NONRELATIVISTIC LIMIT OF BOSE-EINSTEIN CONDENSATION IN A STATIC SPACETIME

by

İslam Çağrı İşeri

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ABSTRACT

NONRELATIVISTIC LIMIT OF BOSE-EINSTEIN CONDENSATION IN A STATIC SPACETIME

Nonrelativistic limit of thermodynamics of noninteracting Bose field in curved background is investigated via a post-Newtonian approach. In large but finite c limit matter becomes nonrelativistic whereas the geometry stays unchanged. Mellin transform and heat kernel techniques lead to asymptotics of thermodynamic variables that are used to analyze the gravitational effects on Bose Einstein Condensation in static background. For finite volume, particle density-temperature relation is obtained which include gravitational corrections to the classical expression. Boundary effects due to gravitation are also examined and the results are found to be reduced to usual Minkowski space ones in the absence of gravitation in the thermodynamic limit.

ÖZET

DURAĞAN BİR UZAYZAMANDA BOSE-EINSTEIN YOĞUŞMASININ GÖRESİZ LİMİTİ

Eğik uzayda etkileşimsiz bir Bose alanının termodinamik özelliklerinin göresiz limiti post-Newton bir yaklaşımla incelenmiştir. Yüksek ancak sonlu c limiti alınmak suretiyle maddenin göresiz hale gelmesi temin edilirken aynı zamanda uzayzamanın geometrisi de muhafaza edilmiştir. Kütleçekimin, durağan uzayda Bose-Einstein yoğuşmasına etkilerini çözümlemek için kullandığımız termodinamik değişkenlerin asimtot davranışları Mellin dönüşümü ve ısı çekirdeği yöntemleri kullanılarak elde edilmiştir. Sonlu bir hacim için hesapladığımız parçacık yoğunluğu-sıcaklık ilişkisi kütleçekim etkilerinin gözardı edildiği durumlar için hâlihazırda bilinen matematiksel ifadeye ek olarak yeni terimler de içermektedir. Ayrıca kütleçekimin Bose-Einstein yoğuşması üzerindeki yüzey etkileri de termodinamik limit alınarak incelenmiş ve kütleçekimin olmadığı durumda varılan sonuçların Minkowski uzayındaki bilinen neticelere tekabül ettiği görülmüştür.

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LIST OF SYMBOLS

c	Speed of light in vacuum
$\mathcal{D}$	Measure for functional integration
d	Space dimensions
det	Determinant
e	Exponential function
F	Time Lapse function
$\mathcal{F}$	Free Energy
G	Newton's Gravitational Constant
$\hbar$	Reduced Planck constant
H	Hamiltonian operator
$\mathcal{H}$	Hamiltonian density
j_μ	Conserved current density
k_B	Boltzmann constant
$\mathcal{L}$	Lagrangian density
M	Mass of the gravitational source
m	Mass of Bose particles
$\mathcal{MT}$	Mellin Transform of a function
N	Number of particles
n	Particle density
$\mathcal{S}$	Action
P	Pressure
Q	Conserved Charge
R	Scalar Curvature
Re	Real part
R_+	Holomorphic part
res	Residue
r_s	Schwarzschild radius
T_c	Critical Temperature

Tr	Trace
$\mathcal{U}$	Propagator
Z	Partition function
β	Inverse Temperature
Γ	Gamma function
Δ	Laplacian operator
ζ	Riemann-Zeta function
μ	Chemical Potential
ξ	Curvature Coupling Constant
π	Conjugate Momentum Field
ϕ	Components of Complex Scalar Field
Φ	Complex Scalar Field
∇	Nabla: Differential operator
$\Box$	d'Alembertian operator

LIST OF ACRONYMS/ABBREVIATIONS

BE	Bose-Einstein
BEC	Bose-Einstein Condensation
KG	Klein-Gordon (equation)
NR	Nonrelativistic
QFT	Quantum Field Theory
i.e.	id est (Latin): it is; that is to say

1. INTRODUCTION

Bose–Einstein condensation (BEC) is one of the few quantum phenomena that can be macroscopically observed. When dilute gas of bosons are cooled down to very low temperatures close to absolute zero, since there exists no exclusion principle for bosons and the wave functions are symmetric under exchange of particles, a finite fraction of particles may occupy the same state, i.e. the ground state as the chemical potential reaches the ground state energy (which is zero for free particles) and a macroscopic number of particles can be observed in the ground state. The temperature where the chemical potential becomes zero and the condensation occurs is called the critical temperature.

The first prediction of this phenomenon has come by Albert Einstein upon receiving a letter from Indian physicist Satyendra Nath Bose in 1924. Bose was suggesting a new way to state the formula for black body radiation, yet his letters were rejected by some European journals. However Einstein decided to translate the letter and sent in for publishing. [1] The main idea was to accept the electromagnetic radiation of the black body as particles rather than waves. This approach was in fact consistent with the idea of light quanta formerly proposed by Planck and then used by Einstein in his famous work on photo-electric effect. [2] After the letter, Einstein has worked on the distribution of quantum particles with integer spins and obtained the famous Bose-Einstein distribution function, a quantum generalization of ideal gas distributions such as Maxwell and Boltzmann distributions. Soon he discovered the existence of a condensation below some particular temperature.

For BEC to occur, the density of the gas should be extremely low, for instance about $10^{13} - 10^{15} cm^{-3}$ which is at least ten-thousandth the density of air, and the temperature should be ultra-low, around $10^{-5} K$or less. These conditions were of course very hard to meet at the times when BEC is predicted, however in 1930s, superfluidity of ^4He which is a boson, is discovered. Since it is not straightforward to combine the superfluidity to Bose-Einstein condensation without BCS theory, real macroscopic observation of BEC has come only in 1995 with dilute alkali atoms trapped by magnetic field and laser. [3]

On the other hand, Bose-Einstein condensation in curved geometry is an interesting problem. In the calculation of particle density and critical temperature in Minkowski space, the density of states has a crucial role. However in curved space we cannot properly derive an expression for the density of states over which we take integrals so as to obtain various thermodynamic quantities. There are many different approaches to this problem by applying different techniques.

The purpose of this thesis is to investigate the thermodynamics of a noninteracting Bose gas in curved geometry, particularly in a static spacetime, confined in a large but finite volume B with boundaries ∂B, away from an heavy gravitating object and to analyze the effects of gravitation on Bose-Einstein condensation in the nonrelativistic limit where c, the speed of light is taken large but finite. The field is not assumed to be a weak field, however, the volume is taken far enough from the gravitation source such that there is no pair production, i.e. thermodynamics of particles and antiparticles can be treated separately and we consider the Bose particles only.

In this thesis our main focus is the nonrelativistic limit of Bose field in curved spacetime. For the opposite limit, there are numerous studies in the literature. They concern Bose field in curved spacetimes in the ultrarelativistic regime and their thermodynamics is well known [4–12]. As far as we know, nearly all the work done on the thermodynamics of Bose field in the literature has covered ultrarelativistic limit, but not nonrelativistic one. The main motivation of this thesis is to cover this unexplored area of research.

The term nonrelativistic limit refers to large c limit where c is considered to be finite but also large enough such that matter is considered as heavy and slow. Nonetheless, by the term nonrelativistic we do not imply that the field is weak. In this manner, we make the matter nonrelativistic while the background geometry stays unchanged. One can interpret this limit as a post-Newtonian approach in which the suggested solutions to Einstein field equations are approximated by expansions over Newtonian potentials and the background is exact including the terms with powers of c. However, in contrast to the works making a weak field approximation for post-Newtonian approach, recall that we do not take the field weak.

Nonrelativistic limit can be taken in diverse ways. Since we are dealing with Bose field in curved spacetime, we may consider Klein-Gordon equation and its nonrelativistic limit which is a very complicated and hard problem [13, 14]. For that reason, we will work with the free energy of Bose gas in the grand canonical ensemble and take the nonrelativistic limit of it instead.

In order to obtain free energy expression we start with complex scalar field with a static metric and derive the functional integral representation of partition function which is fully relativistic. For canonical ensemble, a similar analysis is already presented in [15] and following the same path, we generalize their results to grand canonical ensemble with chemical potential included in the Hamiltonian. Functional integral representation of partition function leads to Euclidean action and to a configuration space integral. Nevertheless, the measure in the integral does not match with the one that arises from the inner product in configuration space. At that point a conformal transformation is required to remove the mismatch and that transformation is done by the introduction of optical metric.

Optical metric, which is an ultrastatic metric, can be acquired from static metric by making a conformal transformation and therefore is commonly used in the analysis of quantum field theories in static spacetimes. The chemical potential is generally introduced after the conformal transformation. [4, 7–9, 16, 17]. However we start with the chemical potential in static spacetime which is the physical spacetime of interest and then show that it is invariant under the conformal transformation.

The conformal transformation plays an important role also in taking the nonrelativistic limit. In the technical part, saddle point method is applied to free energy expression in the integral form and generalized Laplace method enables us to take the large but finite c limit explicitly. First we start with ultrastatic spacetime and take the nonrelativistic limit to yield the free energy expression for ideal Bose gas with the Schrödinger operator related to nonrelativistic limit of Klein-Gordon equation. Then by the conformal transformation the static case reduces to ultrastatic one and we have the Schrödinger operator defined on the optical manifold. These results of our method in taking the large but finite c limit justifies our choice of the name 'nonrelativistic limit'. The same terminology is also used for similar processes in the literature. [14,18]

Nonrelativistic limit of free energy can be used to derive several thermodynamic quantities. For example particle number density can be directly found by differentiating the free energy with respect to chemical potential. We use Mellin transform and Heat kernel expansion techniques to attain asymptotic series of free energy with convenient expansion parameters. The dependence of free energy on the temperature is well obtained as an asymptotic series with the dimensionless expansion parameter $\frac{T}{\lambda_0-\mu}$ and the leading order is $\left(\frac{T}{\lambda_0-\mu}\right)^{5/2}$. Since near critical temperature the chemical potential μ is very close to the ground state energy λ_0, smaller positive and negative powers of $\frac{T}{\lambda_0-\mu}$ become negligible around T_c and our asymptotic expansion is well justified. Apart from that, in the ultrarelativistic limit, derived expression for free energy is analogous to the ones in the literature. From the asymptotic expansion, density-temperature relation is obtained and we see that some extra terms that are not present in the flat case arise due to gravitational effects.

For Bose-Einstein condensation (BEC), the system is assumed to be constrained in a large but finite volume and we work explicitly in Schwarzschild background in $3+1$ dimensions. As BEC occurs only at some critical temperature T_c, by the density-temperature relation, T_c is found with some correction terms due to gravitational background. The results tend to yield the usual flat space results as expected when the Schwarzschild radius is taken to 0 ($r_s \to 0$) or when the thermodynamic limit is taken. Gravitational effects on equation of state give some correction terms in curved space, however, they tend to disappear in the thermodynamic limit as they have to. Finite size boundary effects are also important for BEC and there comes some factors representing corrections to the critical temperature and the results again reduce to flat space expressions in the thermodynamic limit. Another interesting result is that there is a divergence in occupation number for excited states in 2-d, i.e. there is no BEC in two dimensions.

One other issue on BEC in curved spacetime is the horizon problem for black holes. We already know from [15, 19] that if the gravitation object is very heavy, for instance a black hole, thermodynamic quantities tend to diverge as one gets closer to the horizon. In fact this can be avoided by the initial assumption that the system is restricted in a volume away from the gravitation source, i.e. away from the horizon. For this reason, throughout this thesis we do not deal with any horizon divergence but as a continuation of this study, the regions close to the horizon where the thermodynamics should be handled with care could be an interesting subject for a new research.

Our study can be further extended to interacting bosons or cosmological backgrounds. For the latter, there are already some efforts concerning nonrelativistic matter [20, 21], nonrelativistic axion dynamics [22, 23] and scalar field dark matter models [24].

The flow of the topics in the thesis is as follows: In Chapter 2 classical and quantum statistics of Bose gas are summarized for both canonical and the grand canonical ensembles and the Bose-Einstein condensation is explained in detail. Density-temperature relations and critical temperature formula are derived. In chapter 3 functional integration methods are introduced. Path integration and functional representation of partition function is illustrated via Euclidean action for both quantum mechanics and scalar fields. In order to absorb chemical potential into the functional integral, complex scalar field is presented with its real components and identities. Chapter 4 contains the derivation of free energy by functional representation of partition function, starting from complex scalar fields in a static spacetime. For the correct functional measure, optical metric is used to make a conformal transformation into the ultrastatic metric. Chapter 5 is the crucial chapter where we take the nonrelativistic limit of free energy by generalized Laplace method and establish the connection between static and ultrastatic spacetimes via optical metric. Chapter 6 contains rather technical work: Mellin transform, Heat kernel expansion, zeta-function analysis along with zeta and gamma functions are put together to obtain asymptotic expansion of free energy. In Chapter 7 gravitational effects on BEC are investigated thanks to the asymptotic expansions and corrections due to gravitation for density-temperature relations and equation of state are obtained explicitly in addition to boundary effects.

This thesis is heavily based on the paper ”Nonrelativistic limit of thermodynamics of Bose field in a static spacetime and Bose–Einstein condensation” [25].

2. BOSE-EINSTEIN CONDENSATION

Quantum statistics of the ideal Bose gas can be best understood by grand canonical ensemble if the total number of particles and the total energy of the system are free to fluctuate. Consider an ensemble of N Bose particles in a volume V which are in an external heat reservoir at constant temperature T and in a particle bath with constant chemical potential μ and let E be the total energy of the N particle system. We call n_r *occupation number* of the state ϵ_r. The system with N particles has the energy spectrum $E_{\{n_r\}}$ where $r = 0, 1, ...$ Then the probability of finding the system with N in energy state $E_{\{n_r\}}$ is given by [26]

$$\mathcal{P}_{\{n_r\}} = \frac{1}{Z} e^{-\beta(E_{\{n_r\}} - \mu N)} \tag{2.1}$$

where Z is the grand canonical partition function

$$Z = \sum_{N,r} e^{-\beta(E_{\{n_r\}} - \mu N)}. \tag{2.2}$$

Total energy of the system is sum of all one particle state energies ϵ_r

$$E_{\{n_r\}} = \sum_r \epsilon_r n_r \quad \text{with} \quad \sum_r n_r = N. \tag{2.3}$$

Since we are considering bosons, n_r can take any positive integer value $n_r = 0, 1, ...$ Substituting in (2.2) yields

$$Z = \prod_r \sum_{n_r} e^{-\beta(\epsilon_r - \mu) n_r} = \prod_r Z_r \tag{2.4}$$

where

$$Z_r = \sum_{n_r} z^{n_r} e^{-\beta \epsilon_r n_r}, \tag{2.5}$$

and $z = e^{\beta\mu}$ is called *fugacity.* The infinite sum $\sum_n x^n$ converges to $\frac{1}{1-x}$ so we have

$$Z = \prod_r \frac{1}{1 - e^{-\beta(\epsilon_r - \mu)}} = \prod_r \frac{1}{1 - z\, e^{-\beta\epsilon_r}}. \tag{2.6}$$

Thus

$$Z_r = \frac{1}{1 - z\, e^{-\beta\epsilon_r}}. \tag{2.7}$$

The probability (2.1) can be written as

$$\mathcal{P}_{\{n_r\}} = \prod_r p_r n_r. \tag{2.8}$$

Here p_r stands for the probability of finding a particle in single particle state ϵ_r and written as

$$p_r = \frac{1}{Z_r}(ze^{-\beta\epsilon_r})^{n_r}. \tag{2.9}$$

The average occupation number for the state ϵ_r is [27]

$$\begin{aligned} \langle n_r \rangle &= \sum_{n_r} p_r n_r = \frac{1}{Z}\sum_{n_r} z^{n_r} \sum_r n_r e^{-\beta\epsilon_r n_r} \\ &= \frac{1}{\beta}\frac{\partial \ln Z_r}{\partial \mu}. \end{aligned} \tag{2.10}$$

Combining all and after some straightforward algebra we have

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}. \tag{2.11}$$

This is the Bose-Einstein distribution. (Note that we have changed the subscript for reasons that will be clear soon.) For any single particle state r, the average number of particles at that state is given by this distribution. For the total number of particles from (2.3) we may write

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}. \tag{2.12}$$

At this point let us clarify the term "condensation": When the temperature is low enough for a given density, we will soon see that a macroscopic fraction of the particles may occupy the same state, i.e. the ground state with $k = 0$ in (2.11). Then the term condensation does not imply a high density presence in position space, but in momentum space. The index k in fact represents the momentum of the state. Consequently the ground state where the condensation occurs is of zero momentum [2].

