H_1(z)H_0(-z)} \begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix}$$

So, to eliminate aliasing and forcing $A(z) = 0$, we require

$$\frac{G_0(z)}{G_1(z)} = \frac{H_1(-z)}{-H_0(-z)} = \frac{-H_1(-z)}{H_0(-z)} \quad (2.62)$$

This can be achieved by the selection

$$G_0(z) = -H_1(-z) \quad (2.63a)$$

$$G_0(w) = -H_1(w + \pi) \quad (2.63b)$$

$$G_1(z) = H_0(-z) \quad (2.63c)$$

$$G_1(w) = H_0(w + \pi) \quad (2.63a)$$

leaving us with

$$T(z) = \frac{1}{2} [H_0(-z)H_1(z) - H_0(z)H_1(-z)] \quad (2.64a)$$

$$T(w) = \frac{\hat{X}(w)}{X(w)} = \frac{1}{2} [H_0(w + \pi)H_1(w) - H_0(w)H_1(w + \pi)] \quad (2.64b)$$

To satisfy the second requirement of the perfect reconstruction property, with $H_0(z)$

and $H_I(z)$ being N -tap (N even) FIR filters, $H_I(z)$ will be chosen as;

$$H_I(z) = z^{-(N-1)} H_0(-z^{-1}) \quad (2.65)$$

This choice gives us

$$T(z) = \frac{1}{2} z^{-(N-1)} [H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1})] \quad (2.66)$$

Therefore, the perfect reconstruction requirement reduces to finding an $H_0(z)$ such that

$$\begin{aligned} Q(z) &= H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = \text{constant} \\ &= R(z) + R(-z) \end{aligned} \quad (2.67)$$

The PR requirement can be alternatively given in time domain. Noting that $R(z)$ is a spectral density,

$$R(z) = \gamma_{N-1}z^{N-1} + \gamma_{N-2}z^{N-2} + \dots + \gamma_0z^0 + \dots + \gamma_{N-1}z^{-(N-1)} \quad (2.68a)$$

Then

$$R(-z) = -\gamma_{N-1}z^{N-1} + \gamma_{N-2}z^{N-2} - \dots + \gamma_0z^0 - \gamma_1z^{-1} \dots - \gamma_{N-1}z^{-(N-1)} \quad (2.68b)$$

Therefore, $Q(z)$ consists only of even powers of z . To force $Q(z) = \text{constant}$, it suffices to make all even-indexed coefficients in $R(z)$ equal to zero except γ_0 . The γ_n coefficients in $R(z)$ being samples of the autocorrelation and hence we need to set the autocorrelation function to set to zero for n even and $n \neq 0$. If the normalization

$$\sum_{k=0}^{N-1} |h(k)|^2 = 1 \quad (2.69)$$

is imposed, one obtains the PR requirement in the time domain as,

$$r(2n) = \sum_{k=0}^{N-1} h(k)h(k+2n) = \delta_n \quad (2.70)$$

Two Dimensional Four-Channel PRFB with Quadratic Sampling

The four-channel filter bank is shown in the below figure where $H_0(\mathbf{z})$, $H_1(\mathbf{z})$, $H_2(\mathbf{z})$, and $H_3(\mathbf{z})$ are analysis filters splitting the spectrum into four equal bands. The vector notation for $\mathbf{n} = [n_1, n_2]$ and $\mathbf{z} = [z_1, z_2]$ is used and the sampling lattice $\mathbf{D}$ is chosen equal here to quadratic sampling case (i.e. $\mathbf{D} = \begin{bmatrix} 20 \\ 02 \end{bmatrix}$). Then according to the Nyquist theorem $m_0(\mathbf{n})$, $m_1(\mathbf{n})$, $m_2(\mathbf{n})$, and $m_3(\mathbf{n})$ are each down sampled 2×2 in the horizontal and vertical

directions to provide the subband signals $v_0(\mathbf{n})$, $v_1(\mathbf{n})$, $v_2(\mathbf{n})$ and $v_3(\mathbf{n})$ as the outputs of the analysis stage. The $v_0(\mathbf{n})$, $v_1(\mathbf{n})$, $v_2(\mathbf{n})$ and $v_3(\mathbf{n})$ are each upsampled 2 by 2 again in the horizontal and vertical directions to give the zero-interlaced signals $y_0(\mathbf{n})$, $y_1(\mathbf{n})$, $y_2(\mathbf{n})$ and $y_3(\mathbf{n})$, which are in turn processed by the interpolation filters $G_0(\mathbf{z})$, $G_1(\mathbf{z})$, $G_2(\mathbf{z})$ and $G_3(\mathbf{z})$, in the synthesis stage to yield the reconstructed signal.

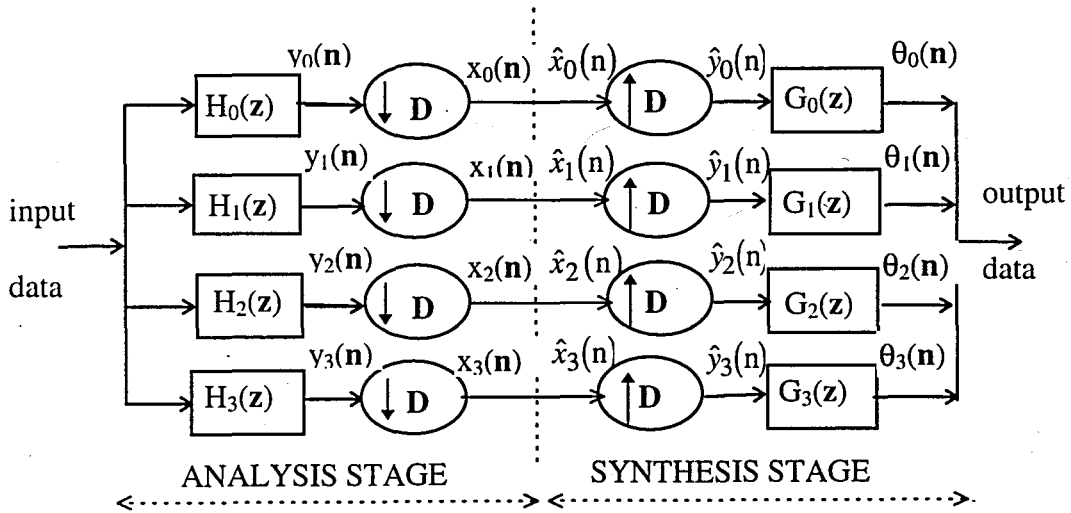


FIGURE 2.21. Two dimensional four-channel filter bank

From the first branch in Figure 2.21.,

$$Y_0(\mathbf{z}) = H_0(\mathbf{z})X(\mathbf{z}) \quad (2.71)$$

$$\theta_0(\mathbf{z}) = G_0(\mathbf{z})\hat{Y}_0(\mathbf{z}) \quad (2.72)$$

as the outputs of the decimation and interpolation filters, while the down sampler and the up sampler in 2-D impose respectively,

$$X_0(\mathbf{z}) = \frac{1}{4} [M_0(z_1^{1/2}, z_2^{1/2}) + M_0(-z_1^{1/2}, z_2^{1/2}) + M_0(z_1^{1/2}, -z_2^{1/2}) + M_0(-z_1^{1/2}, -z_2^{1/2})] \quad (2.73)$$

$$\hat{Y}_0(\mathbf{z}) = X_0(\mathbf{z}^2) \quad (2.74)$$

Combining Equations 2.71, 2.72 and 2.73 gives

$$Y_0(\mathbf{z}) = \frac{1}{4} G_0(z_1, z_2) \begin{bmatrix} H_0(z_1, z_2)X(z_1, z_2) + H_0(-z_1, z_2)X(-z_1, z_2) + \\ H_0(z_1, -z_2)X(z_1, -z_2) + H_0(-z_1, -z_2)X(-z_1, -z_2) \end{bmatrix} \quad (2.75)$$

Similarly,

$$Y_1(\mathbf{z}) = \frac{1}{4} G_1(z_1, z_2) \begin{bmatrix} H_1(z_1, z_2)X(z_1, z_2) + H_1(-z_1, z_2)X(-z_1, z_2) + \\ H_1(z_1, -z_2)X(z_1, -z_2) + H_1(-z_1, -z_2)X(-z_1, -z_2) \end{bmatrix} \quad (2.76)$$

$$Y_2(\mathbf{z}) = \frac{1}{4} G_2(z_1, z_2) \begin{bmatrix} H_2(z_1, z_2)X(z_1, z_2) + H_2(-z_1, z_2)X(-z_1, z_2) + \\ H_2(z_1, -z_2)X(z_1, -z_2) + H_2(-z_1, -z_2)X(-z_1, -z_2) \end{bmatrix} \quad (2.77)$$

$$Y_3(\mathbf{z}) = \frac{1}{4} G_3(z_1, z_2) \begin{bmatrix} H_3(z_1, z_2)X(z_1, z_2) + H_3(-z_1, z_2)X(-z_1, z_2) + \\ H_3(z_1, -z_2)X(z_1, -z_2) + H_3(-z_1, -z_2)X(-z_1, -z_2) \end{bmatrix} \quad (2.78)$$

The Z-transform of the reconstructed signal is then given by,

$$\hat{X}(\mathbf{z}) = \frac{1}{4} \left\{ \begin{aligned} & \left[\begin{aligned} & H_0(z_1, z_2)G_0(z_1, z_2) + H_1(z_1, z_2)G_1(z_1, z_2) + \\ & H_2(z_1, z_2)G_2(z_1, z_2) + H_3(z_1, z_2)G_3(z_1, z_2) \end{aligned} \right] X(z_1, z_2) + \\ & \left[\begin{aligned} & H_0(-z_1, z_2)G_0(z_1, z_2) + H_1(-z_1, z_2)G_1(z_1, z_2) + \\ & H_2(z_1, z_2)G_2(z_1, z_2) + H_3(z_1, z_2)G_3(z_1, z_2) \end{aligned} \right] X(-z_1, z_2) + \\ & \left[\begin{aligned} & H_0(z_1, -z_2)G_0(z_1, z_2) + H_1(z_1, -z_2)G_1(z_1, z_2) + \\ & H_2(z_1, -z_2)G_2(z_1, z_2) + H_3(z_1, -z_2)G_3(z_1, z_2) \end{aligned} \right] X(z_1, -z_2) + \\ & \left[\begin{aligned} & H_0(-z_1, -z_2)G_0(z_1, z_2) + H_1(-z_1, -z_2)G_1(z_1, z_2) + \\ & H_2(-z_1, -z_2)G_2(z_1, z_2) + H_3(-z_1, -z_2)G_3(z_1, z_2) \end{aligned} \right] X(-z_1, -z_2) \end{aligned} \right\} \quad (2.79a)$$

$$= T(z_1, z_2)X(z_1, z_2) + S(z_1, z_2)X(-z_1, z_2) + W(z_1, z_2)X(z_1, -z_2) + R(z_1, z_2)X(-z_1, -z_2) \quad (2.79b)$$

or

$$\hat{X}(w_0, w_1) = \frac{1}{4} \sum_{k=0}^1 \sum_{l=0}^1 X(w_0 + k\pi, w_1 + l\pi) \cdot \sum_{i=0}^1 \sum_{j=0}^1 H_{ij}(w_0 + k\pi, w_1 + l\pi) G_{ij}(w_0, w_1) \quad (2.79c)$$

In this 2-dimensional case, we have three aliasing terms, namely for $(k,l) \neq (0,0)$. In contrast to the 1-dimensional filter bank, one cannot eliminate the aliasing errors by simply choosing certain reconstruction filters $G_{ij}(w_0, w_1)$.

In this case we have the inversion of a 4×4 matrix

$$\frac{1}{\det \mathbf{D}} H_{AC}(\mathbf{z}) \mathfrak{S}(\mathbf{z}) = \begin{bmatrix} T(\mathbf{z}) \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad (2.80a)$$

$$\frac{1}{\det \mathbf{D}} \mathfrak{S}(\mathbf{z}) = H_{AC}^{-1}(\mathbf{z}) \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \mathbf{z}^{-\Delta} \quad (2.80b)$$

$$= \left[1^{st} \text{ column of } H_{AC}^{-1}(\mathbf{z}) \right] \mathbf{z}^{-\Delta} \quad (2.80c)$$

and perfect reconstruction requires the following :

- (1) $S(\mathbf{z}) = 0, W(\mathbf{z}) = 0, R(\mathbf{z}) = 0$, for all $\mathbf{z}$
- (2) $T(\mathbf{z}) = c\mathbf{z}^{-n_0}$, c being a constant

To eliminate aliasing and forcing $S(\mathbf{z}) = 0, W(\mathbf{z}) = 0, R(\mathbf{z}) = 0$ we require

$$\begin{aligned} G_0(z_1, z_2) = & H_0(z_1, z_2) (-H_3(-z_1, z_2) H_1(z_1, -z_2) H_1(-z_1, -z_2) + \\ & H_2(-z_1, z_2) H_3(z_1, -z_2) H_1(-z_1, -z_2) + \\ & H_3(-z_1, z_2) H_1(z_1, -z_2) H_2(-z_1, -z_2) - \\ & H_1(-z_1, z_2) H_3(z_1, -z_2) H_2(-z_1, -z_2) - \\ & H_2(-z_1, z_2) H_1(z_1, -z_2) H_1(-z_1, -z_2) + \\ & H_1(-z_1, z_2) H_1(z_1, -z_2) H_3(-z_1, -z_2)) \end{aligned} \quad (2.81a)$$

$$\begin{aligned} G_1(z_1, z_2) = & H_1(z_1, z_2) (H_3(-z_1, z_2) H_1(z_1, -z_2) H_0(-z_1, -z_2) - \\ & H_2(-z_1, z_2) H_3(z_1, -z_2) H_0(-z_1, -z_2) - \\ & H_3(-z_1, z_2) H_0(z_1, -z_2) H_2(-z_1, -z_2) + \\ & H_0(-z_1, z_2) H_3(z_1, -z_2) H_2(-z_1, -z_2) + \\ & H_2(-z_1, z_2) H_0(z_1, -z_2) H_3(-z_1, -z_2) - \\ & H_0(-z_1, z_2) H_1(z_1, -z_2) H_3(-z_1, -z_2)) \end{aligned} \quad (2.81b)$$

$$\begin{aligned}
G_2(z_1, z_2) = & H_2(z_1, z_2) (-H_3(-z_1, z_2) H_1(z_1, -z_2) H_0(-z_1, -z_2) + \\
& H_1(-z_1, z_2) H_3(z_1, -z_2) H_0(-z_1, -z_2) + \\
& H_3(-z_1, z_2) H_0(z_1, -z_2) H_1(-z_1, -z_2) - \\
& H_0(-z_1, z_2) H_3(z_1, -z_2) H_1(-z_1, -z_2) - \\
& H_1(-z_1, z_2) H_0(z_1, -z_2) H_3(-z_1, -z_2) + \\
& H_0(-z_1, z_2) H_1(z_1, -z_2) H_3(-z_1, -z_2)) \quad (2.81c)
\end{aligned}$$

$$\begin{aligned}
G_3(z_1, z_2) = & H_3(z_1, z_2) (H_2(-z_1, z_2) H_1(z_1, -z_2) H_0(-z_1, -z_2) - \\
& H_1(-z_1, z_2) H_1(z_1, -z_2) H_0(-z_1, -z_2) - \\
& H_2(-z_1, z_2) H_0(z_1, -z_2) H_1(-z_1, -z_2) + \\
& H_0(-z_1, z_2) H_1(z_1, -z_2) H_1(-z_1, -z_2) + \\
& H_1(-z_1, z_2) H_0(z_1, -z_2) H_2(-z_1, -z_2) - \\
& H_0(-z_1, z_2) H_1(z_1, -z_2) H_2(-z_1, -z_2)) \quad (2.81d)
\end{aligned}$$

To satisfy the second condition let choose,

$$H_1(z_1, z_2) = z_1^{-(\det \mathbf{D}-1)} z_2^{-(\det \mathbf{D}-1)} H_0(-z_1^{-1}, -z_2^{-1}) \quad (2.82)$$

so that $H_2(z_1, z_2)$ and $H_3(z_1, z_2)$ can be found accordingly. The resulting synthesis filters will likely be a high order infinite impulse response (IIR) and possibly unstable. The modulation domain analysis is more amenable to design.

As

$$Y(\mathbf{z}) = \frac{1}{\det \mathbf{D}} (G_0(\mathbf{z}) \dots G_{\det \mathbf{D}}(\mathbf{z})) H_m(\mathbf{z}) X_m(\mathbf{z}) \quad (2.83)$$

where $H_m(\mathbf{z})$ contains the modulated versions of all the filters as given in Equation 2.37a. If the modulation matrix is orthogonal, i.e., with $\tilde{H}(\mathbf{z})$ denoting complex conjugate transposition,

$$\sum_{\mathbf{k} \in FPD(\mathbf{D})} H_i(W_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}) \tilde{H}_j(W_{\mathbf{D}^{-1}}(2\pi\mathbf{k}) \circ \mathbf{z}) = \det \mathbf{D} \cdot \delta_{ij} \quad (2.84)$$

which is the Z-transform of

$$r_{ij}(\mathbf{n}) \cdot \sum_{\mathbf{k} \in FPD(\mathbf{D})} \exp(j2\pi\mathbf{k}^T \mathbf{D}^{-1} \mathbf{n}) = \det \mathbf{D} \cdot \delta_{ij} \delta_n \quad (2.85)$$

As,

$$\sum_{\mathbf{k} \in \text{FPD}(\mathbf{D})} \exp(j2\pi \mathbf{k}^T \mathbf{D}^{-1} \mathbf{n}) = \begin{cases} \det \mathbf{D} & \text{if } \mathbf{n} \in \text{LAT}(\mathbf{D}) \\ 0 & \text{otherwise} \end{cases} \quad (2.86)$$

Finally, the orthonormality condition is satisfied by

$$r_{ij}(\mathbf{Dn}) = \langle h_i(\mathbf{k}), h_j(\mathbf{k} + \mathbf{Dn}) \rangle = \delta_{ij} \delta_n \quad (2.87)$$

2.3.5 Wavelet Transform and Subband Decomposition Relations: Bridge between Mathematics and Signal Processing

The idea of multiresolution analysis (MRA) is to analyze the signal at different resolutions and add detail when the need arises for going from coarse to finer resolution. An intuitive explanation can be given when considering images. Assuming that a lowpass filtered version of an image is given which is obviously a lower resolution, or a coarse approximation to the original, to go from this coarse level to the original (finer level), it is necessary to add the removed high frequency part of the image, or equivalently to add detail (roughly edges).

In this subsection, we aim to point out the interrelationships between multiresolution approach, wavelet decomposition and multidimensional filter banks in any dimensions. The relationships are derived for quadratic and cubic subsampling cases in 2 and 3 dimensions similar to the ones corresponding to the 1 dimensional case.

Two- Scale Equations for the 1-D case:

The multiresolution orthogonal representation of a signal consists of,

(i) a family of embedded closed subspaces $V_m \subset L^2(\mathbb{R})$, $m \in \mathbb{Z}$,

$$\dots V_2 \subset V_1 \subset V_0 \subset V_{-1} \dots$$

such that

$$(ii) \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \quad \overline{\bigcup_{m \in \mathbb{Z}} V_m} = L^2(\mathbb{R})$$

$$(iii) f \in V_m \Leftrightarrow f(2 \cdot) \in V_{m-1}$$

To define a multiresolution analysis, one has to introduce a scaling function $\phi \in V_0$ such that for all $m \in \mathbb{Z}$, the $\phi_{m,n}$ constitute an orthogonal basis for V_m , that is ,

(iv) $V_m = \text{linear span}\{\phi_{m,n}, n \in \mathbb{Z}\}$ and there exist $0 < A \leq B < \infty$ such that, for all $(\alpha_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$A \sum_n |\alpha_n|^2 \left\| \sum_n \alpha_n \phi_{mn} \right\|^2 \leq B \sum_n |\alpha_n|^2 \quad (2.88)$$

Moreover, one may, without loss of generality, assume that the $\phi_{0,n}$ are orthonormal (which automatically implies that the $\phi_{m,n}$ are orthonormal for every fixed m)

The dilated and translated versions of the scaling function are denoted as

$$\phi_{m,n}(x) = 2^{-m/2} \phi(2^{-m}x - n) \quad (m, n) \in \mathbb{Z}^2 \quad (2.89)$$

For all m , $(\phi_{m,n})_{n \in \mathbb{Z}}$ constitute an orthonormal base for V_m . Therefore the approximation of f with a resolution 2^{-m} can be written as

$$\text{Proj}_{V_m}(f) = \sum_{n \in \mathbb{Z}} \langle f, \phi_{m,n} \rangle \phi_{m,n} \quad (2.90)$$

Since $\phi \in V_0 \subset V_{-1} = \text{span}\{\phi(2 \cdot -n)\}$, there exist α_n such that for any scale m

$$\phi(x) = \sum_{n \in \mathbb{Z}} \alpha_n \phi(2x - n) \quad (2.91)$$

where the orthogonality of $\phi_{0,k}$ implies that

$$\sum_n \alpha_n \alpha_{n+2k} = 2\delta_{k,0} \quad (2.92)$$

The orthonormal base of wavelets associated to this multiresolution decomposition is then given by

$$\psi(x) = \sum_{n \in \mathbb{Z}} (-1)^n \alpha_{-n+1} \phi(2x - n) \quad (2.93)$$

The $\psi(x)$ set spans the orthogonal complement of V_m in V_{m-1} , i.e.,

$$V_{m-1} = V_m \oplus W_m \quad (2.94)$$

and, given the complementary orthogonality of these two spaces one has:

$$\text{Proj}_{V_{m-1}}(f) = \text{Proj}_{V_m}(f) + \sum_{n \in \mathbb{Z}} \langle f, \psi_{m,n} \rangle \psi_{m,n} \quad (2.95)$$

The function $\phi_{m+1,n}(x) \in V_{m+1} \subset V_m$ can be decomposed on the base of functions $\phi_m(x)$ by the relation

$$\phi_{m+1,n}(x) = \sum_{k=-\infty}^{\infty} \langle \phi_{m+1,n}, \phi_{m,k} \rangle \phi_{m,k}(x) \quad (2.96)$$

where

$$\langle \phi_{m+1,n}, \phi_{m,k} \rangle = \int_{-\infty}^{\infty} 2^{-(m+1)/2} \phi(2^{-(m+1)}x - n) 2^{-m/2} \phi(2^{-m}x - k) dx \quad (2.97)$$

With the change of variable $2^{-(m+1)}x - n = 2^{-1}y$ we obtain then the following relation

$$\langle \phi_{m+1,n}, \phi_{m,k} \rangle = \int_{-\infty}^{\infty} 2^{-1/2} \phi(2^{-1}y) \phi(y - (k - 2n)) dy \quad (2.98)$$

We can then write the following relation meaning that the scalar product is independent of the resolution m

$$\langle \phi_{m+1,n}, \phi_{m,k} \rangle = \langle \phi_{1,0}, \phi_{0,k-2n} \rangle \quad (2.99)$$

Defining

$$h_0(n) = \langle \phi_{1,0}, \phi_{0,n} \rangle = \int_{-\infty}^{\infty} 2^{-1/2} \phi\left(\frac{1}{2}x\right) \phi(x - n) dx \quad (2.100)$$

the impulse response of a lowpass filter, Daubechies has shown that these coefficients $h_0(n)$ corresponds up to a normalization factor $2^{-1/2}$ exactly to the coefficients α_n of Equation 2.91.

If we project a function f on the base functions $\phi_{m+1,n}(x)$ and using Equations 2.96, 2.98, 2.100 one has

$$\langle f, \phi_{m+1,n} \rangle = \sum_{k=-\infty}^{+\infty} \langle f, \phi_{m,k} \rangle \langle \phi_{1,0}, \phi_{0,k-2n} \rangle \quad (2.101)$$

By combining the Equations 2.96, 2.99, 2.100, the following relation for $m=0$ and $n=0$ providing the relation between the filter h and the scaling function ϕ

$$\phi\left(\frac{x}{2}\right) = \sqrt{2} \sum_{k=-\infty}^{+\infty} h(k) \phi(x-k) \quad (2.102)$$

The relations developed for a lowpass filter $h(n)$, are also valid for the highpass $g(n)$. In fact, one can also write

$$\begin{aligned} g(n) &= \int_{-\infty}^{+\infty} 2^{-1/2} \psi\left(\frac{1}{2}x\right) \phi(x-n) dx \\ &= (-1)^n h(-n+1) \end{aligned} \quad (2.103)$$

which gives

$$\psi\left(\frac{x}{2}\right) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^n h(-n+1) \phi(x-n) \quad (2.104)$$

Let $s_{m+1}(n) = \langle f, \phi_{m+1,n} \rangle$ the sampled signal corresponding to the projection of f on $\phi_{m+1,n}(x)$ and $s_m(n) = \langle f, \phi_{m,n} \rangle$ the sampled signal corresponding to the projection of f on $\phi_{m,n}(x)$, the association between orthonormal wavelets and digital filters is presented in Figure 2.22 for a sampled signal s .

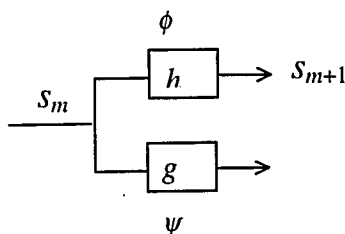


FIGURE 2.22 Orthonormal wavelets and digital filter for 1-D two-band filter bank

For symmetric filters having $h(n)=h(-n)$, the Equations 2.100, 2.101 gives us the possibility to write

$$\begin{aligned} s_{m+1}(n) &= \langle f, \phi_{m+1,n} \rangle = \sum_{k=-\infty}^{+\infty} h(2n-k) \langle f, \phi_{m,k} \rangle \\ &= \sum_{k=-\infty}^{+\infty} h(2n-k) s_m(k) \end{aligned} \quad (2.105)$$

Extension of the two-scale equation to two-dimensions:

To see the connections between filter banks, multiresolutional approach and wavelet theory for 2-D case let us choose the example of 4-band quadratic sampling lattice where downsampling is by a factor of 2 in both directions. An image of size $N \times N$ yields four $N/2 \times N/2$ transform images. One of the branch outputs resembles a low-pass filtered and decimated version and the other branch outputs have the “details” or high-frequency signals in the image.

In the 2-D multiresolutional analysis, the original image is projected on an increasing series of closed vectorial subspaces $(V_m)_{m \in \mathbb{Z}}$ and the lost information between two adjacent scales is retrieved by projection on another series of subspaces $(W_m)_{m \in \mathbb{Z}}$, such that [65, 68, 71],

$$V_m = V_{m-1} \oplus W_{m-1}^1 \oplus W_{m-1}^2 \oplus W_{m-1}^3 \quad (2.106)$$

with $\overline{\bigcup_{m \in \mathbb{Z}} V_m} = L^2(\mathbb{R})$ and $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$. These spaces $(V_m)_{m \in \mathbb{Z}}$ are considered as successive approximation spaces: for a given function r , the successive projections $\text{Proj}_{V_m}(r)$ onto the spaces $(V_m)_{m \in \mathbb{Z}}$ give the approximations of r with a resolution 2^{-m} , so for any $r(x,y) \in V_0$

$$r(x,y) = \text{Proj}_{V_{-1}} r(x,y) + \text{Proj}_{W_{-1}^1} r(x,y) + \text{Proj}_{W_{-1}^2} r(x,y) + \text{Proj}_{W_{-1}^3} r(x,y) \quad (2.107)$$

Let a function $\Phi(x, y)$, called scaling function, providing a MRA such that the projection on $(V_m)_{m \in \mathbb{Z}}$ is characterized by the coordinates on the base $\Phi(x, y)$. The dilated and translated versions of the scaling function are

$$\phi_{m,n} = 2^{-m} \phi(2^{-m} \mathbf{x} - \mathbf{n}) \quad (m, \mathbf{n}) \in \mathbb{Z}^3 \quad (2.108)$$

for all m , $(\phi_{m,n})_{n \in \mathbb{Z}^2}$ constitute an orthonormal base for $(V_m)_{m \in \mathbb{Z}}$. The approximation of r with a resolution of 2^{-m} can be written as,

$$\text{Proj}_{V_m}(r) = \sum_{n \in \mathbb{Z}^2} \sum \langle r, \phi_{m,n} \rangle \phi_{m,n} \quad (2.109)$$

These applications are continuous projections which degrade information in r as m increases and approximate better and better r as m decreases.

The function $\phi_{m+1,n_x,n_y}(x, y) \in V_{m+1} \subset V_m$ can be decomposed on the base of function $\phi_m(x, y)$ by

$$\phi_{m+1,n_x,n_y}(x, y) = \sum_{\mathbf{k}} \sum \langle \phi_{m+1,\mathbf{n}}, \phi_{m,\mathbf{k}} \rangle \phi_{m,\mathbf{k}}(\mathbf{x}) \quad (2.110)$$

where

$$\langle \phi_{m+1,\mathbf{n}}, \phi_{m,\mathbf{k}} \rangle = \iint 2^{-(m+1)} \phi(2^{-(m+1)} \mathbf{x} - \mathbf{n}) 2^{-m} \phi(2^{-m} \mathbf{x} - \mathbf{k}) d\mathbf{x} \quad (2.111)$$

Let us again do a change of variable as $2^{-(m+1)} \mathbf{x} - \mathbf{n} = 2^{-1} \mathbf{y}$. Then we obtain the following relation :

$$\langle \phi_{m+1,\mathbf{n}}, \phi_{m,\mathbf{k}} \rangle = \iint 2^{-1} \phi(2^{-1} \mathbf{y}) \phi(\mathbf{y} - (\mathbf{k} - 2\mathbf{n})) d\mathbf{y} \quad (2.112)$$

Thus the scalar product is independent of the resolution m and therefore one can write:

$$\langle \phi_{m+1,\mathbf{n}}, \phi_{m,\mathbf{k}} \rangle = \langle \phi_{1,0}, \phi_{0,\mathbf{k}-2\mathbf{n}} \rangle \quad (2.113)$$

The function r projected on the base functions $\phi_{m+1,\mathbf{n}}(\mathbf{x})$, using Equation 2.59 becomes:

$$\langle r, \phi_{m+1,\mathbf{n}} \rangle = \sum_{\mathbf{k}} \langle r, \phi_{m,\mathbf{k}} \rangle \langle \phi_{1,0}, \phi_{0,\mathbf{k}-2\mathbf{n}} \rangle \quad (2.114)$$

If one defines the impulse response of the lowpass filter as

$$h_0(\mathbf{n}) = \langle \phi_{1,0}, \phi_{0,\mathbf{n}} \rangle = \iint 2^{-1} \phi(2^{-1}\mathbf{x}) \phi(\mathbf{x} - \mathbf{n}) d\mathbf{x} \quad (2.115)$$

and combining Equations 2.113, 2.114, 2.115 we obtain for $m=0$ and $\mathbf{n}=0$ the relation between the numerical filter $\mathbf{h}$ and scaling function ϕ is given similar to Equation 2.91 in 1-D as,

$$\phi(x, y) = \sum_{n_x} \sum_{n_y} h_0(n_x, n_y) \phi(2x - n_x, 2y - n_y) \quad (2.116)$$

Similar relations for the three wavelets are similarly obtained as in 1-D case shown by except that $h(n_x, n_y)$ is, in its general form, i.e. a nonseparable filter. In image processing the signal is made of samples (pixels). In image processing the space V_0 is the space containing the continuous version of the original image in its given resolution.

Let us now see how an image is decomposed in its subbands. This analysis will provide us the interrelation between MRA, WT and FB approaches. Given a sampled image $I(n_x, n_y)$, let us construct

$$r(x, y) = \sum_{n_x} \sum_{n_y} I(n_x, n_y) \Phi(x - n_x, y - n_y) \quad (2.117)$$

with

$$I(n_x, n_y) = \langle r(x, y), \Phi_{0,n_x,n_y}(x, y) \rangle \quad (2.118)$$

and let us analyze specifically

$$\text{Proj}_{V_m} r(x, y) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \langle r, \phi_{m,\mathbf{n}} \rangle \Phi_{m,\mathbf{n}} \quad (2.119)$$

The coefficients of $a(n_x, n_y)$ which are found by

$$\begin{aligned}
 a(n_x, n_y) &= \left\langle r(x, y), \Phi_{-1, n_x, n_y} \right\rangle \\
 &= 2 \iint r(x, y) \Phi(2^{-1}x - n_x, 2^{-1}y - n_y) \\
 &= 2 \iint \sum_{k_x} \sum_{k_y} I(k_x, k_y) \Phi(x - k_x, y - k_y) \Phi(2^{-1}x - n_x, 2^{-1}y - n_y) dx dy \\
 &= 2 \sum_{k_x} \sum_{k_y} I(k_x, k_y) \iint \Phi(x - k_x, y - k_y) \Phi(2^{-1}x - n_x, 2^{-1}y - n_y) dx dy
 \end{aligned} \tag{2.120}$$

By substituting $x - 2n_x \rightarrow x$ and $y - 2n_y \rightarrow y$ in Equation 2.120,

$$\begin{aligned}
 \Phi(2^{-1}x, 2^{-1}y) &= \sum_{l_x} \sum_{l_y} h_0(l_x, l_y) \Phi(x - l_x, y - l_y) \\
 \Phi(2^{-1}x - n_x, 2^{-1}y - n_y) &= \sum_{l_x} \sum_{l_y} h_0(l_x, l_y) \Phi(x - 2n_x - l_x, y - 2n_y - l_y).
 \end{aligned} \tag{2.121}$$

So,

$$\begin{aligned}
 a(n_x, n_y) &= 2 \sum_{k_x} \sum_{k_y} I(k_x, k_y) \iint \Phi(x - k_x, y - k_y) \sum_{l_x} \sum_{l_y} h_0(l_x, l_y) \Phi(x - 2n_x - l_x, y - 2n_y - l_y) dx dy \\
 &= 2 \sum_{k_x} \sum_{k_y} I(k_x, k_y) \sum_{l_x} \sum_{l_y} h_0(l_x, l_y) \iint \Phi(x - k_x, y - k_y) \Phi(x - 2n_x - l_x, y - 2n_y - l_y) dx dy \\
 &= 2 \sum_{k_x} \sum_{k_y} I(k_x, k_y) \sum_{l_x} \sum_{l_y} h_0(l_x, l_y) \delta(k_x - 2n_x - l_x, k_y - 2n_y - l_y)
 \end{aligned} \tag{2.122}$$

and therefore

$$a(n_x, n_y) = 2 \sum_{k_x} \sum_{k_y} I(k_x, k_y) h_0(k_x - 2n_x, k_y - 2n_y) \tag{2.123}$$

For linear phase symmetric filter $h_0(k_x, k_y) = h_0(-k_x, -k_y)$ the $a(n_x, n_y)$ coefficients can be calculated by first doing convolution of $h_0(n_x, n_y)$ with $I(n_x, n_y)$ and then subsampling by 2. So, schematically one has Figure 2.23:

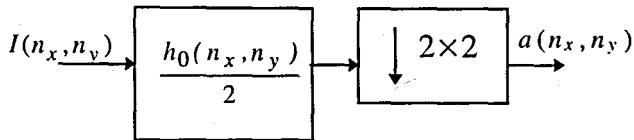


FIGURE 2.23. The output of the lowpass branch in FB using the wavelet filter coefficients

The other three branches in the 2-dimensional 4-band filter bank for quadratic sampling case formed by the three wavelet filters follow the same way of analysis as in Figure 2.24.

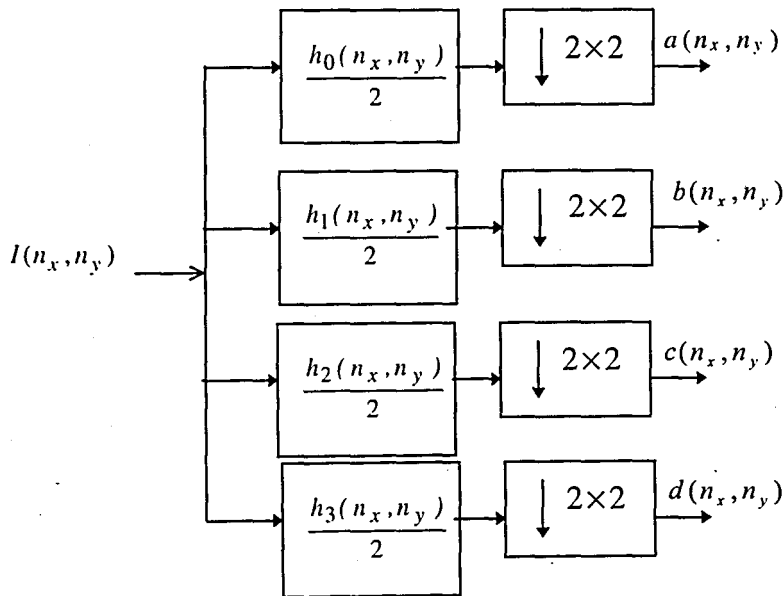


FIGURE 2.24. The output of the branches in FB using the wavelet filter coefficients

For three dimensional cubical sampling example case, similar derivations results in

$$\Phi(x, y, z) = \sum_{n_x} \sum_{n_y} \sum_{n_z} h(n_x, n_y, n_z) \Phi(2x - n_x, 2y - n_y, 2z - n_z) \quad (2.124)$$

and lowpass branch output found as

$$a(n_x, n_y, n_z) = 2\sqrt{2} \sum_{k_x} \sum_{k_y} \sum_{k_z} I(k_x, k_y, k_z) h(k_x - 2n_x, k_y - 2n_y, k_z - 2n_z) \quad (2.125)$$

Let us recall that the iterated function $f^{(i)}(t)$ converges to the scaling function

$$\phi(t) = \lim_{i \rightarrow \infty} f^{(i)}(t), \quad \phi(t) \in L^2 \quad (2.126)$$

and that we can take the limits of both sides of Equation 2.42 given in Section 2.3.3

$$f^{(i)}(t) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_0(\mathbf{k}) \phi(\mathbf{D}t - \mathbf{k})$$

to obtain

$$\phi(t) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_0(\mathbf{k}) \phi(\mathbf{D}t - \mathbf{k}) \quad (2.127)$$

showing that indeed the scaling function satisfies the two-scale Equation 2.43. In the multidimensional case, the change of scale involves the dilation matrix $\mathbf{D}$. The other ($\det \mathbf{D} - 1$) wavelets can be expressed similarly using

$$f_j^{(i)}(t) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_j(\mathbf{k}) \phi(\mathbf{D}t - \mathbf{k}) \text{ for } j = 1, 2, \dots, \det \mathbf{D} - 1$$

as

$$\psi_j(t) = \sqrt{\det \mathbf{D}} \sum_{\mathbf{k}} h_j(\mathbf{k}) \phi(\mathbf{D}t - \mathbf{k}), \quad j = 1, 2, \dots, \det \mathbf{D} - 1 \quad (2.128)$$

The bridge between WT and FB can be seen from the filter impulse responses $h(\mathbf{n})$, two-scale difference equation parameters $a(\mathbf{n})$ s.

2.4 Conclusions

In this Chapter, detailed derivations for general multidimensional operations in filter banks are given. The theory of multidimensional sampling lattices playing fundamental role in multidimensional signal processing, multidimensional sampling lattice structures are classified and the theory behind them is discussed.

Multidimensional multirate filter bank structures and related multirate fundamentals with properties of multidimensional digital filters used in filter bank are restated in a unified approach. Time domain orthonormality requirement of filters is expressed for multidimensional case.

Wavelet transform, multiresolution analysis and subband decomposition being different views of a single theory, the interrelations between them for multidimensional case are derived.

The theories and concepts presented in this Chapter with specific notations and terminology will be used in next chapters.

3 MATCHED FILTER DESIGN

The equivalence between the wavelet transform, quadrature mirror filters, and perfect reconstruction filter banks in general is now well recognized [69-71] as seen in Chapter 2. Thus the wavelet transform (WT) provides a vector space interpretation of a subband decomposition, while the filter bank formulation is an inconvenient description of the fast WT algorithm.

Traditionally, the design of quadrature mirror filter (QMF) banks has been guided by the desire to approximate a perfect half-band filter in the best possible way. More recently, the tendency has shifted towards designing signal-adapted filters and wavelets [72, 73]. Wavelets and other methods of time-frequency analysis in their many practical applications require that the analyzing filter sequence has certain desired properties. Typical of these properties are good time frequency localization, energy compaction, orthogonality and regularity. In addition it may be desired to have an analyzing filter that resembles a given waveform or that statistically matches a process. These wavelets will be referred to as matched wavelets. The "matching" of wavelet filters can be classified in two groups, namely "waveform matching" and "statistical matching". When a given waveform is mapped into a function that possesses perfect reconstruction (PR) and regularity properties, this is called waveform matched wavelet. One technique for obtaining waveform matched wavelets is given by Frazier-Kumar (FK) mapping as detailed in [74]. On the other hand, in the case of statistical matching, the wavelet filter is designed to

optimize its energy compaction or time-frequency resolution performance with respect to a given process, that is with respect to a given autocorrelation function.

In Section 3.1., we will investigate parameter tradeoffs in statistically matched wavelets, and energy compaction performance of two-band schemes vis-à-vis the Karhunen-Loeve Transform (KLT) and in Section 3.2. we will discuss the drawbacks of the FK design technique, an optimization based waveform matched wavelet design, its implemented in frequency domain. We then proposed a time domain version of FK design method and the selection of the reference function in terms of energy compaction.

3.1. Analytical Expressions of Transform Coding Gain for $M \times M$ KLTs and Design with Ideal Filtering Case

The AR(1) source is a crude, first approximation to real world sources such as speech and images. Therefore it is a commonly used artificial source model for analytical performance studies of many signal processing techniques. For analytical tractability the gain analyses are carried out for the AR(1) model only.

Transform coding attains the objective of data compression by decorrelating the signal and repacking the energy among the spectral coefficients. For orthogonal (in general, unitary as explained in Appendix A) transforms, the reconstruction error variance for an

input block is the sum of error variances in the coding (quantization) of individual transform coefficients. If $\mathbf{A}$ is orthogonal ($\mathbf{A}^{-1} = \mathbf{A}^T$) with $\mathbf{A}$ being the transform matrix, the average sum of variances $E[\theta^2(k)] = \sigma_k^2$ of the transform coefficients $\theta(k)$ equals the variance of the input,

$$\sigma_x^2 = \frac{1}{N} \sum_{k=0}^{N-1} \sigma_k^2. \quad (3.1)$$

The question of bit allocation to each coefficient, with R average number of bits per coefficient and R_k the number of bits allocated to the k^{th} coefficient, is solved by ,

$$R_k = R + \frac{1}{2} \log_2 \frac{\sigma_k^2}{\left(\prod_{j=0}^{N-1} \sigma_j^2 \right)^{\frac{1}{N}}}.$$

This bit allocation results in equal quantization error for each coefficient, and thus the distortion is spread out evenly among all the coefficients. Transform coding, which is based on the quantization of these coefficients, can provide an SNR gain over PCM, denoted by G , whose maximum is equal to the ratio of arithmetic to geometric means of transform coefficient variances.

Among all unitary matrices, the KLT transform achieves the minimum rate and minimizes the geometric mean of the subband variances so that [76]

$$G_{\text{KLT}, M} = \frac{\sigma_x^2}{\left(\prod_{k=0}^{M-1} \lambda_k \right)^{1/M}}, \quad (3.2)$$

where σ_x^2 is the input signal variance and λ_k 's are eigenvalues of $M \times M$ KLT. These eigenvalues are also identical to the subband variances, i.e., $\lambda_k = \sigma_k^2$. We have thus, for the 2×2 KLT case, obtained the eigenvalues as for $\mathbf{R}_{xx}$ covayiance matrix defined in (A.7),

$$\mathbf{R}_{xx} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \text{ and } \begin{matrix} \lambda_0 = 1 - \rho \\ \lambda_1 = 1 + \rho \end{matrix} \quad (3.3 \text{ a})$$

so that corresponding transform coding gain is

$$G_{\text{KLT},2} = \frac{1}{(1 - \rho^2)^{1/2}} \quad (3.3 \text{ b})$$

Similarly, for 3×3 KLT one has:

$$\mathbf{R}_{xx} = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} \text{ and } \begin{matrix} \lambda_0 = 1 - \rho^2 \\ \lambda_1 = \frac{2 + \rho^2 + \sqrt{8\rho^2 + \rho^4}}{2} \\ \lambda_2 = \frac{2 + \rho^2 - \sqrt{8\rho^2 + \rho^4}}{2} \end{matrix} \quad (3.4\text{a})$$

$$G_{\text{KLT},3} = \frac{1}{(1 - \rho^2)^{2/3}}, \quad (3.4\text{b})$$

and for 4×4 KLT,

$$\mathbf{R}_{xx} = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \text{ and } \begin{aligned} \lambda_0 &= 1 + \frac{(\rho + \rho^3)}{2} + \sqrt{(\rho + \rho^2)^2 + \frac{1}{4}(\rho - \rho^3)^2} \\ \lambda_1 &= 1 - \frac{(\rho + \rho^3)}{2} + \sqrt{(\rho - \rho^2)^2 + \frac{1}{4}(\rho - \rho^3)^2} \\ \lambda_2 &= 1 + \frac{(\rho + \rho^3)}{2} - \sqrt{(\rho + \rho^2)^2 + \frac{1}{4}(\rho - \rho^3)^2} \\ \lambda_3 &= 1 - \frac{(\rho + \rho^3)}{2} - \sqrt{(\rho - \rho^2)^2 + \frac{1}{4}(\rho - \rho^3)^2} \end{aligned} \quad (3.5a)$$

$$G_{\text{KLT},4} = \frac{1}{(1 - \rho^2)^{3/4}}. \quad (3.5b)$$

In fact, for an arbitrary order M and with a first-order Markov source with adjacent correlation ρ [76] one has:

$$G_{\text{KLT},M} = \left[\prod_{k=1}^{M-1} (1 - \rho^2)^{-1} \right]^{1/M} = (1 - \rho^2)^{-\frac{M-1}{M}}. \quad (3.6)$$

The upper bounds of G for orthonormal block transforms are set by the performance of $M \times M$ KLT. On the other hand, the upper bounds of G with zero aliasing are defined by the performance of the ideal filter banks.

