ON SIMPLE MODULES OVER THE SECTION BURNSIDE FUNCTOR

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ABSTRACT

ON SIMPLE MODULES OVER THE SECTION BURNSIDE FUNCTOR

Biset functors provides a unified approach to module theoretic operations on representations of finite groups. It is well-known that biset functors can be considered as modules over the Green biset functor of the Burnside ring of finite groups. The problem of determining simple modules over a Green biset functor is open. In this thesis, we consider another Green biset functor, the functor of section Burnside ring, introduced by Serge Bouc, and obtain partial results on the parametrization problem of its simple modules.

ÖZET

BURNSİDE BÖLÜM İZLECİNİN BASİT MODÜLLERİ ÜZERİNE

İki etki izleçleri, sonlu öbeklerin adlanımlarının üzerinde tanımlanan modül teoritik işlemlere bütün bir yaklaşım getirir. Bilindiği üzere iki etki izleçleri, sonlu öbeklerin Burnside halkasının Green iki etki izlecinin modülleri gibi görülebilir. Bir Green iki etki izlecinin basit modüllerinin belirlenmesi problemi açıktır. Bu tezde, Serge Bouc tarafından keşfedilmiş başka bir Green iki etki izleci olan Burnside bölüm izlecinin basit modüllerini inceleyeceğiz ve basit modüllerinin belirlenmesi üzerine kısmi sonuçlar elde edeceğiz.

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LIST OF SYMBOLS

$B(G \times H)$	The Burnside group of $(G \times H)$-sets
Ind_H^G	$\left(\frac{G\times H}{\Delta(H),1}\right) \in {}_G\mathbf{set}_H^A$
$\mathrm{Inf}_{G/N}^G$	$\left(\frac{G\times G/N}{\Delta_\pi(G),1}\right) \in {}_G\mathbf{set}_{G/N}^A$
$\mathrm{Def}_{G/N}^G$	$\left(\frac{G/N\times G}{{}_\pi\Delta(G),1}\right) \in {}_{G/N}\mathbf{set}_G^A$
Res_H^G	$\left(\frac{H\times G}{\Delta(H),1}\right) \in {}_H\mathbf{set}_G^A$
$X \times_K Y$	The Mackey product of bisets X and Y over K
S_G	The set of subgroups of G
$\widetilde{S}_G$	A set of representatives of the conjugacy classes of S_G
${}_R\mathrm{Mod}$	The set of R-modules
A	A Green biset functor
A-Mod	The category of A-modules
G-Mor	The category of morphisms of G-sets
G-Mor$^{\mathrm{Gal}}$	The category of Galois morphisms of G-sets
$\Pi(G)$	The set of slices of G
$\Sigma(G)$	The set of sections of G
$\Xi(G)$	The slice Burnside ring of G
$\Gamma(G)$	The section Burnside ring of G
Γ	The section Burnside functor
$\Delta(H)$	$\{(h,h) \mid h \in H\}$

1. INTRODUCTION

There are two ways to let two groups act on a set X. First, suppose that we have two groups G and H. By considering the action of G on the left and the action of H on the right, we may let $G \times H$ act on X. In this case, the set X is called a (G, H)-biset and this leads us to the theory of biset functors introduced by Bouc in [3]. In [6], Bouc defined the notion of a Green biset functor which is a biset functor with a ring structure and for any Green biset functor A, the notion of an A-module.

The Burnside ring of a finite group is an algebraic construction that encodes the different ways the group can act on finite sets. The ideas were introduced by William Burnside at the end of the nineteenth century, but the algebraic ring structure is a more recent development, due to Solomon in [1]. S. Bouc, introduced two variations on Burnside rings of finite groups, called the *slice Burnside ring* and the *section Burnside ring*. Both of them are built as Grothendieck rings of some category of morphisms of finite G-sets, instead of the category of finite G-sets used to build the usual Burnside ring. The difference between those two new Burnside rings is that the slice Burnside ring is built from arbitrary morphisms of finite G-sets, whereas the section Burnside ring uses only Galois morphims of finite G-sets. It turns out that most of the well known properties of the Burnside ring extend to the slice Burnside rings and to the section Burnside ring: both are commutative rings, which are free of finite rank as $\mathbb{Z}$-modules. Finally, both constructions have a natural biset functor structure, for which they also become *Green biset functors*.

The aim of this thesis is to investigate the simple modules over the section Burnside ring with its Green biset functor structure. In order to study the modules over a Green biset functor A, Bouc introduces a category P_A such that A-modules correspond to R-linear functors from P_A to R-Mod. The class of objects of P_A is the same class of groups on which A is defined, and if G and H are groups in P_A, then $\mathrm{Hom}_{P_A}(G, H) = A(H \times G)$. It is a conjecture of Bouc that the simple modules over a Green biset functor A are in correspondence with couples (H, V) for which H is a group such that the quotient algebra $\overline{A}(H)$ is different from zero and V is a simple $\overline{A}(H)$-module. Here $\overline{A}(H)$ denotes the quotient of $A(H \times H)$ over the submodule generated by elements that can be factored through a group of smaller order.

In Chapter 2, we give some preliminary results and introduce some notations. We state the definition of a *Green biset functor* and its module structures from [6]. We also give the definitions and some properties of the section Burnside functor Γ from Bouc [2]. In Chapter 3, we study the structure of the ideal $I_\Gamma(G)$ of $\Gamma(G \times G)$. It is the ideal generated by the elements that can be factored through a group of smaller order. After investigating the conditions step-by-step, we get some results on which type of sections are in the ideal which type of them are not. These results makes us to get closer to understand the structure of the quotient of $\Gamma(G \times G)$ over the ideal $I_\Gamma(G)$. But at least we know that the reduced sections in the last theorem generates a subalgebra in $\overline{A}(G)$. In further studies, we will continue to work on this subalgebra.

2. PRELIMINARIES

2.1. Bisets

In this section, we give the definition and some properties of bisets. Details can be found in [6].

Definition 2.1. *Let H and K be two finite groups. An (H,K)-**biset** X is a set with a left H-action and a right K-action such that*

$$h \cdot (x \cdot k) = (h \cdot x) \cdot k \tag{2.1}$$

for all elements $h \in H$ and $k \in K$ and $x \in X$.

An (H, K)-biset X is called **transitive** if for any elements $x, y \in X$ there exists an element $h \in H$ and an element $k \in K$ such that $h \cdot x \cdot k$ is equal to y.

We can regard any (H, K)-biset X as a right $H \times K$-set with the action given by

$$x \cdot (h, k) = h^{-1} \cdot x \cdot k \tag{2.2}$$

for all $x \in X$ and $h \in H$ and $k \in K$. Clearly, X is a transitive (H, K)-biset if and only if X is a transitive $H \times K$-set. Hence there is a bijective correspondence between

(i) isomorphism classes $[X]$ of transitive (H, K)-bisets,

(ii) conjugacy classes $[L]$ of subgroups of $H \times K$

where the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of a point $x \in X$ is in $[L]$.

Hereafter we denote a transitive biset with a point stabilizer equal to L by $\left(\frac{H\times K}{L}\right)$ and its isomorphism class by $\left[\frac{H\times K}{L}\right]$.

Definition 2.2 (Opposite biset). *If G and H are finite groups, and if U is a finite (H,G)-biset, then let U^{op} denote the opposite biset which is a set equal to U, and the (G,H)-action is given by*

$$\text{for all } h \in H, u \in U, g \in G, \quad g \cdot u \cdot h \ (\text{in } U^{op}) = h^{-1}ug^{-1} \ (\text{in } U)$$

Example 2.1 (opposite subgroup). *If G and H are groups, and L is a subgroup of $H \times G$, then the opposite subgroup $L^{\diamond}$ is the subgroup of $G \times H$ defined by*

$$L^{\diamond} = \{(g,h) \in G \times H |\ (h,g) \in L\} \tag{2.3}$$

With this notation, there is an isomorphism of (G,H)-bisets

$$((H \times G)/L)^{op} \cong (G \times H)/L^{\diamond} \tag{2.4}$$

2.1.1. Double Burnside Group/Ring

Let X and Y be (G,H)-bisets. The disjoint union, called the *coproduct*, $X \sqcup Y$ of X and Y is again (G,H)-biset.

Definition 2.3. *The double Burnside group $B(G,H)$ of G and H is defined as the Grothendieck group of the category of finite (G,H)-bisets: it is the quotient of the free abelian group on the isomorphism classes of transitive (G,H)-bisets by the subgroup generated by the elements of the form*

$$[X \sqcup Y] - [X] - [Y]$$

With the above identification, we have $B(G,H) = B(G\times H)$. We denote by $S_{G\times H}$ the set of subgroups of $G\times H$ and $\widetilde{S}_{G\times H}$ a set of representatives of the conjugacy classes of $S_{G\times H}$. It should be noted that $\left(\frac{G\times H}{L}\right)$, $L \in \widetilde{S}_{G\times H}$ form a $\mathbb{Z}$-basis for $B(G\times H)$. Thus, as abelian groups, we have

$$B(G\times H) = \bigoplus_{L\in\widetilde{S}_{G\times H}} \mathbb{Z}\left[\frac{G\times H}{L}\right] \tag{2.5}$$

2.1.2. Mackey Product

We define a composition product of bisets as follows. Given finite groups H, K, M and an (H,K)-biset X and a (K,M)-biset Y. We define the **Mackey product** $X\times_K Y$ of X and Y as the set

$$X \times_K Y := X \times Y/K \tag{2.6}$$

of K-orbits of the Cartesian product $X\times Y$. Here K acts via $k\cdot(x,y) := (x\cdot k, k^{-1}\cdot y)$. The set $X\times_K Y$ is an (H,M)-biset via

$$h\cdot(x,_K y)\cdot m := (h\cdot x,_K y\cdot m) \tag{2.7}$$

where $h\in H, m\in M$ and $(x,_K y)$ denotes the K-orbit of (x,y). This construction is associative and distributive with respect to disjoint unions. With this definition, we obtain a bilinear map

$$\begin{aligned} -\times_H - : B(G\times H)\times B(H\times K) &\to B(G\times K) \\ ([X],[Y]) &\to [X\times_H Y] \end{aligned} \tag{2.8}$$

An explicit formula for the Mackey product is given in [7]. We will give this formula, but first, we need to introduce some notation.

