DISTRIBUTED CONTROL OF STRINGS OF VEHICLES IN AUTOMATED HIGHWAY SYSTEMS

by

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ABSTRACT

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In this thesis, distributed control of vehicular arrays is studied. The main approach is to design distributed controllers, where the vehicle interconnections repeated in the controllers. The spatially distributed nature of the vehicular array, and the presence of interconnections make the sharing of feedback information necessary and challenging. This factor has motivated new research direction to distributed control for strings of vehicles. The main objective of this thesis is to attenuate disturbance signals, which is imposed by the leader, in the spatial direction. The uniqueness of our design is the communication channel between controllers similar to the idea in distributed control. Well-posedness, stability and dissipativity conditions of interconnected systems are given in terms of LMIs. The controllers are designed such that the $\mathcal{L}_2$-gain of the closed loop system minimized. Also a method for the construction of node by node distributed controllers presented, similar to LPV control synthesis. We designed identical controllers for all interior vehicles, and two different controllers placed at boundaries. And, our simulation results show that the resulting controllers could attenuate disturbance signals in the spatial direction, successfully.

ÖZET

OTOMATİK OTOYOL SİSTEMLERİNDE ARAÇ DİZİLERİNİN DAĞITILMIŞ DENETİMİ

Bu tezde araç dizileri için dağıtılmış denetleyiciler incelenmiştir. Bu tezdeki genel yaklaşım, araçlardaki bağlantıları denetleyicilerde tekrar ederek dağıtılmış denetleyicilerin tasarlamansıdır. Dizilerdeki araçların birbirine bağımlılığı denetleyicilerin de birbirine bağlanmasını gerekli kılıyor ve problemi ilgi çekici bir hale getiriyor. Araç dizisindeki bu bağımlılık bizi dağıtılmış denetleyicilerin tasarlanmasına doğru yöneltmiştir. Burdaki temel amaç, lider araç tarafından sisteme giren bozan etkenin sistemde denetleyici tarafindan bastırılmasıdır. Buradaki tasarımın önceki tasarımlardan temel farkı parçalanmış denetleyiciler arasında bir haberleşme kanalının varlığıdır. Bağımlı sistemlerin iyi sorulmuşluk, kararlılık ve disipatiflik koşulları doğrusal matris eşitşizlikleri olarak verildi. Denetlenen sistemin $\mathcal{L}_2$ kazancı asgariye indirildi. Dağıtılmış denetleyicilerin düğüm düğüm sentezi gösterildi. Sonuçta araç dizisi için özdeş denetleyiciler tasarlandı, ek olarak sınır koşulları için farklı iki denetleyici daha tasarlandı. Elde edilen denetleyicilerin simülasyon sonuçları, denetleyicilerin araç dizisi boyunca aralık hatalarını bastırmakta başarılı olduğu görüldü.

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LIST OF SYMBOLS/ABBREVIATIONS

A_{TT}	State matrix: Temporal Temporal
A_{TS}	State matrix: Temporal Spatial
A_{ST}	State matrix: Spatial Temporal
A_{SS}	State matrix: Spatial Spatial
B_T	Input matrix: Temporal Spatial
B_S	Input matrix: Spatial
B_T	Input matrix: Temporal
C_T	Output matrix: Temporal
C_S	Output matrix: Spatial
D	Direct feedthrough matrix
K	Controller
P	Plant
P_{cl}	Closed Loop Plant
$\mathbf{P}$	Permutation matrix
$\mathcal{P}$	Internal Supply Rates Characterized by IQCs
α	Non-dimensional Weight
γ	Disturbance attenuation level
Δ	Interconnection with non-ideal channels and uncertainty
Δ_0	Interconnection with ideal channels and no uncertainty
Δ_k	Repeated interconnection in the controllers
A/IHS	Automated/Intelligent Highway Systems
IQC	Integral Quadratic Constraint
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LTI	Linear Time-Invariant

1. INTRODUCTION

1.1. Problem Statement and Literature Review

Longitudinal control of platoons of vehicles has attracted much attention over the past decades. Automated or Intelligent highway systems (A/IHS) are still an active area of research in systems and controls area. The attention is focused on developing methods to allow strings of vehicles to automatically move at a desired velocity with a specified safe separation distance between vehicles. Over the past years, there has been a renewed interest in distributed control of large scale systems, which is applicable to the vehicular string problem. The overall complex dynamical behavior is dictated by the dynamic interactions between agents. The presence of interconnections between agents makes the controller synthesis challenging.

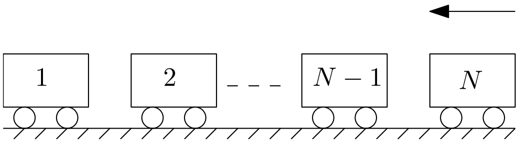

Figure 1.1. Vehicle String

In Figure 1.1, N vehicles are placed on a line and form a vehicle array to maximize traffic throughtput and to reduce traffic congestion. It is easy to see that inter vehicle spacing errors are dependent on both preceding and succeeding vehicle dynamics for the corresponding unit. Hence, spacing errors create a dynamic interaction between each unit. In order to avoid collision, inter vehicle spacing errors are required to be small and inter vehicle spacing errors should decrease along the platoon. We try to accomplish a constant spacing policy where the desired inter vehicular spacing is independent of the velocity.

Earlier work, in vehicular string problem, used optimal centralized control to regulate a small string of vehicles in [1] and [2]. In these works, the objective is to find an optimal controller that all agents controlled by a centralized controller. In this method, computation of controller gets harder, and the size communication system increases enormously as the number of agents increase. Additionally, this method leads to a burdensome data handling problem and increased network requirements. A decentralized control is necessary for simple controllers and for a simple communication system.

For this purpose, a simple decentralized control method proposed which is known as reference cell scheme in [3] by Wilkie. Each controller decides its action by only looking its absolute position which is called moving cell reference. In this method, road serves as a reference for all vehicles. But this method is unsafe because it does not take into account the relative position of vehicle with the preceding one. Relative position information is required because vehicles could be able to avoid collision.

Vehicular string problem can be investigated by two methods. One is to eliminate spatial variable by using $Z-$transform and study the resulting differential equation in Z to t to conclude the stability. The other approach is to eliminate time using I/O properties and study the resulting difference equation in the vehicle index to examine the stability of vehicle of string.

Chu proposes $Z-$transform in order to eliminate the dependence on the vehicle index in [4]. Then they study the resulting differential equation. This method simplifies the problem considerably. In this method, absolute positions of vehicles are available. Every vehicle has the absolute position information of itself and the preceding one. By using this method Chu also showed that constant spacing policy cannot be accomplished under any choice of identical linear controllers.

A variable spacing policy is adopted by Ioannou and Chien in [5] for constant time headway policy in vehicular strings. They showed that constant time headway policy can be accomplished by static feedback controllers. Also, they mentioned slinky effect in constant spacing policy. Slinky-type phenomenon is a consequence of lack of feedforward information (the lead vehicle information), in case for the constant spacing policy.

The second approach is used by Hedrick and Swaroop in [6]. They used relative positions of vehicles. They do not used road as a reference, instead they choose a controller scheme that each vehicles reference is its predecessor. They show that the constant spacing policy cannot be accomplished by using identical linear controllers by employing the difference equations between errors. They show that the lead vehicle information is required in order to suppress disturbance signal effectively. Moreover, alternative methods are presented such as using the preceding vehicle acceleration information, building an observer in order to estimate lead vehicle information. By this way, they try to eliminate the dependance on lead vehicle information in both [7] and [6]. They also show disturbance propagation in vehicular strings, by imitating difference relation to complementary sensitivity function in [8]. They again state that the lead vehicle information is essential. Their work gave us the idea of some degree of extra information required such as lead vehicle information, acceleration or control input information preceding vehicle to be able to employ a semi autonomous control strategy.

Similar to [6], in [9] it is shown that identical PID controllers cannot suppress disturbance along the platoon. They use relative spacing values, and implement identical PID controllers with proper PID gains. They show that the errors are amplified downstream along the platoon. This violates stability in the spatial direction, since the vehicle index goes to infinity the spacing error becomes unbounded. To overcome this problem, PID gains should be tuned as the vehicle index changes. And, it is shown that updating PID gains will prevent amplification of errors further into vehicle platoon. However, this method requires that every vehicle should know its position (index) in the platoon. Hence, controllers should be able to communicate [9]. The idea of scheduling gains via a communication channel along the platoon becomes more apparent.

In [10], a robot formation problem is solved. They used bidirectional controllers with weights as a measure to sensitivity to disturbances. By using bidirectional control law, they alleviate penetration of the disturbance signals into the platoon. And they showed that there is conflicting objectives of stability in the spatial domain (propagation of signals as the vehicle index increases) and disturbance attenuation in the time domain.

A new control design perspective for spatially interconnected systems is presented in [11]. This approach use $Z-$transforms similarly as in [4]. Two sided spatial fourier transform is employed to eliminate spatial variables. Then, they study the resulting equations for all frequencies to conclude stability of whole system.

For our problem, we want that injected disturbance should decrease downstream along the vehicle array. If the disturbance signal further penetrates into the vehicle string, we say that there is an instability in the spatial direction. What we require is that the disturbance signal should be attenuated along the vehicle platoon. From the earlier literature, it is shown that decentralized identical controllers cannot attenuate disturbance along the spatial domain, although they are very good in the time scale. We need non identical controllers such that the gains are varied as the vehicle index changes as in [9]. At this point, we have noticed that, this problem requires smarter controller. Thus we can get better performance in attenuation of the disturbance signal along the vehicle string. From this point, we focused our attention to distributed controllers, because their inherent structure allows more flexible communication structure. Also, they provide tractable solutions to vehicular string problem. We know that, if we want to use spacing error information, the plant dynamics are interconnected. Then, it would be wise to put interconnected controllers which has the same structure as the plant structure.

Now, we should illustrate the fundamental difference between centralized, decentralized and distributed controllers. Centralized controllers are regular controllers for whole system, a large interconnected system is controlled by a large controller. That is to say centralized control approach leads models with large number of states, input and outputs. Central controller decides all actions of the interconnected system, as can be seen in Figure 1.2. Their disadvantages are obvious like computational complexity, commination network requirements and most importantly implementation problem of controllers.

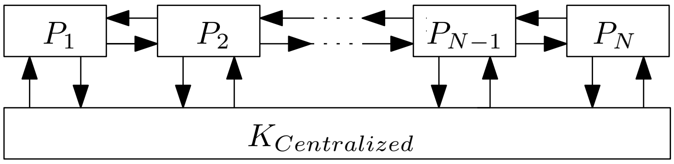

Figure 1.2. Centralized Control

On the other hand, decentralized controllers are a kind of constrained controllers, in which controllers have constrained information to control subsystems independently. Their actions are based on the measurements of individual units and there is no information exchange through controllers. The objective is to design controllers at each node separately as an be seen in Figure 1.3 .

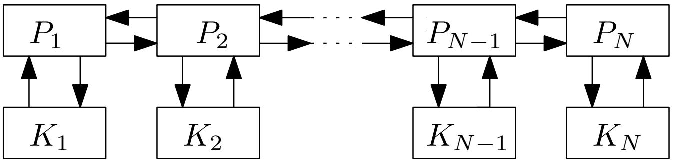

Figure 1.3. Decentralized Control

If we allow an information exchange in controllers through a spatial interconnection channel, we can get distributed controllers. In Figure 1.4, interconnection channel between controllers can be seen easily. Also, different units could be linked to each other arbitrarily by this approach. Hence, incorporation of different units allowed in this framework. The approach is to design simple linked controllers at each node separately.

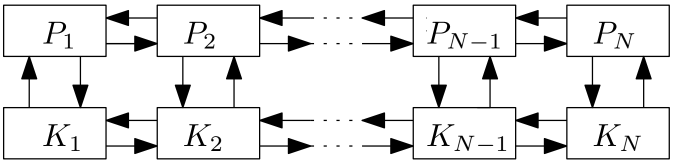

Figure 1.4. Distributed Control

Distributed control approach is fairly new idea with respect to decentralized control. The idea goes back to [12], it is shown that when the underlying dynamics of the system are spatially invariant, then optimal controllers will necessarily have a spatially invariant structure. The optimal controllers have a certain degree of inherent decentralization for such problems. Main advantage is that, one need only design the controller for a single actuator. In [11] and [13], it is shown that such systems could be diagonalized into a decoupled form. The essential idea is, rather than a highly complex centralized controller with information from all the distributed array, it would be desirable to have distributed intelligence, where each actuator runs a local algorithm with information from the neighboring sensors.

These ideas are further exploited in [14] and [15]. In [14], spatially invariant systems are diagonalized by a fourier transform over the spatial domain, the control design problem with quadratic criteria can be decoupled over spatial frequency. The control input $u(t, s)$ for each node is a state feedback control, based on the state information about itself and its direct neighbors. In [15], homogeneous interconnected systems are studied using the $Z-$transform analysis. Furthermore, it is shown that many homogeneous large-scale systems can be reasonably approximated by an infinite number of coupled identical subsystems.

A LMI approach is presented in [16], for decentralized control of distributed parameter systems. They showed that, in order for any optimal control technique to be successful, the structure of the system must be exploited in order to obtain tractable algorithms. The control design problem is generalized to a H_∞ optimization problem by repeating the interconnection structure in the controllers. The ideas are further developed in [17] and [18] for the distributed control design for interconnected systems. In [17], a state space approach with highly structured interconnection topology through spatial variables is presented. Also, implementation of the distributed control strategy explicitly given for spatially invariant systems. For such systems stability, well-posedness and contractiveness conditions are shown in terms of LMIs. These ideas are further exploited and developed in [19], [20], [21], [22] and [23] and also in several articles and proceedings. In [19] and [20], it is shown that framework can be extended to heterogeneous interconnected systems over arbitrary graphs and also quadratic supply rates are defined for the interconnection variables. In [22], integral quadratic constraints are used to model interconnections between subsystems. In [23], the problem transformed into a LPV problem in which subsystems could have parametric uncertainty and online measurement of these variables as in the case gain scheduling. The similarities between LPV control and distributed control are investigated. In a novel paper for gain scheduling, it is shown that each linear fractional parameter must be repeated in the controller analogous to the fact that in H_∞ control in [24]. This approach allows us to treat gain-scheduled controllers for LPV systems.

1.2. Proposed Method

Our motivation is to design controllers for the strings of vehicles in which controllers can schedule their gains via controller communication channel. This approach is called distributed control approach in which interconnected plants are controlled by interconnected controllers. The main idea is to put controllers whose dynamics is similar to the dynamics of the plant. For instance, controller state dimension can be taken to be equal to the plant state dimension. Also, this new communication channel allows us to use more information in the control synthesis, this will bring us more flexible design approach. Hence, by allowing the controller to use extra information channel, we can formulate *convex* conditions for the existence of *output feedback controllers*. Notice that this is equivalent to the online measurement of parameters in LPV controller synthesis. This is the main difference of distributed controllers compared to the decentralized controllers. This method is closely related to IQCs analysis methods, dissipativity theory to large scale systems, in which interconnection supply rates parameterized by IQCs.

Every vehicle must have sensing, actuating, and limited communication capability for distributed control strategy. In our case, we assume that we have identical vehicles which have the sensing, actuating, and limited communication capability. After these assumptions, we can the exploit interconnection structure of the vehicle array and structured interconnections, in order to obtain tractable algorithms. Then, it can be solved similar to LPV synthesis problem, if we capture vehicle interconnections via LFT framework. Moreover, equivalent LMI condition, can be found, and it can be used for the search of the desirable controllers. In short, we can design distributed controllers which has the same interconnection structure as controlled plants.(Interconnection structure of the subsystems should be repeated in the controllers.) By doing this, we allowed to repeat the interconnection structure in the controllers similar to the fact H_∞ control. This is actually sort of gain-scheduled control through interconnection channel.

To sum up, in this work we are trying to implement a distributed control strategy to the vehicular string problem instead of decentralized control. The main objective is to prevent disturbance amplification along the vehicle platoon.

1.3. Notation

We use standard notation, for a real symmetric matrix M is "positive definite" if $p^T M p \succ 0$ for all $p \neq 0$. We will use this condition as $M \succ 0$. For "positive semi definite" matrix operators $(\succ)$ can be replaced by $(\succeq)$. We denote the number of positive, negative, and zero eigenvalues of a real symmetric matrix M as n_+, n_-, and n_0, respectively. Then inertia triplet can be defined as $in(M) = (n_+, n_-, n_0)$. Also, we denote composition of null spaces of the matrix $M^\top$ as $M_\perp$. Equivalently, image of $M_\perp$ spans the kernel of M^T.

The notation is fairly standard in control theory. The LTI system $P := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ represents the realization, where $A \in \mathbb{R}^{m\times m}$ is state, B $\in \mathbb{R}^{m\times n}$ is input, C $\in \mathbb{R}^{p\times m}$ is output and D $\in \mathbb{R}^{p\times n}$ is the feed-through matrix. In shorthand, we can write transfer function of the system as

$$P(s) = C(sI - A)^{-1}B + D.$$

Since we will be dealing assembly of finite number of subsystems, one can concatenate system matrices diagonally as

$$\underset{k\leq i\leq l}{\mathbf{diag}}(A_i) := \begin{pmatrix} A_k & 0 & \dots & 0 \\ 0 & A_{k+1} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & A_l \end{pmatrix}. \tag{1.1}$$

Similarly, we can also concatenate vectors vertically as

$$\underset{k\leq i\leq l}{\mathbf{cat}}(x_i) := \begin{pmatrix} x_k \\ x_{k+1} \\ \vdots \\ x_l \end{pmatrix}. \tag{1.2}$$

Additionally, we need to define p and q as interconnection variables between subsystems, since we will be explicitly using them in state space equations. We will construct stability and well-posedness concepts based on these internal variables.

In certain places where the content of a matrix is obvious, we use a star ($\star$) instead of the actual content for compactness reasons.

1.4. Organization of Thesis

The thesis outline is as follows: in Chapter 1, we provide a brief discussion of literature and then introduce the proposed method and notations used throughout the thesis. In chapter 2, we present spatially interconnected systems and give some examples of such systems. In Chapter 3, we show that vehicular string is a spatially interconnected system. Also, we give spatially interconnected vehicle models. In chapter 4, we analyze interconnected systems via integral quadratic constraints. Also, we will define internal supply rates for each subsystem. Chapter 5 presents controller synthesis methods and resulting LMIs node by node. In Chapter 6, we present a vehicle string model and synthesized distributed controllers. Also, we provide simulations of resulting controllers and we will compare them with decentralized controllers. Finally in Chapter 7, we give a summary of the overall study and we present our conclusions.

2. INTERCONNECTED SYSTEMS

2.1. Arbitrarily Interconnected Systems

We can consider a large global system in which assembly of subsystems connected arbitrarily as in Figure 2.1. In the global system, disturbance to the system d and performance output z can be thought as a composition of vectors. Each sub-system can affect the dynamics of others through interconnection variables. From Figure 2.1, it can be seen that an interconnected system can be interpreted as a large multi input multi output control problem.

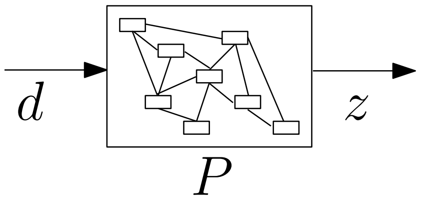

Figure 2.1. Interconnected system

The global system in Figure 2.1 consists of an assembly of L subsystems which are connected arbitrarily. However, if we focus inside composite plant, we can further divide system with interconnections. Here our interpretation is that the whole system can be viewed as a composite system. The interconnected system contains finitely many subsystems P_i which are interacting between themselves as can be seen in Figure 2.2.

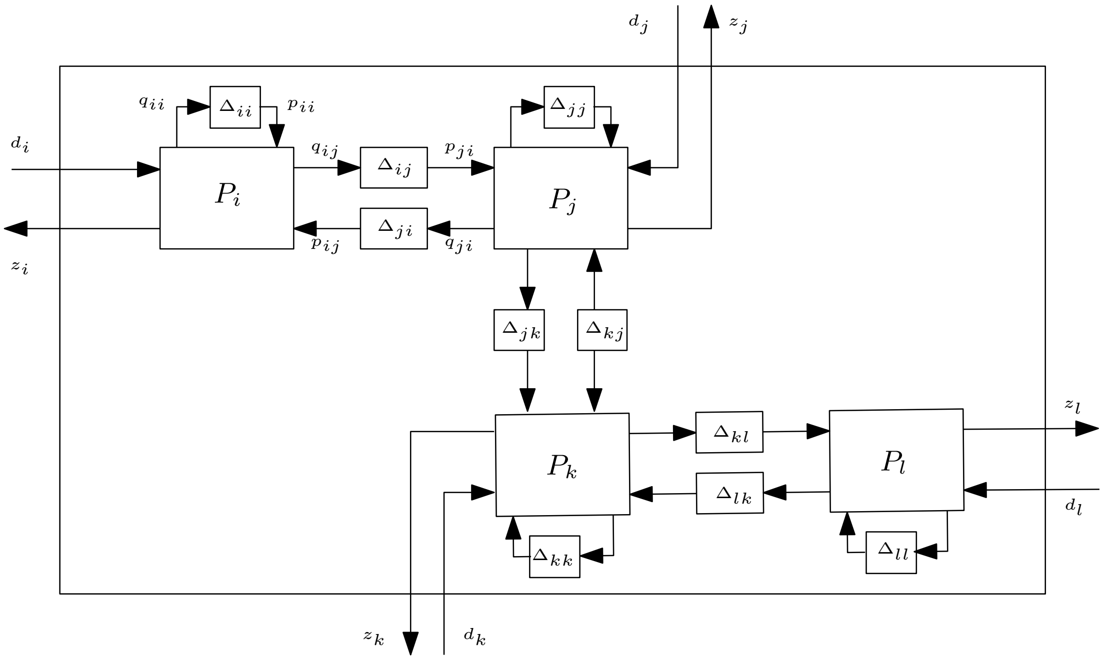

Figure 2.2. Interconnected system focused on internal variables

In Figure 2.2, we are focusing inside the large interconnected plant. Interaction between them are explicitly drawn in Figure 2.2. The interaction between the sub-plants are not restricted, so every sub-plant can affect the dynamics of any sub-plant including itself. How can we formulate these internal variables explicitly? And how can we pull out interconnection terms between units? We begin with writing the sub-system dynamics first, including the interconnection terms. We identify these internal variables as p_i and q_i respectively internal input and internal output of i^{th} sub-system. Sub-unit system dynamics can be written as

$$\begin{aligned}\dot{x}_i &= A^i_{TT} x_i + A^i_{TS} p_i + C^i_T d_i \\ q_i &= A^i_{ST} x_i + A^i_{SS} p_i + C^i_S d_i \\ z_i &= C^i_T x_i + C^i_S p_i + D^i d_i.\end{aligned}$$

By this approach, we decoupled the sub-system equations. Now, we can write sub-systems equations independently. Actually, we opened a new channel for the interconnection variables between sub-plants, since we know that the plants are interacting.