We may take the thermodynamic limit as $V \to \infty$ and $N \to \infty$ together such that the ratio N/V is fixed. Moreover in thermodynamic limit single particle quantum states can be treated as a continuous spectrum. Then it is natural to convert the sums over momentum states k by integrals. We may write (2.12) as

$$N = \int \frac{V}{(2\pi)^3} \frac{d^3k}{e^{\beta[\epsilon(k) - \mu]} - 1}. \tag{2.13}$$

Notice the term $V/(2\pi)^3$ in the integration. Since wave number is given by $k = 2\pi n/V^{1/3}$, there are only $V^{1/3}/2\pi$ wave numbers per unit length in momentum space where V is the volume in position space, and therefore in 3-d the factor $V/(2\pi)^3$ arise in d^3k integral. If we prefer to write N over energy eigenstates then

$$n = \int_0^\infty g(\epsilon) \frac{d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} \tag{2.14}$$

where we have divided the extensive quantity N by another extensive quantity volume V to get the average particle density n which is intensive and does not diverge in the thermodynamic limit. $g(\epsilon)$ is called *density of states* and it reflects the number of states per unit length in energy scale. Density of states is fundamental in the analysis of BEC. Unfortunately, it is not easy to derive $g(\epsilon)$ in many situations such as in the curved spacetime. For this reason in our discussion on BEC, we will use alternative methods to cope with the problem. In 3-dimensions, a free boson has one quantum state per volume $2\pi\hbar^3$ in phase space and $g(\epsilon)$ is calculated easily to yield [3]

$$g(\epsilon) = \frac{Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\,\epsilon^{1/2}. \tag{2.15}$$

After some algebra including gamma function and power series, particle density is given by [2]

$$n(T) = \left(\frac{mT}{2\pi}\right)^{3/2} g_{3/2}(z), \tag{2.16}$$

with the function

$$g_{3/2}(z) = \sum_{p=1}^{\infty} \frac{z^p}{p^{3/2}}. \tag{2.17}$$

Note that the factors k_B and $\hbar$ are omitted in (2.16). This is due to natural units convention $\hbar = k_B = c = 1$. For simplicity, throughout the text we will use natural units unless indicated otherwise. For condensation, the chemical potential should be equal to 0 and the temperature below which this occurs is called the critical temperature T_c. That corresponds to the point $z = 0$ and consequently

$$g_{3/2}(1) = \sum_{p=1}^{\infty} \frac{1}{p^{3/2}} = \zeta(3/2) \tag{2.18}$$

is the Riemann zeta function and its numerical value is 2.612. [2] Substituting in (2.16) and solving for T we arrive at the critical temperature

$$T_c = \frac{2\pi}{m}\left(\frac{n}{\zeta(3/2)}\right)^{2/3}. \tag{2.19}$$

Let us examine further what happens when the temperature is below T_c. If the chemical potential becomes zero, then there will be no cost of putting more particles into the system and all the particles may occupy the ground state. In the thermodynamic limit as $N \to \infty$ there is a finite fraction N_0/N occupying the lowest eigenstate, i.e. $N_0 \propto V$.

For the ground state the distribution in (2.11) is

$$N_0 \;=\; \frac{1}{e^{-\beta\mu}-1}. \tag{2.20}$$

Evidently, when chemical potential reaches to the value 0, N_0 becomes infinite in the thermodynamic limit and we have a condensation at $k = 0$ state. It is convenient to write (2.12) by separating the ground state at $\mu = 0$

$$N \;=\; N_0 + \sum_{k\neq 0} \frac{1}{e^{\beta\epsilon_k}-1}. \tag{2.21}$$

Again applying the same tricks above we get

$$n \;=\; n_0 + \zeta(3/2)\left(\frac{mT}{2\pi}\right)^{3/2} \tag{2.22}$$

for $T < T_c$. Then the fraction of particles in the condensate is given by

$$\frac{n_0}{n} \;=\; 1 - \left(\frac{T}{T_c}\right)^{3/2}. \tag{2.23}$$

From the above equation we may conclude that at $T = 0$, all the particles occupy the ground state, whereas at $0 < T < T_c$ a finite fraction of the particles occupy the ground state in the thermodynamic limit, i.e. condensation occurs partially.

3. FUNCTIONAL INTEGRAL REPRESENTATION

3.1. Functional Integration in Quantum Mechanics

Functional integration has become a fundamental method in Quantum Field Theory (QFT) over the years, especially after the introduction of *Path Integral*, which is a kind of functional integral. The rigorous basics of Path Integral formalism was first developed by Norbert Wiener as the Wiener integral in order to solve Brownian motion and diffusion problems in 1920s [28]. After him, in 1933 Dirac introduced *Lagrangian* into quantum mechanics. [29] However, the completion of the formalism was in 1948 by Richard Feynman. Feynman's path-integral formalism has paved the way for a connection between Classical and Quantum Mechanics. It is an alternative description of quantum mechanics and leads to a deeper and a more general understanding of QFT by generalizing the *action* in classical mechanics to quantum mechanics. Moreover, since it makes use of *Lagrangian* rather than *Hamiltonian*, path integral formalism conserves the symmetries of the system so that it is easier to change coordinates, for instance due to Lorentz invariance of the propagator.

The power of functional integration comes from the fact that it clearly shows the correspondence between classical and quantum mechanics. In quantum mechanics, Hamiltonian for a nonrelativistic particle in 1-dimension is written as

$$H = \frac{p_x^2}{2m} + V(x). \tag{3.1}$$

The probability of this particle to move from some initial point x_0 at time $t_0 = 0$ to some final point x_f in a given time t_f is determined by the transition amplitude $\mathcal{U}(x_0, x_f, t_f)$, the position representation of time evolution operator in Schrödinger picture. $\mathcal{U}$, also known as *propagator*, is written as

$$\mathcal{U}(x_0, x_f, t_f) = \left\langle x_f | e^{-iHt_f} | x_0 \right\rangle . \tag{3.2}$$

On the other hand, any quantum mechanical process that can take place more than one way, has the property that its total amplitude can be expressed as the coherent sum of amplitudes of all the possible ways that lead to the same process. This is also known as the *superposition principle.* Therefore, one can write (3.2) as a sum over all possible ways, i.e. the *paths* or better an integral over all paths. However, the integration over *all paths* cannot be done over a variable, but rather a functional that represents all the functions or say paths starting at point x_0 and ending at point x_f, as follows:

$$\mathcal{U}(x_0, x_f, t_f) = \sum_{\textbf{all paths}} e^{i\mathcal{S}} = \int \mathcal{D}x(t)e^{i\mathcal{S}[x(t)]} \tag{3.3}$$

Here $\mathcal{D}$ symbol, which we call *measure*, represents an integration of a functional over a set of functions. A functional is defined as a map from functions to numbers. This is a generalization of calculus of variables into spaces of functions.

The integral in (3.3) can be evaluated by the method of stationary phase which requires that $\delta\mathcal{S}[x(t)] = 0$. Likewise, the *action* S in classical mechanics satisfies a similar condition $\delta S = 0$ as a consequence of the *least action principle.* Indeed, as we will show later, the phase denoted by the functional $\mathcal{S}[x(t)]$ can be proved to correspond to the action in classical mechanics [30]. The significance of functional formalism comes from this correspondence between classical and quantum mechanics, along with the Lorentz invariance of action. The analogy differs only by the interpretation of action: In classical mechanics, action is minimized only for a single path, yet in quantum mechanics it is a coherent sum of amplitude contributions of all possible paths.

For a more general system described by generalized coordinates $\mathbf{q}$ and conjugate momenta $\mathbf{p}$ with the Hamiltonian $H(\mathbf{q}, \mathbf{p})$, one can derive the transition amplitude (3.2) explicitly. [31] Start with (3.2) and divide the time interval $(t_0 = 0, t_f)$ into N equal time sections Δt so that $\Delta t = t_f/N$. Then

$$e^{-iHt_f} = \prod_{i=1}^{N} e^{-iH\Delta t} = e^{-iH\Delta t}e^{-iH\Delta t} \; ... \; e^{-iH\Delta t} \tag{3.4}$$

N factors of $e^{-iH\Delta t}$ are multiplied consecutively. Between each of these factors, we insert completeness relations of generalized coordinates and conjugate momenta given by

$$\begin{aligned} \int d^d q |q\rangle\langle q| = \mathbf{1}, \\ \frac{1}{2\pi}\int d^d p |p\rangle\langle p| = \mathbf{1}. \end{aligned} \tag{3.5}$$

We also note the orthogonality conditions

$$\begin{aligned} \langle q_k|q_l\rangle = \delta(\mathbf{q}_k - \mathbf{q}_l), \\ \langle p_k|p_l\rangle = \delta(\mathbf{p}_k - \mathbf{p}_l). \end{aligned} \tag{3.6}$$

On the left hand side of operator $e^{-iH\Delta t}$ completeness relation for q is inserted and on the right hand side of it, completeness relation for p is inserted. Therefore total number of N completeness relation pair are inserted in (3.4). Substituting all into (3.2) gives

$$\begin{aligned} \langle q_f|e^{-iHt_f}|q_0\rangle &= \prod_{i=1}^{N}\int \frac{d^d q_i d^d p_i}{2\pi} \\ &\times\langle q_f|p_N\rangle\langle p_N|e^{-iH\Delta t}|q_N\rangle \\ &\times\langle q_N|p_{N-1}\rangle\langle p_{N-1}|e^{-iH\Delta t}|q_{N-1}\rangle \\ &\times\ ... \\ &\times\langle q_2|p_1\rangle\langle p_1|e^{-iH\Delta t}|q_1\rangle \\ &\times\langle q_1|q_0\rangle. \end{aligned} \tag{3.7}$$

The subscripts above refer to different points in time as we have divided t_f into N sections. Notice that from (3.6) we already know the last line

$$\langle q_1|q_0\rangle = \delta(\mathbf{q}_1 - \mathbf{q}_0). \tag{3.8}$$

Yet, by definition $\mathbf{q}_1 = \mathbf{q}_0$, since $\mathbf{q}_1$ comes just before the first time section, i.e. at $t = t_0 = 0$. Thus $\langle q_1|q_0\rangle = \mathbf{1}$. From this definition also note that $\mathbf{q}_{N+1} = \mathbf{q}_f$, as $\mathbf{q}_{N+1}$ corresponds to the final point in time, at $t = t_f$. Now if we look at the first terms in each line, we see that they represent $\langle q_{i+1}|p_i\rangle$. Furthermore we already know from quantum mechanics that

$$\langle q|p\rangle = e^{i\mathbf{p}\cdot\mathbf{q}} \quad \text{and so} \quad \langle q_{i+1}|p_i\rangle = e^{i\mathbf{p}_i\cdot\mathbf{q}_{i+1}}. \tag{3.9}$$

In order to converge to a continuous path, one can take the limit $N \to \infty$ or $\Delta t \to 0$. In that case, we can expand the operator $e^{-iH\Delta t}$ as Δt is small. Considering only first order terms we get

$$e^{-iH\Delta t} \approx 1 - iH\Delta t. \tag{3.10}$$

Then we have

$$\begin{aligned}
\langle p_i|e^{-iH\Delta t}|q_i\rangle &\approx \langle p_i|(1 - iH\Delta t)|q_i\rangle \\
&= (1 - iH_i\Delta t)\langle p_i|q_i\rangle \qquad (3.11)\\
&= e^{-iH_i\Delta t}e^{-i\mathbf{p}_i\cdot\mathbf{q}_i} \qquad (3.12)
\end{aligned}$$

where $H_i = H(\mathbf{q}_i, \mathbf{p}_i)$. Combining all, (3.7) becomes

$$\begin{aligned}
\left\langle q_f|e^{-iHt_f}|q_0\right\rangle &= \lim_{N\to\infty}\prod_{i=1}^{N}\int\frac{d^dq_id^dp_i}{2\pi} \\
&\quad\times\exp\left\{-i\Delta t\sum_{i=1}^{N}\left[H_i(\mathbf{q}_i,\mathbf{p}_i) - \mathbf{p}_i\cdot\frac{\mathbf{q}_{i+1}-\mathbf{q}_i}{\Delta t}\right]\right\}. \qquad (3.13)
\end{aligned}$$

Taking the limit transforms the equation from discretized form into continuous form

$$\mathcal{U}(\mathbf{q}_0, \mathbf{q}_f, t_f) = \int \mathcal{D}q(t)\mathcal{D}p(t)\exp\left[i\int_0^{t_f} dt\,(\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p}))\right] \tag{3.14}$$

with the functional measure

$$\int \frac{d^d q d^d p}{2\pi}, \tag{3.15}$$

the integration over phase space. In fact, one can interpret this method as if we have divided the path from $\mathbf{q}_0$ to $\mathbf{q}_f$ into N paths equal in time, and calculated each propagation amplitude for these path sections and multiplied them in order to get total amplitude. Then in the continuum limit, what we end up with is the most general expression for propagator, (3.14), which is written as an integral over phase space. Further we can substitute the Hamiltonian given in (3.1)

$$\mathcal{U}(\mathbf{q}_0, \mathbf{q}_f, t_f) \;=\; \int \frac{d^d q}{2\pi} \int d^d p \exp\left[i \int_0^{t_f} dt \left(\mathbf{p} \cdot \dot{\mathbf{q}} - \frac{\mathbf{p}^2}{2m} - V(\mathbf{q}) \right) \right] \tag{3.16}$$

where p integrals become Gaussian and can be evaluated easily by completing the square

$$\begin{aligned} \mathcal{U}(\mathbf{q}_0, \mathbf{q}_f, t_f) &= \int d^d q e^{i \int_0^{t_f} dt \left[\frac{m}{2} \dot{\mathbf{q}}^2 - V(\mathbf{q}) \right]} \\ &= \int \mathcal{D}q(t) e^{i \int_0^{t_f} dt L} = \int \mathcal{D}q(t) e^{iS} \end{aligned} \tag{3.17}$$

up to a constant embedded in the measure $\mathcal{D}q(t)$. The integral in the exponent is classical action S, but remember in (3.3), the phase is denoted by the functional $\mathcal{S}[\mathbf{q}(t)]$. Hence the functional $\mathcal{S}$ corresponds to classical action and in fact for any *Weyl ordered* Hamiltonian this derivation is valid [30]. Consequently we will denote *action* by $\mathcal{S}$ from now on.

Finally, the propagator can be written as the configuration space integral

$$\mathcal{U}(\mathbf{q}_0, \mathbf{q}_f, t_f) \;=\; \int \mathcal{D}q(t) e^{i\mathcal{S}[\mathbf{q}(t)]}. \tag{3.18}$$

This is the path integral expression for propagator.

Up to this point, we have described a system with a single particle whose Hamiltonian is given by (3.1). Nevertheless BEC is obviously a many body phenomena and cannot be fully described by single particle quantum mechanics. Statistical mechanics is the main tool to explain BEC and the quantum statistical nature of noninteracting Bose gas in flat space has already been discussed in Section 2. Besides we need a connection between functional formalism and statistical mechanics. Since partition function has a crucial role in deriving thermodynamic variables and analyzing the whole system, one may start with the grand canonical partition function:

$$Z = Tr\, e^{-\beta(H-\mu N)} \tag{3.19}$$

with β, μ and N being inverse temperature, chemical potential and number of particles respectively. For convenience, we include μN term inside H as $H - \mu N \longrightarrow H$. Then the partition function becomes

$$Z = Tr\, e^{-\beta H} = \int dx\, \langle x|e^{-\beta H}|x\rangle. \tag{3.20}$$

Impose the periodicity condition $x_i = x_f$ on (3.2) and then compare it with the expression above. Clearly they are the same expressions except that imaginary unit i times the t is replaced by β. Hence, using (3.18), we may write Z as

$$Z(\beta) = \int_{\mathbf{q}(\beta)} \mathcal{D}q(t)\ e^{-\mathcal{S}_E} \tag{3.21}$$

where $\mathcal{S}_E$ is called *Euclidean action* defined as [32]

$$\mathcal{S}_E \equiv \int dt \left[\frac{1}{2} m\dot{\mathbf{q}}^2(t) + V\big(\mathbf{q}(t)\big)\right]. \tag{3.22}$$

Note that there is no factor of i in front of Euclidean action, which indeed describes a particle moving in an imaginary time. This is why in (3.22) the relative signs of kinetic and potential terms are the same whereas in usual action they are opposite. Euclidean action can be formally derived from the classical action:

$$S = \int dt L = \int dt \left[\frac{1}{2} m \left(\frac{d\mathbf{q}}{dt} \right)^2 - V \right] \tag{3.23}$$

Transform to the imaginary time variable $t' = it$

$$S = \int \frac{dt'}{i} \left[\frac{1}{2} m \left(\frac{id\mathbf{q}}{dt'} \right)^2 - V \right] = i \int dt' \left[\frac{1}{2} m \left(\frac{d\mathbf{q}}{dt'} \right)^2 + V \right]. \tag{3.24}$$

The integral on the right hand side is clearly Euclidean action. Hence, classical and Euclidean actions are related by $S = iS_E$ or equivalently $iS = -S_E$.

Finally we have discovered the connection between functional formalism and statistical mechanics and it is established by Euclidean action. In terms of Euclidean action, we can also write propagator as

$$\mathcal{U}(\mathbf{q}_0, \mathbf{q}_f, t_f) = \int_{\mathbf{q}(0)}^{\mathbf{q}(t_f)} \mathcal{D}q(t) e^{i\mathcal{S}} = \int_{\mathbf{q}(0)}^{\mathbf{q}(t_f)} \mathcal{D}q(t) e^{-\mathcal{S}_E}. \tag{3.25}$$

This is called *Path Integral.* [32] The integration is over all paths starting from $\mathbf{q}(0)$ and ending at $\mathbf{q}(t_f)$.

3.2. Partition Function in Field Formalism

In order to analyze the problems in quantum mechanics better, it is natural to shift to *Field formalism.* Consider real scalar field $\hat{\phi}(\mathbf{x}, t)$ and its conjugate momentum $\hat{\pi}(\mathbf{x}, t)$ in Schrödinger picture. The eigenvalues and the (complete and orthonormal) eigenstates of these operators are defined by

$$\begin{aligned} &\hat{\phi}(\mathbf{x}, t)|\phi_t\rangle = \phi(\mathbf{x}, t)|\phi_t\rangle \quad \text{and} \\ &\hat{\pi}(\mathbf{x}, t)|\pi_t\rangle = \pi(\mathbf{x}, t)|\pi_t\rangle. \end{aligned} \tag{3.26}$$

If one adapts functional integration method to QFT using real scalar field, (3.14) and (3.18) would be still valid due to the fact that they are derived for any general quantum system. However the coordinates $\mathbf{q}$ and its conjugate momenta $\mathbf{p}$ should be replaced by field amplitude $\phi = \phi(\mathbf{x}, t)$ and its conjugate momentum $\pi = \pi(\mathbf{x}, t)$, with the Hamiltonian

$$H = \int d^3x \mathcal{H}(\pi, \phi) \tag{3.27}$$

where $\mathcal{H}$ being the Hamiltonian density, a functional of the field and its conjugate. [31]

On the other hand, in derivation of the above expressions, our starting point is Schrödinger equation in quantum mechanics. Nevertheless, when the particles or fields are relativistic, it is more natural to address to *Klein-Gordon* (KG) equation. Although KG equation itself does not constitute a direct substitute for single particle Schrödinger equation, it is a relativistic wave equation associated with Schrödinger equation and for the quantum theory of special relativity we know that KG equation and its solutions arise in QFT. It would have been also natural to introduce *Dirac Equation* if we had been interested in analyzing fermions in the relativistic case, but of course it is beyond the scope of this thesis since we are dealing with bosons in BEC. For further discussion one may consult to standard QFT textbooks.

