A DETERMINISTIC DEMAND INVENTORY MODEL WITH ADVANCE SUPPLY INFORMATION

by

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ABSTRACT

A DETERMINISTIC DEMAND INVENTORY MODEL WITH ADVANCE SUPPLY INFORMATION

In this study we consider a periodic review, single-item inventory model under non-stationary supply availability with advance supply availability information. We have a dynamic deterministic demand and the objective is to minimize expected cost, including fixed ordering cost, holding and backorder costs, over a finite planning horizon under supply constraints. Supply availability has a binomial structure such that supply is either fully available or completely unavailable. Firstly, the dynamic programming formulation of the model is given and the optimality of state dependent (s, S) policy is shown. A heuristic algorithm for finding a good ordering strategy, which is inspired by Silver-Meal Heuristic, is suggested. Then the model with no fixed ordering cost is analyzed. For this model it is shown that optimal policy is of order-up-to type and various properties of the optimal order-up-to levels are demonstrated. A one-pass algorithm that computes the optimal order-up-to levels over the planning horizon is found. Finally, numerical analysis is given including the value of advance supply information and important managerial insights are provided.

ÖZET

ÖN ARZ BİLGİSİ ALTINDA BELİRLİ TALEBE SAHİP BİR ENVANTER MODELİ

Bu çalışmada dönemsel gözden geçirmeli tek ürünlü bir envanter modeli incelenmiştir. Envanter modeli, belirli dinamik talep ve durağan olmayan arz bulunurluğu altında, ön arz bilgisi ile birlikte modellenmiştir. Amaç, beklenen maliyeti sonlu planlama ufku üzerinden enküçüklemek olup maliyetler sabit sipariş maliyeti, stokta tutma maliyeti ve ardısmarlama maliyetini kapsamaktadır. Arz bulunurluğu iki terimli bir yapıya sahiptir, öyle ki bir dönemde ya tüm siparişi karlayacak kadar arz bulunmaktadır ya da hiç arz yoktur. İlk olarak, modelin dinamik programlama gösterimi verilmiş ve en iyi sipariş politikasının arz durumuna bağlı (s, S) politikası olduğu gosterilmiştir. Silver - Meal sezgisel yonteminden esinlenilerek iyi bir sipariş stratejisi bulmak uzere bir sezgisel yöntem onerilmiştir. Sonrasında, sabit sipariş maliyetini içermeyen model incelenmiştir. En iyi sipariş politikasının hedef stok politikası olduğu gosterilmiştir. Hedef stok değerlerinin çeşitli özellikleri verilmiş ve bu özellikler doğrultusunda en iyi hedef stoklarını planlama ufku uzerinden tek geçişle hesaplayan bir algoritma bulunmuştur. Son olarak sabit sipariş maliyetini içeren ve içermeyen modeller için sayısal analizler yapılmış ve önemli yönetimsel değerlendirmeler verilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

A	Fixed ordering cost
A_i	$A_i = \{(\infty, w) : w \in \Omega_M, w = o_M$ or $K(w) \geq j - i + 1\}$
a_1	A constant greater than or equal to zero, that is used in K-convexity formulation
a_2	A constant greater than zero, that is used in K-convexity formulation
b	Backorder cost
B	Amount of back ordered inventory at the beginning of a period
$C_n(I, z_n)$	minimum expected cost of operating the inventory model through periods $n, n+1, ..., N+1$ when the inventory level at the beginning of period n is I and supply availability information vector is $z_n = (q_n, w_n)$
$C_n(T)$	Cost of placing an order to span the demand of periods n to period T
$C_n(T, B)$	Cost of placing an order to span the demand of periods n to period T when the backorder level is B at the beginning of period n
D_{max}	Maximum demand value in the planning horizon
D_n	Demand of period n
$D(n, k)$	Total demand of periods from period n up to and including period k
$EB_n(T)$	Expected backorder cost of ordering to cover the demand of periods n through T
$EB_n(T, B)$	Expected backorder cost of ordering to cover the demand of periods n through T when the backorder level is B at the beginning of period n
$f^{(i)}(n, A_i)$	The column vector of $\left\{f_z^{(l)}(n, A_i), z \in \Omega_{M+1}\right\}$
$f_z^{(i)}(n, A_i)$	$Pr\left\{Z_{n+1} \notin A_i, Z_{n+2} \notin A_i, ..., Z_{n+l} \notin A_i \mid Z_n = z\right\}$
$G_n(y, w_n)$	Auxiliary function that is defined for the ease of notation as $G_n(y, w_n) = L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})]$

h	Holding cost
I	Inventory level
l	Size of the zero vector o_l
k	The period until where we calculate the expected backorder cost in heuristic algorithm
K	A constant integer used for the number of periods demand of which is covered by the placed order
$K^{'}$	A constant integer used for the number of periods demand of which is covered by the placed order
$L_n(y)$	expected single-period inventory costs incurred at the end of period n where the inventory level after the realization of supply is y
M	Size of the supply availability information vector w_n
n	Index used for periods
n^+	Number of periods of demand that inventory level at the beginning of period n covers
N	Number of periods in the planning horizon
o_l	l dimensional zero vector which consists of the first l elements of vector w_n
$\tilde{o}_l$	l dimensional zero vector which consists of the first l elements of vector $\tilde{w}$
p	Supply availability probability for all periods when the supply availability is stationary
p_n	Supply availability probability of period n
$\mathcal{P}(n)$	Probability transition matrix at period n
$\tilde{\mathcal{P}}^{(i)}(n)$	The matrix $P(n)$ where columns corresponding to A_i are replaced by zeros
$P_{z,z^{'}}(n)$	$P\left\{Z_{n+1} = z^{'} \vert Z_n = z\right\}$
r_{M-l}	$M - l$ dimensional vector which starts with ∞ and consists of the last $M - l$ elements of vector w_n
$\tilde{r}_{t-l}$	$t - l$ dimensional vector which starts with ∞ and consists of the last $t - l$ elements of vector $\tilde{w}$
$R_w(n)$	The first time after n , a supply state $z = (\infty, w)$ is observed.

(s, S)	Optimal policy for a type of models which says that if inventory level is below s increase the inventory level to S, otherwise do not place any order
$s_n(w_n)$	Minimum value of y for which the following holds $G_n(y, w_n) = A + G_n(S_n(w_n), w_n)$
$S_n(w_n)$	y value which minimizes $G_n(y, w_n)$ in the model with non-zero fixed ordering cost.(Note that in the model with no fixed ordering cost $y_n(w_n) = S_n(w_n)$.)
t	The number of periods for which we have supply availability information when we place an order covering the demands of period n to period T. Note that $t = n + M - T$.
T	Last period whose demand is covered by the current order
T^*	Best T value at period n, that is best decision at period n is to place an order to cover the demand of periods from n through T.
q_n	Supply availability state of period n
Q	Supply capacity
Q_n	Unknown supply state of a future period n
u	Ordering amount
$u_n(w_n)$	Optimal ordering quantity at period n given w_n
$V_n(y)$	Auxiliary function that is defined for the ease of notation
% VOI_{0-M}	Percent value of information of $(M + 1)$-period ASI
w	Realization of W_n
w_n	Realization of W_n
W_n	Random vector denoting the supply availability information for periods $n + 1, ..., n + M$, $W_n \in \Omega_M$
$\tilde{w}$	Vector of length t, denoting the sub vector of supply availability information for periods $T + 1, ..., n + M$ which can be written as $\tilde{w} = (\tilde{o}_l, \tilde{r}_{t-l})$
y	Inventory level
$y_n(w_n)$	Optimal order-up-to level of period n when supply availability vector is w_n
y_n^*	Optimal order-up-to level of period n

z	Realization of Z_n
$z^{'}$	Realization of Z_n
z_n	Realization of Z_n
Z_n	Random vector denoting the supply availability information for periods $n, n+1, ..., n+M$, $Z_n \in \Omega_{M+1}$
δ	Used as an indicator function
γ_n	Random variable denoting the first period after n that a full supply availability occurs
η	Used for a small increment/decrement in inventory level
$\eta^{'}$	A small inventory amount that is $0 \leq \eta^{'} \leq \eta$
$\tau_n(\eta)$	The first time after period n that the inventory level is raised to the optimal order-up-to level
Ω_M	M dimensional vector with elements in $\{0, \infty\}$
Ω_{M+1}	$M+1$ dimensional vector with elements in $\{0, \infty\}$
ADI	Advance demand information
ASI	Advance supply information
EOQ	Economic order quantity
G/G/1	A single server queueing system with general arrival and service distributions
VOI	Value of information

1. INTRODUCTION

Due to many uncertainties in the supply process in real life, it is not realistic to assume that desired quantity at the desired time will always be available. Uncertainties in supply can be related to machine breakdowns, strikes, defective items etc. Although there are supply uncertainties, information sharing between supplier and manufacturing may reduce the cost resulting from these uncertainties. The aim of this study is to model and analyze production/inventory related issues under supply uncertainty and the availability of advance supply information (ASI).

In this study we propose a single item, periodic review inventory model under non-stationary supply availability with advance supply information. As in Güllü et al. [1], uncertainty of supply has binomial behavior in that supply is either fully available or completely unavailable, so we prefer to use the term supply availability. Demand has a dynamic structure and is assumed to be deterministic. There is no lead time and costs include holding and back-order costs which are assumed to be linear and there is a fixed cost of ordering which is incurred when an order is placed. Objective is to minimize total expected costs over the planning horizon. ASI is incorporated in the model such that we know the supply state of a determined number of periods.

In this study we make various contributions. For the model with positive fixed cost we show the optimality of the state dependent (s, S) policy. We provide the dynamic programming model, present computational results and propose a heuristic algorithm. For the case without fixed cost we demonstrate the optimality of the order-up-to policy and provide structural properties of the policy. We also develop a simple algorithm that finds the optimal order-up-to levels. Furthermore we provide numerical analysis including the value of advance supply availability information.

The rest of the thesis is organized as follows. In Chapter 2 we provide a brief literature review. In Chapter 3 we present our dynamic programming model with a positive fixed cost and advance supply information and analyze the form of the optimal ordering policy. In Chapter 3 we also present a Silver-Meal based heuristic for the model with non-zero fixed ordering cost. In Chapter 4 we give the model without fixed cost and analyze the optimal policy giving the solution for the optimal order-up-to levels. In Chapter 5 we provide numerical analysis for optimal policy, total expected cost and percent value of advance supply information under different parameter settings and computational results for the performance of the heuristic. We conclude the study, and discuss some extensions in Chapter 6.

2. LITERATURE REVIEW

Earliest study of a random supply inventory model was given by Karlin [2] in the literature. Yano and Lee [3] provide a comprehensive review of literature on determining lot sizes when production or procurement yield are random. Vast majority of our references appear in [3]. Gerchak et al. [4] study a finite horizon problem with stationary, stochastic demand and stationary, stoachastically-proportional random yield. They show that the optimal ordering policy is not a simple order-up-to type. Henig and Gerchak [5] extend [4] in that they work with more general cost structures. They provide qualitative implications of random yield for lot sizing and they show that there exists critical order points for both finite and infinite horizon problems. Note that vast majority of random yield literature assumes stationary and stochastically-proportional random yield.

2.1. Capacity Uncertainty Models

Güllü [6] considers the capacity uncertainty under stochastic demand. He establishes the analogy between the class of base-stock production/inventory policies that operate under demand/capacity uncertainty, and the $G/G/1$ queues. This analogy enables us to transform useful properties of the queueing models into properties of the inventory model. Ciarallo et al. [7] treats the aggregate planning problem for a single product with random demand and random capacity. Available capacity in a given period is a random variable. They first consider the single-period problem which has a unimodal cost function. It is shown that in the presence of the randomized capacity optimal policy is identical to the classic newsboy problem. For the multi-period problems order-up-to policy is still shown to be optimal. In this setting a new ordering decision is made at the beginning of the period. The realizations of capacity and demand occurs during the period. Holding and back-order costs are chosen to be linear. Extending the model in [7], Wang and Gerchak [8] have explored the implications of random yields and variable capacity jointly for a finite-horizon periodic review inventory system. They showed that the objective function is quasi-convex and that the optimal policy is of a 'reorder-point' type.

Iida [9] considers a non-stationary periodic review dynamic production - inventory model with uncertain production capacity and uncertain demand. Uncertainty of the production capacity results from the stochastic behavior of the maximum capacity such as unexpected breakdowns or unplanned maintenance. Different than Ciarallo et al. [7] production cost is also considered and again all costs are assumed to be linear. Demands and production capacities are assumed to be independent across time but not necessarily distribute identically. Besides demands and production capacities are assumed to be independent of each other. Lead time is zero and unsatisfied demand is backordered. Iida considers a modified problem compared to Ciarallo et al [7] in that production capacity is realized before ordering. Upper and lower bounds of optimal policies are developed for infinite horizon non-stationary production-inventory problems with uncertain capacity constraint, and furthermore, their minimal planning horizons. It is discovered that the differences between the upper and lower bounds of optimal policies decrease as the length of planning horizons become longer.

2.2. Supply Uncertainty Models

Another group of random yield literature appears under the name of supply uncertainty. Considering continuous review models, Parlar and Berkin [10] incorporate supply disruptions into the classical inventory models analyzing the supply uncertainty for a class of EOQ models. Supply uncertainty is studied such that supply is available only during an interval of random length and unavailable for another interval of random length. Using the renewal reward theorem they find the optimal order quantities. Parlar and Perry [11] extend the model in [10] in that decision maker is not aware of the supply availability state of the supplier and model includes the reorder point as a decision variable. Parlar and Perry [12] generalizes [11] in that the suppliers' market is duopolistic, that is orders can be placed with any of the two suppliers.

Supply uncertainty models are also studied in periodic review setting as our study. Our work is similar to the work of Güllü et al. [1] so we want to give an overview of their paper. They consider a single-item, periodic review, finite horizon, non-stationary supply uncertainty model integrated with a deterministic dynamic demand structure. Supply uncertainty is incorporated into the model with a Bernoulli process in that in a given period, supply is either fully available or fully unavailable. The availability and unavailability probabilities are non-stationary and independent from one period to another. Objective is to minimize total costs, including inventory holding, back-order and purchasing/production costs, over the planning horizon. There is no ordering cost and lead time is assumed to be zero. They present the dynamic programming recursion for the problem. They show the convexity of the cost functions and prove that the optimal ordering policy is of order-up-to type. Optimal order-up-to levels are denoted by y_n^*. If inventory level at the beginning of period is less than or equal to y_n^* an order is placed to increase the inventory level up to y_n^*. Otherwise, no order is placed. Furthermore they explicitly characterize the optimal order-up-to levels. They show that optimal order-up-to level of period n is greater than or equal to its demand, that is $y_n^* \geq D_n$, where D_n is the demand of period n. They also show that $y_n^* \leq D_n + y_{n+1}^*$. These characteristics enable them to write the cost functions in a more compact form. Analysis show that optimal order-up-to levels occur at the cumulative demand points. Central result of their study is the theorem which gives the optimal order-up-to levels. Given that y_{n+1}^* is a K-period demand then y_n^* is K'-period demand where $K' \in \{1, 2, ..., K+1\}$. They give the inequality to determine K' and they provide a one-pass algorithm that determines the optimal order-up-to levels. Afterwards they analyze some special cases one of them being the model with stationary supply uncertainty. They also provide the dynamic programming recursion for the same model with non-zero fixed ordering cost.

Being in a similar path with [1], Güllü et al. [13] work with stationary supply availability and introduce a partial supply availability notion. Partial availability means that, given that supply capacity is Q, if an amount of u is ordered and $u > Q$ then there is a positive probability that only the amount Q will be received from the supplier. Only holding and back-order costs are considered and all assumptions are same with Güllü et al. [1]. For the case without partial availability, a newsboy-like formula is developed for the optimal order-up-to levels and for the partial availability case computational results are obtained.

2.2.1. Supply/Capacity uncertainty models with ASI/ACI

The issue of information sharing in supply chain is studied in the literature mostly on the demand side, which is called advance demand information. Tan et al. [14] build a model that incorporates imperfect advance demand information with ordering decisions an they show that optimal policy is of order-up-to type, where order-up-to level depends on the imperfect ADI. Another relevant group of papers on information sharing includes the studies of Lee et al. [15], Cachon and Fisher [16], and Gavirneni et al. [17]. They consider the information provided by the customer to the supplier, that is called downstream information sharing.

Advance supply information (ASI) notion is not heavily studied in the literature. Some of the existing literature considers an implicit supply information using the notion of Markovian availability. Parlar et al. [18] incorporate possibility of Markovian supply availability into a periodic review setting. They use the classical periodic review model with setup cost and zero lead time. Like in [1] and [13], when an order is placed it may be filled entirely or not filled at all. The availability probability depends on whether supply was available in the previous period. This is a kind of perfect supply information concerning the last period's supply availability. They first analyze the finite-horizon case with backlogging, Markovian supply and ordering cost. They prove that optimal policy is a kind of (s, S) policy, where each period has its own s and S values. Besides, s depends on the availability state in the previous period. Finally they discuss conditions for which the properties of the finite horizon solution also hold for the infinite horizon model.

Song and Zipkin [19] also consider a Markovian model of the supply system. The uncertainties in the supply process are incorporated in the model with leadtimes that change according to the supply conditions. They model the supply system as a discrete-time Markov process and this Markov Process describes the current information about the supply system. When fixed cost is positive optimal policy is a state-dependent

(s, S) policy, i.e. reorder point and order-up-to level depend on the current state of the system. For the case with no fixed cost optimal policy reduces to the well known base-stock policy where optimal order-up-to levels depend on the state of the system. Chen and Yu [20] also consider a Markovian leadtime process and they study the value of leadtime information in a single-location inventory model. They provide a numerical study which says that the value of leadtime information can be significant.

Özekici and Parlar [21] consider an infinite horizon periodic review inventory model with a randomly changing environment. State of the environment is modeled as a time-homogeneous Markov chain. They show that under reasonable conditions, the environment dependent (s, S) policy is optimal. They again work with two point supply availability and represent the environmental fluctuations using a two-dimensional stochastic process. Erdem and Özekici [22] develop a rather general periodic-review inventory model that incorporates both random yield and random environment. The problem is analyzed in single, multiple and infinite periods and it is shown that in all cases, the optimal policy is the well-known base-stock policy where the optimal order-up-to levels are state dependent. State of the environment is assumed to be a time-homogeneous Markov chain.

Advance capacity information is studied by Jaksic et al. [23], who basically build their inventory model on the model of Ciarallo et al. [7] under stochastic demand and stochastic limited supply. They study the benefits of obtaining advance capacity information (ACI) about future uncertain supply capacity. They propose that, by receiving ACI supply capacity uncertainty can be lowered. They show that, optimal ordering policy under ACI is a state-dependent modified order-up-to policy where the optimal base stock level depends on the advance capacity information vector and base-stock level is a decreasing function of the ACI size. Afterwards they develop a capacitated inventory system with advance demand information (ADI) which is shown to be closely related to the ACI model.