Let $L \leq H \times K$ be a subgroup. Then, for $i = 1, 2$, we write $p_i(L)$ for the projection of L into H and K, respectively. We also define

$$k_1(L) = \{h \in H \mid (h, 1) \in L\} \text{ and } k_2(L) = \{k \in K \mid (1, k) \in L\} \tag{2.9}$$

Now, suppose $N \leq K \times M$ is a subgroup. Then we define the subgroup $L * N$ of $H \times M$ by

$$L * N = \{(h, m) \mid \text{ for some } k \in K,\ (h, k) \in L \text{ and } (k, m) \in N\} \tag{2.10}$$

Proposition 2.4. *The Mackey product of transitive bisets* $\left(\frac{H \times K}{L}\right)$ *and* $\left(\frac{K \times M}{N}\right)$ *is explicitly given by*

$$\left(\frac{H \times K}{L}\right) \times_K \left(\frac{K \times M}{N}\right) = \sum_{x \in [p_2(L) \backslash K / p_1(N)]} \left(\frac{H \times M}{L * {}^{(x,1)}N}\right)$$

Then, $B(G \times G)$ becomes a ring by this multiplication. Its identity element is equal to $[G] = \left[\frac{G \times G}{\Delta(G)}\right]$. It is called the double Burnside ring of G.

2.1.3. Decomposition of Transitive Bisets

Let G be a finite group, H be a subgroup of G and N be a normal subgroup of G. We define the induction, restriction, inflation and deflation bisets, respectively, as

(i) **Induction** $(G, H) -$ **biset** : $\mathrm{Ind}_H^G := \left(\frac{G \times H}{D}\right)$ where $D = \{(h, h) : h \in H\}$
(ii) **Inflation** $(G, G/N) -$ **biset** : $\mathrm{Inf}_{G/N}^G := \left(\frac{G \times G/N}{I}\right)$ where $I = \{(g, gN) : g \in G\}$
(iii) **Deflation** $(G/N, G) -$ **biset** : $\mathrm{Def}_{G/N}^G := \left(\frac{G/N \times G}{J}\right)$ where $J = \{(gN, g) : g \in G\}$
(iv) **Restriction** $(G, H) -$ **biset** : $\mathrm{Res}_H^G := \left(\frac{H \times G}{R}\right)$ where $R = \{(h, h) : h \in H\}$

Moreover, if G' is a finite group and $\phi : G' \to G$ is a group isomorphism, then we define the **isomorphism** (G, G') – **biset** as

$$\mathrm{Iso}^{\phi}_{G,G'} := \left(\frac{G \times G'}{C^{\phi}}\right) \text{ where } C^{\phi} = \{(\phi(g'), g') : g' \in G'\}$$

Remark 2.1. Throughout the thesis, we use notations $\mathrm{Ind}^G_H \mathrm{Res}^G_J, \mathrm{Inf}^K_J \mathrm{Res}^G_J$ etc. instead of $\mathrm{Ind}^H_J \times_J \mathrm{Res}^G_J, \mathrm{Inf}^K_J \times_J \mathrm{Res}^G_J$ etc.

The following theorem explicitly shows that any transitive biset is equal to a Mackey product of the above basic bisets.

Theorem 2.5 (Bouc)**.** *Let L be any subgroup of $H \times K$. Then*

$$\left(\frac{H \times K}{L}\right) = \mathrm{Ind}^H_{p_1(L)} \mathrm{Inf}^{p_1(L)}_{p_1(L)/k_1(L)} \mathrm{Iso}^{\phi}_{p_1(L)/k_1(L), p_2(L)/k_2(L)} \mathrm{Def}^{p_2(L)}_{p_2(L)/k_2(L)} \mathrm{Res}^K_{p_2(L)}$$

where the isomorphism

$$\phi : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$$

is the one given by associating $lk_2(L)$ to $mk_1(L)$ where for a given element $l \in p_2(L)$ we let m be the unique element in $p_1(L)$ such that $(m, l) \in L$.

2.2. Green biset functors

2.2.1. Biset functors

The definitions appearing in this section, as well as more results and theory around them, can be found in Bouc [6].

Definition 2.6. *Let R be a commutative ring. Fix a set $\mathcal{G}$ of representatives of isomorphism classes of finite groups. The biset category of finite groups $\mathcal{C} := \mathcal{C}^{\mathcal{G}}_R$ is the category defined as follows:*

- *The objects of $\mathcal{C}$ are the groups in $\mathcal{G}$.*
- *If G and H are finite groups, then* $\mathrm{Hom}_{\mathcal{C}}(G,H) = RB(H\times G) = R\otimes_{\mathcal{Z}} B(G\times H)$.
- *If K is another finite group and $x \in RB(H\times G)$ and $y \in RB(K\times H)$, then the composition of x and y is $y \circ x := y \cdot_H x$ where*

$$- \cdot_H - : RB(K\times H) \times RB(H\times G) \to RB(K\times G)$$

is the R-linear extension of the map induced by the Mackey product of bisets.

Definition 2.7. *A* biset functor *over R is an R-linear functor from $\mathcal{C}_R^{\mathcal{G}}$ to the R-linear category ${}_R\mathrm{Mod}$.*

Since ${}_R\mathrm{Mod}$ is abelian, biset functors form an abelian category with pointwise constructions. We denote this category by Ω_R. This allows us to define subfunctors, quotient functors, simple functors, etc.

From now on, let $\mathcal{Z}$ be a class of groups closed under subquotients and direct products. We will write $\Omega_{R,\mathcal{Z}}$ for the biset category with objects in $\mathcal{Z}$. There are two equivalent definition of Green biset functor. The first one is given in terms of tensor product of biset functors. In [6], it is shown that this definition is equivalent to the following.

Definition 2.8. *A* Green biset functor *A is a biset functor equipped with bilinear products*

$$\begin{aligned} A(G)\times A(H) &\to A(G\times H) \\ (a,b) &\to a\times b \end{aligned} \tag{2.11}$$

for each groups $G, H \in \mathcal{Z}$ and an element $\varepsilon_A \in A(1)$, satisfying the following conditions.

- *Associativity : Let G, H and K be groups in $\mathcal{Z}$. If $\alpha_{G,H,K}$ is the canonical group isomorphism from $G\times(H\times K)$ to $(G\times H)\times K$, then for any $a\in A(G), b\in A(H)$*

and $c \in A(K)$*, we have*

$$(a \times b) \times c = A(\mathrm{Iso}(\alpha_{G,H,K}))(a \times (b \times c))$$

- *Identity element : Let* G *be a group in* $\mathcal{Z}$*. Let* $\lambda_G : 1 \times G \to G$ *and* $\rho_G : G \times 1 \to G$ *denote the canonical group isomorphisms. Then for any* $a \in A(G)$*, we have*

$$a = A(\mathrm{Iso}(\lambda_G)(\varepsilon_A \times a) = A(\mathrm{Iso}(\rho_G)(a \times \varepsilon_A)$$

- *Functoriality : If* $\varphi : G \to G'$ *and* $\psi : H \to H'$ *are morphisms in* $\Omega_{R,\mathcal{Z}}$ *then for any* $a \in A(G)$ *and* $b \in A(H)$*, we have*

$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b)$$

2.2.2. Modules over Green biset functors

Definition 2.9. *Let* A *be a Green biset functor defined in* $\Omega_{R,\mathcal{Z}}$*. An* A-module *is a biset functor* M *from* $\Omega_{R,\mathcal{Z}}$ *to* $R - Mod$ *together with a morphism, for all groups* $G, H \in \mathcal{Z}$,

$$\begin{aligned} A(G) \times M(H) &\to M(G \times H) \\ (a, m) &\to a \times m \end{aligned} \tag{2.12}$$

fullfilling the following conditions.

- *Associativity : Let* G, H *and* K *be groups in* $\mathcal{Z}$*. If* $\alpha_{G,H,K}$ *is the canonical group isomorphism from* $G \times (H \times K)$ *to* $(G \times H) \times K$*, then for any* $a \in A(G), b \in A(H)$ *and* $m \in M(K)$*, we have*

$$(a \times b) \times m = M(\mathrm{Iso}(\alpha_{G,H,K}))(a \times (b \times m))$$

- *Identity element : Let G be a group in $\mathcal{Z}$. Let $\lambda_G : 1 \times G \to G$ denote the canonical group isomorphism. Then for any $m \in M(G)$, we have*

$$m = M(\mathrm{Iso}(\lambda_G))(\varepsilon_A \times m))$$

- *Functoriality : If $\varphi : G \to G'$ and $\psi : H \to H'$ are morphisms in $\Omega_{R,\mathcal{Z}}$, then for any $a \in A(G)$ and $m \in A(H)$, we have*

$$M(\varphi \times \psi)(a \times m) = A(\varphi)(a) \times M(\psi)(m)$$

As it is seen in Section 8.5 of [6], the class of all A-modules form a category, denoted by A-Mod.

The following proposition is about the category associated to a Green biset functor. It is Proposition 2.11 of [4].

Notation 2.1. *If X is a (G, H)-biset, let $\overrightarrow{X}$ be the $(G \times H, 1)$-biset X with the action*

$$(g, h)x = gxh^{-1} \tag{2.13}$$

for $g \in G$, $h \in H$ and $x \in X$. This construction induces a map

$$\begin{aligned} B(G, H) &\to B(G \times H, 1), \\ a &\mapsto \overrightarrow{a} \end{aligned} \tag{2.14}$$

For a group G, we denote by $\overrightarrow{G}$ the $(G \times G, 1)$-biset $\overrightarrow{G}$. In this case, $\overleftarrow{G}$ will denote the $(1, G \times G)$-biset $\overrightarrow{G}^{op}$.

Proposition 2.10. *Let A be a Green biset functor over $\Omega_{R,H}$ and let $\mathcal{P}_A$ be the following category:*

- *The objects of $\mathcal{P}_A$ are the groups in $\mathcal{Z}$.*
- *If G and H are in $\mathcal{Z}$, then $\mathrm{Hom}_{\mathcal{P}_A}(H, G) = A(G \times H)$.*

- *Let H, G and K be groups in $\mathcal{Z}$. The composition of the elements $\beta \in A(H \times G)$ and $\alpha \in A(G \times K)$ in $\mathcal{P}_A$ is given by:*

$$\beta \circ \alpha = A(H \times \overleftarrow{G} \times K)(\beta \times \alpha).$$

- *If G is in $\mathcal{Z}$, then the identity morphism of G in $\mathcal{P}_A$ is $A(\overrightarrow{G})(\varepsilon_A)$.*

*Then $\mathcal{P}_A$ is an R-linear category and A-*Mod *is equivalent to the category of R-linear functors from $\mathcal{P}_A$ in R-*Mod.

An important problem in the theory of Green biset functors is the classification of simple A-modules for an arbitrary Green biset functor A. The following notation and results let us determine the minimal structure of simple functors and used in the classification of simples in some cases.