We call q_i as the internal output signal, and p_i as the internal input signal at node i. These signals can further divided into elements by indexing where they came from and where they routed. We call q_{ij} as the output signal from i and directed towards to j^{th} element, also we call p_{ij} as the internal input signal to the i^{th} element which is directed from j^{th} subsystem. From Figure 2.2, one can see that q_{ji} can be related to p_{ij} by non-ideal communication channel Δ_{ji} as

$$p_{ij} = \Delta_{ji} q_{ji}.$$

Given non-ideal channel, Δ_{ji}, can represent dynamic interconnections which include loss or delays in the interconnection link [23]. The overall interconnection map between p and q can be defined as

$$\Delta := \mathbf{P} \underset{1 \le i \le L}{\mathbf{diag}} \left(\underset{1 \le j \le L}{\mathbf{diag}} (\Delta_{ji}) \right),$$

where $\mathbf{P}$ is any suitable permutation matrix. Hence, we can represent overall interconnection map as

$$q := \Delta p,$$

where p and q are defined as

$$p := \underset{1 \le i \le L}{\mathbf{cat}} \left(\underset{1 \le j \le L}{\mathbf{cat}} (p_{ij}) \right) \quad \text{and} \quad q := \underset{1 \le i \le L}{\mathbf{cat}} \left(\underset{1 \le j \le L}{\mathbf{cat}} (q_{ij}) \right).$$

We constructed the relation between p and q, we can pullout these variables outside the whole plant. Overall LFT representation can be seen in Figure 2.3.

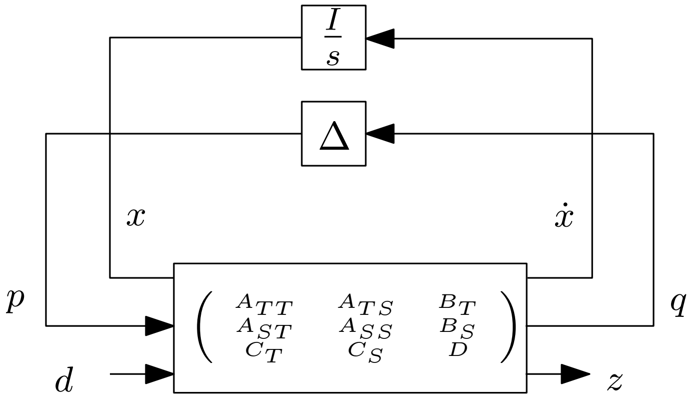

Figure 2.3. Overall system LFT representation

The whole system equations can be written as

$$\begin{pmatrix} \mathbf{cat}_{1\le i\le L}\,(\dot{x}_i) \\ \mathbf{cat}_{1\le i\le L}\,(q_i) \\ \mathbf{cat}_{1\le i\le L}\,(z_i) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{diag}_{1\le i\le L}(A^i_{TT}) & \mathbf{diag}_{1\le i\le L}\,(A^i_{TS}) & \mathbf{diag}_{1\le i\le L}\,(B^i_T) \\ \mathbf{diag}_{1\le i\le L}\,(A^i_{ST}) & \mathbf{diag}_{1\le i\le L}\,(A^i_{SS}) & \mathbf{diag}_{1\le i\le L}\,(B^i_S) \\ \mathbf{diag}_{1\le i\le L}\,(C^i_T) & \mathbf{diag}_{1\le i\le L}\,(C^i_S) & \mathbf{diag}_{1\le i\le L}\,(D^i) \end{pmatrix} \begin{pmatrix} \mathbf{cat}_{1\le i\le L}\,(x_i) \\ \mathbf{cat}_{1\le i\le L}\,(p_i) \\ \mathbf{cat}_{1\le i\le L}\,(d_i) \end{pmatrix}.$$

Or, more compactly,

$$\begin{pmatrix} \dot{x} \\ q \\ z \end{pmatrix} = \begin{pmatrix} A_{TT} & A_{TS} & B_T \\ A_{ST} & A_{SS} & B_S \\ C_T & C_S & D \end{pmatrix} \begin{pmatrix} x \\ p \\ d \end{pmatrix} \tag{2.1}$$

with obvious definitions.

Since, we used block diagonal matrices, we can always decompose system equations node by node. Actually, we have L equations in the form (2.2) for all $i = 1 : L$. We can write them separately, since we can decompose all equations. However, subsystems are still connected by our special state space representation. We call p_i as interconnection input vector composed of signals from other units. More mathematically, we can define $p_i = \mathbf{cat}_{1\le j\le L}(p_{ij})$, if we assume that all plants are connected to each other. If there is no signal between some units, we can skip those signals. Also, output of i^{th} unit can be defined as $q_i = \mathbf{cat}_{1 \le j \le L}(q_{ij})$ where q_{ij} output signal from i to j^{th} unit.

$$\begin{pmatrix} \dot{x}_i \\ q_i \\ z_i \end{pmatrix} = \begin{pmatrix} A^i_{TT} & A^i_{TS} & B^i_T \\ A^i_{ST} & A^i_{SS} & B^i_S \\ C^i_T & C^i_S & D^i \end{pmatrix} \begin{pmatrix} x_i \\ p_i \\ d_i \end{pmatrix} \tag{2.2}$$

In Figure 2.4, we can see subsystem representations.

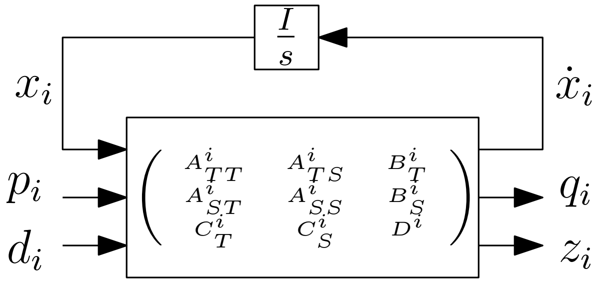

Figure 2.4. Subsystem

The i^{th} subsystem realization can be seen in the Figure 2.5. The first channel is for the internal variables, the second channel is the performance channel.

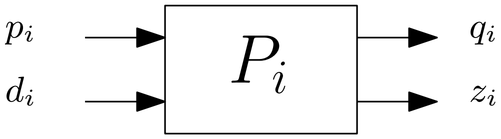

Figure 2.5. Subsystem realization

We will stick to this notation, because we can distinguish state variables, interconnection variables and performance variables easily. Most importantly, we can easily represent sub-system independently as in Equation 2.2.

It will be more clear, when we show a 3 by 3 trivial example. If we consider that we have three subsystems and all interconnections are exist and ideal. First subsystem has 6 interconnection elements 3 of them as input: one is from itself and the the other two is from other elements, other 3 of them as output: one is to itself and the the other two is to other elements. Same is true for other elements. Actually we have $\frac{6\times 3}{2}$ unique signals because all signals are counted 2 times because we used ideal channels as $p_{ij} = q_{ji}$. We can write interconnection signals as

$$p_i = \begin{pmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Same applies to q. After all, we can relate $p = (\mathbf{P}I)q$ where $\mathbf{P}$ is any suitable permutation matrix for ideal channels. More precisely, $p = \Delta q$ can be given as

$$q = \left(\begin{array}{c} q_{11} \\ q_{12} \\ q_{13} \\ \hdashline q_{21} \\ q_{22} \\ q_{23} \\ \hdashline q_{31} \\ q_{32} \\ q_{33} \end{array}\right) = \left(\begin{array}{ccc:ccc:ccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hdashline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hdashline 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{c} p_{11} \\ p_{12} \\ p_{13} \\ \hdashline p_{21} \\ p_{22} \\ p_{23} \\ \hdashline p_{31} \\ p_{32} \\ p_{33} \end{array}\right) = \mathbf{P}p.$$

At subsystem basis, we have 3 equations in the form of Equation (2.2), or we have the whole system equation in the form in Equation (2.1) and the relation between p and q.

For example, one can consider the following composite system where three systems dynamics are connected to each other

$$\dot{x}_1 = -x_1 + 0.1x_2 + 0.2x_3 + u_1 \tag{2.3a}$$

$$\dot{x}_2 = -1.2x_2 + 0.3x_1 + 0.4x_3 + u_2 \tag{2.3b}$$

$$\dot{x}_3 = -1.5x_3 + 0.2x_1 + 0.5x_2 + u_3 \tag{2.3c}$$

We can rewrite system dynamics simply by adding interconnection channels.

$$\dot{x}_1 = -x_1 + 0.1p_{12} + 0.2p_{13} + u_1 \tag{2.4a}$$

$$q_{12} = x_1 \tag{2.4b}$$

$$q_{13} = x_1 \tag{2.4c}$$

$$\dot{x}_2 = -1.2x_2 + 0.3p_{21} + 0.4p_{23} + u_2 \tag{2.4d}$$

$$q_{21} = x_2 \tag{2.4e}$$

$$q_{23} = x_2 \tag{2.4f}$$

$$\dot{x}_3 = -1.5x_3 + 0.2p_{31} + 0.5p_{32} + u_3 \tag{2.4g}$$

$$q_{31} = x_3 \tag{2.4h}$$

$$q_{32} = x_3 \tag{2.4i}$$

In this trivial example, we allowed an interconnection structure through spatial channel. By doing so, we are able to write systems of equations independently for each subsystem. This will allow us to design controllers independently which has the same interconnection structure. If the interconnection is ideal $p_{ij} = q_{ji}$, we can relate q to p by simple permutation matrix which actually defines the interconnection structure.

2.2. Spatially Interconnected Systems

Spatially interconnected systems have a fixed interconnection structure in which units takes information from their direct neighbors in a spatial manner. All vehicles are indexed based on their spatial location. For instance, if we assume that i^{th} unit is incorporating with only its local neighbors $(i-2)^{\text{th}}$, $(i-1)^{\text{th}}$, $(i+1)^{\text{th}}$, and $(i+2)^{\text{th}}$ units. This type of interconnection is a fixed structure.

In this situation, incoming interconnection signal to i^{th} element p_i can be written as

$$p_i = \begin{pmatrix} p_{i,i-2} \\ p_{i,i-1} \\ p_{i,i+1} \\ p_{i,i+2} \end{pmatrix}. \tag{2.5}$$

Same holds for interconnection output signal q_i from i^{th} element. Also different spatial interconnections can be defined further by increasing the number of agents which are incorporating. We can also expand these type of interconnection structure to more spatial variables.

2.2.1. Spatially Invariant Interconnected System

Since our main problem is closely related distributed control synthesis of spatially invariant interconnected system, we should show similarities between our approach and the approach between [17]. Actually, we are using the same approach, however we are not using spatial shift operators which is dependent on bilateral discrete fourier transformation on the spatial domain. Hence, we are not dealing with infinite number of agents. In [17], by defining spatial shift operators infinite dimensional stability can be reduced to finite sized LMI problem. In [17], spatially invariant structure of the distributed system exploited. In [17], a hybrid stability condition is presented, which guarantees both continuous stability for temporal variables and also discrete stability for spatial variables. For such systems dynamics at node i can be written as

$$\begin{aligned} \dot{x}_i &= A^i_{TT} x_i + A_{TS} p_i \\ q_i &= A^i_{ST} x_i + A_{SS} p_i. \end{aligned}$$

Due to spatial invariance, we can write $q_i = p_{i+1}$. Hence, the problem becomes a hybrid stability problem whose dynamics are given by

$$\dot{x}_i = A^i_{TT} x_i + A^i_{TS} p_i$$

$$p_{i+1} = A^i_{ST} x_i + A^i_{SS} p_i.$$

After discrete to continuous transformation in the spatial domain, system dynamics are all converted to continuous state variables. Then one could design controllers for the resulting continuous model. Details for the control synthesis for such systems can be found in [17].

We want to focus on periodic ideal interconnections, we can always represent spatial shift operators by a simple permutation matrix. We can show that q and p are coupled to each other. This permutation matrix is similar to the shift operators used in [17]. For infinite case and more abstract ideas we can refer to [25]. We observe that infinite dimensional shift operator shifts each signal in the backward direction. We can also define forward shifts in a similar manner. Details can be found in [25] and [17].

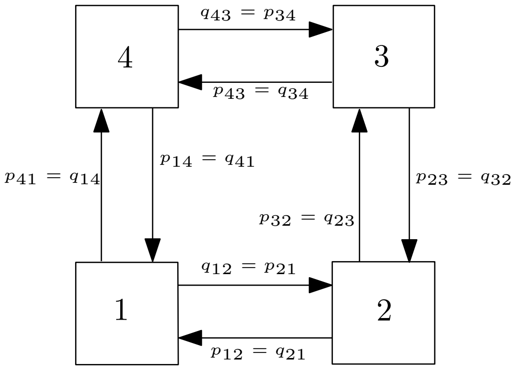

Figure 2.6. Finite Periodic Interconnection

By exploiting the ideal interconnection structure in Figure 2.6, we can write following relation between p and q as

$$\begin{pmatrix} p_{14} \\ p_{12} \\ \hdashline p_{21} \\ p_{23} \\ \hdashline p_{32} \\ p_{34} \\ \hdashline p_{43} \\ p_{41} \end{pmatrix} = \left(\begin{array}{cc:cc:cc:cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} q_{14} \\ q_{12} \\ \hdashline q_{21} \\ q_{23} \\ \hdashline q_{32} \\ q_{34} \\ \hdashline q_{43} \\ q_{41} \end{pmatrix}.$$

This permutation matrix represents the graph structure or interconnection map of the whole system. Hence, we show that for ideal channels p and q can be related as

$$q = \mathbf{P}p.$$

2.2.2. Spring Mass Damper system

For clarity, we can investigate the following spring mass damper system as an example of interconnected system. Let a set of spring mass dampers are ordered such that in Figure 2.7 to form an array. It is obvious that the plant dynamics are interconnected. How can we write system of equations in our description and interconnection structure? We will give a clear example of an interconnected spatially invariant array.

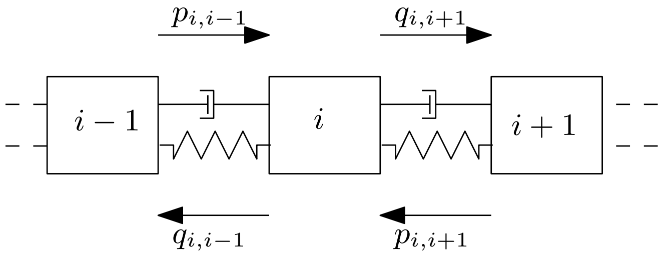

Figure 2.7. Spring mass dampers

The dynamics of subsystem indexed at i is both dependent to its direct neighbors $i-1$ and $i+1$, and obviously itself. The equation of motion for the i^{th} subsystem can be written as

$$m_i\ddot{x}_i = k(x_{i+1}-x_i) + k(x_{i-1}-x_i) + c(\dot{x}_{i+1}-\dot{x}_i) + c(\dot{x}_{i-1}-\dot{x}_i). \tag{2.6}$$

For simplicity, we can take $m = 1$, $k = 1$ and $c = 1$. The equation (2.6) further reduces to

$$\ddot{x}_i = x_{i+1} + x_{i-1} - 2x_i + \dot{x}_{i+1} + \dot{x}_{i-1} - 2\dot{x}_i. \tag{2.7}$$

From the Equation in (2.7), we see that the dynamics are coupled. In order to represent dynamics of i^{th} subsystem, position and velocity information is required from its direct neighbors. We can write system of equations at node i just by calling

$$p_i = \begin{pmatrix} p_{i,i-1} \\ \hline p_{i,i+1} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ \dot{x}_{i-1} \\ \hline x_{i+1} \\ \dot{x}_{i+1} \end{pmatrix} \quad \text{and} \quad q_i = \begin{pmatrix} q_{i,i-1} \\ \hline q_{i,i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ \dot{x}_i \\ \hline x_i \\ \dot{x}_i \end{pmatrix}.$$

By allowing this information structure, we allow that i^{th} subsystem takes position and velocity information as input from its direct neighbors and sends its position and velocity information to its direct neighbors.

It is clear that, at node i, we can write the following relation

$$\begin{pmatrix} \dot{x}_i \\ \ddot{x}_i \\ \hline x_i \\ \dot{x}_i \\ \dot{x}_i \\ \dot{x}_i \end{pmatrix} = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x_i \\ \dot{x}_i \\ \hline x_{i-1} \\ \dot{x}_{i-1} \\ x_{i+1} \\ \dot{x}_{i+1} \end{pmatrix}, \tag{2.8}$$

more compactly as

$$\left(\begin{array}{c} \dot{x}_i \\ \hdashline q_i \end{array}\right) = \left(\begin{array}{c:c} A^i_{TT} & A^i_{TS} \\ \hdashline A^i_{ST} & A^i_{SS} \end{array}\right) \left(\begin{array}{c} x_i \\ \hdashline p_i \end{array}\right). \tag{2.9}$$

2.2.3. Two Dimensional Heat Equation

For further discussion, we can consider heat equation in two dimensional plate. Heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function $u(x, y, t)$ of two spatial variables (x, y) and the time variable t, the heat equation can be described as

$$\begin{aligned} \frac{\partial U}{\partial t} &= \kappa \nabla^2 U + Q \\ &= \kappa \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) + Q. \end{aligned}$$

For the mathematical treatment, it is sufficient to consider the case $\kappa = 1$. We try to represent this system as interconnected system in a similar manner in [17]. A similar example and a more detailed discussion can be found in [26].

We can discretize the governing PDE as

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &\cong \frac{U(x+h_1, y) + U(x-h_1, y) - 2U(x, y)}{h_1} \\ \frac{\partial^2 U}{\partial y^2} &\cong \frac{U(x+h_2, y) + U(x, y-h_2) - 2U(x, y)}{h_2}, \end{aligned}$$

by gridding the two dimensional plate and treating nodes as individual subsystems, we can employ a distributed control strategy. However, the main point is that each node has sensing and actuating capability. And also, we need limited communication capability between controllers. Assume that $h_1 = 1$ and $h_2 = 1$ for simplicity. Dynamics at node (s_1, s_2) can be written as

$$\begin{aligned}\dot{U}(t, s_1, s_2) &= -4U(t, s_1, s_2)\\ &\quad + U(t, s_1 + 1, s_2) + U(t, s_1 - 1, s_2) + U(t, s_1, s_2) + U(t, s_1, s_2) + Q(t, s_1, s_2),\end{aligned}$$

where $U(t, s_1, s_2)$ is temperature of at that node (s_1, s_2) and time t and the related 2-D grid is shown in Figure 2.8 .

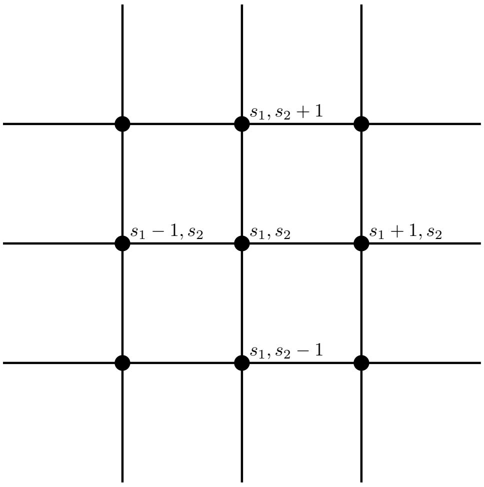

Figure 2.8. Two Dimensional Heat Grid

System dynamics at node (s_1, s_2) can be given as

$$\left(\begin{array}{c} \dot{U}(t, s_1, s_2) \\ \hline q(t, s_1, s_2) \\ \hline U(t, s_1, s_2) \end{array}\right) = \left(\begin{array}{c|cccc|c} -4 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} U(t, s_1, s_2) \\ \hline p(t, s_1, s_2) \\ \hline Q(t, s_1, s_2) \end{array}\right), \tag{2.10}$$

where p and q are defined as

$$q(t, s_1, s_2) = \begin{pmatrix} U(t, s_1, s_2) \\ U(t, s_1, s_2) \\ U(t, s_1, s_2) \\ U(t, s_1, s_2) \end{pmatrix} \quad \text{and} \quad p(t, s_1, s_2) = \begin{pmatrix} U(t, s_1 + 1, s_2) \\ U(t, s_1 - 1, s_2) \\ U(t, s_1, s_2 + 1) \\ U(t, s_1, s_2 - 1) \end{pmatrix}.$$

At last, we have showed that one can easily represent spatially interconnected systems by this approach. Also, we have showed that discretized approximations of partial differential equations are well suited to this framework. Now, we are ready to build interconnected models of vehicle platoons.