KG Equation is a relativistic wave equation, that is the d'Alembertian $\Box$ plus Einstein's mass energy term, m^2. The field $\phi(t, \mathbf{x})$ in Minkowski spacetime (due to relativistic convention, t and $\mathbf{x}$ were exchanged in parenthesis from now on) satisfies Klein-Gordon Equation:

$$(\Box + m^2)\phi = 0 \tag{3.28}$$

with d'Alembertian $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ where $\eta^{\mu\nu}$ is the inverse Minkowski metric with signature {+,-,-,-}. (Note that natural units, $c = \hbar = k_B = 1$ are used throughout the text and are therefore omitted unless indicated otherwise). This is in fact classical field theory and $\phi(t, \mathbf{x})$ can be a classical field satisfying the wave equation, (3.28). Yet the theory can be later quantized in QFT, where $\phi(t, \mathbf{x})$ would be quantum field and m would be considered as the mass of the field quanta. [33]

In describing a relativistic system, it is more convenient to start with writing the Lagrangian density $\mathcal{L}$, instead of the Hamiltonian density defined in (3.27). Lagrangian density is formally defined in terms of Lagrangian L as

$$L = \int d^3x \mathcal{L} \tag{3.29}$$

whereas in terms of action

$$\mathcal{S} = \int dt \int d^3x \mathcal{L} \quad \text{or} \quad \mathcal{S} = \int d^4x \mathcal{L}. \tag{3.30}$$

The reason Lagrangian density is more preferable in functional approach comes from the fact that it preserves the symmetries of the system explicitly under Lorentz transformations. Indeed, the action defined in (3.30) is a Lorentz scalar and due to least action principle, minimizing action gives the equations of motion or evolution of the system in field language. The Hamiltonian formalism itself is not manifest invariant, i.e. time and space components are not treated equally and the way we write the total energy of the system explicitly is not the same in every reference frame, whereas in the Lagrangian formalism it is. As functional integration takes Lagrangian density as the fundamental object of the theory, it becomes a powerful tool and superior to writing down the Hamiltonian directly. Besides, KG equation can be obtained directly from Lagrangian density via least action principle $\delta\mathcal{S}[\phi(t, \mathbf{x})] = 0$.

For a single scalar field one can write the *Lagrangian density* most generally as [31]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - V. \tag{3.31}$$

The momentum conjugate of the field $\phi(t, \mathbf{x})$ is defined by

$$\pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \dot{\phi} \tag{3.32}$$

and the Hamiltonian density can be obtained from Lagrangian density via *Legendre transformation*

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = \pi \dot{\phi} - \mathcal{L}. \tag{3.33}$$

Using the standard metric signature {+,-,-,-}, we arrive at

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V. \tag{3.34}$$

The first term above is the kinetic term and the second term can refer to the potential energy of the field. Third one is Einstein's mass energy in KG Equation and the last term V may include both field dependent terms $V(\phi)$ and external potential V_{ext} depending on the nature of the problem we are interested in. For convenience, we are leaving it as V in the most general form at the moment.

Starting from the very beginning, it is straightforward to show that partition function $Z(\beta)$, in field language, can be formally derived in the same way as (3.21) is obtained in the previous section. Following the same steps and again applying our standard tricks $it \to \beta$ and periodicity $\phi_0(0, \mathbf{x}) = \phi_f(\beta, \mathbf{x})$, we end up with

$$Z(\beta) = \int \mathcal{D}\phi e^{-\mathcal{S}_E} \tag{3.35}$$

where Euclidean action is written explicitly as

$$\mathcal{S}_E = \int_0^\beta dt \int d^3x \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V \right]. \tag{3.36}$$

Note that once more the bizarre constant in (3.35) is thrown in measure $\mathcal{D}\phi$. Alternatively in phase space integral formalism partition function is

$$Z(\beta) = \int \mathcal{D}\phi\mathcal{D}\pi \times \exp\left[\int_0^\beta dt \int d^3x \left(i\pi\frac{\partial\phi}{\partial t} - \mathcal{H}(\pi,\phi)\right)\right]. \tag{3.37}$$

One question at this point may arise over the validity of these expressions in curved spacetime since they are indeed derived for flat case. Following the same procedure in the derivation of (3.35) and (3.37), one can verify that these results are general and safe to be used in curved geometry. [15, 31] Therefore (3.35) and (3.37) are important and legitimate formulations and can be used to derive several thermodynamic variables such as free energy in curved spacetime.

3.3. Complex Scalar Field and BEC

In order to describe Bose-Einstein condensation discussed in Chapter 2 in the field language, it will be convenient first to look at the charged scalar field, which is actually a complex field whereas bosons with negative and positive charges are antiparticles of each other. [31] The corresponding charged field, i.e. complex scalar field Φ and its complex conjugate Φ^* fully represent the charged system and the Lagrangian density given in (3.31) can be rewritten as

$$\mathcal{L} = \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - V. \tag{3.38}$$

Lagrangian density $\mathcal{L}$ clearly has $U(1)$ symmetry as $\Phi \to \Phi e^{-i\theta}$. According to Noether's theorem, for every differentiable symmetry of the Lagrangian there is a conserved current. [30, 34] The conserved current density can be derived from equation of motion from the Lagrangian density in (3.38) and found to be

$$j_\mu = i(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*). \tag{3.39}$$

For convenience complex scalar field can be divided into its real and imaginary parts as ϕ_1 and ϕ_2 respectively, both being real scalar fields, satisfying

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{3.40}$$

where the factor $1/\sqrt{2}$ comes from normalization. For the conjugate momenta defined in (3.32), calculation is straightforward and similarly yields

$$\pi_1 = \partial_0\phi_1 \qquad \text{and} \qquad \pi_2 = \partial_0\phi_2. \tag{3.41}$$

Here π_1 and π_2 are real and imaginary components of the momentum conjugate of complex scalar field Φ. Substituting them in (3.38) yields

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i - \frac{1}{2}V \tag{3.42}$$

with $i = 1, 2$ and combining with (3.39), the conserved current density evolves to

$$j_\mu = \phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2. \tag{3.43}$$

Conserved current naturally gives rise to a conserved charge Q:

$$Q = \int d^3x j_0 = \int d^3x(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2) \tag{3.44}$$

Associated with charge Q there is a chemical potential μ similar to the one defined in Chapter 2, which we will include in the Hamiltonian density as $\mathcal{H} \to \mathcal{H} - \mu j_0$. Indeed, the reason we are are dealing with complex scalar field is so as to absorb μ into the Hamiltonian.

All together, functional integration brings out the analogy between Statistical Mechanics and QFT and we benefit from it in our work so as to obtain thermodynamic variables of BEC in curved space.

4. FREE ENERGY

In this chapter we will investigate the thermodynamics of a Bose field in a static spacetime and derive an expression for *Free Energy.* Free energy of such fields is already studied by Alwis and Ohta in 1995. [15] Our aim is to generalize their results to grand canonical ensemble and introduce the chemical potential μ.

4.1. Chemical Potential

We commence with the static metric

$$ds^2 = -F(x)dt^2 + h_{ij}(x)dx^i dx^j, \tag{4.1}$$

where $g_{\mu\nu}$ is the spacetime metric on a (d+1)-dimensional manifold $\mathcal{M}$ and h_{ij} is the spatial part of the metric. Both *time lapse function* F and h_{ij} are positive definite, $F > 0$, and we use the metric signature $\{-,+,+,+\}$ throughout the rest of the text. Hence $g = \det g_{\mu\nu}$ and $|g| = -g$.

We work with the complex scalar field introduced in the last section of chapter 3 so as to include the chemical potential μ in our analysis. For convenience we divide complex scalar field into its real and imaginary parts ϕ_1, ϕ_2 as we did in (3.40) and corresponding conjugate momenta π_1, π_2 are defined by (3.41).

In curved spacetime, Lagrangian density given in (3.42) becomes [33]

$$\mathcal{L} = \sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_a + \frac{1}{2}V\phi_a\phi_a\right], \tag{4.2}$$

where $a = 1, 2$. Remember that we have switched to spacelike metric convention $\{-,+,+,+\}$ and therefore the signs of potential and kinetic terms are different than usual. The potential

$$V = m^2 + \xi R + V_{ext}, \tag{4.3}$$

contains mass energy term m^2, the scalar curvature R of the metric $g_{\mu\nu}$, curvature coupling constant ξ and external potential V_{ext} if present. Action $\mathcal{S}$ corresponding to Lagrangian density $\mathcal{L}$ given in (4.2) is expressed via (3.30) as

$$\mathcal{S} = \int dt \int d^d x \sqrt{|g|} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a + \frac{1}{2} V \phi_a \phi_a \right]. \tag{4.4}$$

Applying least action principle $\delta \mathcal{S}[x(t)] = 0$ gives scalar field equation which is the Klein-Gordon (KG) equation in curved spacetime [33]

$$\begin{aligned} [\Box - (m^2 + \xi R)]\phi = 0 \\ \text{or} \qquad [\Box - V]\phi = 0, \end{aligned} \tag{4.5}$$

where $\Box$ is the d'Alembertian

$$\Box = g^{\mu\nu} \partial_\mu \partial_\nu. \tag{4.6}$$

In the presence of external potential V_{ext}

$$[\Box - (m^2 + \xi R + V_{ext})]\phi = 0. \tag{4.7}$$

Conserved U(1) current density (3.43) is then given by

$$j^\mu = \sqrt{|g|} g^{\mu\nu} (\phi_2 \partial_\nu \phi_1 - \phi_1 \partial_\nu \phi_2), \tag{4.8}$$

and the conjugate momenta (3.41) are written as

$$\pi_a \;\; = \;\; \sqrt{|g|} g^{00} \partial_0 \phi_a, \tag{4.9}$$

with the volume element

$$\sqrt{|g|} = \sqrt{|detg^{\mu\nu}|} = \sqrt{g_{00}h} = \sqrt{Fh}. \tag{4.10}$$

Here $h = \det h_{ij}$ and inverse metric $g^{00} = F^{-1}$. The Hamiltonian density can be found via Legendre transformation (3.33) and we should also add the chemical potential term inside so that $\mathcal{H} \to \mathcal{H} - \mu j_0$:

$$\mathcal{H} = \partial_0\phi_a\pi_a - \mu j_0 + \sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_a + \frac{1}{2}V\phi_a\phi_a\right] \tag{4.11}$$

Notice the sign of Lagrangian since we use spacelike convention. Combining all together

$$\mathcal{H} = \frac{1}{2}\sqrt{\frac{F}{h}}\pi_a\pi_a - \mu(\pi_1\phi_2 - \pi_2\phi_1) + \sqrt{Fh}\left[\frac{1}{2}h^{ij}\partial_i\phi_a\partial_j\phi_a + \frac{1}{2}V\phi_a\phi_a\right]. \tag{4.12}$$

Second quantized field Hamiltonian $\hat{H}$ which corresponds to (4.12) can be obtained via (3.27). Then the partition function defined by (2.2) can be written for the quantum field as

$$Z(\beta) = \operatorname{Tr} e^{-\beta\hat{H}}. \tag{4.13}$$

Remember that we use natural units ($c = \hbar = k_B = 1$) and $\beta = T^{-1}$ is the inverse temperature. The phase space integral representation of $Z(\beta)$ is derived in chapter 3 and is found to be (3.37). Accordingly for the quantum field ϕ_a

$$Z(\beta) = \int \mathcal{D}\pi\mathcal{D}\phi\, e^{-\int_0^\beta dt\int d^dx(\mathcal{H} - i\pi_a\partial_0\phi_a)} \tag{4.14}$$

in $d+1$ dimensions, with the functional measure

$$\mathcal{D}\pi\mathcal{D}\phi = \prod_{t,x}\prod_{a=1}^{2} d\pi_a(t,x)d\phi_a(t,x). \tag{4.15}$$

Remind the periodicity condition on fields in the derivation of partition function: $\phi_a(0,x) = \phi_a(\beta,x)$. Arranging the terms, the integrand in the exponent is

$$\begin{aligned}\mathcal{H} - i\pi_a\dot{\phi}_a &= \frac{1}{2}\sqrt{\frac{F}{h}}\pi_1^2 + (-\mu\phi_2 - i\dot{\phi}_1)\pi_1 + \frac{1}{2}\sqrt{\frac{F}{h}}\pi_2^2 + (\mu\phi_1 - i\dot{\phi}_2)\pi_2 \\ &\quad + \sqrt{Fh}\left[\frac{1}{2}h^{ij}\partial_i\phi_a\partial_j\phi_a + \frac{1}{2}V\phi_a\phi_a\right]. \qquad (4.16)\end{aligned}$$

The integral over π_a can be easily evaluated by Gaussian integration and it should yield *Euclidean action*. Remember (3.35)

$$Z(\beta) = \int \mathcal{D}\phi\, e^{-S_E},$$

where the phase is found to be Euclidean action. Now completing the square in (4.16) and evaluating π integral, in the exponent we have the integral

$$\begin{aligned}\int dt \int d^dx \quad &\times \quad \Bigg\{ -\sqrt{\frac{h}{4F}}\left[(-\mu\phi_2 - i\dot{\phi}_1)^2 + (\mu\phi_1 - i\dot{\phi}_2)^2\right] \\ &\qquad + \sqrt{Fh}\left[\frac{1}{2}h^{ij}\partial_i\phi_a\partial_j\phi_a + \frac{1}{2}V\phi_a\phi_a\right]\Bigg\} \\ = \int dt \int d^dx \quad &\times \quad \Bigg\{ \sqrt{\frac{h}{F}}\left[-\frac{1}{2}(\phi_a\mu\phi_a) - i\mu(\dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1) + \frac{1}{2}(\dot{\phi}_a\dot{\phi}_a)\right] \\ &\qquad + \sqrt{Fh}\left[\frac{1}{2}h^{ij}\partial_i\phi_a\partial_j\phi_a + \frac{1}{2}V\phi_a\phi_a\right]\Bigg\}. \qquad (4.17)\end{aligned}$$

Note that there is a factor of $\sqrt{h/F}$ in front of the exponential coming from the Gaussian integration and is absorbed into the measure. The importance of this factor will be clear shortly. Now introducing the Riemann metric

$$(g_E)_{\mu\nu} = \begin{pmatrix} F & 0 \\ 0 & h_{ij} \end{pmatrix}, \qquad (4.18)$$

we are left with Euclidean action [31]

$$\mathcal{S}_E = \int dt \int d^d x \sqrt{g_E} \quad \Bigg[\quad \frac{1}{2} g_E^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a + \frac{1}{2} \phi_a \left(V - \mu^2 F^{-1} \right) \phi_a$$
$$- i F^{-1} \mu (\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1) \quad \Bigg], \tag{4.19}$$

as anticipated. The first term can be integrated by parts

$$\int_B dt d^d x \, g_E^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a = g_E^{\mu\nu} \phi_a \partial_\nu \phi_a \Big|_{\partial B} - \int_B dt d^d x \, \phi_a g_E^{\mu\nu} \partial_\mu \partial_\nu \phi_a. \tag{4.20}$$

In this work we confine the system in a finite region B in space with boundary ∂B and assume that the field ϕ satisfies either Neumann or Dirichlet boundary conditions. [25] Neumann boundary condition requires that the normal derivative of field should vanish at boundary, i.e. $N^\mu \partial_\mu \phi|_{\partial B} = 0$ where $N^\mu \partial_\mu$ is the unit vector field normal to the boundary ∂B and points towards inside. If the field satisfies Dirichlet boundary condition, then it should vanish at the boundary, i.e. $\phi|_{\partial B} = 0$

For both boundary conditions, it is obvious that the boundary term in (4.20) vanishes. So we are left with

$$S_E = \int dt d^d x \sqrt{g_E} \left[\frac{1}{2} \phi_a \left(-\Delta + V - \mu^2 F^{-1} \right) \phi_a - i F^{-1} \mu (\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1) \right], \tag{4.21}$$

with Δ being the Laplacian for $(g_E)_{\mu\nu}$.

The measure $\mathcal{D}\phi$, after the integration over conjugate momenta, can be found to be

$$\mathcal{D}\phi = \prod_{t,x} \prod_{a=1}^{2} \left(\frac{g_E(x)}{F^2(x)} \right)^{1/4} d\phi_a(t,x), \tag{4.22}$$

as it is also given in [15]. The factor $\left(\frac{g_E(x)}{F^2(x)} \right)^{1/4}$ inside the product comes from the Gaussian integration.

On the other hand, the measure can be also calculated using the standard inner product of fields on configuration space. The kinetic term of the action can be used to define the inner product as

$$\langle \delta\phi | \delta\psi \rangle = \int dt d^d x \sqrt{g_E} \delta_{ab} \delta\phi_a \delta\psi_b. \tag{4.23}$$

Then the corresponding measure would be trivial

$$\prod_{t,x} \prod_{a=1}^{2} \left[g_E(x) \right]^{1/4} d\phi_a(t,x). \tag{4.24}$$

Apparently the two measures differ by a factor $F^{-1/2}$ unless $F = 1$. In fact when $F = 1$, this would be a special case meaning that we are in an *ultarastatic spacetime.* Ultrastatic metric is then written as

$$ds^2 = -dt^2 + h_{ij}(x) dx^i dx^j. \tag{4.25}$$

In order to avoid the side effects of this mismatch, it would be convenient to make a conformal transformation such that the metric becomes ultrastatic.