Definition 2.11. *Let M be a biset functor defined on $\Omega_{R,\mathcal{Z}}$. A group H in $\mathcal{Z}$ is called minimal for M if $M(H) \neq 0$ and $M(K) = 0$ for any K in $\mathcal{Z}$ with $|K|<|H|$.*

This motivates the following definition.

Definition 2.12. *Let A be a Green biset functor and H be a group in $\mathcal{Z}$. We will write $I_A(H)$ for the submodule of $A(H \times H)$ generated by the elements of the form $a \circ b$ where a is in $A(H \times K)$, b is in $A(K \times H)$ and K is a group in $\mathcal{Z}$ of order smaller than $|H|$. We will denote by $\overline{A}(H)$ the quotient $A(H \times H)/I_A(H)$.*

If S is a simple A-module, then $S \neq 0$, so taking H as a group of minimal order such that $S(H) \neq 0$, we have a minimal group for S. The next result is from [4].

Lemma 2.13. *Let S be a simple A-module for a Green biset functor A defined in $\Omega_{R,Z}$. If $H \in \mathcal{Z}$ is a minimal group for S, then $\overline{A}(H) \neq 0$. Furthermore, $S(H)$ is a simple $\overline{A}(H)$-module.*

Proof. Suppose we have $\overline{A}(H) = 0$, then, in particular,

$$A(\overrightarrow{H})(\varepsilon_A) = \sum_{i=1}^{r} \alpha_i \circ \beta_i \tag{2.15}$$

where $\alpha_i \in A(H \times K_i)$ and $\beta_i \in A(K_i \times H)$ with K_i a group of order smaller than $|H|$. But $S(\alpha_i \circ \beta_i) = 0$ because H is minimal for S, thus $S(A(\overrightarrow{H})(\varepsilon_A)) = 0$, a contradiction.

Now, let $x \in I_A(H)$ and assume that $x = y \circ z$ where $y \in \Gamma(H \times K)$ and $z \in \Gamma(K \times H)$ for some group K of order smaller than H and let α be in $S(H)$, then

$$x \cdot \alpha = (y \circ z) \cdot \alpha = y \cdot (z \cdot \alpha) = y \cdot 0 = 0 \tag{2.16}$$

since $z \cdot \alpha \in S(K)$ and $S(K) = 0$ by the minimality of H. Hence, $I_A(H)$ annihilates $S(H)$. Therefore, $S(H)$ becomes an $\overline{A}(H)$-module, so it suffices to show that it is a simple $A(H \times H)$-module. Let W be an $A(H \times H)$-submodule of $S(H)$, define

$$T(G) = \{m \in S(G) \mid \text{for all } X \in A(H \times G), S(X)(m) \in W\} \tag{2.17}$$

It is easy to see that T is a subfunctor of S together with the induced actions. Thus, since S is simple, we should have $T = 0$ or $T = S$. Besides, $T(H) = W$. Thus $T \neq 0$. So $T = S$ and we have $T(H) = W = S(H)$. So $S(H)$ is a simple $A(H \times H)$-module. □

2.3. The slice and section Burnside functors

Details on the definitions and some properties of the slice and section Burnside functors of summarized in this section can be found in [2]. Furthermore, we give some proofs of Bouc in more detail.

2.3.1. Morphisms of G-sets

Definition 2.14. *Let G be a group. If $f : X \to Y$ and $f' : X' \to Y'$ are morphisms of G-sets, a* ***morphism*** *from f to f' is a pair of morphisms of G-sets $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

is commutative.
*Morphisms of morphisms of G-sets can be composed in the obvious way. This composition endows the class of morphisms of G-sets with a structure of a category, denoted by G-*Mor.

Proposition 2.15. *The disjoint union of G-sets induces a coproduct*

$$(X \xrightarrow{f} Y, X' \xrightarrow{f'} Y') \to (X \sqcup X' \xrightarrow{f \sqcup f'} Y \sqcup Y')$$

*in the category G-*Mor*. Similarly, the direct product of G-sets, with the diagonal action of G, induces a product*

$$(X \xrightarrow{f} Y, X' \xrightarrow{f'} Y') \to (X \times X' \xrightarrow{f \times f'} Y \times Y')$$

*in the category G-*Mor*.*

2.3.2. The slice Burnside functor

In this section, we consider the Grothendieck ring of the category G-Mor. It turns out that it can be realized as a ring on the set of all slices of G. We start with some definitions.

Definition 2.16. *Let G be a group. A* slice *of G is a pair (T, S) of subgroups of G with $S \leq T$. A* section *of G is a slice (T, S) with $S \trianglelefteq T$.*

Notation 2.2. *Let* $\Pi(G)$ *denote the set of slices of* G*, and* $\Sigma(G)$ *denote the set of sections of* G*. When* $(T,S) \in \Pi(G)$*, denote by* $G/S \to G/T$ *the projection morphism.*

Definition 2.17. *Let* G *be a finite group. The slice Burnside group* $\Xi(G)$ *of* G *is the quotient of the free abelian group on the set of isomorphism classes* $[X \xrightarrow{f} Y]$ *of morphisms of finite* G*-sets, by the subgroup generated by the elements of the form*

$$[(X_1 \sqcup X_2) \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f_1(X_1)] - [X_2 \xrightarrow{f_2} f_2(X_2)]$$

whenever $X \xrightarrow{f} Y$ *is a morphism of finite* G*-sets with a decompositon* $X = X_1 \sqcup X_2$ *as a disjoint union of* G*-sets, where* $f_1 = f_{|X_1}$ *and* $f_2 = f_{|X_2}$*.*

When $f : X \to Y$ *is a morphism of finite* G*-sets, let* $\pi(f)$ *denote the image in* $\Xi(G)$ *of the isomorphism class of* f*. If* $S \leq T$ *are subgroups of* G*, set*

$$\langle T, S\rangle_G = \pi(G/S \to G/T)$$

Lemma 2.18. *Let* G *be a finite group.*

(i) $\pi(\emptyset \to \emptyset) = 0$

(ii) Let $X \xrightarrow{f} Y$ *be a morphism of finite* G*-sets. Then*

$$\pi(X \xrightarrow{f} Y) = \pi(X \xrightarrow{f} f(X))$$

(iii) Let $X \xrightarrow{f} Y$ *and* $X' \xrightarrow{f'} Y'$ *be morphims of finite* G*-sets. Then*

$$\pi((X \sqcup X') \xrightarrow{f'} (Y \sqcup Y')) = \pi(X \xrightarrow{f} Y) + \pi(X' \xrightarrow{f'} Y')$$

Lemma 2.19. *We have* $f : X \to Y$ *be a morphism of finite* G*-sets. Then, in the group* $\Xi(G)$*, the following equality holds.*

$$\pi(X \xrightarrow{f} Y) = \sum_{x \in [G \backslash X]} \langle G_{f(x)}, G_x\rangle_G$$

Proof. Indeed, $X \cong \bigsqcup_{x\in[G\backslash X]} G/G_x$ and the image $f(G\cdot x)$ of the G-orbit of x is equal to the G-orbit of $f(x)$. Moreover the morphisms $f_{|G\cdot x} : G\cdot x \to G\cdot f(x)$ and $G/G_x \to G/G_{f_x}$ are isomorphic via the pair (α,β) for which the diagram

$$\begin{array}{ccc} G\cdot x & \xrightarrow{f_{|G\cdot x}} & G/G_x \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ G/G_x & \longrightarrow & G/G_{f_x} \end{array}$$

is commutative where $\alpha : g\cdot x \mapsto gG_x$ and $\beta : gf(x) \mapsto gG_{f(x)}$. Thus the claimed formula follows by Lemma 2.18(ii), as

$$\begin{aligned} \pi(X \xrightarrow{f} Y) &= \pi(\bigsqcup_{x\in[G\backslash X]} G/G_x \xrightarrow{f} Y) = \sum_{x\in[G\backslash X]} \pi(G\cdot x \xrightarrow{f_{|G\cdot x}} G\cdot f(x)) \\ &= \sum_{x\in[G\backslash X]} \pi(G/G_x \to G/G_{f(x)}) = \sum \langle G_{f(x)}, G_x\rangle_G \end{aligned} \tag{2.18}$$

□

Corollary 2.20. *The group $\Xi(G)$ is generated by the elements $\langle T,S\rangle_G$ where (T,S) runs through a set $[\Pi(G)]$ of representatives of conjugacy classes of slices of G.*

Remark 2.2. *It can be shown that this generating set is actually a basis of $\Xi(G)$.*

Proposition 2.21. *The product of morphisms induces a commutative unital ring structure on $\Xi(G)$. The identity element for multiplication is the image of the class $[\bullet \to \bullet]$, where $\bullet$ denotes a G-set of cardinality 1.*

Proof. If we can show that the product of morphisms induces a well defined bilinear product $\Xi(G)\times\Xi(G)\to\Xi(G)$, it will be clear that this product is associative, commutative, and admits $[\bullet\to\bullet]$ as an identity element. Hence the only point to check is that the product preserves the defining relations of $\Xi(G)$. This is clear, since if $g : Z \to T$ is a morphism of finite G-sets, and if $X_1 \sqcup X_2 \xrightarrow{f_1\sqcup f_2} Y$ is a morphism, setting $X = X_1 \sqcup X_2$, the domain of the morphism

$$h : Z\times X \xrightarrow{g\times(f_1\sqcup f_2)} T\times Y$$

has a disjoint union decomposition $Z \times X = (Z \times X_1) \sqcup (Z \times X_2)$, and moreover the restriction of $g \times f$ to $Z \times X_1$ is $g \times f_1$. Thus

$$\begin{aligned}\pi(h) &= \pi((Z \times X_1) \xrightarrow{g \times f_1} (g(Z) \times f_1(X_1))) + \pi((Z \times X_2) \xrightarrow{g \times f_2} (g(Z) \times f_2(X_2))) \\ &= \pi((Z \times X_1) \xrightarrow{g \times f_1} (T \times f_1(X_1))) + \pi((Z \times X_2) \xrightarrow{g \times f_2} (T \times f_2(X_2)))\end{aligned}$$

where the last equality follows from Lemma 2.18. □

The following Proposition describes the above product in terms of the basis of $\Xi(G)$.