3. VEHICLE PLATOONS AS INTERCONNECTED SYSTEMS

Vehicle platoons can be seen as an example of interconnected system, since the relative spacing error is dependent both on absolute position of the vehicle itself and the preceding vehicle. Hence, we can say the dynamics of each subsystem are coupled. Similar to the spring mass damper example in the previous chapter, we can represent system of equations at each node independently.

The vehicles are placed on a line and form an array. Hence, when we say strings of vehicles, we refer an array of units lined up sequentially. Here s represents the vehicle index and t stands for time. Hence, when we call $x_s(t)$, we refer t as time and s as vehicle index. We need to define errors based on the definitions of absolute positions. The vehicles in the string are unmanned so their actions solely depend on controller actions, except than the lead vehicle. The main point is the leader imposes some spacing error due to an unexpected velocity variation in the lead vehicle. We may index the lead vehicle as zero $x_0(t)$.

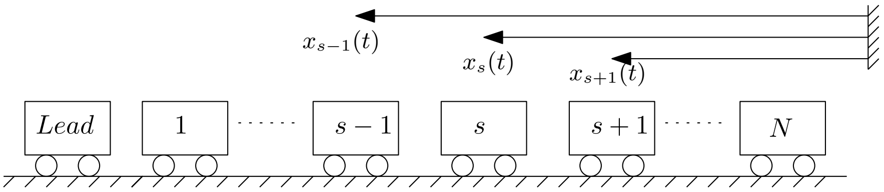

Figure 3.1. Vehicle Platoon

Here our aim is to decide control actions at each unit as $u_s(t)$. Acceleration of each unit, $\ddot{x}_s(t)$, is controlled by $u_s(t)$ as in the double integrator case. We assumed that $m = 1$ for simplicity and we use $u_s(t)$ and $a_s(t)$ interchangeable. One can always choose to work with $m \neq 1$ by defining the relation $u_s(t) = m\ddot{x}_s(t)$.

3.1. One Directional Models

Our original idea is to build interconnected vehicle models, then design controllers similar to plant structure. At first, one directional models are built. We define spacing error and take derivative of it, as can be seen in (3.1).

$$\begin{aligned} & e_s(t) := x_{s-1}(t) - x_s(t) - L_{s_{des}} && (3.1\text{a}) \\ \Rightarrow\ & \dot{e}_s(t) = \dot{x}_{s-1}(t) - \dot{x}_s(t) && (3.1\text{b}) \\ \Rightarrow\ & \ddot{e}_s(t) = \ddot{x}_{s-1}(t) - \ddot{x}_s(t) && (3.1\text{c}) \end{aligned}$$

It is obvious that the vehicle dynamics are coupled, if we consider spacing error as a state variable.

3.1.1. Control Inputs Transferred/Incorporated

We can build one directional models with input variable transfer through spatial variable. We have the following spacing error definition and its derivative relations as

$$\begin{aligned} & e_s(t) := x_{s-1}(t) - x_s(t) - L_{s_{des}} \\ \Rightarrow\ & \dot{e}_s(t) = \dot{x}_{s-1}(t) - \dot{x}_s(t) \\ \Rightarrow\ & \ddot{e}_s(t) = \ddot{x}_{s-1}(t) - \ddot{x}_s(t) \\ & \qquad\quad = u_{s-1}(t) - u_s(t). \end{aligned}$$

We defined state and interconnection variables as

$$\begin{aligned} & x_s(t) := \begin{pmatrix} e_s(t) \\ \dot{e}_s(t) \end{pmatrix} && (3.2\text{a}) \\ & p_s(t) := u_{s-1}(t) && (3.2\text{b}) \\ & q_s(t) := u_s(t). && (3.2\text{c}) \end{aligned}$$

The resulting dynamics at node s can be given compactly as

$$\left(\begin{array}{c} \dot{e}_s(t) \\ \ddot{e}_s(t) \\ \hdashline u_s(t) \\ \hdashline y_s(t) \end{array}\right) = \left(\begin{array}{cc:c:c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \hdashline 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} e_s(t) \\ \dot{e}_s(t) \\ \hdashline u_{s-1}(t) \\ \hdashline u_s(t) \end{array}\right). \tag{3.3}$$

This formulation shows us inputs are coupled. Hence, we have a spatially interconnected model in which preceding vehicle acceleration information required in order to represent subsystem dynamics.

3.1.2. Absolute Velocity Transferred/Incorporated

Also, another formulation brings us absolute positions coupled. In this model, we required the absolute velocity transfer through spatial channel. We have the following spacing error definition and its derivative relation as

$$\begin{aligned} e_s(t) &= x_{s-1}(t) - x_s(t) - L_{s_{des}} \\ \Rightarrow \dot{e}_s(t) &= \dot{x}_{s-1}(t) - \dot{x}_s(t). \end{aligned}$$

We defined state and interconnection variables as

$$x_s(t) := \begin{pmatrix} e_s(t) \\ v_s(t) \end{pmatrix} \tag{3.4a}$$

$$p_s(t) := \dot{x}_{s-1}(t) \tag{3.4b}$$

$$q_s(t) := \dot{x}_s(t). \tag{3.4c}$$

The resulting dynamics at node s can be given as

$$\begin{pmatrix} \dot{e}_s(t) \\ \ddot{x}_s(t) \\ \hdashline \dot{x}_s(t) \\ \hdashline y_s(t) \end{pmatrix} = \left(\begin{array}{cc:c:c} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} e_s(t) \\ \dot{x}_s(t) \\ \hdashline \dot{x}_{s-1}(t) \\ \hdashline u_s(t) \end{pmatrix}. \tag{3.5}$$

This formulation shows us vehicle array can be modeled as spatially interconnected system, if we transfer absolute velocity through spatial channel.

3.2. Two Directional Models

Also bilateral models can be built by introducing two sided error formulation. Here, we need to define spacing errors slight differently. We defined e_{s_f} as forward spacing error and e_{s_b} as backward spacing error in (3.6).

$$e_{s_f}(t) := x_{s-1}(t) - x_s(t) - L_{des} \tag{3.6a}$$

$$e_{s_b}(t) := x_{s+1}(t) - x_s(t) + L_{des} \tag{3.6b}$$

We can define combined spacing error as and its derivative relation in (3.7). Then, we can combine these errors by introducing a factor α between $(0, 1)$. For simplicity, we assumed that desired inter spacings are equal.

$$e_s(t) := \alpha e_{s_f}(t) + (1-\alpha) e_{s_b}(t) \tag{3.7a}$$

$$= \alpha x_{s-1}(t) + (1-\alpha) x_{s+1}(t) - x_s(t) \tag{3.7b}$$

$$\Rightarrow \dot{e}_s(t) = \alpha v_{s-1}(t) + (1-\alpha) v_{s+1}(t) - v_s(t) \tag{3.7c}$$

We can select $\alpha = \frac{1}{2}$. By taking derivative of combined spacing error, we can get the following relations in (3.8).

$$e_s(t) = \frac{1}{2} x_{s-1}(t) + \frac{1}{2} x_{s+1}(t) - x_s(t) \tag{3.8a}$$

$$\Rightarrow \dot{e}_s(t) = \frac{1}{2}v_{s-1}(t) + \frac{1}{2}v_{s+1}(t) - v_s(t) \tag{3.8b}$$

$$\Rightarrow \ddot{e}_s(t) = \frac{1}{2}a_{s-1}(t) + \frac{1}{2}a_{s+1}(t) - a_s(t) \tag{3.8c}$$

3.2.1. Control Inputs Transferred/Incorporated

We can build two sided models, as we did in one directional case. We defined the interconnection variables as

$$q_s(t) := \begin{pmatrix} u_s(t) \\ u_s(t) \end{pmatrix}$$

$$p_s(t) := \begin{pmatrix} u_{s-1}(t) \\ u_{s+1}(t) \end{pmatrix}.$$

Two sided interconnected model of the vehicle at node s can be given as

$$\left(\begin{array}{c} \dot{e}_s(t) \\ \ddot{e}_s(t) \\ \hdashline q_s(t) \\ \hdashline y_s = e_s(t) \end{array}\right) = \left(\begin{array}{cc:cc:c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ \hdashline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} e_s(t) \\ \dot{e}_s(t) \\ \hdashline p_s(t) \\ \hdashline u_s(t) \end{array}\right) \tag{3.9}$$

where the inputs coupled.

3.2.2. Absolute Velocities Transferred/Incorporated

Also, absolute velocity information could be exploited through spatial channel. We defined state and interconnection variables as

$$q_s(t) := \begin{pmatrix} v_s(t) \\ v_s(t) \end{pmatrix}$$

$$p_s(t) := \begin{pmatrix} v_{s-1}(t) \\ v_{s+1}(t) \end{pmatrix}.$$

Two sided interconnected model of the each vehicle in the string can be given as

$$\left(\begin{array}{c} \dot{e}_s(t) \\ \dot{v}_s(t) \\ \hdashline q_s(t) \\ \hdashline y_s = e_s(t) \end{array}\right) = \left(\begin{array}{cc:cc:c} 0 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} e_s(t) \\ v_s(t) \\ \hdashline p_s(t) \\ \hdashline u_s(t) \end{array}\right). \tag{3.10}$$

We modeled identical vehicles which are incorporating with their direct neighbors, in order to ensure stability and attenuation of disturbance along the platoon. A slightly different two sided vehicular string model can be found in [21]. By this way, we can represent each vehicle independently.

4. SYSTEM ANALYSIS VIA IQCs

We have L subunits connected to each other, we try to analyze stability, well-posedness and dissipativity for such systems.

4.1. Composite System Equations

In order to represent subunits and interconnection elements, we can write system equations of i^{th} subsystem as

$$\begin{pmatrix} \dot{x}_i \\ q_i \\ z_i \end{pmatrix} = \begin{pmatrix} A^i_{TT} & A^i_{TS} & B^i_T \\ A^i_{ST} & A^i_{SS} & B^i_S \\ C^i_T & C^i_S & D^i \end{pmatrix} \begin{pmatrix} x_i \\ p_i \\ d_i \end{pmatrix},$$

where p_i is the input interconnection signal to the i^{th} subsystem, q_i is the output of interconnection signal to the i^{th} subsystem on the spatial channel, d_i is disturbance to the i^{th} subsystem and z_i is the output of the i^{th} subsystem on performance channel.

Throughout this chapter, we will be using block diagonal system matrices which are constructed by diagonally concatenating sub-system matrices. Also, vectors are concatenated vertically to form x, d, z, p and q. By using this notation, we can compactly write interconnected system as

$$\begin{pmatrix} \dot{x} \\ q \\ z \end{pmatrix} = \begin{pmatrix} A_{TT} & A_{TS} & B_T \\ A_{ST} & A_{SS} & B_S \\ C_T & C_S & D \end{pmatrix} \begin{pmatrix} x \\ p \\ d \end{pmatrix}.$$

By this approach, we can always write decoupled system equations compactly. Whole system can be represented as follows in Figure 4.1, where P composed of sub units P_i and i varies between 1 and L. Δ includes both interconnection map, loss and delays in the commination network and also linear parameter changes in the i^{th} node throughout Δ_{ii} for all i from 1 to L.

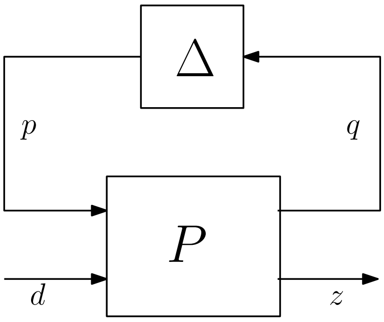

Figure 4.1. LPV like representation

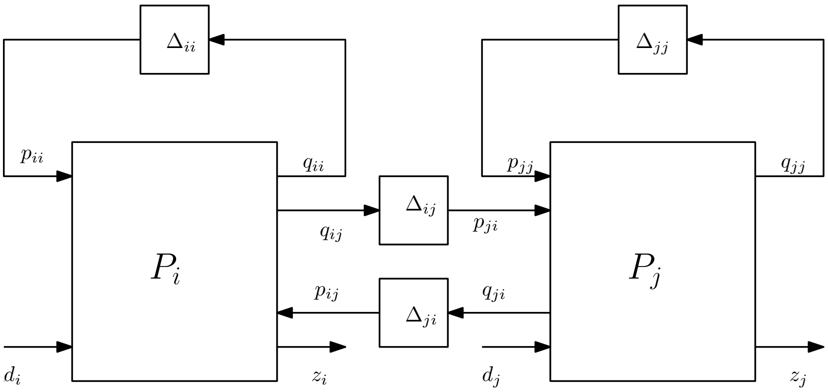

Figure 4.2. Non-ideal arbitrary interconnection between i and j

For non-ideal interconnection structure in Figure 4.2 [23], we can always relate p and q as

$$p = \Delta q.$$

And by suitable permutation of variables in p and q, we can always define $\tilde{\Delta}$ such that self supply variables and interconnection variables decomposed as

$$\tilde{p} =: \tilde{\Delta}\tilde{q},$$

where

$$\tilde{\Delta} = \begin{pmatrix} \Delta_u & 0 \\ 0 & \Delta_0 \end{pmatrix}.$$

Here, we defined Δ_u as

$$\Delta_u := \underset{1 \le i \le L}{\mathbf{diag}}(\Delta_{ii}),$$

which is similar to uncertainty block in LPV control, Δ_{ii} actually represents the linear parameter changes in corresponding subsystem, and Δ_0 is the unusual variable which represents the interconnection structure. In case for ideal channels with no parametric uncertainty. Δ reduces to a certainty block which represents the interconnection topology.

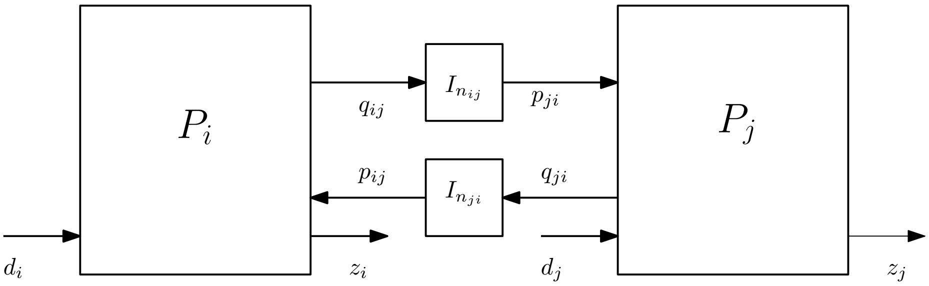

Figure 4.3. Ideal arbitrary interconnection between arbitrary units i and j

From the Figure 4.3, we can say that

$$p_{ij} = q_{ji}.$$

For spatially interconnected periodic systems, it is more like a permutation matrix. If the interconnection is ideal, we can relate p and q, by a certain block Δ_0 as

$$p = \Delta_0 q.$$

Here ideal interconnections constitute the permutation Δ_0, and the whole problem can be cast as seen in Figure 4.4.

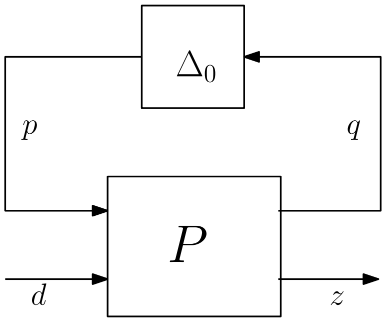

Figure 4.4. LPV like representation with fixed interconnection Δ_0

4.2. Interconnection Supply Rates

Previously, we defined interconnection vectors as

$$p = \underset{1\le i\le L}{\mathbf{cat}}(p_i) = \underset{1\le i\le L}{\mathbf{cat}}\left(\underset{1\le j\le L}{\mathbf{cat}}(p_{ij})\right)$$

and

$$q = \underset{1\le i\le L}{\mathbf{cat}}(q_i) = \underset{1\le i\le L}{\mathbf{cat}}\left(\underset{1\le j\le L}{\mathbf{cat}}(q_{ij})\right).$$

Most generally, we can define the interconnection supply rate $\mathcal{P}$ as a function of interconnection variables p and q as

$$\mathcal{P} := \mathcal{P}(p,q).$$

4.2.1. Quadratic Supply Rates

In particular, a quadratic interconnection supply rate is defined in [22] as

$$\mathcal{P} := \begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

This definition will be useful in the analysis of interconnected systems. We will be using quadratic supply rates. And we will call then interconnection supply rate. Next, we will switch to proper subclass of these supply rates and we will call them node by node supply rates in which matrix variables are constrained to be block diagonal.

4.2.2. Quadratic Supply Rates Node by Node

Let $\mathcal{P}_i$ a supply rate function at node i. We can constraint $\mathcal{P}$ as the sum of all supply rates at each node. Then, $\mathcal{P}$ can be written as

$$\mathcal{P}(p,q) := \sum_{i=1}^{L} \mathcal{P}_i(p_i,q_i).$$

And also we can further constraint internal supply rates signal by signal $\mathcal{P}_i$ as the sum of all supply rates for each signal as

$$\mathcal{P}_i(p_i,q_i) := \sum_{j=1}^{L} \mathcal{P}_{ij}(p_{ij},q_{ij}).$$

We will be using these as interconnection supply according to the definitions above. Depending on the situation, we will be using node by node supply rates or signal by signal supply rates. For decomposition of sub-systems node by node supply rates are satisfactory. Since, we are especially interested in quadratic supply rates node by node basis. We can define $\mathcal{P}$ as quadratic interconnection supply rates as

$$\mathcal{P}(p,q) := \sum_{i=1}^{L} \begin{pmatrix} p_i \\ q_i \end{pmatrix}^T \begin{pmatrix} Q_i & S_i \\ S_i^T & R_i \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix}.$$

Similarly, quadratic supply rates can be defined signal by signal as

$$\mathcal{P}_i(p_i,q_i) := \sum_{j=1}^{L} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}^T \begin{pmatrix} Q_{ij} & S_{ij} \\ S_{ij}^T & R_{ij} \end{pmatrix} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}.$$

However, this particular choice will put a heavy constraint on system and not required for decomposition purposes.

We used node by node supply rates in which the sum of all $\mathcal{P}_i$ constitutes the overall $\mathcal{P}$. Based on the block diagonal structure of Q, S and R, we can get rid of summation over i as

$$\begin{aligned}\mathcal{P}(p,q) &= \sum_{i=1}^{L} \begin{pmatrix} p_i \\ q_i \end{pmatrix}^T \begin{pmatrix} Q_i & S_i \\ S_i^T & R_i \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix} \\ &= \begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.\end{aligned}$$

Here, we restricted IQCs node by node basis. Actually, we designed IQCs are ordered in subsystem basis. IQCs for different sub-systems are not interrelated. Since we used block diagonal IQCs, we can directly decouple whole system to a node by node basis.

One can choose to work with full block multipliers. In that case, analysis conditions whole system cannot decoupled to individual units. Hence, whole system of LMIs should be solved simultaneously which will bring us a computational cost similar to centralized control computational cost. Actually, solving whole system equations and synthesizing controllers will lead to large centralized controllers.

To be clear about the internal energy supplies, we must repeat the following multiplier definitions which we used in particularly for node by node basis supplies as

$$\begin{aligned} Q &= \underset{1\le i\le L}{\mathbf{diag}}(Q_i) \\ R &= \underset{1\le i\le L}{\mathbf{diag}}(R_i) \\ S &= \underset{1\le i\le L}{\mathbf{diag}}(S_i). \end{aligned}$$

4.3. Analysis via IQCs

In this section, we provide an analysis test that is based on finding constant quadratic Lyapunov function and suitable quadratic supply rate, in order to guarantee well-posedness of the linear fractional transformation, stability and performance, specified as quadratic performance for the system given in (4.1).

$$\begin{pmatrix} \dot{x} \\ q \\ z \end{pmatrix} = \begin{pmatrix} A_{TT} & A_{TS} & B_T \\ A_{ST} & A_{SS} & B_S \\ C_T & C_S & D \end{pmatrix} \begin{pmatrix} x \\ p \\ d \end{pmatrix} \tag{4.1a}$$

$$p = \Delta q. \tag{4.1b}$$

Theorem 4.1. *[23] The system given in* (4.1) *is well-posed, stable, and dissipative, if there exists*

$$X_T = X_T^T \succ 0,$$

$$R_p = R_p^T \succeq 0,$$

and $Q = Q^T$, S, $R = R^T$ *such that*

$$\mathcal{P}_\Delta := \begin{pmatrix} \Delta \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad \forall \Delta \in \boldsymbol{\Delta},$$

and

$$J := \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT} & A_{TS} & B_T \\ \hdashline 0 & I & 0 \\ A_{ST} & A_{SS} & B_S \\ \hdashline 0 & 0 & I \\ C_T & C_S & D \end{array}\right)^T \left(\begin{array}{cc:cc:cc} 0 & X_T & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT} & A_{TS} & B_T \\ \hdashline 0 & I & 0 \\ A_{ST} & A_{SS} & B_S \\ \hdashline 0 & 0 & I \\ C_T & C_S & D \end{array}\right) \prec 0.$$

Proof. We give the proof in three steps. [21]

(i) Well-posedness: The system given in equation (4.1), $d \mapsto z$ admits the alternative representation by using the relation in (4.1b), and suppose there exist positive internal supply rates $Q = Q^T$, S and $R = R^T$ such that $\mathcal{P}_\Delta \succeq 0$ for all $\Delta \in \boldsymbol{\Delta}$ and $J \prec 0$. Apply upper LFT to the overall system in (4.1a). We have dynamical system as function of Δ.