The notion of "advance supply information" (ASI) is introduced by Altuğ and Muharremoğlu [24]. For finite and infinite horizon settings, they show the optimality of state-dependent base stock policies under advance supply information. They analyzed the value of information sharing between the manufacturer and the supplier. The information sharing is such that supplier provides a capacity forecast vector to the manufacturer. Getting this forecast vector manufacturer makes an ordering decision where the base-stock level depends on the entire forecast vector. They also provide easily implementable heuristics based on the relationship between the base-stock levels and the forecast vector.

There is no explicit computations of order-up-to levels in the supply uncertainty literature with advance supply information. Our study contributes to the literature in that we give the full characterization of order-up-to levels when there is no fixed ordering cost. We worked with deterministic demand which makes the computations easier and facilitates analysis.

3. MODEL WITH NON-ZERO FIXED ORDERING COST

In this chapter we study the model with non-zero fixed ordering cost. Dynamic programming formulation of the model is given and it is shown that the optimal policy is a state dependent (s, S) policy. Since the dynamic programming problem is time consuming and difficult to solve when state space gets larger we propose a Silver-Meal based heuristic for finding a good ordering strategy.

3.1. Problem Formulation

The notation in Table 3.1 is to be used throughout the thesis but new notation will be introduced as need arises.

Table 3.1. Notation

N:	number of periods in the planning horizon
D_n:	demand in period n
$D(n, k)$:	total demand of periods from period n up to and including period k, that is $D(n, k) = \sum_{i=n}^{k} D_i$
h:	holding cost per unit per period
b:	backorder cost per unit per period
A:	fixed ordering cost that is incurred when an order is placed (variable ordering cost is assumed to be zero)
I:	inventory level at the beginning of the related period
M:	Size of the supply availability information vector excluding the current period
p_n:	probability that the supply is available, i.e. there is ∞ capacity, in period n which means that with probability $1 - p_n$ there will be no supply

Let $L_n(y)$ denote the expected single-period inventory costs incurred at the end of period n where the inventory level after the realization of supply is y.

$$L_n(y) = h \ \max(0, y - D_n) + b \ \max(0, D_n - y). \tag{3.1}$$

It is seen that $L_n(y)$ is the sum of two convex functions, so it is also convex itself.

Define Ω_M and Ω_{M+1} as M and $M+1$ dimensional vectors respectively, with elements in $\{0, \infty\}$. That is,

$$\begin{aligned} \Omega_M &= \left\{(q_1, ..., q_M) : q_i \in \{0, \infty\}, i = 1, 2, ..., M\right\}, \\ \Omega_{M+1} &= \left\{(q_1, ..., q_{M+1}) : q_i \in \{0, \infty\}, i = 1, 2, ..., M+1\right\}. \end{aligned} \tag{3.2}$$

Define Z_n as the random vector denoting the supply availability information for periods $n, n+1, ..., n+M$. Note that $Z_n \in \Omega_{M+1}$. Also let W_n be the random vector denoting the supply availability information for periods $n+1, ..., n+M$, and note that $W_n \in \Omega_M$. By denoting Q_n as the supply information for the current period n, $Z_n = (Q_n, W_n)$. Let z_n, w_n and q_n be the realizations of Z_n, W_n and Q_n respectively.

Notice that w_n can be written as $w_n = (o_l, r_{M-l})$ where o_l is a vector of length l with all entries being zero and r_{M-l} a vector of length $M-l$ with the first entry being ∞.

For $n = 1, 2, .., N+1$ we define $C_n(I, z_n)$ as the minimum expected cost of operating the inventory model through periods $n, n+1, ..., N+1$ when the inventory level at the beginning of period n is I and supply availability information vector is z_n.

The dynamic programming recursion for this problem is

$$\begin{aligned} C_n(I, z_n) &= \min_{I \leq y \leq I+q_n} \{A\ \delta(y - I) + L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})]\} \\ &= \min_{I \leq y \leq I+q_n} \{A\ \delta(y - I) + L_n(y) \\ &\quad + p_{n+M+1}\ C_{n+1}(y - D_n, w_n, \infty) + (1 - p_{n+M+1})\ C_{n+1}(y - D_n, w_n, 0)\} \end{aligned} \tag{3.3}$$

where $\delta(y - I)$ is 1 when $y > I$, and zero otherwise; Q_{n+M+1} is the unknown supply state for period $n + M + 1$. Stochasticity is due to the unknown supply state of period $n + M + 1$. It will be ∞ with probability p_{n+M+1} and 0 with probability $1 - p_{n+M+1}$.

The sequence of events is as follows. (1) At the beginning of period n we receive the new supply information for period $n + M$ so have a new supply availability information vector $z_n = (q_n, w_n)$. (2) The ordering decision is made according to z_n. (3) At the end of the period demand realization occurs.

For the ease of notation an auxiliary function $G_n(y, w_n)$ is defined as

$$G_n(y, w_n) = L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})]. \tag{3.4}$$

The dynamic programming recursion then becomes

$$C_n(I, z_n) = \min_{I \leq y \leq I+q_n} \{A\delta(y - I) + G_n(y, w_n)\} \tag{3.5}$$

where the recurrence relation is considered within G_n.

When we have no available supply in the current period we can not place an order so cost function is:

$$C_n(I, 0, w_n) = G_n(I, w_n).$$

On the other hand, if we have ∞ supply in the current period cost function becomes:

$$C_n(I,\infty,w_n) = \min_{y\geq I}\left\{A\delta(y-I) + G_n(y,w_n)\right\}.$$

3.2. Optimality of State Dependent (s,S) Policy

We see from the numerical results that optimal policy is of state dependent (s,S) type. The proof of this can be done with the analysis of *K-convexity* using the theory provided by Bertsekas [25].

Theorem 3.1. *For* $n = 1,2,...,N$.

- *(i)* $G_n(y,w_n)$ *is A-convex in* y *for all* w_n.
- *(ii) The optimal ordering policy is a state dependent* $(s_n(w_n), S_n(w_n))$ *policy where* $S_n(w_n)$ *minimizes* $G_n(y,w_n)$ *and* $s_n(w_n)$ *is the smallest value of* y *for which* $G_n(y,w_n) = A + G_n(S_n(w_n), w_n)$.
- *(iii)* $C_n(I,z_n)$ *is A-convex in* I *for all* z_n *and it is minimized at* $S_n(w_n)$.

Proof. We know that $C_{N+1} = 0$. Therefore, $G_N(y,w_N) = L_N(y)$ which is convex. So $G_N(y,w_N)$ is A-convex as a property of K-convexity for $A \geq 0$. We know that $L_N(y)$ attains its minimum at D_N and $L_N(D_N) = 0$. $S_N(w_N)$ is defined as the minimizer of $G_N(y,w_N)$ so $S_N(w_N) = D_N$. Suppose the assertions hold for $n+1$.

We know that

$$C_{n+1}(I,z_{n+1}) = \min_{I\leq y\leq I+q_{n+1}}\left\{A\delta(y-I) + G_{n+1}(y,w_{n+1})\right\}. \tag{3.6}$$

If $q_{n+1} = 0$, $C_{n+1}(I,0,w_{n+1}) = G_{n+1}(I,w_{n+1})$ so C_{n+1} is A-convex since (i) holds for $n+1$. If $q_{n+1} = \infty$ on the other hand, by the assumption that (ii) holds for $n+1$,

$C_{n+1}(I,\infty,w_{n+1})$ can be written as

$$C_{n+1}(I,\infty,w_{n+1}) = \begin{cases} A + G_{n+1}(S_{n+1}(w_{n+1}), w_{n+1}) & \text{if } I < s_{n+1}(w_{n+1}) \\ G_{n+1}(I, w_{n+1}) & \text{if } I \geq s_{n+1}(w_{n+1}) \end{cases} . \tag{3.7}$$

To show that $C_{n+1}(I,\infty,w_{n+1})$ is A-convex we must verify that

$$\begin{aligned} A + C_{n+1}(y+a_1,\infty,w_{n+1}) \geq{}& C_{n+1}(y,\infty,w_{n+1}) \\ &+ a_1\left[\frac{C_{n+1}(y,\infty,w_{n+1}) - C_{n+1}(y-a_2,\infty,w_{n+1})}{a_2}\right] \end{aligned} \tag{3.8}$$

for all $a_1 \geq 0$, $a_2 > 0$, y.

We distinguish three cases:

Case 1: $y \geq s_{n+1}(w_{n+1})$

If $y - a_2 \geq s_{n+1}(w_{n+1})$, then in this region for all values of a_1, a_2, y, we know that $C_{n+1}(I,\infty,w_{n+1}) = G_{n+1}(I,w_{n+1})$ for all I and w_{n+1}. Therefore in this case C_{n+1} is A-convex.

If $y - a_2 < s_{n+1}(w_{n+1})$, then in the view of Equation (3.7) we can write Equation (3.8) as

$$\begin{aligned} A + G_{n+1}(y+a_1,w_{n+1}) \geq{}& G_{n+1}(y,w_{n+1}) \\ &+ a_1\left[\frac{G_{n+1}(y,w_{n+1}) - G_{n+1}(s_{n+1}(w_{n+1}),w_{n+1})}{a_2}\right]. \end{aligned} \tag{3.9}$$

Now if y is such that $G_{n+1}(y,w_{n+1}) \geq G_{n+1}(s_{n+1}(w_{n+1}),w_{n+1})$ then by the A-convexity of G_{n+1} we have

$$\begin{aligned} A + G_{n+1}(y + a_1, w_{n+1}) &\geq G_{n+1}(y, w_{n+1}) \\ &\quad + a_1 \left[\frac{G_{n+1}(y, w_{n+1}) - G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})}{y - s_{n+1}(w_{n+1})} \right] \\ &\geq G_{n+1}(y, w_{n+1}) \\ &\quad + a_1 \left[\frac{G_{n+1}(y, w_{n+1}) - G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})}{a_2} \right] \qquad (3.10) \end{aligned}$$

since $y - a_2 < s_{n+1}(w_{n+1})$. Thus Equation (3.9) and hence also Equation (3.8) holds.

If y is such that $G_{n+1}(y, w_{n+1}) < G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})$ then we have

$$\begin{aligned} A + G_{n+1}(y + a_1, w_{n+1}) &\geq A + G_{n+1}(S_{n+1}(w_{n+1}), w_{n+1}) = G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1}) \\ &> G_{n+1}(y, w_{n+1}) \\ &\geq G_{n+1}(y, w_{n+1}) \\ &\quad + a_1 \left[\frac{G_{n+1}(y, w_{n+1}) - G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})}{a_2} \right]. \qquad (3.11) \end{aligned}$$

So for this case Equation (3.9) and hence also Equation (3.8) holds.

Case 2: $y \leq y + a_1 \leq s_{n+1}(w_{n+1})$

In this region, according to Equation (3.7) C_{n+1} is always A-convex because it is always equal to $A + G_{n+1}(S_{n+1}(w_{n+1}), w_{n+1})$, which is the sum of a constant and an A-convex function, so it always obeys Equation 3.8.

Case 3: $y < s_{n+1}(w_{n+1}) < y + a_1$

For this case in the view of Equation (3.7) we can write Equation (3.8) as

$$
\begin{aligned}
A + G_{n+1}(y + a_1, w_{n+1}) \geq G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1}) \\
+ a_1 \left[\frac{G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1}) - G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})}{a_2} \right].
\end{aligned}
$$

Therefore $A + G_{n+1}(y + a_1, w_{n+1}) \geq G_{n+1}(s_{n+1}(w_{n+1}), w_{n+1})$ which always holds by the definition of $s_{n+1}(w_{n+1})$.

We have showed that C_{n+1} is A-convex by using A-convexity of G_{n+1}.

$$
\begin{aligned}
G_n(y, w_n) &= L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})] \\
&= L_n(y) + p_{n+M+1} C_{n+1}(y - D_n, w_n, \infty) \\
&+ (1 - p_{n+M+1}) C_{n+1}(y - D_n, w_n, 0).
\end{aligned}
$$

By the definition of K-convexity if $C_{n+1}(y, z_{n+1})$ is A-convex, $p_{n+M+1}C_{n+1}(y - D_n, w_n, \infty) + (1 - p_{n+M+1})C_{n+1}(y - D_n, w_n, 0)$ is also A-convex. So G_n is also A-convex for all w_n. With the same procedure if G_n is A-convex, C_n is also A-convex.

$$
C_n(I, z_n) = \begin{cases} G_n(I, w_n) & \text{if } I \geq s_n(w_n) \\ G_n(I, w_n) & \text{if } q_n = 0 \\ A + G_n(S_n(w_n), w_n) & \text{if } q_n = \infty, I < s_n(w_n) \end{cases}. \tag{3.12}
$$

If we set $I = S_n(w_n)$ in Equation (3.12) we get $C_n(S_n(w_n), z_n) = G_n(S_n(w_n), w_n)$ since $S_n(w_n) \geq s_n(w_n)$. As $G_n(S_n(w_n), w_n) \leq G_n(I, w_n)$ for all I then we can directly say that $C_n(I, z_n)$ is minimized at $S_n(w_n)$ □

So we know that $G_n : R \to R$ is a continuous A-convex function and $G_n(y, w_n) \to \infty$ as $|y| \to \infty$, then there exists $s_n(w_n)$, $S_n(w_n)$ values that give the optimal policy.

Remark 3.1. Note that given a supply information vector $z_n = (q_n, w_n)$ at the beginning of period n, q_n has no relevance in determining the inventory policy for period

n. This is evident from the definition of $G_n(y, w_n)$. Moreover the knowledge of the current supply state Q_n does not yield any value of information on the average. If we do not know the current supply state the expected minimum cost of the system from period n onward is $E_{Q_n}\left[C_n(I, Q_n, w_n)\right] = p_n\ C_n(I, \infty, w_n) + (1 - p_n)\ C_n(I, 0, w_n)$. If we know the current supply state $Q_n = q_n$ we will have either $C_n(I, \infty, w_n)$ or $C_n(I, 0, w_n)$ with probabilities p_n and $(1 - p_n)$ respectively.So the expected cost is again $p_n\ C_n(I, \infty, w_n) + (1 - p_n)\ C_n(I, 0, w_n)$.

The dynamic programming problem for the model with non-zero fixed ordering cost is difficult to solve when the state space of the problem is large. For a large state space, a huge memory is needed and the problem is time-consuming which are undesired. Therefore we propose a Silver-Meal based heuristic for finding a good ordering decision in a short time.

3.3. A Silver-Meal Based Heuristic for Non-zero Fixed Cost Model

In this section our aim is to develop a heuristic algorithm for finding a good ordering strategy under non-zero fixed cost and advance supply information.

The suggested method is a forward method, like Silver-Meal Heuristic [26], which requires determining the average cost per period as a function of the number of periods the current order is to span. The number of periods to span is increased until the average cost per period starts increasing. Because of the stochastic nature of supply availability, we need to consider the expected backorder costs on top of the holding and set-up costs in the classical Silver-Meal Heuristic. Besides we have supply availability information for a number of periods so we also need to incorporate this information into the method.

Say that we are at the beginning of period n. We have an on-hand inventory of $D(n, n+n^+-1)$, where $n^+ = 0, 1, ..., N-n+1$. Note that $D(n, n-1) = 0$. That means we have an on-hand inventory corresponding to the demand of n^+ periods. Besides, we have a negative inventory of B. Note that by the definition of the terms n^+ and B, at least one of them should be zero.

At the beginning of period n we have a supply availability information of $z_n = (q_n, w_n)$ and w_n can be written as $w_n = (o_l, r_{M-l})$. (Note that the definition of l and M in this context are given in Section 3.1.) In this decision epoch we can order nothing or place an order to cover the demand of a number of periods. Assume that in this decision epoch the demand of periods $n+n^+-1, n+n^+, ..., T$ will be ordered, T being our decision variable. If we place an order of $D(n,T)$, which means that we cover the demands of periods n through T, we will incur a cost of $C_n(T,B)$. Note that $T = n+n^+-1$ corresponds to placing no orders.

For period n, first we successively calculate the average cost per period, that is $C_n(T,B)$ for $T = n+n^+, n+n^++1, ..., N$. (Note that for $T = n+n^+, n+n^++1, ..., N$ we will clear the negative inventory at once, so B does not affect the calculations and we can write the cost as $C_n(T)$.) Throughout the cost calculations for $T = n+n^+, n+n^++1, ..., N$ stop as soon as $C_n(T+1) > C_n(T)$ and say that T is the best among $n+n^+, n+n^++1, ..., N$. Secondly we calculate the average cost per period when we order nothing, that is $C_n(T = n+n^+-1, B)$. If $C_n(T) < C_n(T = n+n^+-1, B)$ we say that at period n best decision is to place an order to cover the demand of periods from n through T, that is $T^* = T$. Otherwise if $C_n(T) \geq C_n(T = n+n^+-1, B)$ we say that at period n, best decision is to order nothing, that is $T^* = n+n^+-1$.

As mentioned above, first we need to iteratively calculate the average cost per period, $C_n(T)$ for $T = n+n^+, n+n^++1, ..., N$.

We distinguish two cases in order to be able to define $C_n(T)$.

Case 1: If $T < n+M$, then it means that we still have supply availability information for periods $T+1, ..., T+M$. That is, we have supply availability information for $t = n+M-T$ periods. Let $\tilde{w} = (\tilde{o}_l, \tilde{r}_{t-l})$ be a vector of length t, denoting the sub vector of supply availability information for periods $T+1, ..., n+M$. $C_n(T)$ depends on $\tilde{w}$ in this case.

If $\tilde{w}$ starts with ∞, ($q_{T+1} = \infty$ and therefore $l = 0$), we do not need to consider any backorder cost incurred. So we just consider the fixed cost and the holding cost and $C_n(T)$ can be written as

$$\frac{A \ + \ h \displaystyle\sum_{i=1}^{T-n} i \ D_{n+i}}{T - n + 1}. \tag{3.13}$$

Equation (3.13) says that we will incur the fixed cost and the holding costs corresponding to periods $n + 1, ..., T$. To be able to find the per period cost, we divide the cost by the number of periods of demand that will be ordered, that is, $T - n + 1$.

If $\tilde{w}$ starts with 0 ($q_{T+1} = 0$ and therefore $l > 0$) but includes at least one ∞ supply, that is $l < t$, then on top of fixed and holding costs, we need to consider the backorder costs that will be incurred. Then $C_n(T)$ can be written as

$$\frac{A \ + \ h \displaystyle\sum_{i=1}^{T-n} i \ D_{n+i} \ + \ b \displaystyle\sum_{i=1}^{l} (l - i + 1) \ D_{T+i}}{T - n + l + 1}. \tag{3.14}$$

Equation (3.14) consists of fixed cost, holding costs regarding the periods $n + 1, ..., T$ and the backorder costs regarding the periods $T + 1, ..., T + l$. To be able to find the per period cost, we divide the cost by $T - n + l + 1$. As in Equation (3.13), $T - n + 1$ is the number of periods of demand that will be ordered in this decision epoch . We have an additional l periods in this case which represents the number of periods that have no available supply and we will not be able to place an order in these periods.