Proposition 2.22. *Let (T, S) and (Y, X) be slices of G. Then, in $\Xi(G)$, we have*

$$\langle T, S\rangle_G \langle Y, X\rangle_G = \sum_{g \in [S \backslash G/X]} \langle T \cap {}^gY, S \cap {}^gX\rangle_G$$

Proof. Consider the map

$$\begin{aligned}\bigsqcup_{g \in [S \backslash G/X]} (G/(S \cup {}^gX)) &\to (G/S) \times (G/X) \\ u(S \cup {}^gX) &\mapsto (uS, ugX)\end{aligned} \tag{2.19}$$

for any $u \in G$. We have $(G/S) \times (G/X) \cong \bigsqcup_{g \in [S \backslash G/X]} (G/(S \cup {}^gX))$. The image of (S, gX) by the map $(G/S) \times (G/X) \to (G/T) \times (G/Y)$ is the pair (T, gY) whose stabilizer in G is $T \sqcup {}^gY$. Then, by Lemma 2.19, we get

$$\begin{aligned}\langle T, S\rangle_G \langle Y, X\rangle_G &= \pi((G/S) \times (G/X) \to (G/T) \times (G/Y)) \\ &= \sum_{g \in [S \backslash G/X]} \langle G_{(T,gY)}, G_{(S,gX)}\rangle_G \\ &= \sum_{g \in [S \backslash G/X]} \langle T \cap {}^gY, S \cap {}^gX\rangle_G\end{aligned} \tag{2.20}$$

□

2.3.3. Galois morphisms of G-sets

In this section, we define *Galois morphisms* of finite G-sets. As we built the slice Burnside functor from arbitrary morphisms of finite G sets, we construct the section Burnside functor by using only *Galois morphisms* of finite G-sets.

Definition 2.23. *Let G be a group. A morphism $f : X \to Y$ of G-sets is a* ***Galois morphism*** *if for any $x, x' \in X$ such that $f(x) = f(x')$, there exists $\varphi \in \mathrm{Aut}_G(X)$ such that $f \circ \varphi = f$ and $\varphi(x) = x'$.*

Example 2.2. *Any injective morphism of G-sets is a Galois morphism since there will be no $x \neq x'$ such that $f(x) = f(x')$, in this case we take $\varphi = \mathrm{Id}_{|X}$.*

Proposition 2.24. *Let $f : X \to Y$ be a morphism of G-sets. The following conditions are equivalent :*

(i) f is a Galois morphism.
(ii) for all $x, x' \in X$, if $f(x) = f(x')$ then $G_x = G_{x'}$.

Then, a straightforward consequence follows.

Corollary 2.25. *Let $f : X \to Y$ be a Galois morphism of G-sets. If X_1 is a G-subset of X, the restricted morphism $f|_{X_1} : X_1 \to f(X_1)$ is a Galois morphism of G-sets.*

The next result also comes from Proposition 2.24.

Proposition 2.26. *A morphism $f : X \to Y_1 \sqcup Y_2$ is a Galois morphism if and only if the restricted morphisms $f^{-1}(Y_1) \to Y_1$ and $f^{-1}(Y_2) \to Y_2$ are Galois morphims.*

Let G and H be groups. A morphism $f : U \to U'$ of (H, G)-bisets is called a Galois morphism if it is a Galois morphism of $(H \times G^{op})$-sets.

Proposition 2.27. *Let G, H and K be groups. If $f : U \to U'$ is a Galois morphism of (H, G)-bisets, and $g : V \to V'$ is a Galois morphism of (K, H)-bisets, then the map $g \times_H f : V \times_H U \to V' \times_H U'$ is a Galois morphism of (K, G)-bisets.*

Proof. Let $(v,_H u)$ and $(v',_H u')$ be elements of $V \times_H U$ such that

$$(g \times_H f)((v,_H u)) = (g \times_H f)((v',_H u')) \tag{2.21}$$

It means that there exists $h \in H$ such that

$$g(v') = g(v)h = g(vh) \text{ and } f(u') = h^{-1}f(u) = f(h^{-1}u) \tag{2.22}$$

As g is a Galois morphism, the stabilizers of v' and vh in $K \times H^{op}$ are equal. Similarly, as f is a Galois morphism, the stabilizers of u' and $h^{-1}u$ in $H \times G^{op}$ are equal.

Now let $(k, g) \in K \times G^{op}$ such that $k(v',_H u')g^{-1} = (v',_H u')$. It means that there exists $a \in H$ such that $kv'a^{-1} = v'$ and $aug^{-1} = u'$. Hence

$$kvha^{-1} = vh \text{ and } ah^{-1}ug^{-1} = h^{-1}u \tag{2.23}$$

It follows that

$$\begin{aligned} k(v,_H u)g^{-1} &= (kv,_H ug^{-1}) = (vhah^{-1},_H ug^{-1}) \\ &= (v,_H hah^{-1}ug^{-1}) = (v,_H hh^{-1}u) = (v,_H u) \end{aligned} \tag{2.24}$$

By symmetry, the stabilizers of $(v,_H u)$ and $(v',_H u')$ in $K \times G^{op}$ are equal. Thus $(g \times_H f)$ is a Galois morphism of (K, G)-bisets. □

Corollary 2.28. *Let G and H be finite groups, and U be an (H, G)-biset. If $f : X \to Y$ is a Galois morphism of G-sets, then $U \times_G f : U \times_G X \to U \times_G Y$ is a Galois morphism of H-sets.*

Proof. The claim on $U \times_G f$ is the special case of Proposition 2.27, with the following changes in the assumptions: the group G becomes the trivial group, the group H becomes G, the group K becomes H, the biset U becomes X, the biset V becomes U, and the map g becomes the identity map of U. □

Corollary 2.29. *Let $f : X \to Y$ be a Galois morphism of G-sets. If H is a subgroup of G, the restriction $\mathrm{Res}^G_H f : \mathrm{Res}^G_H X \to \mathrm{Res}^G_H Y$ is a Galois morphism of H-sets.*

Proof. In Corollary 2.28, set $U = G$, viewed as a (H, G)-biset for left and right multiplication. □

Corollary 2.30. *Let G and H be groups. If $f : X \to Y$ is a Galois morphism of G-sets and $g : Z \to T$ is a Galois morphism of H-sets, then $f \times g : X \times Z \to Y \times T$ is a Galois morphism of $(G \times H)$-sets.*

Proof. Consider f and g as a morphism of $(G, 1)$-bisets and $(1, H^{op})$-bisets, respectively. □

Notation 2.3. *Let G-$\mathrm{Mor}^{\mathrm{Gal}}$ denote the full subcategory of G-*Mor *consisting of Galois morphisms of G-sets.*

2.3.4. The section Burnside functor

As in Section 2.3.2, we consider the Grothendieck ring of the category G-Mor^{Gal}. This time, it is related to sections of G. The following proposition shows that the product and coproduct defined on G-Mor restricts to a product and a coproduct on G-$\mathrm{Mor}^{\mathrm{Gal}}$.

Proposition 2.31. *If $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ are Galois morphisms of G-sets, then $X \sqcup X' \xrightarrow{f \sqcup f'} Y \sqcup Y$ and $X \times X' \xrightarrow{f \times f'} Y \times Y'$ are Galois morphisms of G-sets.*

Proof. The case of the disjoint union follows from Proposition 2.26 by taking the set $X = f^{-1}(Y) \sqcup f'^{-1}(Y') = X_1 \sqcup X_2$. Moreover, Corollary 2.30 shows us that if $f : X \to Y$ and $f' : X' \to Y'$ are Galois morphisms of G-sets, then

$$f \times f' : X \times X' \to Y \times Y'$$

is a Galois morphism of $G\times G$-sets. By Corollary 2.29, the restriction of this morphism to the diagonal $G \cong \Delta(G) \leq G \times G$ is a Galois morphism of G-sets. □

Definition 2.32. *Let G be a finite group. The* section Burnside group $\Gamma(G)$ *of G is the subgroup of the slice Burnside ring $\Xi(G)$ generated by the classes of Galois morphims of G-sets. By Proposition 2.32, the group $\Gamma(G)$ is actually a subring of $\Xi(G)$, called the section Burnside ring of G.*

The following lemma determines the image of a Galois morphism in $\Gamma(G)$.

Lemma 2.33. *Let $f : X \to Y$ be a Galois morphism of finite G-sets. Then in the group $\Gamma(G)$, we have*

$$\begin{aligned}\pi(X \xrightarrow{f} Y) &= \sum_{x\in[G\backslash X]} \langle G_{f(x)}, G_x\rangle_G \\ &= \sum_{y\in[G\backslash Y]} |G_y\backslash f^{-1}(y)|\langle G_y, G_{x_y}\rangle_G\end{aligned}$$

where x_y is chosen in $f^{-1}(y)$, for each $y \in [G\backslash f(X)]$.

Proof. The first formula follows from Lemma 2.19. For the second one, write

$$\pi(X \xrightarrow{f} Y) = \sum_{y\in[G\backslash Y]} \sum_{x\in[G_y\backslash f^{-1}(y)]} \pi(G/G_x \to G/G_y) \tag{2.25}$$

and $G_x = G_{x'}$ since $f(x) = f(x')$ for any $x \in [G_y\backslash f^{-1}(y)]$. Hence, the last equality follows. □

Corollary 2.34. *The elements $\langle T, S\rangle_G$, where (T, S) runs through a set $[\Xi(G)]$ of representatives of conjugacy classes of sections of G, form a basis of $\Gamma(G)$.*

Remark 2.3. *This also shows that $\Gamma(G)$ is the quotient of the free abelian group on the set of isomorphism classes $[X \xrightarrow{f} Y]$ of Galois morphisms of finite G-sets, by the subgroup generated by the elements of the form*

$$[(X_1 \sqcup X_2) \xrightarrow{f_1\sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f_1(X_1)] - [X_2 \xrightarrow{f_2} f_2(X_2)]$$

whenever $f : X \to Y$ *is a Galois morphism of finite* G*-sets with a decomposition* $X = X_1 \sqcup X_2$ *as a disjoint union of* G*-sets, where* $f_1 = f_{|X_1}$ *and* $f_2 = f_{|X_2}$.

Remark 2.4. *Since* $\Gamma(G)$ *is a subring of* $\Xi(G)$*, the product formula given in Proposition 2.22 also expresses the product of two sections* $\langle T, S\rangle_G$ *and* $\langle Y, X\rangle_G$ *of* G.

Finally, the section Burnside ring constructed above has a natural biset functor structure, for which it becomes a *Green biset functor* by the following theorem. Then, it is called the section Burnside functor.