The ideal filter banks are optimal since they provide perfect interband decorrelation for any signal source as well as alias-free frequency characteristics for multirate signal processing. But, this perfect frequency localization implies infinite duration time functions.

Optimality of ideal filters for wavelet signal approximation is discussed in [77] and it is proven that, for a stationary process with power spectral density (PSD) $S_{xx}(\exp(jw))$, the residual error is minimized if the frequency response of the filter $H(\exp(jw))$ is such that for $w \in [0, \pi/2]$,

$$|H(\exp(jw))| = \begin{cases} \sqrt{2}, & \text{if } w \in \Omega_1 = \left\{ w \in \left[0, \frac{\pi}{4}\right) : S_{xx}(\exp(jw)) \geq S_{xx}\left(\exp\left(w + \frac{\pi}{2}\right)\right) \right\} \\ & \cup \left\{ w \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] : S_{xx}(\exp(jw)) > S_{xx}\left(\exp\left(w + \frac{\pi}{2}\right)\right) \right\} \\ 0, & \text{if } w \in \Omega_2 = \left\{ w \in \left[0, \frac{\pi}{4}\right) : S_{xx}(\exp(jw)) < S_{xx}\left(\exp\left(w + \frac{\pi}{2}\right)\right) \right\} \\ & \cup \left\{ w \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] : S_{xx}(\exp(jw)) \leq S_{xx}\left(\exp\left(w + \frac{\pi}{2}\right)\right) \right\} \end{cases} \quad (3.7)$$

where Ω_1 and Ω_2 are two sets of measure $\pi/4$ that partition the frequency interval $[0, \pi/2]$, with the property

$$w \in \Omega_1 \Leftrightarrow \frac{\pi}{2} - w \in \Omega_2$$

The optimal choice is therefore to select an ideal filter with a bandpass region Ω_1 and a stopband Ω_2 . This implies that the half-band ideal lowpass filter is optimal whenever

$$\forall w \in \left[0, \frac{\pi}{4}\right], S_{xx}(\exp(jw)) \geq S_{xx}\left(\exp\left(w + \frac{\pi}{2}\right)\right)$$

In particular, above condition is satisfied when $S_{xx}(\exp(jw))$ is a nonincreasing function for $w \in [0, \pi/2]$.

The corresponding performance upper bound is calculated using the variances in the G formula of Equation 3.2 as,

$$G_{\text{ideal}} = \frac{\sigma_x^2}{\left[\prod_{i=1}^M \sigma_i^2 \right]^{1/M}} \quad (3.8)$$

and for AR(1) model this equals to

$$G_{\text{ideal, 2-band}} = \frac{1}{\left[1 - \left(\frac{4}{\pi} \text{ArcTan}[\rho]\right)^2\right]^{1/2}} \quad (3.9)$$

as given in Appendix B.

The comparison in terms of G for $M \times M$ KLT ($M=2, 3, 4, 6$) and ideal 2-band PR-QMF is given in Appendix B with the graphically obtained result that :

for $0 < \rho < 1$

$$G_{\text{KLT},2} < G_{\text{ideal, 2-band}}$$

for $0.78 \leq \rho < 1$

$$G_{\text{KLT},3} > G_{\text{ideal, 2-band}}$$

for $0.5 \leq \rho < 1$

$$G_{\text{KLT},4} > G_{\text{ideal, 2-band}}$$

for $0.1 \leq \rho < 1$

$$G_{\text{KLT},6} > G_{\text{ideal, 2-band}}$$

(3.10)

These results in Equation 3.10 explained in detail in Appendix B are obtained graphically. To see how the optimal design of 2-band PR-QMFs satisfy for $0.5 \leq \rho < 1$,

$G_{\text{KLT},4} > G_{\text{ideal, 2-band}}$ relation, PR-QMF bounds will be given in Section 3.2.2.

3.2. Statistically Matched Wavelets

3.2.1. Optimal 2-band QMF design based on energy compaction

The design of statistically matched wavelet filter banks has been extensively studied by Çağlar [46, 78, 79], where it has been formulated as a constrained optimization problem.

In this section, we want to assess the coding gain performance of matched filter banks vis-à-vis that of the KLT, and then we want to evaluate the parametric analysis.

Eigenvalue problem

The matched wavelet that maximizes the energy compaction can be formulated in terms of the objective function J :

$$\max \{J\} = \mathbf{h}^T \mathbf{R}_{xx} \mathbf{h} + \lambda_0 [1 - \mathbf{h}^T \mathbf{h}] + \mu_1 [\mathbf{h}^T \mathbf{C}_1 \mathbf{h}] + \dots + \mu_{N-1} [\mathbf{h}^T \mathbf{C}_{N-1} \mathbf{h}] \quad (3.11)$$

where $\mathbf{h}$ of size $2N$ consists of the filter coefficients, $\mathbf{C}_i$ denote the i^{th} ($i=1, \dots, N-1$) shift matrix linearly shifting a vector by $2i$ samples and μ_i 's denote Lagrange multipliers. Several other constraints can be imposed on $\mathbf{h}$, such as, step response behavior, penalty for deviations from phase linearity, minimum aliasing energy etc. Obviously, the filter bank design with the minimum set of constraints, i.e., with the orthogonality constraint only, as in Equation 3.11, should provide the highest coding gain because they constitute the least set of constraints guaranteeing PR and energy compaction. The optimal filter is found by:

$$\frac{\partial J}{\partial \mathbf{h}} = 0 \quad (3.12)$$

which can be written as an eigenvalue-like problem (with unknown Lagrange coefficients),

$$\mathbf{R}_{xx} \mathbf{h} + \mu_1 \mathbf{C}_1 \mathbf{h} + \dots + \mu_{N-1} \mathbf{C}_{N-1} \mathbf{h} = \lambda_0 \mathbf{h},$$

or alternatively

$$\mathbf{R} \mathbf{h} = \lambda_0 \mathbf{h}, \quad (3.13)$$

where the terms in the left have been combined into a new matrix $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{R}_{xx} + \mu_1 \mathbf{C}_1 + \dots + \mu_{N-1} \mathbf{C}_{N-1}. \quad (3.14)$$

The vector $\mathbf{h}$ which satisfies Equation 3.13 is the optimal PR-QMF halfband lowpass filter. Equation 3.13 looks like a classical eigenvalue problem but here the matrix $\mathbf{R}$ has unknown parameters in it.

One can note that the KLT solution corresponds to the unconstrained case in Equation 3.13, i.e., where $\mathbf{R} = \mathbf{R}_{xx}$. One can thus argue that the 2×2 KLT solution is a two tap two-band matched wavelet solution.

In the following section we will compare the compaction performance of KLT (the unconstrained case) vis-à-vis that of the matched PR-QMF filter.

3.2.2. PR-QMF Bounds using Gerschgorin Discs

Our goal is now to invoke the parameter bounds in optimal design of finite-tap 2-band QMF filters and to observe bounds on eigenvalues with respect to these bounds.

Due to the energy compaction objective of unitary 2-band PR-QMF design case, it is well known that its maximum eigenvalue must be close to 2 and it must be real, positive. Using Gerschgorin discs given in Appendix A the maximum eigenvalue of 2×2 KLT is bounded by ,

$$\lambda_{\text{KLT},2}^{\max} \leq 1 + |\rho| \quad (3.16)$$

However, we know from Equation 3.10 and from the study in Appendix B specially from Figure B.1 that in 2-band optimal PR-QMF filter bank we obtain better performance in terms of energy compaction than 2×2 KLT with

$$1 + |\rho| < \lambda_{\text{PR-QMF}} \approx 2 \quad (3.17)$$

However, the maximum eigenvalue of 4×4 KLT is always greater than the maximum eigenvalue of 2-band PR-QMF as shown in Figure B.1 whatever the number of taps used in design.

We will use maximum eigenvalues of 4×4 KLT as upper bounds to the 2-band PR-QMF ones in finding Lagrange coefficients μ_i 's of Equation 3.11 in the matched wavelet design maximizing energy compaction.

The two possible bounds on 4-band 4×4 KLT eigenvalues found using Gerschgorin discs are given by

$$|\lambda_{\text{KLT},4} - 1| \leq |\rho| + |\rho^2| + |\rho^3| \quad (3.18a)$$

$$|\lambda_{\text{KLT},4} - 1| \leq 2|\rho| + |\rho^2| \quad (3.18b)$$

An example for $\rho=0.95$ shows that the maximum eigenvalue of 4×4 KLT can be only be bounded as in Equation 3.18b,i.e.,

$$|\lambda_{\text{KLT},4}^{\max} - 1| \leq 2|\rho| + |\rho^2|$$

with $\lambda_{\text{KLT},4}^1 = 3.7568$, $\lambda_{\text{KLT},4}^2 = 0.1627$, $\lambda_{\text{KLT},4}^3 = 0.0506$, $\lambda_{\text{KLT},4}^4 = 0.0300$ and $1 + |\rho| + |\rho^2| + |\rho^3| = 3.7099$, $1 + 2|\rho| + |\rho^2| = 3.8025$.

When using Gerschgorin discs, there will be $N/2$ different bounds for 2-band N -tap PR-QMF design in $N \times N$ symmetric $\mathbf{R}$ given in Equation 3.14. The bounds on the Lagrange coefficients when designing a wavelet are found as,

$$\left\{ \begin{array}{ll} -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k + \frac{\rho^2}{2} + \frac{\rho^{i+1}}{2} & \text{for } i=2 \\ -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k - \left(\frac{-\rho^2 - 2 \sum_{k=3}^i \rho^k - \rho^{i+1} - \mu_{\lfloor \frac{i}{2} \rfloor + 1}}{2} \right) - \sum_{k=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \mu_k & \text{for } i=3 \dots N-1 \\ & i \text{ odd} \\ -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k - \left(\frac{-\rho^2 - 2 \sum_{k=3}^i \rho^k - \rho^{i+1}}{2} \right) - \sum_{k=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \mu_k & \text{for } i=3 \dots N-1 \\ & i \text{ even} \end{array} \right.$$

We realize an induction based proof to obtain bounds on Lagrange coefficients :

Proof :

Step 1: Initialization

4-tap 2-band PR-QMF design

Applying the Gerschgorin discs to the matrix given by Equation D.2 in Appendix D, one obtains the two (4-tap/2=2 implying 2 bounds) bounds as,

$$\left| \lambda_{\text{PR-QMF}} - 1 \right| \leq |\rho| + |\rho^2 + \mu_1| + |\rho^3|,$$

and

(3.19)

$$\left| \lambda_{\text{PR-QMF}} - 1 \right| \leq 2|\rho| + |\rho^2 + \mu_1|.$$

To find bounds on Lagrange coefficient μ_1 , we will use the fact that maximum eigenvalue of the 4×4 KLT is always greater than the maximum eigenvalue of PR-QMF design using ideal filters. The maximum eigenvalue of the 4×4 KLT can only be bounded for all $0 < \rho < 1$ values by the second bound given in Equation 3.18b as $2\rho + \rho^2$.

The bigger of the two bounds in Equation 3.19 is $2|\rho| + |\rho^2 + \mu_1|$ and for $0 < \rho < 1$, one has the result that

$$2\rho + |\rho^2 + \mu_1| < 2\rho + \rho^2,$$

which in turn indicates that μ_1 must be a negative number. If we use the triangular inequality,

$$|2\rho + \rho^2 + \mu_1| \leq 2\rho + |\rho^2 + \mu_1| \quad (3.20a)$$

we can use the left bound in Equation 3.20a to search for possible μ_1 values. Finally, all these bounds, can be combined as in

$$\rho < |2\rho + \rho^2 + \mu_1| \leq 2\rho + |\rho^2 + \mu_1| < 2\rho + \rho^2. \quad (3.20b)$$

To continue the analysis, let us analyze the two cases of the $2\rho + \rho^2 + \mu_1$ value separately,

a) if $2\rho + \rho^2 + \mu_1 > 0$, then $-\rho - \rho^2 < \mu_1 < 0$ is obtained.

b) if $2\rho + \rho^2 + \mu_1 < 0$, then as $\mu_1 < -2\rho - \rho^2$ conflicts $2\rho + |\rho^2 + \mu_1| < 2\rho + \rho^2$

there is no negative solution for μ_1 and for $\mu_1 = 0$, $|2\rho + \rho^2 + \mu_1|$ is a strict upper bound of eigenvalues as it will be $2\rho + \rho^2$ found using Equation 3.20b.

We have found bounds on μ_1 for 4-tap design where we have 2 multiple eigenvalues and 1 unknown Lagrange coefficient μ_1 . Let us move to 6-tap design case, where we have 3 multiple eigenvalues and 2 Lagrange coefficients μ_1 and μ_2 .

Step 2. When increasing the number of taps, there will always be only 2 inequalities to check : one of them is the maximum of the bounds for eigenvalues of 2-band PR-QMF with ideal filters being less than maximum eigenvalue of 4×4 KLT and the second one is the system inequality.

a) 6-tap 2-band PR-QMF design

Bounds now become,

1st inequality:

$$2\rho + 2(\rho^2 + \mu_1) + \rho^3 < 2\rho + \rho^2$$

2nd inequality:

$$\rho < 2\rho + \rho^2 + \mu_1 + \rho^3 + \rho^4 + \mu_2 < 2\rho + \rho^2$$

When combining them with $-\rho - \rho^2 < \mu_1 < 0$, bounds on μ_2 is obtained as

$$-\rho^3 - \rho^4 < \mu_2 < \frac{\rho^2}{2} + \frac{\rho^3}{2} - \rho^3 - \rho^4. \quad (3.21)$$

b) 8-tap 2-band PR-QMF design

Bounds now become,

1st inequality:

$$2\rho + 2(\rho^2 + \mu_1) + 2\rho^3 + \rho^4 + \mu_2 < 2\rho + \rho^2$$

2nd inequality:

$$\rho < 2\rho + \rho^2 + \mu_1 + \rho^3 + \rho^4 + \mu_2 + \rho^5 + \rho^6 + \mu_3 < 2\rho + \rho^2$$

When combining them with

$$-\rho - \rho^2 < \mu_1 < 0$$

and

$$-\rho^3 - \rho^4 < \mu_2 < \frac{\rho^2}{2} + \frac{\rho^3}{2} - \rho^3 - \rho^4,$$

bound on μ_3 is obtained as

$$-\rho^5 - \rho^6 < \mu_3 < \frac{\rho^2}{4} + \frac{\rho^3}{4} - \rho^5 - \rho^6. \quad (3.22)$$

c) 10-tap 2-band PR-QMF design

Bounds now become,

1st inequality:

$$2\rho + 2(\rho^2 + \mu_1) + 2\rho^3 + 2(\rho^4 + \mu_2) + \rho^5 < 2\rho + \rho^2$$

2nd inequality:

$$\rho < 2\rho + \rho^2 + \mu_1 + \rho^3 + \rho^4 + \mu_2 + \rho^5 + \rho^6 + \mu_3 + \rho^7 + \rho^8 + \mu_4 < 2\rho + \rho^2$$

When combining them with

$$-\rho - \rho^2 < \mu_1 < 0$$

and ,

$$-\rho^5 - \rho^6 < \mu_3 < \frac{\rho^2}{4} + \frac{\rho^3}{4} - \rho^5 - \rho^6,$$

bound on μ_4 is obtained as

$$-\rho^7 - \rho^8 < \mu_4 < \frac{\rho^2}{4} - \frac{\rho^3}{4} + \frac{\rho^5}{2} - \rho^7 - \rho^8. \quad (3.23)$$

d) 12-tap 2-band PR-QMF design

Bounds now become,

1st inequality:

$$2\rho + 2(\rho^2 + \mu_1) + 2\rho^3 + 2(\rho^4 + \mu_2) + 2\rho^5 + \rho^6 + \mu_3 < 2\rho + \rho^2$$

2nd inequality:

$$\rho < 2\rho + \rho^2 + \mu_1 + \rho^3 + \rho^4 + \mu_2 + \rho^5 + \rho^6 + \mu_3 + \rho^7 + \rho^8 + \mu_4 + \rho^9 + \rho^{10} + \mu_5 < 2\rho + \rho^2$$

When combining them with

$$-\rho - \rho^2 < \mu_1 < 0$$

and

$$-\rho^5 - \rho^6 < \mu_3 < \frac{\rho^2}{4} + \frac{\rho^3}{4} - \rho^5 - \rho^6,$$

$$-\rho^7 - \rho^8 < \mu_4 < \frac{\rho^2}{4} - \frac{\rho^3}{4} + \frac{\rho^5}{2} - \rho^7 - \rho^8,$$

bound on μ_5

$$-\rho^9 - \rho^{10} < \mu_5 < \frac{\rho^2}{8} + \frac{\rho^3}{8} - \frac{\rho^6}{2} - \rho^9 - \rho^{10} \quad (3.24)$$

is obtained.

Step 3.

Finally, as the bounds grow by adding always $\rho^{2i-1} + \rho^{2i} + \mu_i$ for $i=1, N-1$ and $2N$ tap filters ;

$$\left\{ \begin{array}{ll} -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k + \frac{\rho^2}{2} + \frac{\rho^{i+1}}{2} & \text{for } i=2 \\ -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k - \left(\frac{-\rho^2 - 2 \sum_{k=3}^i \rho^k - \rho^{i+1} - \mu_{\lfloor \frac{i}{2} \rfloor + 1}}{2} \right) - \sum_{k=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \mu_k & \text{for } i=3 \dots N-1 \\ & \text{i odd} \\ -\rho^{2i-1} - \rho^{2i} < \mu_i < -\sum_{k=3}^{2i} \rho^k - \left(\frac{-\rho^2 - 2 \sum_{k=3}^i \rho^k - \rho^{i+1}}{2} \right) - \sum_{k=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \mu_k & \text{for } i=3 \dots N-1 \\ & \text{i even} \end{array} \right. \quad (3.25)$$

Q.E.D.

These bounds on Lagrange coefficients for $2N$ tap 2-band PR-QMF design satisfy the transform coding gain results that for $0.5 \leq \rho < 1$, $G_{TC}^{KLT_4} > G_{TC}^{\text{ideal 2-band}}$ as given in Appendix B.

3.4. Waveform Matched Wavelets

Wavelet filter banks can also be designed by matching to a specific waveform. Here the goal is not to maximize the energy compaction with respect to a process model as was the case in statistically matched wavelet filters, but starting from a given reference waveform, to map it to a closest function that possesses wavelet properties. A mapping operator, which for a given reference function f , finds a wavelet filter ψ that is closest to it in the mean square norm, is given by Frazier-Kumar (FK) technique. In the FK technique the wavelet function $\psi(t)$ is related to the reference function as follows [74] :

$$\Psi(z) = \frac{|F(z)|}{\sqrt{|F(z)|^2 + |F(-z)|^2}}$$

where $F(z)$ denotes the z -transform of the reference function $f(n)$ and $\Psi(z)$ is the z -transform of the wavelet filter $\psi(t)$. The resulting filters are of infinite impulse response type, hence must be windowed. The windowing and truncation invariably result in some loss of orthogonality or perfect reconstruction property. In this section, the loss in PR when using circularly orthogonal IIR type wavelet filter design implemented in frequency domain is incrementally decreased by using firstly linearly orthogonal IIR type wavelet filter design gain in frequency domain and secondly linearly orthogonal FIR type wavelet filter design implemented in time domain.

The FT of the wavelet filter and the scaling function are denoted by $\psi(m)$ and $H(m)$ respectively, as found in the FK design technique:

$$\Psi(m) = \sqrt{1 + H(m)} \exp(j\theta(m)) \quad m \in \mathbb{Z}_{N/2}$$

$$= \frac{\sqrt{2}F(m)}{\sqrt{|F(m)|^2 + \left|F\left(m + \frac{N}{2}\right)\right|^2}} \quad m \in \mathbb{Z}_{N/2}$$

(3.27 a)

$$\text{with } H(m) = \frac{a^2(m) - b^2(m)}{a^2(m) + b^2(m)} = \frac{|F(m)|^2 - \left|F\left(m + \frac{N}{2}\right)\right|^2}{|F(m)|^2 + \left|F\left(m + \frac{N}{2}\right)\right|^2} \quad m \in \mathbb{Z}_{N/2}$$

(3.27 b)

and $\mathbb{Z}_N$ ring of integers $\{0, \dots, N-1\}$.

For $N/2 \leq m < N$, we have

$$H(m) = -H\left(m - \frac{N}{2}\right) \quad (3.28a)$$

and by $\mathbb{Z}_N$ arithmetic

$$H\left(m - \frac{N}{2}\right) = -H\left(m + \frac{N}{2}\right) \quad (3.28b)$$

so the expressions for $H(m)$ and $\Psi(m)$ hold for $m \in \mathbb{Z}_N$.

We can have PR only with symmetric sequences which hold for $m \in \mathbb{Z}_N$ or $m \in \mathbb{Z}_{N/2}$. We have zero mean high pass filter by $H(0)=1$ as seen in Table 3.1. for symmetric sequences only.

TABLE 3.1. $|F(m)|$, $\rho(w)$ and $H(m)$ values for different sizes of the reference Gauss function in circularly orthogonal FK design method: (a) for symmetric sequence, (b) and (c) for nonsymmetric sequences where $H(0)$ is indicated by arrow $\uparrow$

(a)	For $x=(-7, -5, -3, -1, 1, 3, 5, 7)$ supported Gauss reference function $ F(m) =(0.4928, 0.4505, 0.3359, 0.177, 0, 0.177, 0.3359, 0.4505)$ $\rho(\text{degree}) = (0, -157, 45, -112, 0, 112, -45, 157)$ $H(m) = (1, 0.73, 0, -0.73, -1, -0.73, 0, 0.73)$ $\uparrow$
(b)	For $x=(-4, -3, -2, -1, 0, 1, 2, 3)$ supported Gauss reference function $ F(m) =(0.9368, 0.7347, 0.291, 0.062, 0.014, 0.062, 0.291, 0.7347)$ $\rho(\text{degree}) = (0, -180, 0, -180, 0, 180, 0, 180)$ $H(m) = (0.9996, 0.9854, 0, -0.9854, -0.9996, -0.9854, 0, 0.9854)$ $\uparrow$
(c)	For $x=(-5, -4, -3, -2, -1, 0, 1, 2)$ supported Gauss reference function $ F(m) =(0.9954, 0.7379, 0.2911, 0.0598, 0.0187, 0.0598, 0.2911, 0.7379)$ $\rho(\text{degree}) = (0, 135.24, -90.87, 48., 180., -48., 90.87, -135.24)$ $H(m) = (0.9993, 0.9869, 0, -0.9869, -0.9993, -0.9869, 0, 0.9869)$ $\uparrow$

The obtained wavelet filter, scaling function, their corresponding frequency domain magnitude and phase plots, and plot of PR control are given in Figures 3.1-3.3.

Linearly orthogonality is needed where 1-D FK does not work, i.e., for nonsymmetric functions and it provides the computational ease of using block convolutions in the system.

So, the main idea to use spectral factorization is to double the support of the circularly orthogonal functions and so obtain linearly orthogonal ones.

In 2-D, as factorization can be accomplished using only approximation methods [75], we cannot use them in the design of PRFB.

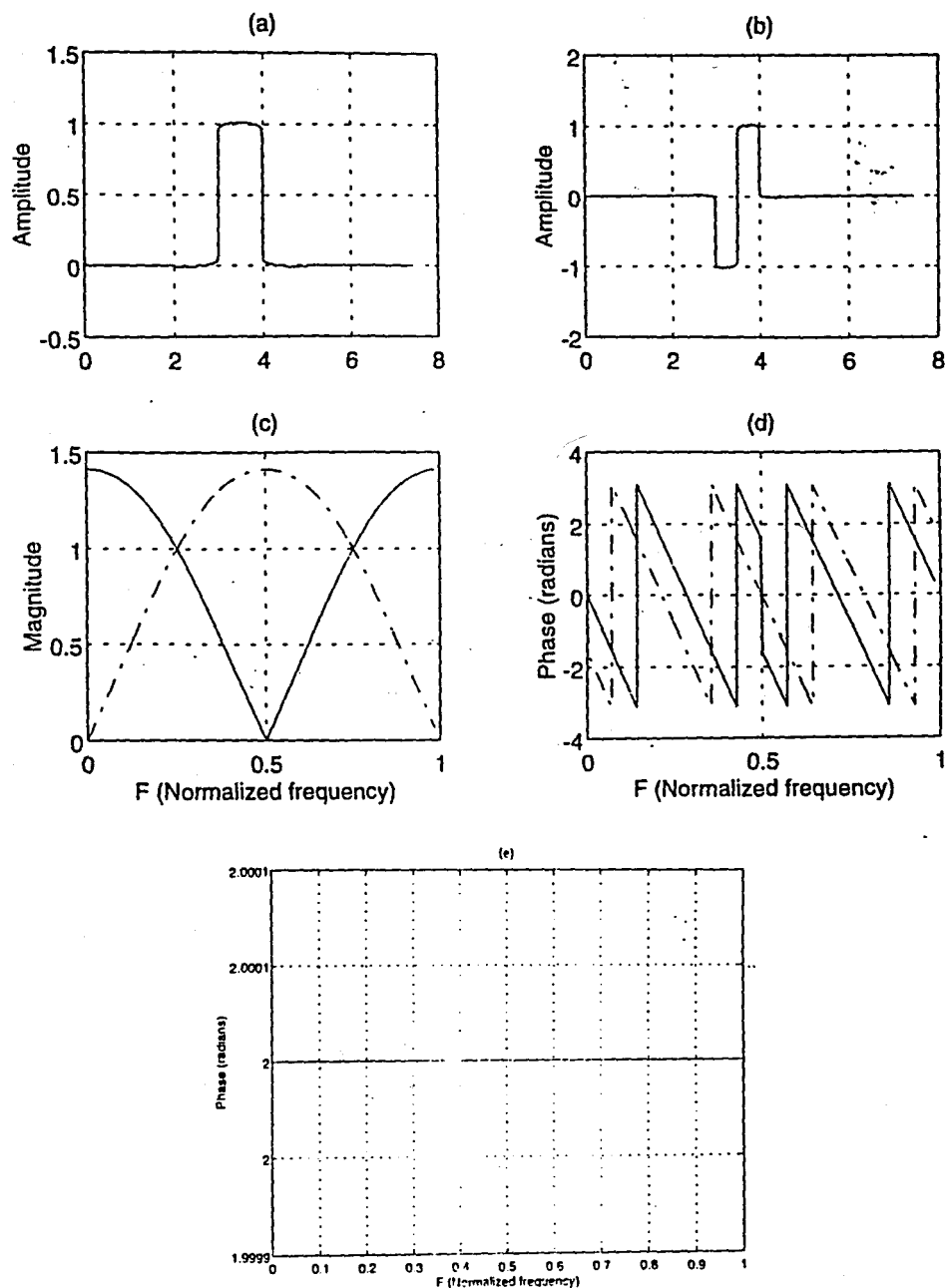


FIGURE 3.1. Sketch of the (a) scaling function, (b) wavelet filter, (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w+\pi)|^2$ for the $(-7 \leq 2n \leq 7)$ supported Gauss reference function in 1-D FK using circular orthogonality based on frequency domain design method

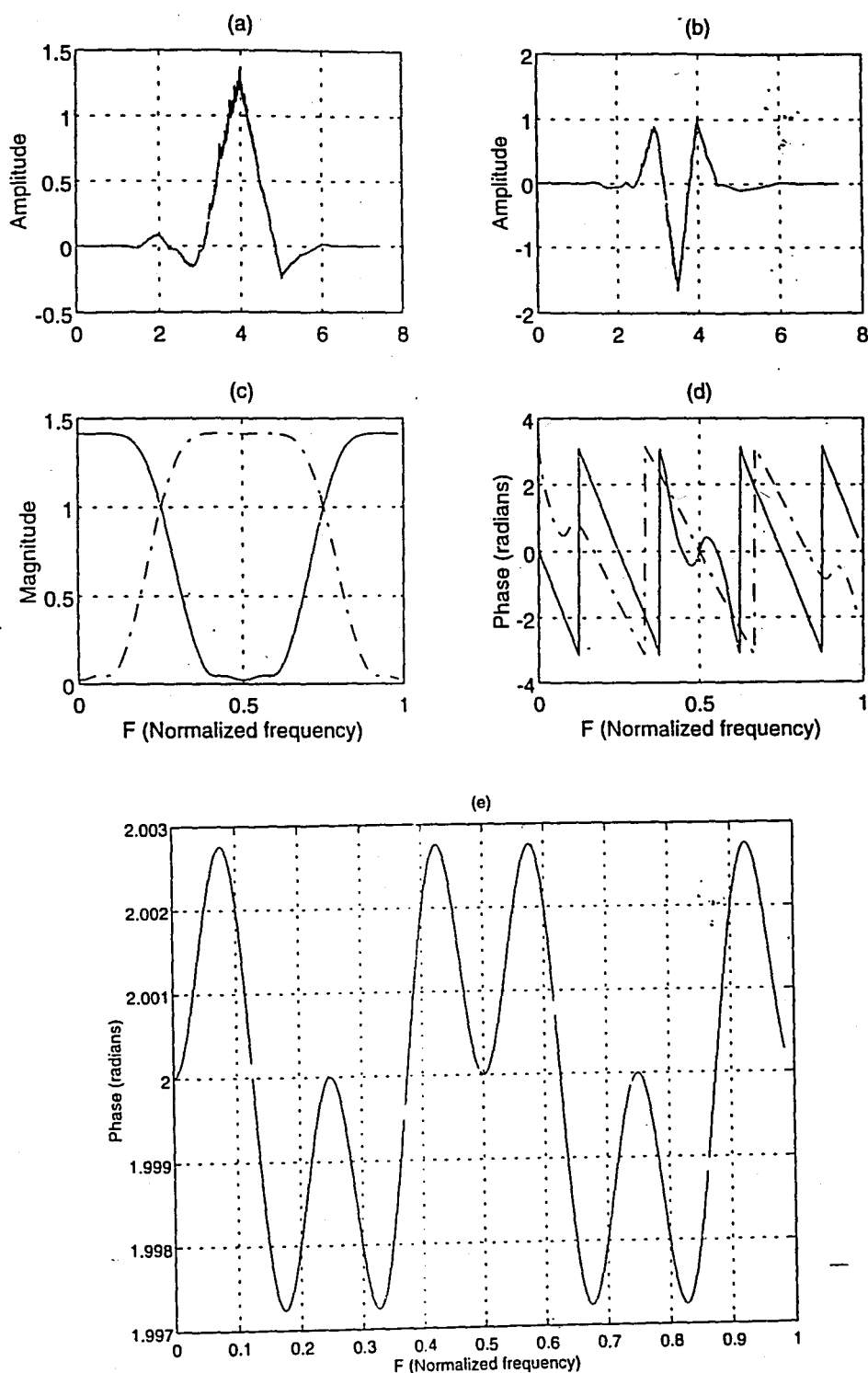


FIGURE 3.2. Sketch of the (a) scaling function, (b) wavelet filter, (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w+\pi)|^2$ for the $(-4 \leq n \leq 3)$ supported Gauss reference function in 1-D FK using circular orthogonality based on frequency domain design method

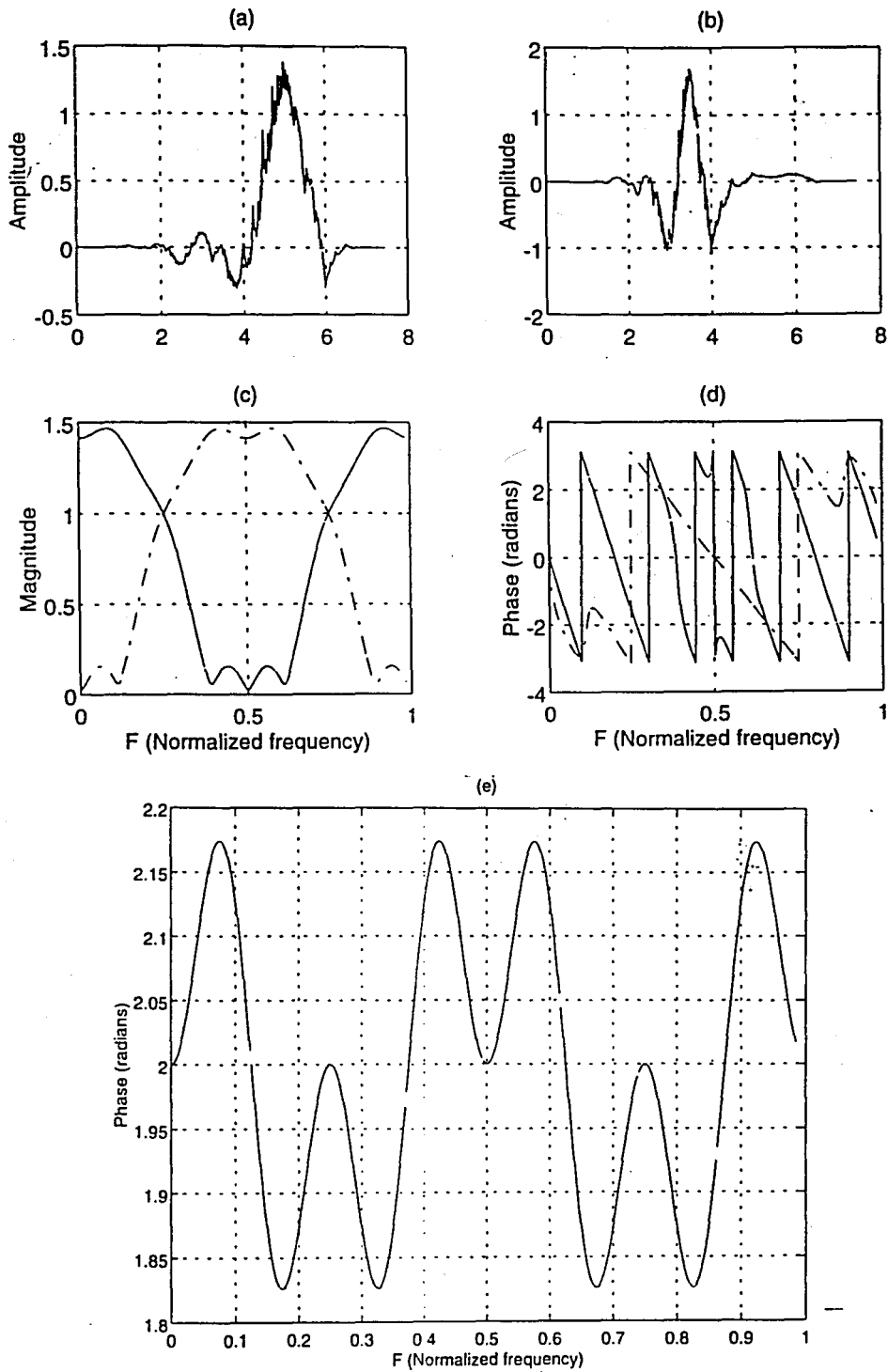


FIGURE 3.3. Sketch of the (a) scaling function, (b) wavelet filter , (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w+\pi)|^2$ for the $(-5 \leq n \leq 2)$ supported Gauss reference function in 1-D FK using circular orthogonality based on frequency domain design method

3.3.1. Spectral Factorization and Design of Linearly Orthogonal Filters

Since for 1-D FK wavelet filter design method, only symmetric reference functions can satisfy PR, for nonsymmetric reference functions we can improve the SNR by using spectral factorization.

We know from Equation 3.26 that

$$\psi(z) = \frac{\sqrt{2}F(z)}{\sqrt{|F(z)|^2 + |F(-z)|^2}}$$

and from [74] that,

$$|F(z)|^2 = 1 + H(z) \quad (3.29)$$

So we can write

$$\begin{aligned} F(z)F(z^{-1}) &= 1 + H(z) \\ Y(z)Y(z^{-1}) &= \frac{2F(z)F(z^{-1})}{F(z)F(z^{-1}) + F(-z)F(-z^{-1})} \end{aligned}$$

where $F(z)F(z^{-1}) = f(n)*f(-n)$ and $F(-z)F(-z^{-1}) = (-1)^n f(n)*(-1)^n f(-n)$. Once we have the nonsymmetric reference function $f(n)$ at hand, we implement the convolutions $f(n)*f(-n)$ and $(-1)^n f(n)*(-1)^n f(-n)$ to evaluate

$$H(m) = \frac{|F(m)|^2 - \left|F\left(m + \frac{N}{2}\right)\right|^2}{|F(m)|^2 + \left|F\left(m + \frac{N}{2}\right)\right|^2} \quad m \in \mathbb{Z}_N. \quad (3.30)$$

Then, we take inverse Fourier Transform (IFFT) of $H(k)$ and put $h(0)=1$ to obtain the centered (0,0 at the middle) autocorrelation sequence. The roots possessing minimum phase of the obtained polynomial from the autocorrelation sequence are chosen in forming a new polynomial which is then normalized.

One dimensional FK can give us linear phase circularly orthogonal wavelet filters. They are always close to two tap solutions of course (since there are no orthogonal and

from symmetric reference functions defines also the quality of the result in terms of PR.

The results obtained for 1-D circularly orthogonal FK are given in Figures 3.4.-3.5. The ratio of $\frac{SNR_{LO}}{SNR_{CO}} = 5.15$ ($SNR_{LO}=309$ dB and $SNR_{CO}=60$ dB) for $n \in \{-4, \dots, 3\}$ supported Gauss reference function indicates the improvement of SNR measure when using LO design for nonsymmetric reference functions.

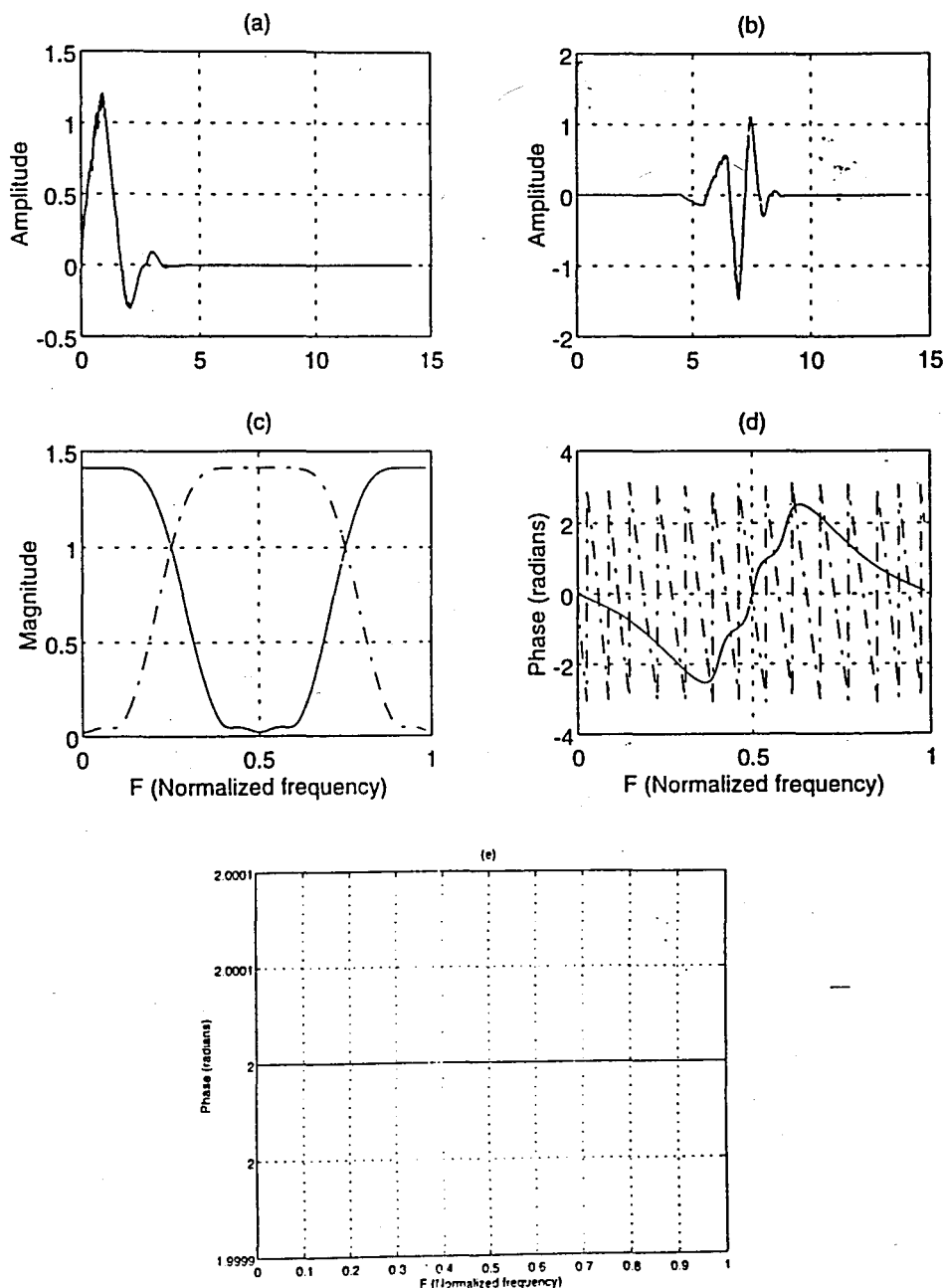


FIGURE 3.4. Sketch of the (a) scaling function, (b) wavelet filter , (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w + \pi)|^2$ for the $(-4 \leq n \leq 3)$ supported Gauss reference function which will result after linear orthogonalization by doing spectral factorization in 1-D FK

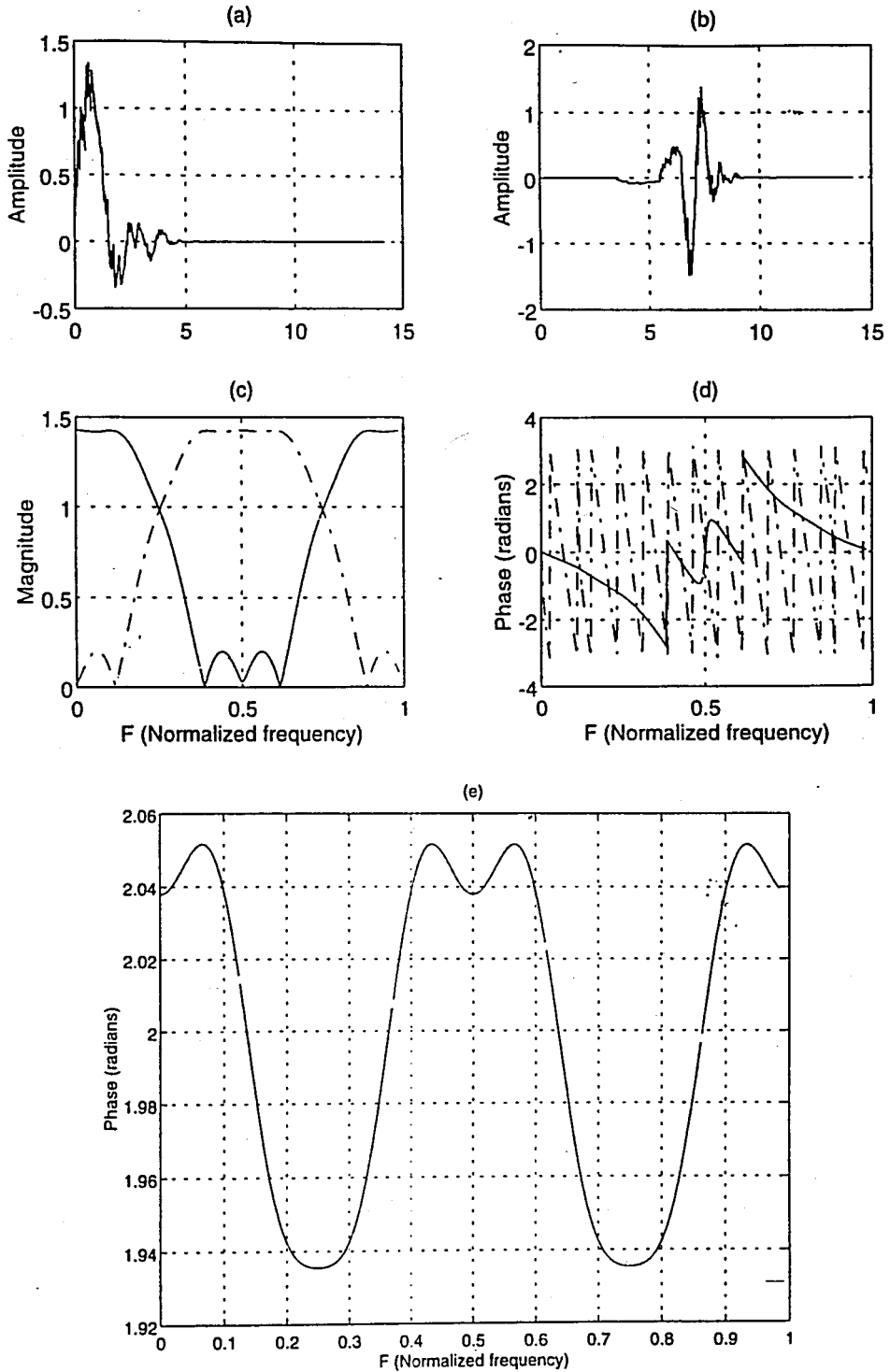


FIGURE 3.5. Sketch of the (a) scaling function, (b) wavelet filter, (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w + \pi)|^2$ for the $(-5 \leq n \leq 2)$ supported Gauss reference function which will result after linear orthogonalization by doing spectral factorization in 1-D FK

3.3.2. Time domain interpretation

An alternative waveform matched wavelet design technique has been formulated in the time domain such as to minimize

$$J = \|\psi(k) - f(k)\|^2 \quad (3.31)$$

under the constraints of unit norm and perfect reconstruction. The advantage of the time domain solution with respect to the Frazier-Kumar technique is that complicated inversion procedures are not needed and both phase and magnitude characteristics of the waveform are taken into account.