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{A}(\Delta) & \mathbf{B}(\Delta) \\ \mathbf{C}(\Delta) & \mathbf{D}(\Delta) \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}(\Delta) & \mathbf{B}(\Delta) \\ \mathbf{C}(\Delta) & \mathbf{D}(\Delta) \end{pmatrix} =$$

$$= \begin{pmatrix} A_{TT} + A_{TS}\Delta(I - A_{SS}\Delta)^{-1}A_{ST} & B_T + A_{TS}\Delta(I - A_{SS}\Delta)^{-1}B_S \\ C_T + C_S\Delta(I - A_{SS}\Delta)^{-1}A_{ST} & D + C_S\Delta(I - A_{SS}\Delta)^{-1}B_S \end{pmatrix}$$

The function above is continuous on $\boldsymbol{\Delta}$ if

$$(I - A_{SS}\Delta) \tag{4.2}$$

is invertible $\forall \Delta \in \mathbf{\Delta}$. Thus we need show $\begin{pmatrix} I & \Delta \\ A_{SS} & I \end{pmatrix}$ is nonsingular $\forall \Delta \in \mathbf{\Delta}$. By introducing the concept of topological separator, the feedback system can be said well-posed if there exists $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$ such that $\mathcal{P}_\Delta \succeq 0$ for all $\Delta \in \mathbf{\Delta}$ and

$$\begin{pmatrix} I \\ A_{SS} \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ A_{SS} \end{pmatrix} \prec 0. \tag{4.3}$$

Since $J \prec 0$ particularly for $x(0) = 0$, and $d(t) = 0$ implies inequality (4.4).

$$\begin{pmatrix} I \\ A_{SS} \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ A_{SS} \end{pmatrix} + C_S^T R_p C_S \prec 0 \tag{4.4}$$

We use the fact $R_p \succeq 0$, then (4.3) satisfied. This completes the proof of well-posedness.

(ii) Stability: Suppose the interconnection is well-posed and suppose there exists $X_T = X_T^T \succ 0$. We have two set of equations for the unforced dynamics of the system. The first equation $J \prec 0$ implies that if we pre and post multiply by a nonzero vector $\begin{pmatrix} x \\ p \\ 0 \end{pmatrix}$ and its transpose

$$0 > \star^T \star^T \left(\begin{array}{cc|cc|cc} 0 & X_T & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_{TT} & A_{TS} \\ \hline 0 & I \\ A_{ST} & A_{SS} \\ \hline 0 & 0 \\ C_T & C_S \end{array}\right) \begin{pmatrix} x \\ p \end{pmatrix}$$

$$= \star^T \left(\begin{array}{cc|cc} 0 & X_T & 0 & 0 \\ X_T & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{c} x \\ \dot{x} \\ \hline p \\ q \end{array}\right) + z^T R_p z \prec 0$$

Alternatively, we have

$$0 \succ \left(\begin{array}{c} x \\ \dot{x} \\ \hline p \\ q \end{array}\right)^T \left(\begin{array}{cc} I & 0 \\ A_{TT} & A_{TS} \\ \hline 0 & I \\ A_{ST} & A_{SS} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & X_T & 0 & 0 \\ X_T & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_{TT} & A_{TS} \\ \hline 0 & I \\ A_{ST} & A_{SS} \end{array}\right) \left(\begin{array}{c} x \\ \dot{x} \\ \hline p \\ q \end{array}\right)$$

$$+ \begin{pmatrix} x \\ p \end{pmatrix}^T \begin{pmatrix} C_T \\ C_S \end{pmatrix} R_p \begin{pmatrix} C_T & C_S \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

$$= x^T X_T \dot{x} + \dot{x}^T X_T x + q^T \begin{pmatrix} \Delta \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} q + z^T R_p z$$

Since, we know $\mathcal{P}_\Delta$ is positive $\forall \Delta \in \mathbf{\Delta}$, and $R_p \succeq 0$, the above equation reduces to

$$\Rightarrow x^T X_T \dot{x} + \dot{x}^T X_T x \prec 0.$$

This completes the stability part.

(iii) Performance: Finally, when a disturbance applied as d, we pre and post multiply J by $\begin{pmatrix} x \\ p \\ d \end{pmatrix}$ and its transpose.

We have the following inequality as

$$
\begin{aligned}
0 > \star^T \star^T &\left(\begin{array}{cc|cc|cc} 0 & X_T & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT} & A_{TS} & B_T \\ \hline 0 & I & 0 \\ A_{ST} & A_{SS} & B_S \\ \hline 0 & 0 & I \\ C_T & C_S & D \end{array}\right) \begin{pmatrix} x \\ p \\ d \end{pmatrix} \\
= \star^T &\left(\begin{array}{cc|cc|cc} 0 & X_T & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{c} x \\ \dot{x} \\ \hline p \\ q \\ \hline d \\ z \end{array}\right) \\
= &x^T X_T \dot{x} + \dot{x}^T X_T x + \mathcal{P} + \begin{pmatrix} d \\ z \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix}
\end{aligned}
$$

If there exists multipliers $Q = Q^T$, S and $R = R^T$ such that $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$ is positive or equal to zero on $\begin{pmatrix} \Delta \\ I \end{pmatrix}$. Then, we use the fact $\mathcal{P}_\Delta \succeq 0 \quad \forall \Delta \in \mathbf{\Delta}$, we get

$$x^T X_T \dot{x} + \dot{x}^T X_T x + d^T Q_p d + d^T S_p z_i + z^T S_p^T d + z^T R_p z \prec 0 \tag{4.5}$$

Then, we can guarantee that sum of potential rate and supply rate is negative. Now we have proved performance part.(Similar to KYP Lemma)

□

Remark 4.1. Special cases for quadratic performance index [27].

(i) Select performance index P_p as

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix}$$

guarantees that the $\mathcal{L}_2$-gain of the whole system $d \mapsto z$ is smaller than γ.

(ii) Select performance index P_p as

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

guarantees dissipativity of the system $d \mapsto z$, generalizing positive realness.

(iii) We can always prove contractiveness by choosing performance index P_p as

$$\begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

Decomposition of whole system LMIs could be done by exploiting the succeeding observation. The following inequality holds

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \succ 0,$$

if and only if

$$X_1 \succ 0$$
$$X_2 \succ 0$$

hold.

For example, we can always reduce the following LMI

$$\left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline A^1 & 0 \\ 0 & A^2 \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & X_2 \\ \hline X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline A^1 & 0 \\ 0 & A^2 \end{array}\right) \prec 0$$

to a smaller set of LMIs as

$$\begin{pmatrix} I \\ A^1 \end{pmatrix} \begin{pmatrix} 0 & X_1 \\ X_1 & 0 \end{pmatrix} \begin{pmatrix} I \\ A^1 \end{pmatrix} \prec 0$$
$$\begin{pmatrix} I \\ A^2 \end{pmatrix} \begin{pmatrix} 0 & X_2 \\ X_2 & 0 \end{pmatrix} \begin{pmatrix} I \\ A^2 \end{pmatrix} \prec 0$$

Since, we used block diagonal matrix variables. Using the fact above, we can decompose the whole LMIs from $i = 1 : L$ as

$$\left(\begin{array}{ccc} I & 0 & 0 \\ A^i_{TT} & A^i_{TS} & B^i_T \\ \hline 0 & I & 0 \\ A^i_{ST} & A^i_{SS} & B^i_S \\ \hline 0 & 0 & I \\ C^i_T & C^i_S & D^i \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & X^i_T & 0 & 0 & 0 & 0 \\ X^i_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q_i & S_i & 0 & 0 \\ 0 & 0 & S_i^T & R_i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{p_i} & S_{p_i} \\ 0 & 0 & 0 & 0 & S^T_{p_i} & R_{p_i} \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A^i_{TT} & A^i_{TS} & B^i_T \\ \hline 0 & I & 0 \\ A^i_{ST} & A^i_{SS} & B^i_S \\ \hline 0 & 0 & I \\ C^i_T & C^i_S & D^i \end{array}\right) \prec 0.$$

Proposition 4.1. *[19] Let the quadratic form $\mathcal{P}$ be defined on $\mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}}$ by*

$$\mathcal{P}(p,q) = \sum_{i=1}^{L} \mathcal{P}_i(p_i, q_i) \tag{4.6}$$

then $\mathcal{P}$ vanishes on the interconnection subspace $\mathcal{S}_{\mathcal{I}}$ which is defined as

$$\mathcal{S}_{\mathcal{I}} := \left\{(p,q) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}} : q_{ij} = p_{ij} \quad \forall i,j = 1 : L\right\} \tag{4.7}$$

for ideal connections.

So by the use of Preposition 4.1, we can always guarantee that the quadratic supply of interconnection channel is always zero or positive on $\mathcal{S}_{\mathcal{I}}$ and negative on $\mathcal{S}_{\mathcal{B}}$ which is defined as

$$\mathcal{S}_{\mathcal{B}} := \left\{ (p,q) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}} : \begin{pmatrix} p \\ q \end{pmatrix} \in \mathcal{S}_{\mathcal{B}} \quad \forall i = 1 : L \right\},$$

also equivalently

$$\mathcal{S}_{\mathcal{B}} = \operatorname{im} \begin{pmatrix} I \\ A_{SS} \end{pmatrix}. \tag{4.8}$$

Remark 4.2. For ideal channels Δ is a permutation matrix. By the use of Preposition 4.1, we can say that the interconnection subspace $\mathcal{S}_{\mathcal{I}}$ defined on $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \Delta \\ I \end{pmatrix} q$ and $\mathcal{S}_{\mathcal{B}}$ defined on $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} I \\ A_{SS} \end{pmatrix} p$. By using the above arguments, we can argue that $\mathcal{S}_{\mathcal{I}}$ and $\mathcal{S}_{\mathcal{B}}$ topologically separated. We know that Preposition 4.1, $\mathcal{P}$ vanishes on the interconnection subspace [19]. This result implies that

$$\mathcal{S}_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{B}} = \{0\}.$$

This result is useful for both stability, cause it proves that quadratic supply for interconnection variables is always greater than or equal to 0, and well-posedness from the view of topological separation.

For the perfect interconnection case, we have $p_{ij} = q_{ji}$ we can relate $p = \Delta_0 q$ by simple permutation matrix Δ_0. We can write quadratic supply for the interconnection signals more explicitly as

$$\mathcal{P} = \sum_{1\leq i\leq L}\sum_{1\leq j\leq L} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}^T \begin{pmatrix} Q_{ij} & S_{ij} \\ S_{ij}^T & R_{ij} \end{pmatrix} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}.$$

Using the symmetry, we can further put constraints to our system.

$$\mathcal{P} = \sum_{1\leq i\leq L}\sum_{i\geq j} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}^T \begin{pmatrix} Q_{ij}+R_{ji} & S_{ij}+S_{ji}^T \\ S_{ij}^T+S_{ji} & R_{ij}+Q_{ji} \end{pmatrix} \begin{pmatrix} p_{ij} \\ q_{ij} \end{pmatrix}.$$

We can choose Q, S and R such that

$$Q_{ij} + R_{ji} = 0$$

and

$$S_{ij} + S_{ji}^T = 0.$$

With this specific parametrization, we can always guarantee $\mathcal{P} = 0$. By this approach, we can provide neutral interconnection for ideal interconnections. Hence, this constraint imposes Q_i and R_i are coupled to each other. And S_{ij} is skew symmetric [22].

Remark 4.3. For neutral (ideal) channels, we can restrict $\mathcal{P}$ as

$$\begin{pmatrix} \Delta_0 \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta_0 \\ I \end{pmatrix} = 0.$$

Hence, we can use the following interconnection supply $Q_{ij} + R_{ji} = 0$ and $S_{ij} + S_{ji}^T = 0$ [22]. We must notice this particular choice will certainly introduces conservatism. However, in this particular choice, we do not need to solve interconnection relation which is imposed by the interconnection map.

4.4. An Example With Two Subsystems

We can give an example with two subsystems. Lets consider that we have two subsystems that are interconnected via an ideal channel, and each subsystem has linear parameter changes as can be seen in Figure 4.5. This particular example will increase our understanding for such systems.

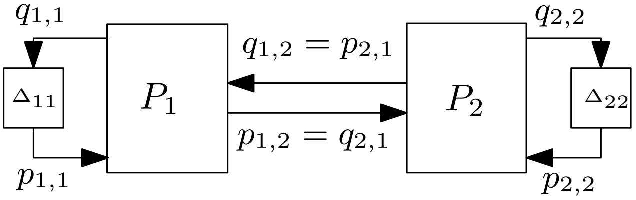

Figure 4.5. Two by Two example

We defined internal supply rate for whole system as follows

$$\begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{4.9}$$

If we restrict our supply choice as we did in signal by signal basis such that $Q_i = \mathbf{diag}_j(Q_{ij})$. For both $i = 1, 2$ We have the following interconnection supplies. First consider the element at node $i = 1$, we have the following supply $\mathcal{P}_1$ as

$$\begin{pmatrix} p_{11} \\ p_{12} \\ q_{11} \\ q_{12} \end{pmatrix}^T \left(\begin{array}{cc:cc} Q_{11} & 0 & S_{11} & 0 \\ 0 & Q_{12} & 0 & S_{12} \\ \hdashline S_{11}^T & 0 & R_{11} & 0 \\ 0 & S_{12}^T & 0 & R_{12} \end{array}\right) \begin{pmatrix} p_{11} \\ p_{12} \\ q_{11} \\ q_{12} \end{pmatrix}. \tag{4.10}$$

Now consider the element at node $i = 2$, we have the following supplies $\mathcal{P}_2$ as

$$\begin{pmatrix} p_{21} \\ p_{22} \\ q_{21} \\ q_{22} \end{pmatrix}^T \left(\begin{array}{cc|cc} Q_{21} & 0 & S_{21} & 0 \\ 0 & Q_{22} & 0 & S_{22} \\ \hline S_{21}^T & 0 & R_{21} & 0 \\ 0 & S_{22}^T & 0 & R_{22} \end{array}\right) \begin{pmatrix} p_{21} \\ p_{22} \\ q_{21} \\ q_{22} \end{pmatrix}. \tag{4.11}$$

We can write sum of the supplies as

$$\left(\begin{array}{c} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \\ \hline q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{array}\right)^T \left(\begin{array}{cccc|cccc} Q_{11} & 0 & 0 & 0 & S_{11} & 0 & 0 & 0 \\ 0 & Q_{12} & 0 & 0 & 0 & S_{12} & 0 & 0 \\ 0 & 0 & Q_{21} & 0 & 0 & 0 & S_{21} & 0 \\ 0 & 0 & 0 & Q_{22} & 0 & 0 & 0 & S_{22} \\ \hline S_{11}^T & 0 & 0 & 0 & R_{11} & 0 & 0 & 0 \\ 0 & S_{12}^T & 0 & 0 & 0 & R_{12} & 0 & 0 \\ 0 & 0 & S_{21}^T & 0 & 0 & 0 & R_{21} & 0 \\ 0 & 0 & 0 & S_{22}^T & 0 & 0 & 0 & R_{22} \end{array}\right) \left(\begin{array}{c} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \\ \hline q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{array}\right) \succ 0. \tag{4.12}$$

Use the interconnection relation as

$$p = \Delta q,$$

and particularly as

$$\begin{pmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{pmatrix} = \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{pmatrix}. \tag{4.13}$$

We require that

$$\Delta^T Q \Delta + S\Delta + \Delta^T S^T + R \succeq 0 \tag{4.14}$$

This constraint yields the following inequality

$$\star^T \left(\begin{array}{cccc:cccc} \Delta_{11}^T Q_{11}\Delta_{11} & 0 & 0 & 0 & \Delta_{11}^T S_{11} & 0 & 0 & 0 \\ 0 & Q_{21} & 0 & 0 & 0 & 0 & S_{12} & 0 \\ 0 & 0 & Q_{12} & 0 & 0 & S_{21} & 0 & 0 \\ 0 & 0 & 0 & \Delta_{22}^T Q_{22}\Delta_{22} & 0 & 0 & 0 & \Delta_{22}^T S_{22} \\ \hdashline S_{11}^T\Delta_{11} & 0 & 0 & 0 & R_{11} & 0 & 0 & 0 \\ 0 & 0 & S_{12}^T & 0 & 0 & R_{12} & 0 & 0 \\ 0 & S_{21}^T & 0 & 0 & 0 & 0 & R_{21} & 0 \\ 0 & 0 & 0 & S_{22}^T\Delta_{22} & 0 & 0 & 0 & R_{22} \end{array}\right) \left(\begin{array}{c} q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \\ \hdashline q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{array}\right) \succeq 0 \tag{4.15}$$

If we select Q, S, and R such that

$$\begin{aligned} Q_{ij} + R_{ji} &= 0 \quad \forall i \neq j \\ S_{ij} + S_{ji}^T &= 0 \quad \forall i \neq j. \end{aligned}$$

By choosing above parametrization, we can always guarantee $\mathcal{P} \succ 0$. Only by solving

$$\begin{aligned} &\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}^T \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix} + \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix}^T \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \\ &+ \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}^T \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} + \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix} \succeq 0 \quad \forall \Delta_{11} \in \boldsymbol{\Delta_{11}}, \Delta_{22} \in \boldsymbol{\Delta_{22}}. \end{aligned}$$

We explicitly show that Q_{ij} is related to R_{ji} for $i \neq j$. By this particular choice we can directly get rid of interconnection variables and reduce the problem to the LPV problem. However, this approach introduces certain level of conservatism in expense of computational convenience.

5. DISTRIBUTED CONTROL OF INTERCONNECTED SYSTEMS VIA IQCs

In the control synthesis procedure, our objective is to find an interconnected dynamic controller such that the closed loop system should satisfy the LMIs given in Theorem 4.1. Our synthesis method solely depend on feasibility of LMIs and the controllers are synthesized by using linear matrix variables. We will be extracting controllers similar to LPV control synthesis. Details can be found in [28]. A more close approach can be found in [23].

We restricted our matrix variables as block diagonal, so we can always decompose system. Decoupled LMIs are used, in order to synthesize controllers separately. One might choose to work with full matrix variables for whole system equations to synthesize controllers. This will bring a computational complexity and will lead to centralized controllers. However, the search for such stable controllers will be certainly in a larger set. Hence, by restricting matrix variables to be block diagonal, we avoid computational complexity and we are searching controllers in a smaller set. We are trying to find simple controllers which can be implemented to each vehicle separately node by node basis.

5.1. Closed Loop

Plant P has the following dynamics

$$\begin{pmatrix} q \\ z \\ y \end{pmatrix} = \left[\begin{array}{c|ccc} A_{TT} & A_{TS} & B_T & B_{TU} \\ \hline A_{ST} & A_{SS} & B_S & B_{SU} \\ C_{PT} & C_{PS} & D & E \\ C_T & C_S & F & 0 \end{array}\right] \begin{pmatrix} p \\ d \\ u \end{pmatrix},$$

where y represents the measurement and u represent the control input.

The controller K has the same structure as the plant

$$\begin{pmatrix} q^k \\ u \end{pmatrix} = \left[\begin{array}{c|cc} A^k_{TT} & A^k_{TS} & B^k_{TY} \\ \hline A^k_{ST} & A^k_{SS} & B^k_{SY} \\ C^k_T & C^k_S & D^k \end{array}\right] \begin{pmatrix} p^k \\ y \end{pmatrix}. \tag{5.1}$$

Controller decides the control actions u based on the measured output y. The controller communication channel uses p^k and q^k as spatial controller variables. The relation between p^k and q^k is a replica of the relation between p and q. This leads to a distributed controller which shares its interconnection topology similar to the plant interconnections. The relation between them could be written as

$$\begin{aligned} p &= \Delta q \\ p^k &= \Delta_k q^k. \end{aligned}$$

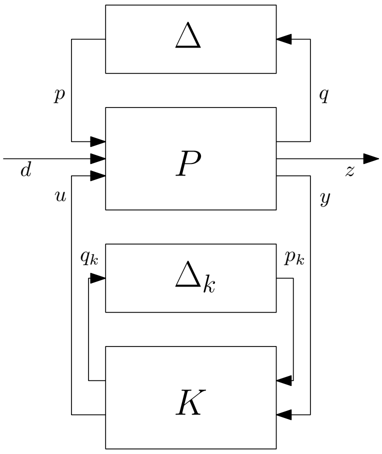

Figure 5.1. Open Loop Plant and Controller structure

The controller relates measured output y to the control input u. By adding the controller dynamics, we can write closed loop equations as

$$\begin{pmatrix} \dot{x}_c \\ q_c \\ z \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{\mathcal{TT}} & \mathcal{A}_{\mathcal{TS}} & \mathcal{B}_{\mathcal{T}} \\ \mathcal{A}_{\mathcal{ST}} & \mathcal{A}_{\mathcal{SS}} & \mathcal{B}_{\mathcal{S}} \\ \mathcal{C}_{\mathcal{T}} & \mathcal{C}_{\mathcal{S}} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x_c \\ p_c \\ d \end{pmatrix}, \tag{5.2}$$

where

$$x_c := \begin{pmatrix} x \\ x^k \end{pmatrix}, \quad p_c := \begin{pmatrix} p \\ p^k \end{pmatrix} \quad \text{and} \quad q_c := \begin{pmatrix} q \\ q^k \end{pmatrix}.$$

After applying a lower LFT to representation given in Figure 5.1 , we can get the closed loop system P_{cl}, which depends on both plant and controller dynamics as can be seen in Figure 5.2.