If $\tilde{w}$ consists of all zeros (that is, $l = t$) then we need to consider expected backorder costs ($EB_n(T)$), since we do not have supply availability information. $EB_n(T)$ is the expected backorder cost of ordering to cover the demand of periods n through T and it is calculated up to a point where we expect to observe the first full supply availability. Then in this case, $C_n(T)$ can be written as

$$\frac{A \; + \; h \sum_{i=1}^{T-n} i \; D_{n+i} \; + \; b \sum_{i=1}^{l} (l-i+1) \; D_{T+i} + EB_n(T)}{T-n+l+1}. \tag{3.15}$$

Equation (3.15) consists of fixed cost, holding costs regarding the demand of periods $n+1,...,T$ and the backorder cost regarding the periods $T+1,...,T+l$ and the expected backorder cost that will be incurred after period T. As in Equation (3.14) we divide the cost by $T-n+l+1$ to find the per period cost.

As we have said before $EB_n(T)$ is the expected backorder cost of ordering to cover the demand of periods n through T. If the first expected period with full supply availability is greater than N, then we calculate the expected backorder cost until the end of the planning horizon. Otherwise we calculate the expected backorder cost until the first expected period with full supply availability. So we define this period, denoted by k, as

$$k = \min(N \; , \; T + t + \lceil E\left[\gamma_{n+M}\right] \rceil - 1) \tag{3.16}$$

where γ_{n+M} is the random variable denoting the first period after $n+M$ that a full supply availability occurs.

So we can write the expected backorder cost as

$$\begin{aligned} EB_n(T) = b \; &\sum_{i=1}^{k-T-t} Pr(\gamma_{n+M} = i) \; \sum_{j=1}^{i-1} (i-j) \; D_{T+t+j} \\ &+ b \; Pr(\gamma_{n+M} = \infty) \; \sum_{j=1}^{k-T-t} \; (k-T-t-j+1) \; \left[D_{T+t+j} + B\right] \end{aligned} \tag{3.17}$$

where B is the backorder amount which is equal to $\sum_{i=1}^{l} D_{T+i}$ if $l = t$. The probabilities in Equation (3.17) are defined as

$$\begin{aligned} P(\gamma_n = 1) &= p_{n+1}, \\ P(\gamma_n = i) &= p_{n+i} \prod_{j=1}^{i-1} (1 - p_{n+j}) \quad \text{for } 1 < i \leq k-n, \\ P(\gamma_n = \infty) &= \prod_{j=1}^{k-n} (1 - p_{n+j}). \end{aligned} \tag{3.18}$$

Note that $P(\gamma_n = \infty)$ represents the probability that none of the periods from period $n+M+1$ through k will have full supply availability. The definition of this probability term ensures that probabilities sum up to 1.

Combining the Equations (3.13), (3.14), and (3.15) we can write $C_n(T)$ for the case $T < n+M$ (that is, $t > 0$) as

$$C_n(T) = \begin{cases} \dfrac{A + h\sum_{i=1}^{T-n} iD_{n+i}}{T-n+1} & \text{if } l = 0, \\[2ex] \dfrac{A + h\sum_{i=1}^{T-n} iD_{n+i} + b\sum_{i=1}^{l} (l-i+1)D_{T+i}}{T-n+l+1} & \text{if } 0 < l < t, \\[2ex] \dfrac{A + h\sum_{i=1}^{T-n} iD_{n+i} + b\sum_{i=1}^{l} (l-i+1)D_{T+i} + EB_n(T)}{T-n+l+1} & \text{if } l = t. \end{cases} \tag{3.19}$$

Case 2: If $T \geq n+M$ (that is, $t \leq 0$), it means that we do not have any information on hand regarding periods after T. We incur fixed cost, holding cost and expected backorder costs as shown in Equation (3.20)

$$C_n(T) = \frac{A + h\sum_{i=1}^{T-n} i\, D_{n+i} + EB_n(T)}{T-n+1}. \tag{3.20}$$

Equation (3.20) consists of fixed cost, holding costs regarding the demand of periods

$n+1, ..., T$ and the expected backorder costs after period T. As in Equation (3.17) we need to define the expected backorder cost $EB_n(T)$. In this case we have no supply availability information regarding periods after T so k is defined as

$$k = \min(N, T + \lceil E\left[\gamma_T\right] \rceil - 1).$$

Therefore we can write $EB_n(T)$ as

$$\begin{aligned} EB_n(T) = b \sum_{i=1}^{k-T} Pr(\gamma_T = i) \sum_{j=1}^{i-1} (i-j) D_{T+j} \\ + bPr(\gamma_T = \infty) \sum_{j=1}^{k-T} (k - T - j + 1) D_{T+j}. \end{aligned} \tag{3.21}$$

Note that the probability terms are defined in Equation (3.18).

Afterwards we calculate the cost of placing no orders, that is $C_n(T = n + n^+ - 1, B)$. While calculating this cost, backorder level B affects the calculations since we do not order anything. If $n^+ = 0$, we can think as if there is an additional 0-supply at the beginning of the supply availability information vector because we will not order anything. For this reason we define δ, which is 1 if $n^+ = 0$ and 0 if $n^+ > 0$.

We distinguish two cases in order to be able to define $C_n(T = n + n^+ - 1, B)$.

Case 1: If $T + \delta < n + M$, we still have supply availability information for periods $T + \delta + 1, ..., T + \delta + M$. That is, we have supply availability information for $t = n + M - T - \delta$ periods. Let $\tilde{w} = (\tilde{o}_l, \tilde{r}_{t-l})$ be a vector of length t, denoting the sub vector of supply availability information for periods $T + \delta + 1, ..., T + \delta + M$. $C_n(T = n + n^+ - 1, B)$ depends on $\tilde{w}$ in this case.

If $\tilde{w}$ starts with ∞, (therefore $l = 0$), additional to B, we will incur backorder cost regarding D_n if $n^+ = 0$. On the other hand, if we have an on-hand inventory more than the current demand, we will incur holding costs. To be able to find the per period cost, we divide the cost by the number of periods, that is, $T - n + \delta + 1 = n^+ + \delta$. This means that, we divide the cost by n^+ (that is, the number of periods that our on-hand inventory covers) or we divide by 1 if we have nothing on hand. Therefore $C_n(T = n + n^+ - 1, B)$ can be written as

$$\frac{h\sum_{i=1}^{T-n} iD_{n+i} + B\ b + b\ \delta D_n}{T - n + \delta + 1}. \tag{3.22}$$

If $\tilde{w}$ starts with 0 (therefore $l > 0$) but includes at least one ∞ supply, that is $l < t$, we need to consider the backorder costs that will be incurred during the periods with 0-supply, and the holding costs if we have an on-hand inventory more than the current demand. To be able to find the per period cost, we divide the cost by $T-n+l+\delta+1 = n^+ + l + \delta$. This means that, if we have on-hand inventory we divide the cost by $n^+ + l$ and we divide by $l + \delta$ otherwise. Then $C_n(T = n + n^+ - 1, B)$ can be written as

$$\frac{h \sum_{i=1}^{T-n} i\ D_{n+i}\ + B\ b\ (l + \delta) +\ b \sum_{i=1}^{l+\delta} (l + \delta - i + 1)\ D_{T+i}}{T - n + l + \delta + 1}. \tag{3.23}$$

If $\tilde{w}$ consists of all zeros (that is, $l = t$) then we need to also consider expected backorder costs ($EB_n(T, B)$), since we do not have supply availability information. $EB_n(T, B)$ is the expected backorder cost of ordering to cover the periods n through T and it is calculated up to a point where we expect to observe the first full supply availability. To be able to find the average cost per period we divide by $T-n+l+\delta+1 = n^+ + l + \delta$ which is $n^+ + l$ when we have on-hand inventory and $l + \delta$ otherwise. Then $C_n(T = n + n^+ - 1, B)$ can be written as

$$\frac{h\sum_{i=1}^{T-n} iD_{n+i}\ + B\ b\ (l + \delta) + b\sum_{i=1}^{l+\delta} (l + \delta - i + 1)D_{T+i} + EB_n(T, B)}{T - n + l + \delta + 1}. \tag{3.24}$$

$EB_n(T, B)$ is different than $EB_n(T)$ in Equation (3.17) in that, now we may have a negative inventory of B and the cost depends on this B. Note that before calculating the expected backorder cost we need to update the backorder level as $B = B + D(T + 1, T + l + \delta)$. For this case k value (that is, the period up to which we will calculate the expected backorder cost) is defined as

$$k = \min(N\ ,\ T + t + \delta + \lceil E\left[\gamma_{n+M}\right]\rceil - 1).$$

So in this case we can write $EB_n(T, B)$ as

$$\begin{aligned} EB_n(T,B) = b \sum_{i=1}^{k-T-t-\delta} Pr(\gamma_{n+M} = i) \left[(i-1)B + \sum_{j=1}^{i-1} (i-j) D_{T+t+j}\right] \\ + bPr(\gamma_{n+M} = \infty) \left[(k - T - t - \delta)B + \sum_{j=1}^{k-T-t-\delta} (k - T - t - \delta - j + 1) D_{T+t+j}\right] \end{aligned} \tag{3.25}$$

where the probability terms are defined in Equation 3.18.

Combining the equations (3.22), (3.23), and (3.24) we can write $C_n(T = n + n^+ - 1, B)$ for the case $T + \delta < n + M$, that is $t > 0$ as

$$C_n(T = n + n^+ - 1, B) = \begin{cases} \dfrac{h\sum_{i=1}^{T-n} iD_{n+i} + Bb + b\delta D_n}{T-n+\delta+1} & \text{if } l = 0, \\[2ex] \dfrac{h\sum_{i=1}^{T-n} iD_{n+i} + Bb(l+1) + b\sum_{i=1}^{l+\delta} (l+\delta-i+1) D_{T+i}}{T-n+l+\delta+1} & \text{if } l < t, \\[2ex] \dfrac{h\sum_{i=1}^{T-n} iD_{n+i} + Bb(l+1) + b\sum_{i=1}^{l+\delta} (l+\delta-i+1) D_{T+i} + EB_n(T,B)}{T-n+l+\delta+1} & \text{if } l = t. \end{cases} \tag{3.26}$$

Case 2: If $T + \delta \geq n + M$ (that is, $t \leq 0$, $n^+ \geq M + 1$ and $\delta = 0$), meaning that we do not have any information on hand regarding periods after T, we incur, holding cost and expected backorder costs. We divide by $T-n+\delta+1=n^{+}+\delta$ (that is n^{+} if we have on-hand inventory and 1 otherwise) as shown in Equation (3.27)

$$C_n(T)=\frac{h\sum_{i=1}^{T-n} iD_{n+i}+EB_n(T,B)}{T-n+1}. \tag{3.27}$$

As in Equation (3.25) we need to define the expected backorder cost $EB_n(T,B)$. In this case we have no supply availability information regarding periods after T and $EB_n(T,B)$ differs from Equation 3.21 in that we may have negative inventory which is updated as $B=B+D(T+1,T+l)$. For this case k value (that is, the period up to which we will calculate the expected backorder cost) is defined as

$$k=\min(N\ ,\ T+\lceil E\left[\gamma_{n+M}\right]\rceil-1). \tag{3.28}$$

Therefore we can write $EB_n(T)$ as

$$\begin{aligned}EB_n(T)&=b\sum_{i=1}^{k-T}Pr(\gamma_T=i)\left[(i-1)B+\sum_{j=1}^{i-1}(i-j)D_{T+j}\right]\\&\quad+bPr(\gamma_T=\infty)\left[(k-T)B+\sum_{j=1}^{k-T}(k-T-j+1)D_{T+j}\right].\end{aligned} \tag{3.29}$$

Note that the probability terms are defined in Equation (3.18).

As an example, say that we have an inventory system under stationary supply availability with an availability probability of 0.5 (that is, $p_n=p=0.5\ \forall\ n$) and our planning horizon consists of 5 periods ($N=5$). Our supply availability information includes the supply states of the current period and the next period, that is $M=1$. Say that we have 10 units of demand for all periods ($D_n=10\ \forall n$). We have no on-hand inventory at the beginning of the planning horizon. Holding, backorder costs are 1 and 5 respectively and we have fixed ordering cost of 20.

As an example, we want to find the best ordering strategy, according to the heuristic procedure, for period 1 when the supply availability information vector is $z_1 = (\infty, 0)$. Note that $n = 1$, $M = 1$, $w_1 = (0)$, $l = 1$, $I = 0$, $B = 0$, and $n^+ = 0$.

We start with $T = 1$. $T < n + M$, $t = n + M - T = 1$, $\tilde{w} = (0)$ so we have $l = t = 1$. Then we will use the Equation (3.15) to find the cost $C_1(1)$.

$$\frac{20 + 1\sum_{i=1}^{0} iD_{1+i} + 5\sum_{i=1}^{1}(2-i)D_{1+i} + EB_1(1)}{2}$$

To be able to calculate the expected backorder cost $EB_1(1)$ we need to find k which is the period we expect to have available supply.

$$k = \min(5, 2 + \lceil E\left[\gamma_2\right] \rceil - 1)$$

In our case we have stationary supply availability, therefore we expect to have available supply after $\gamma_2 = 1/p = 2$ periods. So $k = 3$. We will find the expected backorder cost for $k = 3$ with the Equation 3.17

$$\begin{aligned} EB_1(1) = 5\ &\sum_{i=1}^{1} Pr(\gamma_2 = i)\ \sum_{j=1}^{i-1} (i-j)\ D_{2+j} \\ &+ 5\ Pr(\gamma_2 = \infty)\ \sum_{j=1}^{1} (2-j)\ \left[D_{2+j} + D_2\right] \end{aligned}$$

$Pr(\gamma_2 = 1) = p = 0.5$ and $Pr(\gamma_2 = \infty) = 1 - p = 0.5$. Therefore $EB_1(1) = 5 * 0.5 * (D_3 + D_2) = 50$. With this we can write $C_1(1)$ as

$$C_1(1) = \frac{20 + 5D_2 + EB_1(1)}{2} = 60.$$

Similarly we will find the cost for $T = 2$, which is $C_1(2)$. Now $T = n + M$ so $t = 0$

so we are in Case 2 and we will use the Equation (3.20) and the expected backorder cost $EB_1(2)$ will be calculated according to Equation (3.21)

$$\begin{aligned} EB_1(2) = 5 \sum_{i=1}^{1} Pr(\gamma_2 = i) \sum_{j=1}^{i-1} (i-j)\ D_{2+j} \\ + 5\ Pr(\gamma_2 = \infty) \sum_{j=1}^{1} (2-j)\ D_{2+j}, \end{aligned}$$

where $Pr(\gamma_2 = i) = 0.5$ and $Pr(\gamma_2 = \infty) = 0.5$ so $EB_1(2) = 2.5\ D_3 = 25$. Therefore $C_1(2)$ is

$$C_1(2) = \frac{20\ +\ D_2\ +\ EB_1(2)}{2} = 27.5.$$

$C_1(2) < C_1(1)$ so we continue with $T = 3$ and find $C_1(3)$ with the Equation (3.20) Now k is find as 4 and $EB_1(3) = 25$.

$$C_1(3) = \frac{20\ +\ 1 \sum_{i=1}^{2} i\ D_{1+i}\ +\ EB_1(3)}{3} = 25.$$

$C_1(3) < C_1(2)$ so we continue with $T = 4$ and find $C_1(4)$ with the Equation (3.20) Now k is find as 5 and $EB_1(4) = 25$.

$$C_1(4) = \frac{20\ +\ 1 \sum_{i=1}^{3} i\ D_{1+i}\ +\ EB_1(4)}{4} = 26.25.$$

$C_1(4) > C_1(3)$ so we stop and we need to check the cost of not ordering anything that is $C_1(0,0)$ since $B = 0$ and $T = n + n^+ - 1 = 0$. We have $n^+ = 0$ so $\delta = 1$. $T + \delta < n + M$ so we are in Case 1. $t = n + M - T - \delta = 1$ and $\tilde{w} = (0)$ so we have $l = t = 1$. Then we will use the Equation (3.24) to find the cost $C_1(0,0)$. We need to update the backorder level B as $B = B + D(T+1, T+l+\delta) = B + D_1 + D_2 = 20$.

$$C_1(0,0) = \frac{b \sum_{i=1}^{2} (3-i)\ D_i + EB_1(0,B)}{2}.$$

$EB_1(0,B)$ can be calculated as in Equation (3.25) with $k = \min(N, T+t+\delta+\lceil E[\gamma_{n+M}]\rceil - 1) = 3$.

$$\begin{aligned} EB_1(0,B) &= b\ \sum_{i=1}^{1} Pr(\gamma_2 = i)\ \left[(i-1)\ B + \sum_{j=1}^{i-1} (i-j)\ D_{2+j}\right] \\ &+ b\ Pr(\gamma_2 = \infty)\ \left[(1)\ B + \sum_{j=1}^{1} (2-j)\ D_{2+j}\right] \\ &= b\ Pr(\gamma_2 = \infty)\ [B +\ D_3] = 75. \end{aligned}$$

Then,

$$C_1(0,0) = \frac{b\ (2\ D_1 + D_2) + EB_1(0,B)}{2} = 112.5.$$

$C_1(3) < C_1(0,0)$ so $T^* = 3$, that is, best decision at period 1 is to place an order of 30 units which the cumulative demand of periods 1,2, and 3.

Order of complexity is $O(N^2)$ for the heuristic algorithm which can be checked from the pseudo-code in Figure 3.1. Note that it does not depend on the size of the advance supply information vector M.

The performance of the heuristic algorithm is analyzed in Chapter 5, Section 5.4.

```
Repeat the below procedure for a defined number of iterations.
Set $I = 0$,  Total Cost $= 0$, $n^+ = 0$, $B = 0$.
Get the supply availability information vector $z_1 = (q_1, w_1)$.
FOR $n = 1$ to $N$
     IF $q_n = \infty$ THEN
     Find $T^*$ as explained in the above algorithm
          IF $I < D_n(n, T^*)$ THEN
               set $I = D(n, T^*)$, $n^+ = T^* - n + 1$
               set  Total Cost $=$ Total Cost $+A$.
          END IF
     END IF
     Set $I = I - D_n$.
     Set $B = \max(0, -I)$.
     Set $n^+ = \max(0, n^+ - 1)$.
     Set  Total Cost $=$ Total Cost $+h \max(0, I) + b \max(0, -I)$.
     Get supply availability information for period $n + M + 1$.
     Update the supply availability information vector.
END FOR
```

Figure 3.1. Pseudo-code for heuristic algorithm

4. MODEL WITH NO FIXED ORDERING COST

In this chapter we study the same model in Chapter 3, however there is no fixed ordering cost. Firstly the model of the new problem is given. Then it is shown that optimal policy is of order-up-to type where order-up-to level depends on the advance supply information. The characterization of the order-up-to levels is provided and a simple algorithm is given for finding the optimal order-up-to levels.