Theorem 2.35. *Let* G *and* H *be finite groups. Then,*

(i) Let U *be a finite* (H, G)*-biset. The functor*

$$(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)$$

from $G\text{-Mor}^{\mathrm{Gal}}$ *to* $H\text{-Mor}^{\mathrm{Gal}}$ *induces a group homomorphism*

$$\Gamma(U) : \Gamma(G) \to \Gamma(H).$$

(ii) The correspondence $G \mapsto \Gamma(G)$ *is a Green biset functor.*

Proof. For Assertion (i), the only thing to check is that the defining relations of $\Gamma(G)$ are mapped to the relations in $\Gamma(H)$. But if

$$X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$$

is a Galois morphism of finite G-sets, then

$$U \times_G (X_1 \sqcup X_2) \cong (U \times_G X_1) \sqcup (U \times_G X_2)$$

Moreover the image of the map $U \times_G f_1$ is equal to $U \times_G f_1(X_1)$. It follows that the relation

$$[X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f}_1 f_1(X_1)] - [X_2 \xrightarrow{f_2} f_2(X_2)]$$

in $\Gamma(G)$ is mapped to the relation

$$[UX_1 \sqcup UX_2 \xrightarrow{Uf_1 \sqcup Uf_2} UY] - [UX_1 \xrightarrow{Uf_1} Uf_1(UX_1)] - [UX_2 \xrightarrow{Uf_2} (Uf_2)(UX_2)]$$

where $U\times_G$ is abbreviated to U. It is now clear that the correspondence sending a finite group G to $\Gamma(G)$ and a finite (H,G)-biset U to $\Gamma(U)$ endows Γ with a structure of a biset functor. Moreover, if G, G' are finite groups, $f : X \to Y$ is a Galois morphism of finite G-sets and $f' : X' \to Y'$ is a Galois morphism of finite G'-sets, then

$$f \times f' : X \times X' \to Y \times Y'$$

is a Galois morphism of $(G \times G')$-sets, by Corollary 2.30. This induces a product

$$\Gamma(G) \times \Gamma(G') \to \Gamma(G \times G')$$

which is associative in the obvious sense. Moreover, the morphism $\bullet \to \bullet$ of $\mathbf{1}$-sets is obviously an identity element for this product, up to identification $G \times \mathbf{1} = G$. Finally, if G, G', H, H' are finite groups, if U is a finite (H,G)-biset, if U' is a finite (H', G')-biset, it is clear that the morphisms

$$(U \times U') \times_{G \times G'} (f \times f')$$

and

$$(U \times_G f) \times (U' \times_{G'} f')$$

are isomorphic Galois morphisms of (H, H')-sets. Thus Γ is a Green biset functor.(see [6] Section 8.5) $\square$

3. SIMPLE SECTION FUNCTORS

In this chapter, we are aiming to study the structure of simple modules over the section Burnside functor. As remarked in the previous chapter, first we need to understand the quotient algebra $\overline{\Gamma}(G)$ for any group G. Although we couldn't fully describe the structure, we have determined a subalgebra of it. We have also shown that the module category of Γ is a new counter example to Bouc's conjecture.

3.1. The Product of Sections

We have seen in the first section that if S is a simple Γ-module then $S(G)$ is a simple $\Gamma(G \times G)$-module. Moreover, when G is a minimal for S, $S(G)$ is a simple $\Gamma(G \times G)/I_\Gamma(G)$-module by Lemma 2.13 where

$$I_\Gamma(G) = \langle\ x \in \Gamma(G \times G) : x = y \cdot_H z,\ \ y \in \Gamma(G \times H),\ \ z \in \Gamma(H \times G),\ \ |H|<|G|\ \rangle.$$

In this section, our main goal is to determine the quotient algebra $\Gamma(G \times G)/I_\Gamma(G)$. Thus we have to understand precisely the elements of the ideal $I_\Gamma(G)$. First, the product of sections that also appears in the definition of $I_\Gamma(G)$ is explicitly given by the following lemma.

Lemma 3.1. *Let G and H be two finite groups. For any $\langle K, N\rangle \in \Gamma(G \times H)$ and $\langle L, M\rangle \in \Gamma(H \times G)$, we have*

$$\langle K, N\rangle_{G\times H} \cdot \langle L, M\rangle_{H\times G} = \sum_{x\in[p_2(N)\backslash H/p_1(M)]} \langle K *^{(x,1)} L, N *^{(x,1)} M\rangle$$

where $[p_2(N)\backslash H/p_1(M)]$ is a complete set of representatives of the double cosets of $p_2(N)$ and $p_1(M)$ in H.

Proof. By the definition of Γ and Proposition 2.4, we have $\langle K, N\rangle_{G\times H} \cdot \langle L, M\rangle_{H\times G}$ is equal to

$$\begin{aligned}
&= \left[\left(\frac{G\times H}{N}\right) \xrightarrow{f} \left(\frac{G\times H}{K}\right)\right] \cdot_H \left[\left(\frac{H\times G}{M}\right) \xrightarrow{g} \left(\frac{H\times G}{L}\right)\right] \\
&= \left[\left(\frac{G\times H}{N}\right) \times_H \left(\frac{H\times G}{M}\right) \xrightarrow{f\times_H g} \left(\frac{G\times H}{K}\right) \times_H \left(\frac{H\times G}{L}\right)\right] \\
&= \left[\sum_{x\in[p_2(N)\backslash H/p_1(M)]} \left(\frac{G\times G}{N *^{(x,1)} M}\right) \xrightarrow{f\times_H g} \sum_{y\in[p_2(K)\backslash H/p_1(L)]} \left(\frac{G\times G}{K *^{(y,1)} L}\right)\right]
\end{aligned} \tag{3.1}$$

Thus, it remains to show that for any $x \in [p_2(N)\backslash H/p_1(M)]$, we have

$$(f \times_H g)\left(\frac{G\times G}{N *^{(x,1)} M}\right) = \frac{G\times G}{K *^{(x,1)} L} \tag{3.2}$$

Then, by Lemma 2.18(ii), we can conclude that

$$\begin{aligned}
\langle K, N\rangle_{G\times H} \cdot \langle L, M\rangle_{H\times G} &= \left[\sum_{x\in[p_2(N)\backslash H/p_1(M)]} \left(\frac{G\times G}{N *^{(x,1)} M} \xrightarrow{f\times_H g} \frac{G\times G}{K *^{(x,1)} L}\right)\right] \\
&= \sum_{x\in[p_2(N)\backslash H/p_1(M)]} \langle K *^{(x,1)} L, N *^{(x,1)} M\rangle
\end{aligned} \tag{3.3}$$

as required. We first consider the case $x = 1$, we have

$$(f \times_H g)\left(\frac{G\times G}{N * M}\right) = \{\ (f \times_H g)[(g, g')(N * M)] : (g, g') \in G\times G\ \}. \tag{3.4}$$

First, we should note that $(g, 1)N * (1, g')M$ is equal to

$$\begin{aligned}
&= \{\ (x, y) \in G\times G \mid (x, h) \in (g, 1)N,\ (h, y) \in (1, g')M \text{ for some } h \in H\} \\
&= \{\ (x, y) \in G\times G \mid (g^{-1}x, h) \in N,\ (h, (g')^{-1}y) \in M \text{ for some } h \in H\} \\
&= \{\ (x, y) \in G\times G \mid (g^{-1}x, (g')^{-1}y) \in N * M\ \} \\
&= \{\ (x, y) \in G\times G \mid (x, y) \in (g, g')(N * M)\ \} \\
&= (g, g')(N * M)
\end{aligned} \tag{3.5}$$

for any subgroups N and M. Also, for any $x \in [p_2(N)\backslash H/p_1(M)]$, we have

$$
\begin{aligned}
(f \times_H g)\left(\frac{G \times G}{N *^{(x,1)} M}\right) &= \{(f \times_H g)((g,g')(N *^{(x,1)} M)) \ : (g,g') \in G \times G \ \} \\
&= \{ \ (f \times_H g)((g,1)N * (1,g')^{(x,1)} M) \ : (g,g') \in G \times G \ \} \\
&= \{ \ f((g,1)N) * g((1,g')^{(x,1)} M) \ : (g,g') \in G \times G \ \} \\
&= \{ \ f((g,1)N) *^{(x,1)} g((1,g')M) \ : (g,g') \in G \times G \ \} \\
&= \{ \ (g,1)K * (1,g')^{(x,1)} L \ : (g,g') \in G \times G \ \} \\
&= \{ \ ((g,g')(K *^{(x,1)} L) \ : (g,g') \in G \times G \ \} \\
&= \frac{G \times G}{K *^{(x,1)} L}
\end{aligned}
$$

□

Therefore, there is a map

$$\Gamma(G \times H) \times \Gamma(H \times G) \to \Gamma(G \times G)$$

and the ideal $I_\Gamma(G)$ that we are looking for can be seen as the sum of images of these maps as H runs over the set of groups of order smaller than $|G|$. Moreover, $\Gamma(G \times G)$ is endowed with a ring structure by the product defined above.

If a section of G factors through a group of smaller order, we call it decomposable, otherwise it is called indecomposable. Thus $I_\Gamma(G)$ consists of all decomposable sections in $\Gamma(G \times G)$. We also call the elements of the quotient $\Gamma(G \times G)/I_\Gamma(G)$ reduced sections.

3.1.1. Decomposable Sections

Remember that our first aim is to describe the elements of $I_\Gamma(G)$ precisely. At first, we fix a section $\langle T, S\rangle \in \Gamma(G \times G)$. Given that $S \trianglelefteq T$ and they are both subgroups of $G \times G$. By Theorem 2.5, we already know that

$$\left(\frac{G \times G}{T}\right) = \mathrm{Ind}^{G}_{p_1(T)} \mathrm{Inf}^{p_1(T)}_{p_1(T)/k_1(T)} \mathrm{Iso}^{\phi}_{p_1(T)/k_1(T), p_2(T)/k_2(T)} \mathrm{Def}^{p_2(T)}_{p_2(T)/k_2(T)} \mathrm{Res}^{G}_{p_2(T)} \quad (3.6)$$

$$\left(\frac{G \times G}{S}\right) = \mathrm{Ind}^{G}_{p_1(S)} \mathrm{Inf}^{p_1(S)}_{p_1(S)/k_1(S)} \mathrm{Iso}^{\varphi}_{p_1(S)/k_1(S), p_2(S)/k_2(S)} \mathrm{Def}^{p_2(S)}_{p_2(S)/k_2(S)} \mathrm{Res}^{G}_{p_2(S)} \quad (3.7)$$

Therefore, we can identify each decomposition with a quintuple

$$(p_1(T), k_1(T), \phi(T), p_2(T), k_2(T))$$
$$(p_1(S), k_1(S), \varphi(S), p_2(S), k_2(S))$$

respectively, such that $p_i(S) \trianglelefteq p_i(T)$ and $k_i(S) \trianglelefteq k_i(T)$ for $i = 1, 2$. Moreover, if a decomposition for $\left(\frac{G \times G}{S \trianglelefteq T}\right)$ exists, it must be determined by these two quintuples. Hence, it is sufficient to check the conditions on these five special subgroups of G.