4.2. Ultrastatic Spacetime

We use *optical metric* to switch from static spacetime to an ultrastatic one. Optical metric, which was first introduced by Walter Gordon in 1923, is of the following form

$$ds^2 = -dt^2 + \frac{h_{ij}}{F} dx^i dx^j, \tag{4.26}$$

and can be obtained via dividing spacetime interval by the time lapse function F and evidently it is an ultrastatic metric (4.25). Define a conformal transformation

$\mathcal{M} \to \mathcal{M}' = \mathbf{R} \times \mathbf{M}'$ with global time $\mathbf{R}$ as

$$\overline{\phi} = F^{\frac{d-1}{4}} \phi, \tag{4.27}$$

with $\overline{\phi}$ being the new field corresponding to the optical metric

$$\overline{g}_{\mu\nu} = F^{-1}(g_E)_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}, \quad \gamma_{ij} = \frac{h_{ij}}{F}. \tag{4.28}$$

The new metric $\overline{g}_{\mu\nu}$ is clearly an ultrastatic one and differs from the static metric by only a scalar factor in front and are said to be conformally related. [17] The volume element is then

$$\sqrt{\overline{g}} = \sqrt{\gamma} = F^{-\frac{d+1}{2}} \sqrt{|g|}. \tag{4.29}$$

Now we should revise the boundary conditions. Since, by our initial assumption, the field ϕ satisfies Dirichlet boundary condition $\phi|_{\partial B} = 0$ then obviously does $\overline{\phi}$, i.e. $\overline{\phi}\big|_{\partial B} = 0$ since

$$\phi|_{\partial B} = F^{-\frac{d-1}{4}} \, \overline{\phi}\big|_{\partial B} = 0. \tag{4.30}$$

Nevertheless, although we impose that ϕ is supposed to satisfy Neumann boundary condition $N^\mu \partial_\mu \phi|_{\partial B} = 0$, we have

$$N^\mu \partial_\mu \left[F^{-\frac{d-1}{4}} \overline{\phi} \right]_{\partial B} = 0,$$

$$F^{-\frac{d-1}{4}} N^\mu \partial_\mu \overline{\phi} \bigg|_{\partial B} + \left(N^\mu \partial_\mu F^{-\frac{d-1}{4}} \right) \bigg|_{\partial B} \overline{\phi} = 0,$$

$$N^\mu \partial_\mu \overline{\phi}|_{\partial B} - \left(\frac{d-1}{4} N^\mu \partial_\mu (\log F) \right) \overline{\phi} \bigg|_{\partial B} = 0. \tag{4.31}$$

This is called generalized Neumann boundary condition or Robin boundary condition. Remember that $N^\mu \partial_\mu$ is the unit vector field normal to the boundary ∂B and its direction is inwards.

We should rewrite Klein-Gordon equation after the conformal transformation (4.27) as $\overline{\phi}$ is no longer a solution of (4.5). For $\overline{g}$, which is conformally related to the metric g_E by (4.28), KG Equation is given as

$$\left[\Box_\gamma - F(m^2 + \xi R) - \frac{d-1}{4d}(R_\gamma - FR)\right]\overline{\phi} = 0, \tag{4.32}$$

with $\Box_\gamma$ being d'Alembertian for optical metric. Then, $\overline{\phi} = F^{(d-1)/4}\phi$ would be a solution of (4.32). Note that conformal transformation brings extra terms of both R_γ and R and there are special circumstances when $\xi = 0$ and $\xi = (d-1)/4d$. They are called minimally coupled case and conformally coupled case, respectively. [17, 33] Applying the conformal transformation over (4.21) and performing the momenta integration we have

$$\begin{aligned} S_E \;=\; & \int dt d^d x F^{\frac{d+1}{2}} \sqrt{|g|} \times \left[\frac{1}{2} F^{-\frac{d-1}{4}} \overline{\phi}_a \Big(-\partial_0^2 - \Delta_\gamma + m^2 + \xi R \right. \\ & + \frac{d-1}{4d}(F^{-1}R_\gamma - R) + V_{ext} - \mu^2 F^{-1} \Big) F^{-\frac{d-1}{4}} \overline{\phi}_a \\ & \left. - iF^{-1}\mu \left(F^{-\frac{d-1}{4}} \dot{\overline{\phi}}_1 F^{-\frac{d-1}{4}} \overline{\phi}_2 - F^{-\frac{d-1}{4}} \dot{\overline{\phi}}_2 F^{-\frac{d-1}{4}} \overline{\phi}_1 \right) \right]. \end{aligned} \tag{4.33}$$

Here Δ_γ is the Laplace-Beltrami operator on $\mathbf{M}'$

$$\Delta_\gamma = \frac{1}{\sqrt{|\overline{g}|}} \partial_i \sqrt{|\overline{g}|} \gamma^{ij} \partial_j. \tag{4.34}$$

For reasons that will be apparent shortly, let us define a new potential $U = V - (mc)^2$. Then

$$U = \frac{d-1}{4d} R_\gamma + F\left(\xi - \frac{d-1}{4d}\right) R + (F-1)m^2c^2 + FV_{ext}. \tag{4.35}$$

At this point writing the factors c explicitly is beneficial.

$$S_E = -\int dt\, d^dx \sqrt{\overline{g}} \frac{1}{2} \Big[\overline{\phi}_a (-c^{-2}\partial_0^2 - \Delta_\gamma + U + (mc)^2 - \mu^2) \overline{\phi}_a \\ -2ic^{-2}\mu(\overline{\phi}_2 \partial_0 \overline{\phi}_1 - \overline{\phi}_1 \partial_0 \overline{\phi}_2) \Big]. \tag{4.36}$$

We can write (4.36) in a more compact form if we define

$$\mathcal{A} = \begin{pmatrix} -c^{-2}\partial_0^2 + A - \mu^2 c^{-2} & 2ic^{-2}\mu\partial_0 \\ -2ic^{-2}\mu\partial_0 & -c^{-2}\partial_0^2 + A - \mu^2 c^{-2} \end{pmatrix} \tag{4.37}$$

with

$$A = -\Delta_\gamma + m^2c^2 + U. \tag{4.38}$$

Then we have

$$S_E = \int dt\, d^dx \sqrt{\overline{g}} \frac{1}{2} \overline{\phi}_a \mathcal{A}_{ab} \overline{\phi}_b, \tag{4.39}$$

and the partition function is

$$Z = \int \mathcal{D}\overline{\phi_1} \mathcal{D}\overline{\phi_2}\, e^{-\int dt\, d^dx \sqrt{\overline{g}} \frac{1}{2} \overline{\phi}_a \mathcal{A}_{ab} \overline{\phi}_b}. \tag{4.40}$$

Obviously the measure becomes

$$\mathcal{D}\overline{\phi_a} = \prod_{t,x} \prod_{a=1}^{2} (\overline{g}(x))^{1/4}\, d\overline{\phi}_a(t,x). \tag{4.41}$$

Now we have the same measure calculated both by Gaussian integration and by standard inner product. Thence the mismatch in (4.22) has gone thanks to the conformal transformation.

4.3. Free Energy

We proceed further to obtain an expression for free energy. In statistical mechanics, free energy $\mathcal{F}$ is given by [26]

$$\mathcal{F}(\beta) = -\frac{1}{\beta}\log Z(\beta). \tag{4.42}$$

The integral in (4.40) is a Gaussian and can be evaluated to yield free energy. However, Gaussian integral is well defined only if operator in the exponent is positive definite. Hence the operator $\mathcal{A}$ defined by (4.37) and (4.38) must be positive, or consequently $-c^{-2}\partial_0^2 + A - \mu^2 c^{-2}$ needs to be a positive operator. $-\partial_0^2$ is evidently a positive operator, and it has been shown that $A - \mu^2 c^{-2}$ is positive in some particular circumstances. [25] As a result of these circumstances, we get a condition for the chemical potential: The lowest eigenvalue ϵ_0 of the operator $c^2 A$ needs to satisfy $|\mu| < \epsilon_0$. This is an important requirement for the validity of rest of our analysis. Before attempting to integrate the Gaussian, let us rescale the fields so that $\overline{\phi}_a \to c\overline{\phi}_a$ for simplicity. Then (4.37) turns into

$$\mathcal{A} = \begin{pmatrix} -\partial_0^2 + c^2 A - \mu^2 & 2i\mu\partial_0 \\ -2i\mu\partial_0 & -\partial_0^2 + c^2 A - \mu^2 \end{pmatrix}. \tag{4.43}$$

In order to evaluate Gaussian integral, we should consider a unitary transformation which diagonalizes $\mathcal{A}$

$$D = \mathbf{U}^\dagger \mathcal{A} \mathbf{U} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{4.44}$$

Here λ_1, λ_2 are the eigenvalues of $\mathcal{A}$. The unitary matrix $\mathbf{U}$ which diagonalizes $\mathcal{A}$ can be determined via the eigenvectors of $\mathcal{A}$. The Gaussian integral of the form

$$\int d^d u \exp\left\{-\frac{1}{2}(u^T \mathcal{A} u)\right\}, \tag{4.45}$$

evolves into

$$\begin{aligned}\int d^d u' \exp\left\{-\frac{1}{2}\sum_i \lambda_i u_i^2\right\} &= \prod_i \lambda_i^{-1/2} \\ &= [\det(D)]^{-1/2}, \end{aligned} \tag{4.46}$$

up to a constant where $u' = \mathbf{U}^T u$. As the determinant is invariant under unitary transformation, we finally have

$$\int d^d u \exp\left\{-\frac{1}{2}(u^T \mathcal{A} u)\right\} = [\det(\mathcal{A})]^{-1/2}. \tag{4.47}$$

Combining all together, the Gaussian integration in (4.40) yields

$$Z = \left[\det\begin{pmatrix} -\partial_0^2 + c^2 A - \mu^2 & 2i\mu\partial_0 \\ -2i\mu\partial_0 & -\partial_0^2 + c^2 A - \mu^2 \end{pmatrix}\right]^{-1/2}. \tag{4.48}$$

The eigenvalues of operator $-\partial_0$ are called Matsubara frequencies and we denote them by $\omega_n$whereas ϵ_σ^2 are the eigenvalues of $c^2 A$. Note that these two operators commute with each other. Then the determinant (4.48) can be written as

$$Z = \left\{\prod_n \prod_\sigma \left[(\omega_n^2 + \epsilon_\sigma^2 - \mu^2)^2 + 4\mu^2\omega_n^2\right]^{-1/2}\right\}. \tag{4.49}$$

Since free energy expression (4.42) contains logarithm, it is useful to factorize Z:

$$\begin{aligned}(\omega_n^2 + \epsilon_\sigma^2 - \mu^2)^2 + 4\mu^2\omega_n^2 &= \omega_n^4 + (\epsilon_\sigma^2 - \mu^2)^2 + 2\omega_n^2\epsilon_\sigma^2 + 2\omega_n^2\mu^2 \\ &= \left[\omega_n^2 + (\epsilon_\sigma - \mu)^2\right]\left[\omega_n^2 + (\epsilon_\sigma + \mu)^2\right]. \end{aligned} \tag{4.50}$$

Finally we arrive at the free energy

$$\begin{aligned}\mathcal{F} &= -\frac{1}{\beta}\log Z = \frac{1}{2\beta}\sum_n\sum_\sigma \left\{\log[\omega_n^2 + (\epsilon_\sigma - \mu)^2] + \log[\omega_n^2 + (\epsilon_\sigma + \mu)^2]\right\} \\ &= \frac{1}{2\beta}\sum_n \left\{\log\det[\omega_n^2 + (c\sqrt{A} - \mu)^2] + \log\det[\omega_n^2 + (c\sqrt{A} + \mu)^2]\right\}. \end{aligned} \tag{4.51}$$

This result is very much alike the free energy expression in Minkowski case where we exactly know the density of states of A operator and therefore also its spectrum ϵ_σ. Nonetheless, in curved spacetime we do not know either. It would be comfortable to make an analysis of free energy using zeta function techniques.

4.4. Zeta Function Analysis

Begin with the spectral zeta function identity

$$\log \det O = -\frac{d}{ds}\left[\zeta_O(s)\right]_{s=0}, \tag{4.52}$$

where

$$\zeta_O(s) = \frac{1}{\Gamma(s)}\int_0^\infty dy\, y^{s-1}\mathrm{Tr}\, e^{-yO} = Tr\, O^{-s}, \tag{4.53}$$

for the positive operator O. We can express the terms in (4.51) as

$$\sum_n \log\det(\omega_n^2 + (c\sqrt{A}\pm\mu)^2) = -\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty dy y^{s-1} \times \sum_n e^{-y\omega_n^2}\mathrm{Tr}\, e^{-y(c\sqrt{A}\pm\mu)^2}\right]_{s=0}. \tag{4.54}$$

Secondly we use the Poisson summation

$$\sum_n e^{-4\pi^2\frac{s}{\beta^2}n^2} = \frac{\beta}{\sqrt{4\pi s}}\sum_n e^{-\frac{\beta^2 n^2}{4s}}. \tag{4.55}$$

Since Bosonic Matsubara frequency is given by $\omega_n = 2\pi n/\beta$, (4.54) produces

$$\frac{1}{2\beta}\sum_n \log\det(\omega_n^2 + (c\sqrt{A}\pm\mu)^2) = -\frac{1}{2\sqrt{4\pi}}\sum_n \frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty dy\, y^{s-\frac{3}{2}}e^{-\frac{\beta^2 n^2}{4y}} \times\mathrm{Tr}\, e^{-y(c\sqrt{A}\pm\mu)^2}\right]_{s=0}. \tag{4.56}$$

Notice that around $s = 0$ integration over y converges and it is also analytic. Then we can perform the s differentiation safely to yield

$$\frac{1}{2\beta}\sum_n \log\det(\omega_n^2 + (c\sqrt{A} \pm \mu)^2) = -\sum_n \int_0^\infty \frac{dy}{y^{3/2}} \frac{1}{2\sqrt{4\pi}} e^{-\frac{\beta^2 n^2}{4y}} \mathrm{Tr}\, e^{-y(c\sqrt{A}\pm\mu)^2}. \quad (4.57)$$

In the limit $T \to 0$ in above equation only $n = 0$ terms would exist and partition function becomes

$$\mathcal{F}(T=0) \;=\; -\int_0^\infty \frac{dy}{y^{3/2}} \frac{1}{2\sqrt{4\pi}} \left[\mathrm{Tr}\, e^{-y(c\sqrt{A}-\mu)^2} + \mathrm{Tr}\, e^{-y(c\sqrt{A}+\mu)^2}\right]. \quad (4.58)$$

Using the identity (4.53) we get

$$Tr\sqrt{O} = \zeta_O\left(-\frac{1}{2}\right) = \frac{1}{\Gamma\left(-\frac{1}{2}\right)} \int_0^\infty \frac{dy}{y^{3/2}} \mathrm{Tr}\, e^{-yO}. \quad (4.59)$$

Hence

$$\begin{aligned} \mathcal{F}(T=0) &= \frac{1}{2}\left[\mathrm{Tr}\,(c\sqrt{A}-\mu) + \mathrm{Tr}\,(c\sqrt{A}+\mu)\right] \\ &= \mathrm{Tr}\,(c\sqrt{A}) = \sum_\sigma \epsilon_\sigma. \end{aligned} \quad (4.60)$$

This result is well expected since as $T \to 0$ the thermal excitation between the states disappears and the total free energy becomes just a sum over the spectrum. On the other hand for $T \neq 0$ case, the contribution to the free energy comes from $n \neq 0$ terms in the summation. Let us denote this contribution by $\mathcal{F}' = \mathcal{F} - \mathcal{F}(T=0)$ and introduce subordination identity [35]

$$e^{-b\sqrt{x}} \;=\; \frac{b}{\sqrt{4\pi}} \int_0^\infty \frac{dy}{y^{3/2}}\, e^{-\frac{b^2}{4y}}\, e^{-yx}. \quad (4.61)$$

Substitution in (4.57) yields the following for $\mathcal{F}'$

$$\frac{1}{\beta}\sum_{n=1}^\infty \log\det(\omega_n^2 + (c\sqrt{A}\pm\mu)^2) = -\frac{1}{\beta}\sum_{n\neq 0} \frac{1}{n} \mathrm{Tr}\, e^{-\beta n(c\sqrt{A}\pm\mu)}. \quad (4.62)$$

In fact since the contribution coming from $T \to 0$ is constant at β, μ or other thermodynamic variables, in the derivation of thermodynamic quantities such as number of particles or entropy, which include derivatives with respect to these variables, we can safely replace $\mathcal{F}'$ by $\mathcal{F}$ and the thermodynamics does not change. Thus free energy is

$$\begin{aligned} \mathcal{F} &= -\frac{1}{\beta}\sum_{n\neq 0}\frac{1}{n}\left[\operatorname{Tr} e^{-\beta n(c\sqrt{A}-\mu)} + \operatorname{Tr} e^{-\beta n(c\sqrt{A}+\mu)}\right] && (4.63)\\ &= \frac{1}{\beta}\left[\operatorname{Tr}\log(1-\ e^{-\beta(c\sqrt{A}-\mu)}) + \operatorname{Tr}\log(1-\ e^{-\beta(c\sqrt{A}+\mu)})\right]. && (4.64) \end{aligned}$$

This is the expected expression for Free Energy. [27, 31] Alternatively again using subordination identity (4.61) and (4.63) we can write

$$\begin{aligned} \mathcal{F} &= \sum_{n=1}^{\infty}\left[(e^{n\beta\mu} + e^{-n\beta\mu})\right]\frac{c}{\sqrt{4\pi}}\int_0^{\infty}\frac{du}{u^{3/2}}e^{\frac{(nc\beta)^2n^2}{4u}}\operatorname{Tr} e^{-uA}\\ &= \sum_{n=1}^{\infty}\left[(e^{n\beta\mu} + e^{-n\beta\mu})\right]\frac{c}{\sqrt{4\pi}}\int_0^{\infty}\frac{du}{u^{3/2}}e^{-m^2c^2\left(\frac{(\beta m^{-1})^2n^2}{4u}+u\right)}\operatorname{Tr} e^{-u(-\Delta_\gamma+U)}. \end{aligned} \tag{4.65}$$

This last expression will be very useful while taking the nonrelativistic limit for free energy in the next chapter. Moreover (4.63) is common in literature in expanding free energy in the ultrarelativistic limit [8, 9].