4.1. The Model and the Optimal Ordering Policy

As in the model in Chapter 3, $C_n(I, z_n)$ is the minimum expected cost of operating the inventory model through periods $n, n+1, ..., N+1$ when the inventory level at the beginning of period n is I and $M+1$-period supply information is z_n. The sequence of events is the same and the dynamic programming recursion for this problem is

$$C_n(I, z_n) = \min_{I \leq y \leq I+q_n} \{L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})]\} \tag{4.1}$$

where Q_{n+M+1} is the unknown supply state of period $n+M+1$. It will be ∞ with probability p_{n+M+1} and 0 with probability $1 - p_{n+M+1}$

To simplify the notation we define the auxiliary function $G_n(y, w_n) := L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})]$. Hence we can write Equation (4.1) as follows

$$C_n(I, z_n) = \min_{I \leq y \leq I+q_n} \{G_n(y, w_n)\} \tag{4.2}$$

where the recurrence relation is considered within G_n.

Theorem 4.1. *For $n = 1, 2, ..., N$:*

- *(i) $G_n(y, w_n)$ is convex in y. Let the minimum of $G_n(y, w_n)$ be at $y_n(w_n)$,*
- *(ii) $C_n(I, z_n) = C_n(I, q_n, w_n)$ is convex in I and it is minimized at $I = y_n(w_n)$,*
- *(iii) the optimal ordering policy is of order-up-to type. The ordering quantity at the*

beginning of the period is $u_n(w_n) = \max\{y_n(w_n) - I, 0\}$.

Proof. In Section 3.2, Theorem 3.1 we have showed that $G_n(y, w_n)$ is A-convex in y and $C_n(I, z_n)$ is A-convex in I which are both minimized at $S_n(w_n)$. By the property of K-convexity we know that for $A = 0$ these functions are 0-convex, that is convex. So (i) and (ii) holds. When $A = 0$, $S_n(w_n) = s_n(w_n)$ so the policy is of optimal order-up-to type which says that if inventory level is below $S_n(w_n)$ order up to $S_n(w_n)$ otherwise do not order anything. So (iii) also holds trvially. Here we use the notation $y_n(w_n)$ for the optimal point of $G_n(y, w_n)$. □

4.2. Characterization of the Optimal Order-up-to Levels

Proposition 4.1. $y_n(w_n) \geq D_n$ *for all* $n = 1, 2, ..., N$.

Proof. For $n = N$, $y_N(w_N) = D_N$ and the assertion holds. Assume that, the statement is true for $n+1$. Since $C_{n+1}(I, z_{n+1})$ is convex and it is minimized at $I = y_{n+1}(w_{n+1})$, it follows that for $y < D_n$, $C_{n+1}(y - D_n, z_{n+1}) \geq C_{n+1}(0, z_{n+1}) \geq C_{n+1}(y_{n+1}(w_{n+1}), z_{n+1})$ for any z_{n+1} since $y_{n+1}(w_{n+1}) \geq D_{n+1} \geq 0 \geq y - D_n$. We also know that $L_n(y)$ is minimized at D_n so $L_n(y) \geq L_n(D_n)$.

$$\begin{aligned}
G_n(y, w_n) &= L_n(y) + E[C_{n+1}(y - D_n, w_n, Q_{n+M+1})] \\
&= L_n(y) + p_{n+M+1}\ C_{n+1}(y - D_n, w_n, \infty) \\
&\quad + (1 - p_{n+M+1})\ C_{n+1}(y - D_n, w_n, 0) \\
&\geq L_n(D_n) + p_{n+M+1}\ C_{n+1}(D_n - D_n, w_n, \infty) \\
&\quad + (1 - p_{n+M+1})\ C_{n+1}(D_n - D_n, w_n, 0) = G_n(D_n, w_n).
\end{aligned}$$

Therefore, for $y < D_n$ we have $G_n(y, w_n) \geq G_n(D_n, w_n)$ and this completes the proof. □

Remark 4.1. If we have a supply availability information vector which is composed of zeros we can write $w_n = o_M$ because we know that $l = M$. Similarly if the supply availability information vector starts with ∞ we can write $w_n = r_M$ because we know that $l = 0$.

Proposition 4.2. *Let* $w_n = (o_l, r_{M-l})$ *for some* $l \in 0, 1, ..., M$ *then* $y_n(w_n)$ *is minimized at* $l = 0$*. That is* $y_n(r_M) = D_n \leq y_n(w_n)$ *, for any* w_n*,* $\forall n = 1, 2, ..., N$.

Proof. It is sufficient to show that $y_n(r_M) = D_n$ because the second part is already satisfied by Proposition 4.1. For $n = N$, $y_N(w_N) = D_N$ for any w_N and the assertion trivially holds. Assume that, it also holds for $n+1$, i.e. $y_{n+1}(r_M) = D_{n+1}$. Then,

$$\begin{aligned} G_n(y, r_M) = L_n(y) &+ p_{n+M+1} C_{n+1}(y - D_n, r_M, \infty) \\ &+ (1 - p_{n+M+1}) C_{n+1}(y - D_n, r_M, 0). \end{aligned} \tag{4.3}$$

Define $\eta \in (0, D_{n+1})$. For $y \leq D_n$, $y - D_n \leq 0 \leq y_{n+1}(w_{n+1})$ and $G_n(y, r_M) = L_n(y) + E\left[G_{n+1}(y_{n+1}(w_{n+1}), w_{n+1})\right]$. For $D_n < y \leq D_n + \eta$, $y - D_n \leq \eta < D_{n+1} \leq y_{n+1}(w_{n+1})$ and $G_n(y, r_M) = L_n(y) + E\left[G_{n+1}(y_{n+1}(w_{n+1}), w_{n+1})\right]$.

Note that $E\left[G_{n+1}(y_{n+1}(w_{n+1}), w_{n+1})\right]$ is constant. Therefore $G_n(y, r_M)$ is minimized by minimizing $L_n(y)$. We know that $L_n(y)$ is minimized at D_n. So we say that $G_n(y, r_M)$ is minimized at D_n that is $y_n(r_M) = D_n$.

By Proposition 4.1 we know that $y_n(w_n) \geq D_n$ so we can conclude that $y_n(r_M) = D_n \leq y_n(w_n)$. □

Proposition 4.3. *Let* $w = (o_l, r_{M-l})$ *for some* $l \in 1, 2, ..., M-1$*. Then there exists* $K(w) \in 1, 2, ..., l+1$ *such that* $y_n(w) = D(n, n + K(w) - 1)$*. Moreover,*

$$K(w) = \begin{cases} 1 & \text{if } \; h \geq l\; b, \\ j & \text{if } \; h \in \left\{ \frac{l-j+1}{j}\; b\;,\; \frac{l-j+2}{j-1}\; b \right\} \quad j = 2, ..., l, \\ l+1 & \text{if } \; h < \frac{1}{l}\; b. \end{cases} \tag{4.4}$$

Proof. First note that since $w = (o_l, r_{M-l})$ for some $l \in 1, 2, ..., M-1$

$$\begin{aligned} G_n(y,w) = L_n(y) + \sum_{j=1}^{l} L_{n+j}(y - D(n, n+j-1)) \\ + E\left[C_{n+l+1}(y - D(n, n+l), \infty, W_{n+l+1})\right]. \end{aligned}$$

Define $V_n(y) = L_n(y) + \sum_{j=1}^{l} L_{n+j}(y - D(n, n+j-1))$. First note that $V_n(y)$ increases for $y > D(n, n+l)$. Also, since the supply state in period $n+l+1$ is ∞, that is, since we can raise the inventory level of period $n+l+1$ to $y_{n+l+1}(w_{n+l+1})$ as long as $y - D(n, n+l) < y_{n+l+1}(w_{n+l+1})$, $C_{n+l+1}(y - D(n, n+l), \infty, W_{n+l+1})$ is non-decreasing for $y > D(n, n+l)$. Therefore at optimality we have $y_n(w) \leq D(n, n+l)$.

Moreover, for $y \leq D(n, n+l)$, $y - D(n, n+l) \leq 0 \leq y_{n+l+1}(w_{n+l+1})$. Hence,

$$\begin{aligned} E\left[C_{n+l+1}(y - D(n, n+l), \infty, W_{n+l+1})\right] \\ = E\left[G_{n+l+1}(y_{n+l+1}(W_{n+l+1}), W_{n+l+1})\right] \end{aligned}$$

which is a constant. Then $G_n(y, w)$ is minimized by minimizing

$$V_n(y) = L_n(y) + \sum_{j=1}^{l} L_{n+j}(y - D(n, n+j-1)).$$

$V_n(y)$ is not differentiable at $D(n,n), D(n, n+1), ..., D(n, n+l)$ but for $y \in (D(n, n+i-1), D(n, n+i))$, $i = 1, 2, ..., l$ we can take the derivative as $V_n^{'}(y, w) = i\ h - (l - i + 1)\ b$. Then

$$y_n(w) = \min\left\{y : V_n^{'}(y, w) \geq 0\right\}.$$

Then we can write $y_n(w) = D(n, n + K(w) - 1)$ where

$$K(w) = \min\left\{i \in 1, 2, ..., l : i\ h - (l - i + 1)\ b \geq 0\right\},$$

and let $K(w) = l + 1$ if no such $K(w)$ exists. $\square$

Corollary 4.1. *$y_n(o_{l_1}, r_{M-l_1}) \leq y_n(o_{l_2}, r_{M-l_2})$ for $l_1 < l_2$ for all $n = 1, 2, ..., N$.*

Proof. $K(w)$ in Proposition 4.3 increases in l, that is $G_n^{'}(y, w_1) \leq G_n^{'}(y, w_2)$ for $w_1 = (o_{l_1}, r_{M-l_1})$ and $w_2 = (o_{l_2}, r_{M-l_2})$ where $l_1 < l_2$. $\square$

Proposition 4.4. *$y_n(w) \leq y_n(o_M)$ for any $w = (o_l, r_{M-l})$ where $l < M$, for $n = 1, 2, ..., N$.*

Proof. For any $w = (o_l, r_{M-l})$ where $l < M$, $y_n(w) = D(n, n + K(w) - 1)$ for $K(w) \in \{1, 2, ..., l + 1\}$ (note that $K(w) \leq M$). Proposition 4.3 states that if $h \in \left\{\frac{(l-j+1)}{j}\ b, \frac{(l-j+2)}{j-1}\ b\right\}$ for $j \in \{2, ..., l + 1\}$ then $K(w) = j$. This means that, for any $K(w) \in \{2, ..., l + 1\}$ we need to have $h \in \left\{\frac{(l-K(w)+1)}{K(w)}\ b, \frac{(l-K(w)+2)}{K(w)-1}\ b\right\}$. Notice that we can write the upper bound on h as $h\ (K(w) - 1) < (l - K(w) + 2)\ b$.

We can write $G_n(y, o_M)$ as

$$\begin{aligned} G_n(y, o_M) = L_n(y) + \sum_{j=1}^{M} L_{n+j}(y - D(n, n + j - 1)) \\ + E\left[C_{n+M+1}(y - D(n, n + M), Z_{n+M+1})\right]. \end{aligned} \tag{4.5}$$

Choose $y \in (D(n, n + K(w) - 2), D(n, n + K(w) - 1))$. Then $y - D(n, n + M) < D(n, n + K(w) - 1) - D(n, n + M) < 0 < y_{n+M+1}(w_{n+M+1})$ since $K(w) \leq M$.

Define $V_n(y) = L_n(y) + \sum_{j=1}^{M} L_{n+j}(y - D(n, n + j - 1))$ for the ease of notation. Select η in such a way that $y + \eta \in (D(n, n + K(w) - 2), D(n, n + K(w) - 1))$. Then we can write

$$\begin{aligned} G_n(y + \eta, o_M) - G_n(y, o_M) = V_n(y + \eta) - V_n(y) \\ + E\left[C_{n+M+1}(y + \eta - D(n, n + M), Z_{n+M+1})\right] \\ - E\left[C_{n+M+1}(y - D(n, n + M), Z_{n+M+1})\right]. \end{aligned} \tag{4.6}$$

We know that $C_{n+M+1}(y, z_{n+M+1})$ is convex in y and minimized at $y_{n+M+1}(w_{n+M+1}) \geq D_{n+M+1} \geq 0$. Then $C_{n+M+1}(0, Z_{n+M+1}) < C_{n+M+1}(y + \eta - D(n, n + M), Z_{n+M+1}) < C_{n+M+1}(y - D(n, n + M), Z_{n+M+1})$ for any such selected η. Therefore $E\left[C_{n+M+1}\right.$ $\left.(y + \eta - D(n, n + M), Z_{n+M+1}) - C_{n+M+1}(y - D(n, n + M), Z_{n+M+1})\right] < 0$.

Also we can write $V_n(y + \eta) - V_n(y) = (K(w) - 1)\ h\ \eta - (M - K(w) + 2)\ b\ \eta$ since under both y and $y + \eta$ we have on-hand inventory for periods $n, n + 1, ..., n + K(w) - 2$ and insufficient inventory for periods $n + K(w) - 1, n + K(w), ..., n + M$. We know from Proposition 4.3 that $h\ (K(w) - 1) < (l - K(w) + 2)\ b$. Therefore $h\ (K(w) - 1) < (M - K(w) + 2)\ b$ since $l < M$. Therefore $V_n(y + \eta) - V_n(y) \leq 0$.

As a result $G_n(y + \eta, o_M) - G_n(y, o_M) \leq 0$ which means that $y_n(o_M) \geq D(n, n + K(w) - 1) = y_n(w)$ for any $w = (o_l, r_{M-l})$ with $l < M$. ☐

Proposition 4.5. *$y_n(w_n) \leq D_n + y_{n+1}(o_M)$ for all w_n, $n = 1, 2, ..., N$.*

Proof. It is sufficient to show that $y_n(o_M) \leq D_n + y_{n+1}(o_M)$ since we already know that $y_n(w_n) \leq y_n(o_M)$ for all w_n. Assertion holds for $n = N$, because $y_N(w_N) = D_N$ and $y_{N+1}(w_{N+1}) = 0$ for all w_{N+1}.

Suppose $y > D_n + y_{n+1}(o_M)$, we need to show that $G_n(y, o_M) \geq G_n(D_n + y_{n+1}(o_M), o_M)$

- $L_n(y) \geq L_n(D_n + y_{n+1}(o_M))$ because we are already larger than D_n and if we increase the inventory level further cost will also increase.
- $C_{n+1}(y - D_n, o_M, \infty) \geq C_{n+1}(y_{n+1}(o_M), o_M, \infty)$ because we know that $y_{n+1}(o_M) \geq y_{n+1}(o_{M-1}, \infty)$. So by convexity of C_{n+1}, $C_{n+1}(y - D_n, o_M, \infty) \geq C_{n+1}(D_n + y_{n+1}(o_M) - D_n, o_M, \infty) \geq C_{n+1}(y_{n+1}(o_{M-1}, \infty), o_M, \infty)$.
- $C_{n+1}(y - D_n, o_M, 0) \geq C_{n+1}(y_{n+1}(o_M), o_M, 0)$ obviously because $y_{n+1}(o_M)$ is the optimal point of $C_{n+1}(y, o_M, 0)$.

Putting everything together:

$$\begin{aligned}
G_n(y, o_M) &= L_n(y) + p_{n+M+1}C_{n+1}(y - D_n, o_M, \infty) + (1 - p_{n+M+1})C_{n+1}(y - D_n, o_M, 0) \\
&\geq L_n(D_n + y_{n+1}(o_M)) + p_{n+M+1}C_{n+1}(y_{n+1}(o_M), o_M, \infty) \\
&\qquad + (1 - p_{n+M+1})C_{n+1}(y_{n+1}(o_M), o_M, 0) \\
&= G_n(D_n + y_{n+1}(o_M), o_M).
\end{aligned}$$

So we can conclude that $G_n(y, o_M) \geq G_n(D_n + y_{n+1}(o_M), o_M)$. $\square$

As shown in Proposition 4.3, whenever $w \neq o_M$, the order-up-to level of any period n can be written as $y_n(w) = D(n, n + K(w) - 1)$. Note that $K(w)$ does not depend on n. In what follows, we characterize the optimal order-up-to level of a period when $w = o_M$. For $z_n = (q_n, w_n)$ define

$$R_w(n) = \min \left\{k : k \in \{1, 2, ..., N - n\}, Z_{n+k} = (\infty, w)\right\}$$

given that $w_n = o_M$.

If no such k exists, set $R_w(n) = \infty$. Note that $R_w(n)$ is the first time after n , a supply state $z = (\infty, w)$ is observed. In particular whenever $w = o_M$

$$R_{o_M}(n) = \min \left\{k : k \in \{1, 2, ..., N - n\}, Z_{n+k} = (\infty, o_M)\right\}.$$

Proposition 4.6. *Assume that $y_{n+1}(o_M) = D(n+1, n+K)$ (K-period demand) for some $n \in 1, 2, ...N - 1$ and $1 \leq K \leq N - n$. Then, for $j = 1, 2, ..., K$,*

$$G_n(D(n, n+j), o_M) \leq G_n(D(n, n+j) - \eta, o_M) \ \ \forall\ 0 \leq \eta \leq D_{n+j}$$

if and only if

$$\sum_{i=j}^{N-n} Pr\left\{R_w(n) > i; \forall w\right\} \frac{b}{h} \geq$$

$$1 + \sum_{i=1}^{j-1} Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j - i + 1\right\} \quad (4.7)$$

where $K(w)$ is defined in Proposition 4.3 and

$$Pr\left\{R_w(n) > i; \forall w\right\} = \begin{cases} 1 & \textit{if } \; i < M+1 \\ \prod_{k=M+1}^{i} (1 - p_{n+k}) & \textit{if } \; i \geq M+1 \end{cases} .$$

Proof. Suppose that $y_{n+1}(o_M) = D(n+1, n+K)$ and for $j \in \{1, 2, .., K\}$ set $y = D(n, n+j) - \eta$ for some $\eta \in [0, D_{n+j}]$. The evolution of the inventory level in periods $n+1, n+2, ...$ depends on the choice of η.

For a fixed $j \in \{1, 2, .., K\}$ define $\tau_n(\eta)$ as the first time after n that the inventory level is raised to the optimal order-up-to level (possibly by ordering zero) when the inventory level after ordering at the beginning of period n is equal to y.

First note that $\tau_n(\eta) = i$ implies that (1) the supply availability information in period $n+i$ should be of the form $z_{n+i} = (\infty, w)$ for some w and (2) the inventory level before ordering at period $n+i$ should be less than $y_{n+i}(w)$. In periods prior to $n+i$ it is either not possible to order (that is, supply is not available), or the inventory level is above the period's respective order-up-to level.