The resulting spectra and wavelets from the two design methods are shown in Figure 3.6 -3.7. Notice that the optimization scheme in Equation 3.31 achieves PR property, while the FK method does not, due to the inherent approximation.

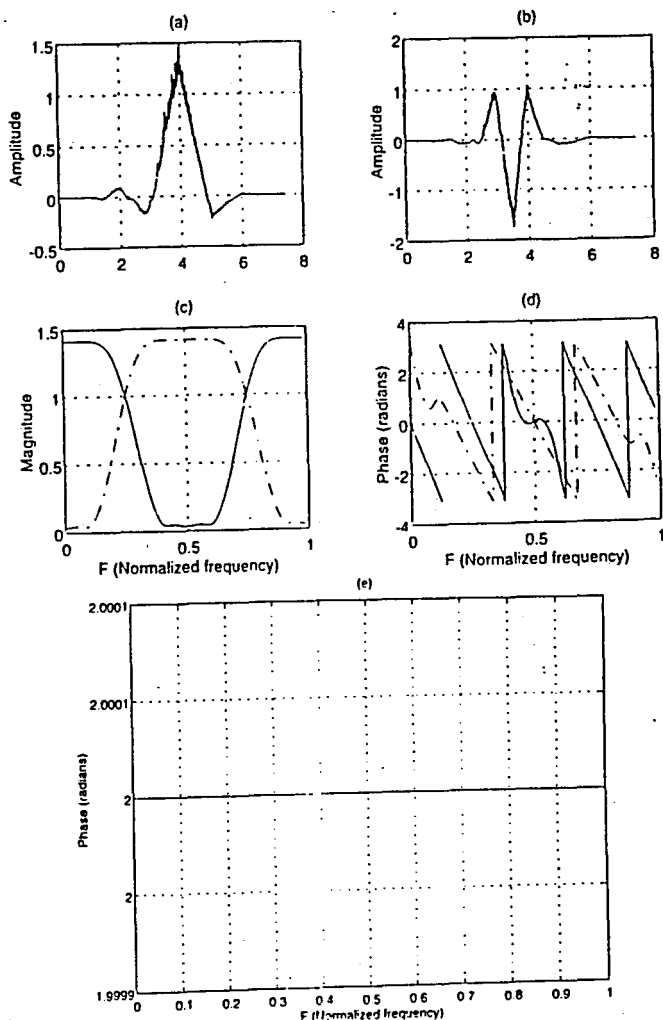


FIGURE 3.6. Sketch of the (a) scaling function, (b) wavelet filter, (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w+\pi)|^2$ in the $(-4 \leq n \leq 3)$ supported Gauss reference function for 1-D time domain method with the filter obtained after 228 iterations

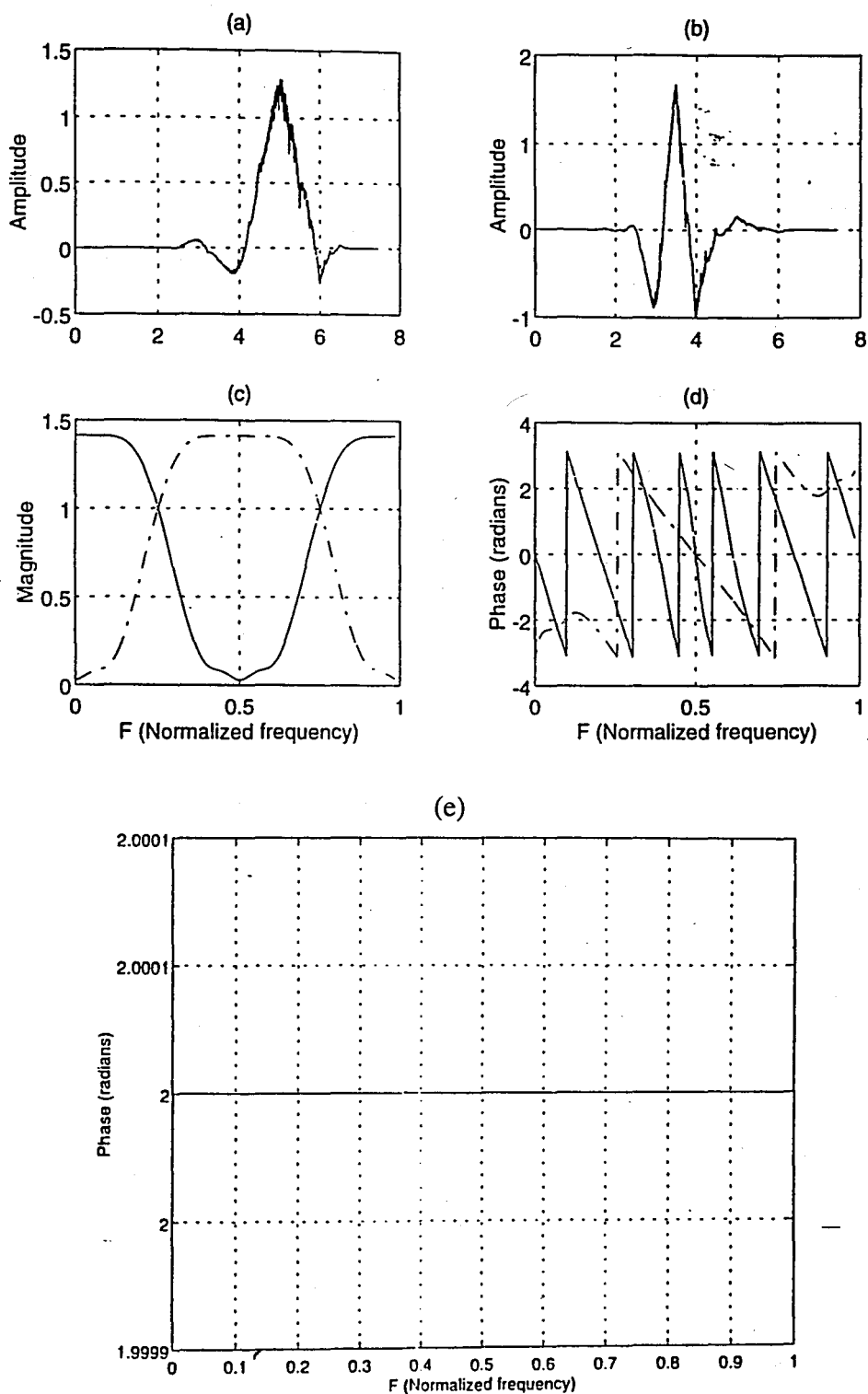


FIGURE 3.7. Sketch of the (a) scaling function, (b) wavelet filter, (c) corresponding magnitude function of (a), (b), then (d) corresponding phase function of (a), (b) and (e) control of PR by $|H(w)|^2 + |H(w+\pi)|^2$ in the $(-5 \leq n \leq 2)$ supported Gauss reference function for 1-D time domain method with the filter obtained after 264 iterations

3.3.3. Analysis of 1-D Frazier Kumar design in terms of energy compaction

The reference function $f(t)$, in the FK methodology can be selected in a number of different ways. If the aim is energy compaction, then $f(t)$ can be selected using the eigenfunctions of the process. In this section, the selection of a proper reference function to maximize the energy compaction is investigated as a contribution to the FK methodology.

It is suggested that the reference function can be chosen as a combination of the eigenspectra of a process. In other words the reference function in Equation 3.29 can be chosen as

$$F(\exp(jw)) = \sum_{k=1}^{N/2} a_k u_k(\exp(jw)), \quad (3.32a)$$

where the k^{th} eigenspectrum is given by

$$u_k(\exp(jw)) = \sum_{j=0}^{N-1} u_{kj} \exp(2\pi j/N), \quad (3.32b)$$

with u_k denoting an eigenvector of the $N \times N$ covariance matrix of a process. Since we deal with a two-band scheme, the reference function is constituted with the eigenspectra corresponding to the $N/2$ largest eigenvalues λ_i for $i=1, \dots, N/2$ (From Property 2). Finally, in Equation 3.32a $a(k)$'s denote the combiner coefficients. The choice of $a(k)$ s is done to maximize the coding gain of the N -tap 2-band filters. However after the FK mapping operation as given by Equation 3.26, there is no guarantee that the energy compaction performance will still be optimum. This second method can now be formulated as follows:

$$\max \int_{-p}^p \left| \sum a_k u_k(\exp(jw)) \right|^2 S_{xx}(\exp(jw)) dw \quad (3.33)$$

for the inequality constraints given in Table 3.2 as related to the number of taps N, and $S_{xx}(\exp(jw))$ is the PSD of the input signal.

TABLE 3.2. Constraint inequalities for different number of taps (4, 6, 8) of the reference function in the analysis of FK design method maximizing energy compaction

h	Constraint Inequalities
4-tap $(\alpha_1 > \alpha_2)$	$\alpha_1 + \alpha_2 = 1$ $\alpha_1 < (2\lambda_1 / \lambda_2) \alpha_2$
6-tap $(\alpha_1 > \alpha_2 > \alpha_3)$	$\alpha_1 + \alpha_2 + \alpha_3 = 1.5$ $\alpha_2 < (2\lambda_2 / \lambda_3) \alpha_3$ $\alpha_1 \equiv (\lambda_1 / \lambda_3) \alpha_3$ $\alpha_1 \equiv (\lambda_1 / \lambda_2) \alpha_2$
8-tap $(\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$ $\alpha_2 < (2\lambda_2 / \lambda_3) \alpha_3$ $\alpha_3 \equiv 2(\lambda_1 / \lambda_3) \alpha_4$ $\alpha_1 \equiv (\lambda_1 / \lambda_2) \alpha_2$

Energy compaction comparison of reference filters, $f(n)$, is summarized in Table 3.3. where in the first choice the weights corresponding to decreasing eigenvalues λ_i for $i=1,...,N/2$ in FK design method which maximizes the coding gain are taken into design whereas in the second choice only the first coefficient corresponding to the largest eigenvalue is taken into account. In this table we compare the performance of 4, 6, and 8 tap filters designed according to an AR(1) model found at the end of optimization given in Equation 3.32 a and b.

TABLE 3.3. Compaction performance of wavelets designed with the FK method for two choices of combiner coefficients

h	AR(1) $\rho=0.95$	α_1	α_2	α_3	α_4	choice 1	choice 2
						σ_{LP}^2 for α_i $i=1,..., N/2$	for $\alpha_1 \neq 0$, $\alpha_i = 0$ $i=2,..., N/2$
4-tap	0.95	0.98	0.023			1.9618	1.95
6-tap	0.95	1.33	0.1675	0.025		1.9646	1.9612
8-tap	0.95	1.8	0.16	0.03	0.01	1.9658	1.9645

An optimization method is thus provided to direct the selection of the reference function in maximizing energy compaction.

3.5. Conclusions

In this Chapter 3, design issues related to statistically and waveform matched wavelet design are discussed.

Firstly, energy compaction performance of two-band schemes vis-à-vis the Karhunen Loeve transform is investigated by the:

- Derivation of transform coding gain performance of $G_{\text{ideal},2\text{-band}}$ vs. $G_{\text{KLT},2}$, $G_{\text{KLT},4}$, $G_{\text{KLT},6}$ for AR(1) model by the Equation 3.10,
- Derivation of Lagrange coefficients bounds:

Given $G_{\text{KLT},2} < G_{\text{ideal},2\text{-band}} < G_{\text{KLT},4}$ in the Equation 3.10 and some ρ , we find bounds on Lagrange coefficients as in Equation 3.25.

Secondly, parameters tradeoffs in statistically matched wavelets is investigated:

- One dimensional Frazier Kumar matched wavelet filter design method uses circularly orthogonality of the filters in satisfying PR IIR and so must use always symmetric reference functions. In cases where we want to use nonsymmetric reference functions then we must recourse to the linearly orthogonality of filters to decrease the loss in PR. One dimensional linearly orthogonal matched wavelet filters are obtained using spectral factorization and implemented using block operations,
- For the FK technique the reference function can be obtained using the eigenvectors/eigenspectra of a process. Analysis of 1-D FK design method in terms of energy compaction criteria using input statistics to reach the optimal transform coding gain is done,
- A novel FIR matched wavelet design technique based on time domain approach has been advanced and compared with the so-called Frazier Kumar design technique in terms of flexibility and exactness.

In the next chapter, we will discuss 2-D time domain based design of PR-QMFs.

4 DESIGN OF TWO DIMENSIONAL NONSEPARABLE, PERFECT RECONSTRUCTION, LINEAR PHASE FILTER BANKS USING SIGNED SHUFFLING OPERATIONS

4.1 Introduction

A novel design method for two dimensional M -band nonseparable, linear phase, perfect reconstruction filter banks in the time domain [80-85] is given in this Chapter. The design is based on first finding a suitable lowpass filter, while the remaining filters in the filter bank are obtained through use of signed shuffling operations of that filter. This leads to a set of filters that use the same coefficients in different shift positions and with different signs, allowing efficient VLSI realization of the filter bank. Filter bank requirements such as orthogonality, regularity, as well as optimality criteria such as maximization of coding gain, are then all imposed on the coefficients of the lowpass section. The suggested method is general enough that it can be applied to filters on all the sampling lattices as given in Section 2.2.1.1. The only requirement for the proposed method is that the unit sample responses of the filters have centrosymmetry property which itself provides linearity of the phase of the filters.

The plan of Chapter 4 is as follows. We first explicate the desired properties of the filter banks. In section 4.2, shuffling based design for quadratic sampling case (see Section 2.2.1.1) is presented. It is shown that this method works for all filter sizes, in other words no additional constraint equations are needed other than the self-orthogonality equations of the lowpass filter, which suffice to provide unique filter bank solutions for the properties as nonseparability, linear phase and perfect reconstruction. The only limitation of this shuffling method is that it allows only partial control on the spectra of the other bands. This limitation is overcome in the implementations by iterating the tree at least three times so that the controllable bands (bands that we can control the spectra in the design) have very high variances when compared to other bands.

The extension of this quadratic sampling design case to M -band solutions is presented in section 4.2.2. The advantage of our method is that it provides a solution for the PRFB with NS filters. Nonseparable filters in turn can achieve band selection with fewer coefficients as compared to their separable realization.

Design examples for quadratic sampling cases are provided in Section 4.4. The results show that nonseparable filters found after optimization using energy compaction performance measure outperforms the separable filtering based designs by 50% in terms of transform coding gain. Other NSLPPRFB design cases form a subset of the proposed signed shuffling design method that they use only special case of symmetries. Thus a generalization of the nonseparable solutions in terms of symmetries is realized in this thesis. All the referred NSLPPRFB design methods [82-90] that provides some special solutions are given in the polyphase domain differing to the proposed one in this thesis given in the time domain.

4.1.1 Filter Bank Design Requirements

In this Section, we explicate some of the desired properties of the two-dimensional filter banks. These properties are orthonormality, nonseparability and linear phase of the filters.

4.1.1.1 Orthonormality

An important property of the paraunitary systems is the orthonormality property. A consequence of the paraunitary analysis/synthesis system is that the analysis and synthesis filters are simply time-reversed conjugate versions of each other, and in the FIR case they are of the same length. Since the filter bank calculates projections onto orthogonal subspaces a perfect reconstruction system is obtained when the synthesis filters are within shift-reversal the same as the analysis ones.

The self-orthonormality and cross-orthonormality conditions on any lattice can be written as

$$\mathbf{r}_{i,j}(\mathbf{Dn}) = \langle \mathbf{h}_i(\mathbf{m}), \mathbf{h}_j(\mathbf{m} + \mathbf{Dn}) \rangle = \delta_{i,j} \delta_{n_1} \delta_{n_2}$$

or more precisely

$$\sum_{m_1, m_2} \mathbf{h}_i(m_1 + d_{00}n_1, m_2 + d_{01}n_1 + d_{11}n_2) \mathbf{h}_j(m_1, m_2) = \delta_{i-j} \delta_{n_1} \delta_{n_2} \quad (4.1)$$

with $\mathbf{h}_i(\cdot)$ being the i^{th} filter's impulse response, $\mathbf{D}$ the sampling lattice, $\mathbf{n}^T = [n_1, n_2]$, $\mathbf{m}^T = [m_1, m_2]$ are integer vectors.

Let us use two different sampling lattice for 2-dimension, namely, quadratic and hexagonal ones. For the two-dimensional quadratic sampling case, the orthonormality equations become

$$\sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \mathbf{h}_i(k_1, k_2) \mathbf{h}_j(k_1 + 2n_1, k_2 + 2n_2) = \delta(n_1, n_2) \quad (4.2)$$

and Equation 4.2 can be best illustrated by means of filter masks as shown in Figure 4.1. The mask products can alternately be represented as a table of inner products for different shift of filter unit sample responses.

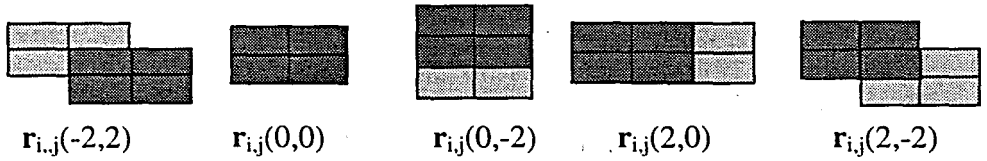


FIGURE 4.1. Mask product of two 4x4 (minimum size of filters for 4-band quadratic case) filters in 4-band quadratic sampling case to satisfy PR. Each subblock is 2x2.

The design procedure that will be explained in the next section is mainly based on using the Equation 4.1. The constraint equations for the specific example case of 4-band quadratic case are given in Table 4.1 using Figure 4.1.

TABLE 4.1. Constraint equations to be satisfied to guarantee PR for 4×4 sized filters in 4-band

quadratic case(for filter structure $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$)

$r_{ij}(.,.)$	-3	-2	-1	0	1	2	3
-3							
-2		$2(ch+dg)$		$2(ah+bg + cf+de)$		$2(ch+dg)$	
-1							
0		$2(ac+bd + eg+fh)$		$2(a^2+b^2+c^2 + d^2+e^2+f^2+ g^2+h^2)$		$2(ac+bd + eg+fh)$	
1							
2		$2(af+be)$		$2(ah+bg + cf+de)$		$2(af+be)$	
3							

For two dimensional hexagonal sampling case, the self-orthonormality equations become

$$\sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} h(k_1, k_2) h(k_1 + 2n_1, k_2 + n_1 + 2n_2) = \delta(n_1, n_2) \quad (4.3)$$

and Equation 4.3 can be best illustrated by means of filter masks shown in Figure 4.2.

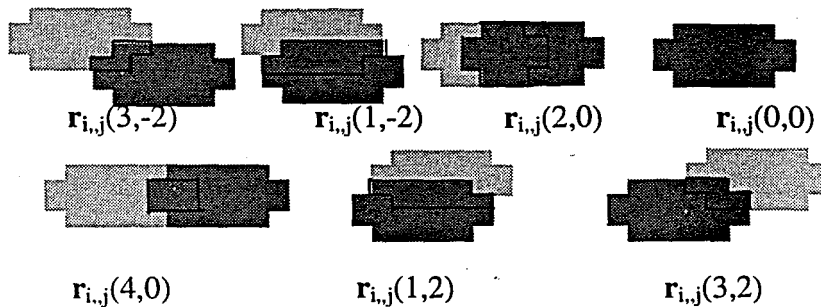


FIGURE 4.2. Mask product of two 4×6 (minimum size of filters for 4-band hexagonal case most resembling a hexagon) filters in 4-band hexagonal sampling case to satisfy PR

4.1.1.2. Nonseparability

A p -dimensional, even sized separable filter for M -band was expressed in time domain as,

$$h_i(n_1, \dots, n_p) = h^{(1)}(n_1) \circ \dots \circ h^{(p)}(n_p) \quad i = 1, \dots, M$$

Let us give the example of a $N \times N$ separable 2-D filters formed by using the outer product of a 1-D filter and its transpose:

$$\begin{bmatrix} a \\ b \\ c \\ d \\ \vdots \\ k \end{bmatrix} \begin{bmatrix} a & b & c & d & \dots & k \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac & ad & \dots & ak \\ ba & b^2 & bc & bd & \dots & bk \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ ka & kb & kc & \dots & \dots & k^2 \end{bmatrix} = h_0(n_1, n_2)$$

On the other hand take a general $N \times N$ centrosymmetric matrix and test its nonseparability applying it on the above separable matrix model, by equating one by one their respective terms.

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta & . & . & . & \chi \\ \varepsilon & \varphi & \theta & \rho & . & . & . & \xi \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \xi & . & . & . & \rho & \theta & \varphi & \xi \\ \chi & . & . & . & \delta & \gamma & \beta & \alpha \end{bmatrix}$$

The term by term correspondence yields the constraints below :

$$\alpha = a^2, \beta = ab, \gamma = ac, \delta = ad, \dots, \chi = ak$$

and for the last term of the last row $\alpha = k^2$ thus $\chi = ak = \sqrt{\alpha} \cdot \sqrt{\alpha} = \alpha$.

Hence, the last term of the arbitrary chosen first row, becomes equal to its first term, destroying in this way its independence and generality. This result being contrary to the initial assumption, proves that the considered last matrix does not coincide with the separable model and therefore it is nonseparable. We can now deduce, that a general centrosymmetric matrix is always nonseparable. This property may also be proved in a different way :

Continuing the correspondence for the following rows, we can write :

$$\begin{aligned} \xi &= ba = \beta = \left(\frac{\beta}{\alpha}\right) \cdot \alpha; \varphi = b^2 = \frac{\beta^2}{a^2} = \frac{\beta^2}{\alpha} = \left(\frac{\beta}{\alpha}\right) \cdot \beta \\ \theta &= bc = \frac{\beta}{a} \cdot \frac{\gamma}{a} = \left(\frac{\beta}{\alpha}\right) \cdot \gamma; \rho = bd = \frac{\beta}{a} \cdot \frac{\delta}{a} = \left(\frac{\beta}{\alpha}\right) \cdot \delta \\ \dots\dots\dots \xi &= bk = \frac{\beta}{a} \cdot \frac{\chi}{a} = \left(\frac{\beta}{\alpha}\right) \cdot \chi \end{aligned}$$

Thus the matrix becomes :

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta & \dots & \chi \\ \left(\frac{\beta}{\alpha}\right) \cdot \alpha & \left(\frac{\beta}{\alpha}\right) \cdot \beta & \left(\frac{\beta}{\alpha}\right) \cdot \gamma & \left(\frac{\beta}{\alpha}\right) \cdot \delta & \dots & \left(\frac{\beta}{\alpha}\right) \cdot \chi \\ \left(\frac{\gamma}{\alpha}\right) \cdot \alpha & \left(\frac{\gamma}{\alpha}\right) \cdot \beta & \left(\frac{\gamma}{\alpha}\right) \cdot \gamma & \left(\frac{\gamma}{\alpha}\right) \cdot \delta & \dots & \left(\frac{\gamma}{\alpha}\right) \cdot \chi \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \left(\frac{\chi}{\alpha}\right) \cdot \alpha & \vdots & \vdots & \vdots & \dots & \left(\frac{\chi}{\alpha}\right) \cdot \chi \end{bmatrix}$$

The smallest row is equal to the first one multiplied by the factor $\left(\frac{\beta}{\alpha}\right)$ and the last row should be equal to the first one multiplied by the factor $\left(\frac{\chi}{\alpha}\right)$. The last term of the last row becomes $\left(\frac{\chi}{\alpha}\right) \cdot \chi$, which should be equal to α , but $\left(\frac{\chi}{\alpha}\right) \cdot \chi \neq \alpha$, because generally $\alpha \neq \chi$, thus the rank rule, stating that all separable filters have rank one, i.e. all rows are linear combination of each other does not hold anymore in the half lower part of the matrix, that is, the general filter matrix is non separable. Consequently the above deduction is once again stressed.

The frequency response requirement being in contradiction with the short kernel requirement, the nonseparability of the filters provide us better designs in frequency domain of filters when compared to separable filtering cases.

4.1.1.3 Linear Phase

The linear phase used in the proposed design method is based on the following centrosymmetry :

$$h(n_1, n_2) = \pm h(N-1-n_1, N-1-n_2)$$

with $0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1$

The symmetric/antisymmetric two dimensional sequences can be divided into four major types as [86],

Type 1. Even symmetry in both the n_1 and n_2 directions,i.e.,

$$\begin{aligned} h(n_1, n_2) &= h(N-1-n_1, n_2) \\ &= h(n_1, N-1-n_2) \end{aligned} \quad 0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1$$

Type 2. Even symmetry in the n_1 direction and odd symmetry in the n_2 direction,i.e.,

$$\begin{aligned} h(n_1, n_2) &= h(N-1-n_1, n_2) \\ &= -h(n_1, N-1-n_2) \end{aligned} \quad 0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1$$

Type 3. Odd symmetry in the n_1 direction and even symmetry n_2 directions,i.e.,

$$\begin{aligned} h(n_1, n_2) &= -h(N-1-n_1, n_2) \\ &= h(n_1, N-1-n_2) \end{aligned} \quad 0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1$$

Type 4. Odd symmetry in both the n_1 and n_2 directions,i.e.,

$$\begin{aligned} h(n_1, n_2) &= -h(N-1-n_1, n_2) \\ &= -h(n_1, N-1-n_2) \end{aligned} \quad 0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1$$

Since the centro-even symmetric 2-dimensional sequence being combination of type 1 and type 4 symmetries and centro-odd symmetric 2-dimensional sequence being combination of type 2 and type 3 symmetries, the proposed method should naturally work for the types 1, 2, 3, and 4 in particular.

The advantages of using linear phase in PRFBs was discussed in Section 2.3.4.2. When compared to the 1-dimensional case the linear phase requirement is less constrained since linear phase in multiple dimensions just means centrosymmetry of the filters's impulse responses[87].

The types of symmetry used for the prototype filter h_0 is given in Table 4.2.

TABLE 4.2 .Examples of types of symmetries for 4×4 sizes of filters and corresponding number of free variables in the general $N \times N$ filter size case

Centrosymmetry (type1+type4 or type2+type3) (Number of free variables: $\frac{N^2}{2}$)	$\begin{matrix} & n_2 \\ & \left[\begin{array}{cc cc} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{array} \right] \\ n_1 \end{matrix}$ Examples to: type1 + type4 $\left[\begin{array}{cc cc} 5 & 5 & 1 & -1 \\ 5 & -1 & -9 & 3 \\ 3 & -9 & -1 & 5 \\ -1 & 1 & 5 & 5 \end{array} \right] = \left[\begin{array}{cc cc} 2 & 3 & 3 & 2 \\ 4 & -5 & -5 & 4 \\ 4 & -5 & -5 & 4 \\ 2 & 3 & 3 & 2 \end{array} \right] + \left[\begin{array}{cc cc} 3 & 2 & -2 & -3 \\ 1 & 4 & -4 & -1 \\ -1 & -4 & 4 & 1 \\ -3 & -2 & 2 & 3 \end{array} \right]$ type2 + type3 $\left[\begin{array}{cc cc} 3 & -5 & 7 & 1 \\ -2 & 7 & -1 & -8 \\ 8 & 1 & -7 & 2 \\ -1 & -7 & 5 & -3 \end{array} \right] = \left[\begin{array}{cc cc} 1 & -6 & 6 & -1 \\ 3 & 4 & -4 & -3 \\ 3 & 4 & -4 & -3 \\ 1 & -6 & 6 & -1 \end{array} \right] + \left[\begin{array}{cc cc} 2 & 1 & 1 & 2 \\ -5 & 3 & 3 & -5 \\ 5 & -3 & -3 & 5 \\ -2 & -1 & -1 & -2 \end{array} \right]$
Full horizontal - Full vertical symmetry (type 1, type 2, type 3, type 4) (Number of free variables: $\frac{N^2}{4}$)	$\left[\begin{array}{cc cc} a & b & b & a \\ c & d & d & c \\ c & d & d & c \\ a & b & b & a \end{array} \right]$ Examples to: type 1 type 2 $\left[\begin{array}{cc cc} 2 & 3 & 3 & 2 \\ 4 & -5 & -5 & 4 \\ 4 & -5 & -5 & 4 \\ 2 & 3 & 3 & 2 \end{array} \right] \left[\begin{array}{cc cc} 1 & -6 & 6 & -1 \\ 3 & 4 & -4 & -3 \\ 3 & 4 & -4 & -3 \\ 1 & -6 & 6 & -1 \end{array} \right]$ type 3 type 4 $\left[\begin{array}{cc cc} 2 & 1 & 1 & 2 \\ -5 & 3 & 3 & -5 \\ 5 & -3 & -3 & 5 \\ -2 & -1 & -1 & -2 \end{array} \right] \left[\begin{array}{cc cc} 3 & 2 & -2 & -3 \\ 1 & 4 & -4 & -1 \\ -1 & -4 & 4 & 1 \\ -3 & -2 & 2 & 3 \end{array} \right]$

4.2. Shuffling Based Design Technique in 2-Dimensions

The proposed design method is based on signed shuffling operations to realize two dimensional nonseparable, linear phase, perfect reconstruction filter banks. As we will see, the signed shuffling operations satisfying the orthogonality conditions necessitate linear phase structure of all the impulse responses in the filter bank. The M band design method first finds a suitable lowpass filter, while the remaining M-1 filters are obtained through use of signed shuffling operations on the coefficients of that filter. In other words the shuffling operations are sufficient to guarantee the orthogonality between the bands. Centrosymmetry imposes least number of constraints on the filter coefficients as compared to (Table 4.2) full symmetry. This in turn allows more freedom for optimal design. These centrosymmetric filters are nonseparable without having to impose any additional constraint equations (see Section 4.1.1.2).

A time-domain proof that signed permutations do indeed provide orthogonality is given in Section 4.2.1.1 for the 4×4 filter case. The generalization to the $N \times N$ dimensional case is given in section 4.2.1.2. A frequency domain interpretation of these filters is combined in Section 4.2.1.3. The regularity conditions for the shuffling-based filters is expressed in the Z- domain in Section 4.2.1.4. Various other details such as comparison of the number of unknowns, cross and self orthogonality constraints are given in Section 4.2.1.5.

Shuffling based design techniques for two dimensional M band cases are provided in section 4.2.2. Several design results and performance comparisons for the proposed design method and vis-à-vis existing ones are given in Section 4.3.

4.2.1. Shuffling Based Design Technique for 4-band 'Quadratic' Sampling

A method to design 2-D nonseparable, 4-band linear phase, Perfect Reconstruction (PR) filter banks can be based on signed shuffling operations. Besides providing a viable design tool, this method serves to reduce significantly the number of unknowns, since

the coefficients of the bandpass filters are the same as those of the lowpass filter. Thus rather than having $4N^2$ coefficients in the most general case, one ends up with only N^2 coefficients, furthermore if linear phase is desired, then the impulse response of the lowpass filter must be centrosymmetric, which reduces the variables by another factor of 2. Thus rather than having $4N^2$ coefficients in the most general case, one ends up with only $N^2 / 2$ coefficients. This reduction of the variables by a factor of 8 alleviates the design by optimization search.

4.2.1.1 Time Domain Permutation-Elimination Design Method

The design method will be explained step by step by first forming the design for the smallest sized filter and then generalizing it to all sizes.

The impulse responses of the filters are of the form,

$$h_i = \begin{bmatrix} h_i(0,0) & h_i(0,1) & . & . & . & h_i(0, N-1) \\ h_i(1,0) & h_i(1,1) & . & . & . & h_i(1, N-1) \\ . & . & . & . & . & . \\ h_i\left(\frac{N}{2}, 0\right) & h_i\left(\frac{N}{2}, 1\right) & . & . & . & h_i\left(\frac{N}{2}, N-1\right) \\ h_i\left(\frac{N}{2}, N-1\right) & . & . & . & h_i\left(\frac{N}{2}, 1\right) & h_i\left(\frac{N}{2}, 0\right) \\ . & . & . & . & . & . \\ h_i(1, N-1) & . & . & . & h_i(1,1) & h_i(1,0) \\ h_i(0, N-1) & . & . & . & h_i(0,1) & h_i(0,0) \end{bmatrix} \quad (4.5)$$

Lemma. $x, y \in \mathbb{R}^2$ being a row vector, the kernels $q_i(x)$ with $i = 0,1$ can be written as

$$q_0(x) = xI, \quad q_1(x) = xI_2, \quad r_0(x) = xI_3, \quad r_1(x) = xJ$$

using bases of the form

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In this case, the solution can be given in terms of blocks as

$$X_{k_1, k_2}^{(0)} = \begin{cases} \begin{bmatrix} x_{k_1, k_2} \\ y_{k_1, k_2} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < T/2 \\ 0 \leq k_2 < T \end{matrix} \\ JX_{T-1-k_1, T-1-k_2}^{(i)} J & \begin{matrix} T/2 \leq k_1 < T \\ 0 \leq k_2 < T \end{matrix} \end{cases} \quad (4.6)$$

and

$$X_{k_1, k_2}^{(i)} = \begin{cases} \begin{bmatrix} q_i(x_{k_1, k_2}) \\ q_i(y_{k_1, k_2} J) J \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 0, 1 \\ JX_{N-1-k_1, N-1-k_2}^{(i)} J & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 0, 1 \\ \begin{bmatrix} r_{i-2}(x_{k_1, k_2}) \\ -r_{i-2}(y_{k_1, k_2} J) J \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 2, 3 \\ -JX_{N-1-k_1, N-1-k_2}^{(i)} J & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 2, 3 \end{cases} \quad (4.7)$$

$$\text{or } \mathbf{X}_{k_1, k_2}^{(i)} = \begin{cases} \begin{bmatrix} \mathbf{x}_{k_1, k_2} \\ \mathbf{y}_{k_1, k_2} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < T/2 \\ 0 \leq k_2 < T \end{matrix} & i=0 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{I}_1 \\ -\mathbf{y}_{k_1, k_2} \mathbf{J} \mathbf{I}_1 \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < T/2 \\ 0 \leq k_2 < T \end{matrix} & i=1 \\ \mathbf{J} \mathbf{X}_{T-1-k_1, T-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} T/2 \leq k_1 < T \\ 0 \leq k_2 < T \end{matrix} & i=0,1 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{J} \\ -\mathbf{y}_{k_1, k_2} \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < T/2 \\ 0 \leq k_2 < T \end{matrix} & i=2 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{I}_2 \\ \mathbf{y}_{k_1, k_2} \mathbf{J} \mathbf{I}_2 \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < T/2 \\ 0 \leq k_2 < T \end{matrix} & i=3 \\ -\mathbf{J} \mathbf{X}_{T-1-k_1, T-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} T/2 \leq k_1 < T \\ 0 \leq k_2 < T \end{matrix} & i=2,3 \end{cases} \quad (4.8)$$

Proof :

The proof will be based on the time domain orthonormality conditions in designing perfect reconstruction filter banks as given in Equation 4.2.

We can partition (the procedure will be explained in the next section) the $\mathbf{h}_i$ filters in the

filter bank for $2 \times T = N$ and $0 \leq k_1, k_2 \leq T-1$ as

$$\mathbf{h}_i = \left[\left(\mathbf{X}_{k_1, k_2}^{(i)} \right)_{2 \times 2} \right]_{T \times T} \text{ for}$$

$$\mathbf{X}_{k_1, k_2}^{(i)} = \begin{bmatrix} h_i(2k_1, 2k_2) & h_i(2k_1, 2k_2 + 1) \\ h_i(2k_1 + 1, 2k_2) & h_i(2k_1 + 1, 2k_2 + 1) \end{bmatrix}$$

Step 1 : For $N=4$, the lowpass filter has the form of,

$$\mathbf{h}_0 = \begin{bmatrix} h_0(0,0) & h_0(0,1) & h_0(0,2) & h_0(0,3) \\ h_0(1,0) & h_0(1,1) & h_0(1,2) & h_0(1,3) \\ h_0(1,3) & h_0(1,2) & h_0(1,1) & h_0(1,0) \\ h_0(0,3) & h_0(0,2) & h_0(0,1) & h_0(0,0) \end{bmatrix}$$

or for ease of notation,

$$\mathbf{h}_0 = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$$

To impose all of the PR constraints on the lowpass filter's coefficients, we have to find structures for the other three filters in 4-band case. So, let us search for combinations of coefficients in the filter structures that will satisfy PR condition.:

Let us begin by the first subblock $X_{0,0}^{(0)}$, i.e.

$$X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$$

with the coefficient set $\{a,b,e,f\}$ composed of $n=4$ elements, such that without any requirements imposed as PR and LP, there are $n!-1=4!-12=3$ other possible placements of $\{a,b,e,f\}$ in a 2×2 matrix format. The possible structures which will be used in PRFB will be chosen from this set of placements after imposing orthonormality requirement on the filters. Grouping them according to their first elements, we obtain,

$$\begin{aligned} \begin{bmatrix} a. \\ .. \end{bmatrix} &\rightarrow \begin{bmatrix} ab \\ ef \end{bmatrix}, \begin{bmatrix} ab \\ fe \end{bmatrix}, \begin{bmatrix} ae \\ bf \end{bmatrix}, \begin{bmatrix} ab \\ fb \end{bmatrix}, \begin{bmatrix} af \\ be \end{bmatrix}, \begin{bmatrix} af \\ eb \end{bmatrix} \\ \begin{bmatrix} b. \\ .. \end{bmatrix} &\rightarrow \begin{bmatrix} ba \\ ef \end{bmatrix}, \begin{bmatrix} ba \\ fe \end{bmatrix}, \begin{bmatrix} be \\ af \end{bmatrix}, \begin{bmatrix} be \\ fa \end{bmatrix}, \begin{bmatrix} bf \\ ae \end{bmatrix}, \begin{bmatrix} bf \\ ea \end{bmatrix} \\ \begin{bmatrix} e. \\ .. \end{bmatrix} &\rightarrow \begin{bmatrix} ea \\ bf \end{bmatrix}, \begin{bmatrix} ea \\ fb \end{bmatrix}, \begin{bmatrix} eb \\ af \end{bmatrix}, \begin{bmatrix} eb \\ fa \end{bmatrix}, \begin{bmatrix} ef \\ ab \end{bmatrix}, \begin{bmatrix} ef \\ ba \end{bmatrix} \\ \begin{bmatrix} f. \\ .. \end{bmatrix} &\rightarrow \begin{bmatrix} fa \\ be \end{bmatrix}, \begin{bmatrix} fa \\ eb \end{bmatrix}, \begin{bmatrix} fb \\ ae \end{bmatrix}, \begin{bmatrix} fb \\ ea \end{bmatrix}, \begin{bmatrix} fe \\ ba \end{bmatrix}, \begin{bmatrix} fe \\ ab \end{bmatrix} \end{aligned}$$

When one begins with the first $X_{0,0}^{(0)}$ block and has the goal of satisfying PR, the other three filter's $X_{1,1}^{(0)}$ last blocks (satisfying $r_{0j}(2,2)$ in Equation 4.2) can be obtained at the end of three steps of statements:

(Statement 1) Started with a special initial block, other $X_{1,1}^{(0)}$ blocks cannot have the first element of the initial block in their first place because that this will cause a squared term in constraint equations which cannot satisfy PR i.e. Equation 4.2.

This statement 1 implies that for an initial block in h_0 , one will have only three other possible configurations which begin with other than “a” coefficient at their first places. For example, with the first block $X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ the

$$\begin{bmatrix} a & . \\ . & . \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ f & e \end{bmatrix}, \begin{bmatrix} a & e \\ b & f \end{bmatrix}, \begin{bmatrix} a & b \\ f & b \end{bmatrix}, \begin{bmatrix} a & f \\ b & e \end{bmatrix}, \begin{bmatrix} a & f \\ e & b \end{bmatrix}$$

$(n-1)!-1=(4-1)!-1=5$ blocks are cancelled.

(Statement 2) Once $X_{0,0}^{(0)}$ and its first element is fixed, the number of $X_{1,1}^{(0)}$ blocks having any element other than the first one in its original place, should be cancelled. This corresponds to cancel,

$$(n-1) \left[(n-1)! - \left\{ \binom{n}{1}(n-2)! - \binom{n-1}{2}(n-3)! + \dots \right\} \right]$$

or

$$(n-1) \left[(n-1)! - \left\{ \sum_{i=1}^{n-2} (-1)^{(i+1)} \binom{n-1}{i} (n-1-i)! \right\} \right]$$

blocks.

Referring to $X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ with the first element “a”, the other blocks should not have $\{b, e, f\}$ coefficients in $\begin{bmatrix} . & b \\ e & f \end{bmatrix}$ places, so that 9 blocks

$$\begin{aligned}
\begin{bmatrix} b & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} b & a \\ e & f \end{bmatrix}, \begin{bmatrix} b & e \\ a & f \end{bmatrix}, \begin{bmatrix} b & f \\ e & a \end{bmatrix} \\
\begin{bmatrix} e & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} e & a \\ b & f \end{bmatrix}, \begin{bmatrix} e & b \\ a & f \end{bmatrix}, \begin{bmatrix} e & b \\ f & a \end{bmatrix} \\
\begin{bmatrix} f & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} f & a \\ e & b \end{bmatrix}, \begin{bmatrix} f & b \\ a & e \end{bmatrix}, \begin{bmatrix} f & b \\ e & a \end{bmatrix}
\end{aligned}$$

will be cancelled.

(Statement 3) For the first places of the $X_{1,1}^{(0)}$ blocks fixed to one of $n-1$ elements (i.e. one of $\{b,e,f\}$ other than $\{a\}$), one will have a fixed term $\{a\}$ times one of the $n-1$ coefficients when satisfying PR for $r_{0,j}(2,2)$ $j=1,2,3$. This implies that when satisfying PR by using sign operations, there must be another term equal to term $\{a\}$ times one of the coefficients from the set which can be obtained by placing $\{a\}$ in places of $\{b,e,f\}$ for $X_{1,1}^{(0)}$ blocks beginning with $\{b,e,f\}$ respectively. This implies that in general,

$$(n-1)(n-2)(n-3)! = (n-1)!$$

should be cancelled.

For $X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$,

$$\begin{aligned}
\begin{bmatrix} b & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} b & e \\ f & a \end{bmatrix}, \begin{bmatrix} b & f \\ a & e \end{bmatrix} \\
\begin{bmatrix} e & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} e & a \\ f & b \end{bmatrix}, \begin{bmatrix} e & f \\ b & a \end{bmatrix} \\
\begin{bmatrix} f & . \\ . & . \end{bmatrix} &\dashrightarrow \begin{bmatrix} f & a \\ b & e \end{bmatrix}, \begin{bmatrix} f & e \\ a & b \end{bmatrix}
\end{aligned}$$

6 other combinations are eliminated by using (Statement 3).

So, for $X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$, first subblock in h_0 , one can have only

$$\begin{bmatrix} b & a \\ f & e \end{bmatrix}, \begin{bmatrix} e & f \\ a & b \end{bmatrix}, \begin{bmatrix} f & e \\ b & a \end{bmatrix}$$

as their last block $X_{1,1}^{(0)}$ for the other three filters.

Processing in the same way for $r_{0,i}(2,-2)$ and using now $X_{0,1}^{(0)} = \begin{bmatrix} c & d \\ g & h \end{bmatrix}$ one can obtain, possible centrosymmetric structures as,

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix} \begin{bmatrix} b & a & d & c \\ f & e & h & g \\ g & h & e & f \\ c & d & a & b \end{bmatrix} \begin{bmatrix} e & f & g & h \\ a & b & c & d \\ d & c & b & a \\ h & g & f & e \end{bmatrix} \quad (4.9a)$$

A structure as,

$$\begin{bmatrix} f & e & h & g \\ b & a & d & c \\ c & d & a & b \\ g & h & e & f \end{bmatrix} \quad (4.9b)$$

cannot exist in the set since it will introduce additional constraint equations other than the selforthogonality constraints on the lowpass filter which cannot be cancelled to satisfy PR by applying sign operators on them. This is caused by the fact that the coefficients of the block $X_{0,0}^{(0)}$ of the lowpass filter and block $X_{1,1}^{(0)}$ of the other filters are same. The first block in the lowpass filter and the last block in the other filters cannot be the same for that the sum of squared terms does not come as constraint equations in PRFB design. This structure, on the other hand, will provide a possible structure for 3 dimensional NSLPPRFB design.

$$\text{For } X_{0,0}^0 = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \rightarrow X_{1,1}^0 = \left\{ \begin{bmatrix} f & e \\ b & a \end{bmatrix}, \begin{bmatrix} e & f \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ f & e \end{bmatrix} \right\}$$

three possibilities can be used when properly signing to obtain PR. So, let now find the possible sign operations to guarantee the orthornormality of the filter banks. The possible sign configuration for zero Schur product of the matrices in each case are listed out exhaustively below:

i) For $X_{0,0}^0 = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ and $X_{1,1}^0 = \begin{bmatrix} f & e \\ b & a \end{bmatrix}$; The possible combinations of $X_{1,1}^0$ is a set of signed blocks as

$$A_j = \{ A_1 = \begin{bmatrix} f & e \\ -b & -a \end{bmatrix}, A_2 = \begin{bmatrix} -f & e \\ -b & a \end{bmatrix}, A_3 = \begin{bmatrix} -f & -e \\ b & a \end{bmatrix}, A_4 = \begin{bmatrix} f & -e \\ b & -a \end{bmatrix} \} j=1,2,3,4$$

ii) For $X_{0,0}^0 = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ and $X_{1,1}^0 = \begin{bmatrix} e & f \\ a & b \end{bmatrix}$,

$$C_j = \{ C_1 = \begin{bmatrix} e & -f \\ -a & b \end{bmatrix}, C_2 = \begin{bmatrix} -e & -f \\ a & b \end{bmatrix}, C_3 = \begin{bmatrix} -e & f \\ a & -b \end{bmatrix}, C_4 = \begin{bmatrix} e & f \\ -a & -b \end{bmatrix} \} j=1,2,3,4$$

iii) For $X_{0,0}^0 = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ and $X_{1,1}^0 = \begin{bmatrix} b & a \\ f & e \end{bmatrix}$,

The sets $\mathbf{C}_j$ and $\mathbf{D}_j$ cannot be put together in the solution of the filter bank, because of the fact that the coefficients in blocks $X_{0,0}^{(0)}$ and $X_{0,0}^{(0)}$ being same additional constraint equations come in design.