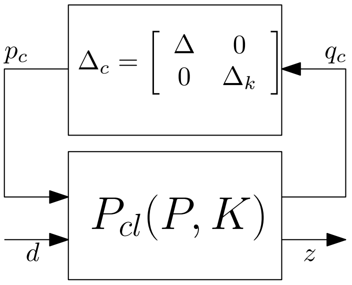

Figure 5.2. Closed Loop Plant

After simple algebraic operations, we could give closed loop system as follows

$$\left(\begin{array}{c:c:c} \mathcal{A}_{\mathcal{T}\mathcal{T}} & \mathcal{A}_{\mathcal{T}\mathcal{S}} & \mathcal{B}_{\mathcal{T}} \\ \hdashline \mathcal{A}_{\mathcal{S}\mathcal{T}} & \mathcal{A}_{\mathcal{S}\mathcal{S}} & \mathcal{B}_{\mathcal{S}} \\ \hdashline \mathcal{C}_{\mathcal{T}} & \mathcal{C}_{\mathcal{S}} & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc:cc:c} A_{TT} & 0 & A_{TS} & 0 & B_T \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline A_{ST} & 0 & A_{SS} & 0 & B_S \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline C_{PT} & 0 & C_{PS} & 0 & D \end{array}\right)$$

$$+ \begin{pmatrix} 0 & 0 & B_{TU} \\ I & 0 & 0 \\ 0 & 0 & B_{SU} \\ 0 & I & 0 \\ 0 & 0 & E \end{pmatrix} \begin{pmatrix} A^k_{TT} & A^k_{TS} & B^k_{TY} \\ A^k_{ST} & A^k_{SS} & B^k_{SY} \\ C^k_T & C^k_S & D^k \end{pmatrix} \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_T & 0 & C_S & 0 & F \end{pmatrix}$$

$$=: U_A + U_B K U_C.$$

The closed loop system should satisfy the requirements in Theorem 4.1. Here, the only unknown is the controller K. The variables U_A, U_B and U_C is dependent only plant dynamics. By eliminating the dependance on K, we could apply Theorem 4.1 for the closed loop system. We should use parameter elimination lemma, in order to eliminate the dependance on K. This can be done by pre-and post multiplying $(U_C^T)_\perp$. Then transpose of closed loop, pre-and-post multiplied by $(U_B)_\perp$ in order to get dual equation. Details can be found in [28].

Theorem 5.1. *[28] The following statements are equivalent:*

(i) There exists a controller (5.1) *such that the closed loop system* (5.2) *admits a symmetric* $\mathcal{X}^e > 0$, *and a symmetric multipliers* $\begin{pmatrix} \mathcal{Q}_e & \mathcal{S}_e \\ \mathcal{S}_e^T & \mathcal{R}_e \end{pmatrix} \succeq 0$ *on* $\begin{pmatrix} \Delta_c \\ I \end{pmatrix}$ *for all* $\Delta \in \mathbf{\Delta}$, *that satisfy the LMI*

$$\left(\begin{array}{ccc} \mathcal{A}_{\mathcal{TT}} & \mathcal{A}_{\mathcal{TS}} & \mathcal{B}_{\mathcal{T}} \\ \mathcal{A}_{\mathcal{ST}} & \mathcal{A}_{\mathcal{SS}} & \mathcal{B}_{\mathcal{S}} \\ \mathcal{C}_{\mathcal{T}} & \mathcal{C}_{\mathcal{S}} & \mathcal{D} \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right)^T \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & \mathcal{X}^e & 0 & 0 \\ 0 & \mathcal{R}^e & 0 & 0 & (\mathcal{S}^e)^T & 0 \\ 0 & 0 & \mathcal{R}_p^e & 0 & 0 & (\mathcal{S}_p^e)^T \\ \hline \mathcal{X}^e & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{S}_p^e & 0 & 0 & \mathcal{Q}^e & 0 \\ 0 & 0 & \mathcal{S}_p^e & 0 & 0 & \mathcal{Q}_p^e \end{array}\right) \left(\begin{array}{ccc} \mathcal{A}_{\mathcal{TT}} & \mathcal{A}_{\mathcal{TS}} & \mathcal{B}_{\mathcal{T}} \\ \mathcal{A}_{\mathcal{ST}} & \mathcal{A}_{\mathcal{SS}} & \mathcal{B}_{\mathcal{S}} \\ \mathcal{C}_{\mathcal{T}} & \mathcal{C}_{\mathcal{S}} & \mathcal{D} \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right) \prec 0.$$

(ii)There exists symmetric X_T, *and* Y_T *and multipliers* $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$, *and* $\begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix}$ *with*

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0$$

and

$$\begin{pmatrix} I \\ -\Delta^T \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} \preceq 0$$

for all $\Delta \in \mathbf{\Delta}$, *that satisfy the LMIs*

$$\star^T \star^T \left(\begin{array}{cc|cc|cc} 0 & X_T & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT} & A_{TS} & B_T \\ \hline 0 & I & 0 \\ A_{ST} & A_{SS} & B_S \\ \hline 0 & 0 & I \\ C_{PT} & C_{PS} & D \end{array}\right) \Psi \prec 0,$$

$$\star^T \star^T \left(\begin{array}{cc|cc|cc} 0 & Y_T & 0 & 0 & 0 & 0 \\ Y_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{array}\right) \left(\begin{array}{ccc} -A_{TT}^T & -(A_{ST}^T & -C_{PT}^T \\ I & 0 & 0 \\ \hline -A_{TS}^T & -A_{SS}^T & -C_{PS}^T \\ 0 & I & 0 \\ \hline -B_T^T & -B_S^T & -D^T \\ 0 & 0 & I \end{array}\right) \Phi \succ 0,$$

and

$$\begin{pmatrix} X_T & I \\ I & Y_T \end{pmatrix} \succ 0.$$

where Φ *and* Ψ *are defined as*

$$\Phi := \begin{pmatrix} B_{TU} \\ B_{SU} \\ E \end{pmatrix}_{\perp} \quad \textit{and} \quad \Psi := \begin{pmatrix} C_T^T \\ C_S^T \\ F^T \end{pmatrix}_{\perp} .$$

Proof and more detailed treatment can be found in [28].

5.2. Closed Loop Subsystem

We can decompose the composite system into the subunits, since we restricted linear matrix variables to be block diagonal. Then synthesize distributed controllers at each node separately. Plant dynamics at node i, as P_i

$$\begin{pmatrix} \dot{x}_i \\ q_i \\ z_i \\ y_i \end{pmatrix} = \begin{pmatrix} A^i_{TT} & A^i_{TS} & B^i_T & B^i_{TU} \\ A^i_{ST} & A^i_{SS} & B^i_S & B^i_{SU} \\ C^i_{PT} & C^i_{PS} & D^i & E^i \\ C^i_T & C^i_S & F^i & 0 \end{pmatrix} \begin{pmatrix} x_i \\ p_i \\ d_i \\ u_i \end{pmatrix}$$

where y_i represents the local measurement at node i and u_i represent the local control input for the i subsystem. Controller K^i has the same structure at node i as

$$\begin{pmatrix} \dot{x}^k_i \\ q^k_i \\ u^k_i \end{pmatrix} = \begin{pmatrix} (A^i_{TT})^k & (A^i_{TS})^k & (B^i_{TY})^k \\ (A^i_{ST})^k & (A^i_{SS})^k & (B^i_{SY})^k \\ (C^i_T)^k & (C^i_S)^k & (D^i)^k \end{pmatrix} \begin{pmatrix} x^k_i \\ p^k_i \\ y_i \end{pmatrix} \tag{5.3}$$

Hence, we know relation between u_i and y_i, we can give the closed loop dynamics at node i as

$$\left(\begin{array}{c:c:c} {\mathcal{A}_{\mathcal{T}\mathcal{T}}}^i & {\mathcal{A}_{\mathcal{T}\mathcal{S}}}^i & {\mathcal{B}_{\mathcal{T}}}^i \\ \hdashline {\mathcal{A}_{\mathcal{S}\mathcal{T}}}^i & \mathcal{A}_{\mathcal{S}\mathcal{S}} & {\mathcal{B}_{\mathcal{S}}}^i \\ \hdashline {\mathcal{C}_{\mathcal{T}}}^i & {\mathcal{C}_{\mathcal{S}}}^i & \mathcal{D}^i \end{array}\right) = \left(\begin{array}{cc:cc:c} A^i_{TT} & 0 & A^i_{TS} & 0 & B^i_T \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline A^i_{ST} & 0 & A^i_{SS} & 0 & B^i_S \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline C^i_{PT} & 0 & C^i_{PS} & 0 & D^i \end{array}\right)$$

$$+ \begin{pmatrix} 0 & 0 & B^i_{TU} \\ I & 0 & 0 \\ 0 & 0 & B^i_{SU} \\ 0 & I & 0 \\ 0 & 0 & E^i \end{pmatrix} \begin{pmatrix} (A^i_{TT})^k & (A^i_{TS})^k & (B^i_{TY})^k \\ (A^i_{ST})^k & (A^i_{SS})^K & (B^i_{SY})^k \\ (C^i_T)^k & (C^i_S)^k & (D^i)^k \end{pmatrix} \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C^i_T & 0 & C^i_S & 0 & F^i \end{pmatrix}$$

$$=: U^i_A + U^i_B K^i U^i_C.$$

We want to further emphasize some points regarding variables $p_i^k := \mathbf{cat}_{1\leq j\leq L}(p_{ij}^k)$ and $q_i^k := \mathbf{cat}_{1\leq j\leq L}(q_{ij}^k)$ in the spatial controller channel. Controller communication channel has the same structure as the plant interconnection. That is to say, if there exists an interconnection signal between i and j, such that $p_{ij} = \Delta_{ij} q_{ji}$. That interconnection signal is repeated in the controllers between i and j, such that $p_{ij}^k = (\Delta_{ij})_k q_{ji}^k$. Hence, when the plant dynamics of i^{th} unit affected by the j^{th} unit, same interconnection is repeated between controller units i and j.

5.3. Solvability Conditions and Controller Construction

We can decompose the overall LMIs in Theorem 5.1 into the smaller LMIs at each node, since we restricted linear matrix variables to be block diagonal.

Corollary 5.1. *[23] There exists a distributed controller defined by* (5.3) *which achieves quadratic performance for the closed loop system, if there exists* $X_T = X_T^T = \mathbf{diag}_{i=1:L}(X_T^i)$, $Y_T = Y_T^T = \mathbf{diag}_{i=1:L}(Y_T^i)$, $Q = Q^T = \mathbf{diag}_{i=1:L}(Q_i)$, $S = \mathbf{diag}_{i=1:L}(S_i)$ *and* $R = R^T = \mathbf{diag}_{i=1:L}(R_i)$ *such that* $\mathcal{P}_\Delta \succeq 0$ *for all* $\Delta \in \mathbf{\Delta}$ *and*

$$\star^T \star^T \left(\begin{array}{cc:cc:cc} 0 & X_T^i & 0 & 0 & 0 & 0 \\ X_T^i & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_i & S_i & 0 & 0 \\ 0 & 0 & S_i^T & R_i & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & Q_{p_i} & S_{p_i} \\ 0 & 0 & 0 & 0 & S_{p_i}^T & R_{p_i} \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT}^i & A_{TS}^i & B_T^i \\ \hdashline 0 & I & 0 \\ A_{ST}^i & A_{SS}^i & B_S^i \\ \hdashline 0 & 0 & I \\ C_{PT}^i & C_{PS}^i & D^i \end{array}\right) \Psi_i \prec 0, \tag{5.4a}$$

$$\star^T \star^T \left(\begin{array}{cc|cc|cc} 0 & Y_T^i & 0 & 0 & 0 & 0 \\ Y_T^i & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q}_i & \tilde{S}_i & 0 & 0 \\ 0 & 0 & \tilde{S}_i^T & \tilde{R}_i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q}_{p_i} & \tilde{S}_{p_i} \\ 0 & 0 & 0 & 0 & \tilde{S}_{p_i}^T & \tilde{R}_{p_i} \end{array}\right) \left(\begin{array}{ccc} -(A_{TT}^i)^T & -(A_{ST}^i)^T & -(C_{PT}^i)^T \\ I & 0 & 0 \\ \hline -(A_{TS}^i)^T & -(A_{SS}^i)^T & -(C_{PS}^i)^T \\ 0 & I & 0 \\ \hline -(B_T^i)^T & -(B_S^i)^T & -(D^i)^T \\ 0 & 0 & I \end{array}\right) \Phi_i \succ 0, \tag{5.4b}$$

$$\begin{pmatrix} X_T^i & I \\ I & Y_T^i \end{pmatrix} \succ 0 \quad \forall i = 1 : L, \tag{5.4c}$$

where Φ^i *and* Ψ^i *are defined as* $\Phi^i := \begin{pmatrix} B_{TU}^i \\ B_{SU}^i \\ E^i \end{pmatrix}_\perp$ *and* $\Psi^i := \begin{pmatrix} (C_T^i)^T \\ (C_S^i)^T \\ (F^i)^T \end{pmatrix}_\perp$ $\forall i = 1 : L.$

We give Corollary 5.1 as an extension of Theorem 5.1 by exploiting the block diagonal structure. If the above decomposed LMIs solvable from $i = 1 : L$, we say there exists distributed controllers K_i at each node. Solving these inequalities in Corollary 5.1 is equivalent to search stable controllers for the corresponding unit. Then, we can construct controllers node by node.

We can construct controller in the same way as in LPV control. Construction of the controllers node by node can be found in [23], also more detailed and general discussion can be found in [28].

We want that the resulting closed system should satisfy the LMI in (5.5) for all

i from 1 to L.

$$\star^T \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & X_i^e & 0 & 0 \\ 0 & R_i^e & 0 & 0 & (S_i^e)^T & 0 \\ 0 & 0 & R_{p_i}^e & 0 & 0 & (S_{p_i}^e)^T \\ \hline X_i^e & 0 & 0 & 0 & 0 & 0 \\ 0 & S_i^e & 0 & 0 & Q_i^e & 0 \\ 0 & 0 & S_{p_i}^e & 0 & 0 & Q_{p_i}^e \end{array}\right) \left(\begin{array}{ccc} \mathcal{A}_{\mathcal{T}\mathcal{T}}{}^i & \mathcal{A}_{\mathcal{T}\mathcal{S}}{}^i & \mathcal{B}_{\mathcal{T}}{}^i \\ \mathcal{A}_{\mathcal{S}\mathcal{T}}{}^i & \mathcal{A}_{\mathcal{S}\mathcal{S}} & \mathcal{B}_{\mathcal{S}}{}^i \\ \mathcal{C}_{\mathcal{T}}{}^i & \mathcal{C}_{\mathcal{S}}{}^i & \mathcal{D}^i \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{5.5}$$

The construction of the controller is straight forward and includes some manipulation of the variables. After having solved the all LMIs form 1 to L, the only unknown remaining is the controller. We can construct the controllers directly by following the procedure below.

- We simplify the inequality in (5.5) as

$$\begin{pmatrix} U_A^i + U_B^i K^i U_C^i \\ I \end{pmatrix}^T \Pi_e^i \begin{pmatrix} U_A^i + U_B^i K^i U_C^i \\ I \end{pmatrix} \prec 0. \tag{5.6}$$

- Open up the above the inequality as

$$\begin{pmatrix} K^i U_C^i \\ I \end{pmatrix}^T \begin{pmatrix} U_B^i & U_A^i \\ 0 & I \end{pmatrix}^T \Pi_e^i \begin{pmatrix} U_B^i & U_A^i \\ 0 & I \end{pmatrix} \begin{pmatrix} K^i U_C^i \\ I \end{pmatrix} \prec 0. \tag{5.7}$$

- Define Γ^i as

$$\Gamma^i := \begin{pmatrix} U_B^i & U_A^i \\ 0 & I \end{pmatrix}^T \Pi_e^i \begin{pmatrix} U_B^i & U_A^i \\ 0 & I \end{pmatrix}.$$

Then the inequality in (5.7) reduces to

$$\begin{pmatrix} K^i U_C^i \\ I \end{pmatrix}^T \Gamma^i \begin{pmatrix} K^i U_C^i \\ I \end{pmatrix} \prec 0.$$

- By using the inertia arguments in [28]. We can write the dual inequality as, in order to put K^i terms outside,

$$\begin{pmatrix} I \\ -(U_C^i)^T(K^i)^T \end{pmatrix}^T (\Gamma^i)^{-1} \begin{pmatrix} I \\ -(U_C^i)^T(K^i)^T \end{pmatrix} \succ 0 \tag{5.8}$$

$$\Leftrightarrow \begin{pmatrix} I \\ (K^i)^T \end{pmatrix}^T \begin{pmatrix} I & 0 \\ 0 & -(U_C^i)^T \end{pmatrix}^T (\Gamma^i)^{-1} \begin{pmatrix} I & 0 \\ 0 & -(U_C^i)^T \end{pmatrix} \begin{pmatrix} I \\ (K^i)^T \end{pmatrix} \succ 0. \tag{5.9}$$

- Define Λ^i as

$$\Lambda^i := \begin{pmatrix} I & 0 \\ 0 & -(U_C^i)^T \end{pmatrix}^T (\Gamma^i)^{-1} \begin{pmatrix} I & 0 \\ 0 & -(U_C^i)^T \end{pmatrix}.$$

  Now, we can simplify (5.9) to

$$\begin{pmatrix} I \\ (K^i)^T \end{pmatrix}^T \Lambda^i \begin{pmatrix} I \\ (K^i)^T \end{pmatrix} \succ 0. \tag{5.10}$$

  In the inequality (5.10), the only unknown is the K^i term. The objective is to find a proper K^i for the given condition.

- Hence solving the inequality

$$\begin{pmatrix} I \\ (K^i)^T \end{pmatrix}^T \Lambda^i \begin{pmatrix} I \\ (K^i)^T \end{pmatrix} \succ 0$$

  will result a stable controller.

- Then, take positive eigenspace of Λ as $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$.

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \Lambda^i \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \succ 0 \tag{5.11}$$

Then, $K^i = (V_2^i(V_1^i)^{-1})^T$ is one possible choice for the controller matrix for the i^{th} unit.

At the end of this section, we provided a clear method which can be used in node by node basis control synthesis. Now, we are ready to design distributed controllers for the vehicle platoon models given in Chapter 3.

6. DISTRIBUTED CONTROL OF VEHICLE PLATOONS

6.1. Resulting LMIs and Actual controllers

6.1.1. Finite Interconnected Vehicle Array Representative model

We are interested in the idealized problem in which injected disturbance should be suppressed along the platoon. There is no disturbance on agents, the only source of disturbance is lead vehicle. We can model all interior vehicles as shown in the Figure 6.1. It is assumed that all interior vehicles are identical, and we want to design identical controllers for all interior vehicles. However, we need to model the first vehicle(follower of the lead) and the last vehicle independently. Hence, boundaries of the vehicle string should be designed separately.

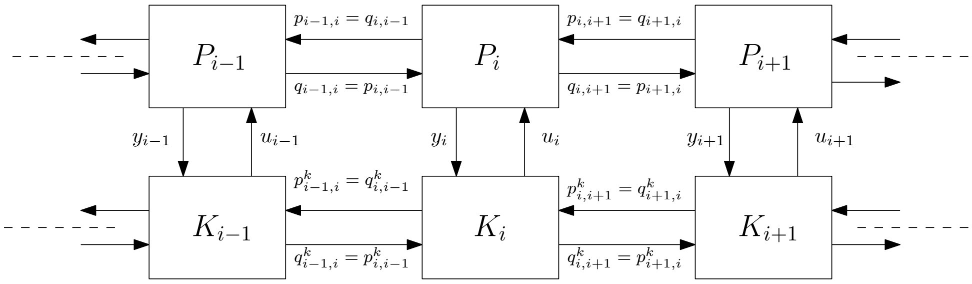

Figure 6.1. Stability problem with control no disturbance:interior vehicles

In Figure 6.2 a representative model is given. First vehicle represents the follower of the leader, second vehicle represents all interior vehicles and 3rd vehicle represents the last vehicle in the vehicle string. Disturbance d enters the first vehicle and the performance output z from the third vehicle. Since, we are interested in the idealized problem. As shown Figure 6.2, we model all interior vehicles as a single vehicle. The main reason is obviously the implementation problem, we want to design identical controllers for all interior vehicles. By this way we can employ same strategy along the vehicle platoon for all vehicles. The key point is IQCs are constrained such that, no matter how many interior vehicles are in the string, the stability analysis is still valid by exploiting the interconnection structure for all interior vehicles as in Figure 6.1.

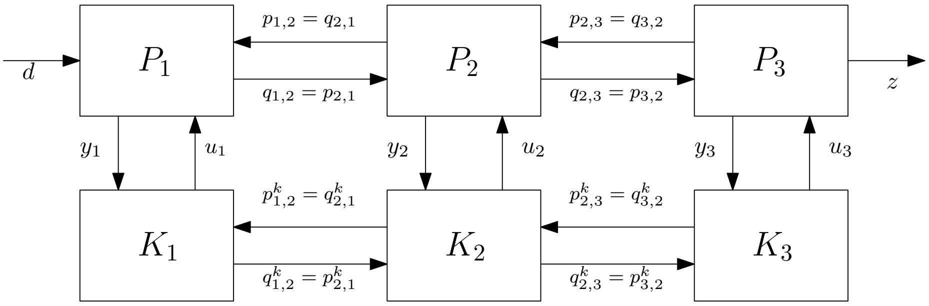

Figure 6.2. Representative Model

The representative model given in Figure 6.2 can be transformed to LPV like model as in Figure 6.3.