With this definition of $\tau_n(\eta)$ we can write $G_n(y)$ for $y = D(n, n+j) - \eta$ as

$$G_n(y, o_M) = L_n(y) + \sum_{i=1}^{N-n} Pr\left\{\tau_n(\eta) > i\right\}\ L_{n+i}(y - D(n, n+i-1)) \\ + \sum_{i=1}^{N-n} \sum_{w} Pr\left\{\tau_n(\eta) = i, W_{n+i} = w\right\}\ G_{n+i}(y_{n+i}(w), w). \tag{4.8}$$

The second term in Equation (4.8) stands for the case that ordering occurs at a period later than $n+i$ and hence the starting inventory level of period $n+i$ is $y - D(n, n+i-1)$, and a cost of $L_{n+i}(y - D(n, n+i-1))$ is incurred in period $n+i$.

The third term in Equation (4.8) is due to the fact that if $\tau_n = i$ and $W = w$, then the starting inventory level of period $n+i$ after ordering is $y_{n+i}(w)$ and the expected cost incurred is $G_{n+i}(y_{n+i}(w), w)$.

It can be shown that the distribution of $\tau_n(\eta^{'})$ is the same for all $0 \leq \eta^{'} \leq \eta$ and in particular for $\eta^{'} = 0$. Note that if it is not possible to order prior to period $n+i$ for a particular $\eta > 0$, then in the same periods an ordering can not occur for $0 \leq \eta^{'} \leq \eta$.

Moreover, if an ordering occurs in period $n+i$ (by raising the inventory position up to $y_{n+i}(w)$) with $\eta > 0$, then for any $0 \leq \eta^{'} \leq \eta$ an ordering occurs in period $n+i$ by raising the inventory level to the same order-up-to level $y_{n+i}(w)$.

Suppose that $\{\tau_n(\eta) = i, W = w\}$ with $w = o_M$ (the case $w \neq o_M$ is similar). Note that $y = D(n, n+j) - \eta \leq D_n + y_{n+1}(o_M)$ since $y_{n+1}(o_M) = D(n+1, n+K)$ and $j \in \{1, 2, .., K\}$. By Proposition 4.5 we know that $y_{n+1}(o_M) \leq D_{n+1} + y_{n+2}(o_M)$ so we can write $y_{n+1}(o_M) \leq D(n+1, n+i-1) + y_{n+i}(o_M)$ (that is, $y_{n+i}(o_M) \geq D(n+i, n+K)$). Therefore $y = D(n, n+j) - \eta \leq D(n, n+i-1) + y_{n+i}(o_M)$. Note that for $\eta = 0$ we will have an inventory of $D(n, n+j)$ at the beginning of period n. Having no available supply till period $n+i$, at the beginning of period $n+i$ we will have an inventory of $D(n, n+j) - D(n, n+i-1) = D(n+i, n+j) \leq D(n+i, n+K) \leq y_{n+i}(o_M)$. This enables us to drop η from $\tau_n(\eta)$ and write Equation (4.8) as:

$$\begin{aligned} G_n(y, o_M) = L_n(y) + \sum_{i=1}^{N-n} Pr\left\{\tau_n > i\right\}\ L_{n+i}(y - D(n, n+i-1)) \\ + \sum_{i=1}^{N-n} \sum_{w} Pr\left\{\tau_n = i, W_{n+i} = w\right\}\ G_{n+i}(y_{n+i}(w), w). \end{aligned} \tag{4.9}$$

Therefore we can write the difference $G_n(y, o_M) - G_n(D(n, n+j), o_M)$ as

$$\begin{aligned} G_n(y, o_M) - G_n(D(n, n+j), o_M) = L_n(y) - L_n(D(n, n+j)) \\ + \sum_{i=1}^{N-n} Pr\left\{\tau_n > i\right\}\ (L_{n+i}(y - D(n, n+i-1)) - L_{n+i}(D(n+i, n+j))). \end{aligned} \tag{4.10}$$

For $i \geq j$, $y - D(n, n+i-1) = D(n+i, n+j) - \eta \leq D_{n+i}$. We know from Proposition 4.1 that $y_{n+i}(w) \geq D_{n+i}$ so we can write

$$Pr\left\{\tau_n > i\right\} = Pr\left\{R_w(n) > i, \forall\ w\right\}.$$

For $i < j$, the inventory level is raised to $y_{n+i}(o_M)$ whenever $Z_{n+i} = (\infty, o_M)$. If $Z_{n+i} = (\infty, w)$ for $w \neq o_M$ then an order is placed if $y - D(n, n+i-1) \leq y_{n+i}(w)$. Therefore

$$\begin{aligned} Pr\left\{\tau_n > i\right\} &= Pr\left\{R_w(n) > i : y - D(n, n+i-1) \leq y_{n+i}(w)\right\} \\ &= Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, y - D(n, n+i-1) \leq y_{n+i}(w).\right\} \end{aligned}$$

For $w \neq o_M$, $y - D(n, n+i-1) \leq y_{n+i}(w)$ implies that $D(n, n+j) - \eta - D(n, n+i-1) = D(n+i, n+j) - \eta \leq y_{n+i}(w) = D(n+i, n+i+K(w)-1)$. Therefore $n+j \leq n+i+K(w)-1$ which implies $K(w) \geq j-i+1$. Hence for $i < j$

$$Pr\left\{\tau_n > i\right\} = Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M,\ K(w) \geq j-i+1\right\}$$

where $K(w)$ is defined as in Proposition 4.3.

Now putting things together Equation (4.10) can be written as

$$\begin{aligned} G_n(y, o_M) - G_n(D(n, n+j), o_M) &= -h\ \eta \\ &- h\ \eta \sum_{i=1}^{j-1} Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M,\ K(w) \geq j - i + 1\right\} \\ &+ b\ \eta \sum_{i=j}^{N-n} Pr\left\{R_w(n) > i;\ \forall\ w\right\}. \end{aligned} \tag{4.11}$$

Therefore $G_n(D(n, n+j), o_M) \leq G_n(y, o_M)$ if and only if

$$\sum_{i=j}^{N-n} Pr\left\{R_w(n) > i; \forall w\right\} \frac{b}{h} \geq$$
$$1 + \sum_{i=1}^{j-1} Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j - i + 1\right\}$$

The probability term $Pr\left\{R_w(n) > i;\ \forall\ w\right\}$ is in fact the probability that there is no available supply in one of the periods from period $n+M+1$ (note that we know the supply state of the periods $n, n+1, ..., n+M$) up to and including period $n+i$. Therefore $Pr\left\{R_w(n) > i;\ \forall\ w\right\} = 1$ for $i < M+1$ since we know that in periods $n+1, ..., n+M$ supply is unavailable. For $i \geq M+1$, $Pr\left\{R_w(n) > i;\ \forall\ w\right\} = \prod_{k=M+1}^{i} (1 - p_{n+k})$. Remember that p_{n+k} is the probability of having ∞ supply at period $n+k$. $\square$

$Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j - i + 1\right\}$ can be found by using the first hitting time probability notion.

For a fixed $j \in \{1, 2, ..., K\}$ define

$$A_i = \{(\infty, w) : w \in \Omega_M, w = o_M \text{ or } K(w) \geq j - i + 1\}$$

for $i \in \{1, 2, ..., j-1\}$.

Define:

$$f_z^{(l)}(n, A_i) = Pr\left\{Z_{n+1} \notin A_i, Z_{n+2} \notin A_i, ..., Z_{n+l} \notin A_i \mid Z_n = z\right\}.$$

Therefore

$$Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j - i + 1\right\} = f_z^{(i)}(n, A_i).$$

Define $f^{(l)}(n, A_i)$ as the column vector of $\left\{f_z^{(l)}(n, A_i), z \in \Omega_{M+1}\right\}$.

For all $z \in \Omega_{M+1}$ and $z' \in \Omega_{M+1}$ define $P_{z,z'}(n) = P\left\{Z_{n+1} = z' | Z_n = z\right\}$, let $\mathcal{P}(n)$ be the respective matrix of $\left\{P_{z,z'}(n), z, z' \in \Omega_{M+1}\right\}$.

Let $\tilde{\mathcal{P}}^{(i)}(n)$ be the matrix $\mathcal{P}(n)$ where columns corresponding to A_i are replaced by zeros.

$$f_z^{(1)}(n, A_1) = Pr\left\{Z_{n+1} \notin A_1 | \; Z_n = z\right\} = \sum_{z' \notin A_1} Pr\left\{Z_{n+1} = z' | \; Z_n = z\right\}.$$

Therefore $f^{(1)}(n, A_1) = \tilde{\mathcal{P}}^{(i)}(n) \; \underline{1}$, where $\underline{1}$ is the column vector whose entries are all 1's and whose size matches $\tilde{\mathcal{P}}^{(i)}(n)$.

We can find $f_z^{(2)}(n, A_2)$ as:

$$\begin{aligned} f_z^{(2)}(n, A_2) &= Pr\left\{Z_{n+1} \notin A_2, Z_{n+2} \notin A_2 | \; Z_n = z\right\} \\ &= \sum_{z' \notin A_2} P_{z,z'(n)} Pr\left\{Z_{n+2} \notin A_2 | \; Z_{n+1} = z'\right\} \\ &= \tilde{P}^{(2)}(n) f^{(1)}(n+1, A_2) \end{aligned}$$

since $Pr\left\{Z_{n+2} \notin A_2 | \; Z_n = z'\right\} = f_{z'}^{(1)}(n+1, A_2)$.

In general,

$$\begin{aligned} f_z^{(i)}(n, A_i) &= Pr\left\{Z_{n+1} \notin A_i, ..., Z_{n+i} \notin A_i \middle| \; Z_n = z\right\} \\ &= \sum_{z' \notin A_i} P_{z,z'(n)} Pr\left\{Z_{n+2} \notin A_i, ..., Z_{n+i} \notin A_i \middle| \; Z_{n+1} = z^{'}\right\} \end{aligned}$$

where $Pr\left\{Z_{n+2} \notin A_i, ..., Z_{n+i} \notin A_i | \; Z_{n+1} = z^{'}\right\}$ is equal to $f_{z^{'}}^{(i-1)}(n+1, A_i)$. Therefore,

$$f^{(i)}(n, A_i) = \tilde{\mathcal{P}}^{(i)}(n) f^{(i-1)}(n+1, A_i)$$

for $i = 1, 2, ..., j-1$ with $f^{(0)} \equiv \underline{1}$.

Finally, setting $w_n = o_M$ we can write the probability term in Equation 4.7 as follows:

$$\begin{aligned} &Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j-i+1\right\} \\ &\qquad = f_{(0,o_M)}^{(i)}(n, A_i)\ (1-p_n) + f_{(\infty,o_M)}^{(i)}(n, A_i)\ p_n \end{aligned} \tag{4.12}$$

where $f_{(0,o_M)}^{(i)}(n, A_i)$ and $f_{(\infty,o_M)}^{(i)}(n, A_i)$ are obtained as explained above.

As an example, for $M = 1$ (that is, 2-period ASI), state space is $\Omega_{M+1} = \{(\infty, 0), (0, \infty), (0, 0), (\infty, \infty)\}$ and the probability transition matrix is

$$\mathcal{P}(n) = \begin{bmatrix} 0 & p_{n+2} & 1-p_{n+2} & 0 \\ 1-p_{n+2} & 0 & 0 & p_{n+2} \\ 0 & p_{n+2} & 1-p_{n+2} & 0 \\ 1-p_{n+2} & 0 & 0 & p_{n+2} \end{bmatrix}.$$

If $A_1 = \{(\infty, 0)(\infty, \infty)\}$ then $\tilde{\mathcal{P}}^{(1)}(n)$ and $f^{(1)}(n, A_1)$ are defined as

$$\tilde{\mathcal{P}}^{(1)}(n) = \begin{bmatrix} 0 & p_{n+2} & 1 - p_{n+2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p_{n+2} & 1 - p_{n+2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$f^{(1)}(n, A_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let's compute $f^{(4)}(n, A_4)$ with the assumption that $A_i = \{(\infty, 0), (\infty, \infty)\}$.

$$\begin{aligned} f^{(4)}(n, A_4) &= \tilde{\mathcal{P}}^{(4)}(n) f^{(3)}(n+1, A_4), \\ f^{(3)}(n+1, A_4) &= \tilde{\mathcal{P}}^{(4)}(n+1) f^{(2)}(n+2, A_4), \\ f^{(2)}(n+2, A_4) &= \tilde{\mathcal{P}}^{(4)}(n+2) f^{(1)}(n+3, A_4), \\ f^{(1)}(n+3, A_4) &= \tilde{\mathcal{P}}^{(4)}(n+3)\ \underline{1}. \end{aligned}$$

Therefore

$$f^{(1)}(n+3, A_4) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$f^{(2)}(n+2,A_4)=\begin{bmatrix}1-p_{n+4}\\0\\1-p_{n+4}\\0\end{bmatrix},$$

$$f^{(3)}(n+1,A_4)=\begin{bmatrix}(1-p_{n+3})(1-p_{n+4})\\0\\(1-p_{n+3})(1-p_{n+4})\\0\end{bmatrix},$$

$$f^{(4)}(n,A_4)=\begin{bmatrix}(1-p_{n+2})(1-p_{n+3})(1-p_{n+4})\\0\\(1-p_{n+2})(1-p_{n+3})(1-p_{n+4})\\0\end{bmatrix}.$$

Notice that the inequality established in Proposition 4.6 does not depend on the demand sequence and η. It depends on the supply availability probabilities and the cost parameters. Note that, $K(w) \geq j-i+1$ condition is determined by the holding and backorder costs and $R_w(n) > i$ just depend on the supply availability probabilities. So Proposition 4.6 guarantees that the optimal order-up-to levels always occur in one of the cumulative demand points. If there is a cost benefit of increasing (decreasing) the order-up-to level a small amount η, it should be increased (decreased) up to the next cumulative demand point. Corollary 4.2 states this observation.

Corollary 4.2. $y_n(w) \in \{D_n, D(n,n+1),...,D(n,n+N)\}$ *for* $n=1,2,...,N.$

Theorem 4.2. *The optimal order-up-to level for period* $n \in \{1,2,...,N\}$ *is equal to* K_n *period demand* $D(n,n+K_n-1)$ *for some* $1 \leq K_n \leq N-n+1$ *with* $K_N=1$*. Given that* $y_{n+1}(o_M) = D(n+1,n+K)$ *(*K*-period demand) for some* $1 \leq K \leq N-n+1$

and $n = N-1, N-2, ..., 1$. *Then* $y_n(o_M) = D(n, n+K^{'})$ *where*

$$\begin{aligned} K^{'} = \max \{ & j = 1, 2, ..., K : \\ & \sum_{i=j}^{N-n} Pr\left\{R_w(n) > i; \forall w\right\} \frac{b}{h} \geq \\ & \quad 1 + \sum_{i=1}^{j-1} Pr\left\{R_{o_M}(n) > i, R_w(n) > i; w \neq o_M, K(w) \geq j-i+1\right\} \end{aligned} \tag{4.13}$$

where $K(w)$ *is defined as in Proposition 4.3 and*

$$Pr\left\{R_w(n) > i; \forall w\right\} = \begin{cases} 1 & if \;\; i < M+1 \\ \displaystyle\prod_{k=M+1}^{i} (1 - p_{n+k}) & if \;\; i \geq M+1 \end{cases},$$

$$\begin{aligned} & Pr\left\{R_{o_M}(n) > i, R_w(n) > i : \forall w : w \neq o_M, K(w) \geq j-i+1\right\} \\ & \qquad = f^{(i)}_{(0,o_M)}(n, A_i)\ (1-p_n) + f^{(i)}_{(\infty,o_M)}(n, A_i)\ p_n. \end{aligned}$$

Proof. Since $G_N(y,w) = L_N(y)$, $y_N(w) = D_N$ and therefore $K_N = 1$ as desired. Assume that the assertions hold for $n+1$, and in particular $y_{n+1}(o_M) = D(n+1, n+K)$. Note that $y_n(o_M) \geq D_n$ by Proposition 4.1 and $y_n(o_M) \leq D_n + y_{n+1}(o_M) = D(n, n+K)$ by Proposition 4.5. These observations, together with Corollary 4.2, assure that the minimum of $G_n(y,w)$ will occur on $\{D_n, D(n,n+1), ..., D(n,n+K)\}$. Since $G_n(y,w)$ is convex, the minimum of it is equal to $D(n, n+K^{'})$, where $K^{'}$ is the greatest number satisfying

$$G_n(D(n, n+K^{'}), w) \leq G_n(D(n, n+K^{'}-1), w),$$

which is found by Equation (4.13). If $K^{'} = K$, then $G_n(y,w)$ is a decreasing convex function and the order-up-to level for period n is set to its highest possible value ($K+1$

- period demand). However if no such $K^{'}$ exists then $G_n(y, w)$ is an increasing convex function and hence $y_n(o_M) = D_n$. $\square$

Optimal order-up-to levels when $w_n = o_M$ are determined by the following algorithm.

Table 4.1. Algorithm for optimal order-up-to levels

```
Step 0.  $K = 1$  $(y_N(o_M) = D_N)$
Step 1.  For $n = N - 1$ to 1, find $K^{'}$ satisfying Equation (4.13).
         Set $y_n(o_M) = D(n, n + K^{'})$ and $K = K^{'} + 1$.
         If no such $K^{'}$ exists, set $y_n(o_M) = D_n$ and $K = 1$.
```

As an example, say that we have an inventory system under stationary supply availability with an availability probability of 0.5 (that is, $p_n = p = 0.5 \ \forall \ n$) and our planning horizon consists of 5 periods ($N = 5$). Our supply availability information includes the supply states of the current period and the next period, that is $M = 1$ and $\Omega_{M+1} = \{(\infty, 0), (0, \infty), (0, 0), (\infty, \infty)\}$.

We have no fixed ordering cost and per unit backorder and holding costs are 10 and 1 respectively. We are given that, $y_2(0) = D(2, 4)$ meaning that when we have $w_2 = (0)$ at period 2, optimal order-up-to level is the cumulative demand of $K = 3$ periods.

Given all these, we want to find the optimal order-up-to level of period 1 when $w_1 = (0)$, that is $y_1(0)$.

We will use Theorem 4.2 to find $K^{'}$ that satisfies the inequality (4.13). In this example $w \neq o_M$ means $w = (\infty)$. As it was shown in Proposition 4.2, $K(w) = 1$ for $w \neq o_M$ and therefore $Pr\left\{R_{o_M}(n) > i, R_w(n) > i; w \neq o_M, K(w) \geq j - i + 1\right\}$ simplifies into $Pr\left\{R_{o_M}(n) > i\right\}$ (note that $j - i + 1 \geq 2$ which is always strictly greater than K(w)). For this example we can write the inequality (4.13) as

$$K^{'} = \max \left\{ j = 1,2,3 : \sum_{i=j}^{4} Pr\left\{R_w(1) > i; \forall w\right\} \frac{b}{h} \geq 1 + \sum_{i=1}^{j-1} Pr\left\{R_0(1) > i\right\} \right\}$$

Start with $j = 3$.