From $r_{0,j}, j=1,2,3$ one has two possible sets in forming FB : $\mathbf{h}_0$ and $\{\mathbf{A}_j, \mathbf{C}_j, j=1,2,3,4\}$ or $\mathbf{h}_0$ and $\{\mathbf{A}_j, \mathbf{D}_j, j=1,2,3,4\}$.

For $r_{0j}(2,0)$, there are $[2 \times 4]$ (or for $r(0,2)$, $[4 \times 2]$) blocks consisted of two $[2 \times 2]$ subblocks (coming from $r_{0j}(2,2)$ or $r_{0j}(2,-2)$ constraints) that fixes signs of two elements in the $[2 \times 4]$, i.e.,

i) For $X_{0,0}^0 = \begin{bmatrix} a & b \\ e & f \end{bmatrix}$ and $X_{1,1}^0 = \begin{bmatrix} f & e \\ b & a \end{bmatrix}$, i.e., so for $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ we have

$$\begin{bmatrix} +h & +g \\ -d & -c \end{bmatrix} \begin{bmatrix} f & e \\ -b & -a \end{bmatrix} \begin{bmatrix} -h & +g \\ -d & +c \end{bmatrix} \begin{bmatrix} -f & e \\ -b & a \end{bmatrix} \begin{bmatrix} -h & -g \\ +d & +c \end{bmatrix} \begin{bmatrix} -f & -e \\ b & a \end{bmatrix} \begin{bmatrix} +h & -g \\ +d & -c \end{bmatrix} \begin{bmatrix} f & -e \\ b & -a \end{bmatrix}$$

or for

$$X_{0,0}^{(0)} = \begin{bmatrix} a & b \\ e & f \\ h & g \\ d & c \end{bmatrix} \text{ we have,}$$

$$\begin{bmatrix} +c & +d \\ -g & -h \\ f & e \\ -b & -a \end{bmatrix} \begin{bmatrix} -c & d \\ -g & h \\ -f & e \\ -b & a \end{bmatrix} \begin{bmatrix} -c & -d \\ g & h \\ -f & -e \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ g & -h \\ f & -e \\ b & -a \end{bmatrix}$$

All cases are formed for ii) and iii).

At the end, for

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$$

the selection for the other three filters are done from the possible set as

i) AAAj=

$$\{AAA1 = \begin{bmatrix} a & b & c & d \\ -e & -f & -g & -h \\ h & g & f & e \\ -d & -c & -b & -a \end{bmatrix}, AAA2 = \begin{bmatrix} a & -b & c & -d \\ e & -f & g & -h \\ h & -g & f & -e \\ d & -c & b & -a \end{bmatrix}, AAA3 = \begin{bmatrix} -a & -b & -c & -d \\ e & f & g & h \\ -h & -g & -f & -e \\ d & c & b & a \end{bmatrix}, AAA4 = \begin{bmatrix} -a & b & -c & d \\ -e & f & -g & h \\ -h & g & -f & e \\ -d & c & -b & a \end{bmatrix}\}$$

where anyone in the set cannot be put together to satisfy PR property since the correlation terms are not equal to zero.

ii) CCCj=

$$\{CCC1 = \begin{bmatrix} b & -a & d & -c \\ -f & e & -h & g \\ g & -h & e & -f \\ -c & d & -a & b \end{bmatrix}, CCC2 = \begin{bmatrix} -b & -a & -d & -c \\ f & e & h & g \\ -g & -h & -e & -f \\ c & d & a & b \end{bmatrix}, CCC3 = \begin{bmatrix} -b & a & -d & c \\ f & -e & h & -g \\ -g & h & -e & f \\ c & -d & a & -b \end{bmatrix}, CCC4 = \begin{bmatrix} b & a & d & c \\ -f & -e & -h & -g \\ g & h & e & f \\ -c & -d & -a & -b \end{bmatrix}\}$$

can be put together other than the combinations CCC₁-CCC₃, CCC₂-CCC₄ since they are same except a (-1) multiplication of all coefficients in the structures.

iii) DDDj=

$$\{DDD1 = \begin{bmatrix} e & -f & g & -h \\ -a & b & -c & d \\ d & -c & b & -a \\ -h & g & -f & e \end{bmatrix}, DDD2 = \begin{bmatrix} -e & f & -g & h \\ -a & b & -c & d \\ -d & c & -b & a \\ -h & g & -f & e \end{bmatrix}, DDD3 = \begin{bmatrix} -e & f & -g & h \\ a & -b & c & -d \\ -d & c & -b & a \\ h & -g & f & -e \end{bmatrix}, DDD4 = \begin{bmatrix} e & -f & g & -h \\ a & -b & c & -d \\ d & -c & b & -a \\ h & -g & f & -e \end{bmatrix}\}$$

can be put together other than the combinations DDD1-DDD3, DDD2-DDD4 since they are same except a (-1) multiplication of all coefficients in the structures.

We know that with AAAj s we cannot put any other AAAj s in the filter bank and CCCj's or DDDj's cannot be put together three times, resulting in the fact that there must be at least one AAAj for 4-band PRFB. The possible combinations for the filter bank are given in the Figure 4.3 below:

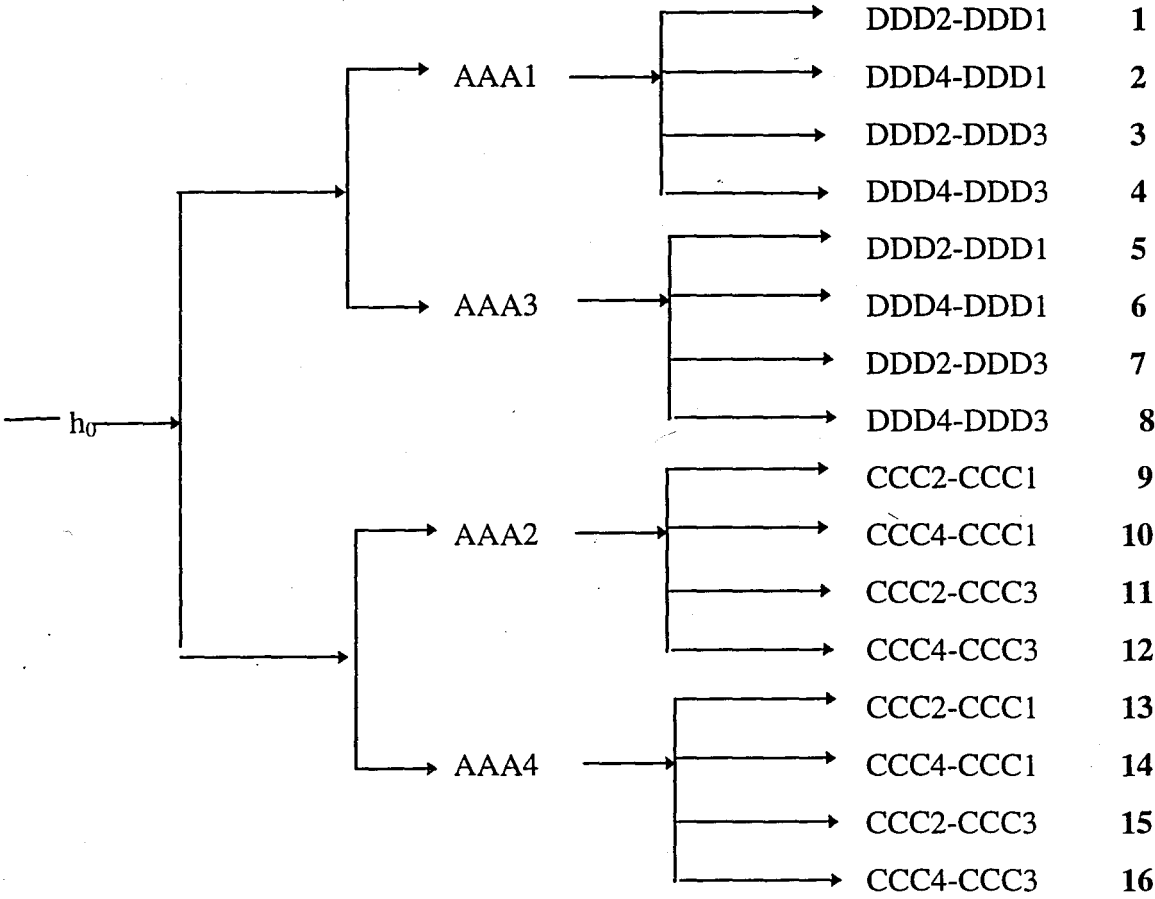


FIGURE 4.3 All possible combinations for the shuffle based filter structures in design of PRFB

The group {1,2,3,4} corresponds to row shufflings while the group {9,10,11,12} corresponds to column shufflings and the ones {5,6,7,8},{13,14,15,16} are only (-1) multiplied the solutions in {1,2,3,4} and {9,10,11,12}.

Step 2:

All the statements in step 1 was formed for $N=4$, that is the smallest filter size for quadratic sampling case. The generalization of the proof to the $N \times N$ sized filters is straightforward with the fact that the 2×2 subblocks have to satisfy the orthomormality conditions to satisfy PR. This extension will be checked in the next section. Q.E.D.

The signed column shufflings and signed row shufflings lead to a shifted version in the frequency domain so that there is a unique solution in terms of frequency interpretation for 4 band quadratic sampling case in two dimensional PRFB design. The frequency interpretation will be studied in 4.2.1.3.

4.2.1.2 Proof of the Filter Bank Solutions for Arbitrary Sized Filters

Letting $h_i = \left[\left(X_{k_1, k_2}^{(i)} \right)_{2 \times 2} \right]_{N \times N}$ for $0 \leq k_1 \leq N-1, 0 \leq k_2 \leq N-1$ and so ,

$$h_i = \left[\begin{array}{c|ccc|c} X_{0,0}^{(i)} & X_{0,1}^{(i)} & . & . & . & X_{0,N-1}^{(i)} \\ \hline . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ \hline X_{N-1,0}^{(i)} & . & . & . & . & X_{N-1,N-1}^{(i)} \end{array} \right]$$

and

$$X_{k_1, k_2}^{(i)}(m, n) = h_i(2k_1 + m, 2k_2 + n) \quad \text{for } m, n=0,1$$

By definition, $x, y \in \mathbb{R}^2$ being a row vector, the kernels $q_i(x)$ with $i = 0,1$ can be written as

$$q_0(x) = xI, \quad q_1(x) = xI_2, \quad r_0(x) = xI_3, \quad r_1(x) = xJ$$

using bases of the form

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In this case, the solution can be given as in Equation 4.6, 4.7 and 4.8 in terms of blocks as

$$\mathbf{X}_{k_1, k_2}^{(0)} = \begin{cases} \begin{bmatrix} \mathbf{x}_{k_1, k_2} \\ \mathbf{y}_{k_1, k_2} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} \\ \mathbf{J} \mathbf{X}_{N-1-k_1, N-1-k_2}^{(0)} \mathbf{J} & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} \end{cases}$$

and

$$\mathbf{X}_{k_1, k_2}^{(i)} = \begin{cases} \begin{bmatrix} q_i(x_{k_1, k_2}) \\ q_i(y_{k_1, k_2} \mathbf{J}) \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 0, 1 \\ \mathbf{J} \mathbf{X}_{N-1-k_1, N-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 0, 1 \\ \begin{bmatrix} r_{i-2}(x_{k_1, k_2}) \\ -r_{i-2}(y_{k_1, k_2} \mathbf{J}) \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 2, 3 \\ -\mathbf{J} \mathbf{X}_{N-1-k_1, N-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 2, 3 \end{cases}$$

or

$$\mathbf{X}_{k_1, k_2}^{(i)} = \begin{cases} \begin{bmatrix} \mathbf{x}_{k_1, k_2} \\ \mathbf{y}_{k_1, k_2} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 0 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{I}_1 \\ -\mathbf{y}_{k_1, k_2} \mathbf{J} \mathbf{I}_1 \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 1 \\ \mathbf{J} \mathbf{X}_{N-1-k_1, N-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 0, 1 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{J} \\ -\mathbf{y}_{k_1, k_2} \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 2 \\ \begin{bmatrix} \mathbf{x}_{k_1, k_2} \mathbf{I}_2 \\ \mathbf{y}_{k_1, k_2} \mathbf{J} \mathbf{I}_2 \mathbf{J} \end{bmatrix} & \begin{matrix} 0 \leq k_1 < N/2 \\ 0 \leq k_2 < N \end{matrix} & i = 3 \\ -\mathbf{J} \mathbf{X}_{N-1-k_1, N-1-k_2}^{(i)} \mathbf{J} & \begin{matrix} N/2 \leq k_1 < N \\ 0 \leq k_2 < N \end{matrix} & i = 2, 3 \end{cases}$$

The uniqueness of the solution in this quadratic case is proven in the way that the orthogonality conditions

- (1) $\langle q_i(x), q_j(y) \rangle + \langle q_i(y), q_j(x) \rangle = 2\delta_{i-j} \langle x, y \rangle$
- (2) the condition (1) for $r_i(x)$
- (3) $\langle q_i(x), r_j(y) \rangle - \langle q_i(y), r_j(x) \rangle = 0$

are satisfied only by the block sized bases used in the $q_i(x)$ and $r_i(x)$ kernels.

The structures found for this “quadratic” case can be then extended for all sizes of filters in filter banks.

Proof:

Let us partition h_i as $h_i = \left[\begin{pmatrix} X_{k_1, k_2}^{(i)} \end{pmatrix}_{2 \times 2} \right]_{N \times N}$ where

$$X_{k_1, k_2}^{(i)} = \begin{bmatrix} h_i(2k_1, 2k_2) & h_i(2k_1, 2k_2 + 1) \\ h_i(2k_1 + 1, 2k_2) & h_i(2k_1 + 1, 2k_2 + 1) \end{bmatrix}$$

for $0 \leq k_1, k_2 \leq N-1$. We have for $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, the orthogonality conditions as

$$\sum_{m_1, m_2} h_i(m_1 + 2n_1, m_2 + 2n_1) h_j(m_1, m_2) = \delta_{i-j} \delta_{n_1} \delta_{n_2}.$$

So, the sum $\sum_{k_1, k_2} \langle X_{k_1+n_1, k_2+n_2}^{(i)}, X_{k_1, k_2}^{(j)} \rangle = \sum_{\substack{0 \leq k_1 < N-n_1 \\ 0 \leq k_2 < N-n_2}} \langle X_{k_1+n_1, k_2+n_2}^{(i)}, X_{k_1, k_2}^{(j)} \rangle$ is equal to

$$\sum_{\substack{0 \leq k_1 < N-n_1 \\ 0 \leq k_2 < N-n_2}} \langle X_{N-1-k_1, N-1-k_2}^{(i)}, X_{N-1-k_1-n_1, N-1-k_2-n_2}^{(j)} \rangle$$

or by combining the two,

$$u_{i,j} = \frac{1}{2} \sum_{k_1, k_2} \left\{ \left\langle X_{k_1+n_1, k_2+n_2}^{(i)}, X_{k_1, k_2}^{(j)} \right\rangle + \left\langle X_{N-1-k_1, N-1-k_2}^{(i)}, X_{N-1-k_1-n_1, N-1-k_2-n_2}^{(j)} \right\rangle \right\}$$

Case 1: $i, j \in \{0, 1\}$

$$X_{k_1, k_2}^{(i)} = J X_{N-1-k_1, N-1-k_2}^{(i)}$$

for all k_1, k_2 where $i=0$ or $i=1$. Then

$$\begin{aligned} u_{i,j} &= \frac{1}{2} \sum_{k_1, k_2} \left\{ \left\langle X_{k_1+n_1, k_2+n_2}^{(i)}, X_{k_1, k_2}^{(j)} \right\rangle + \left\langle X_{k_1, k_2}^{(i)}, X_{k_1+n_1, k_2+n_2}^{(j)} \right\rangle \right\} \quad (4.9c) \\ &= \frac{1}{2} \sum_{\substack{0 \leq k_1+n_1 < \frac{N}{2} \\ k_2}} \left\{ \left\langle q_i(x_{k_1+n_1, k_2+n_2}), q_j(x_{k_1, k_2}) \right\rangle + \left\langle q_i(y_{k_1+n_1, k_2+n_2} \cdot \mathbf{J}), q_j(y_{k_1, k_2} \mathbf{J}) \right\rangle + \right. \\ &\quad \left. \left\langle q_i(x_{k_1, k_2}), q_j(x_{k_1+n_1, k_2+n_2}) \right\rangle + \left\langle q_i(y_{k_1+n_1, k_2+n_2} \cdot \mathbf{J}), q_j(y_{k_1+n_1, k_2+n_2} \mathbf{J}) \right\rangle \right\} \\ &+ \frac{1}{2} \sum_{\substack{0 \leq k_1 < \frac{N}{2} \\ k_2}} \left\{ \left\langle q_i(y_{N-1-k_1-n_1, N-1-k_2-n_2} \cdot \mathbf{J}), q_j(x_{k_1, k_2}) \right\rangle + \left\langle q_i(x_{N-1-k_1-n_1, N-1-k_2-n_2} \mathbf{J}), q_j(y_{k_1, k_2} \mathbf{J}) \right\rangle + \right. \\ &\quad \left. \left\langle q_i(x_{k_1, k_2}), q_j(y_{N-1-k_1-n_1, N-1-k_2-n_2} \cdot \mathbf{J}) \right\rangle + \left\langle q_i(y_{k_1, k_2} \cdot \mathbf{J}), q_j(x_{N-1-k_1-n_1, N-1-k_2-n_2} \mathbf{J}) \right\rangle \right\} \\ &+ \frac{1}{2} \sum_{\substack{\frac{N}{2} \leq k_1 \\ k_2}} \left\{ \left\langle q_i(y_{N-1-k_1-n_1, N-1-k_2-n_2} \cdot \mathbf{J}), q_j(y_{N-1-k_1, N-1-k_2} \cdot \mathbf{J}) \right\rangle + \right. \\ &\quad \left. \left\langle q_i(x_{N-1-k_1-n_1, N-1-k_2-n_2} \cdot \mathbf{J}), q_j(y_{N-1-k_1-n_1, N-1-k_2-n_2}) \right\rangle + \right. \\ &\quad \left. \left\langle q_i(y_{N-1-k_1, N-1-k_2} \cdot \mathbf{J}), q_j(y_{N-1-k_1-n_1, N-1-k_2-n_2}) \right\rangle + \right. \\ &\quad \left. \left\langle q_i(x_{N-1-k_1, N-1-k_2} \mathbf{J}), q_j(x_{N-1-k_1-n_1, N-1-k_2-n_2}) \right\rangle \right\} \end{aligned}$$

Using (1),

$$u_{i,j} = \delta_{i-j} \sum_{k_1, k_2} \left\langle X_{k_1+n_1, k_2+n_2}^{(0)}, X_{k_1, k_2}^{(0)} \right\rangle = \delta_{i-j} \delta_{n_1} \delta_{n_2}$$

The indices in the expressions are arranged as depicted in Figure 4.4.

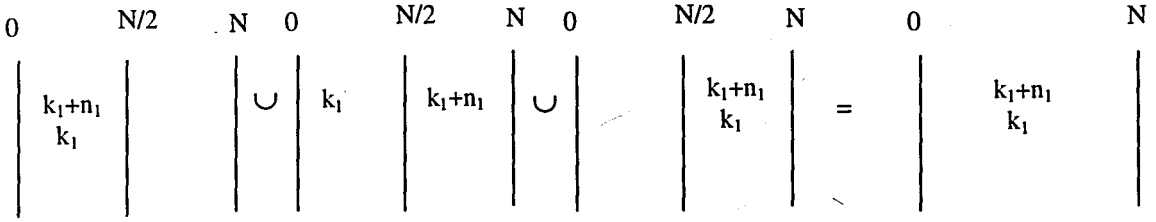


FIGURE 4.4. Indices placements for the Equation 4.9

All of the steps are identical for

Case 2: $i, j \in \{2, 3\}$ and Case 3: $i \in \{0, 1\}$ $j \in \{2, 3\}$ or vice versa $\square$

4.2.1.3 Frequency Domain Interpretation

Assuming that the first filter $h_0(n_1, n_2)$ has the following structure,

$$h_0(n_1, n_2) = \begin{matrix} (0,0) & \xrightarrow{n_2} & -\infty & - \\ \downarrow & & & \\ -\infty & n_1 & \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix} \end{matrix}$$

and with the transforms $h_0(n_1, n_2) \leftrightarrow H_0(z_1, z_2)$, the other 3 filters frequency domain expressions can be expressed for 4-band quadratic sampling design case as,

$$H_1(z_1, z_2) = H_0(z_1, -z_2) \quad (4.10)$$

With $z_i = e^{j\omega_i}$, $i = 1, \dots, p$;

$$\begin{aligned} TT1(z_1, z_2) &= \frac{1}{2} z_2^{-1} [H_0(z_1, z_2) + H_0(z_1, -z_2)] + \frac{1}{2} z_2 [H_0(z_1, z_2) - H_0(z_1, -z_2)] \\ &= H_0(z_1, z_2) \cos(\omega_2) - jH_0(z_1, -z_2) \sin(\omega_2) \end{aligned} \quad (4.11)$$

$$\begin{aligned} H_2(z_1, z_2) &= TT1(-z_1, z_2) \\ &= H_0(-z_1, z_2) \cos(\omega_2) - jH_0(-z_1, -z_2) \sin(\omega_2) \end{aligned} \quad (4.12)$$

Similarly, the fourth filter is in the form of,

$$\begin{aligned} H_3(z_1, z_2) &= TT1(-z_1, -z_2) \\ &= -H_0(-z_1, -z_2) \cos(\omega_2) + jH_0(-z_1, z_2) \sin(\omega_2) \end{aligned} \quad (4.13)$$

or in block diagram representation:

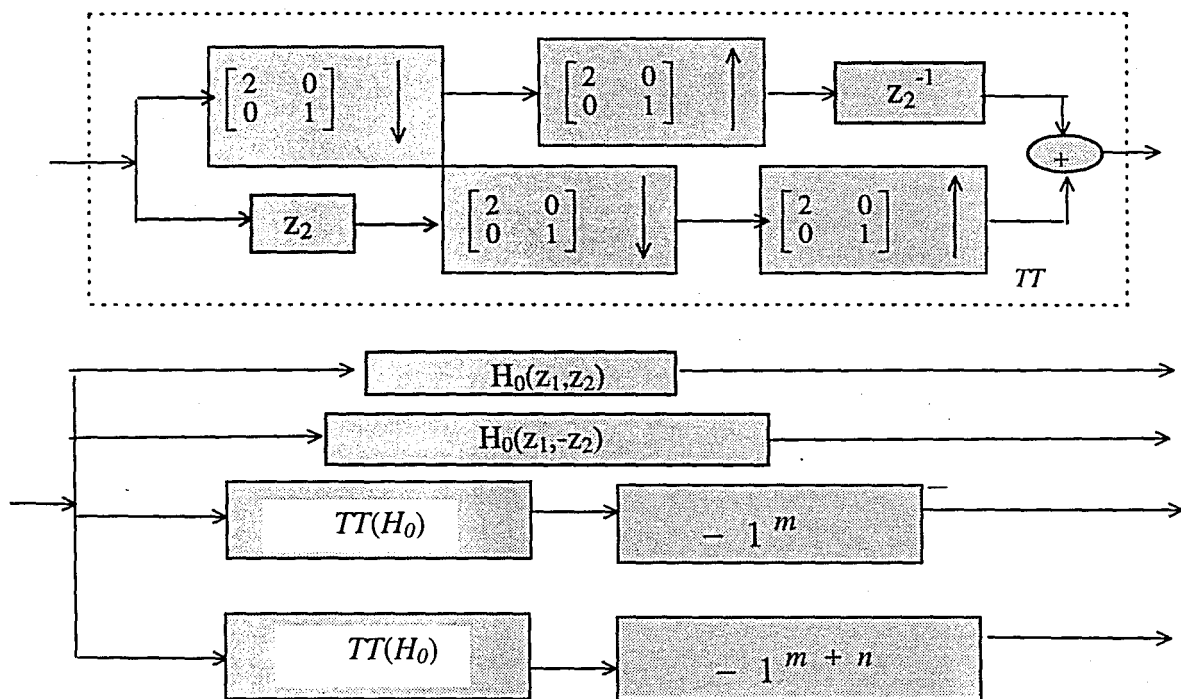


FIGURE 4.5. Block diagram representation of the frequency representation for 2 dimensional quadratic sampling case design

4. 2.1.4. Analysis of Regularity Conditions

Although regularity has been reported in 1-D case, it is still an open problem for 2-D processing as previously discussed in Chapter 2. For 1-D case, the investigation on LPFs

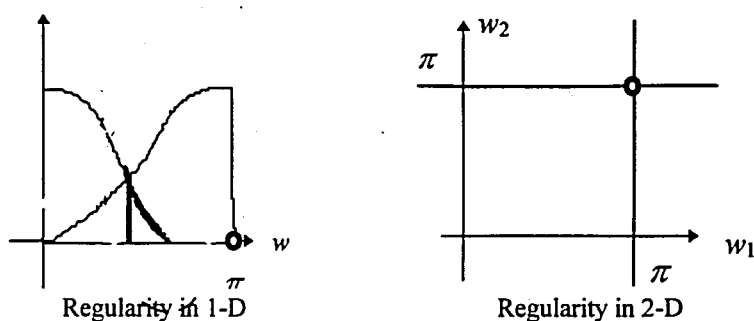


FIGURE 4.6 1-D regularity concept and a 2-D similar one

yields results such as the fact that an L -th order regular LPF passes through signal component which are polynomial of degree less than $N/2$.

Similar to the 1-D case, to obtain a filter bank with regularity in 2-D, one can start imposing zeroes at $(w_1 = \pi, w_2 = \pi)$ on the filter $H_0(z_1, z_2)$, and the bandpass sections $\{H_i(z_1, z_2), i = 1, 2, 3\}$, i.e. zeroes is required at the center frequency ($w_1 = 0, w_2 = 0$). Thus the regularity for 2-D filters becomes relegated to the number of vanishing moments of the bandpass filters at DC. Furthermore since the bandpass sections were derived from the lowpass section via signed shuffling operations, these vanishing moment conditions can be

formulated as additional moment conditions of the lowpass section itself. As seen in section 4.2.1.3, the filters obtained by shuffling based design method can be expressed in the frequency domain as

$$H_1(z_1, z_2) = H_0(-z_1, z_2)$$

$$H_2(z_1, z_2) = H_0(z_1, -z_2) \left[\frac{z_1 + z_1^{-1}}{2} \right] + H_0(-z_1, -z_2) \left[\frac{-(z_1 - z_1^{-1})}{2} \right]$$

$$H_3(z_1, z_2) = H_0(z_1, -z_2) \left[\frac{z_1 - z_1^{-1}}{2} \right] + H_0(-z_1, -z_2) \left[\frac{-(z_1 + z_1^{-1})}{2} \right]$$

For $n=m=0$, i.e. zeroth order moments,

1. The only additional constraint equation comes from H_3 (zeroes located as 0 in frequency domain)
2. Other zeroes indicated as ● comes directly from centrosymmetry property in design

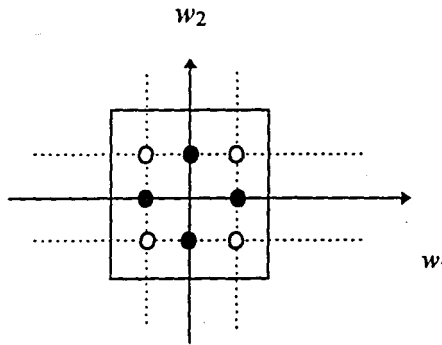


FIGURE 4.7 The placement of zeros when zeroth order moments are considered.

So the placement of a zero at $(w_1 = \pi, w_2 = \pi)$ is necessary in determining regularity when zeroth order moments are considered.

In fact the $n \times m$ 'th order vanishing moment conditions of the $H_1(z_1, z_2), H_2(z_1, z_2), H_3(z_1, z_2)$ filters become for filters of sizes $N \times N$

$$\frac{d^{n+m}}{dz_1^n dz_2^m} H_1(z_1, z_2)|_{z_1=1, z_2=1} = -\frac{d^{n+m}}{dz_1^n dz_2^m} H_0(-z_1, z_2)|_{z_1=1, z_2=1} \quad (4.14)$$

$$\frac{d^{n+m}}{dz_1^n dz_2^m} H_2(z_1, z_2)|_{z_1=1, z_2=1} = -\frac{d^{n+m}}{dz_1^n dz_2^m} H_0(z_1, -z_2)|_{z_1=1, z_2=1} \quad (4.15)$$

$$\frac{d^{n+m}}{dz_1^n dz_2^m} H_3(z_1, z_2)|_{z_1=1, z_2=1} = -\frac{d^{n+m}}{dz_1^n dz_2^m} H_0(-z_1, -z_2)|_{z_1=1, z_2=1} \quad (4.16)$$

all set equal to zero, and where

$$\frac{d^{n+m}}{dz_1^n dz_2^m} H_0(z_1, z_2) = \sum_{j=i=1}^M \sum_{i=1}^M \frac{(i+m-1)!}{(i-1)!} \frac{(j+n-1)!}{(j-1)!} h_0(i, j) z_1^{-(i+n)} z_2^{-(j+m)} \quad (4.17)$$

The above equations can be incorporated in the PRFB design procedure as additional constraint equations depending on the degree of freedom. As will be seen in section 4.2.1.5, there are $N-1$ possible additional constraint equations which can be fulfilled in the optimization procedure. These $N-1$ freedom can be used for the regularity constraints of the filters.

4.2.1.5. Results for Design of Two Dimensional Quadratic Sampling Case

In this section, our aim is to comment on the resulted filter structures obtained for two dimensional quadratic sampling case. As we have seen in the previous section, orthonormality of the filter bank needs only the lowpass selforthogonality constraint equations to be satisfied. This result directs us to compare results with the one dimensional 4 band cases where again no additional crossorthogonality constraint equations come in design. We see at the end that with proper placements of the filter coefficients, obtained structures have common properties.

Let us give the possible solution in a matrix format to be able to compare the results with the one dimensional solutions given in Appendix E:

Denoting the given 2-D unit sample $h_0(k_1, k_2)$ as the matrix $\mathbf{h}_0$, shuffling matrix $\mathbf{S}$ with

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 1 & 0 & \\ & 0 & & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix} \quad (4.18)$$

and sign change matrix $\mathbf{P}$, a diagonal matrix with alternating +1's and -1's on its main diagonal,

$$\mathbf{P} = [p_{jk}], \quad p_{jk} = \pm (-1)^j \delta_{j-k} \quad (4.19)$$

where both $\mathbf{P}$ and $\mathbf{S}$ are $N \times N$, the two feasible PR-QMF solutions being equivalent representations in terms of frequency domain interpretation, are given in matrix format as follows

$$\{ \mathbf{h}_0, \mathbf{h}_0 \mathbf{P}, \mathbf{Ph}_0 \mathbf{S}, \mathbf{Ph}_0 \mathbf{SP} \} \quad (4.20)$$

corresponding to column shufflings and

$$\{ \mathbf{h}_0, \mathbf{Ph}_0, \mathbf{Sh}_0 \mathbf{P}, \pm \mathbf{PSh}_0 \mathbf{P} \} \quad (4.21)$$

corresponding to row shufflings.

As an example of 4×4 sized filter bank design, the lowpass section (low-low=LL) has a centrosymmetric impulse response and is given by $\mathbf{h}_0 = \{h_0(m, n)\}$,

$$\mathbf{h}_0 = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$$

The second filter is found as $h_1(m, n) = (-1)^n h_0(m, n)$, and the other two by using shuffling and proper sign change operations either on rows or columns. These are obtained as following,

$$\mathbf{h}_1 = \begin{bmatrix} a & -b & c & -d \\ e & -f & g & -h \\ h & -g & f & -e \\ d & -c & b & -a \end{bmatrix}, \mathbf{h}_2 = \begin{bmatrix} b & a & d & c \\ -f & -e & -h & -g \\ g & h & e & f \\ -c & -d & -a & -b \end{bmatrix}, \mathbf{h}_3 = \begin{bmatrix} b & -a & d & -c \\ -f & e & -h & g \\ g & -h & e & -f \\ -c & d & -a & b \end{bmatrix}$$

The results of filter banks for 1 dimensional shuffle based design on the other hand has the following structure,

$$\begin{aligned}
\mathbf{h}_0 &= [a \quad b \quad e \quad f \quad c \quad d \quad g \quad h \quad h \quad g \quad d \quad c \quad f \quad e \quad b \quad a] \\
\mathbf{h}_1 &= [b \quad -a \quad -f \quad e \quad d \quad -c \quad -h \quad g \quad g \quad -h \quad -c \quad d \quad e \quad -f \quad -a \quad b] \\
\mathbf{h}_2 &= [a \quad -b \quad e \quad -f \quad c \quad -d \quad g \quad -h \quad h \quad -g \quad d \quad -c \quad f \quad -e \quad b \quad -a] \\
\mathbf{h}_3 &= [b \quad a \quad -f \quad -e \quad d \quad c \quad -h \quad -g \quad g \quad h \quad -c \quad -d \quad e \quad f \quad -a \quad -b]
\end{aligned}$$

Let $\mathbf{x} = [a, b]$, $\mathbf{y} = [e, f]$, $\mathbf{z} = [c, d]$, $\mathbf{t} = [g, h]$ and define

$$Q_1(a, b) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad Q_2(a, b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then in Hadamard matrix format,

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix} = \begin{bmatrix} Q_1(\mathbf{x}) & Q_1(\mathbf{yJ})\mathbf{J} & Q_1(\mathbf{z}) & Q_1(\mathbf{tJ})\mathbf{J} & Q_1(\mathbf{tJ}) & Q_1(\mathbf{zJ}) & Q_1(\mathbf{yJ}) & Q_1(\mathbf{xJ}) \\ Q_2(\mathbf{x}) & -Q_2(\mathbf{yJ})\mathbf{J} & Q_2(\mathbf{z}) & -Q_2(\mathbf{tJ})\mathbf{J} & Q_2(\mathbf{tJ}) & -Q_2(\mathbf{zJ}) & Q_2(\mathbf{yJ}) & -Q_2(\mathbf{xJ}) \end{bmatrix}$$

4 × 4K
(4.22)

The PR conditions necessitate that with $\mathbf{W}$ being the $N \times N$ one block shift matrix,

$$\mathbf{H}\mathbf{W}^k\mathbf{H}^T = \delta_k \mathbf{I}, \quad k=0,1,\dots,K-1$$

which in turn implies that all necessary PR conditions for the $4 \times 4K$ filter bank design are reduced to the orthogonality of $\mathbf{h}_0$ by shifts of 4 given as,

$$\mathbf{h}_0\mathbf{W}^k\mathbf{h}_0^T = \delta_k, \quad k=0,1,\dots,K-1$$

and in two dimension 4×4 filter size using Equation 4.6,7,8, we have

$$\mathbf{X}_{k_1, k_2}^{(0)} = \begin{cases} \begin{bmatrix} \mathbf{x}_{k_1, k_2} \\ \mathbf{y}_{k_1, k_2} \end{bmatrix} & 0 \leq k_1 < 1 \\ & 0 \leq k_2 < 2 \\ \mathbf{J} \mathbf{X}_{1-k_1, 1-k_2}^{(0)} \mathbf{J} & 1 \leq k_1 < 2 \\ & 0 \leq k_2 < 2 \end{cases} \quad \text{and } \mathbf{X}_{k_1, k_2}^{(i)} = \begin{cases} \begin{bmatrix} q_i(x_{k_1, k_2}) \\ q_i(y_{k_1, k_2} \mathbf{J}) \mathbf{J} \end{bmatrix} & 0 \leq k_1 < 1 \\ & 0 \leq k_2 < 2 & i = 0, 1 \\ \mathbf{J} \mathbf{X}_{1-k_1, 1-k_2}^{(i)} \mathbf{J} & 1 \leq k_1 < 2 \\ & 0 \leq k_2 < 2 & i = 0, 1 \\ \begin{bmatrix} r_{i-2}(x_{k_1, k_2}) \\ -r_{i-2}(y_{k_1, k_2} \mathbf{J}) \mathbf{J} \end{bmatrix} & 0 \leq k_1 < 1 \\ & 0 \leq k_2 < 2 & i = 2, 3 \\ -\mathbf{J} \mathbf{X}_{1-k_1, 1-k_2}^{(i)} \mathbf{J} & 1 \leq k_1 < 2 \\ & 0 \leq k_2 < 2 & i = 2, 3 \end{cases}$$

For 4-band quadratic case and $2T \times 2T$ supported filters in PRFB we have always $\mathbf{X}_{0,0}^{(0)}$ to satisfy Equation 4.5 so that the structure above extend straightforwardly to $2T \times 2T$ case as shown in Section 4.2.1.2.

This shuffling mechanism is general in that any filter size can be designed, though the number of equations to obtain PR conditions will be different. In fact for an $N \times N$ impulse response, there will be $N^2 / 2$ unknowns and $N^2 / 2 - (N - 1)$ equations due to PR conditions, while the rest of $N - 1$ equations are provided by regularity constraints.

The example of the set of filters in PRFB can be given for 6×6 sizes case:

$$\begin{bmatrix} a & b & c & d & e & f \\ g & h & j & k & l & m \\ n & o & p & q & r & s \\ s & r & q & p & o & n \\ m & l & k & j & h & g \\ f & e & d & c & b & a \end{bmatrix} \begin{bmatrix} a & -b & c & -d & e & -f \\ g & -h & j & -k & l & -m \\ n & -o & p & -q & r & -s \\ s & -r & q & -p & o & -n \\ m & -l & k & -j & h & -g \\ f & -e & d & -c & b & -a \end{bmatrix}$$

$$\begin{bmatrix} b & a & d & c & f & e \\ -h & -g & -k & -j & -m & -l \\ o & n & q & p & s & r \\ -r & -s & -p & -q & -n & -o \\ l & m & j & k & g & h \\ -e & -f & -c & -d & -a & -b \end{bmatrix} \begin{bmatrix} b & -a & d & -c & f & -e \\ -h & g & -k & j & -m & l \\ o & -n & q & -p & s & -r \\ -r & s & -p & q & -n & o \\ l & -m & j & -k & g & -h \\ -e & f & -c & d & -a & b \end{bmatrix} =$$

It is intriguing to investigate all shuffling rules that lead to interband orthogonality. It was shown however that for the case of the downsampling matrix $D=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, shufflings only between neighboring columns (rows) are allowed, namely shufflings of the columns 1 and 2, 3 and 4, etc. and it is unique in design presented in Section 4.2.1.1.

In the 4-band quadratic sampling case all of the constraint equations come from the selforthogonality on h_0 . The number of unknowns, cross and self orthogonality constraints for 4 band centrosymmetric shuffled design are shown in Table 4.3.

TABLE 4.3. Comparison of number of unknowns, cross and self orthogonality constraints for 4-band centrosymmetric shuffled design and nonshuffled design cases

	Number of unknowns	Number of constraint equations to satisfy cross-orthogonality	Number of constraint equations to satisfy self-orthogonality
Design with shuffling for quadratic ($d_{00}=2$) sampling case	$\frac{N^2}{2}$ instead of $4N^2$ as in case of design without shuffling	---	$\frac{N^2}{2} - (N-1)$ N-1 additional freedom in design

Karlsson et.al. [88], Kovacevic et. al. [36] provided solutions for NS,LP,PRFBs in two dimension using designs in polyphase domain. The solution they give for two dimensional quadratic sampling case when the filters are of the same size correspond to the full horizontal-full vertical symmetric case as depicted in Table 4.2. But this symmetric structure of the filters does not extend for arbitrary $N \times N$ size to satisfy PR condition. As

many authors wonder if the found solutions cover all the search space, the one proposed in this thesis for quadratic sampling case prove itself to be unique with design considerations. It remains to investigate the shuffling operations allowed with other downsampling schemes, or M -band two dimensional filter banks which will be considered in the next section.

The signed shufflings are critical in satisfying the PR conditions given in Equation 4.5. There are in general case $\min[d_{00}, d_{11}]$ possible structure on which the sign operator can be applied in FB. For a given sampling lattice $\mathbf{D}$ in FB, the minimum sized filters are of $2(d_{11} \times (d_{00} + d_{01}))$ and there are four $d_{11} \times (d_{00} + d_{01})$ sized subblocks in each of them.

Hence, from

$$\prod_{t=0}^{\left(\frac{K}{2}-1\right)} (2t+1) \quad (4.23)$$

possible column/row shuffling structure, K being the number of free variables in a subblock, $\min[d_{00}, d_{11}]$ structure will be chosen. The advantage of using signed shufflings on the prototype filter in finding the remaining filters is the reduction of number of unknowns and orthogonality constraints.

4.2.2. Shuffling Based Design Technique for M-band,i.e. for Variations of D with $M=\det D$

The analysis filters divide the signal $x(n)$ into subbands in p-dimensional case. The output of each analysis filter is decimated by D . Since the decimator reduces the sample density by $\det D$, we use a total of $\det D$ analysis filters so that the sampling density is preserved by the complete analysis bank. The decimated subband signals are then expanded D -fold and passed through the $\det D$ synthesis filters. The outputs of these filters are added to obtain the reconstructed signal. Let now construct filters for M -band in PRFB design.

4.2.2.1. Design for ‘Quadratic’ Sampling; example for $D=\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $M=16$ band case

The matrix which can provide 16 filters in the proposed design is of size 8×8 . There are here $\min[4,4]=4$ structures and

$$\mathbf{h}_{i,0} = \left[\begin{array}{c|c} (\mathbf{X}_{0,0}^{(i)})_{4 \times 4} & (\mathbf{X}_{0,1}^{(i)})_{4 \times 4} \\ \hline (\mathbf{X}_{1,0}^{(i)})_{4 \times 4} & (\mathbf{X}_{1,1}^{(i)})_{4 \times 4} \end{array} \right], i=0,4,8,12 \text{ can be shown as}$$

$$\mathbf{h}_{0,0} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{17} & x_{18} & x_{19} & x_{20} & x_{21} & x_{22} & x_{23} & x_{24} \\ x_{25} & x_{26} & x_{27} & x_{28} & x_{29} & x_{30} & x_{31} & x_{32} \\ x_{32} & x_{31} & x_{30} & x_{29} & x_{28} & x_{27} & x_{26} & x_{25} \\ x_{24} & x_{23} & x_{22} & x_{21} & x_{20} & x_{19} & x_{18} & x_{17} \\ x_{16} & x_{15} & x_{14} & x_{13} & x_{12} & x_{11} & x_{10} & x_9 \\ x_8 & x_7 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \end{bmatrix}, \quad \mathbf{h}_{4,0} = \begin{bmatrix} x_2 & x_1 & x_4 & x_3 & x_6 & x_5 & x_8 & x_7 \\ x_{10} & x_9 & x_{12} & x_{11} & x_{14} & x_{13} & x_{16} & x_{15} \\ x_{18} & x_{17} & x_{20} & x_{19} & x_{22} & x_{21} & x_{24} & x_{23} \\ x_{26} & x_{25} & x_{28} & x_{27} & x_{30} & x_{29} & x_{32} & x_{31} \\ x_{31} & x_{32} & x_{29} & x_{30} & x_{27} & x_{28} & x_{25} & x_{26} \\ x_{23} & x_{24} & x_{21} & x_{22} & x_{19} & x_{20} & x_{17} & x_{18} \\ x_{15} & x_{16} & x_{13} & x_{14} & x_{11} & x_{12} & x_9 & x_{10} \\ x_7 & x_8 & x_5 & x_6 & x_3 & x_4 & x_1 & x_2 \end{bmatrix}$$

$$\mathbf{h}_{8,0} = \begin{bmatrix} x_3 & x_4 & x_1 & x_2 & x_7 & x_8 & x_5 & x_6 \\ x_{11} & x_{12} & x_9 & x_{10} & x_{15} & x_{16} & x_{13} & x_{14} \\ x_{19} & x_{20} & x_{17} & x_{18} & x_{23} & x_{24} & x_{21} & x_{22} \\ x_{27} & x_{28} & x_{25} & x_{26} & x_{31} & x_{32} & x_{29} & x_{30} \\ x_{30} & x_{29} & x_{32} & x_{31} & x_{26} & x_{25} & x_{28} & x_{27} \\ x_{22} & x_{21} & x_{24} & x_{23} & x_{18} & x_{17} & x_{20} & x_{19} \\ x_{14} & x_{13} & x_{16} & x_{15} & x_{10} & x_9 & x_{12} & x_{11} \\ x_6 & x_5 & x_8 & x_7 & x_2 & x_1 & x_4 & x_3 \end{bmatrix}, \quad \mathbf{h}_{12,0} = \begin{bmatrix} x_4 & x_3 & x_2 & x_1 & x_8 & x_7 & x_6 & x_5 \\ x_{12} & x_{11} & x_{10} & x_9 & x_{16} & x_{15} & x_{14} & x_{13} \\ x_{20} & x_{19} & x_{18} & x_{17} & x_{24} & x_{23} & x_{22} & x_{21} \\ x_{28} & x_{27} & x_{26} & x_{25} & x_{32} & x_{31} & x_{30} & x_{29} \\ x_{29} & x_{30} & x_{31} & x_{32} & x_{25} & x_{26} & x_{27} & x_{28} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{17} & x_{18} & x_{19} & x_{20} \\ x_{13} & x_{14} & x_{15} & x_{16} & x_9 & x_{10} & x_{11} & x_{12} \\ x_5 & x_6 & x_7 & x_8 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

where for each of the structures the sign operation is applied as

$$\mathbf{h}_{i,1} = \begin{bmatrix} (\mathbf{X}_{0,0}^{(i)})_{4 \times 4} & -(\mathbf{X}_{0,1}^{(i)})_{4 \times 4} \\ (\mathbf{X}_{1,0}^{(i)})_{4 \times 4} & -(\mathbf{X}_{1,1}^{(i)})_{4 \times 4} \end{bmatrix}, \quad \mathbf{h}_{i,2} = \begin{bmatrix} (\mathbf{X}_{0,0}^{(i)})_{4 \times 4} & (\mathbf{X}_{0,1}^{(i)})_{4 \times 4} \\ -(\mathbf{X}_{1,0}^{(i)})_{4 \times 4} & -(\mathbf{X}_{1,1}^{(i)})_{4 \times 4} \end{bmatrix}, \quad \mathbf{h}_{i,3} = \begin{bmatrix} -(\mathbf{X}_{0,0}^{(i)})_{4 \times 4} & (\mathbf{X}_{0,1}^{(i)})_{4 \times 4} \\ (\mathbf{X}_{1,0}^{(i)})_{4 \times 4} & -(\mathbf{X}_{1,1}^{(i)})_{4 \times 4} \end{bmatrix}$$

Contrarily to the 4-band quadratic case, all of the constraint equations to satisfy PR in FB design do not come only from the self-orthogonality equations on $\mathbf{h}_{0,0}$ but for $l_1 = 0, 1, 2, 3$ five additional constraint equations come from cross-orthogonality constraint equations between $\mathbf{h}_{0,0}$ and $\mathbf{h}_{l_1,0}$ and for $l_2 = 0, 2, 3, 4$ one additional constraint equation come from cross-orthogonality constraint equations between $\mathbf{h}_{0,0}$ and $\mathbf{h}_{l_2,0}$. Hence, the proposed design method can provide practical solutions only at the minimum sized filters as in the example of 8×8 for 16 band quadratic case where d_{00} is even.