5. NONRELATIVISTIC LIMIT

In the previous section we have derived an expression for the free energy of Bose field in a static spacetime making use of a conformal transformation from static metric into an ultrastatic one. In this section we investigate the nonrelativistic limit of thermodynamics as this limit is the main area of research in this thesis. We take the nonrelativistic limit of free energy (4.65) in such a way that only the matter is made nonrelativistic but the background geometry stays unchanged. This can be achieved by imposing large but finite c limit and investigate the large c asymptotics of free energy via *generalized Laplace method.*

Let us start with free energy expression (4.65) and write the trace term in the path integral representation as (see (3.20),(3.21) and (3.22))

$$Tr\, e^{-u(-\Delta_\gamma+U)} \quad = \quad \int \mathcal{D}x e^{-s(u,c)}, \tag{5.1}$$

where the integration is over closed paths and

$$s(u,c) = \int_0^u d\tau \left[\frac{1}{4}\gamma_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + U(x(\tau))\right]. \tag{5.2}$$

As we are investigating large c limit, all the factors of c are shown explicitly throughout this chapter. Then (4.65) can be written as

$$\mathcal{F} = \sum_{n=1}^{\infty}[2\cosh(n\beta\mu)]\int\mathcal{D}x\int_0^\infty \frac{cdu}{u^{3/2}}\frac{1}{\sqrt{4\pi}}e^{-m^2c^2\left(\frac{(\beta m^{-1})^2n^2}{4u}+u\right)}e^{-s(u,c)}. \tag{5.3}$$

Here we have an integral over u of the following form

$$\int_0^\infty \frac{cdu}{\sqrt{4\pi}u^{3/2}}\, e^{-s(u,c)}e^{-c^2r(u)}, \tag{5.4}$$

with

$$r(u) = m^2 \left(\frac{n^2(\beta m^{-1})^2}{4u} + u \right). \tag{5.5}$$

For large c, integral (5.4) can be evaluated using Laplace method: Consider the integral

$$\int_a^b dx\, g(x)\, e^{-\alpha f(x)}. \tag{5.6}$$

Assume that $g(x)$ is a smooth function and $f(x)$ is differentiable at least up to second order, i.e. $f'(x)$ and $f''(x)$ are well defined in the interval (a, b). When α is large, the integral above can be approximated by Laplace method as follows [36]: Expand $f(x)$ by Taylor series up to the second order around its global minimum x_0 in the interval (a, b):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + O(x^3) \tag{5.7}$$

Since α is large, most of the contribution to the integral (5.6) originates from the neighborhood of the point x_0 and this point is given by $f'(x_0) = 0$, which means the second term in Taylor expansion vanishes. Thus

$$\begin{aligned} \int_a^b dx\, g(x)\, e^{-\alpha f(x)} \quad &\approx \quad e^{-\alpha f(x_0)}\, g(x_0) \int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha}{2} f''(x_0)(x-x_0)^2} \\ &\quad + e^{-\alpha f(x_0)}\, g'(x_0) \int_{-\infty}^{\infty} dx (x - x_0)\, e^{-\frac{\alpha}{2} f''(x_0)(x-x_0)^2}. \end{aligned} \tag{5.8}$$

Note that we have replaced (a, b) interval by $(-\infty, \infty)$ as the peak of the integrand is concentrated around x_0 and therefore we only get a negligible error by doing so. Moreover $g(x)$ is a smooth function and approximated linearly around x_0. Since x_0 is a minimum, $f''(x_0)$ is positive, hence the first integral in (5.8) is clearly a Gaussian whereas the second one vanishes. Evaluating the Gaussian we end up with

$$\int_a^b dx\, g(x)\, e^{-\alpha f(x)} \sim e^{-\alpha f(x_0)}\, g(x_0) \sqrt{\frac{2\pi}{\alpha f''(x_0)}}. \tag{5.9}$$

We can use this result in order to attain the large c asymptotics of free energy (5.3). First start with the ultrastatic spacetime where the metric is

$$ds^2 = -c^2 dt^2 + h_{ij}(x) dx^i dx^j, \tag{5.10}$$

where $F = 1$ and therefore $\gamma_{ij} = h_{ij}$. Metric h_{ij} generally does not depend on c, hence we may write $s(u, c) = s(u)$. Apply Laplace method with

$$g(u) = \frac{c}{\sqrt{4\pi} u^{3/2}} e^{-s(u)}, \tag{5.11}$$

and

$$f(u) = r(u) \quad , \quad u_0 = \overline{u} \quad , \quad \alpha = c^2. \tag{5.12}$$

Using the result in (5.9), we get

$$\int \frac{c du}{\sqrt{4\pi} u^{3/2}} e^{-s(u)} e^{-c^2 r(u)} \sim \frac{c}{\sqrt{4\pi} \overline{u}^{3/2}} e^{-s(\overline{u})} e^{-c^2 r(\overline{u})} \sqrt{\frac{2\pi}{c^2 r''(\overline{u})}}, \tag{5.13}$$

for the terms in (5.3) in the large c limit. We can find the saddle point by differentiating $r(u)$ defined by (5.5)

$$\frac{dr(u)}{du} = \frac{d}{du} \left[\frac{(n\beta m^{-1})^2}{4u} + u \right] = 0. \tag{5.14}$$

Solving this equation we find

$$\overline{u} = \frac{n\beta}{2m}, \tag{5.15}$$

as the saddle point. Hence

$$r(\overline{u}) = n\beta m, \qquad r''(\overline{u}) = \frac{(n\beta)^2}{2\overline{u}^3}. \tag{5.16}$$

We can verify by inserting $\overline{u}$ above that $r''(\overline{u})$ is positive, which fits our initial assumption that $\overline{u}$ is a minimum. Combining all we get

$$\begin{aligned}\int \mathcal{D}x \int \frac{cdu}{\sqrt{4\pi}u^{3/2}}\, e^{-s(u)}e^{-c^2 r(u)} &\sim \frac{e^{-n\beta mc^2}}{n\beta}\int \mathcal{D}x e^{-s(\overline{u})} \\ &\sim \frac{e^{-n\beta mc^2}}{n\beta} Tr\, e^{-n\beta\left[\frac{1}{2m}(-\Delta_h+U)+mc^2\right]}. \end{aligned} \tag{5.17}$$

Now if we look at (4.65), the first factor represents both particle and antiparticle contributions to free energy. For reasons that will be discussed at the end of the chapter we accept that there is no particle-antiparticle creation and we can treat the thermodynamics of them individually. In fact here we think of a background which is not strong enough to enable pair production. [37] Thus, free energy of particles in the nonrelativistic limit is

$$\mathcal{F}_{NR} = -\sum_{n=1}^{\infty} \frac{1}{n\beta}\, Tr\, e^{-n\beta(L-\mu)}, \tag{5.18}$$

with the operator

$$L = \frac{1}{2m}(-\Delta_h + U) + mc^2, \tag{5.19}$$

whose eigenvalues of are λ_σ. Then

$$\mathcal{F}_{NR} = \frac{1}{\beta}\sum_{\sigma} \log(1 - e^{-\beta(\lambda_\sigma - \mu)}). \tag{5.20}$$

Free energy for antiparticles can be obtained in the same way. Notice that the condition $\mu < \lambda_0$ should be met by the chemical potential μ where λ_0 is the lowest eigenvalue of L, which is indeed the Schrödinger operator H_{NR} for the nonrelativistic limit of the Klein-Gordon (KG) equation in an ultrastatic spacetime. Nevertheless, we are interested in the nonrelativistic limit of Bose gas and its thermodynamics in a static spacetime where taking the nonrelativistic limit of KG equation is not straightforward. [13, 14]

In order to study the static case let us restate the *metric* concept. A *metric* of a spacetime is a solution of *Einstein Field Equations* (EFE)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{5.21}$$

first introduced by Einstein in 1916 [38] with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{5.22}$$

being Einstein tensor. Here $R_{\mu\nu}$ is the Ricci curvature tensor, $T_{\mu\nu}$ is the stress–energy tensor and Λ is the cosmological constant. [17, 39] Newton's gravitational constant is taken as $G = 1$ in natural units. In fact metric determines the structure of the spacetime, however not all the field equations can be solved to yield a metric. Exactly solvable ones of (5.21) generally include symmetries. Let us consider the gravitational field of a slowly rotating spherical mass with neutral electrical charge and let $\Lambda = 0$. Such a system is a good model for the gravitational field outside an heavy astronomical object, for example the Sun. The solution in that case is known as the Schwarzschild metric and given by

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.23}$$

where r_s is called the Schwarzschild radius

$$r_s = \frac{2MG}{c^2}. \tag{5.24}$$

Time lapse for Schwarzschild metric is then

$$.F = 1 - \frac{2GM}{c^2 r} \tag{5.25}$$

We can again make use of Laplace method to take the nonrelativistic limit. Yet, we should first observe the behavior of $s(u, c)$ since it does depend on c now as (5.2) contains optical metric γ_{ij} corresponding to (5.23) which can be written as

$$\gamma_{ij} = \mathrm{diag}\left(\frac{1}{\left(1-\frac{r_s}{r}\right)^2}, \frac{r^2}{\left(1-\frac{r_s}{r}\right)}, \frac{r^2 \sin^2\theta}{\left(1-\frac{r_s}{r}\right)} \right). \tag{5.26}$$

Expanding γ_{ij} in the large c limit by the series

$$\frac{1}{(1-x)^2} = \sum_{i=1}^{\infty} n x^{n-1}, \tag{5.27}$$

the order in spherical coordinates is found to be

$$\gamma_{ij} = \mathrm{diag}\left(1+O(c^{-2}), r^2[1+O(c^{-2})], r^2\sin^2\theta[1+O(c^{-2})]\right), \tag{5.28}$$

or alternatively in Cartesian coordinates we have

$$\gamma'_{ij} = \delta_{ij} + O(c^{-2}). \tag{5.29}$$

The scalar curvature of optical metric is

$$R_\gamma = -\frac{6M^2}{c^4 r^4} = O(c^{-4}), \tag{5.30}$$

and from (5.25) time lapse is $F = 1 + O(c^{-2})$. Therefore

$$(F-1)(mc)^2 = 2m^2\Phi = O(c^0), \tag{5.31}$$

with $\Phi = -M/r$ being the usual Newtonian gravitational potential ($G = 1$). For Schwarzschild metric, Ricci curvature tensor can be calculated using Christoffel symbols to yield $R_{\mu\nu} = 0$ and so the scalar curvature is $R = 0$. [39] Combining (5.30), (5.31) and $R = 0$ with (4.35) we get

$$U = O(c^{-4}) \tag{5.32}$$

Together with (5.29)

$$s(u,c) = \int d\tau \left[\frac{1}{4}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + 2m^2\Phi\right] + O(c^{-2}) = s(u, c=\infty) + O(c^{-2}). \qquad (5.33)$$

Compare the orders of $s(u,c)$ and $c^2r(u)$. Obviously $c^2r(u)$ term is of higher order in c. Then the integral is concentrated in the neighborhood of $\overline{u}$ as it is also the case in ultrastatic metric, and consequently we are safe to use generalized Laplace method in the same manner as we used to approximate the integral in (5.13) [36]:

$$\int \frac{c du}{\sqrt{4\pi}u^{3/2}} e^{-s(u,c)} e^{-c^2 r(u)} \sim \frac{c}{\sqrt{4\pi}\overline{u}^{3/2}} e^{-s(\overline{u},c)} e^{-c^2 r(\overline{u})} \sqrt{\frac{2\pi}{c^2 r''(\overline{u})}} \qquad (5.34)$$

This is the same result as (5.13) in ultrastatic case except the term $s(u,c)$ which contains the metric and c dependence in it. An important distinction here should be made clear while taking the large c limit. If we take c directly to infinity as $c = \infty$, Einstein Field Equations (5.21) reduce to Newton's law of gravitation and this corresponds to a weak field approximation, similar to the one in [18]. We can observe this in (5.33) by substituting $s(u,c)$ for $s(u, c=\infty)$. Then we would be left with the thermodynamics of classical noninteracting Bose gas governed by an external Newtonian potential $m\Phi$. In this case we loose the geometry and turn back to flat space which is something we want to stay away from.

Instead, we prefer a post-Newtonian approach where the geometry stays unchanged while taking large but finite c limit. In this context (5.34) can be thought as an improved asymptotics for free energy where the factors of c are preserved up to order $O(c^{-1})$. In fact only the matter is taken to the nonrelativistic limit, which means we have "heavy" particles with small velocities, but still a curved space.

One other consequence of our method is that although we have taken Schwarzschild metric into account in arguments above, large c asymptotics in (5.34) can be proven to be true also for other static metrics satisfying (5.31).

Again we define the operator L

$$L \;=\; \frac{1}{2m}\left(-\Delta_\gamma + U\right) + mc^2, \tag{5.35}$$

which has a similar appearance with (5.19). Of course the spectrum $\{\lambda_\sigma\}$ has changed now as Laplacian and potential terms differ from the ultrastatic case. So for free energy we have

$$\mathcal{F}_{NR} \;=\; -\sum_{n=1}^{\infty}\frac{1}{n\beta}\,Tr\,e^{-n\beta(L-\mu)} = \frac{1}{\beta}Tr\log\left[1-e^{-\beta(L-\mu)}\right]. \tag{5.36}$$

This is the same expression as (5.18) with different operators of L as expected in the nonrelativistic limit. Thus, we can conclude that the thermodynamics of Bose gas in static spacetime is ruled by operator L constructed in optical metric. However, unlike the ultrastatic case, this time the operator L does not directly coincide with the Schrödinger operator H_{NR} in the nonrelativistic limit of Klein-Gordon equation. In order to construct the connection between the two, let us write down Klein-Gordon equation

$$(-\Delta_g + \xi R + (mc)^2 + V_{ext})\phi = 0. \tag{5.37}$$

For ultrastatic case where $F = 1$, we can expand the terms and express this as

$$-\partial_0^2\phi = (-\Delta_h + \xi R + V_{ext} + (mc)^2)\phi, \tag{5.38}$$

or

$$-\partial_0^2\phi = (-\Delta_h + U + (mc)^2)\phi, \tag{5.39}$$

with $U = V - (mc)^2 = \xi R + V_{ext}$. Nonrelativistic limit may be obtained by formally taking the square root of the operator on the right hand side of the equation above and expanding the result in powers of $(mc)^{-2}$. As Klein-Gordon equation is derived by taking the square of the energy operator in Schrödinger equation, what we get would be

$$i\partial_0\phi = H_{NR}\phi, \tag{5.40}$$

Schrödinger equation itself. Taking the square root of (5.38) we have

$$\begin{aligned}\sqrt{c^2[-\Delta_h + \xi R + V_{ext} + (mc)^2]} &= mc^2\sqrt{\frac{-\Delta_h + \xi R + V_{ext}}{(mc)^2} + 1} \\ &= mc^2\left[1 + \frac{1}{2}\frac{-\Delta_h + \xi R + V_{ext}}{(mc)^2}\right].\end{aligned} \tag{5.41}$$

Then we arrive at operator H_{NR} in (5.40)

$$H_{NR} = \frac{1}{2m}(-\Delta_h + U) + mc^2. \tag{5.42}$$

Note that as expected, this result is identical to the operator L defined in (5.19). Now let us try to apply the same steps to a general static metric for which the explicit form of KG Equation is

$$-\partial_0^2\phi = -\left(\frac{F}{\sqrt{|g|}}\partial_i\sqrt{|g|}h^{ij}\partial_j + F(mc)^2 + \xi FR + FV_{ext}\right)\phi = 0. \tag{5.43}$$

Unlike the ultrastatic case, we are unable to proceed further because not all the terms in (5.43) commute with each other and one cannot simply multiply them to expand in series. As mentioned before taking nonrelativistic limit of KG equation is not straightforward in static spacetimes. Nevertheless, generalized Laplace method utilized to derive large c asymptotics make it fortunately uncomplicated. Let us write the right hand side of (5.43) as

$$c^2\left[-\frac{F}{\sqrt{|g|}}\partial_i\sqrt{|g|}h^{ij}\partial_j + (F-1)(mc)^2 + \xi RF + (mc)^2\right]. \tag{5.44}$$

Making use of (5.31) we may conclude that all the terms other then $(mc)^2$ have negative powers of c and can be represented by $O(c^0)$ and now the term $(mc)^2$ commutes with other terms. Then taking the square root of this expression and again expanding in $(mc)^{-2}$ we get the Schrödinger operator

$$H_{NR} = mc^2 + \frac{1}{2m}\left[-\frac{F}{\sqrt{|g|}}\partial_i\sqrt{|g|}h^{ij}\partial_j + U\right] + \dots . \tag{5.45}$$

$$\tag{5.46}$$

Define a new operator A_1

$$A_1 = \left[-\frac{F}{\sqrt{|g|}}\partial_i\sqrt{|g|}h^{ij}\partial_j + FV\right] . \tag{5.47}$$

Then (5.45) becomes the root of this expression and again expanding in $(mc)^{-2}$ we get the Schrödinger operator

$$H_{NR} = mc^2 + \frac{1}{2m}\left(c^{-2}A_1 - (mc)^2\right) + O((c^{-2}A_1)^2), \tag{5.48}$$

and one can show that A_1 can be obtained from A given in (4.38) by a similarity transformation $A = F^{\frac{d-1}{4}}A_1F^{-\frac{d-1}{4}}$. Applying the same similarity transformation to H_{NR} we get the operator L defined by (5.35):

$$F^{\frac{d-1}{4}}H_{NR}F^{-\frac{d-1}{4}} = mc^2 + \frac{1}{2m}\left(c^{-2}A - (mc)^2\right) = mc^2 + \frac{1}{2m}\left(-\Delta_\gamma + U\right) = L. \tag{5.49}$$

Therefore in contrast to the ultrastatic case, in static spacetimes L does not correspond to Schrödinger operator H_{NR} in the nonrelativistic limit, but they are connected via a similarity transformation.