Firstly we need to find $\sum_{i=3}^{4} Pr\left\{R_w(1) > i; \forall w\right\}$ by using the definition provided in Proposition 4.6.

$$Pr\left\{R_w(n) > i; \forall w\right\} = \begin{cases} 1 & \text{if } i < M+1 \\ \prod_{k=M+1}^{i} (1 - p_{n+k}) & \text{if } i \geq M+1 \end{cases}$$

Therefore $Pr\left\{R_w(1) > 3; \forall w\right\} = \prod_{k=2}^{3}(1-p) = (1-p)^2$ and $Pr\left\{R_w(1) > 4; \forall w\right\} = \prod_{k=2}^{4}(1-p) = (1-p)^3$. Sum of the two makes $\sum_{i=3}^{4} Pr\left\{R_w(1) > i; \forall w\right\}$ which is $(1-p)^2 + (1-p)^3$.

Secondly we need to find $\sum_{i=1}^{j-1} Pr\left\{R_0(1) > i\right\}$. For $j = 3$ we have to sum up the probabilities $Pr\left\{R_0(1) > 1\right\}$ and $Pr\left\{R_0(1) > 2\right\}$. According to Equation (4.12) we can find these probabilities.

$$Pr\left\{R_0(1) > 1\right\} = f^{(1)}_{(0,0)}(n, A_1)\ (1-p) + f^{(1)}_{(\infty,0)}(n, A_1)\ p$$
$$Pr\left\{R_0(1) > 2\right\} = f^{(2)}_{(0,0)}(n, A_2)\ (1-p) + f^{(2)}_{(\infty,0)}(n, A_2)\ p$$

where $A_i = \{(\infty, 0)\}\ \forall i$ and the corresponding probability transition matrix for the state space $\Omega_{M+1} = \{(\infty,0), (0,\infty), (0,0), (\infty,\infty)\}$ is

$$\mathcal{P} = \begin{bmatrix} 0 & p & 1-p & 0 \\ 1-p & 0 & 0 & p \\ 0 & p & 1-p & 0 \\ 1-p & 0 & 0 & p \end{bmatrix},$$

and the corresponding $\tilde{\mathcal{P}}$ is

$$\tilde{\mathcal{P}}^{(i)} = \begin{bmatrix} 0 & p & 1-p & 0 \\ 0 & 0 & 0 & p \\ 0 & p & 1-p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \quad \forall\ i.$$

Note that, since we have stationary availability, transition probabilities does not depend on n. We can write

$$\begin{aligned} f^{(1)}(n, A_1) &= \tilde{\mathcal{P}}^{(1)}\ \underline{1} \\ &= \begin{bmatrix} 1 \\ p \\ 1 \\ p \end{bmatrix}. \end{aligned}$$

Therefore $f^{(1)}_{(0,0)}(n, A_1) = 1$ and $f^{(1)}_{(\infty,0)}(n, A_1) = 1$ as the first and third entries of $f^{(1)}(n, A_1)$ respectively. This says that from state $(0, 0)$ or $(\infty, 0)$ it is not possible to move to state $(\infty, 0)$ in one-step which is obvious. Then, $Pr\left\{R_0(1) > 1\right\} = 1$.

$$\begin{aligned} f^{(2)}(n, A_2) &= \tilde{\mathcal{P}}^{(2)}\ f^{(1)}(n, A_2) \\ &= \begin{bmatrix} 1-p+p^2 \\ p^2 \\ 1-p+p^2 \\ p^2 \end{bmatrix}. \end{aligned}$$

Therefore $f^{(2)}_{(0,0)}(n, A_2) = 1-p+p^2$ and $f^{(1)}_{(\infty,0)}(n, A_1) = 1-p+p^2$ as the first and third entries of $f^{(2)}(n, A_2)$ respectively. Then, $Pr\left\{R_0(1) > 2\right\} = 1-p+p^2$. As a result $\sum_{i=1}^{j-1} Pr\left\{R_0(1) > i\right\} = 2-p+p^2$.

Therefore we need to check the inequality (4.13) for $j = 3$ which is

$$\begin{aligned}
\sum_{i=3}^{4} Pr\left\{R_w(1) > i; \forall w\right\} 5 &\geq 1 + \sum_{i=1}^{2} Pr\left\{R_0(1) > i\right\} \\
\left((1-p)^2 + (1-p)^3\right) 10 &\geq 3 - p + p^2 \\
3.75 &\geq 2.75.
\end{aligned}$$

We have found out that $j = 3$ satisfies the inequality (4.13). As it is the maximum possible value for $K^{'}$ we say that $K^{'} = 3$ and therefore $y_1(0) = D(1, 4)$ (4-period demand).

5. NUMERICAL ANALYSIS

In this chapter we present our numerical analysis regarding the dynamic programming model and the heuristic algorithm. Firstly we give the parameter setting and the methodology for the numerical study. Then we provide the analysis of the optimal policy and total expected cost under different parameter setting to provide managerial insights. Afterwards, value of information (VOI) is analyzed for the models with non-zero fixed ordering cost and no fixed ordering cost. Value of information is investigated for different sizes of the information vector. The behavior of the system under different parameter settings is analyzed and interpretation of the results is given. Finally, the performance of the heuristic is examined for the same parameter settings.

Dynamic programming model is coded in *C#*. Planning horizon is selected to be 12 periods. It is assumed that at the beginning of the planning horizon the initial on-hand inventory is zero. For numerical analysis we used stationary availability ($p_n = p \ \forall n$) with three levels of supply availability: low availability ($p = 0.1$), moderate availability ($p = 0.5$), and high availability ($p = 0.9$). Parameter values for the holding, backorder and fixed ordering cost is given in Table 5.1.

Table 5.1. Cost parameters

Cost parameter	**Unit cost**
h	1
b	(5,10)
A	(5b, 10b)

We generated demand values by discretizing the Gamma distribution. We used 9 Gamma distributions having different mean and coefficient of variation values which can be seen in Table 5.2.

Discretization of the Gamma distribution is done in such a way that probability of having a Gamma random variable in ±0.5 neighborhood of an integer is assigned to the probability of that integer. For example, probability of being smaller than 0.5

is assigned to the probability of having 0 demand, probability of being between 0.5 and 1.5 is assigned to the probability of having 1 unit demand. For each gamma distribution we have generated 100 different demand sequence. So we have worked with 900 different demand sequences.

Table 5.2. Gamma distributions used for demand generation

demand distributions	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
mean value	5	10	15	5	10	15	5	10	15
coefficient of variation	0.1	0.1	0.1	0.5	0.5	0.5	1	1	1

5.1. Optimal Policy for the Model with Non-zero Fixed Ordering Cost

In this section, optimal policy is analyzed under different parameter settings as this gives valuable managerial insights. As shown in Section 3.2 when there is fixed ordering cost the optimal policy is a state dependent $(s_n(w_n), S_n(w_n))$ policy. That is, if the inventory level at the beginning of period n (supply availability information vector is w_n) is lower than $s_n(w_n)$ it is optimal to order up to $S_n(w_n)$, otherwise no order is placed.

In Table 5.3 optimal re-order points $(s_1(w_1))$ and optimal order-to-levels $(S_1(w_1))$ are given for the model with 2-period ASI ($M = 1$) under a stationary demand of 10 units per period, that is $D_n = 10\ \forall n$.

When the results in Table 5.3 are analyzed following observations can be made.

- The order-up-to level when $w_n = 0$ is greater than or equal to the order-up-to level when $w_n = \infty$, that is $S_n(0) \geq S_n(\infty)$. When supply is unavailable in the next period, it may be better to order up to a higher level compared to the case when supply is available in the next period. For example, in Table 5.3 for the low availability case($p = 0.1$) it is seen that $S_1(0) > S_1(\infty)$. When fixed cost is high they may be equal not to give extra fixed cost. For instance, in Table 5.3 it is seen that for moderate availability case ($p = 0.5$), $S_1(0) = S_1(\infty)$ for $b = 5$, $A = 50$ and $b = 10$, $A = 100$ where fixed cost is relatively high.

Table 5.3. Optimal policy under different parameter settings

p=0.1	**b=5, A=25**	**b=5, A=50**	**b=10, A=50**	**b=10, A=100**
$S_1(0)$	90	100	110	110
$S_1(\infty)$	10	10	10	10
$s_1(0)$	80	76	93	89
$s_1(\infty)$	5	0	6	0
p=0.5	**b=5, A=25**	**b=5, A=50**	**b=10, A=50**	**b=10, A=100**
$S_1(0)$	40	40	50	60
$S_1(\infty)$	20	40	20	60
$s_1(0)$	25	20	29	26
$s_1(\infty)$	6	3	6	5
p=0.9	**b=5, A=25**	**b=5, A=50**	**b=10, A=50**	**b=10, A=100**
$S_1(0)$	20	30	30	40
$S_1(\infty)$	20	30	30	40
$s_1(0)$	15	12	17	14
$s_1(\infty)$	7	6	8	7

- The re-order point when $w_n = 0$ is greater than or equal to the re-order point when $w_n = \infty$, that is $s_n(0) \geq s_n(\infty)$. In Table 5.3 for all cases $s_1(0) > s_1(\infty)$. When supply is available in the next period re-order point is lower because placing no orders is less risky compared to the case when next period's supply is zero.

- For the same backorder cost and same level of the supply availability, order-up-to levels monotonically increase when fixed ordering cost increases. When there is a high fixed cost the orders are placed in large quantities to pay less number of fixed costs. For example in Table 5.3 for $p = 0.5$ and $b = 10$, both $S_1(0)$ and $S_1(\infty)$ increases when fixed cost increases from 50 to 100. For the low availability case ($p = 0.1$) optimal order-up-to levels stays same since they are already high because of the low supply availability.

- For the same backorder cost and same level of the supply availability, re-order points monotonically decrease when fixed ordering cost increases. The reason is that optimal policy is to avoid the number of ordering points so no ordering occurs until the inventory level is considerably low and the ordering occurs in large quantities as mentioned previously. It is seen in Table 5.3 that for all cases $s_1(w_1)$ is lower when fixed cost is higher at for same backorder cost and same supply availability level.

- For the same fixed ordering cost and same level of supply availability, $S_n(0)$ and $s_n(0)$ monotonically increase as backorder cost increases. When backorder cost increases, optimal policy avoids possible backorders by ordering in large amounts and by ordering before the inventory level gets too low when the next period's supply is zero. This is more evident when supply availability is low. For example, in Table 5.3 it is seen that, for $A = 50$ and $p = 0.1$, $S_1(0)$ increases from 100 to 110 when backorder cost increases from 5 to 10. Similarly $s_1(0)$ increases from 76 to 93. These observations also hold for $s_n(\infty)$ but not for $S_n(\infty)$. In Table 5.3 for $A = 50$ and $p = 0.5$, $S_1(\infty)$ decreases from 40 to 20 when backorder cost increases from 5 to 10. It is difficult to give interpretations for some of these results due to the complex nature of the problem.

- For the same fixed ordering cost and backorder cost, as the availability probability increases $S_n(0)$ and $s_n(0)$ monotonically decreases. This is because the environment becomes less risky so there is no need to place order in large quantities and to take action in the early steps (that is when the inventory is still high). In Table 5.3 it is seen that for all cases $S_1(0)$ and $s_1(0)$ decreases when availability increases. It is again difficult to make the same interpretations for the case with $w_n = \infty$. $S_1(\infty)$ and $s_1(\infty)$ do not behave in a monotonic way.

5.2. Total Expected Cost

In this section we analyze the total expected cost under different parameter values. As the optimal policy, total expected cost is important in terms of managerial aspects.

The models with no ASI ($M = 0$) and with 2-period ASI ($M = 1$) are considered and the same parameter values are used that are given at the beginning of this chapter. Note that models with no ASI ($M = 0$) and with 2-period ASI ($M = 1$) behaves in the same direction in terms of the total expected cost. Results with a backorder cost of 5 and 10 can be seen in Appendix A, in Tables A.1 and A.2 respectively.

Analyzing the total expected cost of the system for the selected parameters we can make the following observations.

- For the same coefficient of variation of demand, total expected cost monotonically increases with increasing mean value of demand. We have 3 demand distribution groups having the same mean value which are (1 - 4 - 7), (2 - 5 - 8), and (3 - 6 - 9). In all these groups costs increase when we move to the right since coefficient of variation increases in the same direction. Cost also monotonically increases with increasing coefficient of variation at the same mean value of demand. We have also 3 groups of demand distributions having the same coefficient of variation which are (1 - 2 - 3), (4 - 5 - 6), and (7 - 8 - 9). In all these groups costs increase when we move to the right since mean value of the demand increases in the same direction.

- For the same backorder cost and the same demand distribution, total expected cost monotonically increases with increasing fixed ordering cost. In Tables A.1 and A.2 this situation can be seen for all cases. This situation is more when the availability probability is high ($p = 0.9$) since there are more number of ordering points in this case. For the low availability case ($p = 0.1$) cost increase is not so significant since the orders are placed in larger amounts and so less frequently.

- For the same fixed ordering cost and same demand distribution, total expected cost monotonically increases with increasing backorder cost. This situation can be seen in Tables A.1 and A.2 for $A = 50$ case. The cost increase is more obvious for the low availability case ($p = 0.1$) since possibility of backorder is high in this case. For the high availability case ($p = 0.9$) cost increase is not so high since the possibility of backorder is low.

- Total expected cost monotonically decreases with increasing availability probability. When supply availability probability is high, the problem environment becomes less risky and as a result incurred costs decrease.

5.3. Value of Information

In this section we analyze the value of information for different parameter settings and for different sizes of the advance supply information vector. The model has advance supply information for $M + 1$ periods including the current period. That is for an M value we have $M + 1$-period ASI. To be able to calculate the value of information we need to obtain the results for the model with no ASI. Note that, as mentioned previously, knowing the current supply state, q_n, does not change the expected cost. Therefore the models with no ASI and 1-period ASI are equivalent in terms of our perspective and both models are denoted with $M = 0$.

5.3.1. Effect of the size of advance supply information vector

In this part we analyze the value of information with different sizes of advance information vector. For this purpose models with 2-period ASI ($M = 1$), 3-period ASI ($M = 2$), and 4-period ASI ($M = 3$) are compared with the model with no ASI ($M = 0$). In the analysis, holding and backorder costs are fixed to 1 and 5 respectively.

When we have non-zero fixed ordering cost ($A > 0$) the interactions between the fixed ordering cost, ASI, and the supply availability probabilities make it difficult to draw conclusions. For the model with no fixed ordering cost ($A = 0$), the conclusions are more clear.

Percent value of information of $(M+1)$-period ASI is denoted by $\%\ \text{VOI}_{0-M}$ and calculated by the equation

$$\%\ \text{VOI}_{\ 0-M} = \frac{\text{Cost with no ASI} - \text{ Cost with } (M+1) \text{ -period ASI}}{\text{Cost with no ASI}} * 100 \tag{5.1}$$

In addition to the value of information, marginal percent value of information is also presented. The aim is to see whether the marginal percent VOI decreases when the size of the advance supply information vector increases. We denote the marginal percent VOI of 3-period ASI ($M = 2$) as % VOI_{1-2} which gives the additional value resulting from 3-period ASI. Similarly, the marginal percent VOI of 4-period ASI ($M = 3$) as % VOI_{2-3} which gives the additional value resulting from 4-period ASI. To find marginal percent VOI, the following equations are used.

$$\%\ \mathrm{VOI}_{\ 1-2} = \frac{\text{Cost with 2-period ASI} - \text{ Cost with 3-period ASI}}{\text{Cost with 2-period ASI}} * 100$$

$$\%\ \mathrm{VOI}_{\ 2-3} = \frac{\text{Cost with 3-period ASI} - \text{ Cost with 4-period ASI}}{\text{Cost with 3-period ASI}} * 100 \tag{5.2}$$

Table 5.4. Marginal percent VOI with low availability

p=0.1	**h=1, b=5, A=0**								
Demand	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Cost for M=0	951.4	1894.4	2829.9	959.6	1965.7	2805.6	919.2	1822.8	3121.0
Cost for M=1	949.1	1889.9	2823.1	957.3	1961.0	2799.0	917.0	1818.4	3113.8
% $\mathbf{VOI}_{0-1}$	0.240	0.240	0.240	0.237	0.238	0.237	0.244	0.241	0.230
Cost for M=2	945.3	1882.4	2811.9	953.5	1953.3	2788.1	913.3	1811.2	3102.1
% $\mathbf{VOI}_{0-2}$	0.636	0.635	0.636	0.627	0.630	0.627	0.646	0.635	0.607
% $\mathbf{VOI}_{1-2}$	0.396	0.396	0.396	0.391	0.393	0.390	0.404	0.395	0.378
Cost for M=3	941.1	1874.0	2799.4	949.4	1944.7	2775.8	909.1	1803.3	3089.0
% $\mathbf{VOI}_{0-3}$	1.077	1.076	1.077	1.063	1.066	1.062	1.101	1.071	1.027
% $\mathbf{VOI}_{2-3}$	0.444	0.444	0.444	0.438	0.439	0.438	0.457	0.439	0.422

In Tables 5.4, 5.5, 5.6 the costs for the models with no ASI ($M = 0$), 2-period ASI ($M = 1$), 3-period ASI ($M = 2$), and 4-period ASI ($M = 3$) can be seen. Besides percent VOI for 2, 3, and 4-period ASI and the marginal percent VOI for 3 and 4-period ASI are given. Note that percent VOI and marginal percent VOI for 2-period ASI are equivalent.

Table 5.5. Marginal percent VOI with moderate availability

p = 0.5	**h=1, b=5, A=0**								
Demand	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Cost for M=0	158.5	315.7	472.3	160.2	327.8	467.4	153.6	302.7	516.9
Cost for M=1	132.7	264.6	395.9	134.6	274.9	392.5	128.7	252.8	435.1
% $\mathbf{VOI}_{0-1}$	16.25	16.20	16.19	16.00	16.12	16.02	16.18	16.50	15.83
Cost for M=2	111.7	222.7	333.4	113.6	231.9	331.5	108.5	212.2	369.0
% $\mathbf{VOI}_{0-2}$	29.53	29.47	29.43	29.08	29.26	29.06	29.37	29.92	28.62
% $\mathbf{VOI}_{1-2}$	15.86	15.83	15.79	15.57	15.67	15.53	15.74	16.07	15.20
Cost for M=3	102.0	203.5	304.6	104.0	212.1	303.5	99.2	193.7	339.0
% $\mathbf{VOI}_{0-3}$	35.63	35.56	35.50	35.07	35.30	35.06	35.41	36.03	34.42
% $\mathbf{VOI}_{2-3}$	8.65	8.64	8.61	8.44	8.54	8.46	8.54	8.72	8.13

Analyzing the results in Tables 5.4, 5.5, 5.6 we can say the following.