For d_{00} odd, i.e. odd quadratic M-band, the middle term in each subblock implies that the sum of squares of the coefficients to be zero in order to satisfy PR. So, only

$$\mathbf{h}_{0,0} = \begin{bmatrix} a & b & c & d & e & f \\ g & 0 & j & k & 0 & m \\ n & o & p & q & r & s \\ s & r & q & p & o & n \\ m & 0 & k & j & 0 & g \\ f & e & d & c & b & a \end{bmatrix}$$

structure for the lowpass filter can be used.

4.2.2.2 Design for 'Rectangular' Sampling; example for $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ case

The structural elements in this design case are 2×3 sized subblocks. Thus, from possible 15 structures, $\min[3,2]=2$ ones are selected for required 6 band filters and when $i=0,2$ these required 6 band filters are obtained by

$$\mathbf{h}_{i,0} = \left[\begin{array}{c|c} (\mathbf{X}_{0,0}^{(i)})_{2 \times 3} & (\mathbf{X}_{0,1}^{(i)})_{2 \times 3} \\ \hline (\mathbf{X}_{1,0}^{(i)})_{2 \times 3} & (\mathbf{X}_{1,1}^{(i)})_{2 \times 3} \end{array} \right]$$

So,

$$\mathbf{h}_{0,0} = \left[\begin{array}{c|c} a & b & c & d & e & f \\ g & h & j & k & l & m \\ m & l & k & j & h & g \\ f & e & d & c & b & a \end{array} \right], \quad \mathbf{h}_{2,0} = \left[\begin{array}{c|c} c & b & a & f & e & d \\ j & h & g & m & l & k \\ k & l & m & g & h & j \\ d & e & f & a & b & c \end{array} \right]$$

$$\mathbf{h}_{0,1} = \left[\begin{array}{c|c} (\mathbf{X}_{0,0}^{(0)})_{2 \times 3} & (\mathbf{X}_{0,1}^{(0)})_{2 \times 3} \\ \hline -(\mathbf{X}_{1,0}^{(0)})_{2 \times 3} & -(\mathbf{X}_{1,1}^{(0)})_{2 \times 3} \end{array} \right], \quad \mathbf{h}_{0,2} = \left[\begin{array}{c|c} -(\mathbf{X}_{0,0}^{(0)})_{2 \times 3} & (\mathbf{X}_{0,1}^{(0)})_{2 \times 3} \\ \hline -(\mathbf{X}_{1,0}^{(0)})_{2 \times 3} & (\mathbf{X}_{1,1}^{(0)})_{2 \times 3} \end{array} \right],$$

$$\mathbf{h}_{2,1} = \left[\begin{array}{c|c} (\mathbf{X}_{0,0}^{(2)})_{2 \times 3} & (\mathbf{X}_{0,1}^{(2)})_{2 \times 3} \\ \hline -(\mathbf{X}_{1,0}^{(2)})_{2 \times 3} & -(\mathbf{X}_{1,1}^{(2)})_{2 \times 3} \end{array} \right], \quad \mathbf{h}_{2,2} = \left[\begin{array}{c|c} -(\mathbf{X}_{0,0}^{(2)})_{2 \times 3} & (\mathbf{X}_{0,1}^{(2)})_{2 \times 3} \\ \hline -(\mathbf{X}_{1,0}^{(2)})_{2 \times 3} & (\mathbf{X}_{1,1}^{(2)})_{2 \times 3} \end{array} \right]$$

Since there are other constraint equations than the ones obtained from orthonormality conditions on $\mathbf{h}_{0,0}$ the found solution can only be practical for the minimum sized filters ,i.e. 4×6 in 6 band rectangular design case.

4.2.2.3. Design for 'Nonrectangular' Sampling; $D=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ case

For regular 'hexagonal' sampling case, i.e. a commonly used sampling structure providing the tightest packing of energy, the minimum sizes of filters are $2(2 \times 3) = 4 \times 6$. The obtained filters are as follows:

$$\mathbf{h}_0 = \begin{bmatrix} a & b & c & d \\ e & f & g & h & j & k \\ k & j & h & g & f & e \\ d & c & b & a \end{bmatrix}, \mathbf{h}_1 = \begin{bmatrix} -a & -b & -c & -d \\ -e & -f & -g & -h & -j & -k \\ k & j & h & g & f & e \\ d & c & b & a \end{bmatrix}, \mathbf{h}_2 = \begin{bmatrix} -b & -a & -d & -c \\ f & e & h & g & k & j \\ j & k & g & h & e & f \\ -c & -d & -a & -b \end{bmatrix}, \mathbf{h}_3 = \begin{bmatrix} -b & -a & -d & -c \\ f & e & h & g & k & j \\ -j & -k & -g & -h & -e & -f \\ c & d & a & b \end{bmatrix}$$

As all of the constraint equations to satisfy PR in FB design do not come only from the self-orthogonality equations on $\mathbf{h}_{0,0}$ but seven additional constraint equations come from cross-orthogonality constraint equations between $\mathbf{h}_{0,0}$ and $\mathbf{h}_{1,0}$ and three additional constraint equation come from cross-orthogonality constraint equations between $\mathbf{h}_{0,0}$ ile $\mathbf{h}_{2,0}$, the proposed design method can provide practical solutions only at the minimum sized filters in this sampling case also.

4.3. Design Comparisons

Various nonseparable linear phase filters were designed using the shuffling method, and based on the maximization of the coding gain performance with respect to a generalized correlation model as in [89]

$$R_{xx}(m,n) = \exp \left\{ - \left[(\alpha m^n)' + (\beta n^r)' \right]^{1/r} \right\} \quad (4.24)$$

$$\exp(-\alpha) = \rho_h, \quad \exp(-\beta) = \rho_v$$

and with respect to nonseparable Gaussian model

$$R_{xx}(m,n) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left(\frac{n_1^2}{\sigma_1^2} + \frac{n_2^2}{\sigma_1^2} - 2\rho \frac{n_1 n_2}{\sigma_1 \sigma_2} \right) \right\} \quad (4.25)$$

The transform coding gain can be expressed as

$$G_{TC} = \frac{\frac{1}{N^2} \sum_{i,j=1}^N \sigma^2(i,j)}{\left[\prod_{i,j=1}^N \sigma^2(i,j) \right]^{\frac{1}{N^2}}} \quad (4.26)$$

where $\sigma^2(i,j)$ denotes the variance of the $(i,j)^{\text{th}}$ band, given by

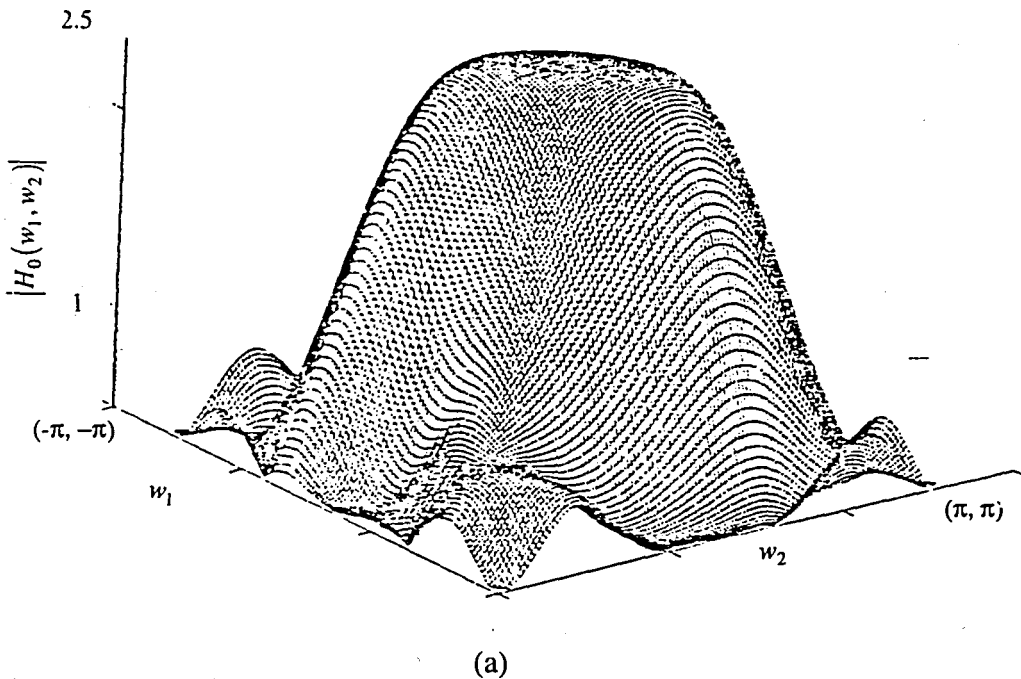
$$\sigma^2(i,j) = \sum_{m,n=0}^{N-1} \sum_{l=m,k=n}^{N-1} h_{ij}(m,n) h_{ij}(m,n) R_{xx}(m-l, n-k) \quad (4.27)$$

and where $h_{ij}(m,n)$ is the impulse response of the $(i,j)^{\text{th}}$ band.

The PR property of $h_0(m,n)$ is to be satisfied while the PR property of the other $\{h_i(m,n), i = 1,2,3\}$ filters are guaranteed by the definitions given in Equation 4.5. This underconstrained set of equations can be solved via an optimization scheme such as coding gain maximization. The underconstraint can either be used for optimization freedom or alternatively, the rest can be assigned to regularity, namely to obtain zeroes at (π, π) .

The main findings can be summarized as follows:

In Figure 4.8, the mesh plot and contour plot of a 8×8 wavelet filter matched to Equation 4.24 with correlation coefficients being $\rho_h = 0.9753$ and $\rho_v = 0.9812$, $r_1 = 1.137$, $r_2 = 1.09$ and $t = \sqrt{2}$ are shown. It is interesting to note that the contours are diagonally oriented, which would not have been feasible with a separable solution. The scaling function obtained by iterating on the lowpass branch is shown in Figure 4.9.



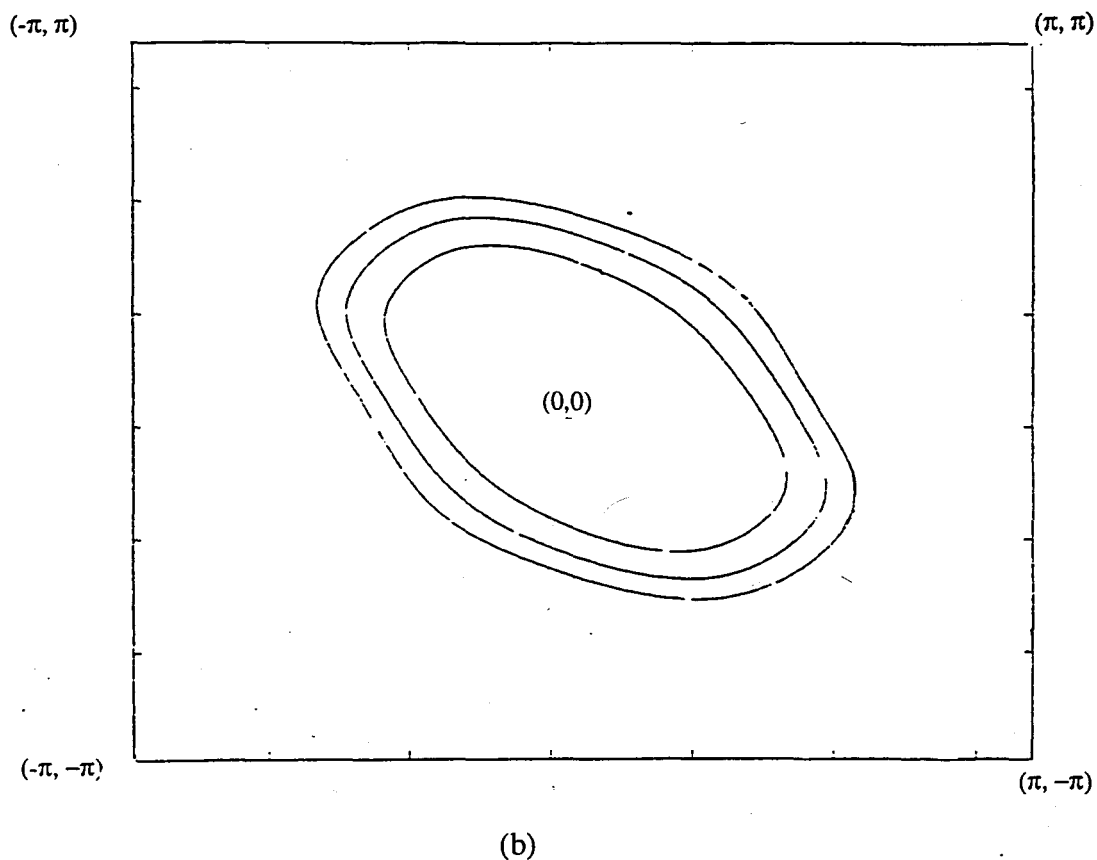


FIGURE 4.8. The effect of directionality in nonseparable linear phase filter bank design using generalized correlation model with horizontal correlation model $\rho_h=0.9753$ and the vertical one being $\rho_v=0.9812$ a) Spectrum of H_0 for 128×128 points b) Contour plot of H_0

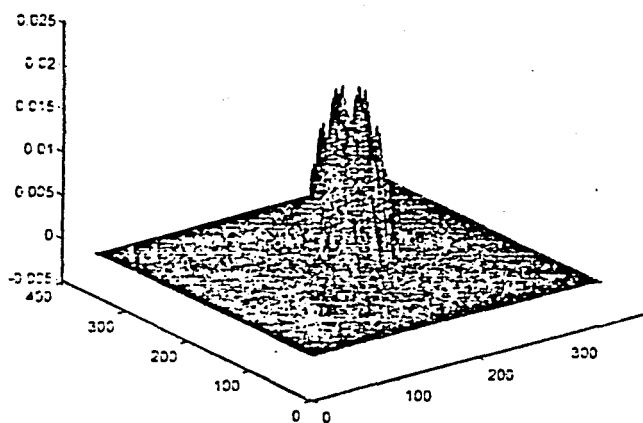


FIGURE 4.9. Scaling function corresponding to the lowpass filter in Figure 4.8 by iterating it 5 times

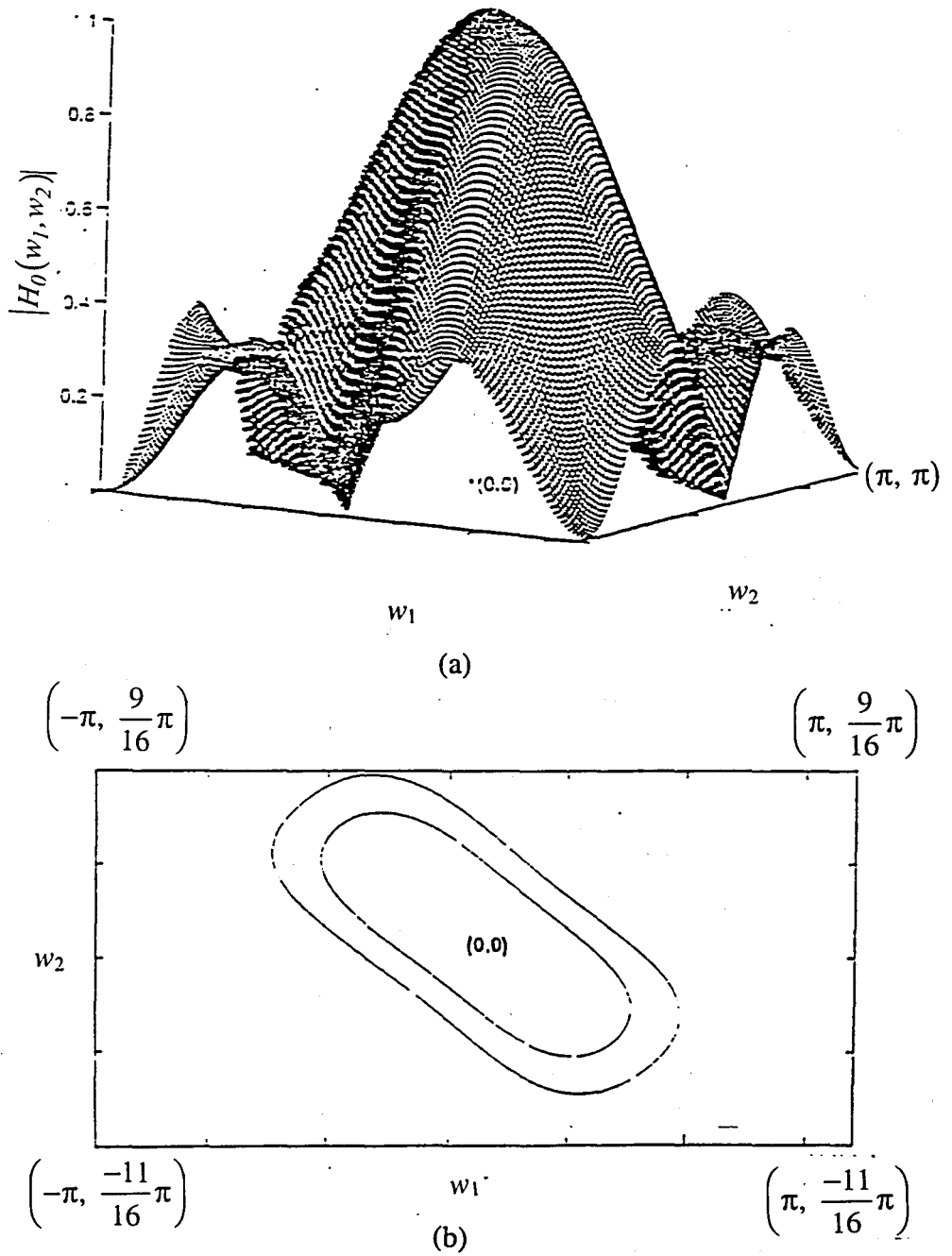


FIGURE 4.10. The effect of directionality in nonseparable linear phase and regular filter bank design using nonseparable Gaussian model with $\rho = 0.5$ a) Spectrum of H_0 (Low-Low portion) for 128×128 points b) Contour plot of H_0

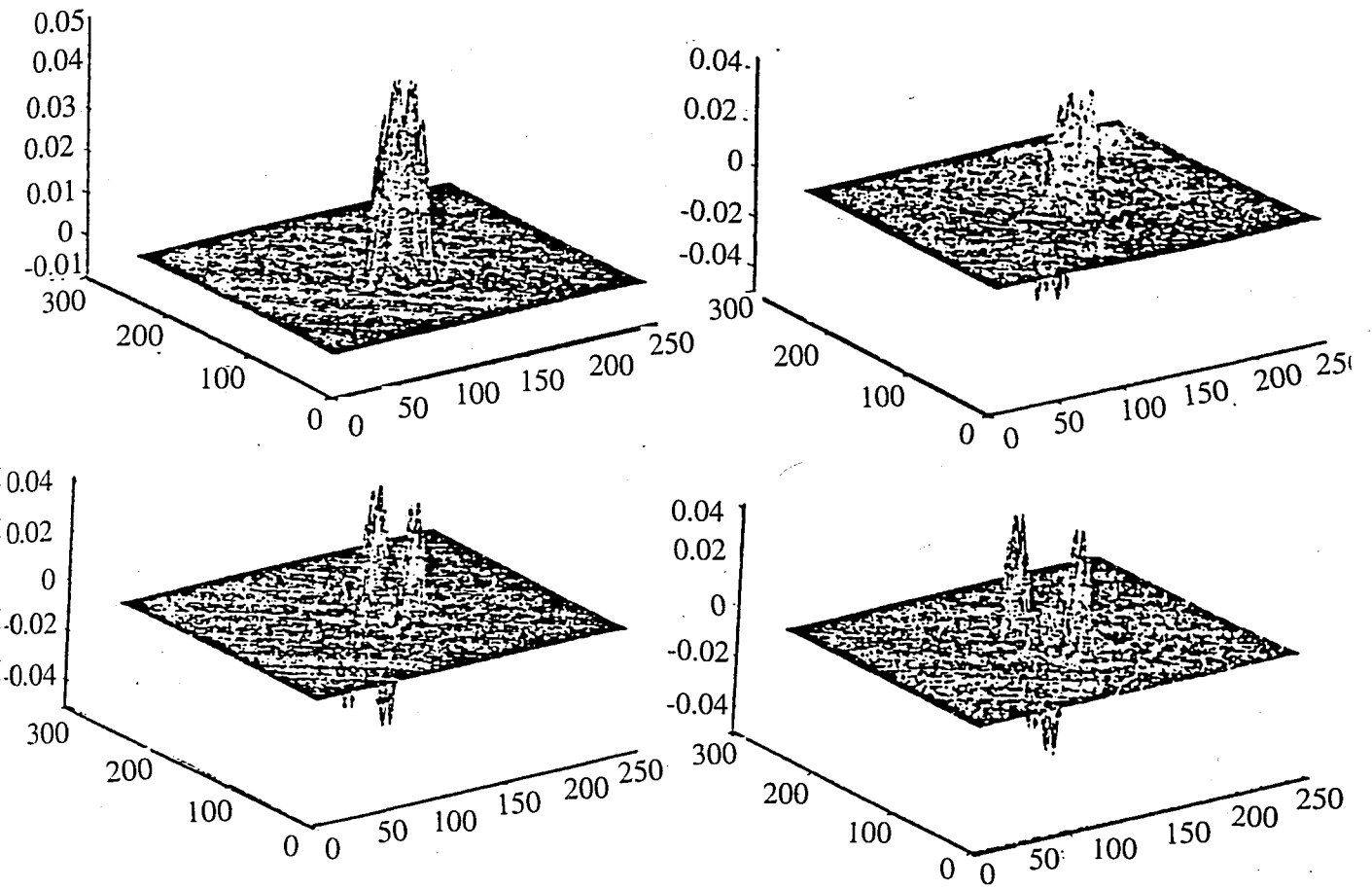


FIGURE 4.11. Scaling function and wavelets corresponding to the regular 8×8 LPF in Figure 4.10 by iterating it 3 times

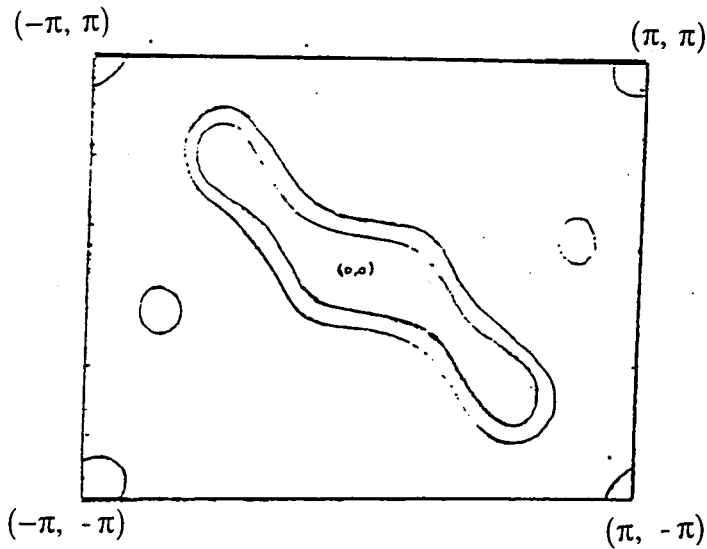


FIGURE 4.12. Contour plot of 4×6 sized filter for 128×128 points at the hexagonal sampling case matched to the generalized correlation model with horizontal correlation model $\rho_h=0.9753$ and the vertical one being $\rho_v=0.9812$

We have also compared the compression performance on a real image (256×256 Lenna) in Table 4.4. The gain of 8×8 DCT is tabulated against those achieved with various filters obtained by using the proposed design method. To obtain a fair comparison the image is decomposed with these filters into 64 subbands. These results show that the proposed design method provides higher transform coding gain vis-à-vis DCT by as much as 50 %. The superiority of the 2-D nonlinear scheme holds true both with real images (Lenna) as well as theoretical correlation models.

TABLE 4.4 Compression performance based on gain of transform coding is done between 8×8 DCT and 8×8 filters formed by using $\rho_h=0.9753$, $\rho_v=0.9812$, $r_1=1.137$, $r_2=1.09$, $t=\sqrt{2}$ coefficients in Equation 4.24 and no regularity constraints is imposed in the optimization

			G_{TC}
Lenna	Opt. Des.	4x4	10.5243
		6x6	14.42
		8x8	18.23
	DCT		12.11
Gen. Cor. Model	Opt. Des.	6x6	28.2182
		8x8	34.269
	DCT		23.4346

The results being very surprising as seen from Table 4.4 the coding gain performance is recalculated by applying the proposed design method for different bands on an image test bank. The obtained results are shown in Table 4.5 where we can see the fact that the proposed design method outperforms the other techniques especially when the design is made for 64-bands.

TABLE 4. 5. Transform coding gain results using nonseparable signed shuffle based filters and many separable ones as Daubechies-32, Johnston-16, Smith-Barnwell 16, (a) for 64-band case and (b) 16-band case as given in Table H.2.

	FILTERS	OBTAINED	USING	SHUFFLING
64-band	Lenna	World	Sea	Cameraman
	18.23	31.57	9.99	21.42
	DAUBECHIES-32			
64-band	Lenna	World	Sea	Cameraman
	13.17	28.09	6.75	19.62
	JOHNSTON-16			
64-band	Lenna	World	Sea	Cameraman
	12.96	27.63	6.65	19.28
	SMITH BARNWELL 16			
64-band	Lenna	World	Sea	Cameraman
	13.16	27.96	6.72	19.43

(a)

G in 16-band for Lenna image	
Shuffle based NS filters	12.58
Daubechies-32	10.96
Johnston-16	10.85
Smith-Barnwell 16	10.99

(b)

4.4. Conclusions

In this work, a novel method for the design of nonseparable 2-D linear phase wavelet filters is presented. This design method could be of interest for its following advantages : linear phase is guaranteed, the resulting filters reflect the dominant spectral directions, fast implementation with much fewer multiplications is achieved and the storage requirements decrease because of the fact that the only retained coefficients are the lowpass filter coefficients.

The proposed design method is provided in time domain. It first finds for all sampling lattices (or dilation matrices) a suitable lowpass filter, while the remaining $M-1$ filters are obtained through use of signed shuffling operations on the coefficients of that filter. All filter requirements are imposed on the lowpass filter. The method is computationally efficient by only storing the coefficients of the lowpass filter. A fast implementation with much fewer multiplications is achieved. The viability of the method is based on shuffling and is due to reduction of unknowns by a factor of 8 and reduction of constraint equations and thus optimization techniques become practical. This special structure (i.e. repeated coefficients) mitigates the computational load brought in by the non-separable structure.

The proposed design method provides a complete analysis for nonseparable sampling/nonseparable filtering, but separable 4-band sampling/nonseparable filtering case can provide desired results and the obtained filters in this case can be extended to all dimensions when compared to other design in the literature [88, 36].

Linear phase property is accomplished by centrosymmetry of filters in FB which provides everytime true nonseparable filters. An increase in the number of free variables can be obtained when compared to other symmetries [88, 36]. This centrosymmetric design of PRFBs works also for all other kind of symmetries.

The proposed method provides a complete search for possible combinations of filter structures in 2-D M -band nonseparable linear phase PRFBs design when compared to other designs in the literature [88, 36].

Nonseparable linear phase filters were designed using the shuffling method, and based on the maximization of the coding gain performance with respect to a generalized correlation model and with respect to nonseparable Gaussian model. These results show that the proposed design method provides higher transform coding gain vis-a-vis DCT by as much as 50 %. The superiority of the 2-D nonlinear scheme holds true both with real images(Lenna) as well as theoretical correlation models. The proposed design method outperforms the other techniques especially when the design is made for 64-bands (see Table 4.5).

5. DESIGN OF THREE DIMENSIONAL NONSEPARABLE FILTER BANKS

5.1. Introduction

Visual communication services require the transmission of image sequences. In an image sequence compression, there are two reasons why the information to be transmitted can be reduced. The statistical redundancy in the image sequence is removed by exploiting the spatial and the temporal correlation. The information can be reduced even further by transmitting only the information relevant to the application. An important property of an image sequence is the spatial correlation between the pixels[1]. If the spatial correlation is large, the bandwidth of the signal is narrow, and consequently the spatial sampling frequency can be reduced. In the extreme case that an image sequence consists of images with a constant intensity value, only one pixel per image is necessary to completely describe the image sequence. Another property is the amount of motion within the image sequence, as it determines the temporal correlation between the pixels. The temporal correlation is maximal if the contents of the scene are not moving. In that case, the temporal sampling frequency, i.e. image rate can be reduced. Besides the correlation, the variance of the image also plays an important role. It requires more information to represent a signal with a high variance than a signal with a low variance given the same amount of permissible distortion. The quality and the compression factor are controlled by the application. Some factors which are important in this case are the image dimensions, the bandwidth of the communication channel and the end-user of the image sequence. For instance, in a HDTV application, the quality is more important than in a videophone application. HDTV images also have larger dimensions than videophone images.

Three dimensional subband coding without requiring a separate motion estimation stage is a simple, low complexity method being scalable both spatially and temporally and

it is almost always free from blocking artifacts even at low bit rates, problem for 3 dimensional DCT based coding methods. When compared to motion compensated methods, 3 dimensional SBD based methods are simpler and lower complexity methods. The natural ordering in importance of the subbands lead to be easily used in a joint source-channel coding environment. The high frequency subbands consisting of edge information, slight motion errors lead to large prediction errors so that it is difficult to perform motion compensation which requires motion estimation. The motion compensation and SBD integrated hybrid coding methods are difficult to practice[1, 44].

The video is decomposed into various properly subsampled component video signals [95]. These various component video signals can be encoded independently using algorithms adapted to the statistical and psychovisual properties of the respective spatiotemporal frequency bands. Compression can be achieved by appropriate quantization of the various components and entropy coding of the quantized values.

The most popular approach is to divide the space time volume of the data into two separate spatial and time components. The redundancies are then reduced in different ways in space and in time. Very short filters are used along time (such as Haar filters) since long filters need huge storage, implementation time and will smear motion in the lowpass channel and create artificial high frequencies in the highpass channel. The lowpass output of the 3 dimensional subband decomposition is similar to the original and most of the energy will be contained in the band that has gone through lowpass filtering in all three directions thus iterating the decomposition on this band is most natural. The HLL branch, i.e. highpass filtered over time followed by lowpass filters on two spatial dimensions contains energy every time there is motion so that it can be used as a motion indicator. Another approach which seems more natural is to apply a three dimensional operation directly to the data. Hence, the spatio-temporal redundancies will be reduced together in contrast to the previous approach. But for a good efficiency, it is necessary to perform the 3 dimensional operation on an important number of frames increasing the amount of the memory needed for encoding and decoding. Most 3 dimensional subband decomposition schemes utilize 2 or 4 frame temporal blocks at a time due to practical implementations. The 3 dimensional approach however, represents a simpler architecture when compared to hybrid techniques using motion compensation techniques. In fact, the motion estimation is

one of the most complex parts in any hybrid encoder. The correlation between images is very high if the motion in the image sequence is moderate. In this case, the low frequency components of an image being reconstructed can easily be “picked up” from previous images in a sequence, since the difference information between subsequent images in a sequence is largely of a high frequency nature.

Image sequences being three-dimensional(3-D) signals, i.e. video signals, are generally processed by separately filtering each dimension. However, processing and coding of video signals by true 3-D filtering, one dimension being time, has some advantages.

The advantages of nonseparable filtering versus separable one in the example of subband coding of video sequences can be listed as: i) true 3-D structure is used, ii) they can provide better design possibilities by possessing more free variables in the same dimension, iii) linear phase can be obtained even in 8 band design.

In this chapter, a new design method of 3-D nonseparable, linear phase and perfect reconstruction filter banks is proposed and discussed[103]. Derived filter structures are solved using an optimization procedure and applied to digital video signal. The proposed design method is based on first finding a suitable prototype filter by an optimization procedure and the remaining (M-1) filter impulse responses are found by using proper signed shufflings on the coefficients of that filter to satisfy orthogonality requirements. The $N \times N \times N$ prototype filter is symmetric to guarantee linear phase and $N^3 / 4$ unknown coefficients are found by an optimization procedure. In the proposed design method : i) number of free variables and constraint equations are greatly decreased by using signed shuffling operations, ii) linear phase is obtained directly at the beginning, iii) nonseparability of filters is guaranteed at the very beginning by a centrosymmetric filter structure without any additional constraint on the coefficients.

5.2. Filter Bank Design Requirements

Necessary and sufficient conditions in the design of perfect reconstruction filters is reduced to a set of orthogonality conditions in time domain. These conditions, can be written with $\mathbf{h}_i(\cdot)$ impulse response of the i^{th} filter, $\mathbf{D}$ subsampling matrix, $\mathbf{h}_0$ prototype filter mask and $\mathbf{n}^T = [n_1, n_2, n_3]$, $\mathbf{m}^T = [m_1, m_2, m_3]$ integer vectors in 3-D as,

$$r_{i,j}(\mathbf{Dn}) = \langle \mathbf{h}_i(\mathbf{m}), \mathbf{h}_j(\mathbf{m} + \mathbf{Dn}) \rangle = \delta_{i,j} \delta_{n_1} \delta_{n_2} \delta_{n_3} \quad (5.1)$$

The prototype filter mask $\mathbf{h}_0$ has to be symmetric in order to satisfy the perfect reconstruction conditions given in Equation 5.1 when using signed shuffling operations. This centrosymmetry is also obligatory if linear phase is required in design procedure. The inner product of $4 \times 4 \times 4$ filter impulse responses to provide perfect reconstruction for 3-D cubic 8-band design is shown in Figure 5.1. These inner products are realized by summing the one-to-one product of the the filter masks elements.

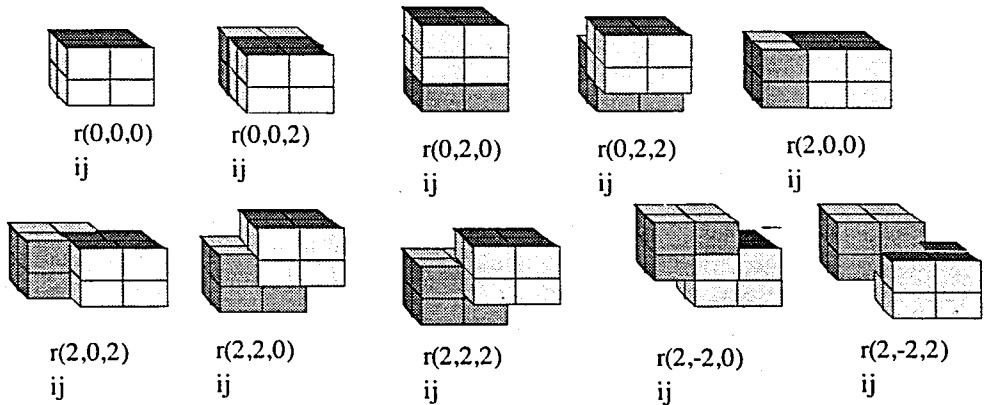


FIGURE 5.1. Inner product realizations of $4 \times 4 \times 4$ supported filter impulse responses for cubic 8-band design

The cubic filter while being symmetric with respect to its center, it will also be centrosymmetric for each of its slices. The even/odd symmetry conditions can therefore be written as,

$$h(n_1, n_2, n_3) = \pm h(N-1-n_1, N-1-n_2, n_3) = \pm h(N-1-n_1, N-1-n_2, N-1-n_3)$$

for

$$0 \leq n_1 \leq N-1, 0 \leq n_2 \leq N-1, 0 \leq n_3 \leq N-1.$$

These centrosymmetric filters are checked to be nonseparable at the beginning of the design.

5.3. Design based on signed shuffling operations for cubic sampling case

Design method in 3-D based on signed shuffling operations is developed in a similar manner as in the design given for 2-D case [79, 83, 84]. The difficulty in the design of 3-D PRFBs was in the implementation of the optimization procedure due to the increase of free variables and constraint equations when compared to the 2-D design case. Let us now investigate the design method based on signed shuffling operations for 3-D case. The freedom in the structures to obtain PR for 3 dimensional design being similar to 2 dimensional design presented in Chapter 4, is obtained by employing the structure given in Equation 4.9b not used in 2-D design case. The proper sign operations make this structure a possible one in satisfying PR for FB.

Let us decompose the impulse response of the $\mathbf{h}_i$ filter into $2 \times 2 \times 2$ blocks denoted as $\mathbf{X}_{k_1, k_2, k_3}^{(i)}$ for the design of 3-D cubic sampling case. Here (k_1, k_2, k_3) denotes the position of the block. Thus, the description of a block in terms of filter's elements and description of the filter in terms of blocks, for $2 \times T = N$ sizes and $0 \leq k_1, k_2, k_3 < T$ position, can be expressed respectively as,

$$\mathbf{X}_{k_1, k_2, k_3}^{(i)}(m_1, m_2, m_3) = \mathbf{h}_i(2k_1 + m_1, 2k_2 + m_2, 2k_3 + m_3), \quad m_1, m_2, m_3 = 0, 1$$

and,

of the filter in terms of blocks, for $2 \times T = N$ sizes and $0 \leq k_1, k_2, k_3 < T$ position, can be expressed respectively as,

$$\mathbf{X}_{k_1, k_2, k_3}^{(i)}(m_1, m_2, m_3) = \mathbf{h}_i(2k_1 + m_1, 2k_2 + m_2, 2k_3 + m_3), \quad m_1, m_2, m_3 = 0, 1$$

and

$$\mathbf{h}_i = \left[\mathbf{X}_{k_1, k_2, k_3}^{(i)} \right]_{T \times T \times T} \quad (5.2)$$

The impulse response of the lowpass filter being $\mathbf{h}_0$, impulse responses of the other $\mathbf{h}_i$ filters can be obtained from $\mathbf{h}_0$ using the following P_i transformations:

$$\mathbf{X}_{k_1, k_2, k_3}^{(i)} = P_i(\mathbf{X}_{k_1, k_2, k_3}^{(0)}) \quad (5.3)$$

For a given $2 \times 2 \times 2$ $\mathbf{X}$ block $P_i(\mathbf{X})$ is as follows:

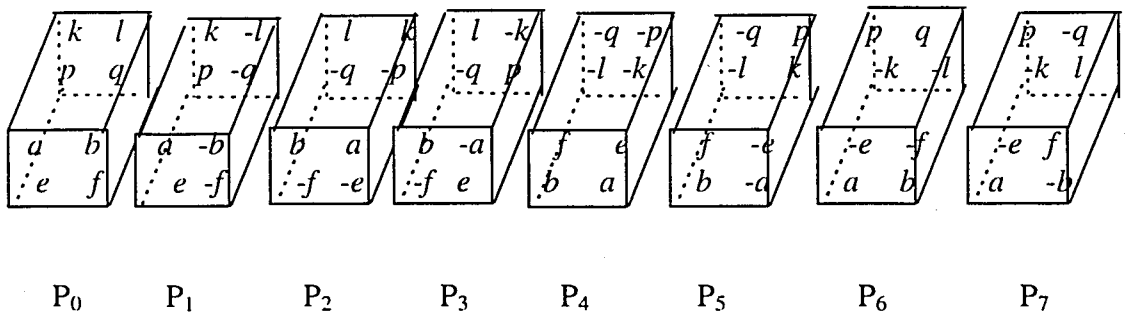


FIGURE 5.2 The $P_i(\mathbf{X})$, $i = 0, \dots, 7$ transformations for a given $2 \times 2 \times 2$ block $\mathbf{X}$ in 3-D cubic sampling design case

Thus the design for 3-D cubic sampling is based on decomposing the filter into $2 \times 2 \times 2$ supported segments whatever its dimension is and other bands filter responses are obtained by applying the proper signed shuffling operations shown in Figure 5.2.

An example to the general solution for 3-D cubic sampling design case is given by the centrosymmetric $4 \times 4 \times 4$ filter impulse responses in Figure 5.3. The centrosymmetry

used in this study is with respect to the center of the cube and also with respect to the center of the centers of slices shown in Figure 5.3.

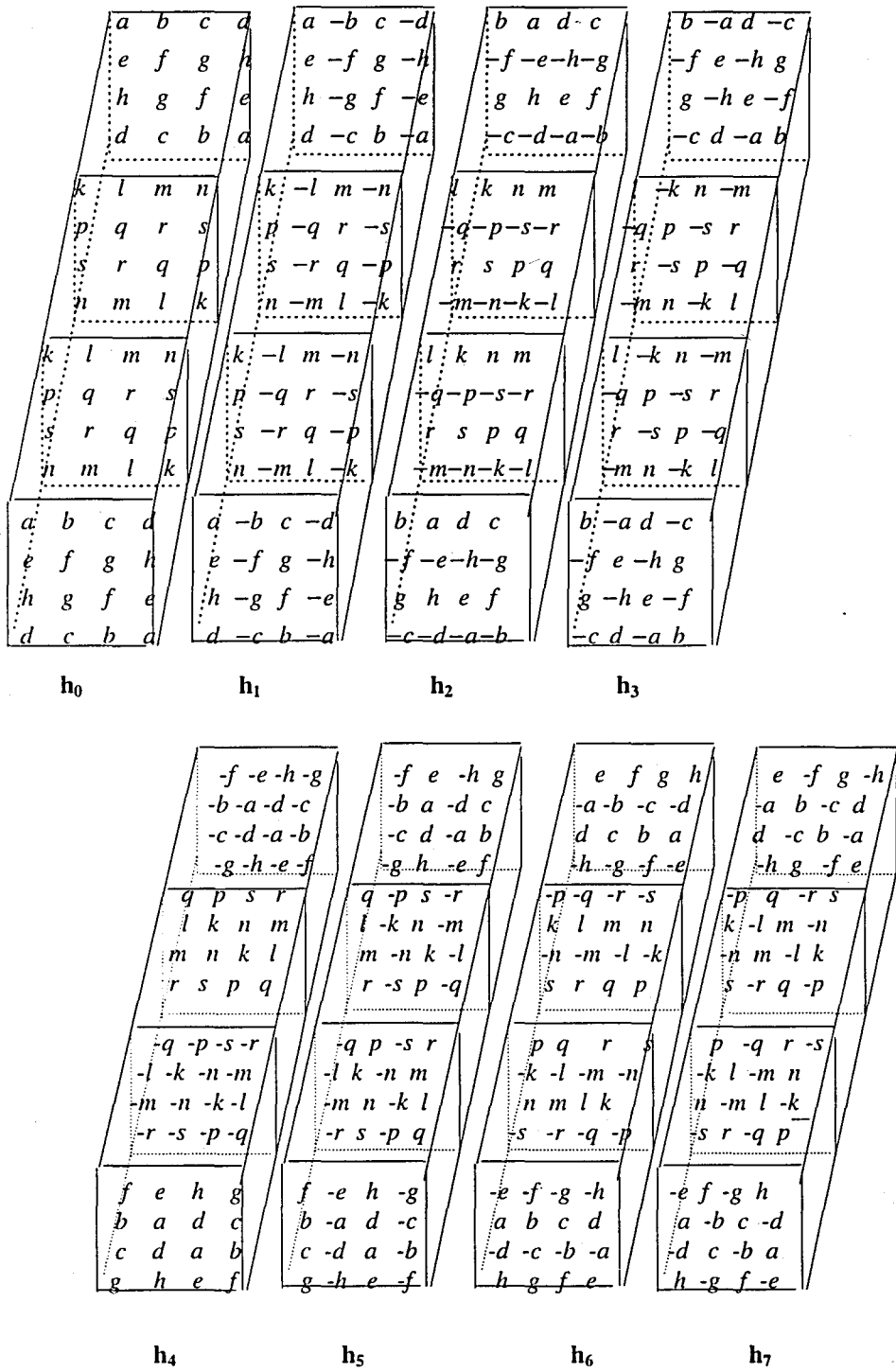


FIGURE 5.3. Centrosymmetric 4x4x4 filter impulse responses for 3-D cubic sampling design case

The structures formed by Equation 5.3 satisfy PR of the FB by needing additional crossorthogonality conditions on the filters beside selforthogonality condition imposed on the lowpass filter. This result impose the design of filters to be changed as in the Figure 5.4 by taking into account only two slices of the filters in the design which provides removal of the additional crossorthogonality conditions. Thus we have at hand a design method for $N \times N \times 2$ sized filters. This causes to take two by two the frames in the sequence, a desired property for 3-D applications reducing the amount of memory and computational complexity.