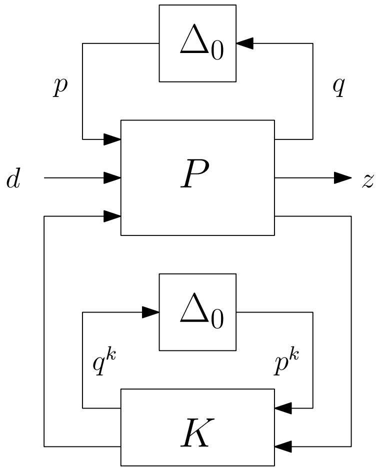

Figure 6.3. LPV like model Representative Model

In this problem, we have fixed interconnections, and there is no loss or delay in the transferred signal. By using the relation $p_{ij} = q_{ji}$, we can write Δ as

$$\left(\begin{array}{c} p_{12} \\ \hdashline p_{21} \\ p_{23} \\ \hdashline p_{32} \end{array}\right) = \left(\begin{array}{c:cc:c} 0 & I & 0 & 0 \\ \hdashline I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hdashline 0 & 0 & I & 0 \end{array}\right) \left(\begin{array}{c} q_{12} \\ \hdashline q_{21} \\ q_{23} \\ \hdashline q_{32} \end{array}\right).$$

Since, we identified the interconnection map between p and q as Δ_0. We can repeat that interconnection map between p^k and q^k, as Δ_0 in the controllers. We could design distributed controllers for the given 3 subsystems given in Figure 6.2 based on the Corollary 5.1.

At this stage we want to give the three vehicle model explicitly. Model for the first vehicle could be given as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{v}_1 \\ \hdashline q_{1,2} = v_1 \\ \hdashline y_1 = e_1 \end{pmatrix} = \left(\begin{array}{cc:c:c:c} 0 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} e_1 \\ v_1 \\ \hdashline p_{1,2} = v_2 \\ \hdashline d_1 = v_0 \\ \hdashline u_1 \end{pmatrix}, \tag{6.1}$$

if we select disturbance as $d = v_0(t)$. From the first vehicle model, we can see that a velocity change in the leader will introduce a spacing error throughout the vehicle platoon. A local measurement of spacing error is satisfactory in order to decide the control action at that node.

Model for the second vehicle can be given as

$$\begin{pmatrix} \dot{e}_2 \\ \dot{v}_2 \\ \hdashline q_{2,1} = v_2 \\ q_{2,3} = v_2 \\ \hdashline y_2 = e_2 \end{pmatrix} = \left(\begin{array}{cc:cc:c} 0 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} e_2 \\ v_2 \\ \hdashline p_{2,1} = v_1 \\ p_{2,3} = v_3 \\ \hdashline u_2 \end{pmatrix} \tag{6.2}$$

which represents the all interior vehicles. The second vehicle adjust its velocity by looking its direct neighbors and its absolute velocity, and local spacing error measurement.

The last vehicle will not have a backward spacing error, then the local measurement error for the last vehicle is defined similar to the one directional case. Model for the last vehicle could be given as

$$\left(\begin{array}{c} \dot{e}_3 \\ \dot{v}_3 \\ \hline q_{3,2} = v_3 \\ \hline z_3 \\ \hline y_3 = e_3 \end{array}\right) = \left(\begin{array}{cc|c|c} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} e_3 \\ v_3 \\ \hline p_{3,2} = v_3 \\ \hline u_3 \end{array}\right). \tag{6.3}$$

The last vehicle adjust its speed by only looking preceding vehicle velocity and its velocity and local spacing error defined for the last vehicle.

In this model velocity of the zeroth vehicle leader is the source of disturbance and our performance output is the local spacing error of last vehicle $z = e_3$.

6.1.2. Three Vehicle Representative Model:Resulting LMIs

After giving the exact vehicle models, now we are ready to build decomposed LMIs for the each subsystem. We used the Theorem 5.1 and its extension Corollary 5.1 for block diagonal matrix variables.

The first subsystem has no performance output, then we cropped corresponding rows and columns for performance output. As can be seen from (6.4), the resulting LMIs are the same LMIs presented in (5.4). For the first agent at left boundary in Figure 6.2, resulting LMIs are given in equations (6.4).

$$\star^T \star^T \left(\begin{array}{cc|cc|c} 0 & X_T^1 & 0 & 0 & 0 \\ X_T^1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q_1 & S_1 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{TT}^1 & A_{TS}^1 & B_T^i \\ \hline 0 & I & 0 \\ A_{ST}^1 & A_{SS}^1 & B_S^1 \\ \hline 0 & 0 & I \end{array}\right) \Psi_1 \prec 0, \tag{6.4a}$$

$$\star^T \star^T \left(\begin{array}{cc:cc:c} 0 & Y_T^1 & 0 & 0 & 0 \\ Y_T^1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \tilde{Q}_1 & \tilde{S}_1 & 0 \\ 0 & 0 & \tilde{S}_1^T & \tilde{R}_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \tilde{Q}_p \end{array}\right) \left(\begin{array}{cc} -(A_{TT}^1)^T & -(A_{ST}^1)^T \\ I & 0 \\ \hdashline -(A_{TS}^1)^T & -(A_{SS}^1)^T \\ 0 & I \\ \hdashline -(B_T^1)^T & -(B_S^1)^T \end{array}\right) \Phi_1 \succ 0 \tag{6.4b}$$

and

$$\begin{pmatrix} X_T^1 & I \\ I & Y_T^1 \end{pmatrix} \succ 0, \tag{6.4c}$$

where Ψ_1 and Φ_1 are defined as

$$\Psi_1 = \begin{pmatrix} (C_T^1)^T \\ (C_S^1)^T \\ (F^1)^T \end{pmatrix}_{\perp} \quad \text{and} \quad \Phi_1 = \begin{pmatrix} B_{TU}^1 \\ B_{SU}^1 \end{pmatrix}_{\perp} .$$

The middle vehicle has no disturbance and performance output. We can get the resulting LMIs just by cropping the performance rows and columns in Equation (5.4). In this set of LMIs the fundamental difference is the internal supply rates Q_2, S_2, R_2 are constrained such that $S_2 = 0$ and $Q_2 = -R_2$, since we want that adding an extra identical interior vehicle will not effect the analysis conditions. For the middle vehicle in Figure 6.2, the resulting LMIs are given in Equations (6.5).

$$(\Psi_2)^T \left(\begin{array}{cc} I & 0 \\ A_{TT}^2 & A_{TS}^2 \\ \hdashline 0 & I \\ A_{ST}^2 & A_{SS}^2 \end{array}\right)^T \left(\begin{array}{cc:cc} 0 & X_T^2 & 0 & 0 \\ X_T^2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_2 & 0 \\ 0 & 0 & 0 & -Q_2 \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_{TT}^2 & A_{TS}^2 \\ \hdashline 0 & I \\ A_{ST}^2 & A_{SS}^2 \end{array}\right) \Psi_2 \prec 0, \tag{6.5a}$$

$$\star^T \star^T \left(\begin{array}{cc|cc} 0 & Y_T^2 & 0 & 0 \\ Y_T^2 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q}_2 & 0 \\ 0 & 0 & 0 & -\tilde{Q}_2 \end{array}\right) \left(\begin{array}{cc} -(A_{TT}^2)^T & -(A_{ST}^2)^T \\ I & 0 \\ \hline -(A_{TS}^2)^T & -(A_{SS}^2)^T \\ 0 & I \end{array}\right) \Phi_2 \succ 0 \tag{6.5b}$$

and

$$\begin{pmatrix} X_T^2 & I \\ I & Y_T^2 \end{pmatrix} \succ 0, \tag{6.5c}$$

where Ψ_2 and Φ_2 are defined as

$$\Psi_2 = \begin{pmatrix} (C_T^2)^T \\ (C_S^2)^T \end{pmatrix}_{\perp} \quad \text{and} \quad \Phi_2 = \begin{pmatrix} B_{TU}^2 \\ B_{SU}^2 \end{pmatrix}_{\perp}$$

We used identical internal supply rates for mid vehicle. Q_2 is constrained as follows since there two incoming signals to the i^{th} unit from units $i-1$ and $i+1$. We define Q_2 and R_2 on a signal by signal basis as

$$Q_2 = \begin{pmatrix} Q^i & 0 \\ 0 & Q^i \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} R^i & 0 \\ 0 & R^i \end{pmatrix}$$

There is always permutation between these variables, we can guarantee that supply rate $\mathcal{P}_i$ is equal to 0 for all interior vehicles by the particular choice such as

$$\begin{pmatrix} Q_i & 0 \\ 0 & -Q_i \end{pmatrix} = \left(\begin{array}{cc|cc} Q^i & 0 & 0 & 0 \\ 0 & Q^i & 0 & 0 \\ \hline 0 & 0 & -Q^i & 0 \\ 0 & 0 & 0 & -Q^i \end{array}\right)$$

This particular choice guarantees that middle model can represent all interior vehicles. For a detailed discussion one can look Appendix C. Since we put same variables on these vehicles, we can synthesize identical distributed controllers for all interior vehicles.

At the end of string, there is no disturbance, however there is a performance output z. Again, we crop the corresponding rows and columns. Then, the resulting LMIs are given in equations (6.6).

$$\star^T \star^T \left(\begin{array}{cc:cc:c} 0 & X_T^3 & 0 & 0 & 0 \\ X_T^3 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_3 & S_3 & 0 \\ 0 & 0 & S_3^T & R_3 & 0 \\ \hdashline 0 & 0 & 0 & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_{TT}^3 & A_{TS}^3 \\ \hdashline 0 & I \\ A_{ST}^3 & A_{SS}^3 \\ \hdashline C_{PT}^3 & C_{PS}^3 \end{array}\right) \Psi_3 \prec 0, \tag{6.6a}$$

$$\star^T \star^T \left(\begin{array}{cc:cc:c} 0 & Y_T^3 & 0 & 0 & 0 \\ Y_T^3 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \tilde{Q}_3 & \tilde{S}_3 & 0 \\ 0 & 0 & \tilde{S}_3 & \tilde{R}_3 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \tilde{R}_p \end{array}\right) \left(\begin{array}{ccc} -(A_{TT}^3)^T & -(A_{ST}^3)^T & -(C_{PT}^3)^T \\ I & 0 & 0 \\ \hdashline -(A_{TS}^3)^T & -(A_{SS}^3)^T & -(C_{PS}^3)^T \\ 0 & I & 0 \\ \hdashline 0 & 0 & I \end{array}\right) \Phi_3 \succ 0, \tag{6.6b}$$

and

$$\begin{pmatrix} X_T^3 & I \\ I & Y_T^3 \end{pmatrix} \succ 0 \tag{6.6c}$$

where Ψ_3 and Φ_3 are defined as

$$\Psi_3 = \begin{pmatrix} (C_T^3)^T \\ (C_S^3)^T \end{pmatrix}_{\perp} \quad \text{and} \quad \Phi_3 = \begin{pmatrix} B_{TU}^3 \\ B_{SU}^3 \\ E^3 \end{pmatrix}_{\perp}$$

As a quadratic performance we select Q_p as $-\gamma I$, and R_p as $\gamma^{-1} I$. In our synthesis we select $\gamma = 1$.

Additionally, we need to find $Q = Q^T$, S and $R = R^T$ such that $\mathcal{P}_\Delta \succeq 0$. More clearly,

$$\mathcal{P}_\Delta = \begin{pmatrix} \Delta_0 \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta_0 \\ I \end{pmatrix},$$

where Δ defined as $p = \Delta_0 q$, particularly

$$\Delta_0 = \left(\begin{array}{c:cc:c} 0 & I & 0 & 0 \\ \hdashline I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hdashline 0 & 0 & I & 0 \end{array}\right).$$

6.2. Numerical Results

All controller conditions have been expressed in terms of LMIs. Efficient commercial and free LMI solvers are available to find a feasible solution. We use the MATLAB toolbox, YALMIP as parser and SeDuMi as solver for our calculations. Resulting distributed controllers have the following numerical values

$$\left(\begin{array}{c} \dot{x}_1^k \\ \hdashline q_{12}^k \\ \hdashline u_1 \end{array}\right) = \left(\begin{array}{cc:c:c} -2.8822 & 0.9848 & 2.5898 & -13.5460 \\ 1.9970 & -2.2716 & -2.6310 & 14.3097 \\ \hdashline -0.0653 & 0.2805 & 0.0083 & -0.6461 \\ \hdashline -0.5846 & 0.5284 & 0.4994 & -3.9245 \end{array}\right) \left(\begin{array}{c} x_1^k \\ \hdashline p_{12}^k \\ \hdashline y_1 \end{array}\right) \tag{6.7a}$$

$$\left(\begin{array}{c} \dot{x}_2^k \\ \hdashline q_{21}^k \\ q_{23}^k \\ \hdashline u_2 \end{array}\right) = \left(\begin{array}{cc:cc:c} -3.0152 & 1.0785 & 2.6917 & 2.6928 & -18.8216 \\ 2.0999 & -2.3834 & -2.7335 & -2.7340 & 19.8834 \\ \hdashline -0.0333 & 0.2015 & 0.0074 & 0.0073 & -0.6419 \\ -0.0312 & 0.1925 & 0.0065 & 0.0064 & -0.6118 \\ \hdashline -0.4528 & 0.4426 & 0.3825 & 0.3827 & -4.1444 \end{array}\right) \left(\begin{array}{c} \dot{x}_2^k \\ \hdashline p_{21}^k \\ p_{23}^k \\ \hdashline y_2 \end{array}\right) \tag{6.7b}$$

$$\begin{pmatrix} \dot{x}_3^k \\ \hdashline p_{32}^k \\ \hdashline u_3 \end{pmatrix} = \left(\begin{array}{cc:c:c} -3.3662 & 2.2311 & 6.5566 & -34.3457 \\ 3.3063 & -4.2487 & -9.3005 & 49.2642 \\ \hdashline -0.0986 & 0.3136 & 0.0142 & -0.8310 \\ \hdashline -0.8698 & 0.8922 & 1.9638 & -12.0167 \end{array}\right) \begin{pmatrix} x_3^k \\ \hdashline p_{32}^k \\ \hdashline y_3 \end{pmatrix} \tag{6.7c}$$

Linear models of the vehicle string for various platoon sizes extracted by using linmod command in MATLAB. Bode magnitude plots disturbance to last spacing error are plotted for various string sizes in Figure 6.4. Also, bode magnitude plots disturbance to spacing errors are plotted for fixed string sizes ($N = 6$) in Figure 6.5. Also, various disturbance profiles applied to resulting closed loop, then spacing error values plotted in Figures 6.7, 6.8, and 6.9.

We see that the disturbance is attenuated from d to z bode magnitude plots for various platoon sizes for $N = 3$, $N = 6$, $N = 9$, $N = 12$ and $N = 18$ vehicles in Figure 6.4. We see that the peak of the bode plot does not changed as the string size increased. From this figure we see that adding an extra interior vehicle does not change frequency response characteristic of the overall system.

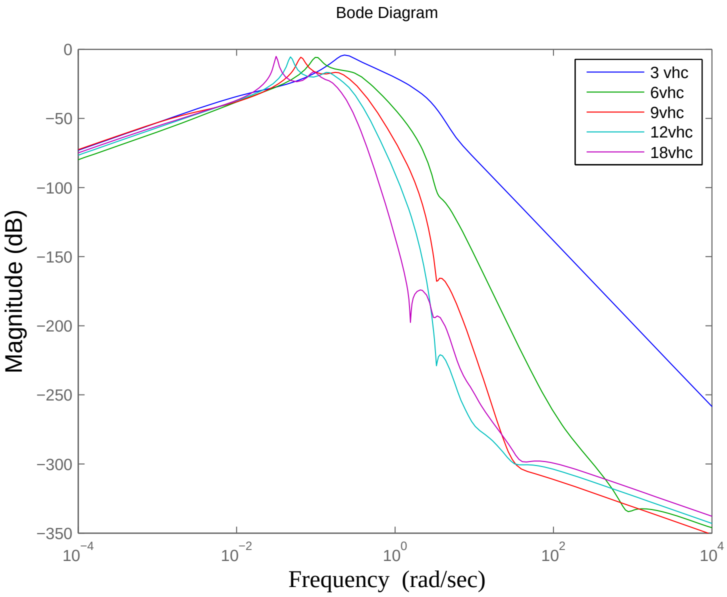

Figure 6.4. Bode magnitude plots d to z for various platoon sizes

Bode magnitude plots disturbance to spacing error e_1, e_2, e_3, e_4, e_5, and e_6 are plotted. In Figure 6.5, we see that the designed controllers could attenuate all spacing errors in a fixed string size $N = 6$ closed loop model. From this figure, we see that the designed controllers are effective to suppress errors for low and high frequency signals. However, the signals with $\omega = 0.1$ rad/sec are more critical.

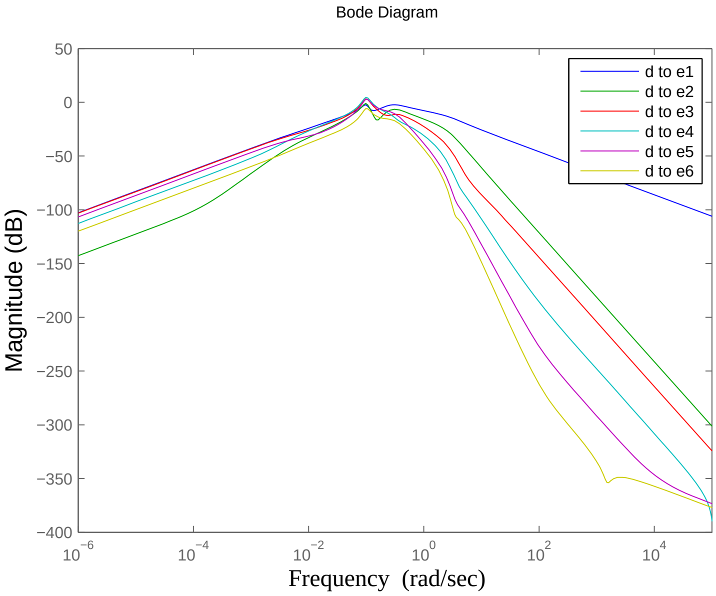

Figure 6.5. Bode magnitude plots d to e_1, e_2, e_3, e_4, e_5, e_6

Now, we can switch to time domain responses of the closed loop system. For computational purposes, we fixed our vehicle string model as $N = 9$, since our aim is to observe general trend in the spacing errors along the platoon. First, we consider the scenario where the lead vehicle changes the its absolute velocity as can be seen in Figure 6.6.

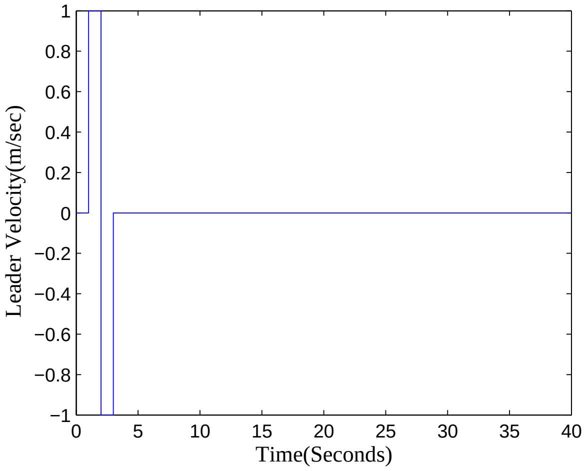

Figure 6.6. Input profile:Velocity variation of the leader

We applied a velocity variation in the leader as disturbance in Figure 6.6. Then, we plotted all spacing errors except e_9, since its definition is different than the previous ones. The spacing errors are all attenuated as can be seen in Figure 6.7. Furthermore, we want to point out that the spacing errors along the platoon has a decreasing trend. Hence, we see that as vehicle index increases spacing errors are not amplified in the spatial domain. We conclude that the designed controllers shows excellent properties in handling velocity variations of lead vehicle in Figure 6.6.

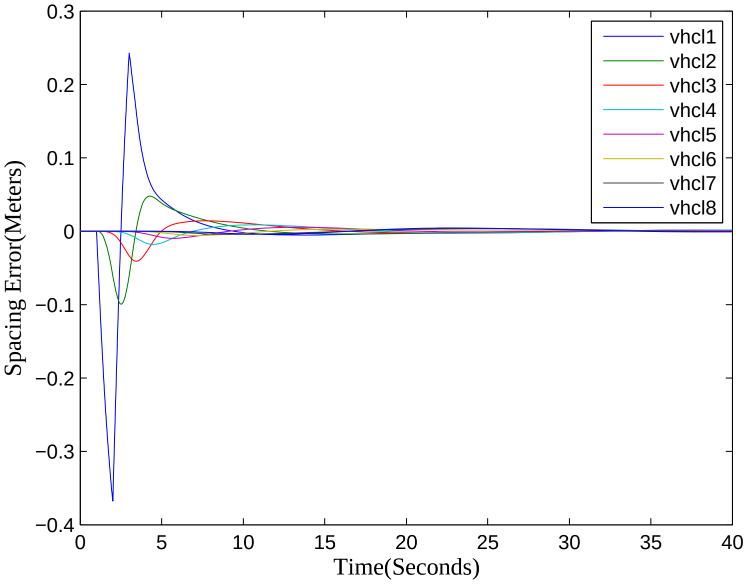

Figure 6.7. Spacing errors versus time in response to Figure 6.6

Now consider a the second scenario in which leader moves at a constant velocity then stops at 10 seconds later. If we apply a pulse for 10 seconds as disturbance. The response of the controlled system can be seen in Figure 6.8. The controllers could attenuate errors effectively. Also spacing errors along the platoon still has a decreasing trend. However, as the pulse time increases our performance characteristics regarding the spacing errors along the platoon derogates. This is simply due to the fact that the disturbance suppression along the platoon is vulnerable to frequency content of the disturbance.