To make our calculations easy, we shall use some abbreviations for these special subgroups. From now on, $p_1(T) = P_T$, $k_1(T) = K_T$, $p_2(T) = Q_T$ and $k_2(T) = L_T$. Similarly, for S.

To obtain all decomposable sections, we proceed step-by-step and determine conditions on the above quintuples.

Step 1 : Assume that P_T is a proper subgroup of G. In this case, we can rewrite the decomposition (3.7) as

$$\begin{aligned} \left(\frac{G \times G}{S}\right) &= \mathrm{Ind}^{G}_{P_T} \mathrm{Ind}^{P_T}_{P_S} \mathrm{Inf}^{P_S}_{P_S/K_S} \mathrm{Iso}^{\varphi}_{P_S/K_S, Q_S/L_S} \mathrm{Def}^{Q_S}_{Q_S/L_S} \mathrm{Res}^{G}_{Q_S} \\ &= \frac{G \times P_T}{D} \times \frac{P_T \times G}{\alpha(S,T)} \end{aligned} \quad (3.8)$$

where $\alpha(S,T) = D * I * C^{\varphi} * J * R$.

On the other hand, by the decomposition (3.6), we have

$$\left(\frac{G\times G}{T}\right)=\frac{G\times P_T}{D}\times\frac{P_T\times G}{\alpha(T,T)} \tag{3.9}$$

where $\alpha(T,T) = I * C^{\phi} * J * R$. We need to check that the above decompositions induces a decomposition in the section Burnside algebra. It is sufficient to check that $\alpha(S,T) \trianglelefteq \alpha(T,T)$. First notice that

$$\begin{aligned} D * I &= \{(p_s, p_s) :\ p_s \in P_S\} * \{(p'_s, p'_s K_S) :\ p_s \in P_S\} \\ &= \{(p_s, p_s K_S) :\ p_s \in P_S\} \end{aligned} \tag{3.10}$$

Then, since $C^{\phi} = \{(\phi(q_s L_S), q_s L_S) :\ q_s L_S \in Q_S/L_S\}$, we have

$$(D * I) * C^{\phi} = \{((p_s, q_s L_S) \in P_S \times Q_S/L_S :\ \varphi(q_s L_S) = p_s K_S\} \tag{3.11}$$

On the other hand,

$$\begin{aligned} J * R &= \{(q'_s L_S, q'_s) :\ q'_s \in Q_S\} * \{(q_s, q_s) :\ q_s \in Q_S\} \\ &= \{(q'_s L_S, q_s) \in Q_S/L_S \times Q_S :\ q_s L_S = q'_s L_S\} \end{aligned} \tag{3.12}$$

Finally, we get

$$\begin{aligned} \alpha(S,T) &= (D * I * C^{\phi}) * (J * R) \\ &= \{(p_s, q_s) \in P_S \times Q_S :\ \varphi(q_s L_S) = p_s K_S\} \leq P_T \times G \end{aligned} \tag{3.13}$$

Similar calculations show that

$$\alpha(T,T) = \{(p,q) \in P_T \times Q_T :\ \phi(q L_T) = p K_T\} \leq P_T \times G \tag{3.14}$$

But, by equations (3.6) and (3.7), we have

$$\begin{aligned} T &= \{(p,q) \in P_T \times Q_T : \ \phi(qK_T) = pK_T\} \leq G \times G \\ S &= \{(p_s,q_s) \in P_S \times Q_S : \ \varphi(q_sL_S) = p_sK_S\} \leq G \times G \end{aligned} \tag{3.15}$$

Therefore, it immediately follows that $\alpha(T,T) = T$ and $\alpha(S,T) = S$ as sets. Note that these resulting structures differ by the groups in which they live. Therefore, since $S \trianglelefteq T$, we get $\alpha(S,T) \trianglelefteq \alpha(T,T)$.

Under these assumptions we can factor the section by Lemma 3.1,

$$\frac{G \times P_T}{D \trianglelefteq D} \times \frac{P_T \times G}{\alpha(S,T) \trianglelefteq \alpha(T,T)}$$

$$= \sum_{x \in [p_2(D) \backslash P_T / p_1(D)]} \frac{G \times G}{(D * {}^{(x,1)}\alpha(S,T)) \trianglelefteq (D * {}^{(x,1)}\alpha(T,T))} \tag{3.16}$$

But, $[p_2(D) \backslash P_T / p_1(D)]$ is just the identity set. Thus, the equality becomes

$$\begin{aligned} \frac{G \times P_T}{D \trianglelefteq D} \times \frac{P_T \times G}{\alpha(S,T) \trianglelefteq \alpha(T,T)} &= \frac{G \times G}{(D * \alpha(S,T)) \trianglelefteq (D * \alpha(T,T))} \\ &= \frac{G \times G}{S \trianglelefteq T} \end{aligned} \tag{3.17}$$

Therefore, $\langle T,S \rangle \in I_\Gamma(G)$ if $P_T \lneq G$. For the next steps, we assume that $P_T = G$. Furthermore, by the definition of the opposite subgroup in Example 2.1, for a given section $\langle T,S \rangle \in \Gamma(G \times H)$, we define its **opposite section** as

$$\left(\frac{G \times H}{S \trianglelefteq T}\right)^{op} = \frac{H \times G}{S^\diamond \trianglelefteq T^\diamond} \tag{3.18}$$

The opposite section of $\langle T, S\rangle$ is denoted by $\langle T^{\diamond}, S^{\diamond}\rangle$. Therefore, it can be concluded that if $\langle T, S\rangle \in I_{\Gamma}(G)$, then $\langle T^{\diamond}, S^{\diamond}\rangle \in I_{\Gamma}(G)$. From now on, when we have conditions on the subgroups P_T, K_T, we conclude that the same conditions for the subgroups Q_T, L_T hold, respectively. Similar arguments apply to the special subgroups of S.

Step 2 : Assume that $P_T = G = Q_T$. Further assume $P_S \neq G$ and $K_S \neq 1$. Then, in particular, $L_S \neq 1$. First, we prove the following observation.

Claim : The subgroup K_S is normal in G.

Proof : Let $h \in K_S$ and $g \in G$. We want to show that $ghg^{-1} \in K_S$. Note that $(h, 1) \in S$ and $(g, g') \in T$ for some $g' \in T$ since $P_T = G$. It follows that ${}^{(g,g')}(h, 1) \in S$ since $S \trianglelefteq T$. That means $(ghg^{-1}, 1) \in S$ so that $ghg^{-1} \in K_S$.

Now, we can write decompositions as follows;

$$\begin{aligned}\left(\frac{G\times G}{T}\right) &= \mathrm{Inf}^{G}_{G/K_S}\mathrm{Inf}^{G/K_S}_{G/K_T}\mathrm{Iso}^{\phi}_{G/K_T,G/L_T}\mathrm{Def}^{G}_{G/L_T} \\ &= \frac{G\times (G/K_S)}{I}\times\frac{(G/K_S)\times G}{\alpha(T,S)}\end{aligned} \tag{3.19}$$

$$\begin{aligned}\left(\frac{G\times G}{S}\right) &= \mathrm{Inf}^{G}_{G/K_S}\mathrm{Ind}^{G/K_S}_{P_S/K_S}\mathrm{Iso}^{\varphi}_{P_S/K_S,Q_S/L_S}\mathrm{Def}^{Q_S}_{Q_S/L_S}\mathrm{Res}^{G}_{Q_S} \\ &= \frac{G\times (G/K_S)}{I}\times\frac{(G/K_S)\times G}{\alpha(S,S)}\end{aligned} \tag{3.20}$$

where $\mathrm{Inf}^{G/K_S}_{G/K_T} = \mathrm{Inf}^{G/K_S}_{(G/K_S)/(K_T/K_S)}\mathrm{Iso}^{\alpha'^{-1}}_{(G/K_S)/(K_T/K_S),G/K_T}$ and α' is obtained by applying the first isomorphism theorem to the homomorphism $\alpha : G/K_S \to G/K_T$, $gK_S \mapsto gK_T$. Then we figure out the following equations.

$$T = \{(g, g') \in G\times G : \ \phi(g'L_T) = gK_T\}, \tag{3.21}$$

$$S = \{(p_s, q_s) \in P_S\times Q_S : \ \varphi(q_S L_S) = p_s K_S\} \tag{3.22}$$

We already know from the previous calculations that

$$J * R = \{(q_s' L_S, q_s) \in (Q_S/L_S) \times Q_S : \ q_s L_S = q_s' L_S\} \tag{3.23}$$

Note also that

$$\begin{aligned} D * C^{\varphi} &= \{(p_s K_S, p_s K_S) : \ p_s \in P_S\} * \{(\varphi(q_s L_S), q_S L_S) : q_s \in Q_S\} \\ &= \{(p_s K_S, q_S L_S) \in (P_S/K_S) \times (Q_S/L_S) : \ \varphi(q_s L_S) = p_s K_S\} \end{aligned} \tag{3.24}$$

Therefore we have

$$\alpha(S,S) = (D * C^{\varphi}) * (J * R) = \{(p_s K_S, q_S) \in (P_S/K_S) \times Q_S : \ \varphi(q_s L_S) = p_s K_S\}$$

Similarly, since $I * C^{\phi} = \{(gK_S, g'L_T) \in (G/K_S) \times (G/L_T) : \ \phi(g'L_T) = gK_T\}$ and $J = \{gL_T, g) : \ g \in G\}$, we obtain

$$\alpha(T,S) = (I * C^{\phi}) * J = \{(gK_S, g') \in (G/K_S) \times G : \ \phi(g'L_T) = gK_T\} \tag{3.25}$$

Now, we need to check that $\alpha(S,S) \subseteq \alpha(T,S)$. That is to show $\phi(q_s L_T) = p_s K_T$ if $\varphi(q_s L_S) = p_s K_S$ for any $(p_s K_S, q_s) \in \alpha(S,S)$. But the required equation holds if and only if there is an extension of φ say $\overline{\varphi} : G/L_S \to G/K_S$ such that the diagram

$$\begin{array}{ccc} G/L_S & \xrightarrow{\overline{\varphi}} & G/K_S \\ \big\downarrow{\scriptstyle \alpha' \circ \pi_L} & & \big\downarrow{\scriptstyle \alpha' \circ \pi_K} \\ G/L_T & \xrightarrow{\phi} & G/K_T \end{array}$$

commutes where $\pi_K : G/K_S \to (G/K_S)/(K_T/K_S)$ and $\pi_L : G/L_S \to (G/L_S)/(L_T/L_S)$ are the projection maps. But this is already satisfied by the assumption that $S \leq T$ where T, S are as in (3.21) and (3.22) respectively. For normality condition of $\alpha(S,S)$ in $\alpha(T,S)$, we need to check that ${}^{(gK_S, g')}(p_s K_S, q_S) \in \alpha(S,S)$ for any $(gK_S, g') \in \alpha(T,T)$, $(p_s K_S, q_S) \in \alpha(S,S)$. That is $\varphi(g' q_s g'^{-1} L_S) = g p_s g^{-1} K_S$ if $\phi(g' L_T) = gK_T$. But the last condition holds since we have already assumed $S \trianglelefteq T$ and Q_S, P_S are normal in G.