Remember that all the results above are derived with the assumption that there is no pair production due to weakness of the field to create it, otherwise, free energy in (4.65) would not have been reduced to (5.18) or (5.36). Although we do not take a weak field approach, we should explain quantitatively why pair production does not occur. According to (5.33) coupling energy between the matter and the geometry is clearly $m\Phi + O(c^{-2})$. However for pair production this energy must be at least $2mc^2$. If we confine Bose gas in a volume with inner radius r_1 and outer radius r_2 with the gravitating object at the center, then $|m\Phi(r_1)| + O(c^{-2}) < 2mc^2$ in the nonrelativistic limit (large c), as $\Phi \ll c^2$ except for large $|m\Phi(r_1)| = |-mM/r_1|$. If $|m\Phi(r_1)|$ is large, then either we are too close to a gravitation center (small r_1) or the source of gravitation is supermassive (large M). In fact, we assume that the gravitating object is not large enough to create pair production and we confine our system far away from horizon, i.e. r_1 is much larger than r_s. Therefore, one can safely ignore pair production and treat particles and antiparticles separately. One other question at this point may arise for horizon divergences which appear in the expressions of thermodynamic quantities as one gets closer to horizon [19]. Yet, thanks to our assumption that the inner boundary r_1 of quantum field is away from horizon, we should not have any divergences. To define it formally, we may think of a spherical shell B with inner and outer radii r_1 and r_2 respectively, at the center of which a gravitating object is present.

Concluding this chapter, let us remind that the particle number can be derived from free energy by differentiating it with respect to chemical potential as

$$N = -\frac{\partial \mathcal{F}_{\mathcal{NR}}}{\partial \mu} = \sum_{\sigma} \frac{1}{e^{\beta(\lambda_\sigma - \mu)} - 1}. \tag{5.50}$$

The last expression is clearly Bose-Einstein distribution defined in Chapter 2.

6. ASYMPTOTIC EXPANSION OF FREE ENERGY

We commence by rearranging the terms in free energy expression (5.36)

$$\begin{aligned}\mathcal{F}_{NR} &= -\sum_{n=1}^{\infty}\frac{1}{n\beta}\,Tr\,e^{-n\beta(L-\mu)} \\ &= -\sum_{n=1}^{\infty}\frac{1}{n\beta}\,Tr\,e^{-n\beta(\lambda_0-\mu)}e^{-n\beta(\lambda_0-\mu)\frac{(L-\lambda_0)}{\lambda_0-\mu}}.\end{aligned} \tag{6.1}$$

Introducing a new operator

$$\widetilde{L} = \frac{(L-\lambda_0)}{\lambda_0-\mu}, \tag{6.2}$$

we have

$$\mathcal{F}_{NR} = -\sum_{n=1}^{\infty}\frac{1}{n\beta}\,Tr\,e^{-n\beta(\lambda_0-\mu)}e^{-n\beta(\lambda_0-\mu)\widetilde{L}}. \tag{6.3}$$

Note that we drop the subscript NR and write solely $\mathcal{F}$ from now. Let us define the expansion parameter as $x = \beta(\lambda_0-\mu)$. Then one can see that $\mathcal{F}$ is in the form of an harmonic sum

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} h(nx), \tag{6.4}$$

where

$$h(x) = -(\lambda_0-\mu)\,\frac{e^{-x}}{x}\;Tr\,e^{-x\widetilde{L}}. \tag{6.5}$$

An efficient way of investigating small x asymptotics of this harmonic sum is to use *Mellin transform* [12].

6.1. Mellin Transform

Let $g(x)$ be an locally integrable function of a real and positive variable x in the interval $(0, \infty)$. The Mellin transform of $g(x)$ denoted by $\mathcal{MT}\{g(x)\}$ or $\widetilde{g}(s)$ is given as

$$\widetilde{g}(s) = \mathcal{MT}\{g(x)\} = \int_0^\infty dx\, x^{s-1} g(x), \quad s \in \mathbf{C}, \tag{6.6}$$

where s is a complex variable and is restricted for the values where the integral converges. In order to locate those values one should look at the analytic structure of $g(x)$ at the lower and upper bounds. At the lower, the integral is convergent only if the real part of s is greater than for some particular value a and at the lower only if it is smaller than some other value b with $a < b$. In a more formal way, Mellin transform is defined only in a vertical strip $a < Re\{s\} < b$ in the complex plane and $\widetilde{g}(s)$ is an analytical function of s in this strip called the strip of analyticity of the transform $\mathcal{MT}\{g(x)\}$. The largest such strip where is called the *fundamental strip.* [40,41] Note that this strip can also be an half plane.

Let us explore some properties of Mellin transform. Some properties follow from other related transforms such as Laplace and Fourier transform. The connection between these two and Mellin transform can be established by making simple change of variables. For instance $x \to e^{-ix}$ would turn out Fourier transform ($\mathcal{FT}$)

$$\mathcal{FT}\{g(x)\} = \int_{-\infty}^\infty dx\, e^{-ixs} g(x), \tag{6.7}$$

and $x \to e^{-x}$ would give (bilateral) Laplace Transform ($\mathcal{LT}$)

$$\mathcal{LT}\{g(x)\} = \int_{-\infty}^\infty dx\, e^{-xs} g(x). \tag{6.8}$$

Therefore the inverse Mellin transform can be deduced from inverse Laplace transform

$$g(x) = \mathcal{LT}^{-1}\{g(s)\} = \frac{1}{2\pi i}\int_{\delta-\infty}^{\delta+\infty} ds\, e^{xs} g(s), \tag{6.9}$$

and the result is

$$g(x) = \mathcal{MT}^{-1}\{g(s)\} = \frac{1}{2\pi i}\int_{\delta-\infty}^{\delta+\infty} ds\, x^{-s} g(s). \tag{6.10}$$

The integration path constitutes a line in the imaginary direction in the complex plane, and it is inside the strip of analyticity. One can transform from $g(z)$ using the inversion formula (6.10) and reach $g(x)$ uniquely. [42] Note that the inversion formula is also analogous to that of Fourier transform.

The first and the most trivial property of Mellin transform is its linearity, i.e. $\mathcal{MT}\{a\, g(x)\} = a\, \mathcal{MT}\{g(x)\}$, which is obvious from its definition. One important property is called rescaling rule. If we make a change of variable $x \to \alpha x$, then the Mellin transform of $g(\alpha x)$ is

$$\mathcal{MT}\{g(\alpha x)\} = \alpha^{-s}\, \mathcal{MT}\{g(x)\}. \tag{6.11}$$

This can be verified easily by changing variable in (6.6). Mellin transform is especially useful in evaluating integrals including harmonic sums as

$$G = \sum_n \chi_n\, g(\alpha_n x), \tag{6.12}$$

where $g(x)$ is the base function, χ_n and α_n are named as amplitudes and frequencies respectively. If we take the Mellin transform of this harmonic sum, making use of the rescaling rule (6.11) and the linearity property gives the separation property:

$$\mathcal{MT}\{G(x)\} \;=\; \sum_n \chi_n \alpha_n^{-s}\, \mathcal{MT}\{g(x)\} \tag{6.13}$$

As can be seen from above equation, amplitude and frequency terms are separated from the base function $g(x)$ when Mellin transform of an harmonic series is evaluated. Setting $\chi_n = 1$ and $\alpha_n = n$ we get the *Riemann-Zeta function*:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{6.14}$$

There are other useful functions which can be generated by Mellin transform. For instance if we take the Mellin transform of e^{-x}, what we get will be the definition of *Gamma function*:

$$\Gamma(s) = \mathcal{MT}\{e^{-x}\} = \int_0^{\infty} dx\, x^{s-1} e^{-x} \tag{6.15}$$

By the definition of Mellin transform, for the integral to be convergent at x=0, we require $Re\{s\} > 0$ and no other restriction is necessary. Therefore, the domain of the gamma function is then the whole half complex plane on the right hand side. Now divide the integral in (6.15) as

$$\Gamma(s) = \int_0^{1} dx\, x^{s-1} e^{-x} + \int_1^{\infty} dx\, x^{s-1} e^{-x}. \tag{6.16}$$

Expand the exponential in the first integral and then integrate to get

$$\begin{aligned} \Gamma(s) &= \left(\frac{x^s}{0!\,(s)} - \frac{x^{s+1}}{1!\,(s+1)} + \frac{x^{s+2}}{2!\,(s+2)} + ... \right) + \int_1^{\infty} dx\, x^{s-1} e^{-x} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{s+n} + \int_1^{\infty} dx\, x^{s-1} e^{-x}. \end{aligned} \tag{6.17}$$

The right hand side of (6.17) is evidently meromorphic with poles at nonpositive integers and in fact (6.17) constitutes the analytic continuation of (6.15) from the right hand side of complex plane to the whole. Moreover, the singular part of the gamma function is straightforward. Near the poles $z = 0, -1, ..., -n, ...$ most of the contribution to gamma function comes from the first term of (6.17). Thus the singular part of

$\Gamma(x)$ is

$$\Gamma(s) \asymp \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{1}{s+l}, \tag{6.18}$$

where $\asymp$ stands for to the singular part of $\Gamma(s)$. When working with Mellin transforms gamma function is often encountered and it plays an important role in deriving the asymptotic behavior of free energy as we will see shortly. A crucial feature of Mellin transform is as follows:

If a function $g(x)$ has a meromorphic extension with the singular part

$$\widetilde{g}(s) \asymp \sum_{w,k} \frac{res(w,k)}{(s-w)^{k+1}}, \tag{6.19}$$

(as $\Gamma(s)$ does) then small x asymptotics of $g(x)$ is given by the dictionary [41, 43, 44]

$$g(x) \sim \sum_{w,k} res(w,k) \frac{(-1)^k}{k!} x^{-w} (\log x)^k. \tag{6.20}$$

which means the behavior of the asymptotic expansion of the function is solely determined by the poles of its Mellin transform in complex plane.

Now we can come back to free energy and take its Mellin transform. Since $\mathcal{F}$ in the form of an harmonic sum (6.4) we have

$$\widetilde{\mathcal{F}}(s) = \zeta(s)\widetilde{h}(s), \tag{6.21}$$

with zeta function (6.14) and

$$\widetilde{h}(s) = -(\lambda_0 - \mu) \int_0^{\infty} dx\, x^{s-2}\, e^{-x} Tr\, e^{-x\widetilde{L}}. \tag{6.22}$$

The trace term above can be expended via *Heat Kernel* expansion of $\widetilde{L}$.

6.2. Heat Kernel Expansion

Let us start with the *heat diffusion equation*

$$(\partial_t - \Delta)u(x,t) = 0, \tag{6.23}$$

where $u(x,t)$ is the heat density (or alternatively temperature). The equation involves both time derivative ∂_t and Laplacian Δ of the heat density. In other words heat diffusion equation states that time evolution of the heat density depends on its curvature. The fundamental solution to this equation with the initial condition

$$\lim_{t \to 0} K(x, x', t) = \delta(x - x'), \tag{6.24}$$

is called *heat kernel*. In Euclidean space $\mathbb{R}^d$ of d-dimensions, for $x \in X$ with $X \subset \mathbb{R}^d$ heat kernel is given by [45]

$$K(x, x', t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-x'|^2}{4t}}. \tag{6.25}$$

The limit in (6.24) should be handled with care. Since $\delta(x - x')$ is a distribution, then the limit can be expressed by a test function $f_t(x')$ as

$$\lim_{t \to 0} \int_{\mathbf{R}^d} K(t, x, x') f_t(x')\, dx' = f_t(x) = u(x, 0), \tag{6.26}$$

where the test function is found to be the initial condition $u(x,0)$ for all points x at time $t = 0$. One can interpret this solution as follows. At time $t = 0$ a sudden heat increase at point x' starts to diffuse along the x-axis and (6.24) along with (6.26) refers to initial condition for all points along the axis at $t = 0$. Then the time evolution of heat density is given by the convolution

$$u(x,t) = \int_{\mathbf{R}^d} K(t, x, x') u(x', t)\, dx' \quad \text{for} \quad t > 0. \tag{6.27}$$

The above equation gives the solution of heat equation in flat space without spatial boundaries. On the other hand we are interested in a finite volume in curved spacetime. In the presence of boundary conditions, the solution is given by short time asymptotics of heat kernel and is also called heat kernel expansion. It is an important tool in mathematical physics and also a widely used one for calculations involving the spectrum of Laplace operator.

First let us introduce the spectral representation of heat kernel on a Riemann manifold M

$$Tre^{t\Delta} = \sum_{\sigma} e^{-t\epsilon_\sigma}, \tag{6.28}$$

for the static metric and associated Laplacian Δ whose spectrum is ϵ_σ. In the form of integral transform heat kernel is written as

$$Tre^{t\Delta} = \int_M d^d x' K(t,x,x'). \tag{6.29}$$

When the small t limit is considered, we have the short time asymptotics of above equation by [46]

$$\int_M d^d x' K(t,x,x') \sim a_0 t^{-\frac{d}{2}} + a_1 t^{-\frac{d-1}{2}} + a_2 t^{-\frac{d-2}{2}} + \ldots, \tag{6.30}$$

where as are coefficients to be defined. Combining all we can write the heat kernel expansion for the optical metric as

$$Tr\, e^{-t(-\Delta_\gamma + U)} = \sum_{j=0}^{\infty} a_{j/2} t^{\frac{j-d}{2}}, \tag{6.31}$$

and $a_{j/2}$ are called heat kernel coefficients corresponding to the operator

$$-\Delta_\gamma + U. \tag{6.32}$$

First few of $a_{j/2}$ are given below [47, 48]

$$\begin{aligned} a_0 &= \frac{1}{(4\pi)^{d/2}} \int_B dB_\gamma, \\ a_{1/2} &= \pm \frac{1}{4(4\pi)^{\frac{d-1}{2}}} \int_{\partial B} d(\partial B_\gamma), \\ a_1 &= \frac{1}{6(4\pi)^{d/2}} \left[\int_B dB_\gamma(-6U + R_\gamma) + 2 \int_{\partial B} d(\partial B_\gamma)\, K \right] \end{aligned} \tag{6.33}$$

Here R_γ is Ricci scalar and K is the extrinsic curvature of ∂B and $a_{1/2}$ has positive sign for Neumann boundary condition and negative sign for Dirichlet boundary condition.

Now we turn back to free energy. Trace term in (6.22) can be expanded as

$$Tr\, e^{-x\widetilde{L}} = e^{-x'\left(2m^2c^2 - 2m\mu\right)} Tr\, e^{-\frac{\beta}{2m}(-\Delta_\gamma + U)}, \tag{6.34}$$

where

$$x' = \frac{\beta}{2m}. \tag{6.35}$$

We can expand the trace in (6.34) via heat kernel

$$Tr\, e^{-x'(-\Delta_\gamma + U)} = \sum_{j=0}^{\infty} a_{j/2} x'^{\frac{j-d}{2}}, \tag{6.36}$$

with the coefficients $a_{j/2}$ given by (6.33). Then the heat kernel expansion of operator $\widetilde{L}$ is

$$Tre^{-x\widetilde{L}} \sim \sum_{j=0}^{\infty} \widetilde{a}_{j/2} x^{\frac{j-d}{2}}. \tag{6.37}$$

Now $\widetilde{a}_{j/2}$ are the corresponding heat kernel coefficients. Again the actual coefficients can be calculated via [46]. Then we have

$$\begin{aligned} \widetilde{h}(s) &= -(\lambda_0 - \mu) \sum_{j=0}^{\infty} \widetilde{a}_{j/2} \int_0^{\infty} dx\, x^{s-2+\frac{j-d}{2}}\, e^{-x} & (6.38) \\ &= -(\lambda_0 - \mu) \sum_{j=0}^{\infty} \widetilde{a}_{j/2} \Gamma\left(s - 1 + \frac{j-d}{2}\right). & (6.39) \end{aligned}$$

The singular part of the gamma function is given by (6.18). Combining with $\widetilde{h}(s)$ we have

$$\widetilde{h}(s) \asymp -(\lambda_0 - \mu) \sum_{j,l} \widetilde{a}_{j/2} \frac{(-1)^l}{l!} \frac{1}{s - 1 + \frac{j-d}{2} + l}. \tag{6.40}$$

The integral in (6.38) is convergent except the lower limit, in the neighborhood of $x = 0$. As $x \to 0$ heat kernel (6.37) can be approximated as

$$Tr\, e^{-x\widetilde{L}} \sim \widetilde{a}_0 x^{-d/2}. \tag{6.41}$$

Then the integral in (6.38) converges as long as $Re\{s\} > 1 + d/2$. In order to get its analytical continuation one can subtract each term of heat kernel expansion (6.37) and then add the same thing. For the first term in the expansion we write

$$\int_0^{\infty} dx\, x^{s-2}\, e^{-x} \left[Tr\, e^{-x\widetilde{L}} - \frac{\widetilde{a}_0}{x^{d/2}} \right] + \Gamma\left(s - 1 - \frac{d}{2}\right). \tag{6.42}$$

The term in square brackets is obviously at the order of $O(x^{(1-d)/2})$ so the integral now converges providing that $Re\{s\} > 1 + (d-1)/2$ and is called holomorphic in this domain whereas the gamma function in (6.42) is meromorphic. The region of convergence has shifted by 1/2 on the real axis. In this way, we extend the domain of the integral leftwards on complex plane. By adding and subtracting each term one by one, meromorphic extension of $\widetilde{h}$ is obtained on the left hand side of $Re\{s\} > 1+d/2$.