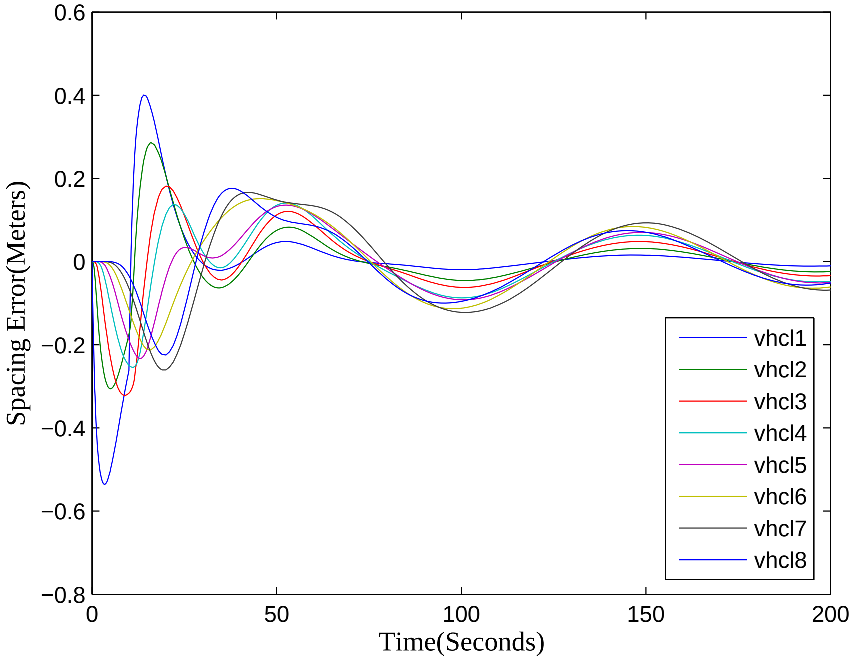

Figure 6.8. Spacing errors versus time in response to 10sec pulse

If a constant disturbance introduced by the leader vehicle, the closed loop system is still stable. However system acts like a lightly damped system. Spacing errors can not penetrate further into the vehicle platoon along the spatial direction. The response of the controlled system can be seen in Figure 6.9. Maximum spacing error is still in the first vehicle. Our interpretation for the degraded performance is the frequency content of the step input.

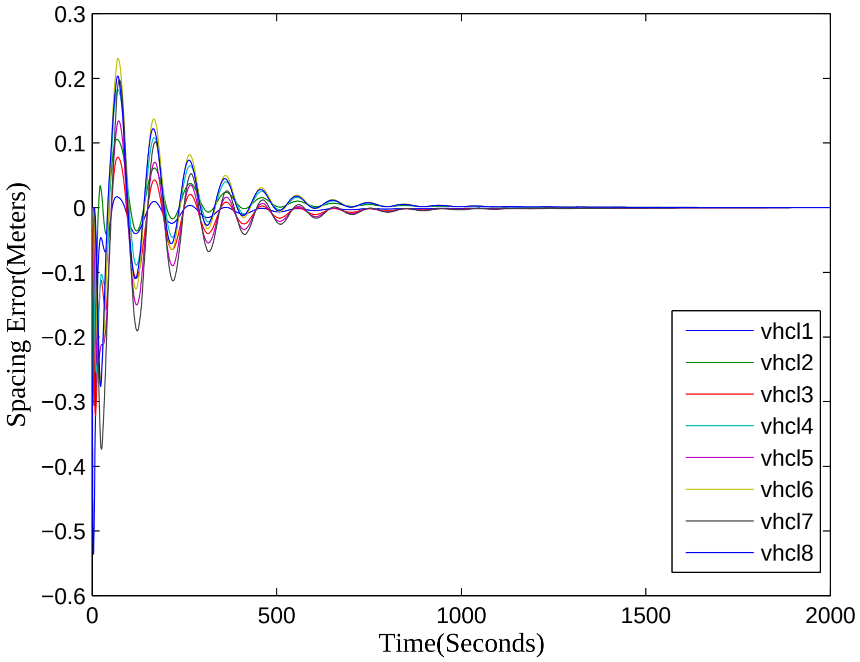

Figure 6.9. Spacing errors versus time when step input applied

When a persistent step input disturbance applied to the system, response of system is not good as the above simulation results. It takes considerable amount of time in order to reach stability for the overall system. The closed loop system damping is very low. We previously saw, from our bode plots, that overall closed loop systems low frequency attenuation is not as good as high frequency attenuation. Hence, low frequency disturbance signals reduces the closed loop system performance. In order to support our claim, we applied pure sinusoid signals as disturbance to our closed loop system. The responses can be seen in Figures 6.10 and 6.11.

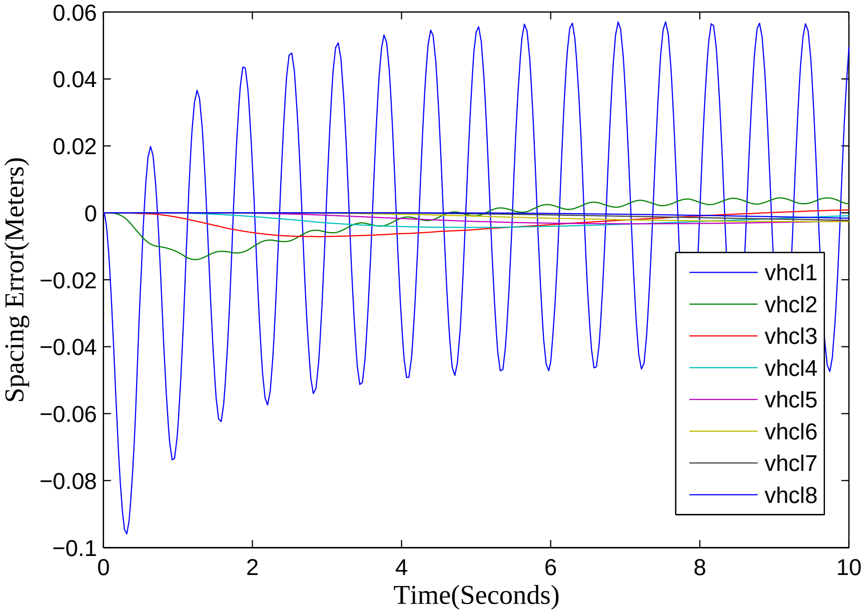

Figure 6.10. Spacing errors in response to $d(t) = sin(10t)$

First, high frequency disturbance injected to our vehicle string. We see disturbance signal attenuated along the platoon as we expected.(See Figure 6.10.)

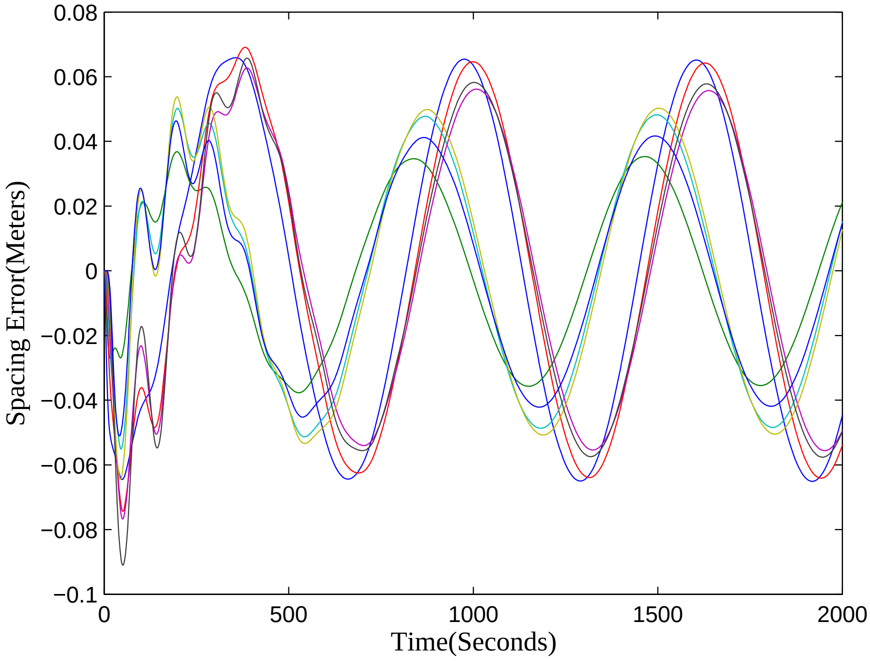

Figure 6.11. Spacing errors in response to $d(t) = sin(0.1t)$

Low frequency disturbance injected to our vehicle string. Spacing error plots show a complex behavior and oscillations along the platoon. Spatial disturbance attenuation is worse as compared to high frequency disturbance.(See Figure 6.11.)

At this point, we may define a translation mode as desired velocity v_d of the whole vehicle array, and we may refer v_L to the actual velocity of the leader. We can always define v_0 as $v_0 := v_L - v_d$, by this approach we avoid from the lower frequency content of the signal. This approach is kind of shifting velocity origin to a desired velocity for the all vehicles. This will eliminate the low frequency content of the disturbance. In this scenario, we assumed that the vehicular arrray is moving at a desired velocity, due to the some reasons the leaders to velocity has changed to v_L around v_d , this will certainly impose a spacing error from leader to end. Our controller scheme excellently handles this kind of scenario. Also, if we model disturbance at the lead vehicle as zero mean white noise we can always suppress this kind of spacing errors due to disturbance. Hence, vehicle string absorbs these kind of spacing errors on bort time domain and spatial domain.

6.3. Variations

We could change the mass of the vehicles. In previous models, we used a unit mass for the vehicle. Now we changed mass $m = 1000$ and repeated the controller synthesis. Bode magnitude plots of disturbance to spacing error are plotted for various string sizes. The bode magnitude plots showed similar characteristics as can be seen in Figure 6.12.

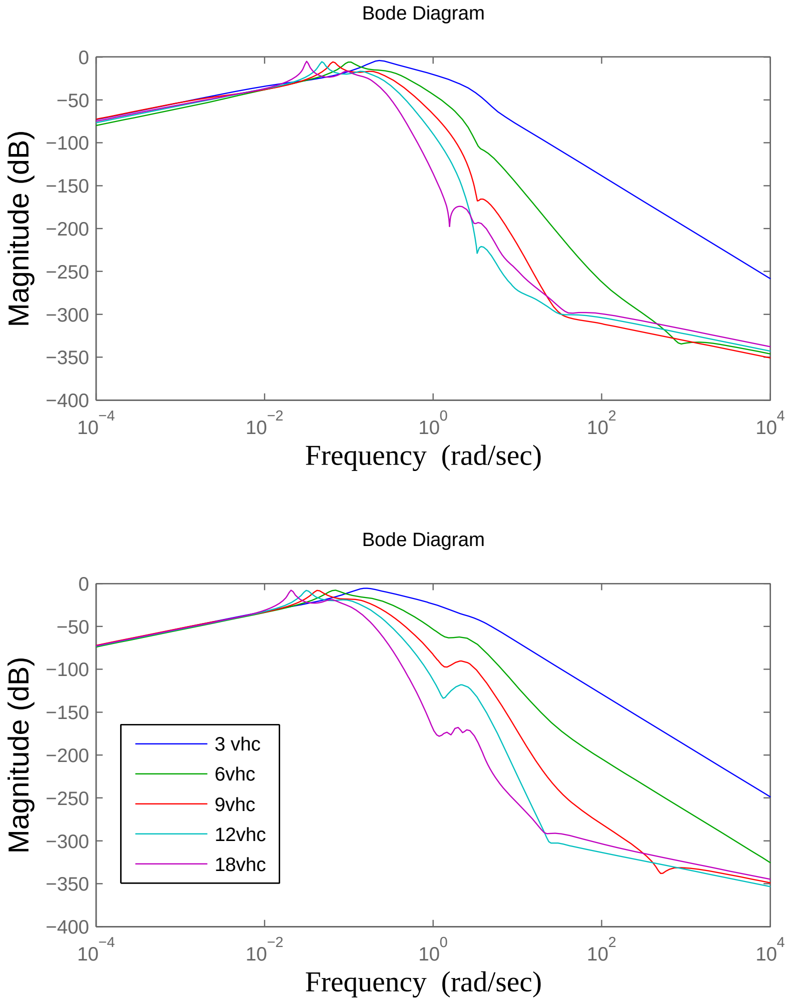

Figure 6.12. Mass Change $m = 1$ and $m = 100$

Also, we can change our model such as defining controlled output as

$$z := \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Then, we can synthesize controllers accordingly. We see that the resulting controllers have similar properties with respect to the previous results. From bode magnitude plot in Figure 6.13, we can see that the the controllers, obtained for $z = e_3$ and $z = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$, show quite similar responses in simulations.

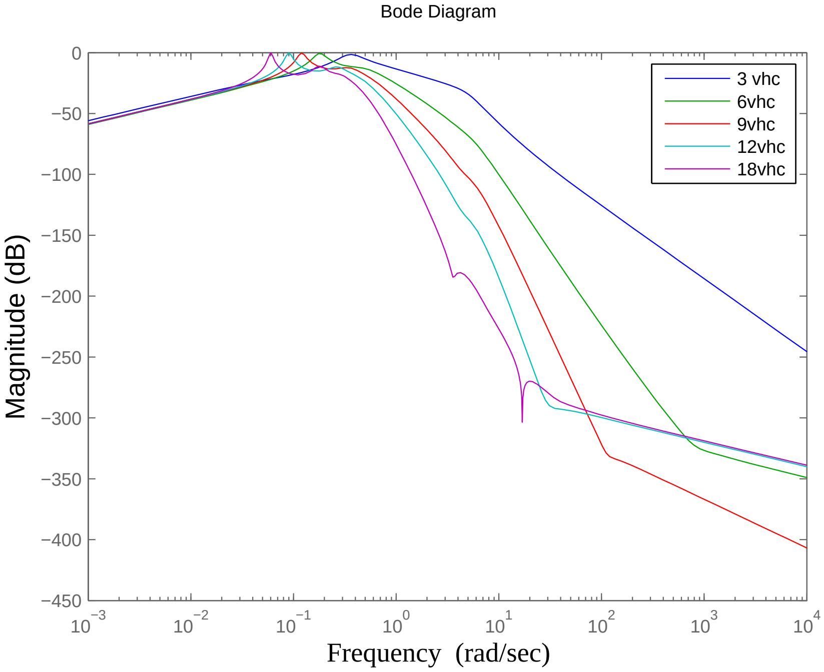

Figure 6.13. Bode plots d to last spacing error for various platoon sizes

More detailed interconnected models can be build. By adding a damping term which is can be seen as resistance forces during operation, under the assumption that resistive forces linearly varies with absolute velocity, we can approximate resistive forces as damping. Then these forces can be added to the model. Also external forces as wind gust or road grade can be seen as disturbance to each sub unit. Non-ideal interconnection channels can be built for more realistic models. Also, an uncertainty in the agent can be modeled similar to the actual LPV synthesis problem by the online measurement of variables. Our approach can cover up all these variations in the problem. However, we left these questions open for future studies.

7. CONCLUSION

In this thesis, we designed spatially invariant and interconnected controllers. The motivation of this study is the fact that constant spacing policy cannot accomplished by identical static feedback controllers. The uniqueness of our design allows us a communication channel between controllers similar to the idea in distributed control. The communication channel between controllers makes the main difference with respect to decentralized identical controllers. Similarly, we designed identical controllers for all interior vehicles, however the fundamental point is that the communication links between controllers make all actions unique. Hence, these links somewhat schedules the gains uniquely. Our simulation results show that employing an extra communication channel between controllers enhances the stability properties in the spatial direction. Since, disturbance signal cannot penetrate further into the vehicle string. Equivalently, we can say that the spacing errors are not amplified as the spatial index increases. Also, we should state that the high frequency disturbance signals excellently attenuated in the spatial domain because each closed loop sub-system acts like a low pass filter. Also, we could design wide variety of controller depending upon the proposed model. Here, our fundamental goal is to show the distributed control design procedure for vehicular string problems.

The advantages of distributed control are obvious. The controllers which are designed in distributed control fashion are well suited to H_∞ framework. In addition, another advantage of our proposed model is, the interconnection structure of the vehicle array can be exploited further, if we allow working on LPV control design framework. By this approach, one can get tractable solutions to the interconnected systems easily by using the similarities between LPV framework and distributed control. We also try to show the fundamental similarities distributed control and LPV control and try to make a link between them. By doing so, we also show that by opening self-scheduling channels, we can make control designs which is robust to bounded parameter changes by allowing online measurement of variables.

APPENDIX A: LMIs in CONTROL

We used Linear Matrix Inequalities (LMIs) extensively to derive well-posedness, stability and dissipativity. A linear matrix inequality can be expressed as

$$F(x) := F_0 + x_1 F_1 + ... + x_n F_n \prec 0 \tag{A.1}$$

where
(i) $x = (x_1, x_2, x_3, ..., x_n)$ are the decision variables.
(ii) F_0,F_1,...,F_n are real symmetric matrices.
(iii) The inequality < 0 is equivalent to say that maximum eigenvalue of $F(x)$ is less than zero.

The condition

$$F(x) \prec 0$$

defines a convex constraint on x. Details can be found in [27].

A.1. Stability Problem with LMIs

Many control problems could be transformed to a LMI feasibility problem. For simplicity, we can investigate stability properties of an autonomy system. We want to analyze stability of the following autonomous system

$$\dot{x} = Ax \tag{A.2}$$

where $A \in R^{n \times n}$ by using LMIs [27].

For such system above we can formulate stability by a quadratic Lyapunov function candidates. The first thing is to define a quadratic function candidate. this quadratic function can be related to potential energy of the system. The potential change rate should be less than zero, so in this way by decaying of the potential the system eventually reaches equilibrium in time. Let $V(x)$ be a candidate Lyapunov funtion.[27]

$$V(x) = x^T X x \succ 0 \tag{A.3}$$

We can take the time derivative of Lyapunov function we get

$$\dot{V}(x) = x^T A^T X x + x^T X A x \prec 0 \tag{A.4}$$

Thus, we are able to transform standard stability problem to a standard LMI feasibility problem by combining the equations (A.3) and (A.4) as

$$\begin{pmatrix} -X & 0 \\ 0 & A^T X + XA \end{pmatrix} \prec 0. \tag{A.5}$$

Many other problems such as μ analysis, singular value minimization, state feedback problems, KYP lemma, bounded real lemma can be transformed to standard LMI problems. Details can be found. [27] Here we want to show how useful LMIs in control theory.

Next, we want to show dissipativity in linear systems since we used systems with input and output. We can consider the following system with input p and output q given in equation (A.6).

$$\dot{x} = Ax + Bp \tag{A.6a}$$

$$q = Cx + Dp \tag{A.6b}$$

The system described in (A.6) is said to be dissipative if there exists $X = X^T > 0$

such that

$$\begin{pmatrix} I & 0 \\ A & B \\ \hdashline 0 & I \\ C & D \end{pmatrix}^T \left(\begin{array}{cc:cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \begin{pmatrix} I & 0 \\ A & B \\ \hdashline 0 & I \\ C & D \end{pmatrix} \prec 0. \tag{A.7}$$

The system admits $V(x) = x^T X x$ as *quadratic storage.* Also, *quadratic storage rate* can be written as

$$\begin{pmatrix} x \\ p \end{pmatrix}^T \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

Also, we could define *quadratic supply rate* as

$$s(p, q) = s(p, Cx + Dp) = \begin{pmatrix} x \\ p \end{pmatrix}^T \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

The idea behind the proof is simple, we require that the sum of quadratic storage rate and quadratic supply rate should be less than zero. By choosing suitable matrices for the unknowns Q, S and R we can state different stability criterions, such as bounded real lemma, positive real lemma only by changing supply rate function. Details can be found [27]. These results will be useful in the analysis section.

APPENDIX B: Parameter Elimination Lemma

Lemma B.1. *[28] Given the following matrices* $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{r\times m}$, *and* $P = P^T \in \mathbb{R}^{(n+m)\times(n+m)}$ *such that* $in(P) = (m, n, 0)$. *There exists an unstructured* $K \in \mathbb{R}^{p\times r}$ *such that*

$$\begin{pmatrix} I \\ A+BKC \end{pmatrix}^{\top} P \begin{pmatrix} I \\ A+BKC \end{pmatrix} \prec 0 \tag{B.1}$$

if and only if

$$(C^{\top})_{\perp}^{\top} \begin{pmatrix} I \\ A \end{pmatrix}^{\top} P \begin{pmatrix} I \\ A \end{pmatrix} (C^{\top})_{\perp} \prec 0 \tag{B.2}$$

$$(B_{\perp})^{\top} \begin{pmatrix} -A^{\top} \\ I \end{pmatrix}^{\top} P^{-1} \begin{pmatrix} -A^{\top} \\ I \end{pmatrix} B_{\perp} \succ 0 \tag{B.3}$$

hold true.

APPENDIX C: Neutral Interconnection for Interior Vehicles

We only consider analysis and synthesis the most general case. At this stage, we need to give particular example and an application of our approach.

Let assume that we have ideal channels and exclude self supply channels. How to select internal supply rates are positive or equal to zero for all interior vehicles?

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^T \begin{pmatrix} \mathbf{diag}(Q_i) & \mathbf{diag}(S_i) \\ \mathbf{diag}(S_i^T) & \mathbf{diag}(R_i \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0$$

We can rewrite the above inequality as

$$\Delta^T Q \Delta + \Delta^T S + S^T \Delta + R \succeq 0 \tag{C.1}$$

where Q_i defined with dimension n_{p_i}(size of the internal input to the corresponding unit). Note this matrices are infinite dimensional. In particular Δ_0 represents the infinite dimensional systems topology.

For our particular interconnection structure, our certainty block can be given as

$$\Delta_0 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & I & 0 & 0 & \cdots \\ \cdots & I & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & I & \cdots \\ \cdots & 0 & 0 & I & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can select supply rates for neutral interconnection similar in [22]. Since, we know the structure of the certainty block which represents the interconnection map.