Hence $\alpha(S,S) \trianglelefteq \alpha(T,S)$, as required.

Finally, by Lemma 3.1, we are to see that

$$\frac{G \times (G/K_S)}{I \trianglelefteq I} \times \frac{(G/K_S) \times G}{\alpha(S,S) \trianglelefteq \alpha(T,S)}$$

$$= \sum_{x \in [p_2(I)\backslash G/K_S/p_1(\alpha(S,S))]} \frac{G \times G}{(I * {}^{(x,1)}\alpha(S,S)) \trianglelefteq (I * {}^{(x,1)}\alpha(T,S))} \tag{3.26}$$

But since $p_2(I) = G/K_S$, the set $[p_2(I)\backslash G/K_S/p_1(\alpha(S,S))]$ contains only the identity element. Therefore, the equation becomes

$$\begin{aligned} \frac{G \times (G/K_S)}{I \trianglelefteq I} \times \frac{(G/K_S) \times G}{\alpha(S,S) \trianglelefteq \alpha(T,S)} &= \frac{G \times G}{(I * \alpha(S,S)) \trianglelefteq (I * \alpha(T,S))} \\ &= \frac{G \times G}{S \trianglelefteq T} \end{aligned} \tag{3.27}$$

As a result, $\langle T, S\rangle \in I_\Gamma(G)$.

Step 3 : We already accept that $P_T = G = Q_T$. Furthermore, assume that $P_S = G$ and $K_S \neq 1$. Then $Q_S = G$, too. In this step, we have the following decompositions

$$\begin{aligned} \left(\frac{G \times G}{T}\right) &= \mathrm{Inf}^{G}_{G/K_S}\mathrm{Inf}^{G/K_S}_{G/K_T}\mathrm{Iso}^{\phi}_{G/K_T,G/L_T}\mathrm{Def}^{G}_{G/L_T} \\ &= \frac{G \times (G/K_S)}{I} \times \frac{(G/K_S) \times G}{\alpha(T,S)} \end{aligned} \tag{3.28}$$

$$\begin{aligned} \left(\frac{G \times G}{S}\right) &= \mathrm{Inf}^{G}_{G/K_S}\mathrm{Iso}^{\varphi}_{G/K_S,G/L_S}\mathrm{Def}^{G}_{G/L_S} \\ &= \frac{G \times (G/K_S)}{I} \times \frac{(G/K_S) \times G}{\alpha(S,S)} \end{aligned} \tag{3.29}$$

where we have

$$\alpha(T,S) = I * C^{\phi} * J = \{(gK_S, g') \in (G/K_S)\times G :\ \phi(g'L_T) = gK_T\}, \tag{3.30}$$

$$\begin{aligned}\alpha(S,S) = C^{\varphi} * J &= \{(\varphi(gL_S), gL_S) : g \in G\} * \{(g'L_S, g') : g' \in G\} \\ &= \{(gK_S, g') \in (G/K_S)\times G : gK_S = \varphi(g'L_S)\}\end{aligned} \tag{3.31}$$

By a similar argument as in Step 2, we conclude that $\alpha(S,S) \trianglelefteq \alpha(T,S)$ since $S \trianglelefteq T$. Therefore, we decompose the section $\langle T, S\rangle$, by Lemma 3.1 as

$$\begin{aligned}\frac{G\times(G/K_S)}{I \trianglelefteq I} \times \frac{(G/K_S)\times G}{\alpha(S,S) \trianglelefteq \alpha(T,S)} &= \frac{G\times G}{(I*\alpha(S,S)) \trianglelefteq (I*\alpha(T,S))} \\ &= \frac{G\times G}{S \trianglelefteq T}\end{aligned} \tag{3.32}$$

Step 4 : In addition to the assumption $P_T = G = Q_T$, we assume that $P_S \neq G$ and $K_S = 1$. Then we also have $Q_S \neq G$ and $L_S = 1$. Thus, we have

$$\left(\frac{G\times G}{T}\right) = \mathrm{Inf}^{G}_{G/K_T}\mathrm{Iso}^{\phi}_{G/K_T,G/L_T}\mathrm{Def}^{G}_{G/L_T} \quad \text{and} \quad \left(\frac{G\times G}{S}\right) = \mathrm{Ind}^{G}_{P_S}\mathrm{Iso}^{\varphi}_{P_S,Q_S}\mathrm{Res}^{G}_{Q_S}$$

Contrary to the previous steps, this step is not straightforward, and we do not have a complete answer. To illustrate the difficulty, note that in this case, even if $G/K_T \cong P_S$, we may not factor $\frac{G\times G}{S\trianglelefteq T}$ through the product of sections. For instance, take a section (T,S) of $Q_8\times Q_8$ such that they satisfy the conditions in Step 4. More explicitly, let $P_S = C_2 = \{1,-1\}$, $K_S = 1$, and $K_T = I = \langle i\rangle$, $P_T = Q_8$. Thus $T = \{(x,y) :\ xI = yI\}$ and $S = \{(1,1),(-1,-1)\}$. Hence $S \trianglelefteq T$.

Since $|G| = 8$, a non-trivial decomposition should be a group H of order smaller than 8. First consider the trivial options of H being the common group $Q_8/I \cong C_2$.

In this case, we can write

$$\left(\frac{G\times G}{T}\right) = \mathrm{Inf}_{Q_8/I}^{Q_8}\mathrm{Def}_{Q_8/I}^{Q_8} = \frac{Q_8\times(Q_8/I)}{I}\times\frac{(Q_8/I)\times Q_8}{J} \tag{3.33}$$

$$\left(\frac{G\times G}{S}\right) = \mathrm{Ind}_{C_2}^{Q_8}\mathrm{Iso}_{C_2,Q_8/I}^{\alpha}\mathrm{Iso}_{Q_8/I,C_2}^{\alpha^{-1}}\mathrm{Res}_{C_2}^{Q_8} = \frac{Q_8\times(Q_8/I)}{D*C^{\alpha}}\times\frac{(Q_8/I)\times Q_8}{C^{\alpha^{-1}}*R} \tag{3.34}$$

where α is the unique isomorphism $\alpha : Q_8/I \to C_2$. This decomposition holds in $\Gamma(Q_8\times Q_8)$ of $(D*C^{\alpha})\trianglelefteq I$ and $(C^{\alpha^{-1}}*R)\trianglelefteq J$. But we have $I=\{(g,gI): g\in Q_8\}$ and

$$\begin{aligned} D*C^{\alpha} &= \{(g,g): g\in C_2\} * \{(\alpha(g'I), g'I): g'\in Q_8\} \\ &= \{\alpha(g'I), g'I): g'\in Q_8\} \\ &= \{(1,I),(-1,jI)\} \end{aligned} \tag{3.35}$$

Since we have $(-1,jI)\notin I$, so the subgroup I does not contain $D*C^{\alpha^{-1}}$. Thus the above decomposition is not valid in $\Gamma(Q_8\times Q_8)$. For the other possibilities, we may try to decompose $\langle T,S\rangle$ through a group of order 4. Let H be a group of order 4. At this point, it can be C_4 or V_4. We need to obtain a decomposition

$$\frac{G\times G}{T} = \left(\mathrm{Inf}_{Q_8/I}^{Q_8}\times X\right)\times_H\left(Y\times\mathrm{Def}_{Q_8/I}^{Q_8}\right) \tag{3.36}$$

$$\frac{G\times G}{S} = \left(\mathrm{Ind}_{C_2}^{Q_8}\times Z\right)\times_H\left(U\times\mathrm{Res}_{C_2}^{Q_8}\right) \tag{3.37}$$

for some appropriate choices of bisets X, Y, Z and U. In order to have the above equalities, we should have

$$X\times_H Y = \mathrm{id} \quad \text{as} \quad (Q_8/I, Q_8/I)-\text{bisets} \tag{3.38}$$

and

$$Z\times_H U = \mathrm{id} \quad \text{as} \quad (C_2, C_2)-\text{bisets} \tag{3.39}$$

Thus, we may assume that

$$ {}_{Q_8/I}X_H = \mathrm{IsoDef}^H_{H/A} \quad \text{and} \quad {}_HY_{Q_8/I} = \mathrm{Inf}^H_{H/A}\mathrm{Iso} \tag{3.40} $$

and

$$ {}_{C_2}Z_H = \mathrm{IsoDef}^H_{H/A} \quad \text{and} \quad {}_HU_{C_2} = \mathrm{Inf}^H_{H/A}\mathrm{Iso} \tag{3.41} $$

Straightforward calculation shows that this decomposition does not lead to a decomposition in $\Gamma(Q_8 \times Q_8)$,too.

It is easy to show that the other orders 1,3,5,6,7 are not possible,too. Thus, $\langle T, S\rangle$ with the above conditions does not factorize in $\Gamma(Q_8 \times Q_8)$.

In order to talk about a decomposition in a more general setting, we are restricted by the conditions as in the following lemma.