- For the same backorder cost, percent VOI is higher when the availability probability is higher. It is seen that, % VOI_{0-1}, % VOI_{0-2}, and % VOI_{0-3} has their maximum when $p = 0.9$. The reason is that, the model with no ASI gives decision as if the supply is fully available and orders are placed up to the demand of the related period. Although it is a very small probability to have unavailable supply, when it occurs model with no ASI pays backorder cost. Since there is no fixed cost, the backorder cost paid by the model with no ASI constitutes the major part of the total expected cost.

- For low availability case ($p = 0.1$), marginal percent VOI increases when size of the advance supply information vector increases. In Table 5.4 is seen that % VOI_{2-3} > % VOI_{1-2} > % VOI_{0-1}. This is because of the fact that when supply availability probability is so low it is important to have advance supply information for a bigger time interval compared to higher availability cases.

- For moderate availability case ($p = 0.5$), marginal percent VOI is highest for 2-period ASI ($M = 1$). In Table 5.5 is seen that % VOI_{0-1} > % VOI_{1-2} > %

Table 5.6. Marginal percent VOI with high availability

p = 0.9	**h=1, b=5, A=0**								
Demand	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Cost for M=0	33.13	65.92	98.60	33.27	68.35	97.30	32.00	63.73	106.92
Cost for M=1	10.75	21.47	32.17	11.03	22.36	31.90	10.43	20.44	35.76
% $\mathbf{VOI}_{0-1}$	67.56	67.42	67.37	66.85	67.28	67.22	67.42	67.92	66.56
Cost for M=2	9.17	18.33	27.48	9.45	19.12	27.32	8.90	17.37	30.73
% $\mathbf{VOI}_{0-2}$	72.33	72.19	72.13	71.60	72.02	71.92	72.17	72.74	71.26
% $\mathbf{VOI}_{1-2}$	14.69	14.62	14.59	14.31	14.49	14.36	14.59	15.03	14.05
Cost for M=3	9.07	18.13	27.18	9.35	18.92	27.02	8.81	17.18	30.42
% $\mathbf{VOI}_{0-3}$	72.63	72.49	72.44	71.90	72.33	72.23	72.48	73.05	71.55
% $\mathbf{VOI}_{2-3}$	1.10	1.10	1.09	1.07	1.08	1.08	1.09	1.12	1.03

VOI_{2-3}. This shows that, when supply availability gets higher the information regarding the further periods becomes less important compared to lower availability cases. It is important to note that % VOI_{1-2} is close to % VOI_{0-1} which indicates that the availability level is not still so high.

- For high availability case ($p = 0.9$),marginal % VOI is again highest for 2-period ASI ($M = 1$). In Table 5.6 is seen that % VOI_{0-1} > % VOI_{1-2} > % VOI_{2-3}. However in this case there is a big difference between % VOI_{1-2} and % VOI_{0-1} and this show that when availability is high supply state of the next period plays the most important role outperforming the supply state of further periods.

5.3.2. Effect of different parameter settings

In this part, we examine the results for the model with non-zero fixed cost in addition to the model with no fixed cost under 2-period ASI ($M = 1$). Results are given for 2 values of backorder cost, that is 5 and 10. Results with a backorder cost of 5 and 10 can be seen in Appendix A, in Tables A.1 and A.2 respectively. For the case with a backorder cost of 5, Figures 5.1, 5.2, and 5.3 are also provided for three levels of supply availability.

Analyzing these results we can make the following observations.

- When fixed is positive, for the same mean value, as the coefficient of variation of the underlying demand distribution increases, percent VOI also increases. As mentioned at the beginning of this chapter, 3 different mean values are used and these three groups are (1 - 4 - 7), (2 - 5 - 8), and (3 - 6 - 9) having a mean value of 5, 10, and 15 respectively. Considering the group (1 - 4 - 7), percent VOI increases from 1 to 7 since coefficient of variation of demand distribution 7 is the highest among the three. Same thing is true for the groups (2 - 5 - 8) and (3 - 6 - 9).

- When fixed cost is positive, for the same coefficient of variation, as the mean value of demand increases percent VOI increases. We have three groups for coefficient of variation (1 - 2 - 3), (4 - 5 - 6), and (7 - 8 - 9) having a coefficient of variation of 0.1, 0.5 and 1 respectively. Consider the group (1 - 2 - 3), it is seen that, percent VOI increases from 1 to 3 since highest mean value if of demand distribution 3 among them. Same thing is true for the other groups (4 - 5 - 6), and (7 - 8 - 9).

- For the same backorder cost and for the same demand distribution, percent VOI decreases as fixed ordering cost increases and percent VOI is highest for the model with no fixed cost for all availability levels. This is because of the fact that as fixed ordering cost increases supply state of the future periods loses its importance in terms of the decision order. Orders are placed in large quantities regardless of the supply states to pay less fixed ordering cost.

- Percent VOI result for different backorder costs is difficult to interpret. For the same fixed ordering cost and for the same demand distribution, as backorder cost increases percent VOI decreases for the cases where supply availability probability is not high. The reason is that, when backorder cost is high, orders are placed in large amounts to get rid of possible backorders when availability probability is low. However, when there is high availability probability this does not hold since possibility of paying backorder cost is insignificant.

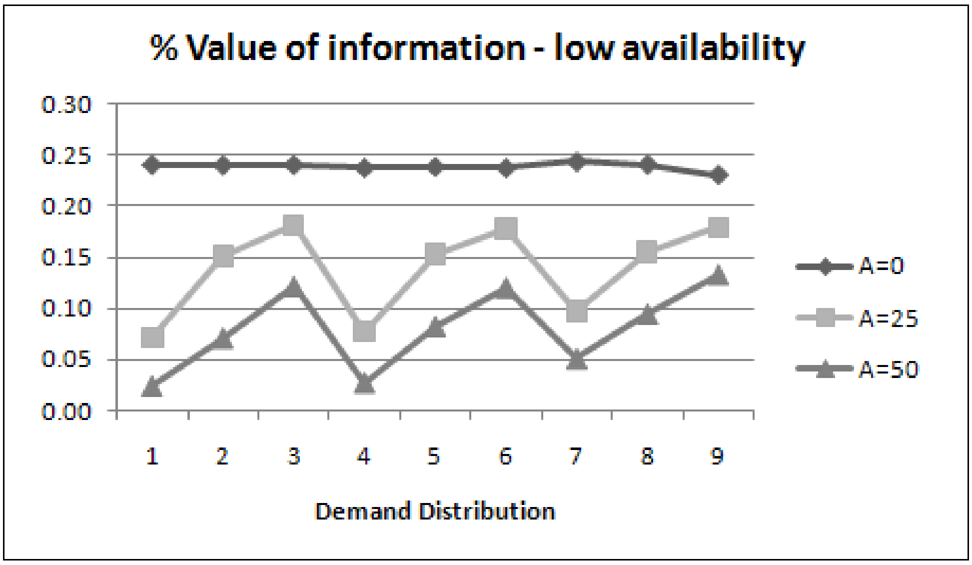

Figure 5.1. Percent VOI with low availability and $b = 5$

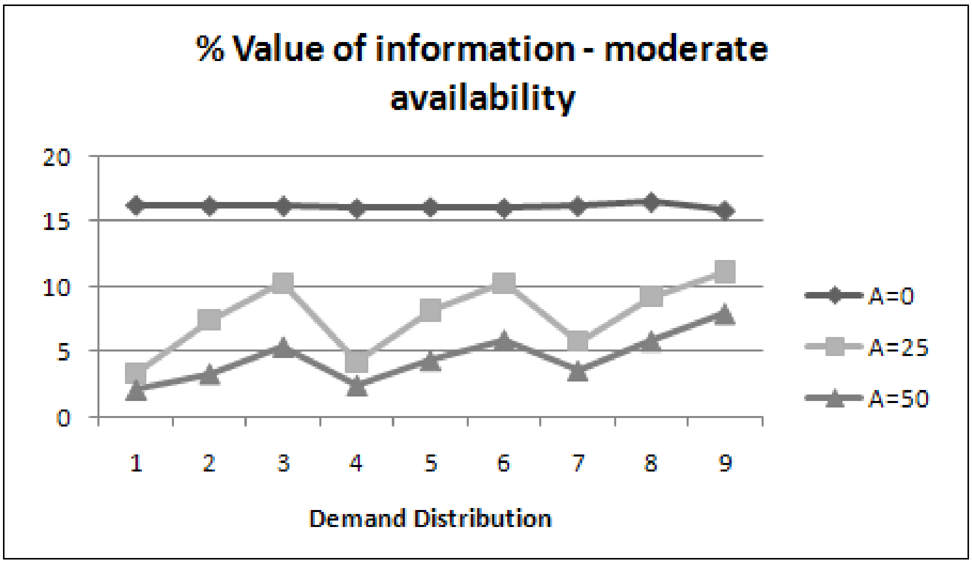

Figure 5.2. Percent VOI with moderate availability and $b = 5$

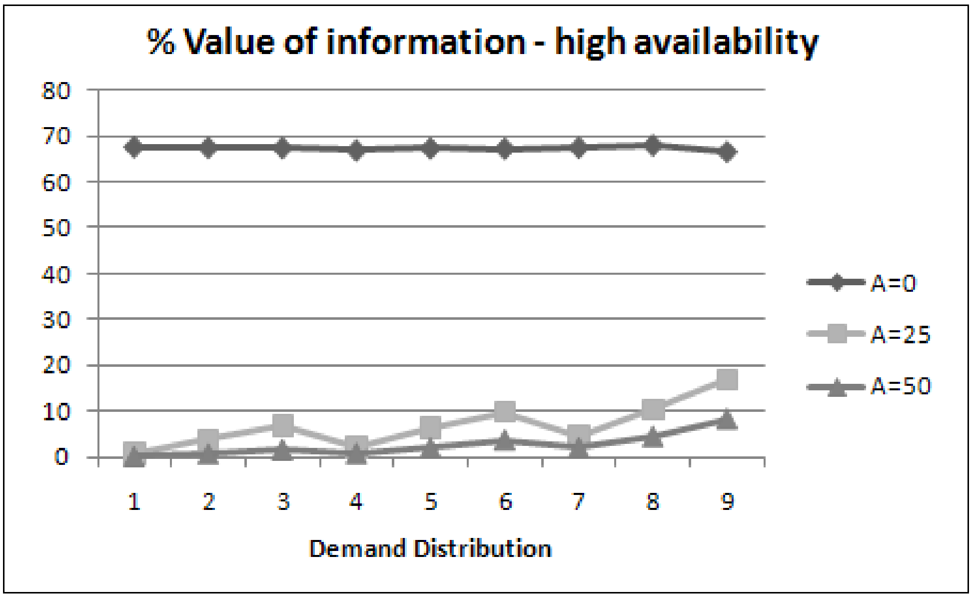

Figure 5.3. Percent VOI with high availability and $b = 5$

5.3.3. Effect of non-stationary supply availability

The original model has non-stationary supply availability and this non-stationarity makes the model more realistic. For example, in Christmas time supply availability is very low in some industries. Similarly, summer months can be problematic or they may be cyclic behavior in the supply states. Therefore, we also analyzed the results under non-stationary supply availability for two different scenarios. Percent value of information of 2-period ASI ($M = 1$) is examined under the parameter settings given at the beginning of this chapter.

The first scenario corresponds to the case where there is very low supply availability in summer time and Christmas. Availability probabilities are selected as $p_1 = 0.9, p_2 = 0.9, p_3 = 0.9, p_4 = 0.9, p_5 = 0.9, p_6 = 0.9, p_7 = 0.1, p_8 = 0.2, p_9 = 0.9, p_{10} = 0.9, p_{11} = 0.1, p_{12} = 0.05$. The results for this scenario can be seen in Appendix A Table A.3. The second scenario corresponds to the case where there is cyclic behavior in the availability of supply. Availability probabilities are selected as $p_1 = 0.4, p_2 = 0.5, p_3 = 0.7, p_4 = 0.9, p_5 = 0.9, p_6 = 0.9, p_7 = 0.6, p_8 = 0.4, p_9 = 0.6, p_{10} = 0.8, p_{11} = 0.4, p_{12} = 0.2$. The results for this scenario can be seen in Appendix A Table A.4.

When the results in Tables A.3 and A.4 analyzed, following observations can be listed.

- It is seen that for these scenarios with non-stationary supply availability advance supply information is important. Therefore in real life instances, advance supply information is quite valuable with considerable cost reductions.

- Percent value of information is highest when there is no fixed ordering cost. When fixed ordering cost increases, percent value of information decreases for the both scenarios.

- Everything being equal, as backorder cost increases, percent value of information increases.

- As it is previously observed, percent value of information increases with increasing mean value and coefficient of variation of demand.

5.4. Performance of the Heuristic Algorithm for Non-zero Fixed Cost Model

In this section we investigate the performance of the suggested heuristic algorithm. As the performance criteria percent deviation from the optimal values is used as in the following equation.

$$\%\ \text{deviation} = \frac{\text{Heuristic Cost} - \text{Optimal Cost}}{\text{Optimal Cost}} * 100 \tag{5.3}$$

The expected total cost of the system when heuristic is applied (that is given as Heuristic Cost in Equation 5.3) is found by Monte Carlo simulation. The supply state sequences are generated randomly for each scenario and this is done for a number of replications and the expected total cost is found by taking the average of the costs of replications. Replication number is selected as 1,000,000 for our case.

Heuristic algorithm is designed for any size of the information vector as seen in Section 3.3. However for the numerical analysis we have used 2-period ASI (that is, $M = 1$). Same parameter settings that are given in Section 5.3 are used. As mentioned previously, for each demand distribution 100 demand sequences are generated. Percent average, percent minimum, and percent maximum deviations from the optimal results are presented for these 100 cases as seen in Appendix B in Tables B.1, B.2, and B.3.

When we analyze the results in Appendix B in Tables B.1, B.2, and B.3 we observe the following.

- Heuristic algorithm performs worse for the cases where the underlying demand distribution has a higher mean. If we think of the groups of demand distributions having the same coefficient of variation it is seen that, in general, as mean value of demand increases percent average deviations increase. For instance demand distributions of 1, 4, and 7 have the same coefficient of variation and their mean values are 5, 10, and 15 respectively. The highest percent average deviation among them is of distribution 7. The reason is that, when demand distribution is high mean, wrong decisions of the heuristic algorithm end with high costs.

- The gap between percent minimum deviation and percent maximum deviation gets bigger when the mean value of demand is higher for the same coefficient of variation. When the demand distributions 1,4, and 7 are analyzed, it is seen in Table B.2 that for the case with $b = 5$ and $A = 25$, the gap between percent minimum deviation and percent maximum deviation is about 1percent for demand distribution 1. The gap increases to 8 percent when the demand is of distribution 4. The gap has its biggest value for the demand distribution 7 which is 11 percent.

- Heuristic method performs best for low availability case ($p = 0.1$) as seen in Table B.1. This is because of the fact that for such a low availability, ordering decision of both heuristic and optimal methods is to place a big amount of order that covers the demand of a several number of periods (sometimes until the end of the planning horizon). Therefore, there is a less number decisions epochs which decreases the possibility of wrong decisions.

- For low availability case ($p = 0.1$), heuristic algorithm has a percent average deviation of 2 percent at most. For moderate availability case($p = 0.5$) the biggest percent average deviation is 14 percent. For high availability case($p = 0.9$) this value is 8 percent at most.

- It is difficult to interpret the results for changing fixed ordering cost or backorder cost. The reason is the complicated interactions among supply availability probability, advance supply information, fixed ordering cost and backorder cost. This is also the reason for the difficulty that is faced during the design of the heuristic algorithm.

Finally it is important to mention the advantage of the heuristic algorithm in terms of computational time. As it is mentioned previously order of complexity for heuristic algorithm is $O(N^2)$. On the other hand dynamic programming has an order of complexity of $O(2^M\ N^3\ D_{max}^2)$, where D_{max} is the maximum demand value in the planning horizon. Therefore the computational time of the dynamic programming depends on the size of the information vector and the maximum value of the demand in addition to N.

Computational time of the optimal dynamic programming algorithm for the model with 2-period ASI ($M = 1$), 3-period ASI ($M = 2$), and 4-period ASI ($M = 3$) can be seen in Table 5.7. Supply availability probability p is 0.5 for all the results and the other parameter values are the same with the values that are given at the beginning of this chapter. As the size of the supply information increases computational time increases. It is important to note that the increase in the computational time grows considerably when we move from the model with $M = 1$ to the model with $M = 3$. The longest computational time for the model with $M = 1$ and the model with $M = 3$ is around 19 seconds and 94 seconds respectively. Moreover, it is seen that for the same coefficient variation of the demand, computational time increases when mean value of demand increases. For the same mean value, computational time increases with increasing coefficient of variation. However, heuristic algorithm solves the same problems in no time (in milliseconds).

To sum up, dynamic programming algorithm has a higher order of complexity than the heuristic algorithm. Computational time for dynamic programming algorithm grows exponentially with M and polynomially with N and D_{max}. Therefore for real life problems with longer planning and information horizon, and higher demand values computational time will be much higher than our experiments. As a result, proposed heuristic algorithm can be an alternative for finding a good ordering strategy.

Table 5.7. Runtime of the dynamic programming algorithm

M=1, p=0.5	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5, A=25	**Runtime (sec.)**	4.0370	7.7820	11.1930	5.6670	9.9340	13.8010	7.2770	10.7340	19.2530
b=5, A=50	**Runtime (sec.)**	4.0580	7.7600	11.2720	5.6440	9.8670	13.7310	7.2210	10.8110	19.1190
b=10, A=50	**Runtime (sec.)**	4.0630	7.7440	11.2190	5.6860	9.7970	13.7670	7.2450	10.7960	19.1040
b=10, A=100	**Runtime (sec.)**	4.0380	7.7530	11.2850	5.7700	9.7980	13.7510	7.2470	10.7370	18.9340
M=2, p=0.5	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5, A=25	**Runtime (sec.)**	8.9130	16.9760	24.9360	12.4130	21.5180	30.0550	15.9570	23.7920	42.4870
b=5, A=50	**Runtime (sec.)**	8.8960	16.9670	24.8580	12.3550	21.5530	30.0290	15.9270	23.6100	42.2830
b=10, A=50	**Runtime (sec.)**	8.8930	16.9980	24.8440	12.4250	21.5590	30.2550	15.9710	23.7320	42.5870
b=10, A=100	**Runtime (sec.)**	8.9610	17.0730	24.8610	12.4970	21.5250	29.9680	15.8570	23.6420	42.2160
M=3, p=0.5	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5, A=25	**Runtime (sec.)**	19.9160	38.1250	56.8880	28.1210	47.9160	67.3890	37.1110	54.9540	94.0280
b=5, A=50	**Runtime (sec.)**	19.7900	37.3640	55.1550	27.8040	47.2230	66.7930	35.9330	53.6560	93.3880
b=10, A=50	**Runtime (sec.)**	19.8090	37.9650	56.5650	28.3150	48.1430	67.7040	36.9450	54.9450	93.8970
b=10, A=100	**Runtime (sec.)**	19.7630	37.2000	55.1960	28.0100	47.3760	66.9330	35.9140	53.4340	93.4260

6. CONCLUSIONS

In this thesis, we analyzed a single-item, periodic review, deterministic demand inventory system under non-stationary supply availability with advance supply information. Supply availability has a binomial structure in that supply is either fully available or completely unavailable. This thesis contributes to the supply uncertainty literature in that, advance supply information is incorporated into an inventory model and near explicit solutions are obtained under advance supply information. We studied two models, one with non-zero and the other with no fixed ordering cost. Main conclusion is that, although the type of optimal policy is of familiar policies (order-up-to or (s, S)), the optimal ordering decision depend on the advance supply information.