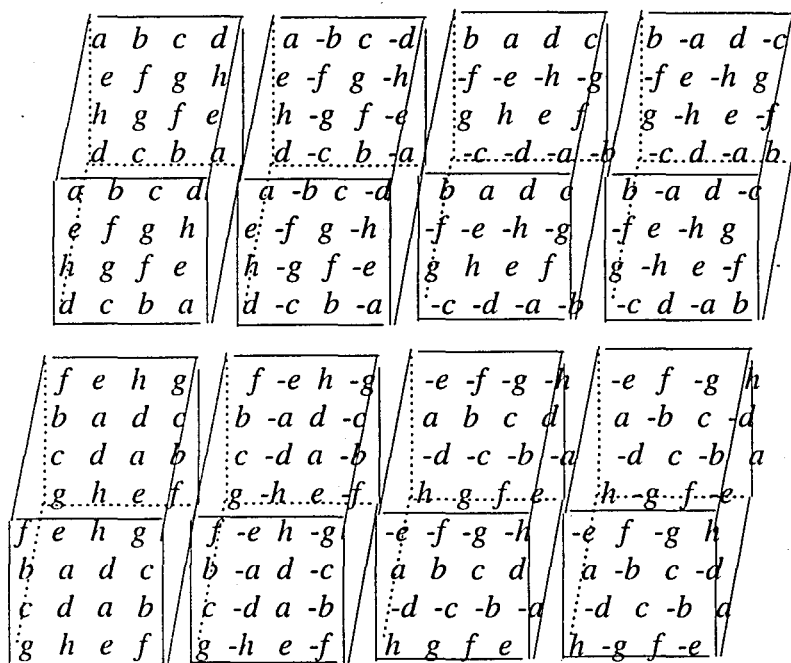


FIGURE 5.4 Centrosymmetric $4 \times 4 \times 2$ filter impulse responses for 3-D cubic sampling design case without requiring any additional constraint equations other than the selforthogonality on the lowpass filter

Expressing these filter structures in terms of Hadamard matrices to be able to compare the results of filter banks for 1 dimensional 8-band shuffle based design case and the three dimensional one,

$$\begin{aligned}
h_0 &= [a \ b \ e \ f \ a \ b \ e \ f \ c \ d \ g \ h \ c \ d \ g \ h \ h \ g \ d \ c \ h \ g \ d \ c \ f \ e \ b \ a \ f \ e \ b \ a] \\
h_1 &= [b \ -a \ -f \ e \ b \ -a \ -f \ e \ d \ -c \ -h \ g \ d \ -c \ -h \ g \ g \ -h \ -c \ d \ g \ -h \ -c \ d \ e \ -f \ -a \ b \ e \ -f \ -a \ b] \\
h_2 &= [a \ -b \ e \ -f \ a \ -b \ e \ -f \ c \ -d \ g \ -h \ c \ -d \ g \ -h \ h \ -g \ d \ -c \ h \ -g \ d \ -c \ f \ -e \ b \ -a \ f \ -e \ b \ -a] \\
h_3 &= [b \ a \ -f \ -e \ b \ a \ -f \ -e \ d \ c \ -h \ -g \ d \ c \ -h \ -g \ g \ h \ -c \ -d \ g \ h \ -c \ -d \ e \ f \ -a \ -b \ e \ f \ -a \ -b] \\
h_4 &= [f \ e \ b \ a \ -f \ -e \ -b \ -a \ h \ g \ d \ c \ -h \ -g \ -d \ -c \ c \ d \ g \ h \ -c \ -d \ -g \ -h \ a \ b \ e \ f \ -a \ -b \ -e \ -f] \\
h_5 &= [-e \ f \ a \ -b \ e \ -f \ -a \ b \ -g \ h \ c \ -d \ g \ -h \ -c \ d \ -d \ c \ h \ -g \ d \ -c \ -h \ g \ -b \ a \ f \ -e \ b \ -a \ -f \ e] \\
h_6 &= [f \ -e \ b \ -a \ -f \ e \ -b \ a \ h \ -g \ d \ -c \ -h \ g \ -d \ c \ c \ -d \ g \ -h \ -c \ d \ -g \ h \ a \ -b \ e \ -f \ -a \ b \ -e \ f] \\
h_7 &= [-e \ -f \ a \ b \ e \ -f \ -a \ -b \ -g \ -h \ c \ d \ g \ h \ -c \ -d \ -d \ -c \ h \ g \ d \ c \ -h \ -g \ -b \ -a \ f \ -e \ b \ a \ -f \ -e]
\end{aligned}
\tag{5.4}$$

Letting $x = [a, b]$, $y = [e, f]$, $z = [c, d]$, $t = [g, h]$ and define

$$Q_1(a, b) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad Q_2(a, b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad Q_3(a, b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad Q_4(a, b) = \begin{bmatrix} -a & b \\ b & a \end{bmatrix}$$

then

$$H = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1(x) & Q_1(y)J & Q_1(x) & Q_1(y)J & Q_1(z) & Q_1(t)J & Q_1(z) & Q_1(t)J & Q_1(t) & Q_1(z)J & Q_1(t) & Q_1(z)J & Q_1(y) & Q_1(x)J & Q_1(x)J \\ Q_2(x) & -Q_2(y)J & Q_2(x) & -Q_2(y)J & Q_2(z) & -Q_2(t)J & Q_2(z) & -Q_2(t)J & Q_2(t) & -Q_2(z)J & Q_2(t) & -Q_2(z)J & Q_2(x) & -Q_2(y)J & -Q_2(x)J \\ Q_1(y)J & Q_3(x)J & -Q_1(y)J & -Q_3(x)J & Q_1(t) & Q_3(z)J & -Q_1(t)J & -Q_3(z)J & Q_3(z) & Q_1(t) & -Q_3(z) & -Q_1(t)J & Q_3(x) & Q_1(y) & -Q_3(x) & -Q_1(y) \\ -Q_2(y)J & Q_4(x)J & Q_2(y)J & -Q_4(x)J & -Q_2(t) & Q_4(z)J & Q_2(t)J & -Q_4(z)J & -Q_4(z) & Q_2(t) & Q_4(z) & -Q_2(t) & -Q_4(x) & Q_2(y) & Q_4(x) & -Q_2(y) \end{bmatrix}
\tag{5.5}$$

In contrast to three dimensional design, there exist additional constraint equations to be satisfied other than the selforthogonality ones imposed on the lowpass filter in 1-D 8-band design case.

5.4. Frequency domain interpretation

Assuming that the first filter $h_0(n_1, n_2, n_3)$ has the following structure,

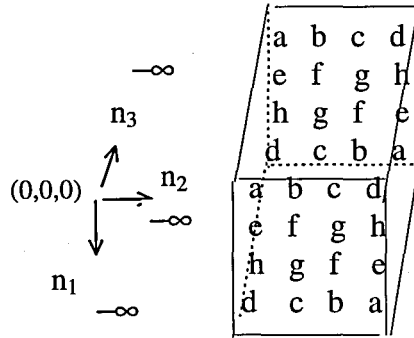


FIGURE 5.5. Structure of 3-D filter

and with the transforms $h_0(n_1, n_2, n_3) \leftrightarrow H_0(z_1, z_2, z_3)$, the other 7 filters frequency domain expressions can be expressed for 8-band cubic sampling design case as,

$$H_1(z_1, z_2, z_3) = H_0(z_1, -z_2, z_3)$$

With $z_i = \exp(jw_i), i = 1, \dots, p$;

$$\begin{aligned}
 TT1(z_1, z_2, z_3) &= \frac{1}{2} z_2^{-1} [H_0(z_1, z_2, z_3) + H_0(z_1, -z_2, z_3)] + \frac{1}{2} z_2 [H_0(z_1, z_2, z_3) - H_0(z_1, -z_2, z_3)] \\
 &= H_0(z_1, z_2, z_3) \cos(w_2) - jH_0(z_1, -z_2, z_3) \sin(w_2)
 \end{aligned}$$

The third filter structure can be found by

$$H_2(z_1, z_2, z_3) = TT1(-z_1, z_2, z_3)$$

$$= H_0(-z_1, z_2, z_3) \cos(w_2) - jH_0(-z_1, -z_2, z_3) \sin(w_2)$$

Similarly, the fourth filter is in the form of,

$$H_3(z_1, z_2, z_3) = TT1(-z_1, -z_2, z_3)$$

$$= -H_0(-z_1, -z_2, z_3) \cos(w_2) + jH_0(-z_1, z_2, z_3) \sin(w_2)$$

The fifth and sixth filter uses a second transformation $TT2$ such that

$$\begin{aligned}
 TT2 &= \frac{1}{2} z_1^{-1} [TT1(z_1, z_2, z_3) + TT1(-z_1, z_2, z_3)] + \frac{1}{2} z_1 [TT1(z_1, z_2, z_3) - TT1(-z_1, z_2, z_3)] \\
 &= \frac{1}{2} z_1^{-1} [(H_0(z_1, z_2, z_3) + H_0(-z_1, z_2, z_3)) \cos(w_2) - j(H_0(z_1, -z_2, z_3) + H_0(-z_1, -z_2, z_3)) \sin(w_2)] + \\
 &\quad + \frac{1}{2} z_1 [(H_0(z_1, z_2, z_3) - H_0(-z_1, z_2, z_3)) \cos(w_2) - j(H_0(z_1, -z_2, z_3) - H_0(-z_1, -z_2, z_3)) \sin(w_2)]
 \end{aligned}$$

$$= H_0(z_1, z_2, z_3) \cos(w_1) \cos(w_2) - jH_0(-z_1, z_2, z_3) \sin(w_1) \cos(w_2) - \\ - jH_0(-z_1, z_2, z_3) \sin(w_1) \cos(w_2) - H_0(-z_1, -z_2, z_3) \sin(w_1) \sin(w_2)$$

$$H_4(z_1, z_2, z_3) = TT2(z_1, z_2, -z_3)$$

$$= H_0(z_1, z_2, -z_3) \cos(w_1) \cos(w_2) - jH_0(-z_1, z_2, -z_3) \sin(w_1) \cos(w_2) - \\ - jH_0(-z_1, z_2, -z_3) \sin(w_1) \cos(w_2) - H_0(-z_1, -z_2, -z_3) \sin(w_1) \sin(w_2)$$

$$H_5(z_1, z_2, z_3) = TT2(z_1, -z_2, -z_3)$$

$$= -H_0(z_1, -z_2, -z_3) \cos(w_1) \cos(w_2) + jH_0(-z_1, -z_2, -z_3) \sin(w_1) \cos(w_2) + \\ + jH_0(-z_1, -z_2, -z_3) \sin(w_1) \cos(w_2) + H_0(-z_1, z_2, -z_3) \sin(w_1) \sin(w_2)$$

The seventh and eight filter uses a third transformation $TT3$ such that

$$TT3(z_1, z_2, z_3) = \frac{1}{2} z_2^{-1} [TT2(z_1, z_2, z_3) + TT2(z_1, -z_2, z_3)] + \frac{1}{2} z_2 [TT2(z_1, z_2, z_3) - TT2(z_1, -z_2, z_3)]$$

$$TT3(z_1, z_2, z_3) =$$

$$= \frac{1}{2} z_2^{-1} \left(\begin{array}{l} H_0(z_1, z_2, z_3) (\cos(w_1) \cos(w_2) - j \cos(w_1) \sin(w_2)) + \\ H_0(-z_1, z_2, z_3) (\sin(w_1) \sin(w_2) - j \cos(w_1) \sin(w_2)) + \\ H_0(z_1, -z_2, z_3) (-\cos(w_1) \cos(w_2) - j \cos(w_1) \sin(w_2)) + \\ H_0(-z_1, -z_2, z_3) (-\sin(w_1) \sin(w_2) + j \sin(w_1) \cos(w_2)) \end{array} \right) + \\ + \frac{1}{2} z_2 \left(\begin{array}{l} H_0(z_1, z_2, z_3) (\cos(w_1) \cos(w_2) + j \cos(w_1) \sin(w_2)) + \\ H_0(-z_1, z_2, z_3) (-\sin(w_1) \sin(w_2) - j \cos(w_1) \sin(w_2)) + \\ H_0(z_1, -z_2, z_3) (\cos(w_1) \cos(w_2) - j \cos(w_1) \sin(w_2)) + \\ H_0(-z_1, -z_2, z_3) (-\sin(w_1) \sin(w_2) - j \sin(w_1) \cos(w_2)) \end{array} \right)$$

$$\begin{aligned}
& H_0(z_1, z_2, z_3) (\cos(w_1) \cos(w_2) \cos(w_2) - \cos(w_1) \sin(w_2) \sin(w_2)) + \\
& + jH_0(-z_1, z_2, z_3) (\sin(w_1) \sin(w_2) \sin(w_2) - \cos(w_1) \cos(w_2) \sin(w_2)) + \\
= & + jH_0(z_1, -z_2, z_3) (\cos(w_1) \cos(w_2) \sin(w_2) - \cos(w_1) \cos(w_2) \sin(w_2)) + \\
& H_0(-z_1, -z_2, z_3) (-\sin(w_1) \sin(w_2) \cos(w_2) - \sin(w_1) \cos(w_2) \sin(w_2))
\end{aligned}$$

$$H_6(z_1, z_2, z_3) = TT3(-z_1, z_2, -z_3)$$

$$\begin{aligned}
& H_0(-z_1, z_2, -z_3) (-\cos(w_1) \cos(w_2) \cos(w_2) + \cos(w_1) \sin(w_2) \sin(w_2)) + \\
& + jH_0(z_1, z_2, -z_3) (-\sin(w_1) \sin(w_2) \sin(w_2) + \cos(w_1) \cos(w_2) \sin(w_2)) + \\
= & + jH_0(-z_1, -z_2, -z_3) (-\cos(w_1) \cos(w_2) \sin(w_2) + \cos(w_1) \cos(w_2) \sin(w_2)) + \\
& H_0(z_1, -z_2, -z_3) (\sin(w_1) \sin(w_2) \cos(w_2) + \sin(w_1) \cos(w_2) \sin(w_2))
\end{aligned}$$

$$H_7(z_1, z_2, z_3) = TT3(-z_1, -z_2, -z_3)$$

$$\begin{aligned}
& H_0(-z_1, -z_2, -z_3) (-\cos(w_1) \cos(w_2) \cos(w_2) + \cos(w_1) \sin(w_2) \sin(w_2)) + \\
& + jH_0(z_1, -z_2, -z_3) (-\sin(w_1) \sin(w_2) \sin(w_2) + \cos(w_1) \cos(w_2) \sin(w_2)) + \\
= & + jH_0(-z_1, z_2, -z_3) (-\cos(w_1) \cos(w_2) \sin(w_2) + \cos(w_1) \cos(w_2) \sin(w_2)) + \\
& H_0(z_1, z_2, -z_3) (\sin(w_1) \sin(w_2) \cos(w_2) + \sin(w_1) \cos(w_2) \sin(w_2))
\end{aligned}$$

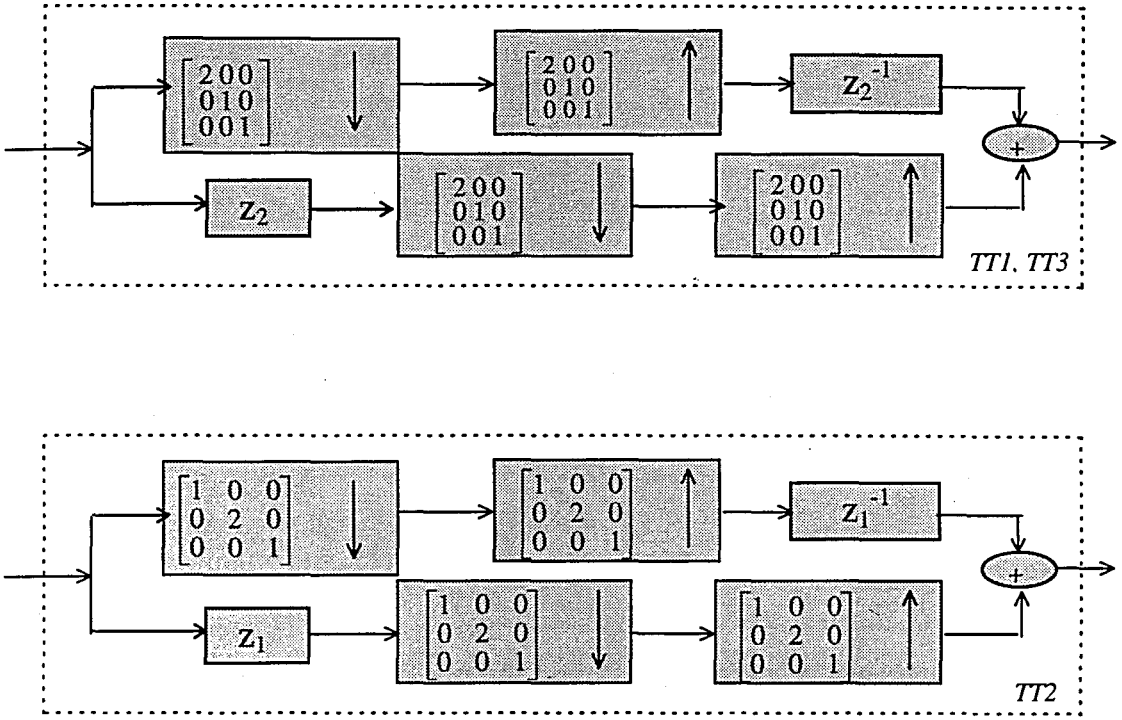


FIGURE 5.5 The structures obtained using shuffling operators

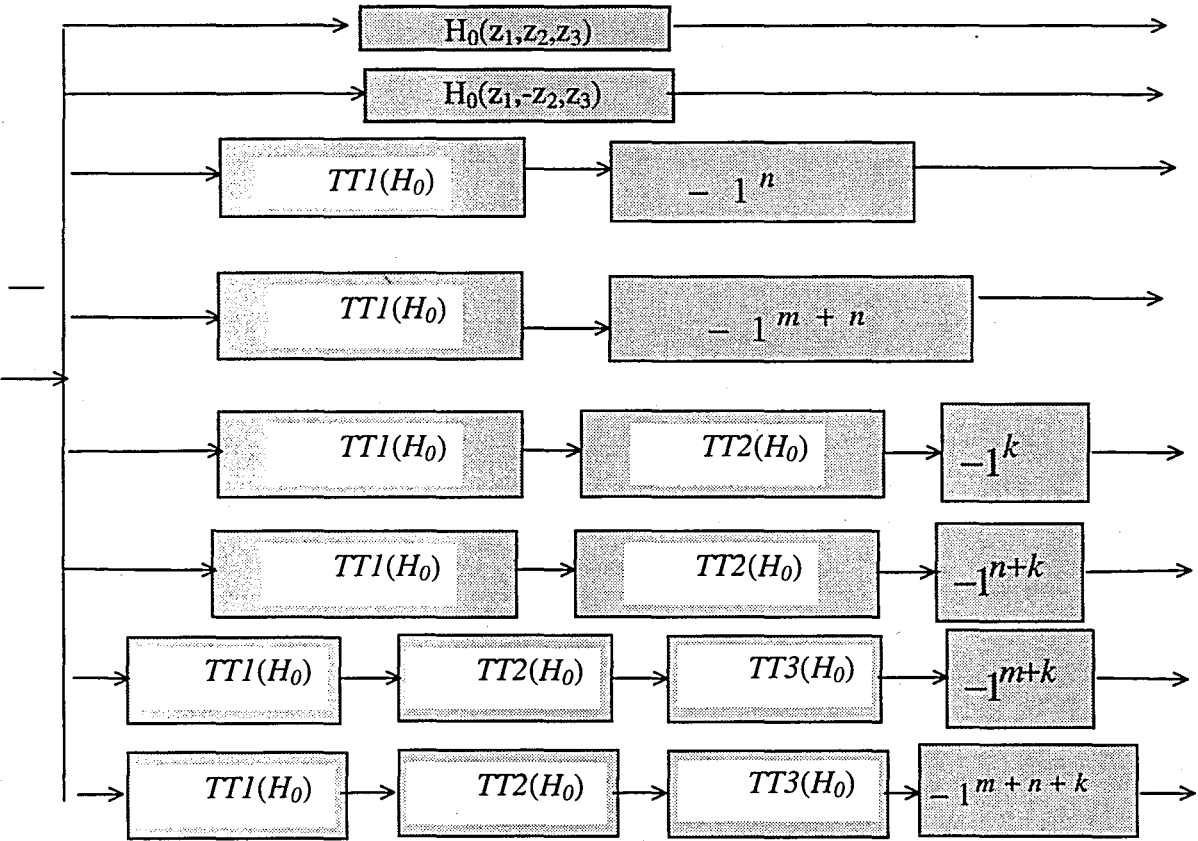


FIGURE 5.6 The frequency domain interpretation of the three dimensional cubic sampling design filters

5.5. Results

We applied our method to the first 8 frames of the well known “MissAmerica” image sequence where each image is of size .

For practical applications, short filters are preferred and so in the implementations, the possible minimum sized $4 \times 4 \times 4$ filter banks with the structures shown in Figure 5.3 for the FB structure of Figure 2.15. a are used. Although the fact of using the same coefficients for all filters in filter bank causes a constraint in the spectral selectivity of the designed filters, it can be overcome by iterating the tree three times. The number of unknowns decreased from $8 \times (4)^3$ to $(4)^3/4$ can be satisfied by 14 constraint equations. The order of the bands in PRFB in terms of energy is found to be $\{1,4,5,2,6,8,3,7\}$ by using notation of the Figure 5.7. In the thesis, first “head and shoulder” video sequence $256 \times 256 \times 8$ “MissAmerica” is decomposed into its subbands, then a “full motion” sequence “Tennis” is processed in the same way. The filtered by $\mathbf{h}_0$, $\mathbf{h}_1$ and subsampled versions of the typical 2^{nd} and 4^{th} frames of the sequence are respectively given in (c) and(d) in Figure 5.7 and Figure 5.8.

The use of signed shuffling operations provides decrease in free variables and constraint equations to satisfy perfect reconstruction. The proposed solution is valid for all sizes of the filters. The computational complexity and memory requirements are decreased by the proposed design method.

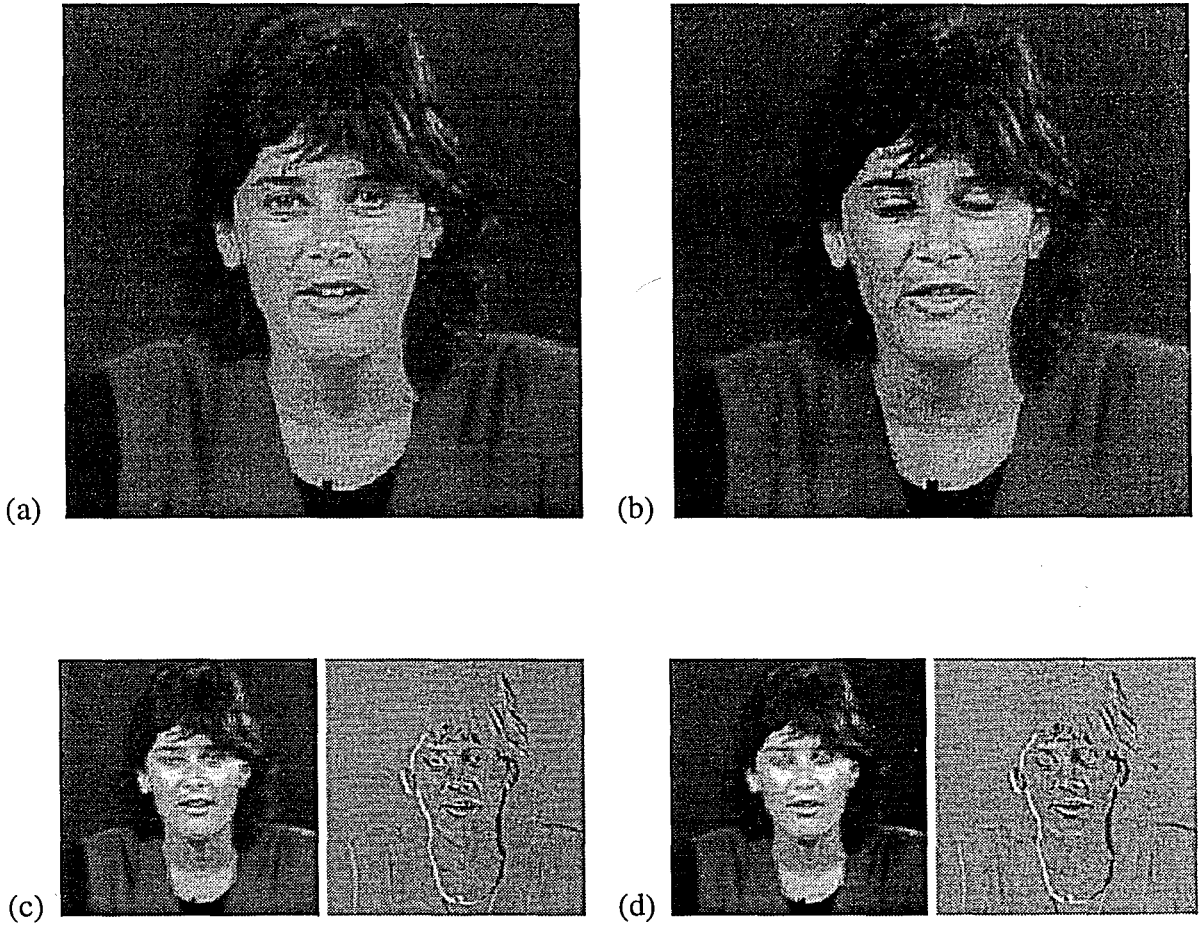


FIGURE 5.7. . Outputs of (a) 2nd and (b) 4th frames of 256×256×8 “MissAmerica” video sequence after filtering by h_0 , h_1 and subsampling operations are respectively given in (c), (d).

(e)



FRAME 3

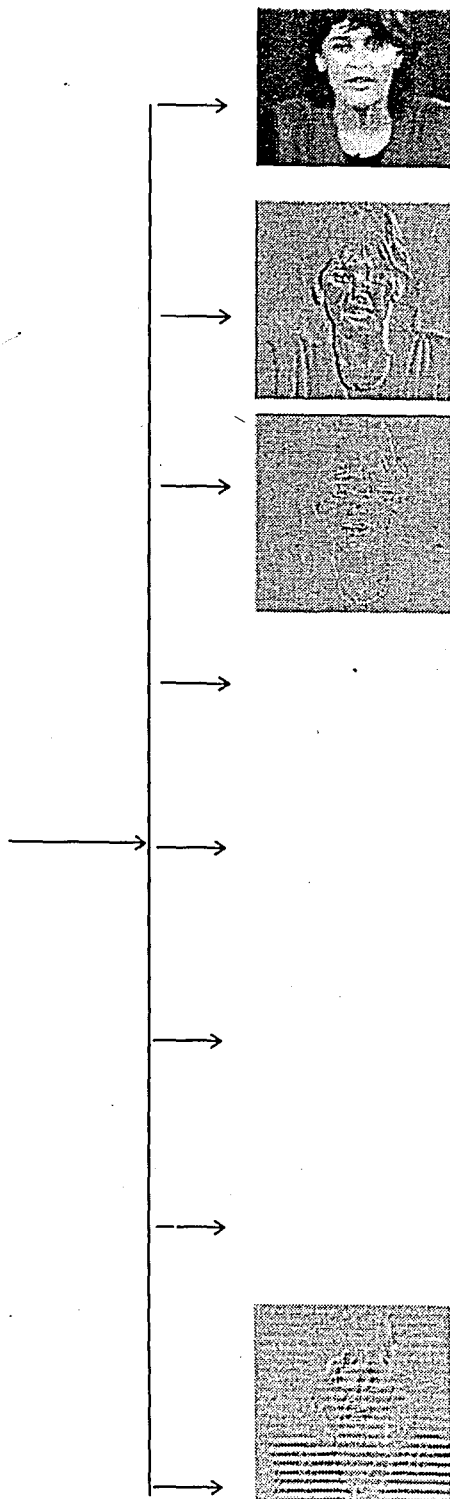


FIGURE 5.7. (Continued) Some subbands for the 3rd frame are given in (e).

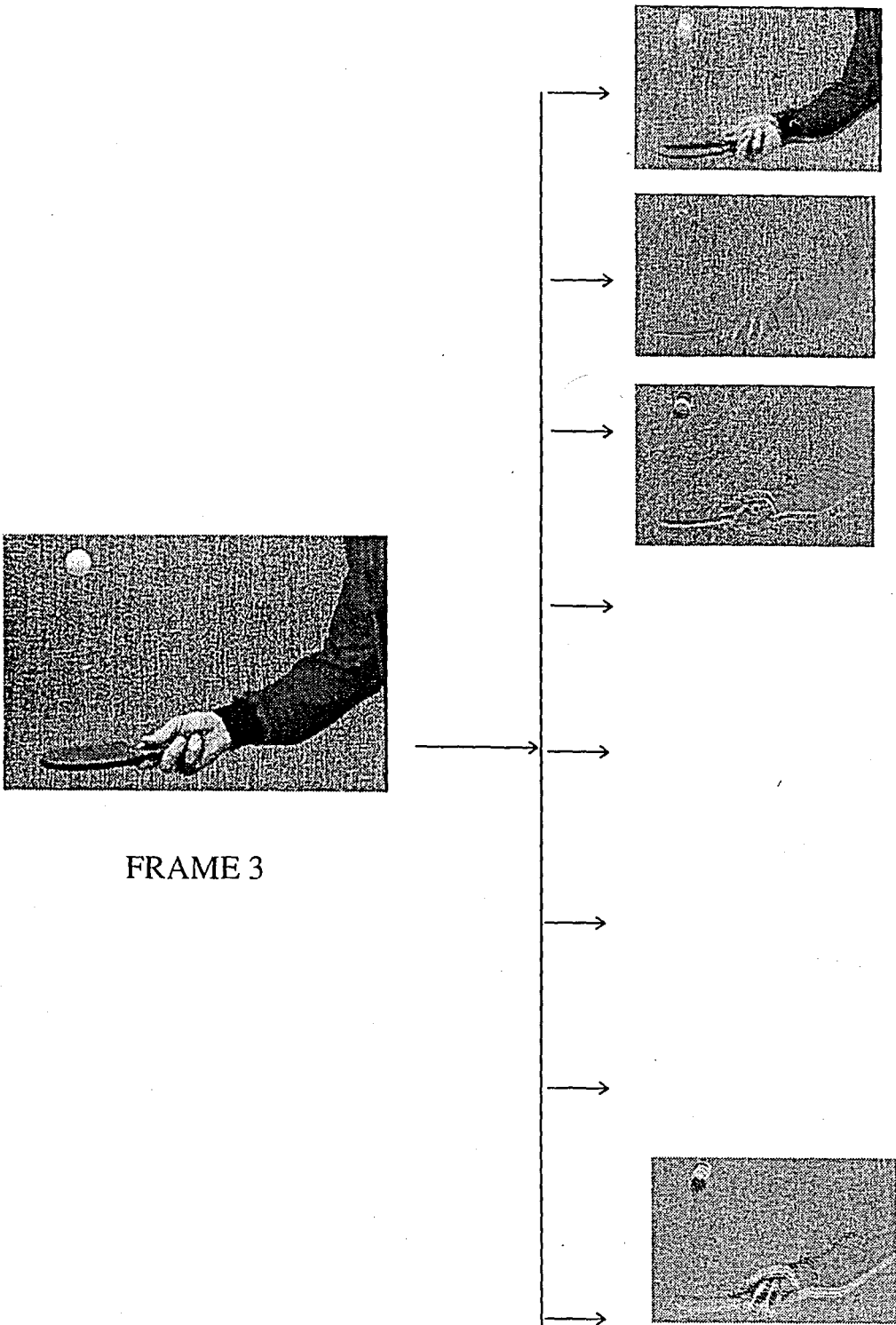


FIGURE 5.8 . Outputs of 3rd frame of 340×560×8 “Tennis” video sequence after filtering by 3-dimensional $h_0, h_1, \dots, h_7$ filters

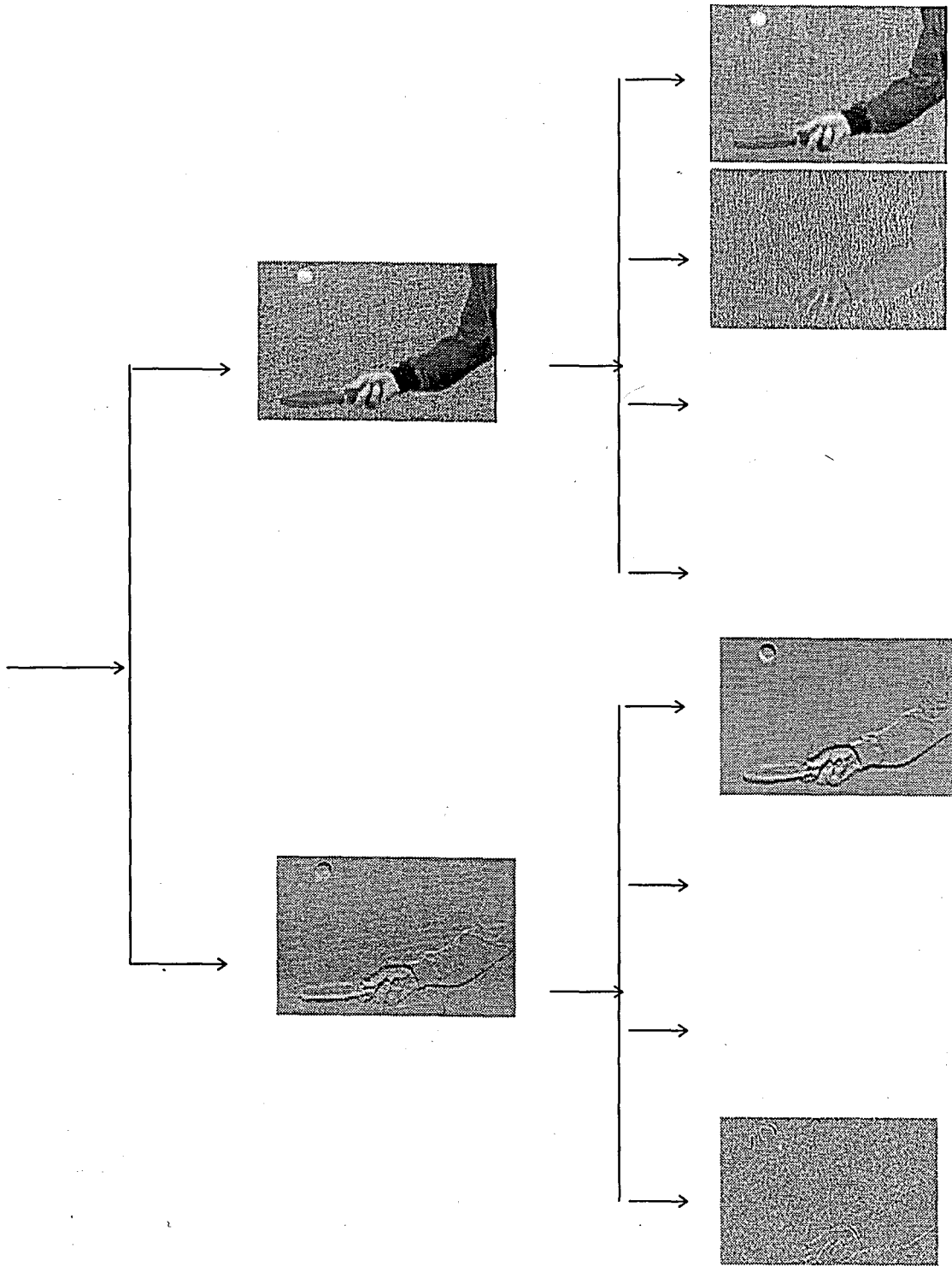


FIGURE 5.9. Outputs of 3rd frame of 340×560×8 “Tennis” video sequence after filtering by two tap Haar filter and then 8×8 two dimensional nonseparable filters of Chapter 4

TABLE 5.1 Subband variances for two different frames when using 3-dimensional nonseparable filtering for "MissAmerica" video sequence

	Frame 3 (mean, variance)	Frame 7 (mean, variance)
Subband 1	(65.013300, 2979.432923)	(65.016900, 3018.313414)
Subband 2	(118.951600, 224.595657)	(114.869300, 141.520618)
Subband 3	(150.110000, 192.293500)	(157.851300, 192.616988)
Subband 4	(140.012800, 188.913236)	(106.620500, 164.156480)
Subband 5	(117.817300, 43.479921)	(110.099000, 145.534399)
Subband 6	(124.679300, 77.359252)	(164.607300, 157.376687)
Subband 7	(139.696300, 279.330466)	(147.996600, 247.670788)
Subband 8	(120.960200, 196.805016)	(113.630300, 221.720022)

As seen from Table 5.1 subband 1 has the most of the energy while remaining subbands's variances are low when compared to it allowing compression possible.

The use of signed shuffling operations provides decrease in free variables and constraint equations to satisfy perfect reconstruction. The computational complexity and memory requirements are decreased by the proposed design method which can work for any sizes of spatial filters.

7. CONCLUSIONS AND FUTURE RESEARCH

Design issues related to statistically matched and waveform matched wavelet design are investigated for 1-D applications. The energy compaction performance of two-band schemes vis-à-vis Karhunen Loeve transform is analyzed. Improvements on the one dimensional Frazier Kumar matched wavelet filter design method including flexibility in the choice of reference function and decrease of computational complexity are given. Analysis of 1-dimensional Frazier Kumar design method in terms of energy compaction criteria using the eigenvectors/eigenspectra of a process as reference function to reach the optimal transform coding gain is done. A new time domain formulation of matched wavelet design has been proposed and compared with Frazier Kumar design technique. This time domain based approach can be easily used in applications since of its exactness and computational ease.

Using a proposed time-domain formulation a new class of efficient filter bank are derived. Different desired properties of this filter bank, such as regularity, linear phase, nonseparability and the degree of match could be attributed in a controlled manner to these filter banks. These filters have a centrosymmetric structure which has three important consequences: 1) filters have linear phase, 2) maximum number of free variables are obtained, 3) filters are nonseparable. The comparison showed that the proposed filter bank is more efficient than other filter banks frequently used in compression applications in terms of computational complexity, while presenting regularity, linear phase of filters. The proposed design method first finds for all sampling lattices, a suitable lowpass filter, while the remaining $M-1$ filters are obtained through use of signed shuffling operations on the coefficients of that filter. All filter requirements are imposed on the lowpass filter (nonseparability, linear phase, regularity, etc.).

A proof of uniqueness of the time-domain shuffle-based multidimensional, nonseparable, linear phase design is also given based on permutation-elimination method. In principle any matched filter bank can be designed by some brute force optimization. In this respect, the use of shuffling has two consequences :

On the positive side it reduces the number of variables in the optimization dramatically, that is, by a factor of 8. With this reduced set of variables implementation of the optimization becomes much more feasible.

On the negative side the spectral liberty of the bands become restricted to the modulations inherent in the coefficient permutations.

The proposed method provides a complete search for possible combinations of filter structures in 2-D M-band nonseparable linear phase PRFBs design under the spectral selection restriction of permutations when compared to other designs in the literature [88, 36].

Computation performance of the proposed filter banks is compared with respect to separable filtering and DCT (Discrete Cosine Transform). Nonseparable linear phase filters designed using the shuffling method, are based on the maximization of the coding gain performance with respect to a generalized correlation model and with respect to nonseparable Gaussian model. These results show that the proposed design method can provide higher transform coding gain vis-à-vis DCT by as much as 50 % when the tree is iterated three times such that the effect of uncontrallable bands diminish. The superiority of the 2-D nonseparable scheme holds true both with real images(Lenna) as well as theoretical correlation models.

The 2-D shuffling -based time-domain approach presented in this thesis is extended also to three dimensional PRFBs. For practical applications, short filters are preferred. Such 3-D filters constitute an interesting alternative to video compression as opposed to motion compensated compression.

These time domain shuffle based nonseparable and linear phase filter design methods can be easily used in many applications since they reduce the computational complexity, processing time and they provide better results when compared to separable filtering by iterating the tree 3 times.

An important issue in subband decomposition is the way in which the frequency decomposition is performed. An adaptive subband decomposition technique using a particular case of packet wavelet decomposition is employed for the still image compression applications. So, an adaptive tree structure using the proposed nonseparable, linear phase filters have been tested to improve subjective quality in image coding applications. The adaptive wavelet packet tree algorithm of Ramchandran et.al. [89] was used. Rate distortion curves for 3 level iterations of the tree and 3 types of MSE measures are studied.

As future research, we can state that:

The nonseparable filter coefficients used in this thesis are found by maximizing the energy compaction. So, there may be improved optimization methods to increase the performance.

The proposed shuffle based design method is very efficient for VLSI implementation. Chip design of this proposed method can be very easily realized.

In quantization step for compression applications, quantizers that can adapt to the spectral controllability of the proposed shuffle based design will improve the compression performance

The computational ease and decrease memory requirements of the proposed shuffle based design can be used in the source coding part of a system for video transmission over a packet-switched network.

Appendix A. Energy compaction and properties of the autocorrelation matrix

A.1. Orthonormality requirements

The orthogonality condition of the filter impulse responses is important particularly in signal coding applications and orthonormality at even shifts is essential for the filter bank. The high-pass filter is assumed to be the mirror of the half-band lowpass filter. Let us denote the filter impulse response $\{h(n)\}$ of length $2N$ which in vector form as $\mathbf{h}$, the high-pass filter is assumed to be the mirror of the half-band lowpass filter. The unitary condition of the filter can be written in inner product form as

$$\mathbf{h}^T \mathbf{h} = 1 \quad (\text{A.1})$$

The perfect reconstruction condition of orthonormal 2-band Perfect Reconstruction Quadrature Mirror Filter (PR-QMF) is found to be in the time domain as:

$$\sum_n h(n)h(n+2k) = \delta(k), \quad k = 0, 1, \dots, (N-1). \quad (\text{A.2})$$

Equations A.1 and A.2 can be combined in the matrix form

$$\mathbf{h}^T \mathbf{C}_k \mathbf{h} = 0 \quad k = 0, 1, 2, \dots, N-1 \quad (\text{A.3})$$

where $\mathbf{C}_0 = \mathbf{I}$ and $\mathbf{C}_k, k > 1$ are the shuffling matrices defined as:

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 \\ 1 & 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}, \dots, C_{N-1} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 1 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 \end{bmatrix} \quad (A.4)$$

These orthonormality requirements in Equation A.3 appear as a constraint term in the filter optimization in Section 3.2.

A.2. Energy Compaction

Energy compaction is a desired feature for any orthonormal signal decomposition technique. This performance measure has been widely used in the literature [89] for the comparison of different signal decomposition techniques.

The output energy of the lowpass filter $\mathbf{h}$ when the input is a zero mean random process with covariance matrix $\mathbf{R}_{xx}$ can be expressed as

$$\sigma_L^2 = \mathbf{h}^T \mathbf{R}_{xx} \mathbf{h} \quad (A.5)$$

It is known that the coding gain in a subband decomposition scheme is maximized when the subband energies are lumped into as few bands as possible. In a two-band scheme this requirement reduces to the condition that the output of one of the filters should be maximized. Without loss of generality we can look for an optimal PR-QMF solution which

maximizes Equation A.5, that is the energy of the lowpass section. It is clear that this will be the sufficient condition to maximize the energy compaction measure, G over PCM, which for the two-band case is given as

$$G = \frac{\sigma_x^2}{(\sigma_L^2 \sigma_H^2)^{1/2}} \quad (\text{A.6})$$

where σ_x^2 , σ_L^2 and σ_H^2 denote respectively, the variance of the total signal, its lowpass subband and its highpass subband.

A.3. Properties of the correlation matrix $\mathbf{R}_{xx}$

It was explained in Section 3.1 that in real world applications AR(1) model is extensively used. Matched wavelet filters were designed for an autoregressive AR(1) model, with autocorrelation function,

$$R_{xx}(k) = \sigma_x^2 \rho^{|k|} \quad k = 0, \pm 1, \pm 2 \quad (\text{A.7})$$

The autocorrelation matrix is of the form $\mathbf{R}_{xx} =$

$$\begin{bmatrix} 1 & \rho & \rho^2 & \cdot & \rho^{N-1} \\ \rho & 1 & \rho & \cdot & \rho^{N-2} \\ \rho^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho^{N-1} & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

and it is a symmetric Toeplitz matrix.