Lemma 3.2. *Suppose $G = H \times A$ is an internal direct product for some subgroups H and A with $A \subseteq K_T$ and $P_S \subseteq H$. Then $\langle T, S\rangle \in I_\Gamma(G)$.*

Proof. The assumption $G = H \times A$ implies that every element of G can uniquely be expressed as a product of an element of H and an element of A. Thus there is a projection map, denoted by $\pi_H : G \to H$ which sends every element g of G to its unique H-component. Then by the first isomorphism theorem, we obtain an isomorphism, say $\alpha : G/A \to H$. Now, we consider the following decompositions;

$$ \begin{aligned} \left(\frac{G\times G}{T}\right) &= \mathrm{Inf}^G_{G/A}\mathrm{Inf}^{G/A}_{G/K_T}\mathrm{Iso}^{\phi}_{G/K_T,G/L_T}\mathrm{Def}^G_{G/L_T} \\ &= \frac{G\times (G/A)}{I} \times \frac{(G/A)\times G}{\alpha(T,T)} \end{aligned} \tag{3.42} $$

$$ \begin{aligned} \left(\frac{G\times G}{S}\right) &= \mathrm{Ind}^G_H\mathrm{Iso}^{\alpha}_{H,G/A}\mathrm{Ind}^{G/A}_{\alpha^{-1}(P_S)}\mathrm{Iso}^{\alpha^{-1}}_{\alpha^{-1}(P_S),P_S}\mathrm{Iso}^{\varphi}_{P_S,Q_S}\mathrm{Res}^G_{Q_S} \\ &= \frac{G\times (G/A)}{D * C^{\alpha}} \times \frac{(G/A)\times G}{\alpha(S,S)} \end{aligned} \tag{3.43} $$

Here instead of inducing directly from P_S to G, we first map to the group H via α^{-1}, induce to G/A and use α to go back to the group G.

Now, we need to check whether this decomposition holds on $\Gamma(G \times G)$ or not. First, we need to check $D * C^\alpha \trianglelefteq I$ but at first we shall check that $D * C^\alpha \subseteq I$. Let $(\alpha(gA), gA) \in (D * C^\alpha) \leq G \times G/A$, then

$$\begin{aligned} (\alpha(gA), gA) \in I &\Leftrightarrow \alpha(gA) \in gA \\ &\Leftrightarrow \alpha(gA) = ga = h \text{ for some } a \in A, h \in H \\ &\Leftrightarrow g = ha^{-1} \text{ for some } a \in A, h \in H \\ &\Leftrightarrow g \in HA \end{aligned}$$

Since we have assumed that $G = H \times A$, the above condition is satisfied for any $g \in G$, and hence $(\alpha(gA), gA) \in I$. In order to show the normality condition, let $(g_1, g_1A) \in I$. Then ${}^{(g_1,g_1A)}(\alpha(gA), gA) \in D * C^\alpha$ if and only if $(g_1\alpha(gA)g_1^{-1}, g_1gg_1^{-1}A) \in D * C^\alpha$. In other words, $\alpha(g_1gg_1^{-1}A) = g_1\alpha(gA)g_1^{-1}$. At this point, we should note that any commutator $[a, h]$ is in $H \cap A$ since they are both normal subgroups of G. But since the intersection is trivial, this implies that H commutes with A. On the other hand we know $g = ha$ and $g_1 = h_1a_1$ for some $a, a_1 \in A$, $h, h_1 \in H$. Therefore,

$$\begin{aligned} \alpha(g_1gg_1^{-1}A) &= \alpha(g_1A)\alpha(gA)\alpha(g_1^{-1}A) \\ &= h_1hh_1^{-1} \\ &= h_1a_1ha_1^{-1}h_1^{-1} \\ &= g_1\alpha(gA)g_1^{-1} \end{aligned} \tag{3.44}$$

as required. Now, it remains to show that $\alpha(S, S) \trianglelefteq \alpha(T, T)$. We know from the equation (3.8) that $\alpha(T, T) = I * C^\phi * J = \{(gA, g') \in (G/A) \times G : \ \phi(g'L_T) = gK_T\}$.

We also have

$$\begin{aligned}
\alpha(S,S) &= (D * C^{\alpha^{-1}}) * (C^{\varphi} * R) \\
&= \{(\alpha^{-1}(p_s), \alpha^{-1}(p_s)) :\ p_s \in P_S\} * \{(\alpha^{-1} \circ \varphi)(q_s), q_s) :\ q_s \in Q_S\} \\
&= \{(\alpha^{-1}(p_s), q_s) \in \alpha^{-1}(P_S) \times Q_S :\ \alpha^{-1}(p_s) = (\alpha^{-1} \circ \varphi)(q_s)\} \\
&= \{(\alpha^{-1}(p_s), q_s) \in \alpha^{-1}(P_S) \times Q_S :\ p_s = \varphi(q_s\} \leq G/A \times G
\end{aligned} \tag{3.45}$$

At the same time, we have

$$T = \{(g, g') \in G \times G :\ \phi(g' L_T) = g K_T\} \tag{3.46}$$

$$S = \{(p_s, q_s) \in P_S \times Q_S :\ \varphi(q_s) = p_s\} \tag{3.47}$$

and $S \trianglelefteq T$. Let $(\alpha^{-1}(p_s), q_s) \in \alpha(S,S)$ then $(p_s, q_s) \in S$, so $(p_s, q_s) \in T$. That is $\phi(q_s L_T) = p_s K_T = p_s A K_T = \alpha^{-1}(p_s) K_T$ since $A \leq K_T$. Hence $(\alpha^{-1}(p_s), q_s) \in \alpha(T,T)$. This proves that $\alpha(S,S) \leq \alpha(T,T)$.

Now, let $(\alpha^{-1}(p_s), q_s) \in \alpha(S,S)$ and $(gA, g') \in \alpha(T,T)$. Then, $(p_s, q_s) \in S$ and $(g, g') \in T$. Since $S \trianglelefteq T$ and $(g, g') \in T$, ${}^{(g,g')}(p_s, q_s) \in S$. i.e. $\phi(g' q_s g'^{-1}) = g p_s g^{-1}$ holds. This also implies that ${}^{(gA,g')}(\alpha^{-1}(p_s), q_s) \in \alpha(S,S)$ since $P_S \trianglelefteq H$. Therefore, it have been shown that $\alpha(S,S) \trianglelefteq \alpha(T,T)$.

Finally, the required decomposition follows as

$$\frac{G \times G}{S \trianglelefteq T} = \frac{G \times (G/A)}{(D * C^{\alpha}) \trianglelefteq I} \times \frac{(G/A) \times G}{\alpha(S,S) \trianglelefteq \alpha(T,T)} \tag{3.48}$$

□

As a result, we have proved the following theorem.

Theorem 3.3. *Let G be a finite group, and $S \trianglelefteq T \leq G \times G$. Then $\langle T, S \rangle$ is decomposable if one of the following conditions hold.*

(i) $P_T < G$ *or* $Q_T < G$.

(ii) $P_T = G = Q_T$ *and* $K_S \neq 1$ *or* $L_S \neq 1$.

(iii) $G = H \times A$ *is an internal direct product for some subgroups* H *and* A *with* $A \subseteq K_T$ *and* $P_S \subseteq H$.

But, since we do not know the opposite direction of the Lemma 3.2, there can still be indecomposable sections under the conditions of Step 4. Hence, we just show some of indecomposable sections in the next part.

3.1.2. Some Indecomposable Sections

Theorem 3.4. *Let* $S \trianglelefteq T$ *in* $G \times G$. *Suppose one of the following three conditions holds:*

(i) $(\frac{G\times G}{T}) = \mathrm{Inf}^G_{G/K_T}\mathrm{Iso}^{\phi}_{G/K_T,G/L_T}\mathrm{Def}^G_{G/L_T}$ *and* $(\frac{G\times G}{S}) = \mathrm{Iso}^{\varphi}_{G,G}$ *such that*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ G/L_T & \xrightarrow{\phi} & G/K_T \end{array} \quad \textit{commutes} \ \ \textit{and} \ L_T \leq Z(G)$$

(ii) $(\frac{G\times G}{T}) = \mathrm{Iso}^{\phi}_{G,G}$ *and* $(\frac{G\times G}{S}) = \mathrm{Ind}^G_{P_S}\mathrm{Iso}^{\varphi}_{P_S,Q_S}\mathrm{Res}^G_{Q_S}$ *such that* $\varphi = \phi_{|Q_S}$ *and* $Q_S \trianglelefteq G$.

(iii) $(\frac{G\times G}{T}) = \mathrm{Iso}^{\phi}_{G,G}$ *and* $(\frac{G\times G}{S}) = \mathrm{Iso}^{\varphi}_{G,G}$ *such that* $\phi = \varphi$.

Then, $\langle T, S\rangle$ *is indecomposable in* $\Gamma(G \times G)$.

Proof. In the first condition (i), we have

$$T = \{(g, g') : \ \phi(g'L_T) = gK_T\} \ \ \textit{and} \ \ S = \{(\varphi(g), g) : \ g \in G\} \tag{3.49}$$

Thus, $S \leq T$ if $\phi(g'L_T) = \varphi(g)K_T$ for any $(\varphi(g), g) \in S$. In other words, the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ G/L_T & \xrightarrow{\phi} & G/K_T \end{array}$$

commutes. Moreover, Moreover, $S \trianglelefteq T$ if ${}^{(g_1,g_2)}(\alpha(g), g) \in S$ for any $(g_1, g_2) \in T$ and $(\alpha(g), g) \in S$. That is $(g_1\alpha(g)g_1^{-1}, g_2gg_2^{-1}) \in S$ if and only if $\varphi(g_2gg_2^{-1}) = g_1\alpha(g)g_1^{-1}$ when $\phi(g_2L_T) = g_1K_T$. Since the diagram commutes, we have $\varphi(L_T) = K_T$. Then,

$$\begin{aligned} g_1K_T = \phi(g_2L_T) = \varphi(g_2)K_T \quad &\Rightarrow \quad g_1^{-1}\varphi(g_2) \in K_T = \varphi(L_T). \\ &\Rightarrow \quad g_1^{-1}\varphi(g_2) = \varphi(l^{-1}) \quad \text{for some } l \in L_T \\ &\Rightarrow \quad g_1 = \varphi(g_2 l) \end{aligned}$$

Hence $g_1\alpha(g)g_1^{-1} = \varphi(g_2l)\alpha(g)\varphi(g_2l)^{-1} = \varphi(g_2lgl^{-1}g_2)$. But $\varphi(g_2lgl^{-1}g_2) = \varphi(g_2gg_2)$ holds if and only if $lgl^{-1} = g$. That is $L_T \leq Z(G)$. In the second one, we have

$$T = \{(\phi(g), g) : \ g \in G\} \ \ \textit{and} \ \ S = \{(\varphi(q_s), q_s) : q_s \in Q_S\} \tag{3.50}$$

Therefore, it is clearly seen that $S \trianglelefteq T$ if $\varphi = \phi_{|Q_S}$ and $Q_S \trianglelefteq G$. In the last condition, we have

$$T = \{(\phi(g), g) : \ g \in G\} \ \ \textit{and} \ \ S = \{(\varphi(g'), g') : g \in G\} \tag{3.51}$$

Hence, it immediately follows that $S \trianglelefteq T$ if $\phi = \varphi$. □

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