Firstly we studied the model with non-zero fixed ordering cost by presenting a dynamic programming formulation. Based on the proofs that establish the A-convexity of the cost functions, the optimal policy is shown to be a state dependent $(s_n(w_n), S_n(w_n))$ policy. That is, if the inventory at the beginning of period n is below $s_n(w_n)$ order is placed up to $S_n(w_n)$), otherwise no ordering occurs. Analytical solutions for the re-order point and the order-up-to level are difficult to obtain and the dynamic programming model becomes hard to solve for a large state space. Therefore a Silver-Meal based heuristic algorithm is suggested for finding a good ordering strategy.

For the model with no fixed ordering cost optimal ordering policy is shown to be of order-up-to type, based on the convexity of the relevant cost functions. Important characteristics of the optimal order-up-to levels are presented. Most important characteristics can be given as: (1) optimal order-up-to level of a period is greater than or equal to the demand of that period, (2) given the information that supply is fully available in the next period order-up-to level is the demand of the current period, (3) given that order-up-to level of period $n + 1$ covers K-period demand, order-up-to level of period n covers at most $K + 1$-period demand. With the help of the characterization of the order-up-to levels a one-pass algorithm is constructed for finding the optimal order-up-to levels for any size of the advance supply information vector.

In addition to the analytical analysis, we also provided numerical analysis that yields important managerial insight. Analysis regarding the advance supply information is important being the core of this study. Advance supply information is useful when (1) the mean value and/or the coefficient of variation of demand is high, (2) fixed cost is not so high compared to other costs, (3) availability probability is not low (because when availability is low the model with ASI orders large amounts similar to the model with no ASI), and (4) there is non-stationary supply availability with periods having very low availability like Christmas time . We observed that the marginal value of information of keeping track of supply information for a few periods is higher as opposed to no information. However this marginal value decreases as the information span increases. When supply availability probability is high, this decrease in the marginal value of information occurs in earlier steps compared to the case where supply availability probability is low.

Proposed heuristic algorithm for the non-zero fixed ordering cost case can be an alternative for finding good ordering strategies when the supply availability is low. Heuristic algorithm runs in milliseconds and maximum percent average deviation from the optimal solution is 14 percent. In 70 percent of the problems the deviation from the optimal solution is less than 10 percent. Considering the interactions among the supply availability, advance supply information and the fixed ordering cost it is a successful algorithm.

In this study we assumed that demand is deterministic. A possible further study can be the same model with a stochastic demand structure. On top of that advance demand information (ADI) can also be incorporated into the model. Another extension can be made on the structure of the supply availability. In this study we assumed that supply is either fully available or completely unavailable. Partial availability can be studied such that only a part of the ordered amount is received. Our advance supply information in this study is perfect information and therefore another extension can be the model with imperfect advance supply information which is can be more realistic in some inventory systems. Therefore, this study can be extended in may directions as a future research.

APPENDIX A: VALUE OF INFORMATION

Table A.1. Percent VOI for 2-period ASI and b=5

p = 0.1	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
A=0	**Cost for M=0**	951.4	1894.4	2829.9	959.6	1965.7	2805.6	919.2	1822.8	3121.0
	Cost for M=1	949.1	1889.9	2823.1	957.3	1961.0	2799.0	917.0	1818.4	3113.8
	% VOI$_{0-1}$	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.23
A=25	**Cost for M=0**	974.9	1919.3	2855.2	982.8	1990.3	2830.8	941.4	1846.3	3145.5
	Cost for M=1	974.2	1916.4	2850.0	982.0	1987.3	2825.7	940.5	1843.5	3139.8
	% VOI$_{0-1}$	0.07	0.15	0.18	0.08	0.15	0.18	0.10	0.16	0.18
A=50	**M=0**	994.4	1941.4	2878.7	1002.5	2012.3	2853.8	960.8	1867.5	3167.8
	Cost for M=1	994.2	1940.0	2875.2	1002.2	2010.6	2850.4	960.3	1865.8	3163.6
	% VOI$_{0-1}$	0.02	0.07	0.12	0.03	0.08	0.12	0.05	0.09	0.13
p = 0.5										
A=0	**Cost for M=0**	158.5	315.7	472.3	160.2	327.8	467.4	153.6	302.7	516.9
	Cost for M=1	132.7	264.6	395.9	134.6	274.9	392.5	128.7	252.8	435.1
	% VOI$_{0-1}$	16.25	16.20	16.19	16.00	16.12	16.02	16.18	16.50	15.83
A=25	**Cost for M=0**	244.7	416.3	584.0	244.2	427.5	575.6	230.2	394.2	617.5
	Cost for M=1	236.5	385.3	524.0	234.0	392.5	516.5	216.8	357.8	548.6
	% VOI$_{0-1}$	3.34	7.42	10.27	4.19	8.17	10.26	5.78	9.23	11.16
A=50	**Cost for M=0**	300.9	488.1	660.9	299.3	497.5	651.9	282.3	459.1	691.2
	Cost for M=1	294.6	471.9	625.1	292.0	475.7	613.2	272.2	432.3	636.1
	% VOI$_{0-1}$	2.08	3.33	5.41	2.44	4.37	5.94	3.58	5.85	7.96
p = 0.9										
A=0	**Cost for M=0**	33.1	65.9	98.6	33.3	68.4	97.3	32.0	63.7	106.9
	Cost for M=1	10.7	21.5	32.2	11.0	22.4	31.9	10.4	20.4	35.8
	% VOI$_{0-1}$	67.56	67.42	67.37	66.85	67.28	67.22	67.42	67.92	66.56
A=25	**Cost for M=0**	167.2	232.9	281.5	160.7	227.0	272.6	144.0	202.5	264.8
	Cost for M=1	166.2	223.9	262.4	157.4	212.8	245.9	137.4	181.5	219.9
	% VOI$_{0-1}$	0.63	3.84	6.77	2.10	6.25	9.77	4.61	10.38	16.93
A=50	**Cost for M=0**	243.9	333.5	402.3	234.6	325.6	385.6	208.6	290.6	368.4
	Cost for M=1	243.8	331.4	396.0	232.8	319.0	371.4	204.4	277.7	337.8
	% VOI$_{0-1}$	0.05	0.64	1.55	0.75	2.01	3.69	2.00	4.44	8.31

Table A.2. Percent VOI for 2-period ASI and b=10

p = 0.1	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
A=0	**Cost for M=0**	1816.87	3618.15	5404.00	1833.52	3756.18	5361.60	1752.88	3481.38	5971.86
	Cost for M=1	1814.34	3613.13	5396.48	1831.00	3750.99	5354.23	1750.43	3476.51	5963.87
	% VOI$_{0-1}$	0.139	0.139	0.139	0.138	0.138	0.137	0.140	0.140	0.134
A=50	**Cost for M=0**	1857.57	3661.46	5448.40	1874.10	3799.23	5405.76	1792.15	3522.68	6014.42
	Cost for M=1	1857.53	3659.68	5444.31	1873.95	3797.25	5401.77	1791.78	3520.72	6009.67
	% VOI$_{0-1}$	0.002	0.049	0.075	0.008	0.052	0.074	0.021	0.056	0.079
A=100	**Cost for M=0**	1893.48	3699.57	5488.71	1910.38	3837.72	5445.93	1828.51	3560.65	6053.56
	Cost for M=1	1893.45	3699.51	5487.25	1910.30	3837.32	5444.41	1828.25	3559.96	6051.13
	% VOI$_{0-1}$	0.001	0.002	0.027	0.004	0.010	0.028	0.014	0.019	0.040
p = 0.5										
A=0	**Cost for M=0**	240.89	480.34	718.97	244.80	499.72	714.03	233.90	458.45	793.50
	Cost for M=1	206.98	412.94	618.23	211.02	430.21	615.45	201.20	392.79	686.15
	% VOI$_{0-1}$	14.077	14.032	14.012	13.798	13.911	13.807	13.977	14.323	13.528
A=50	**Cost for M=0**	373.88	642.04	895.71	375.75	659.18	887.17	356.16	605.07	956.19
	Cost for M=1	363.35	608.43	830.34	364.19	622.19	822.25	341.67	566.31	877.63
	% VOI$_{0-1}$	2.816	5.235	7.298	3.078	5.611	7.318	4.070	6.405	8.216
A=100	**M=0**	464.71	746.12	1019.98	465.18	765.30	1006.25	439.52	705.93	1071.21
	Cost for M=1	456.70	725.34	980.48	456.98	741.11	964.93	428.65	678.56	1014.61
	% VOI$_{0-1}$	1.722	2.784	3.872	1.763	3.161	4.106	2.474	3.877	5.284
p = 0.9										
A=0	**Cost for M=0**	61.29	121.96	182.47	61.62	126.56	180.17	59.14	117.86	198.18
	Cost for M=1	16.52	33.07	49.59	17.11	34.50	49.25	16.07	31.27	55.68
	% VOI$_{0-1}$	73.044	72.885	72.825	72.234	72.741	72.666	72.835	73.470	71.905
A=50	**Cost for M=0**	252.94	355.81	439.03	245.85	352.00	427.63	222.98	321.34	424.50
	Cost for M=1	248.57	339.38	408.16	239.45	329.06	385.26	212.47	288.27	357.04
	% VOI$_{0-1}$	1.726	4.617	7.032	2.605	6.517	9.907	4.712	10.291	15.892
A=100	**Cost for M=0**	357.25	504.70	608.98	351.56	498.94	592.38	325.99	455.71	580.55
	Cost for M=1	356.62	496.27	594.95	348.67	485.29	566.55	319.04	430.82	532.25
	% VOI$_{0-1}$	0.178	1.670	2.305	0.822	2.736	4.361	2.132	5.461	8.321

Table A.3. Percent VOI for non-stationary supply availability - scenario 1

Scenario 1	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5, A=0	**Cost for M=0**	67.99	135.21	202.92	68.59	141.33	201.36	65.04	131.27	214.76
	Cost for M=1	41.55	82.67	124.21	42.63	87.11	124.81	39.60	79.01	131.67
	% VOI$_{0-1}$	38.89	38.85	38.79	37.85	38.36	38.02	39.11	39.81	38.69
b=5, A=25	**Cost for M=0**	172.06	259.04	339.87	170.88	263.10	336.19	159.12	242.33	338.23
	Cost for M=1	170.44	240.44	295.21	164.46	236.71	287.54	148.57	210.58	277.39
	% VOI$_{0-1}$	0.938	7.179	13.138	3.756	10.029	14.471	6.630	13.102	17.989
b=5, A=50	**Cost for M=0**	246.44	343.04	436.62	243.07	345.19	427.17	222.50	319.76	425.68
	Cost for M=1	245.83	339.99	415.02	238.82	331.24	400.01	215.25	297.08	381.00
	% VOI$_{0-1}$	0.246	0.889	4.947	1.746	4.039	6.358	3.260	7.091	10.497
b=10, A=0	**Cost for M=0**	90.98	180.98	271.26	91.99	188.94	269.38	87.18	174.84	290.15
	Cost for M=1	50.27	100.12	150.26	51.67	105.26	150.61	48.31	94.87	161.42
	% VOI$_{0-1}$	44.747	44.679	44.607	43.828	44.287	44.090	44.594	45.741	44.367
b=10, A=50	**Cost for M=0**	258.47	367.22	475.61	257.33	372.41	468.00	239.94	350.87	482.49
	Cost for M=1	252.79	352.58	433.05	247.80	347.41	421.61	226.12	312.88	410.54
	% VOI$_{0-1}$	2.195	3.986	8.949	3.705	6.715	9.912	5.756	10.829	14.912
b=10, A=100	**Cost for M=0**	376.18	515.90	624.70	369.32	519.67	617.60	342.22	484.06	630.46
	Cost for M=1	364.10	504.63	607.19	360.14	499.49	588.93	332.06	453.13	575.62
	% VOI$_{0-1}$	3.211	2.184	2.804	2.486	3.884	4.641	2.967	6.389	8.699

Table A.4. Percent VOI for non-stationary supply availability - scenario 2

Scenario 2	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5, A=0	**Cost for M=0**	108.07	215.41	323.16	110.80	225.41	323.18	105.67	203.93	359.26
	Cost for M=1	79.87	159.43	239.31	82.79	167.36	241.03	78.45	149.33	271.12
	% VOI$_{0-1}$	26.095	25.988	25.946	25.276	25.754	25.420	25.763	26.774	24.535
b=5, A=25	**Cost for M=0**	204.36	327.90	445.63	204.87	335.15	441.68	191.91	304.58	469.49
	Cost for M=1	197.74	303.94	396.73	196.30	306.62	390.96	179.66	272.86	406.95
	% VOI$_{0-1}$	3.235	7.307	10.972	4.188	8.512	11.483	6.381	10.412	13.322
b=5, A=50	**Cost for M=0**	271.62	407.93	534.23	269.36	413.86	528.00	250.45	377.03	551.81
	Cost for M=1	269.29	394.96	505.39	264.02	396.39	495.00	241.28	355.03	505.10
	% VOI$_{0-1}$	0.859	3.181	5.398	1.983	4.221	6.250	3.661	5.835	8.465
b=10, A=0	**Cost for M=0**	167.01	333.45	500.11	172.17	348.94	501.13	163.51	314.04	562.29
	Cost for M=1	127.87	255.62	383.75	133.18	268.28	387.12	126.02	238.33	439.67
	% VOI$_{0-1}$	23.437	23.342	23.267	22.645	23.116	22.751	22.930	24.109	21.808
b=10, A=50	**Cost for M=0**	322.51	515.56	698.23	324.55	528.92	694.45	304.22	481.76	747.73
	Cost for M=1	314.15	486.47	641.95	313.05	492.90	634.16	288.11	441.56	668.46
	% VOI$_{0-1}$	2.594	5.641	8.060	3.543	6.811	8.683	5.295	8.344	10.602
b=10, A=100	**Cost for M=0**	418.60	644.16	843.83	421.55	656.23	836.63	396.60	599.57	883.32
	Cost for M=1	412.79	627.70	805.73	413.90	632.08	794.03	385.51	570.13	822.14
	% VOI$_{0-1}$	1.386	2.556	4.515	1.814	3.680	5.093	2.796	4.910	6.926

APPENDIX B: HEURISTIC RESULTS

Table B.1. Heuristic performance for low availability

p = 0.1	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5	**% av**	0.71	0.83	0.83	0.67	0.75	0.86	2.11	1.69	1.39
A=25	**% min**	0.04	0.18	0.29	0.09	0.20	0.15	0.08	0.03	0.04
	% max	1.20	1.42	1.48	1.42	1.92	1.99	12.67	10.02	7.94
b=5	**% av**	0.72	0.71	0.77	0.63	0.61	0.69	2.10	1.60	1.29
A=50	**% min**	0.37	0.11	0.23	0.12	0.10	0.08	0.03	0.02	0.02
	% max	1.25	1.37	1.34	1.30	1.76	1.52	12.57	10.44	7.75
b=10	**% av**	0.09	0.10	0.14	0.13	0.17	0.17	2.14	1.39	1.07
A=50	**% min**	0.00	0.00	0.01	0.00	0.01	0.01	0.00	0.01	0.01
	% max	0.26	0.23	0.28	0.64	0.57	0.46	15.68	11.74	8.33
b=10	**% av**	0.24	0.15	0.17	0.23	0.18	0.19	2.26	1.42	1.07
A=100	**% min**	0.01	0.00	0.01	0.00	0.00	0.00	0.00	0.01	0.01
	% max	0.57	0.52	0.55	0.60	0.58	0.61	15.56	12.26	8.39

Table B.2. Heuristic performance for moderate availability

p = 0.5	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5	**% av**	0.47	2.12	1.25	2.41	5.24	5.95	4.09	5.15	5.47
A=25	**% min**	0.01	1.50	0.44	0.10	1.64	0.87	0.06	0.93	1.01
	% max	1.58	3.66	4.72	8.67	12.21	13.04	11.41	12.33	13.75
b=5	**% av**	1.85	0.25	3.14	1.77	2.55	3.75	3.29	4.12	4.23
A=50	**% min**	0.37	0.01	1.87	0.24	0.44	0.83	0.01	0.36	0.29
	% max	4.05	1.42	4.10	5.30	7.15	7.96	13.38	10.94	12.17
b=10	**% av**	2.68	6.87	6.15	6.12	10.45	12.22	11.00	13.32	14.47
A=50	**% min**	1.10	5.32	5.22	0.65	4.32	5.57	0.37	1.83	0.62
	% max	5.59	8.41	7.06	15.24	21.70	23.74	33.88	27.86	30.25
b=10	**% av**	2.72	2.47	4.03	4.53	6.81	8.41	9.85	11.63	11.72
A=100	**% min**	1.50	0.99	1.92	0.95	0.76	3.19	0.01	0.16	0.38
	% max	5.67	4.87	6.50	12.29	20.79	16.34	25.20	28.71	25.73

Table B.3. Heuristic performance for high availability

p = 0.9	**Demand**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b=5	**% av**	0.85	1.53	1.09	2.33	3.19	3.63	3.61	3.76	5.13
A=25	**% min**	0.01	0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.04
	% max	4.85	5.04	3.86	12.90	11.97	10.30	14.88	18.15	21.31
b=5	**% av**	5.23	1.52	0.79	2.86	2.21	2.75	3.33	2.78	4.29
A=50	**% min**	0.08	0.00	0.05	0.00	0.00	0.01	0.00	0.00	0.00
	% max	11.74	6.41	3.18	14.29	11.14	11.07	17.61	12.17	17.51
b=10	**% av**	3.22	2.51	0.51	3.16	3.93	5.72	4.47	6.04	7.96
A=50	**% min**	0.04	0.01	0.09	0.00	0.00	0.05	0.00	0.00	0.28
	% max	6.35	6.80	3.12	12.85	18.74	17.20	26.88	21.23	27.98
b=10	**% av**	0.67	3.10	0.22	2.88	2.91	2.58	3.62	4.39	5.36
A=100	**% min**	0.01	0.04	0.01	0.00	0.00	0.00	0.00	0.00	0.01
	% max	4.44	10.15	1.70	20.12	15.68	12.51	23.71	18.54	32.41

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