On the other hand, the singular part of zeta function is unique at $s = 1$, evidently follows from its definition (6.14). The we may write

$$\zeta(s) \sim \frac{1}{s-1} + \gamma, \qquad s \to 1, \tag{6.43}$$

where γ is its finite holomorphic part (γ here has nothing to do with the optical metric). Remember the Mellin transform of free energy (6.21)

$$\widetilde{\mathcal{F}}(s) = \zeta(s)\widetilde{h}(s).$$

Consequently, by (6.18) and (6.43), $\widetilde{\mathcal{F}}(s)$ has a double pole only at $s = 1$ and all other poles are simple ones. The poles can be expressed as $\mathcal{I} = \{(d+2-n)/2 : n = 0, 1, 2, \ldots\}$ and their residues are given by $-(\lambda_0 - \mu)c_{n/2}$ with

$$c_{n/2} = \sum_{l=0}^{[|\frac{n}{2}|]} \frac{(-1)^l}{l!} \widetilde{a}_{\frac{n}{2}-l} \tag{6.44}$$

being the heat kernel coefficients of the expansion

$$\mathrm{Tr} e^{-x(\widetilde{L}+1)} \sim \sum_{n=0}^{\infty} c_{n/2} x^{\frac{-d+j}{2}} \tag{6.45}$$

corresponding to operator $\widetilde{L} + 1$. Note that $[|\frac{n}{2}|]$ denotes the integer part of $n/2$ and by (6.2) we have

$$\widetilde{L} + 1 = \frac{(L - \lambda_0)}{\lambda_0 - \mu} + 1 = \frac{(L - \mu)}{\lambda_0 - \mu}. \tag{6.46}$$

Combining all we are now ready to use the dictionary (6.20) for the asymptotic expansion of free energy $\widetilde{\mathcal{F}}$.

6.3. Asymptotics of Free Energy

Small x asymptotics of free energy is obtained as follows: We take the Mellin transform of $\mathcal{F}(x)$ by using harmonic series and then expand the trace of the operator in the exponential by heat kernel expansion on the transformed complex plane. Mellin transform of free energy, $\widetilde{\mathcal{F}}(s)$, is found to be the product of two functions (see (6.21)) and most of the contribution comes from their poles on complex plane. Then the asymptotics of free energy $\mathcal{F}(x)$ in x space is determined by meromorphic extension of $\widetilde{\mathcal{F}}(s)$ with the singular part (6.19), i.e. its poles in s space, by the dictionary (6.20).

As we have seen above, the two functions in $\widetilde{\mathcal{F}}(s)$ both has poles at $s = 1$ which is a double pole. For convenience, let us separate $s = 1$ and $s \neq 1$ cases and write

$$\mathcal{F}(x) \sim \mathcal{F}_s(x) + \mathcal{F}_d(x). \tag{6.47}$$

Here $\mathcal{F}_s(x)$ contains the contribution from the simple poles where $s \neq 1$ and $\mathcal{F}_d(x)$ stands for the terms coming from the double pole at $s = 1$. For simple poles $k = 0$ in the dictionary and we get

$$\mathcal{F}_s(x) = -(\lambda_0 - \mu) \sum_{n=0}^{\infty}{}' \zeta\left(\frac{d+2-n}{2}\right) c_{n/2} x^{-\frac{d+2-n}{2}}. \tag{6.48}$$

Since the above sum should exclude double pole at $s = 1$, we put a prime on the summation symbol to remark it. The expression for free energy is in fact constitutes an infinite sum in the descending powers of temperature. For example in 3-d ($d = 3$), we may write the first few terms of $\mathcal{F}_s(T)$ as

$$\begin{aligned} \mathcal{F}_s(T) \quad &\sim \quad \widetilde{a}_0 \zeta(5/2) \left(\frac{T}{\lambda_0 - \mu}\right)^{5/2} + \widetilde{a}_{1/2} \zeta(2) \left(\frac{T}{\lambda_0 - \mu}\right)^{2} \\ &. \quad +(\widetilde{a}_1 - \widetilde{a}_0) \zeta(3/2) \left(\frac{T}{\lambda_0 - \mu}\right)^{3/2} + O\left(\frac{T}{\lambda_0 - \mu}\right)^{1/2}. \end{aligned} \tag{6.49}$$

For the double pole we have $s = 1$ and $n = d$ and by the dictionary its contribution to the asymptotic behavior is

$$\begin{aligned}\widetilde{\mathcal{F}}(s) &\sim \left(\frac{1}{s-1} + \gamma\right)\left(\frac{res(1,0)}{s-1} + R_+\tilde{h}(s=1)\right) \\ &\sim \frac{res(1,0)}{(s-1)^2} + \frac{\gamma res(1,0) + R_+\tilde{h}(s=1)}{s-1}. \end{aligned} \tag{6.50}$$

Note that the residue of $\tilde{h}(s)$ at $s = 1$ is $res(1,0) = -(\lambda_0 - \mu)c_{d/2}$ and the holomorphic part of $\tilde{h}(s)$ is denoted by $R_+\tilde{h}(s)$. Then the contribution of double pole to the asymptotic behavior is found to be

$$\mathcal{F}_d(x) \quad = \quad -(\lambda_0 - \mu)\left\{\left[c_{d/2}\gamma - \frac{R_+(\tilde{h}(1))}{\lambda_0 - \mu}\right]x^{-1} - c_{d/2}x^{-1}\log x\right\}. \tag{6.51}$$

Again for 3-dimensions $d = 3$ we may write

$$\mathcal{F}_d(T) \sim \left((\widetilde{a}_{3/2} - \widetilde{a}_{1/2})\gamma - \frac{R_+\tilde{h}(1)}{\lambda_0 - \mu}\right)\frac{T}{\lambda_0 - \mu} - (\widetilde{a}_{3/2} - \widetilde{a}_{1/2})\frac{T}{\lambda_0 - \mu}\log\frac{T}{\lambda_0 - \mu}, \tag{6.52}$$

and combining with (6.49) we have

$$\begin{aligned}\mathcal{F}(T) \quad \sim \quad & \widetilde{a}_0\zeta(5/2)\left(\frac{T}{\lambda_0 - \mu}\right)^{5/2} + \widetilde{a}_{1/2}\zeta(2)\left(\frac{T}{\lambda_0 - \mu}\right)^{2} \\ & +(\widetilde{a}_1 - \widetilde{a}_0)\zeta(3/2)\left(\frac{T}{\lambda_0 - \mu}\right)^{3/2} \\ & +(\widetilde{a}_{3/2} - \widetilde{a}_{1/2})\left[\gamma - \frac{R_+\tilde{h}(1)}{\lambda_0 - \mu} - \log\frac{T}{\lambda_0 - \mu}\right]\left(\frac{T}{\lambda_0 - \mu}\right) \\ & +O\left(\frac{T}{\lambda_0 - \mu}\right)^{1/2}. \end{aligned} \tag{6.53}$$

On the other hand, making use of (4.53) we can write (6.22) as

$$\widetilde{h}(s) = -(\lambda_0 - \mu)\Gamma(s-1)\zeta_{\widetilde{L}+1}(s-1), \tag{6.54}$$

as $s \to 1$ gamma function behaves like

$$\Gamma(s-1) \sim \frac{1}{s-1} - \gamma \qquad s \to 1. \tag{6.55}$$

Hence in the limit $s \to 1$ expanding $\zeta_{\widetilde{L}+1}(s-1)$ around $s=1$ one can write $\mathcal{F}(s)$ as

$$\begin{aligned} \zeta(s)\widetilde{h}(s) &\sim -(\lambda_0-\mu)\left(\frac{1}{s-1}+\gamma\right)\left(\frac{1}{s-1}-\gamma\right)\left(\zeta_{\widetilde{L}+1}(0)+\zeta'_{\widetilde{L}+1}(0)(s-1)+\ldots\right) \\ &\sim \left(\left(\frac{1}{s-1}\right)^2-\gamma^2\right)\left(\zeta_{\widetilde{L}+1}(0)+\zeta'_{\widetilde{L}+1}(0)(s-1)\right) \\ &\sim -(\lambda_0-\mu)\left[\frac{\zeta_{\widetilde{L}+1}(0)}{(s-1)^2}+\frac{\zeta'_{\widetilde{L}+1}(0)}{s-1}\right]. \end{aligned} \tag{6.56}$$

Then the dictionary gives $\mathcal{F}_d(x)$ as

$$\mathcal{F}_d(x) = -(\lambda_0-\mu)\left[\zeta_{\widetilde{L}+1}(0)\frac{T}{\lambda_0-\mu}\log\left(\frac{T}{\lambda_0-\mu}\right)+\zeta'_{\widetilde{L}+1}(0)\frac{T}{\lambda_0-\mu}\right], \tag{6.57}$$

without heat kernel expansion. In fact (6.57) combined with (6.47) is an analogous nonrelativistic expression to the ultrarelativistic, i.e. high temperature expansion of free energy given in [4].

7. BOSE EINSTEIN CONDENSATION

In this chapter we discover the effects of gravitational field on Bose-Einstein condensation (BEC). The main variable for the analysis of BEC is particle density in the grand canonical ensemble. For this reason we should derive the number of particles from the asymptotic expansion of free energy obtained in the previous chapter.

7.1. Asymptotics of Number of Particles

In the grand canonical ensemble, number of particles in a state denoted by N is found by taking the derivative of free energy $\mathcal{F}$ with respect to chemical potential μ. By (6.3) we have

$$N = -\frac{\partial \mathcal{F}}{\partial \mu} = \sum_{p=1}^{\infty} Tr\, e^{-p\beta(\lambda_0-\mu)} e^{-p\beta(\lambda_0-\mu)\widetilde{L}}. \tag{7.1}$$

Similar to the free energy, again the equation above is of the form of an harmonic sum

$$N(x) = \sum_{p=1}^{\infty} n(px), \tag{7.2}$$

with

$$n(x) = e^{-x}\, Tr\, e^{-x\widetilde{L}}. \tag{7.3}$$

We follow the same steps pursued for free energy, in obtaining the asymptotics of particle number. Take the Mellin transform of $N(x)$

$$\widetilde{N}(s) = \zeta(s)\widetilde{n}(s), \tag{7.4}$$

and

$$\widetilde{n}(s) = \int_0^\infty dx\, x^{s-1}\, e^{-x} Tr\, e^{-x\widetilde{L}}. \tag{7.5}$$

Heat kernel expansion yields

$$\widetilde{n}(s) = \sum_{j=0}^{\infty} \widetilde{a}_{j/2} \Gamma\left(s + \frac{j-d}{2}\right). \tag{7.6}$$

Then the singular part of $\widetilde{n}(s)$ is

$$\widetilde{n}(s) \asymp \sum_{j,l} \widetilde{a}_{j/2} \frac{(-1)^l}{l!} \frac{1}{s + \frac{j-d}{2} + l}. \tag{7.7}$$

This time the poles are $\mathcal{J} = \{(d-n)/2 : n = 0, 1, 2, \ldots\}$ and their residues are $c_{n/2}$. Note that we have a double pole again at $s = 1$ and simple poles otherwise. For convenience separate N as

$$N(x) = N_s(x) + N_d(x). \tag{7.8}$$

Remember that the subscripts s and d stand for the contribution of poles with $s \neq 1$ and $s = 1$. Repeating the procedure for free energy in the previous chapter we end up with

$$N_s(x) = \sum_{n=0}^{\infty}{}' \zeta\left(\frac{d-n}{2}\right) c_{n/2} x^{\frac{n-d}{2}}, \tag{7.9}$$

and

$$N_d(x) = [\gamma c_{(d-2)/2} + R_+ \widetilde{n}(s=1)] x^{-1} - c_{(d-2)/2} x^{-1} \log x, \tag{7.10}$$

after straightforward calculations. Remember that the prime on the summation excludes the double pole case, i.e. $s = 1$ or $n = d - 2$. Besides, $c_{n/2}$'s are the heat kernel coefficients of

$$\widetilde{L}+1=\frac{L-\mu}{\lambda_0-\mu}=\frac{-\Delta_\gamma+U+2m^2c^2-2m\mu}{2m(\lambda_0-\mu)},\tag{7.11}$$

but it is more appropriate to use the heat kernel coefficients $a_{n/2}$ of $-\Delta_\gamma+U+2m^2c^2-2m\mu$, since the corresponding operator includes optical metric. By the properties of heat kernel coefficients given in [46] one finds

$$c_{n/2}=\frac{a_{n/2}}{(2m(\lambda_0-\mu))^{\frac{n-d}{2}}}.\tag{7.12}$$

As the asymptotic expansion of N is derived, we can advance further by focusing on Bose-Einstein (BE) condensate in $d=3$ dimensional space in a static background, specifically Schwarzschild metric. We confine the system in a large but finite volume B with the boundary ∂B. One can model the volume as a spherical shell with inner and outer radii r_1 and r_2 respectively. We assume that the system is far enough from the horizon, i.e. $r_1 \gg r_s$ where r_s is defined as the Schwarzschild radius.

In the analysis of BE condensate, the variables in question are generally the densities rather than extensive ones. For instance instead of particle number we will use particle density n (Do not confuse it with index or the function in harmonic sum) and to do so we should divide number of particles by *proper volume* V_{prop}. The term proper volume refers to the volume observed by a static observer [49] and is written as

$$dV_{prop}=\frac{r^2}{\sqrt{F}}\sin\theta dr d\theta d\phi.\tag{7.13}$$

7.2. Leading Order

We consider now the density-temperature relation in the leading order. For this we examine the particle density at excited states, namely the depletion coefficient N_e in a large but finite volume. Then the effects of gravitation on this relation will be shown explicitly. First we separate the occupation numbers of ground state N_0 and the excited state N_e by excluding the ground state in the trace term Tr in (7.1). The resulting trace would be denoted by Tr$'$ and the heat kernel coefficients of Tr$e^{-x\widetilde{L}}$ will be replaced by the coefficients of Tr$'e^{-x\widetilde{L}}$.

Start with (7.9) and write N_e explicitly to the leading order.

$$N_e = a_0\zeta\left(\frac{3}{2}\right)(2mT)^{3/2} = \left(\frac{m}{2\pi}\right)^{3/2}\zeta\left(\frac{3}{2}\right)V_\gamma T^{3/2}. \tag{7.14}$$

Here c_0 is written by (7.12) and a_0 can be found from (6.33) as

$$a_0 = \frac{1}{(4\pi)^{3/2}}\int_B dV_\gamma, \tag{7.15}$$

which contains the volume of the optical metric V_γ given as

$$dV_\gamma = (rF^{-1})^2\sin^2\theta dr d\theta d\phi. \tag{7.16}$$

Schwarzschild metric in (5.23) implies that

$$F = 1 - \frac{r_s}{r}. \tag{7.17}$$

Then

$$V_\gamma \;=\; \int d^3x\,\sqrt{\gamma} = 4\pi\int_{r_1}^{r_2} dr\,\frac{r^2}{F^2}. \tag{7.18}$$

The integration is straightforward, however, as the system is in a large but finite volume, we should better expand the integral asymptotically:

$$V_\gamma \sim 4\pi\int_{r_1}^{r_2} dr\,r^2\left(1+\frac{2r_s}{r}+\dots\right) \;\sim\; 4\pi\left(\frac{r_2^3}{3}+r_s r_2^2+\dots\right). \tag{7.19}$$

Likewise for the proper volume we write

$$V_{prop} = 4\pi \int_{r_1}^{r_2} dr \, \frac{r^2}{\sqrt{F}} \sim 4\pi \left(\frac{r_2^3}{3} + \frac{r_s r_2^2}{4} + \ldots \right). \tag{7.20}$$

Substituting all into (7.14) and rearranging the terms for T, we have

$$\begin{aligned} T &= \frac{2\pi}{m} \left(\frac{N_e}{\zeta\left(\frac{3}{2}\right) V_\gamma} \right)^{2/3} &(7.21)\\ &= \frac{2\pi}{m} \left(\frac{n_e}{\zeta\left(\frac{3}{2}\right)} \right)^{2/3} \left(\frac{V_{prop}}{V_\gamma} \right)^{2/3}. &(7.22) \end{aligned}$$

where $n_e = N_e/V_{prop}$ is the occupation number density. The ratio of proper volume to volume of optical metric may arise in more calculations. Using (7.19) and (7.13) we should calculate it asymptotically

$$\frac{V_{prop}}{V_\gamma} \sim 1 - \frac{9r_s}{4r_2}. \tag{7.23}$$

Now substituting this ratio in (7.21) we get the density-temperature relation as

$$T = \frac{2\pi}{m} \left(\frac{n_e}{\zeta\left(\frac{3}{2}\right)} \right)^{2/3} \left(1 - \frac{3r_s}{2r_2} \right). \tag{7.24}$$

Note that when $r_s = 0$, the above expression reduces to the result for critical temperature in flat space derived in chapter 2 (see (2.19)). Moreover the thermodynamic limit $r_2 \to \infty$ also leads to standard flat space expression [26], meaning that the gravitational effects are wasted away when the volume is too large. This is an expected result, because we know that the asymptotic behavior of spacetime is in fact flat to the leading term.

One other important thermodynamic variable is the pressure P, from which one can deduce the analogous of equation of state $PV = NT$. One can derive pressure from the free energy expression (6.48) since

$$P = -\frac{\partial \mathcal{F}}{\partial V} = -\frac{\partial \mathcal{F}}{\partial V_{prop}}. \tag{7.25}$$

Along with (7.12) we have

$$\begin{aligned} P &= -\frac{\partial \mathcal{F}}{\partial V_{prop}} = \frac{1}{2m}\zeta\left(\frac{5}{2}\right)\frac{\partial a_0}{\partial V_{prop}}(2mT)^{5/2} \\ &= \frac{1}{2m}\zeta\left(\frac{5}{2}\right)\frac{1}{(4\pi)^{3/2}}\frac{dV_\gamma}{dV_{prop}}(2mT)^{5/2}, \end{aligned} \tag{7.26}$$

for the leading order. Therefore

$$\frac{P}{n} = \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\frac{V_{prop}}{V_\gamma}\frac{dV_\gamma}{dV_{prop}}T. \tag{7.27}$$

The ratio of differentials in the asymptotic form is

$$\frac{dV_\gamma}{dV_{prop}} = \frac{dV_\gamma/dr_2}{dV_{prop}/dr_2} = F^{-3/2}(r_2) \sim 1 + \frac{3r_s}{2r_2}. \tag{7.28}$$

Combining all we get the equation of state as

$$\frac{P}{n} = \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}\left(1 - \frac{3}{4}\frac{r_s}{r_2}\right)T. \tag{7.29}$$

Obviously, this expression also reduces to equation of state in flat space as $r_s = 0$ or $r_2 \to \infty$ near T_c. [26].