We could give the following relation for the interior vehicles as

$$\Delta_0^T Q \Delta_0 + \Delta_0^T S + S^T \Delta_0 + R = 0 \tag{C.2}$$

We can define

$$Q_i := \begin{pmatrix} Q_{i,i-1} & 0 \\ 0 & Q_{i,i+1} \end{pmatrix}$$

and $Q_{i,i-1} = Q_{i,i+1}$ from the assumption of identical sub-units. By restricting Q, S and R as in [22]. We do not need to worry about the internal supply rates for all interior vehicles.

$$\begin{pmatrix} \Delta_0 \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta_0 \\ I \end{pmatrix} = 0$$

where Δ_0 is defined as $p = \Delta_0 q$. Explicitly, we restricted Q and R such that

$$Q = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & Q^i & 0 & 0 & 0 & \dots \\ \dots & 0 & Q^i & 0 & 0 & \dots \\ \dots & 0 & 0 & Q^i & 0 & \dots \\ \dots & 0 & 0 & 0 & Q^i & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$R = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & R^i & 0 & 0 & 0 & \dots \\ \dots & 0 & R^i & 0 & 0 & \dots \\ \dots & 0 & 0 & R^i & 0 & \dots \\ \dots & 0 & 0 & 0 & R^i & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since all elements are identical. And they all restricted, $Q = \mathbf{diag}(\mathbf{diag}(Q_{ij}))$ and $R = \mathbf{diag}(\mathbf{diag}(R_{ij}))$ such that $Q_{ij} = Q^i$ and $R_{ij} = Q^i$ for all i and j. Furthermore, we select $S = 0$, the Equation given in (C.2) reduces the Equation (C.3).

$$\Delta_0^T Q \Delta_0 + R = 0 \tag{C.3}$$

We choose Q_i such that overall Q will not affected by permutation of the variables imposed by $\Delta_0^T Q \Delta_0$. One must notice that $\Delta_0^T Q \Delta_0 = Q$. Also, the equation C.3 imposes $Q = -R$ for the interior vehicles.

This will lead to a more conservative solution. However this will reduce the computational effort considerably. By this way infinite sized platoon problem can be transformed to a finite sized problem. So, by this approach we do not need to solve overall system, all we need to solve a representative subsystem and synthesize identical controllers.

$$\mathcal{P}_i = \begin{pmatrix} Q_i & S_i \\ S_i^T & R_i \end{pmatrix} = \begin{pmatrix} Q_i & 0 \\ 0 & -Q_i \end{pmatrix} = \begin{pmatrix} Q^i & 0 & 0 & 0 \\ 0 & Q^i & 0 & 0 \\ 0 & 0 & -Q^i & 0 \\ 0 & 0 & 0 & -Q^i \end{pmatrix} \tag{C.4}$$

We use identical internal supply rates as shown in the Equation (C.4) for interior vehicles with only communication between its closest left and right neighbor. This will guarantee the internal supply is always equal to zero.

Ultimately for representative solution for interior cars in the platoon, we restricted the resulting LMIs accordingly. The primal LMI can be given as

$$(\Psi_i)^T \begin{pmatrix} I & 0 \\ A^i_{TT} & A^i_{TS} \\ \hline 0 & I \\ A^i_{ST} & A^i_{SS} \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & X^i_T & 0 & 0 \\ X^i_T & 0 & 0 & 0 \\ \hline 0 & 0 & Q_i & 0 \\ 0 & 0 & 0 & -Q_i \end{array}\right) \begin{pmatrix} I & 0 \\ A^i_{TT} & A^i_{TS} \\ \hline 0 & I \\ A^i_{ST} & A^i_{SS} \end{pmatrix} \Psi_i \prec 0.$$

We can divide interconnection signals that comes from left and right separately. Hence, we could further divide system as follows

$$\star^T \star^T \left(\begin{array}{cc|cc|cc} 0 & X^i_T & 0 & 0 & 0 & 0 \\ X^i_T & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q^i & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q^i & 0 \\ 0 & 0 & 0 & 0 & 0 & -Q^i \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ A^i_{TT} & A^i_{TS^L} & A^i_{TS^R} \\ \hline 0 & I & 0 \\ A^i_{ST^L} & A^i_{SS^{LL}} & A^i_{SS^{LR}} \\ \hline 0 & 0 & I \\ A^i_{ST^R} & A^i_{SS^{RL}} & A^i_{SS^{RR}} \end{pmatrix} \Psi_i \prec 0.$$

APPENDIX D: Disturbance Propagation along the Vehicle String:Decentralized Control Approach

We can design decentralized controllers with only measurement of spacing error e_i for the vehicle platoon. First, we need to define spacing error as

$$e_i(t) := x_i(t) - x_{i-1}(t) - L_{des},$$

and also we need to define ξ_i as

$$\xi_i := \begin{pmatrix} e_i \\ \dot{e}_i \end{pmatrix}.$$

The system dynamics at the i^{th} unit can be expressed as in Equation (D.1)

$$\dot{\xi}_i = A_i \xi_i + B(u_i - u_{i-1}) \tag{D.1}$$

We applied a decentralized control law as in Equation (D.2) based on the measure local spacing error values.

$$u_i = -k_v \dot{e}_i - k_p e_i \tag{D.2}$$

More clearly we can express $\ddot{e}_i$ as in Equation (D.3).

$$u_i - u_{i-1} = \ddot{e}_i = -k_v \dot{e}_i - k_p e_i + k_v \dot{e}_{i-1} + k_p e_{i-1} \tag{D.3}$$

We can get close loop by using the relation in Equation (D.3). One must notice the interconnection between units i and $i-1$.

$$\begin{pmatrix} \dot{e}_i \\ \ddot{e}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k_{p_i} & -k_{v_i} \end{pmatrix} \begin{pmatrix} e_i \\ \dot{e}_i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_{p_{i-1}} & k_{v_{i-1}} \end{pmatrix} \begin{pmatrix} e_{i-1} \\ \dot{e}_{i-1} \end{pmatrix}$$

Shortly, we can write closed loop equations as,

$$\dot{\xi}_i = A_i \xi_i - A_{i-1} \xi_{i-1}$$

If we assume k_v and k_p are positive, eigenvalues of A_i will be negative. Only source of instability is A_{i-1}, the dependance to the previous vehicle. This coupling term, A_{i-1}, may cause instability for the composite system.

We can give the relation between lead and its follower, and derivative relations as follows

$$\begin{aligned} e_1 &:= x_1 - x_0 + L_1 \\ \Rightarrow \dot{e}_1 &= \dot{x}_1 - \dot{x}_0 \\ \Rightarrow \ddot{e}_1 &= \ddot{x}_1 - \ddot{x}_0. \end{aligned}$$

We can relate second derivative of the spacing error as

$$u_1 - u_0 = \ddot{e}_1.$$

Since, we do not have control over u_0, we take u_{lead} as disturbance. Applying u_{lead} will result a change in the velocity of leader.

$$u_1 = -k_v \dot{e}_i - k_p e_i$$

The dynamics of the first vehicle could be given as

$$\begin{pmatrix} \dot{e}_1 \\ \ddot{e}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k_{p_i} & -k_{v_i} \end{pmatrix} \begin{pmatrix} e_1 \\ \dot{e}_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_{lead}.$$

We applied a disturbance u_0 as can be seen in Figure D.1.

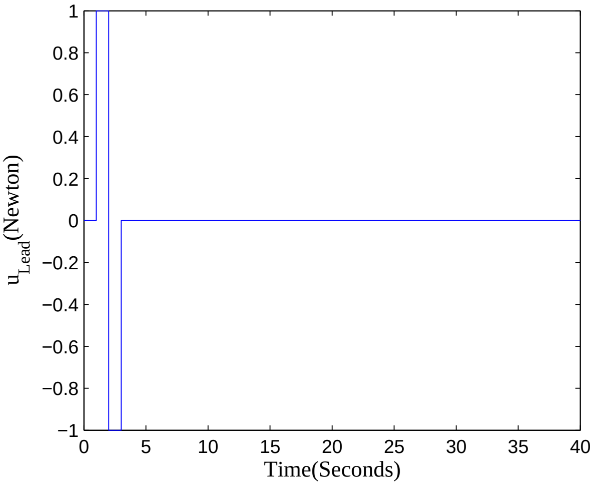

Figure D.1. Input profile:u_0 applied to the closed loop

Each vehicle's reference is its predecessor. Identical controllers used with relative position and relative velocity information similar to the controllers in [6]. Apply a disturbance as in the Figure D.1. The response of the closed loop can be seen in Figure D.2. Although the closed loop system is stable, as the spatial index increases the spacing errors are amplified along the platoon. This is so called "string instability" phenomenon in the vehicle strings.

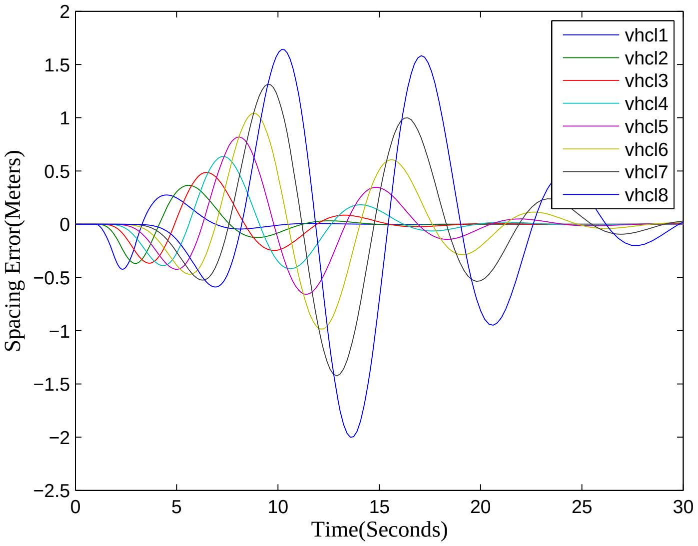

Figure D.2. Identical Decentralized Controllers

Similarly, we can design non-identical controllers. Each vehicle's reference is its predecessor. Non-identical controllers uses relative position and relative velocity information. Static PD controller are designed similar to [9]. Controller gains are tuned as vehicle index changes. Response of the closed loop can be seen in Figure D.3. We see that by tuning gains along the platoon, we can prevent disturbance propagation in the vehicle platoon.

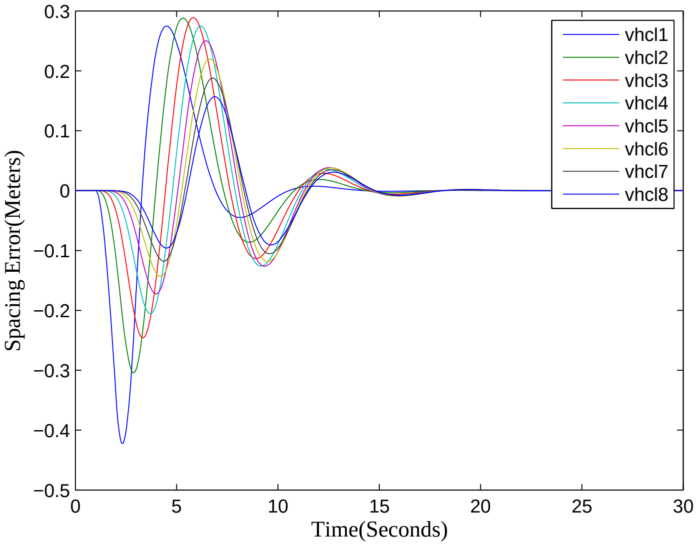

Figure D.3. Non-Identical Decentralized Controllers

Also, we could design decentralized controllers with bidirectional measurement of spacing error $e_i := x(t,i-1) - x(t,i) - L_{des}$ for the vehicle platoon similar to [10]. Bidirectional control law can be given as

$$u_i = -\beta(k_{p_i}e_i + k_{v_i}\dot{e}_i) + (1-\beta)(k_{p_i}e_{i+1} + k_{v_i}\dot{e}_{i+1}).$$

The case where $\beta = 1$ is the same as forward spacing error control. And also the case, where $\beta = 0$, is equivalent to backward spacing error control. However, this control strategy does not take into account forward spacing errors due to the velocity change in the leader. Larger β means that controller is more reactive to disturbances. Conversely, smaller β means that the controller is less concerned about the disturbances. However, controller is more concerned about the backward errors.

Identical controllers with weights on forward (e_i) and backward (e_{i+1}) errors are designed similar to [10]. From Figures D.4 , D.5 and D.6, it is seen that controllers have sluggish performance as β decreasing. Also, in Figures D.4 and D.5, we can see the amplification trend in spacing errors as the vehicle index increase. The case where $\beta = 0.7$ in Figure D.6 did not show an amplification trend in spacing errors, however the overall system response is very slow.

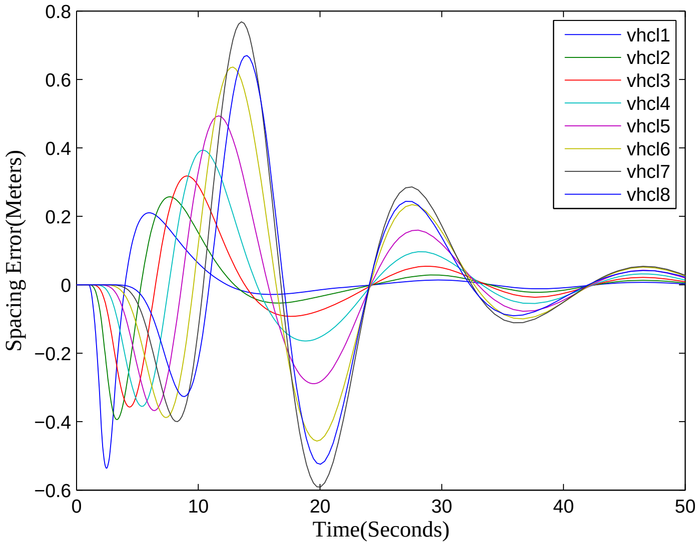

Figure D.4. Bidirectional Identical Decentralized Controllers $\beta = 0.7$

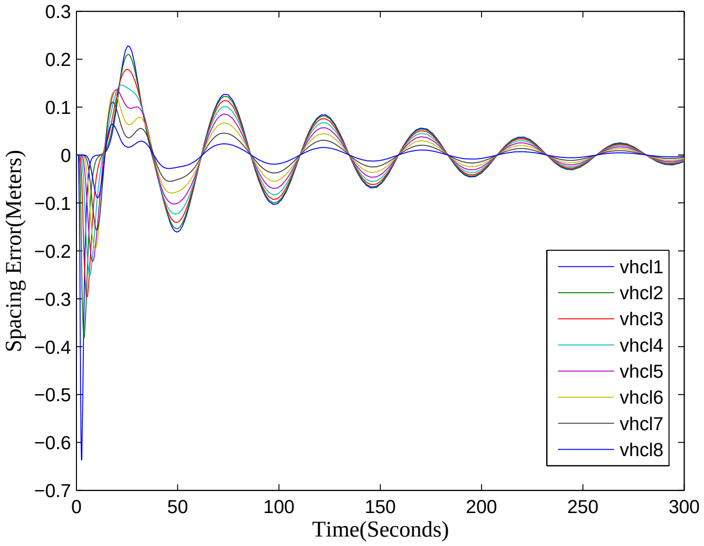

Figure D.5. Bidirectional Identical Decentralized Controllers $\beta = 0.5$

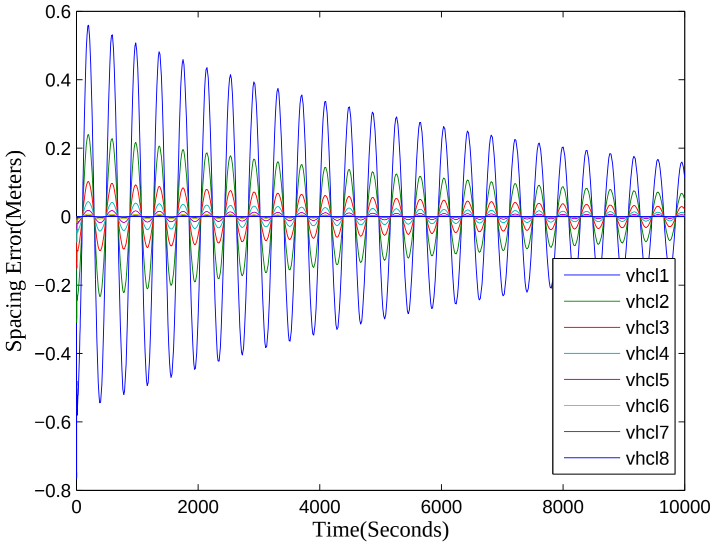

Figure D.6. Bidirectional Identical Decentralized Controllers $\beta = 0.3$

REFERENCES

1. Levine, W. S. and M. Athans, “On the optimal error regulation of a string of moving vehicles,” *IEEE Transactions on Automatic Control*, vol. 11, pp. 355–361, 1966.

2. Melzer, S. and B. Kuo, “A closed-form solution for the optimal error regulation of a string of moving vehicles,” *IEEE Transactions on Automatic Control*, vol. 16, pp. 50–52, 1971.

3. Wilkie, D. F., “A moving cell control control sheme for automated transportation systems,” *Transportation Science*, vol. 4, pp. 347–364, 1970.

4. Chu, K. C., “Decentralized control of high speed vehicular strings,” *Transportation Research*, pp. 361–383, 1974.

5. Ioannou, P. and C. Chien, “Autonomous intelligent cruise control,” *IEEE Transactions on Vehicular Technology*, vol. 42, pp. 657–672, November 1993.

6. Hedrick, J. K. and D. Swaroop, “Constant spacing strategies for platooning in automated highway systems,” *Trsansactions of the ASME*, vol. 121, pp. 462–470, 1999.

7. Swaroop, D., *String Stability of Interconnected Systems:An Application to Automated Highway Systems.* PhD thesis, University of California Berkeley, 1994.

8. Seiler, P., A. Pant and K. Hedrick, “Disturbance propagation in vehicle strings,” *IEEE Transactions on Automatic Control*, vol. 49, pp. 1835–1842, 2004.

9. Khatir, M. and E. Davison, “Decentralized control of a large platoon of vehicles using non-identical controllers,” in *American Control Conference, 2004*, vol. 3, pp. 2769–2776, 2004.

10. Naffin, D. J., M. Akar and G. S. Sukhatme, “Lateral and longitudinal stability for decentralized formation control,” in *Seventh International Symposium on Distributed Autonomous Robotic Systems*, 2004.

11. Bamieh, B., F. Paganini and M. Dahleh, “Optimal control of distributed arrays with spatial invariance,” *Lecture Notes in Control and Information Sciences*, p. 329-343, 1999.

12. Bamieh, B., “The structure of optimal controllers of spatially-invariant distributed parameter systems,” in *Proceedings of the 36th IEEE Conference on Decision and Control, Dec 8-12*, pp. 1056–1061, 1997.

13. Bamieh, B., F. Paganini and M. Dahleh, “Distributed control of spatially invariant systems,” *IEEE Transactions on Automatic Control*, vol. 47, p. 1091 1107, 2002.

14. Fang, H. and P. Antsaklis, “Stability of one-dimensional spatially invariant arrays perturbed by white noise,” in *2006. MED’06. 14th Mediterranean Conference on Control and Automation*, p. 15, 2006.

15. Motee, N. and A. Jadbabaie, “Distributed receding horizon control of spatially invariant systems,” in *American Control Conference*, 2006.

16. D’Andrea, R., “A linear matrix inequality approach to decentralized control of distributed parameter systems,” *Proceedings of the American Control Conference*, 1998.

17. D’Andrea, R. and G. Dullerud, “Distributed control design for spatially interconnected systems,” *IEEE Transactions on Automatic Control*, vol. 48, p. 14781495, 2003.

18. Dullerud, G. E. and R. D’Andrea, “Distributed control of heterogeneous systems,” *IEEE Transactions on Automatic Control*, vol. 49, pp. 2113–2128, 2004.

19. Langbort, C. and R. D’Andrea, “Distributed control of heterogeneous systems

interconnected over an arbitrary graph," No. December, pp. 2835–2840, Ieee, 2003.

20. Langbort, C., R. Chandra, and R. D'Andrea, "Distributed Control Design for Systems Interconnected Over an Arbitrary Graph," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1502–1519, September 2004.

21. Langbort, C. and R. D'Andrea, "Distributed Control of Spatially Reversible Interconnected Systems with Boundary Conditions," *SIAM Journal on Control and Optimization*, vol. 44, no. 1, p. 1, 2005.

22. Fang, H. and P. J. Antsaklis, "Distributed control with integral quadratic constraints," *In Proceedings of the 17th IFAC World Congress*, 2008.

23. Viccione, P., C. W. Scherer and M. Innocenti, "LPV Synthesis with Integral Quadratic Constraints for Distributed Control of Interconnected Systems," 2009.

24. Packard, A.,"Gain scheduling via linear fractional transformations," *Syst. Control Lett.*, vol. 22, no. 2, pp. 79–92, 1994.

25. Dullerud, G. E. and F. G. Paganini, *A Course in Robust Control Theory: a convex approach.* Springer-Verlag New York, Inc., 2000.

26. Curtain, R. F. and H. Zwart, *An introduction to infinite-dimensional linear systems theory.* New York, NY, USA: Springer-Verlag New York, Inc., 1995.

27. Scherer, C. W. and S. Weiland, *Lecture Notes: Linear Matrix Inequalities in Control.* April 1999.

28. Scherer, C. W., "LPV control and full block multipliers," *Automatica*, vol. 37, pp. 361–375, 2001.