Property 1 [90]. If $\mathbf{R}$ is an $M \times M$, $M=2N$, symmetric Toeplitz matrix which is partitioned as

$$\mathbf{R} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{C} \end{bmatrix}$$

then the eigenvectors of $\mathbf{R}$ are of the form

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{J}\mathbf{x} \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{y} \\ -\mathbf{J}\mathbf{y} \end{bmatrix} \quad (\text{A.8})$$

where $\mathbf{x}$ and $\mathbf{y}$ are the eigenvectors of the $N \times N$ matrices, $\mathbf{C} + \mathbf{J}\mathbf{D}$ and $\mathbf{C} - \mathbf{J}\mathbf{D}$ respectively. Here $\mathbf{J}$ is the exchange matrix with ones along the cross diagonal. when $\mathbf{J}$ premultiplies a vector it reverses the order of its elements.

Property 2. For a doubly symmetric matrix of size $M \times M$ and having distinct eigenvalues, there are $\lfloor (M+1)/2 \rfloor$ symmetric eigenvectors and $\lfloor M/2 \rfloor$ skew symmetric eigenvectors ($\lfloor X \rfloor$ denotes the largest integer less than or equal to X).

Gershgorin 's theorem may be restated as follows

Let $\mathbf{A}$ be any M by M complex-valued matrix with element a_{ij} where $i, j = 1, 2, \dots, M$ and let $\lambda_1, \dots, \lambda_M$ be its eigenvalues. Denote by D_i the i^{th} Gershgorin disk with center a_{ii} and radices,

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^M |a_{ij}|, \quad i = 1, 2, \dots, M$$

Denote by Ω the union of these M disks. Then all the eigenvalues of matrix $\mathbf{A}$ lie inside Ω .

Applying this theorem to the special case of the correlation matrix $\mathbf{R}$, whose eigenvalues are real and nonnegative, it is found that the eigenvalues of the correlation matrix $\mathbf{R}$ lie on a strip of the real axis. Thus, the following property results:

Property 3 The largest eigenvalue of the correlation matrix $\mathbf{R}$ is bounded by

$$\lambda_{\max} \leq \sum_{k=0}^{M-1} |r(k)| \quad (\text{A.9})$$

and its smallest eigenvalue is bounded by

$$\lambda_{\min} \leq r(0) - \sum_{k=0}^{M-1} |r(k)|$$

where $r(k)$ for $k = 0, \dots, M - 1$ are entries with values depending on the process in the M by M correlation matrix $\mathbf{R}$.

These properties will be used in finding PR-QMF bounds in 2-band design in Section 3.2.2. and in Appendix C.

Appendix B. Coding gains with ideal filters (infinite number of taps)

The upper bounds for the compaction gain G of a subband decomposition of any order is given by an ideal filter bank, that is filters having an infinite number of taps and no spectral overlap. The ideal filter bank is optimal since they provide perfect interband decorrelation for any signal source. The compaction upper bound is calculated using the powers of the subbands in the formula given as,

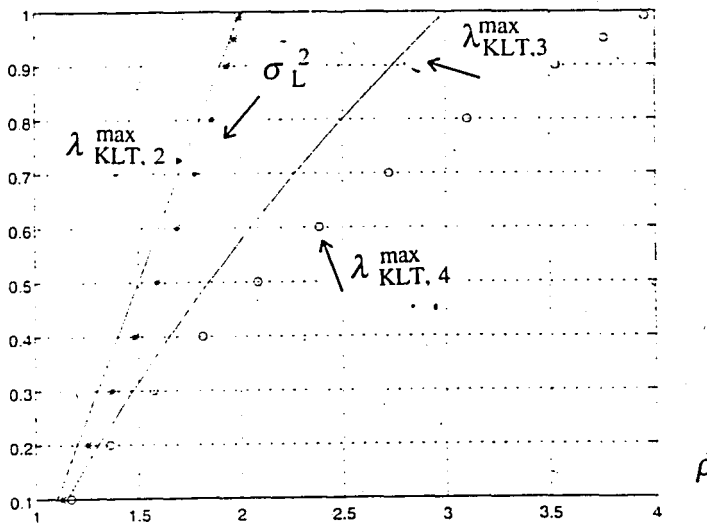
$$G_{\text{ideal}} = \frac{\sigma_x^2}{\left[\prod_{i=1}^M \sigma_i^2 \right]^{1/M}}$$

for σ_i^2 , $i = 0, \dots, M-1$ band variances of the uniformly spaced M -band filter. For AR(1) source model and a two-band design, the variance of the lowpass section of an ideal filter is

found as $\sigma_L^2 = \frac{4}{\pi} \text{Arctan} \left(\frac{1+\rho}{1-\rho} \right)$. It is interesting to compare this result with the maximum

eigenvalues of $L \times L$ $L > 2$. We find that the variance of the 2-band PR-QMF using ideal filters is always smaller than the maximum eigenvalues of $L \times L$, e.g., $L=3, 4$..as seen in

Figure B.1.



(a)

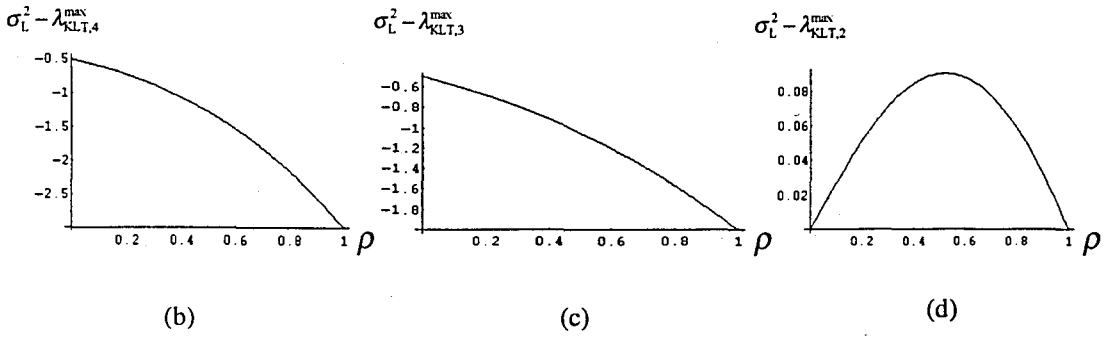


FIGURE B.1. The maximum eigenvalue of 3×3 KLT, 4×4 KLT is always greater than the maximum eigenvalue of two-band PRQMF using ideal filters.

In this Figure B.1, we want to show that $(L+1)$ -band KLT gives always better solutions than L -band ideal filter ones and that for any band L we can pass KLT by using more coefficients.

The G for 2-band PR-QMF with ideal filters is given by,

$$G_{\text{ideal 2-band}} = \frac{1}{2 \left[\left(\frac{2}{\pi} \text{ArcTan} \left[\frac{1+\rho}{1-\rho} \right] \right) \left(1 - \frac{2}{\pi} \text{ArcTan} \left[\frac{1+\rho}{1-\rho} \right] \right) \right]^{1/2}} \quad (\text{B.1})$$

By expanding $\text{ArcTan}[x]$ in series as [B1] without approximations,

$$\text{ArcTan}[x] = \frac{\pi}{4} + \left(\frac{x-1}{x+1} \right) - \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 - +.. \quad |x| \geq 0$$

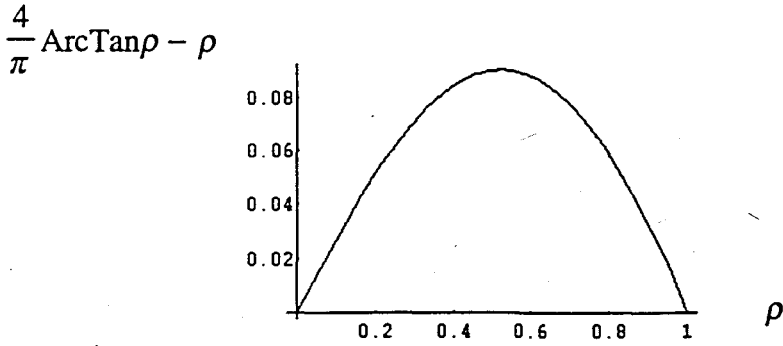
and using $x = \frac{1+\rho}{1-\rho}$, $-1 \leq \rho \leq 1$; $|x| = \left| \frac{1+\rho}{1-\rho} \right| \geq 0 \Rightarrow 1+\rho \geq 0 \quad 1-\rho \leq 0$

$$G_{\text{ideal 2-band}} = \frac{1}{\left[1 - \left(\frac{4}{\pi} \left(\rho - \frac{\rho^3}{3} + .. \right) \right)^2 \right]^{1/2}}$$

$$\text{As } \rho - \frac{\rho^3}{3} + \frac{\rho^5}{5} - \frac{\rho^7}{7} + - = \text{ArcTan}[\rho] \quad |\rho| \leq 1 [90]$$

$$G_{\text{ideal 2-band}} = \frac{1}{\left[1 - \left(\frac{4}{\pi} \text{ArcTan}[\rho]\right)^2\right]^{1/2}}$$

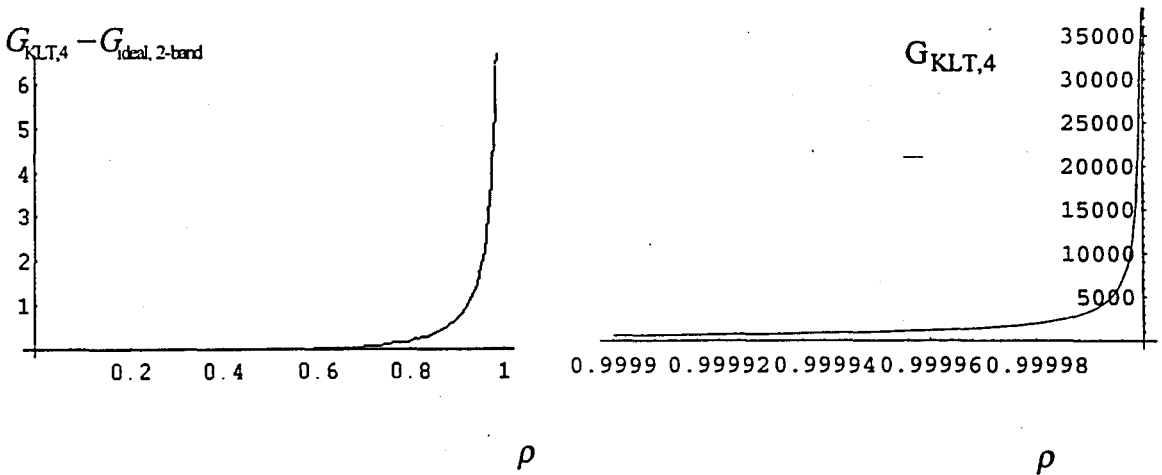
As $\frac{4}{\pi} \text{ArcTan}[\rho] > \rho$ is always true as is shown also graphically below:



we find, as expected

$$G_{\text{KLT}_2} < G_{\text{ideal 2-band}} \quad (\text{B.2})$$

When comparing coding gains of 4×4 KLT and that of ideal filter, we obtain plots as in Figure B.2.



(a)

Continued...

(b)

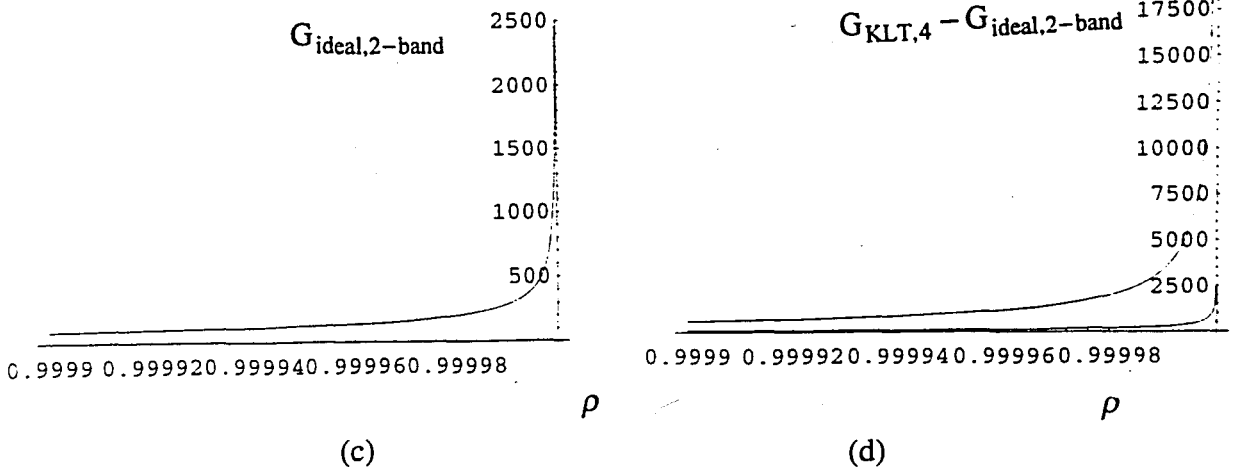


Figure B.2. Comparison of coding gains for 4×4 KLT and that of ideal filter

When comparing coding gains of 3×3 , 4×4 , 5×5 , 6×6 KLTs with respect to that of ideal filters, we obtain zoomed plots as

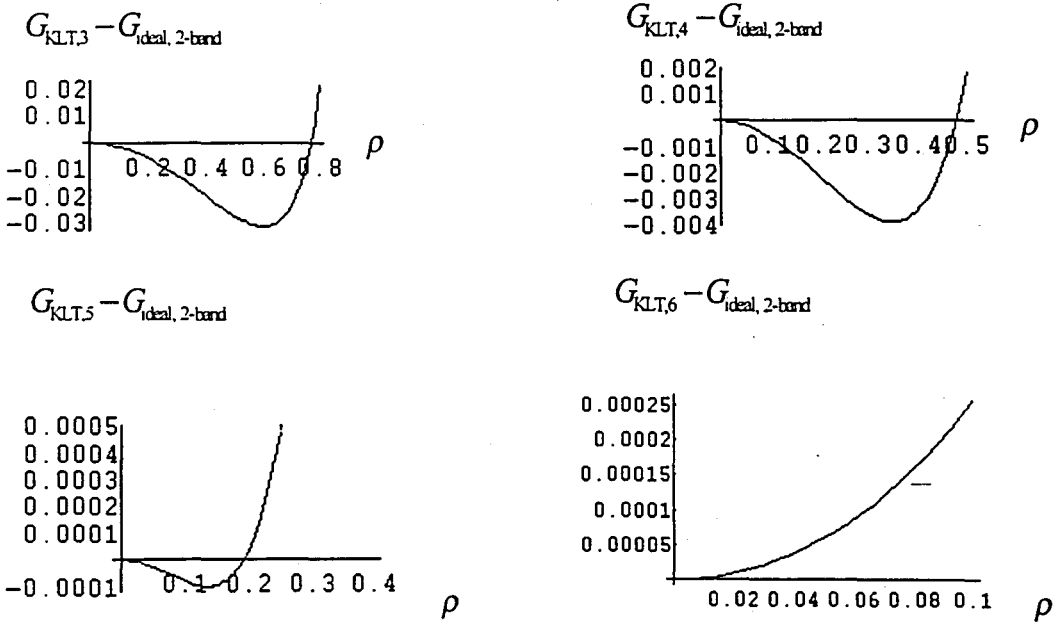


FIGURE B.3. Difference of the transform coding gains of 3×3 , 4×4 and 5×5 , 6×6 KLTs that of two-band PR-QMF using ideal filters.

We can now conclude that

$$\text{for } 0 < \rho < 1 \quad G_{\text{KLT},2} < G_{\text{ideal}, 2\text{-band}}$$

$$\text{for } 0.78 \leq \rho < 1 \quad G_{\text{KLT},3} > G_{\text{ideal}, 2\text{-band}}$$

$$\text{for } 0.5 \leq \rho < 1 \quad G_{\text{KLT},4} > G_{\text{ideal}, 2\text{-band}}$$

$$\text{for } 0.2 \leq \rho < 1 \quad G_{\text{KLT},5} > G_{\text{ideal}, 2\text{-band}}$$

$$\text{for } 0.1 \leq \rho < 1 \quad G_{\text{KLT},6} > G_{\text{ideal}, 2\text{-band}}$$

Appendix C Eigenvalue Bounds and Spectra of R_{xx} using Cauchy Interlace Theorem and Requirements on 2-band QMF Design

There is no linear phase, orthogonal solution for 1-D 2-band QMFs. Linear phase property of filters result in the identities

$$h(n)=h(N-1-n), n = 0, \dots, N-1 \quad (C.1)$$

and orthogonality constraints are given in Equation A.2.

From Property 2 given in Appendix A and the fact that there is no symmetric solution for orthogonal 1-D QMFs, one can conclude that the maximum eigenvalue of the design has to be a multiple one, that is there can be no distinct eigenvalue.

We intend to find bounds for eigenvalues of a principal submatrix in terms of those of the main matrix, we can state Theorem 2 as [92];

Theorem 2. Letting

$$A = \begin{bmatrix} H & B^* \\ B & U \end{bmatrix}$$

be an $n \times n$ matrix, and H an $m \times m$ submatrix with eigenvalues respectively

$$\begin{aligned} \alpha_1 &\leq \alpha_2 \leq \dots \leq \alpha_n \\ \theta_1 &\leq \theta_2 \leq \dots \leq \theta_m \end{aligned}$$

For $j=1, \dots, m$

$$\begin{aligned} \alpha_j &\leq \theta_j \leq \alpha_{j+n-m} \\ \alpha_{-(j+n-m)} &\leq \theta_{-j} \leq \alpha_{-j} \end{aligned}$$

Alternatively, for $k=1, 2, \dots, n$

$$\begin{aligned} \theta_{k-n+m} &\leq \alpha_k \leq \theta_k \\ \theta_{-k} &\leq \alpha_{-k} \leq \alpha_{-(k-n+m)} \end{aligned}$$

In other words $\theta_{\pm j}$ is an inner bound on $\alpha_{\pm j}$ relative to A 's spectrum(i.e. the set of the eigenvalues).

To analyze our 4-tap 2-band QMF design, let consider

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 + \mu_1 & \rho^3 \\ \rho & 1 & \rho & \rho^2 + \mu_1 \\ \rho^2 + \mu_1 & \rho & 1 & \rho \\ \rho^3 & \rho^2 + \mu_1 & \rho & 1 \end{bmatrix} \quad (\text{C.2})$$

from Equation 3.14 in Section 3.2.1. and A.7. For the 2×2 KLT case one has the following correlation matrix and its corresponding eigenvalues,

$$\mathbf{R} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \text{ and } \begin{matrix} \theta_1 = 1 - \rho \\ \theta_2 = 1 + \rho \end{matrix}$$

If we augmented the matrix by one then the corresponding eigenvalues become

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 + \mu_1 \\ \rho & 1 & \rho \\ \rho^2 + \mu_1 & \rho & 1 \end{bmatrix} \text{ and } \begin{matrix} \alpha_1 = \frac{2 + \mu_1 + \rho^2 - \sqrt{8\rho^2 + (\mu_1 + \rho^2)^2}}{2} \\ \alpha_2 = 1 - \mu_1 - \rho^2 \\ \alpha_3 = \frac{2 + \mu_1 + \rho^2 + \sqrt{8\rho^2 + (\mu_1 + \rho^2)^2}}{2} \end{matrix} \quad (\text{C.3})$$

which stand with respect to the eigenvalues of the 2×2 matrix as follows :

$$\begin{array}{ccccccc} \diagdown & | & \diagdown & | & \diagdown \\ \alpha_1(\mu_1) & \theta_1 & \alpha_2(\mu_1) & \theta_2 & \alpha_3(\mu_1) \end{array}$$

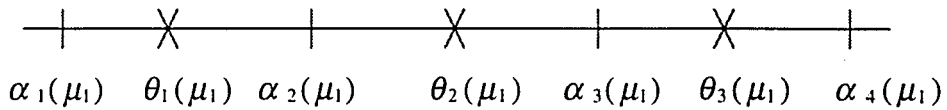
If this 3×3 matrix is augmented to a 4×4 one, one has the correlation matrix with its corresponding eigenvalues as,

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 + \mu_1 & \rho^3 \\ \rho & 1 & \rho & \rho^2 + \mu_1 \\ \rho^2 + \mu_1 & \rho & 1 & \rho \\ \rho^3 & \rho^2 + \mu_1 & \rho & 1 \end{bmatrix}$$

and

$$\begin{aligned} \alpha_1 &= 1 - \frac{(\rho + \rho^3)}{2} - \sqrt{\left((\mu_1 - (\rho - \rho^2))^2 + \frac{1}{4}(\rho - \rho^3)^2\right)} \\ \alpha_2 &= 1 + \frac{(\rho + \rho^3)}{2} - \sqrt{\left((\mu_1 + (\rho + \rho^2))^2 + \frac{1}{4}(\rho - \rho^3)^2\right)} \\ \alpha_3 &= 1 - \frac{(\rho + \rho^3)}{2} + \sqrt{\left((\mu_1 - (\rho - \rho^2))^2 + \frac{1}{4}(\rho - \rho^3)^2\right)} \\ \alpha_4 &= 1 + \frac{(\rho + \rho^3)}{2} + \sqrt{\left((\mu_1 + (\rho + \rho^2))^2 + \frac{1}{4}(\rho - \rho^3)^2\right)} \end{aligned} \quad (\text{C.4})$$

The spectral locations of the eigenvalues for the 3×3 matrix (now denoted as $\theta_1, \theta_2, \theta_3$) and for the 4×4 one denoted with $\alpha_1, \dots, \alpha_n$ are shown graphically below:



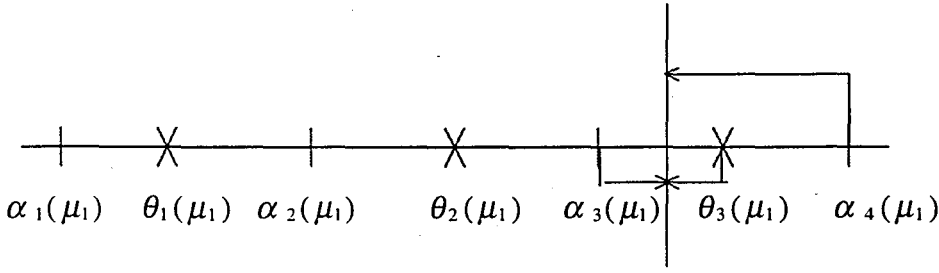
From Property 2 in Appendix A.3, the maximum eigenvalue must be a multiple one so that we have to equate $\alpha_4(\mu_1) = \theta_3(\mu_1) = \alpha_3(\mu_1)$ which results in the equality

$$2 + \rho + \rho^3 + \sqrt{\frac{19\rho^2 - 8\rho^2\sqrt{3+\rho^2} + 7\rho^4 - 8\rho^4\sqrt{3+\rho^2} + 5\rho^6 + \rho^8}{3 + \rho^2}}$$

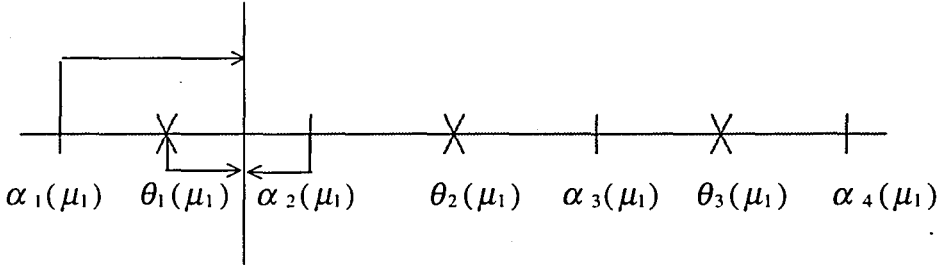
for μ_1 being equal to

$$\mu_1 = -\rho^2 - \frac{(\rho + \rho^3)}{\sqrt{3 + \rho^2}}. \quad (\text{C.5})$$

So, the spectra vary as,



New maximum
multiple
eigenvalue



New minimum
multiple
eigenvalue

For the two-band PR-QMF bank design, the maximum multiple eigenvalue correspond to the lowpass filter's variance and the minimum multiple eigenvalue to the high pass filter's variance. We have shown here how the maximum and minimum eigenvalue spectra varies according to the enlargement in the correlation matrix. The minimum eigenvalue spectra variation follows the same way.

Appendix D. Optimal Design of 4-tap 2-band QMF Filters

D.1. Analytical Expressions for the Eigensystem and Lagrange Coefficient μ_1

In 4-tap 2-band optimal QMF design Equation 3.13 becomes as

$$\begin{bmatrix} 1 & \rho & \rho^2 + \mu_1 & \rho^3 \\ \rho & 1 & \rho & \rho^2 + \mu_1 \\ \rho^2 + \mu_1 & \rho & 1 & \rho \\ \rho^3 & \rho^2 + \mu_1 & \rho & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \lambda_0 \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} \quad (\text{D.1})$$

Our aim is to find the filter coefficients in terms of the correlation values ρ in the AR(1) model, as $\mathbf{h}=\mathbf{t}(\rho)$, where both μ_1 and λ_0 will be solved in terms of ρ as $\mu_1=z(\rho)$ and $\lambda_0=l(\rho)$.

To find λ_0 and μ_1 , we first form the characteristic equation by --

$$\det \begin{bmatrix} 1 - \lambda_0 & \rho & \rho^2 + \mu_1 & \rho^3 \\ \rho & 1 - \lambda_0 & \rho & \rho^2 + \mu_1 \\ \rho^2 + \mu_1 & \rho & 1 - \lambda_0 & \rho \\ \rho^3 & \rho^2 + \mu_1 & \rho & 1 - \lambda_0 \end{bmatrix} = 0$$

whose 4 roots correspond to the 4 eigenvalues of Equation C.1. Thus we have 6 unknowns $(h_1, h_2, h_3, h_4, \lambda_0, \mu_1)$ and 6 equations with the added unit norm and orthogonality constraints. These 6 equations are obtained as follows :

- Using Equation C.4, α_4 is the greatest eigenvalue so that α_4 is set equal to λ_0 such that we obtain an equation of the form:

$$\textcircled{1} \quad \lambda_0 = f(\mu_1; \rho) \quad \rho = [1 \ \rho \ \rho^2 \ \rho^3] \quad (\text{D.2})$$

- Then assuming h_1 known, Equation D.1 is simplified to

$$\begin{bmatrix} 1-\lambda_0 & \rho & \rho^2+\mu_1 \\ \rho & 1-\lambda_0 & \rho \\ \rho^2+\mu_1 & \rho & 1-\lambda_0 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \\ h_4 \end{bmatrix} = h_1 \begin{bmatrix} -\rho \\ -\rho^2-\mu_1 \\ -\rho^3 \end{bmatrix}$$

or in matrix notation

$$\mathbf{T}\mathbf{h} = h_1\mathbf{L}$$

Expressing

$$\mathbf{T}^{-1}\mathbf{L} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad (\text{D.3})$$

or we have at hand 3 equations as,

$$\textcircled{2} \textcircled{3} \textcircled{4} \quad h_i = g_i(\lambda_0, \mu_1, h_1; \rho) \quad i = 2, 3, 4 \quad (\text{D.4})$$

- The normality constraint given in Equation A.1 implies $h_1^2 + h_2^2 + h_3^2 + h_4^2 = 1$

with

$$h_2 = h_1 A$$

$$h_3 = h_1 B$$

$$h_4 = h_1 C$$

using Equation D.3 which results in a form

$$(5) \quad h_1 = \pm \frac{1}{\sqrt{1 + A^2 + B^2 + C^2}} \quad (D.5)$$

- Using (A.3), the PR condition $h_1 h_3 + h_2 h_4 = 0$ yields $h_1^2 (B + AC) = 0$ so that

$$\pm \frac{B + AC}{\sqrt{1 + A^2 + B^2 + C^2}} = 0 \text{ and therefore } B + AC = 0$$

so that one has,

$$(6) \quad k(\lambda_0, \mu_1; \rho) = 0. \quad (D.6)$$

Finally, putting Equation D.2 into Equation D.5;

$$\lambda_0 = l(\rho) \quad \text{and} \quad \mu_1 = z(\rho) \quad (D.7)$$

Going in reverse direction, we obtain

$$\mathbf{h} = \mathbf{t}(\rho) \quad (D.8)$$

Since found polynomials are of high degree and that it takes too many places, we leave Equation D.7 and D.8 in closed-form expressions. As a design example, for $\rho = 0.95$ in 4-tap 2-band design; the maximum multiple eigenvalue is found as $\lambda_0 = 1.9618$ while the corresponding Lagrange coefficient is found as $\mu_1 = -1.818$.

Appendix E Design of M -band One Dimensional Linear Phase, Perfect Reconstruction Filter Banks in Time Domain based on Shuffling Operations

One of the time domain design of M ($M=2^k$) band filter bank based on time domain orthonormality constraints that the filters have to satisfy is presented in [47]. On the other hand, conventional design techniques are based on the frequency domain.

We also pursue a time-domain approach whereby the problem is reduced to finding a suitable prototype filter (generally, a lowpass filter) by some optimization procedures while the remaining $M-1$ filters are obtained through use of shuffling operators on the coefficients of the lowpass filter. This approach is computationally efficient since only the coefficients of the lowpass filter have to be stored and in hardware design the operation of the multipliers can be pipelined. All filter design requirements are imposed on the lowpass filter in optimization procedure provided the filter has enough degrees of freedom. These properties are then automatically guaranteed for the other ($M-1$) bands.

Let the column vector $\mathbf{h}_i(n)$ for $0 \leq n < N$ and $0 \leq i < M$ denote the i^{th} filter impulse response in an M band filter bank. Once a suitable $\mathbf{h}_0$ (generally, lowpass filter) is found, the remaining ($M-1$) filter vectors will be computed based on this lowpass filter coefficients.

Defining $\mathbf{J}_K$ as the $K \times K$ counter identity matrix, $\mathbf{P}_K$ a diagonal matrix with alternating $+1$'s and -1 's, the N th order shuffling matrix of level k is found as with the symbol $\otimes$ denoting Kronecker product of two matrices,

$$\mathbf{A}_k = \mathbf{P}_{N/2^k} \otimes \mathbf{J}_{2^k}$$

The $M-1$ other filter vectors are computed via filter transformation matrices $\mathbf{B}_i$, such that [47, 48]

$$\mathbf{h}_i = \mathbf{B}_i \mathbf{h}_0 \quad i = 1, \dots, M-1$$

where

$$\begin{cases} \mathbf{B}_i = \prod_{k=1}^{\log_2 M - 1} \mathbf{A}_k^{t_{ik}} \\ \mathbf{B}_{M-i} = \mathbf{P}_N \mathbf{B}_i \end{cases} \quad i = 0, \dots, M/2 - 1$$

and the constants t_{ik} take m values from the finite set $\{0, 1\}$ based on the k^{th} least significant bit in the binary representation of the integer i .

A suitable $\mathbf{h}_0$ is found by constrained optimization techniques where the first constraint is the orthonormality of the filters in the filter bank. The orthonormality requirement is satisfied in [47] by using the Equation A.2. The total numbers of equality constraints that the lowpass filter has to satisfy are listed for various values of N and M in Table F.1. For 4-band designs, N/M orthonormality equations coming from the self-orthonormality of $\mathbf{h}_0$ are sufficient and no other constraint equations are necessary. However, for 8-band filter designs, one has $(N/M - 1)$ additional constraints [48].

It is shown in Section 4.2.1.5. in Equation 4.20 that for 4 band two dimensional quadratic sampling case, when the structures of the 1-dimensional filters in the filter bank are rearranged for the two, there are no additional constraint equalities other than self orthogonalities imposed on the lowpass filter as in the 4 band one dimensional case. When the structures of the 1-dimensional filters in the filter bank are rearranged for the three dimensional design cases, it is shown in Equation 5.4 for 8-band three dimensional quadratic sampling case, that there are no additional constraint equalities other than self orthogonalities imposed on the lowpass filter differing from 8-band one-dimensional case.

These results are also expressed using Hadamard array based unitary matrices in Equation 4.22 for two dimensions and in Equation 5.5 for three dimensions. Now, let review Hadamard array based unitary matrices.

Hadamard Array based Unitary Matrices [93]

The mathematical infrastructure of the unitary matrix design as based on Hadamard arrays is detailed as:

Definition: An Hadamard array $H[N, k, \lambda]$, based on the indeterminates

$$\{x[1], x[2], \dots, x[k], k \leq N\}$$

is an $N \times N$ array with entries chosen from

$$\{x[1], -x[1], \dots, x[k], -x[k]\}$$

in such a way that :

- (i) in any row or in any column there are λ repeated (positive or negative) entries for each indeterminate;
- (ii) the rows and columns are, respectively, pairwise orthogonal.

Arrays with single repetition

The Hadamard array $H[N, N, 1]$ with indeterminate vector $\mathbf{x}$, denote by $\mathbf{Q}(\mathbf{x})$, can be seen as a mapping of the N dimensional space into a set of $N \times N$ dimensional matrices $\mathbf{M}_N$, which are generated by signed permutations of the first row:

$$H[N, N, 1] \rightarrow \mathbf{Q}(\mathbf{x}) : \mathbf{R}^N \rightarrow \mathbf{M}_N$$

For a given row vector $\mathbf{x}$ the elements of the matrix $\mathbf{Q}(\mathbf{x}) = (q_i(j))_{N \times N}$ are given as

$$q_i(j) = \alpha_i(j) x[\beta_i(j)]$$

where

$$\alpha_i(j), \alpha_2(j), \dots, \alpha_N(j) \in \{-1, 1\} \quad j=1, 2, \dots, N$$

are the sign change operations and β_i 's denotes permutation functions $i=1, 2, \dots, N$. The permutations and sign changes taking place in a Hadamard array are special in the sense that they have to guarantee pairwise row and column orthogonality. The necessary and sufficient conditions which make the $\mathbf{Q}(\mathbf{x})$ matrix orthogonal are as follows:

Lemma 1. In order to have $\mathbf{Q}(\mathbf{x}) \mathbf{Q}(\mathbf{x})^T = \mathbf{Q}(\mathbf{x})^T \mathbf{Q}(\mathbf{x}) = \mathbf{I}_{N \times N}$, the following conditions are necessary and sufficient:

$$(i) \quad \beta_i^{-1}(\beta_j(k)) = \beta_j^{-1}(\beta_i(k)) \quad 1 \leq i, j, k \leq N$$

$$(ii) \quad \alpha_i(k) \alpha_j(k) + \alpha_i(t) \alpha_j(t) = 2 \delta_{i,j} \quad 1 \leq i, j, k \leq N$$

$$(iii) \quad \mathbf{x} \mathbf{x}^T = \mathbf{I}$$

Lemma 2. Provided that $\mathbf{x} \mathbf{y}^T = 0$, the following holds,

$$\mathbf{Q}(\mathbf{x}) \mathbf{Q}(\mathbf{y})^T + \mathbf{Q}(\mathbf{y}) \mathbf{Q}(\mathbf{x})^T = 0$$

Thus, permutation based 1-D time domain design with Hadamard arrays are used in Chapter 4 and Chapter 5 to see if their counterpart expressions in 2 and 3 dimensions for quadratic and cubical sampling cases respectively can give similar results or where they differ.

Appendix F. NONSEPARABLE WAVELET PACKETS BASED IMAGE CODER WITH SUBJECTIVE DISTORTION MEASURES

F.1. Introduction

The wavelet transform has become recently a cutting edge technology in image compression research[94]. Compression is achieved by applying a wavelet transform to decorrelate the image data, quantizing the resulting transform coefficients, and coding the quantized values. Quantization is lossy in nature and it is where the compressing ability is given to a coder.

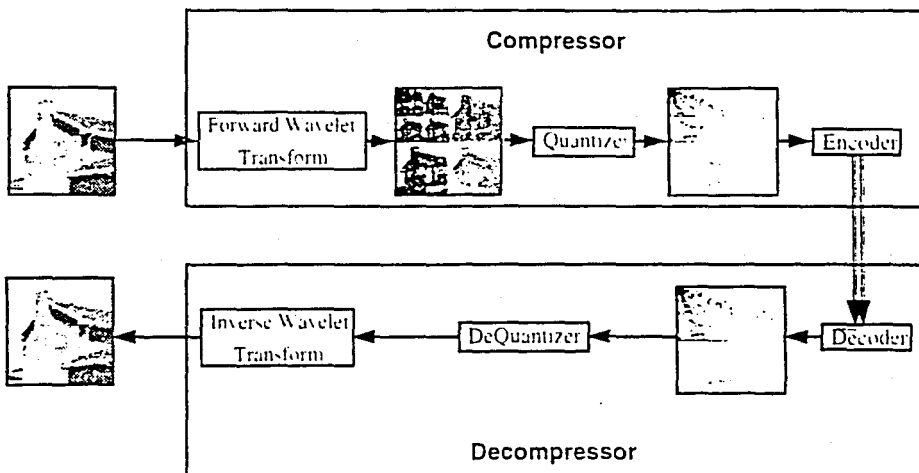


FIGURE F.1. Block diagram of wavelet based image coders

So, in wavelet/subband coding, an image is first filtered to create a set of subimages or subbands, each of which contains a limited range of spatial frequencies. The different subbands can be downsampled due to their lower bandwidths as compared to the original image, keeping the overall data rate the same. The subband samples are then quantized and encoded using one or more coders. Different bitrates or coding techniques can be used for each subband, in order to take advantage of the properties of the individual subbands. For example the coding errors can be distributed over the subbands in a visually optimal manner. The image is reconstructed by upsampling the decoded subbands, interpolating them via appropriate filters and adding the reconstructed subbands together.

Subband coding is motivated by the idea that subbands can be coded more efficiently than the entire full band image. This is because most of the energy in the subband domain is represented by few lowpass coefficients. It is known that for a wavelet/subband coding scheme, the high pass subbands contain a large number of zeroes. For example, the quantized high pass bands may achieve an entropy less than 0.01 bits-per-pixel. Hence, in wavelet based coding, entropy coding (or variable length coding) is almost always used when scalar quantization is employed. We assume implicitly that the coefficients are stationary (spatially) within a band depending on some parameters, such as the variance of the coefficients of that band.

A very important issue in wavelet / subband decomposition, i.e. wavelet approach is the way in which the data is divided into subbands [96-99]. This subject is tightly related to a new concept called packet wavelet theory [100, 101]. After wavelet packets were introduced into signal processing community, various applications of the DWT have been reformulated in terms of the WP approach. Torresani [100] determines the necessity of establishing good criteria to choose a best basis. Ramchandran et al. [54] discuss the selection of the best basis for image coding. They strongly argue for the search of the best WP basis for image compression in a rate distortion sense. The optimization problem is to find the minimum distortion D_t for a given bit budget R . This will also provide us the quantization step sizes which will produce the minimum distortion D_t . This constrained optimization problem can be easily converted to an unconstrained problem by relating the rate and distortion function through the Lagrange cost function.

Each band has its own rate distortion function, which depends on the quantization and coding schemes. It was shown that the overall rate distortion function will be optimal if all the individual bands operate at a constant slope λ on their respective rate distortion functions. The problem of bit allocation is that given a bit budget it must be distributed efficiently among a set of given admissible quantization choices.

In this chapter, we first review the concept of packet wavelets as applied to image compression. Subjective distortion measures are given in section F.4. The best basis search algorithm is given in Section F.5. Performances in terms of PSNR for “Lenna” image by using nonseparable 2 dimensional filters designed in previous chapter and different MSE measures in the best basis search algorithm are presented in Section F.6.

F.2. Packed Wavelets

The theory of packed wavelets was introduced recently by Wickerhauser et. al. [55] in the context of applied mathematics as a natural generalization of the wavelet theory. In contrast with the wavelet theory, where the parameters of the basis of the decomposition are the scale and the position, an additional degree of freedom is added to the wavelet basis, namely, the frequency. We do not necessarily have to split each subspace at every stage. In Figure F.2 we give a schematical representation of a space and its subspaces after using the splitting on three levels. The top rectangle corresponds to a certain subspace of V_3 generated by wavelet packets. The slanted lines between the rectangles indicate the splitting. The dashed rectangles then correspond to the wavelet multiresolution analysis. The bold rectangles correspond to a possible wavelet packet splitting.

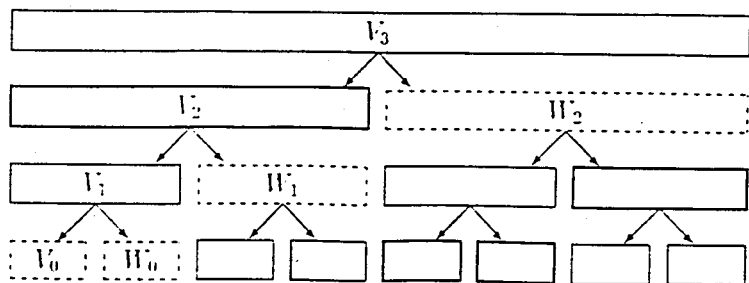


FIGURE F.2. Wavelet packets scheme

Binary tree structure subband decomposition in 1-D which is equivalent to a time-invariant packet wavelet decomposition is shown in Figure F.3. An example of a pruned tree and its corresponding time-frequency tiling is given in the same figure.

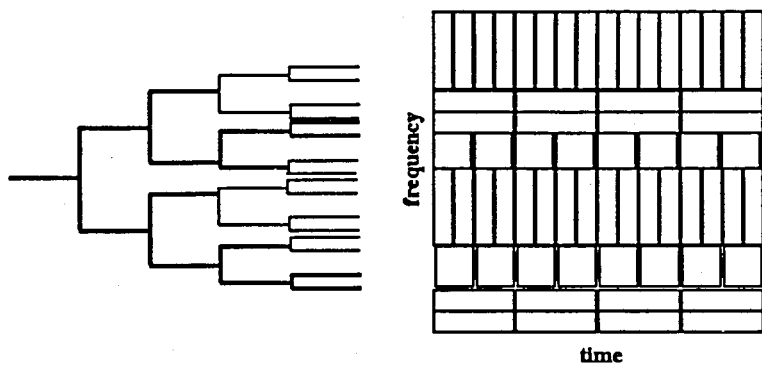


FIGURE F.3. An example of binary tree decomposition form of the 1-D time invariant packet wavelet decomposition.

There exists a variety of possibilities to extend the 1 dimensional packet wavelet decomposition into multidimensional signals. FigureF.4.a shows and example of packet

wavelet decomposition by quadtree subband decomposition. The frequency domain partitioning of the signal is also reported in the same FigureF.4.b.

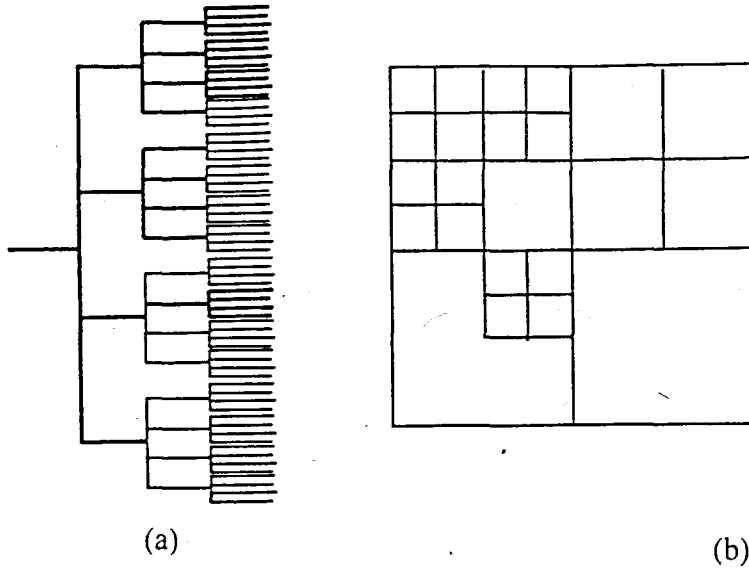


FIGURE F 5 The quadtree subband decomposition form of the 2-D packet wavelet decomposition

The packet wavelet representation is obviously highly redundant because of the additional degree of freedom which is the frequency parameter. For the same reason, there exist an infinite set of position, scale and frequency parameters allowing a packet wavelet decomposition of signals. The problem is to find the best set of parameters according to a given criterion. The best basis wavelet packet algorithm is similar to vector quantization, but has two significant differences. First, the library of vectors is fixed in advance, so there is no need to transmit it to the receiver, nor to enlarge it in an attempt to improve the fit to the signal. Secondly, there is a low complexity algorithm to compute correlations with all the elements in the library, and to select the best fit. To detail this problem in more details, let us concentrate in a particular case of packet wavelet decomposition, the adaptive subband decomposition.