7.3. Boundary Effects

Now we extend our analysis further in subleading terms in the asymptotic expansion. Again we have the occupation number in the excited states N_e, but we now include also the second term:

$$N_e = a_0\zeta\left(\frac{3}{2}\right)(2mT)^{3/2} + a_{1/2}(2mT)\log\left(\frac{T}{\lambda_0 - \mu}\right) \tag{7.30}$$

Notice that the subleading term comes from the double pole contribution N_d. (7.30) is identical to the result found by [50] although the expansion method is different.

At this point let us look at the occupation number N_0 of the ground state. From (7.20) we see that the proper volume is at the order $O(r_2^3)$ in the thermodynamic limit ($r_2 \to \infty$). Since

$$L = -\frac{\delta^{\mu\nu}\partial_\mu\partial_\nu}{2m} + O(c^{-2}), \tag{7.31}$$

the spacing between the eigenvalues of operator L would be at the order of $O(r_2^{-2})$ with correction terms in negative powers of c. If we assume that the small quantity $\lambda_0 - \mu = O(r_2^{-3})$, then using (5.50) we may write

$$N_0 = \frac{1}{e^{\beta(\lambda_0-\mu)} - 1} \simeq \frac{T}{\lambda_0 - \mu}. \tag{7.32}$$

In the thermodynamic limit, we conclude that $N_0/V_{prop} \neq 0$ and consequently $\lambda_\sigma - \mu = (\lambda_0 - \mu) + (\lambda_\sigma - \lambda_0)$ is $O(r_2^{-2})$. Then obviously $N_\sigma/V_{prop} \to 0$ which means the particles are all in the ground state, i.e. BEC occurs in the thermodynamic limit. However, notice that in finite volume there is no condensation as expected since just above T_c, setting $N_0 = 0$ requires $\lambda_0 - \mu$ to be infinite. Now coming back to (7.30), along with the above expression, we may replace the argument of logarithm by N_0 similar to [50]. Nonetheless, if one looks at the second term of N_d, it is clear that direct substitution is not valid since the term with zeta function, $R_+\widetilde{n}(s=1)$ also brings $\log(\lambda_0 - \mu)$ and log terms cancel each other. In order to fix this we should deal with the spectral zeta function and its scaling properties.

Begin with (7.5) (note the prime on trace)

$$\widetilde{n}(s) = \int_0^\infty dx\, x^{s-1}\, Tr'\, e^{-x\frac{L-\mu}{(\lambda_0-\mu)}} = \int_0^\infty dx\, x^{s-1}\, e^{-x} Tr'\, e^{-x\widetilde{L}}. \tag{7.33}$$

The meromorphic continuation around $s = 1$ is illustrated explicitly as

$$\widetilde{n}(s) = \int_0^\infty dx\, x^{s-1}\, e^{-x} \left[Tr'\, e^{-x\widetilde{L}} - \sum_{k=0}^{d} \widetilde{a}_{k/2} x^{\frac{k-d}{2}} \right] + \sum_{k=0}^{d} \widetilde{a}_{k/2} \Gamma\left(s - \frac{d-k}{2} \right). \tag{7.34}$$

The residue at $s = 1$ is

$$c_{(d-2)/2} = \sum_{l=0}^{[|\frac{d-2}{2}|]} \frac{(-1)^l}{l!} \widetilde{a}_{\frac{n}{2}-l}. \tag{7.35}$$

Moreover we have the identity

$$\widetilde{n}(s) = \Gamma(s) \zeta_{\widetilde{L}+1}(s) = \Gamma(s) \zeta_{\frac{L-\mu}{\lambda_0-\mu}}(s). \tag{7.36}$$

Zeta function has the following scaling property [51]

$$\zeta_{\alpha^{-1}A}(s) = \alpha^s \zeta_A(s), \tag{7.37}$$

where we define the variable α as $\alpha = 2mV_\gamma^{2/d}(\lambda_0 - \mu)$ so that

$$\widetilde{n}(s) = \Gamma(s) \zeta_{\alpha^{-1} 2mV_\gamma^{2/d}(L-\mu)}(s) = \alpha^s \Gamma(s) \zeta_{2mV_\gamma^{2/d}(L-\mu)}(s). \tag{7.38}$$

Let us now define

$$f(x, \mu) = Tr'\, e^{-x 2mV_\gamma^{2/d}(L-\mu)}. \tag{7.39}$$

The meromorphic extension of

$$\begin{aligned} \widetilde{f}(s, \mu) &= \int_0^\infty dx\, x^{s-1}\, Tr'\, e^{-x 2mV_\gamma^{2/d}(L-\mu)} && (7.40) \\ &= \int_0^\infty dx\, x^{s-1}\, e^{-x 2mV_\gamma^{2/d}(\lambda_0-\mu)} Tr'\, e^{-x 2mV_\gamma^{2/d}(L-\lambda_0)} && (7.41) \end{aligned}$$

is then given by $\Gamma(s)\zeta_{2mV_\gamma^{2/d}(L-\mu)}(s)$. Combining with (7.38), around $s=1$ we get

$$\widetilde{n}(s) = \alpha\left(1+(\log\alpha)(s-1)+O((s-1)^2)\right)\left[\frac{b_{(d-2)/2}}{s-1}+R_+\widetilde{f}(s,\mu)|_{s\to 1}\right]. \tag{7.42}$$

Here $b_{j/2}$'s denote the heat kernel coefficients of operator $2mV_\gamma^{2/d}(L-\mu)$. On the other hand because

$$2mV_\gamma^{2/d}(L-\mu) = \alpha\frac{L-\mu}{\lambda_0-\mu}, \tag{7.43}$$

one can relate them with $c_{d/2}$ by the scaling property of heat kernel [52]:

$$b_{n/2} = \alpha^{\frac{n-d}{2}}c_{n/2}, \tag{7.44}$$

or consequently

$$b_{(d-2)/2} = \alpha^{-1}c_{(d-2)/2}. \tag{7.45}$$

Combine this with (7.12) to arrive

$$b_{n/2} = (V_\gamma^{2/d})^{(n-d)/2}a_{n/2}. \tag{7.46}$$

Then the holomorphic part of $\widetilde{n}(s)$ becomes

$$\begin{aligned} R_+\widetilde{n}(s=1) &= \alpha\left[\frac{c_{(d-2)/2}}{\alpha}\ln\alpha + R_+\widetilde{f}(s=1,\mu)\right] \\ &= 2mV_\gamma^{2/d}(\lambda_0-\mu)\left[b_{(d-2)/2}\log(2mV_\gamma^{2/d}(\lambda_0-\mu)) + R_+\widetilde{f}(1,\mu)\right]. \end{aligned} \tag{7.47}$$

Substituting the results for $c_{j/2}$ and $R_+\widetilde{n}(s=1)$ in (7.10), the double pole contribution to occupation number evolves into

$$N_d = b_{(d-2)/2}(2mTV_\gamma^{2/d})\log(2mTV_\gamma^{2/d}) + [\gamma b_{(d-2)/2} + R_+\widetilde{f}(1,\mu)](2mTV_\gamma^{2/d}), \tag{7.48}$$

and we see that there is no log term as we have predicted before. By similar arguments one can present N_s in terms of the new heat kernel coefficients $b_{j/2}$:

$$N_s = \sum_{n=0}^{\infty}{}' \zeta\left(\frac{d-n}{2}\right) b_{n/2}(2mTV_\gamma^{2/d})^{\frac{d-n}{2}} \tag{7.49}$$

In 3-dimensions the fixed version of occupation number expression for the excited states is now written as

$$\begin{aligned} N_e &= \zeta\left(\frac{3}{2}\right) b_0(2mTV_\gamma^{2/3})^{3/2} + b_{1/2}(2mTV_\gamma^{2/3}) \log\left(2mTV_\gamma^{2/3}\right) \\ &\quad + [\gamma b_{1/2} + R_+\widetilde{f}(1,\mu)](2mTV_\gamma^{2/3}). \end{aligned} \tag{7.50}$$

Near the critical temperature where $\mu \to \lambda_0$, we conclude from Appendix B of [25] that

$$\widetilde{f}(s,\mu) = \widetilde{f}(s,\lambda_0) + O(V_\gamma^{-1}). \tag{7.51}$$

Now let us we define new variables

$$\begin{aligned} y = 2mTV_\gamma^{2/3} \quad A = \zeta\,(3/2)\, b_0 \\ B = b_{1/2} \quad C = \gamma b_{1/2} + R_+\widetilde{f}(1,\mu), \end{aligned} \tag{7.52}$$

and write (7.50) in the form

$$N_e = Ay^{3/2} + By\log y + Cy. \tag{7.53}$$

Scaling as $y = xN_e^{2/3}$, with $\epsilon = N_e^{-1/3}$ we get

$$Ax^{3/2} - 2B(\epsilon\log\epsilon)x + C\epsilon x = 1. \tag{7.54}$$

This equation can now be solved by perturbation theory by the expansion in perturbation parameter ϵ

$$x = x_0 + (\epsilon \log \epsilon) x_1 + \epsilon x_2 + \ldots , \tag{7.55}$$

where one considers $\epsilon \log \epsilon$ and ϵ as independent from each other. [53] The outcome is

$$x_0 = \frac{1}{A^{2/3}}, \qquad x_1 = \frac{4}{3}\frac{B}{A^{4/3}}, \qquad x_2 = -\frac{2}{3} B A^{-4/3} (\log A^{-2/3} + B^{-1} C). \tag{7.56}$$

The heat kernel coefficient $a_{1/2}$ is given by [47, 48] as

$$a_{1/2} = \frac{\pm 1}{16\pi} A_\gamma . \tag{7.57}$$

Note that $a_{1/2}$ can be also calculated via (6.33). Here ± sign stands for the types of boundary conditions in accordance with (6.33). Finally the temperature expression is written explicitly as

$$\begin{aligned} T \; = \; & \frac{2\pi}{m} \left(\frac{n_e}{\zeta\left(\frac{3}{2}\right)} \right)^{2/3} \left(\frac{V_{prop}}{V_\gamma} \right)^{2/3} \left[1 - \left(\frac{\log N_e^{1/3}}{N_e^{1/3}} \right) \frac{(\pm 1)}{3\zeta^{2/3}\left(\frac{3}{2}\right)} \frac{A_\gamma}{V_\gamma^{2/3}} \right. \\ & \left. - \left(\frac{1}{N_e^{1/3}} \right) \frac{1}{6\zeta^{2/3}\left(\frac{3}{2}\right)} \left(\frac{\pm A_\gamma}{V_\gamma^{2/3}} \log \frac{4\pi e^\gamma}{\zeta^{2/3}\left(\frac{3}{2}\right)} + 16\pi R_+ \widetilde{f}(1, \lambda_0) \right) \right] . \end{aligned} \tag{7.58}$$

Note that we have changed $R_+ \widetilde{f}(1, \mu)$ to $R_+ \widetilde{f}(1, \lambda_0)$ thanks to (7.51) since the term $A_\gamma / V_\gamma^{2/3}$ is at the order $O(V_\gamma^0)$. Clearly there are correction terms to the critical temperature in (7.58), which we call boundary corrections since these corrections have a multiplying factor of $A_\gamma / V_\gamma^{2/3}$ in front, except $R_+ \widetilde{f}(1, \lambda_0)$. It is not trivial to conclude if $R_+ \widetilde{f}(1, \lambda_0)$ only depends on the boundary or not. Now let us discover the flat space where there is no gravity and take the thermodynamic limit.Then in the critical temperature expression above, we would have $V_\gamma = V_{prop} = V$, $A_\gamma = A_{prop} = A$ evidently. One can think of the volume V as a cube of some length scale ℓ, and the boundary A as a square of it, like $V \propto \ell^3$ and $A \propto \ell^2$. In the thermodynamic limit we take the scale to infinity $\ell \to \infty$ whereas the density is held constant N/ℓ^3. Consequently the boundary terms will disappear because $N_e \sim N \to \infty$ in the thermodynamic limit at the critical temperature. Moreover, by dimensional analysis one can conclude that there will be no contribution from $R_+\widetilde{f}(1,\lambda_0)$ when thermodynamic limit is taken. [25] The result is the usual one for critical temperature derived in Section 2:

$$T_c = \frac{2\pi}{m}\frac{n_e}{(\zeta(3/2))^{2/3}}$$

On the contrary, when the gravity is still there, i.e. when the geometry is not flat, then even in the thermodynamic limit, one cannot be sure that the term $R_+\widetilde{f}(1,\lambda_0)$ vanishes. This case requires further analysis with the metric in question.

As a final check let us examine the 2-dimensional case, $d = 2$. After a straightforward algebra, the occupation number in 2 dimensions is found to be

$$N_e = b_0(2mTV_\gamma)\log(2mTV_\gamma) = a_0 2mT\log(2mTV_\gamma), \tag{7.59}$$

and so

$$n_e = \frac{2mT}{4\pi}\log(2mTV_\gamma). \tag{7.60}$$

Evidently this expression diverges in the thermodynamic limit and there is no BEC in 2-dimensions as expected.

7.4. Ultrarelativistic Limit

It will be beneficial to discuss the opposite limit where $\beta m \ll 1$ ($c = 1$). This is called ultrarelativistic regime. The free energy in ultrastatic spacetime is given by (4.63) before taking the nonrelativistic limit. The ultrarelativistic limit of this expression is well studied by [4–6, 9, 11, 12] and we write down their results of $d = 3$ for reference.

Free energy in the form of its asymptotic expansion is

$$\begin{aligned}\mathcal{F} &= \frac{-16}{\sqrt{\pi}}\zeta(4)m^4 a_0(\beta m)^{-4} - 4\zeta(3)m^3 a_{1/2}(\beta m)^{-3} \\ &\quad -\frac{4m^2}{\sqrt{\pi}}\zeta(2)\left[(2\mu^2 - m^2)a_0 + a_1\right](\beta m)^{-2} - \ldots, \end{aligned} \tag{7.61}$$

and the leading term for the net charge Q is

$$Q = -\frac{\partial \mathcal{F}}{\partial \mu} = \frac{4}{\sqrt{\pi}}\zeta(2)4\mu a_0 T^2 = \frac{4}{\sqrt{\pi}}\zeta(2)\frac{4\mu}{(4\pi)^{3/2}}V_\gamma T^2. \tag{7.62}$$

Charge density is obtained as usual

$$q = \frac{Q}{V_{prop}} = \frac{V_\gamma}{V_{prop}}\frac{4}{\sqrt{\pi}}\zeta(2)\frac{4\mu}{(4\pi)^{3/2}}T^2. \tag{7.63}$$

As a result

$$T = q^{1/2}\frac{\pi}{\sqrt{2\mu\zeta(2)}}\left(\frac{V_{prop}}{V_\gamma}\right)^{1/2}. \tag{7.64}$$

Now for BEC, chemical potential has some critical value $\mu_c = \epsilon_0$. Similar to the flat case, the condensation occurs near a critical T_c only if

$$\mu - \mu_c = O\left(\frac{1}{V_{prop}}\right) = O\left(\frac{1}{r_2^3}\right). \tag{7.65}$$

Hence we get the asymptotic expression

$$T \sim q^{1/2}\frac{\pi}{\sqrt{2\mu_c\zeta(2)}}\left[1 - \frac{9}{8}\frac{r_s}{r_2} + \ldots\right]. \tag{7.66}$$

Like in the nonrelativistic regime, temperature expression ends up with the Minkowski space result when $r_s/r_2 \to 0$ [54].

8. CONCLUSION

The problem of taking the nonrelativistic limit of Klein-Gordon equation is an hard one in static spacetimes. In this thesis we have invented a method for taking the nonrelativistic limit of a Bose field in the grand canonical ensemble without dealing with Klein-Gordon equation. Instead, our method has focused on the thermodynamic variables and functions and therefore we have taken the nonrelativistic limit of free energy.

The originality of our study comes from the fact that most of the studies in the literature focus on the opposite limit, i.e. ultrarelativistic Bose field. On the other hand, we take the nonrelativistic limit in such a way that the background geometry remained intact which means the gravitational effects are still present and only the matter has become nonrelativistic. This was a crucial point in our work so as not to loose the effect of gravitation while making the matter heavy and slow. Moreover, this regime is quite realistic when we are not too close to supermassive gravitation centers however the gravitational effects are still important on the thermodynamics of Bose-field.

Although the original problem and so the physics lies in a static spacetime, optical metric is used to make a conformal transformation into ultrastatic spacetime. Generalized Laplace method enabled us to take the nonrelativistic limit in ultrastatic spacetime explicitly. We have seen that the results produced by our method are the same as the ones for the grand canonical ensemble of Bose field in static spacetime except that the two operators governing the fields in both spacetimes are conformally equivalent. These are the Schrödinger operators on space manifolds that correspond, at least by a conformal transformation, to the nonrelativistic limit of Klein-Gordon equation.

For a Bose-Einstein system around the critical temperature in a static spacetime, we have obtained, by the help of Mellin transform and Heat kernel techniques, asymptotic expansions of density-temperature relation and equation of state, along with boundary effects. There we have found some correction terms for the critical temperature and pressure due to gravity of an heavy object in a volume away from the gravitating center. Also for boundary effects there appeared some correction terms disappearing in the thermodynamic limit reducing all to the usual well known flat case knowledge and we have seen that the condensation only occurs when the thermodynamic limit is taken as $V \to \infty$.

To sum up, contrary to the ultrarelativistic limit, the nonrelativistic limit of a Bose system in curved background was an untouched area and our unique way of handling the problem enabled us to investigate the gravitational effects on Bose Einstein condensation and we believe that the results may be of great importance for future studies, such as black hole physics or experimental observations in astrophysics or cosmology. In this work, we have assumed that the system is restricted to a volume away from the heavy gravitating object, if this is a black hole then away from the horizon. Near the horizon since the thermodynamic variables tend to diverge, our calculations should be revised, maybe in a future work. Moreover the analysis in this thesis can be also generalized to a field with external potential or a self interacting Bose field, which can be beneficial for the studies related to dark matter models in the nonrelativistic limit like [20–23].

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