The results may be stored in memory until the possible basis subsets are searched to find the optimal representation. The schematic representation of this algorithm is given as in Figure F 5

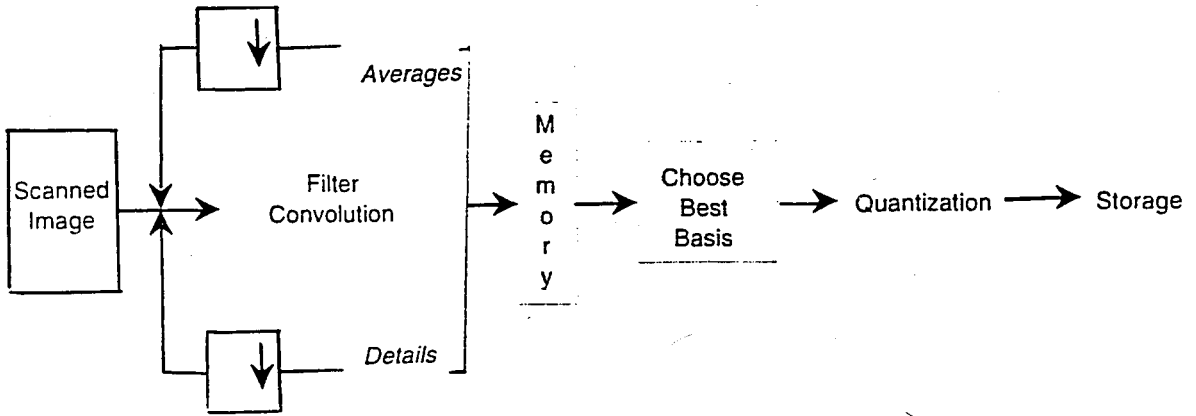


FIGURE F.5 Wavelet packet best basis picture compression.

An important parameter in determination of a specific packet wavelet decomposition is the criterion which is used to continue the decomposition of a branch of the tree into deeper levels. In image compression, the measure of the entropy seems to be a suitable measure for energy compaction.

F.4. Distortion Measures

Source coding schemes are evaluated by using a distortion measure which is essentially a cost function for reproducing the input by an reconstructed output. The choice of a distortion measure depends on various factors. First, it should be easily computable. Secondly, it should be analytically tractable. Finally, it should reflect the characteristics of the human visual system characteristics. The goal of image compression being to reduce

the number of bits in the digital representation of an image with minimum perceived loss of subjective quality, the image quality is ultimately judged by the human observer. Traditionally, the MSE criterion has been used as a measure of reconstructed image quality. However, it has often been found that the MSE do not correlate well with subjective quality assessment. The peak to peak signal to noise ratio as a measure of distortion is widely used in image compression community. It is defined by:

$$\text{PSNR} = 10 \log_{10} \left(\frac{\text{Peak Signal Value}^2}{\text{Mean Square Error}} \right) \quad (\text{F.2})$$

where

$$\text{MSE} = \frac{1}{N \times N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (x_{ij} - y_{ij})^2 \quad (\text{F.3})$$

with x_{ij} original image pixel and y_{ij} reconstructed image pixel.

Modelling the human visual system accurately is difficult task. Hence, one has to resort to some assumptions to simplify the development of a model. First, one assumes that HVS is a linear system which is in general true for low contrast images.

It is also observed that the HVS response is relatively poor at higher frequencies. The eye is less sensitive to noise in a brighter background. Thus, if the background luminance is high, more noise can be introduced without degrading the visual quality. The frequency sensitivity and background luminance provide an account of HVS characteristics for a flat field image. Since, humans are more sensitive to noise in flat fields than in textured regions, texture-masking model improves the coding performance by adding more noise in the coefficients which represent a highly textured region in the original image.

Traditionally, HVS model is assumed to be a combination of different visual channels which act as a bank of spatial frequency filters. It was shown that the visual system analyzes an image in a way that is similar to the multiresolution decomposition of the wavelet function[96].

The demands which we place on perceptual image quality are constantly increasing, and the requirements are becoming more stringent because both the number of users and the average amount of time spent in front of the video display units or TV are constantly

increasing. With TV applications, efforts are clearly directed towards achieving image reproduction which is as naturel as possible and also towards ensuring maximum involvement with the image. With video display units, the emphasis is on visual comfort. From experience we know that a lack of such visual comfort will lead to excessive fatigue.

To evaluate the quantitative relation between MSE and perceived similarity and to examine if more valid error measures could be derived by weighting the error in accordance with assumed properties of the visual system, the error criteria based on a model of perceived image similarity

$$E = \left[\frac{1}{N} \sum_{i=1}^N |Dt_i|^2 \right]^{pw} \frac{1}{pw} \quad (F.4)$$

$$Dt_i = a_i \cdot g(x_i - y_i) \quad (F.5)$$

where g represents some processing function that determines the visibility of the error, a_i is a weight related to the informative value of pixel (i,j) , and pw is a factor that determines the relative importance of used errors. Image properties can be incorporated in subjective error measures.

A number of error measures can be derived from different assumptions about a_i and g [102]. The **unweighted error measure** (MSE1) with

$$Dt_i = x_i - y_i \quad (F.6)$$

is evaluated for $pw=1$ to 6. (This measure includes the MSE measure for $pw=2$).

Many studies have shown that the perceived luminance of the testfield will change with the luminance of the surrounding background. The relevant variable for perceived luminance seems to be the ratio between these luminances, and this fact seems to hold for the visibility of image distortions. This suggest that the critical variable may not the grey-level difference, but rather, the **ratio** (MSE2) between this difference and the background,

assuming the same background level in x_i and y_i . This assumed processing function was tested by specifying Dt_i to be [96]

$$Dt_i = (x_i - y_i) / M_i \quad (F.7)$$

$$Dt_i = 0, \text{ for } Dt_i < 0.1$$

where M_i is the mean level in a 7×7 window.

Another well-known fact is the limited resolution of the eye. The eye is often said to act as a bandpass filter, but for the sake of simplicity this processing function was specified as a simple **mean** (MSE3) filter. The viewing conditions indicated that a window would be appropriate. This fact suggests the following definition of Dt_i , [96]. A table for best fit obtained with different error measure for a set of images is given in [96].

$$Dt_i = Mx_i - My_i \quad (F.8)$$

where Mx_i , My_i are mean levels in a 2×2 window surrounding the pixel in the original and processed image.

Thus, for a specific class of images we have followed this table to choose which error measure to use in order to obtain better results consistent with the Human Visual System. For "Lenna" image as "Face" type image, the Equation F.7 and F.8 type distortion measures are used.

F.5. Best basis search algorithm

In this chapter, an adaptive image compression algorithm is realized using wavelet transforms and the algorithm of Ramchandran et. al. [54]. The algorithm is run for

nonseparable filters that we have obtained in Chapter 4 and different subjective measures are used instead of mean square error.

Outline of the algorithm is as follows:

1. Wavelet transform of the image is taken and 4 subbands are formed.
2. Subbands are also subject to wavelet transform and $4n$ subbands are formed
3. Step 2 is repeated 3 or 4 times (iteration number is 3 or 4 i.e. a predetermined depth)
4. A tree is formed such that each node of this tree represents a subband
5. All subbands are quantized and entropy encoded with varying rates
6. Tree is pruned and best basis tree is formed
7. Best basis tree and accompanying quantized subbands are entropy encoded and stored

After decomposing the image to a predetermined depth, a framework with independent coding of nodes of the subband decomposition is considered. Different mse distortion measures and first order entropy bit rate are used. The hard problem of searching for the best WP basis minimizing a global distortion D_t for a target bit budget is converted to an equivalent unconstrained problem which becomes the minimization of the Lagrangian cost function defined as

$$J(\lambda) = D_t + \lambda R$$

It can be shown that at R-D optimality [54], all leaves of the wavelet packet tree (signal elements) must operate at a constant slope λ on their R-D curves with,

$$\lambda = \left| \frac{\Delta D_t}{\Delta R} \right|$$

Assuming that one knows the optimizing Lagrangian multiplier, then one populates each node with the Lagrangian cost function associated with the best quantizer for that node. The best quantizer for a given node is the one which lives at absolute slope λ on the particular node [54]. Then starts the pruning of the tree where one start from the deepest

level of the tree and come towards the root by comparing each node with the total of its children. At the end of this pruning an optimal tree is obtained.

The desired optimal point λ is not known a priori and one should search for it. It strictly depends on the desired target budget or quality of the reconstructed image. Fortunately, it can be found by fast bisection algorithm given in [54].

F.6. Performance analysis of the nonseparable filters and the subjective MSE measures

The main aim of this Appendix is to present the coding results for nonseparable linear phase perfect reconstruction filters in Chapter 4. The 8×8 filters obtained for 2-dimensional quadratic sampling case are illustrated. The coefficients for the lowpass filter are given in Appendix G. The adaptive wavelet packet tree algorithm of Ramchandran et.al. [54] was used. In the adaptive best wavelet tree algorithm of Ramchandran only MSE is used. Results are provided for different MSE measures adapted to HVS using this adaptive best wavelet tree algorithm.

The quantization levels, variances for the selected subbands using the best wavelet packet basis algorithm when 3 iterations of the tree, MSE2 and 0.6 bits/pixel are used are shown in Table F.1. The reconstructed “Lenna” image for 1.85 bit/pixel, 3 levels of iteration, MSE2 and 8×8 nonseparable filters are shown in Figure F.6.

The results for MSE2 distortion measure are displayed since for “Lenna” image perceived luminance is very important. The distortions seem to be confined to flat areas around edges where the loss of higher frequency subbands is clearly noticeable. Also in busy areas in the “Lenna”, the image has a lowpass character and look somewhat unsharp. The coding results at 1.5 bits/pixel have a high quality and in this case coding errors are hard to find.

PSNR values for different bit budgets, iteration levels and distortion measures are given at Table F.2 when using nonseparable filters presented in Chapter 4 and in Appendix G. We can conclude that 3-level iteration always provides higher PSNR values for all kind of distortion measures when using nonseparable filters of Appendix H. Performance of the obtained images is reflected better when MSE3 type distortion measure is employed.

As in from the below Table F.1, 16 are retained from 64 subbands when using best basis search algorithm. We know from Figure 4.5. that low-low and low-high subbands can also provide good spectral control. If the survival of the nodes in the tree and quantization steps consider this fact then better results in terms of compression performances can be obtained. Recursively scalar quantizer and entropy encoding can be used to do so[104, 107].

TABLE F.1 The selected subbands and their corresponding quanta levels for 0.6 bits/pixel giving a PSNR of 33.6438. While the input image has a mean of 98.96 and variance of 2797.149682, the obtaine dresults when using nonseparable filters gives us a resulted image of mean 95.4388 and variance of 2875.0464. For the separable filtering using Daubechies 4 filters we obtain at the same bitrate 32.2 dB of PSNR.

Selected subbands	Quantization Levels
(1,1)	16
(1,2)	32
(1,3)	32
(2,1)	32
(2,2)	16
(2,3)	16
(3,0)	32
(3,1)	16
(3,2)	16
(3,3)	16

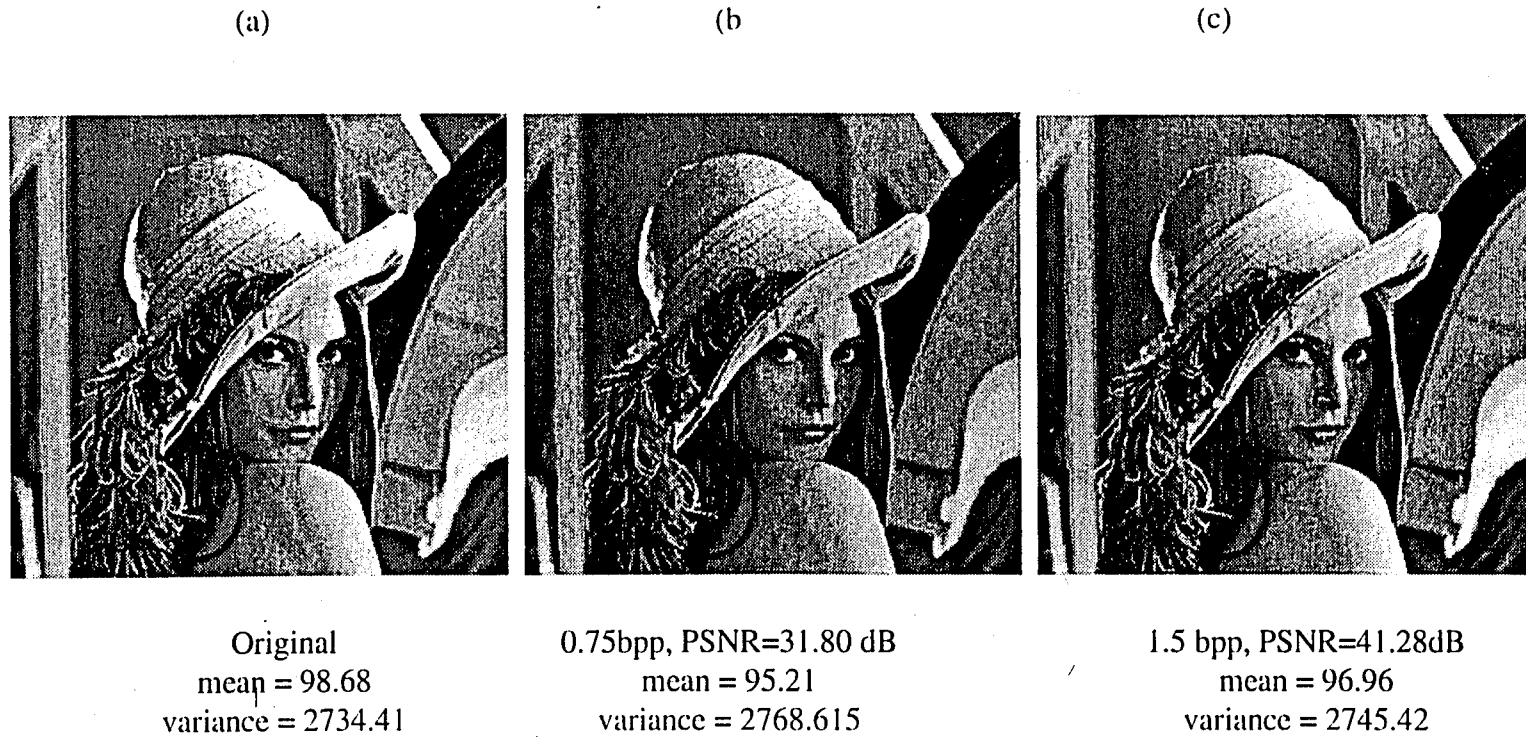


FIGURE F.6 Original “Lenna” image (a) and reconstructed ones for (b) 0.75 bpp(bit/pixel), (c) 1.5 bpp, each using 8×8 NS filters of Table H.1 for 3-level of iterations and MSE2 distortion measure

Table F.2. PSNR values for different bit budgets, iteration levels and distortion measures (MSE s)

1-level MSE1		1-level MSE3
Bitbudget	PSNR(dB)	
0.75 bpp	31.8	36.06
1.5 bpp	37.94	42.15
3-level MSE1		3-level MSE3
Bitbudget	PSNR(dB)	
0.75 bpp	34.97	41.54
1.5 bpp	39.53	47.38

APPENDIX H. TEST IMAGES and FILTER COEFFICIENTS



LENN(256×256)

(a)



WORLD(256×256)

(b)



SEA(256×256)

(c)



CAMERAMAN(256×256)

(d)

FIGURE H.1 Original test images of sizes 256×256 as (a) "Lenna", (b) "World", (c) "Sea", (d) "Cameraman"



FRAME 1



FRAME 2



FRAME 3



FRAME 4



FRAME 5



FRAME 6

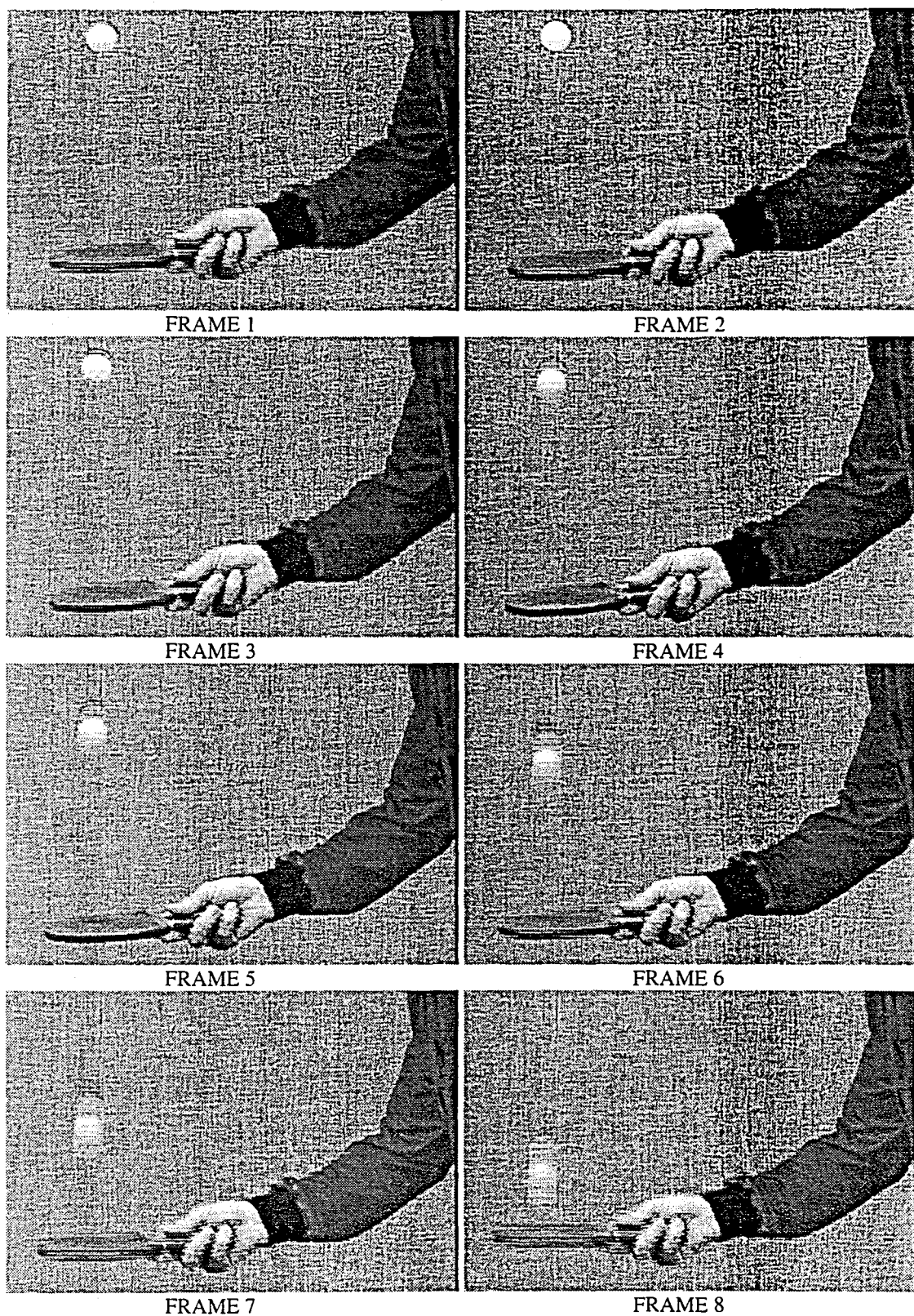


FRAME 7



FRAME 8

FIGURE H.2. Sequential 8 frames of sizes 256×256 taken from original head and shoulder type Miss America sequence



FIGUREH.2.Sequential 8 frames each of size 360×540 taken from original full motion Tennis Table video sequence

Table H.1. 8×8 filter coefficients obtained using the proposed design method
after optimization based on maximizing the coding gain

Filter coefficients (8×8 design example) obtained using the proposed design method after optimization based on maximizing the coding gain							
-0.0039	0.0138	-0.0158	-0.0091	0.0003	0	0.0001	0.0001
0.0026	0.0092	0.0242	0.0424	0.0565	-0.0274	0.0044	-0.0024
-0.0122	0.0407	-0.0491	-0.0999	-0.0032	0.0039	0.0036	0.0018
0.0056	0.0181	-0.0962	-0.4897	-0.4763	0.0374	0.0313	-0.0107
-0.0107	0.0313	0.0374	-0.4763	-0.4897	-0.0962	0.0181	0.0056
0.0018	0.0036	0.0039	-0.0032	-0.0999	-0.0491	0.0407	-0.0122
-0.0024	0.0044	-0.0274	0.0565	0.0424	0.0242	0.0092	0.0026
0.0001	0.0001	0	0.0003	-0.0091	-0.0158	0.0138	-0.0039

Table H.2. Separable filter coefficient of Daubechies-32, Johnston-16, Smith
Barnwell 16

Daubechies-32 even	Daubechies-32 odd	Johnston-16[61]	Smith-Barnwell16
0.0031892209	0.0349077151	0.006525666	0.02193598030
0.1650642871	0.4303127318	-0.020487510	0.00157861649
0.6373563440	0.4402902616	0.001991150	-0.06025449102
-0.0897510983	-0.3270633239	0.046476840	-0.01189065962
-0.0279182122	0.2111906965	-0.026275600	0.13753791563
0.0273402622	-0.1323883101	-0.099295500	0.05745450056
-0.0062397234	0.0759242376	0.117866600	-0.32167029616
-0.0075889748	-0.0368883987	0.472112200	-0.52872027154
0.0102976599	0.0139937691	0.472112200	-0.29577967450
-0.0069900147	-0.0036442797	0.117866600	0.00020431108
0.0031280234	0.0004078969	-0.099295500	0.02906699789
-0.0009410217	0.0001142415	-0.026275600	-0.03533486088
0.0001747872	-0.0000610359	0.046476840	-0.00682104532
-0.0000139456	0.0000113366	0.001991150	0.02606678468
-0.0000010435	-0.0000007363	-0.020487510	0.00103336349
0.0000002308	-0.0000000210	0.006525666	-0.01435930957

APPENDIX I : Analysis by graph theory of the shuffling method

Let us give the example of a 4×4 sized filter

$$h_0 = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$$

which can be regarded as

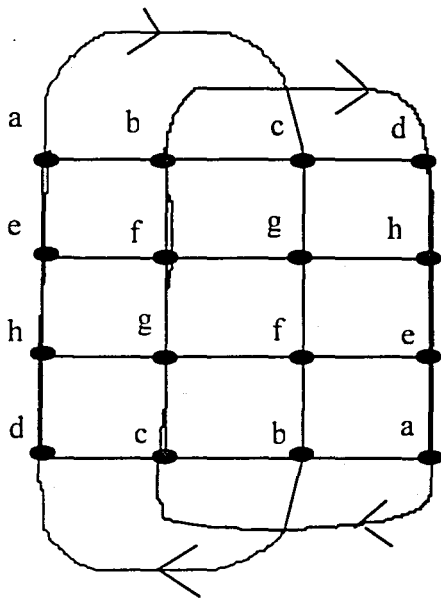


FIGURE I.1. Graph representation for shuffling based 4×4 FB design method

an 8 - colourable graph with vertex set $V = \{a,...,h\}$. Let define an 8 - cycle by $C_1 = \{a,c,g,f,b,d,h,e,a\}$ and another one by $C_2 = \{b,d,h,e,a,c,g,f,b\}$ and clockwise 180° rotation by R . The shuffling operations will correspond to the C_1 and C_2 8-cycles.

REFERENCES

1. Tekalp, M., *Digital Video Processing*, Prentice-Hall, 1995.
2. Lu J., V.R. Algazi and R. R. Estes, Comparison of Wavelet Image Coders using the Picture Quality Scale, Technical Report, Center for Image Processing and Integrated Computing, 1995.
3. Vetterli, M., J. Kovacevic and D. J. Le Gall, "Perfect Reconstruction Filter Banks for HDTV Representation and Coding," *Signal Processing:Image Communication* 2, pp. 349-363, 1990.
4. Cox, R. V., D. E. Bock, K. B. Bauer, J. D. Johnston and J. H. Snyder, "The Analog Voice Privacy System," *Proc. 1977 IEEE Int. Conf. Acoust., Speech, Signal Processing*, pp. 341-344, April 1986.
5. Crochiere, R.E., S.A. Webber, J.L. Flanagan, "Digital Coding of Speech in Subbands," *Bell Sys. Tech. J.*, Vol. 55, pp. 1069-1085, October 1976.
6. Vetterli, M., "Multidimensional Subband Coding: Some Theory and Algorithms," *Signal Processing*, Vol. 6, No. 2, pp. 97-112, February 1984.
7. Gharavi, H., A. Tabatabai, "Subband Coding of Monochrome and Color Images," *IEEE Trans. Circuits Syst.*, Vol. 35, pp. 207-214, February 1988.
8. Guillemot, C., A.E. Çetin, R. Ansari, Nonrectangular Wavelet Representation of 2-D Signals:Application to Image Coding, Technical Report, 1993.
9. Huang, C. L. , C. C. Lien, " The Design and Analysis of Directional 2-D Nonseparable Perfect Reconstruction Filter Banks for Subband Coding of Images and Video Signals," *Visual Communications and Image Processing SPIE*, Vol. 1818, pp. 775-786, 1992.
10. Patti A. J., M. I. Sezan, M. A. Tekalp , "Digital Video Standards Conversion in the Presence of Accelerated Motion", *Signal Processing:Image Communications*, Vol. 6, pp. 213-227, 1994.
11. Soman A. K., P. P. Vaidyanathan, and T. Q. Nguyen, "Linear Phase Paraunitary Filter Banks: Theory, Factorizations and Applications," *IEEE Trans. Signal Proc.*, Vol. 41, No. 12, pp. 3480 - 3496, December 1993.

12. Vetterli, M., "A Theory of Multirate Filter Banks," *IEEE Trans. Acoust., Speech Signal Processing*, Vol. ASSP-35, pp. 356-372, March 1987.
13. Wackersreuther G., "Some New Aspects of Filters for Filter Banks", *IEEE Trans. On ASSP*, Vol. 34, pp. 1182-1200, October 1986.
14. Woods, J.W., S. O'Neil, "Subband Coding of Images," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-34, No. 5, pp. 1278-1288, October 1986.
15. Cassels J.W. , *An Introduction to the Geometry of Numbers*, Berlin: Springer-Verlag, 1971.
16. Vaidyanathan, P. P. , Z. Doğanata, "The Role of Lossless Systems in Modern Digital Signal Processing: A Tutorial," *IEEE Trans. Education*, Vol. 32, No. 3, pp.181-197, August 1989.
17. Dubois, E., "The Sampling and Reconstruction of Time-Varying Imagery with Application in Video Systems," *Proc. IEEE*, Vol. 73, No. 4, pp. 502-522, April 1985.
18. Ansari, R. , S. H. Lee, "Two-Dimensional Nonrectangular Interpolation, Decimation and Filter Banks," in *Proc. IEEE Int. Conf. ASSP*, New York, April 1988.
19. Chen, T., P.P. Vaidyanathan, "Recent Developments in Multidimensional Multirate Systems," *IEEE Trans. on Circuits and Systems for Video Technology*, Vol. 3, No.2, pp. 116-135, April 1993.
20. Chen, T., P.P. Vaidyanathan, 'Multidimensional Multirate Filters derived from One-Dimensional Filters,' *Electron. Lett.*, Vol. 27, No. 3, pp. 225-228, January 1991.
21. Chen, T., P.P. Vaidyanathan, "Commutativity of D-dimensional Decimation and Expansion Matrices, and Application to Rational Decimation Systems," in *Proc. IEEE Int. Conf. ASSP*, San Francisco, Vol. 4, pp. 637-640, March 1992.
22. Chen, T., P.P. Vaidyanathan, "The Role of Integer Matrices in Multidimensional Multirate Systems", *IEEE Trans. Signal Processing*, Vol. 41, No. 3, pp. 1035-1047, March 1993.
23. Chen, T., P.P. Vaidyanathan, "On the Choice of Rational Decimation Systems for Multidimensional Signals," in *Proc. IEEE Int. Conf. ASSP* Minneapolis, MN., April 1993.
24. Evans B. L., R.H. Bamberger and J.H. McClellan, "Rules for Multidimensional Multirate Structures", *IEEE Trans. on Signal Processing*, Vol.42, No.4, pp.762-771, April 1994.

25. Feauveau, J.C., "Analyse Multiresolution pour les Images avec un Facteur de Resolution," *Traitement du Signal*, Vol. 7, No. 2, pp. 117-128, 1990.
26. Karlsson, G., M. Vetterli, "Theory of two-dimensional Multirate Filter Banks," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 38, No. 6, pp. 925-937, June 1990.
27. Smith, M.J.T., T.P. Barnwell III, "Exact Reconstruction for Tree-Structured Subband Coders," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-34, No. 3, pp. 431-441, June 1986.
28. Hitchcock, D., A Development in Nonseparable Multiresolution Methods, Technical Report, 1992.
29. Jain V.K., R.E. Crochiere, "Quadrature Mirror Filter Design in the Time Domain", *IEEE Trans. on ASSP*, Vol. 32, pp. 353-361, April 1984.
30. Smith, M.J.T., T.P. Barnwell III, "A New Filter Bank Theory for Time-Frequency Representation," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-35, No. 3, pp. 314-327, June 1987.
31. Vaidyanathan, P. P., P. Q. Hoang, "Lattice Structures for Optimal Design and Robust Implementation of Two-Channel Perfect Reconstruction Filter Banks," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-36, No. 1, pp. 81-94, 1988.
32. Vaidyanathan P.P., *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.
33. Chen, T., P.P. Vaidyanathan, "Multidimensional Multirate Filters and Filter Banks Derived from One-Dimensional Filters," *IEEE Trans. Signal Processing*, Vol. 41, No. 5, pp. 1749-1765, March 1993.
34. Ansari, R. , "Efficient IIR and FIR Fan Filters," *IEEE Trans. Circuits. Syst.*, Vol. CAS-34, No. 8, pp. 941-945, August 1987.
35. Bamberger, R. H., M. J. T. Smith, " A Filter Bank for the Directional Decomposition of Images: Theory and Design", *IEEE Trans. Signal Processing*, Vol. 40, No. 4, pp. 882-893, April 1992.
36. Kovacevic, J., M. Vetterli, "Nonseparable Multidimensional Perfect Reconstruction Filter Banks and Wavelet Bases for R^n ", *IEEE Trans. on Information Theory*, Vol. 38, No.2, pp. 533-535, March 1992.
37. Kovacevic, J., M. Vetterli, "FCO Sampling of Digital Video using Perfect Reconstruction Filter Banks", *IEEE Trans. on Image Proc.*, Vol.2, No.1, pp. 118-122, January 1993.

38. Kovacevic, J., M. Vetterli, "New Results on Multidimensional Filter Banks and Wavelets," *ICASSP 1993*, pp. 108-111.
39. Pei S-C, J. J. Shyu, "Symmetric Properties of 2-D Sequences and their Applications for Designing Linear Phase 2D FIR Digital Filters," *Signal Processing*, Vol. 42, pp. 261-271, 1995.
40. Hsiao, C. C., "Polyphase Filter Matrix for Rational Sampling Rate Conversions," *Proc. IEEE Int. Conf. ASSP Dallas, TX.*, pp. 2173-2176, April 1987.
41. Ikehara, M., "Cosine-modulated 2-D FIR Filter Banks Satisfying Perfect Reconstruction," *ICASSP' 94*.
42. Kovacevic, J., M. Vetterli, "The Commutativity of Up/Down-Sampling in Two Dimensions," *IEEE Trans. on Information Theory*, Vol. 37, No.3, pp. 695-698, May 1991.
43. Vetterli, M., "Filter Banks allowing Perfect Reconstruction," *Signal Processing*, Vol. 10, No. 3, pp. 219-244, April 1986.
44. Vetterli M., Kovacevic J., *Wavelets and Subband Coding*, Prentice-Hall, 1995.
45. Adnani A. AL., R. Chapman, T. S. Durrani, "Time Domain Design of FIR Multirate Filter Banks", *Electronic Letters*, Vol. 29, No. 9, pp. 752-753, April 1993.
46. Çağlar H., Y. Liu and A.N. Akansu, "Optimal PR-QMF Design for Subband Image Coding", *IEEE Trans. on Signal Processing*, Vol. 4, No. 3, pp. 242 - 253, September 1993.
47. Alkın, O., H. Çağlar, "Design of Efficient M-band Coders with Linear Phase and Perfect Reconstruction Properties," *IEEE Trans. Signal Proc.*, Vol. 43, No. 7, pp. 1579-1590, 1994.
48. Çağlar H., O. Alkın, E. Anarım and B. Sankur, "New Efficient Shuffled Block Transform Design Technique," *Electronics Letters*, Vol. 30, No.11, pp. 834-835, May 1994.
49. Karlsson G., M. Vetterli, "Three Dimensional Subband Coding of Video," *Proc. ICASSP 88*, pp. 1100-1103, April 1988.
50. Kovacevic J., *Filter Banks and Wavelets Extensions and Applications*, Ph.D. Dissertation, Columbia University, October 1991.
51. Vaidyanathan, P. P. , S.K. Mitra, "Polyphase Networks, Block Digital Filtering, LPTV Systems, and Alias-Free QMF Banks: A Unified approach based on

Pseudocirculants," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 36, pp. 381-391, March 1988.

52. Veldhuis R. N. J. , M. Breeuwer and R . G. Van der Waal, "Subband Coding of Digital Audio Signals," *Philips J. Res.*, Vol. 44, pp. 329-343, 1989.

53. Belfor R.A.F., "Subsampling Methods for Image Sequence Coding," Ph.D. Dissertation, Delft University, 1994.

54. Ramchandran K., M. Vetterli, "Best Wavelet Packet Bases in a Rate Distortion Sense," *IEEE Trans. on Image Processing*, Vol.2, pp. 160-173, April 1993.

55. Wickerhauser M. V., Picture Compression by Best Basis Subband Coding, Technical Report, Yale University, 1990.

56. Daubechies, I. , "Orthonormal Bases of Compactly Supported Wavelets," *Commun. Pure Appl. Math.*, Vol. 41, pp. 909-996, Nov. 1988.

57. Rioul O., "Regular Wavelets:A Discrete Time Approach," *IEEE Trans. on Signal Proc.*, Vol 41, No.12, pp. 3572-3579, December 1993.

58. Rioul O., P. Duhamel, "Fast Algorithms for Discrete and Continuous Wavelet Transform," *IEEE Trans. on Information Theory*, Vol.38, No.2, pp.569-586, March 1992.

59. Heller P., H. L, Resnikoff, "Regular M-band Wavelets and Applications," *ICASSP 93*, pp. 229-232, 1993.

60. Roufs J. A.J., "Perceptual Image Quality:Concept and Measurement," *Philips J. Res.*, Vol 47, No. 1, pp.35-62, 1992.

61. Harmancı K., E. Anarım , H. Çağlar and B. Sankur, 2-D Nonseparable Linear Phase PR Filter Bank Design Based on Bernstein Approximation, Technical Report, Boğaziçi University, 1993.

62. Vaidyanathan, P. P. , "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," *Proc. IEEE*, Vol. 78, No.1, pp.56-93, January 1990.

63. Vaidyanathan, P. P. , "Quadrature Mirror Filter Banks, M-Band Extensions, and Perfect-Reconstruction Techniques," *IEEE ASSP Mag.*, Vol. 4, No.3, pp.4-20, July 1987.

64. Vetterli, M., C. Herley, " Wavelets and Filter Banks: Theory and Design," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 40, No.9, pp. 2207 - 2232, September 1992.

65. Vetterli, M., D. J. Le Gall, "Perfect Reconstruction FIR Filter Banks: Some Properties and Factorizations," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 37, No. 7, pp. 1057-1071, July. 1989.
66. Viscito, E., J. P. Allebach, "Design of Perfect Reconstruction Multidimensional Filter Banks using Cascaded Smith Form matrix," *Proc. IEEE Int. Symp. Circuits Syst.*, Espoo, Finland, pp.381-384, June 1988.
67. Vetterli, M., D. J. Le Gall, "Perfect Reconstruction FIR Filter Banks: Lapped Transforms, Pseudo QMF's and Paraunitary Matrices," *Proc. 1988 Int. Symp. Circuits and Systems*, pp.2249-2253, June 1988.
68. Meyer, Y., *Ondelettes*, Paris, Hermann, 1992.
69. Mallat, S., "A Theory of Multiresolution Signal Decomposition: The Wavelet Representation," *IEEE Trans. Pattern Recognition Machine Intell.*, Vol. 11, No.7, pp. 674-693, July 1989.
70. Rioul O., M. Vetterli, "Wavelets and Signal Processing", *IEEE Signal Processing Magazine*, Vol. 8, No.4, pp.14-38, October 1991.
71. Rioul, O., "A Discrete-Time Multiresolution Theory", *IEEE Signal Processsing*, Vol. 41, pp.2591-2606, August 1993.
72. Desarte P., B. Macq and D.J.M. Slock, "Signal Adapted Multiresolution Transform for Image Coding", *IEEE Trans. Inf. Theory*, Vol. 38, pp. 897-904, March 1992.
73. Tewfik A.H., D. Sinha and P. Jorgensen, "On the Optimal Choice of a Wavelet for Signal Representation", *IEEE Trans. Inf. Theory*, Vol. 38, pp. 747-765, 1992.
74. Frazier M., A. Kumar, "The Discrete Orthonormal Wavelet Transform : An Introduction", Washington University, WUCS-090-30, September 20, 1990.
75. Kayran, A. H., Rr. A. King, "Comments on the Least-Square Inverse Polynomials and a Counter Example for Jury's Conjecture", *Electronic Letters*, Vol. 16, pp. 795 - 796, October 1980.
76. Jayant N.S., P. Noll, *Digital Coding of Waveforms*, Englewood Cliffs, NJ:Prentice Hall , 1984.
77. Unser, M., "On the Optimality of Ideal Filters for Pyramid and Wavelet Signal Approximation", *IEEE Trans. On SP*, Vol. 41, No. 12, pp. 3591-3596, December 1993.

78. Çağlar H., "A generalized Parametric PR-QMF / Wavelet Transform Design Approach for Multiresolution Signal Decomposition," Ph.D. Dissertation, New Jersey Institute of Technology, 1992.
79. Çağlar H., Y. Liu and A.N. Akansu, "Optimal PR-QMF Design for Subband Image Coding," *Journal of Visual Comm. and Image Repr.*, Vol. 4, No. 3, pp. 242-253, September 1993.
80. Anarım E., B. Sankur, H. Çağlar and I. Celasun, "Design of Nonseparable 2-D Linear Phase Filter Banks via Shuffling Operations," *Electronics Letters*, Vol. 30, No.11, pp. 843-844, May 1994.
81. Anarım E., B. Sankur, H. Çağlar and I. Celasun, "Shuffling based Design Technique for Nonseparable Linear Phase 2-D Filter Banks", *Proc. of EUSIPCO-94, Seventh European Signal Processing Conference*, pp. 323-326, Edinburgh, Scotland, UK, September 13-16 1994.
82. Celasun I., B. Sankur and E. Anarım, *Optimal Design Methods of 1-D and 2-D PR-QMF and Their Applications*, Technical Report, Boğaziçi University, July 1993.
83. Celasun I, E. Anarım, B. Sankur and H. Çağlar, "İki Boyutlu Doğrusal Evreli Ayrışmaz Tam Geriçatmalı Süzgeç kümelerinin Tasarımı", *SİU-94*, pp 310-313, 8-9 April 1994.
84. Celasun I, E. Anarım, B. Sankur and H. Çağlar "İki Boyutlu Ayrışmaz Süzgeç Kümelerinin Tasarımı," *SİU-95*, pp 164-169, 26-28 April 1995.
85. Celasun I, E. Anarım, B. Sankur, H. Çağlar and S. Güntürk, "Design of 2-D M-Band Nonseparable Linear Phase Perfect Reconstruction Filter Banks," *12th European Conf. on Circ. Theo. and Design, ECCTD-95*, İstanbul, Turkey, Vol. 1, pp. 175-178, August 27-31 1995.
86. Pei S-C, J. J. Shyu, "Symmetric Properties of 2-D Sequences and their Applications for Designing Linear Phase 2D FIR Digital Filters," *Signal Processing*, Vol. 42, pp. 261-271, 1995.
87. Dudgeon, D. E., R.M. Mersereau, *Multidimensional Digital Signal Processing*, Englewood Cliffs, NJ.; Prentice-Hall 1984.
88. Karlsson, G., M. Vetterli and J. Kovacevic, "Nonseparable Two-Dimensional Perfect Reconstruction Filter Banks," *Proc. SPIE Conf. Vis. Commun. Image Processing*, Cambridge, MA, pp. 187-199, November 1988.

89. Akansu A. N. and R. A. Haddad, *Multiresolution Signal Decomposition*, Academic Press, Inc., 1992.
90. Agrawal, B. P., P. Yip, Theory and Computations of Toeplitz Matrices, CRL Report Series, Communications Research Laboratory, McMaster University, Hamilton, Ontario, Canada, July 1980.
91. HÜTTE, Manuel de l'Ingenieur, 1947.
92. Parlett B. N., *The Symmetric Eigenvalue Problem*, Prentice-Hall, 1980.
93. Güntürk, S., H. Çağlar, E. Anarım, B. Sankur, "A New Permutational based Orthogonal Block and Lapped Transform Design", ECCTD'95, pp. 825-828, 1995.
94. Westerink P.H., "Subband Coding of Images," PhD Dissertation, Delft University, 1989.
95. Viscito, E., J. P. Allebach, "The Analysis and Design of Perfect Multidimensional FIR Perfect Reconstruction Filter Banks for Arbitrary Sampling Lattices", *IEEE Trans. Circuits Syst.*, Vol. 38, No. 1, pp. 29-42, January 1991.
96. Bamberger, R. H., "New Subband Decompositions and Coders for Image and Video Compression," *ICASSP* 1992.
97. Barnard H.J., *Image and Video Coding using a Wavelet Decomposition*, Ph.D. Dissertation, Delft University, 1994.
98. Chu C.H., "Data Compression by Multiresolution Tree Search", *Optical Engineering*, Vol.33, No.7, July 1994.
99. Herley C., J. Kovacevic, K. Ramchandran and M. Vetterli, "Tilings of the Time-Frequency Plane: Construction of Arbitrary Orthogonal Bases and Fast Tiling Algorithms," *IEEE Trans. Signal Processing*, Vol.41, No.12, December 1993.
100. Ebrahimi T., "Perceptually Derived Linear Operators- Application to Image Sequence Compression," Ph.D. Dissertation, Laboratoire de Traitement des Signaux, EPFL, Lausanne, 1992.
101. Reynaud P., B. Torresani, "Paquets Continus d'Ondelettes et Decomposition Optimale," Treizieme Colloque Gretsi-Juan-Les-Pins, 16-20 September 1991.
102. Marmolin H., "Subjective MSE Measures," *IEEE Trans. on Systems, Man and Cybernetics*, Vol. 10, No. 3, pp. 486-489, June 1986.
103. Celasun I, B. Sankur, E. Anarım, H. Çağlar and S. Güntürk, "Üç Boyutlu Süzgeç Kümelerinin Tasarımı," *SİU-96*, 5-6 April 1996.

104. Sayood K., *Introduction to Data Compression*, Morgan Kaufmann Publishers, 1996.
105. Beresford N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall Series in Computational Mathematics, 1980.
106. Agrawal, B. P., P. Yip, *Theory and Computations of Toeplitz Matrices*, CRL Report Series, Communications Research Laboratory, McMaster University, Hamilton, Ontario, Canada, July 1980.
107. Sayood K., S. Na, "Recursively Indexed Quantization of Memoryless Sources", *IEEE Trans. On Information Theory*, Vol. 38, pp. 1602-1609, November 1992.

REFERENCES NOT CITED

- Jain A.K., "Image Data Compression : A Review," *Proc. of the IEEE*, Vol. 69, No.3, pp.349-389, March 1991.
- Berger T., *Rate Distortion Theory*, Prentice-Hall, 1971.
- Jain A.K., "Image Data Compression : A Review," *Proc. of the IEEE*, Vol. 69, No.3, pp.349-389, March 1991.
- Haykin, S., *Adaptive Filter Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 07632, 1986.
- Mersereau, R.M., T.C. Speake, "The Processing of Periodically Sampled Multidimensional Signals," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 31, No. 1, pp. 188-194, February 1983.
- Netravali A.N., J.O. Limb, "Picture Coding:A Review", *Proc. of the IEEE*; Vol. 68, No.3, pp.366-406, March 1980.
- Nguyen, T. Q., P. P. Vaidyanathan, "Two Channel Perfect Reconstruction FIR QMF Structures which yield Linear Phase Analysis and Synthesis Filters," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 37, No. 5, pp. 676-690, June 1989.
- Rahgozar M.A., J.P. Allebach, "A General Theory of Time Sequential Sampling", *Signal Processing*, Vol.28, pp.253-270, 1992.
- Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965.
- Rioul, O., "Simple Regularity Criteria for Subdivision Schemes", *SIAM J. Math. Analy.*, Vol. 23, pp. 1544-1576, November 1992.
- Roldan R. R., J. Q. Molina and J.M. Aroza, "Multiresolution Information Analysis for Images," *Signal Processing*, Vol. 24, pp.77-91, .
- Sathe, V., P. P. Vaidyanathan, "Effects of Multirate Systems on the Statistical Properties of Random Inputs," *IEEE Trans. Signal Processing*, Vol. 41, No. 1, pp. 131-146, January 1993.
- Simoncelli E. P., E. H. Adelson, "Nonseparable Extensions of QMFs to Multiple Dimensions", *Proc. of the IEEE*, Vol. 78, No. 4, pp. 652-663, April 1990.

- Strang, G., "Wavelet and Dilation Equations: A Brief Introduction," *SIAM Rev.*, Vol. 31, No. 4, pp. 614-627, December 1990.
- Vaidyanathan, P. P. , " The Role of Smith Form Decomposition of Integer Matrices in Multidimensional Multirate Systems," *Proceedings of IEEE Int. Conf. ASSP*, Toronto, Canada, pp.1777-1780, May. 1991.
- Yoneyama, A., M. Ikehara, "A Design Method of 2-D Perfect Reconstruction Filter Banks," *ECCTD '93*, Circuit Theory and Design, pp.373-378.
- Woods J.W., T. Naveen , "A Filter based Bit Allocation Scheme for Subband Coding of HDTV," *IEEE Trans. on Image Processing*, Vol. 1, No.3, pp. 436-440, July 1992.