OPTION PRICING UNDER STOCHASTIC INTEREST RATE

by

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ABSTRACT

OPTION PRICING UNDER STOCHASTIC INTEREST RATE

This study proposes a newly generated model to price options under a stochastic interest rate environment. The developed model introduces a numerical procedure with the arbitrage-free condition by working on a binomial tree. The model handles price changes in stocks and interest rates. Principally, it is assumed that the movements in the stock prices are defined by the Cox, Ross, Rubinstein (CRR) model. The CRR model proposes a numerical method to price options by assuming the interest rate is constant throughout the option's life. Moreover, the thesis claims that interest rates vary based on the Black, Derman, Toy (BDT) model. The BDT model defines the evolution of interest rates in the future. It presents a numerical procedure by using the binomial tree. Crucially, the interest rate is log-normally distributed in the BDT model; hence, the short rate cannot take a negative value. Also, the BDT model assumes that it has a mean-reverting property which means that the interest rate shows a tendency to converge to the average of interest rates in the long term. Additionally, this study utilizes the CRR and BDT model in order to derive a new option valuation framework. Also, the proposed model gives a numerical solution rather than an analytical formula due to the BDT model's structure. This thesis focuses on pricing the European options that can expire only on the maturity date. Furthermore, a group of options with different strike prices and different maturities is valued according to the developed model and the CRR model to observe interest rate impacts under two parameters: strike price and time-to-maturity. Finally, the estimated prices by both models are compared with actual-market prices to determine the accuracies of the models. Then, it is detected that the effect of the stochastic interest rate behavior on which maturities is significant.

ÖZET

RASSAL FAİZ ORANI ALTINDA OPSİYON FİYATLANDIRMASI

Bu tez çalışması, rassal bir faiz oranı ortamında opsiyonların fiyatlandırılması için yeni oluşturulmuş bir model önermektedir. Geliştirilen model, bir binom ağacı üzerinde çalışarak arbitrajsız sayısal bir prosedür sunar. Model, hisse senedi ve faiz oranlarındaki fiyat değişimlerini ele almaktadır. Prensip olarak hisse senedi fiyatlarındaki hareketlerin Cox, Ross, Rubinstein (CRR) modeli ile tanımlandığı varsayılmıştır. CRR modeli, faiz oranının opsiyonun ömrü boyunca sabit olduğunu kabul ederek opsiyonları fiyatlandırmak için sayısal bir yöntem önerir. Ayrıca bu tez, faiz oranlarının Black, Derman, Toy (BDT) modeline göre belirlendiğini varsaymıştır. BDT modeli, gelecekteki olası faiz oranlarını tanımlar ve binom ağacını kullanarak sayısal bir prosedür sunar. En önemlisi, faiz oranı BDT modelinde log-normal olarak dağılmıştır; bu nedenle, faiz oranı negatif bir değer alamaz. Ayrıca, BDT modeli, faiz oranının uzun vadede ortalama faiz oranına yakınsama eğilimi gösterdiği anlamına gelen ortalamaya dönme özelliğine sahip olduğunu varsayar. Bu tezde, yeni bir opsiyon değerleme çerçevesi türetmek için CRR ve BDT modelleri birlikte kullanılmıştır. Ek olarak önerilen model, BDT modelinin yapısından dolayı analitik bir formülden ziyade sayısal bir çözüm sunmaktadır. Bu tez, Avrupa tipi opsiyonlarının fiyatlandırılmasına odaklanmaktadır. Ayrıca, faiz oranındaki değişimin etkisini ölçmek için farklı vadelerde ve farklı kullanım fiyatlarındaki opsiyonlar için, CRR modeli ve bizim sunduğumuz model ile değerleme yapılmıştır. Daha sonrasında, her iki model tarafından tahmin edilen fiyatlar, modellerin doğruluğunu belirlemek için gerçek piyasa fiyatlarıyla karşılaştırılmış ve son olarak, hangi vadeler için rassal faiz davranışının önemli olduğu tespit edilmiştir.

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LIST OF SYMBOLS

u	Up movement increase rate parameter
d	Down movement increase rate parameter
α	Reversion speed of interest rate
β	Mean level of interest rate in the long term

LIST OF ACRONYMS/ABBREVIATIONS

BDT	Black - Derman - Toy
BS	Black - Scholes
CIR	Cox - Ingersoll - Ross
CRR	Cox - Ross - Rubinstein
MAPE	Mean Absolute Percentage Error
MSE	Mean Square Error
RMSE	Root Mean Square Error

1. INTRODUCTION

One of the essential factors in the economy is financial markets. The financial market is an environment where actors can buy or sell financial instruments such as bonds and stocks. In other words, it allows people to transfer funds with each other. Additionally, the financial market has a crucial impact on economic growth and efficiency. For instance, in developed countries, the financial market performs adequately. On the other hand, one of the reasons why countries are economically weak is the impaired function in the financial market [1]. Furthermore, the information is combined and communicated; also, liquidity and risk-sharing are provided in the financial markets [2].

Even though there are a host of financial markets, the most common types are stock, bond, commodity, and derivatives markets, as illustrated in Figure 1.1. The stock market enables the sellers and the buyers to trade shares. This trading is realized with a specific price decided on the market [3].

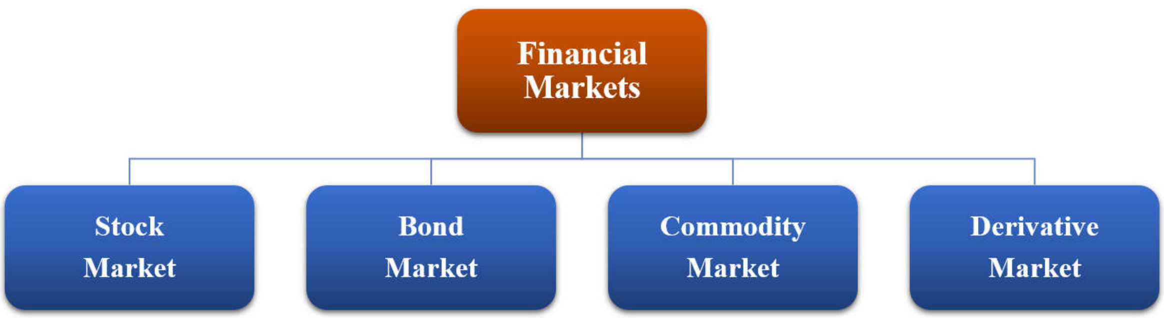

Figure 1.1. Financial Markets Types.

Moreover, the bonds are another financial security: the payment agreement between the holder and the issuer. The issuer makes payments to the holder on particular days. Also, this financial instrument can be issued by economic actors such as governments and corporations. The bonds can be bought, sold, and issued by the actors defined above in the bond markets [4].

In the commodity markets, the buyers and sellers trade commodities such as sugar, oil, and cotton. Also, the major commodities in the financial environment are mainly grouped as precious metals, energy, agricultural instrument, and other metals. Gold and crude oil are the most famous ones in the markets [5].

Besides, the financial derivative is an instrument that takes its value or price from the underlying assets such as index, commodities, and bonds. The prices of the derivatives vary or change in accordance with their underlying assets [6]. Also, the derivatives can be grouped as follows:

(i) Forwards and Futures,
(ii) Swaps,
(iii) Options.

First of all, the forwards and futures enable actors to agree to exchange an asset at a particular time with a specific price. On the other hand, the difference between these two securities is that futures are bought and sold on exchange mainly; however, forward contract agreement can be settled by parties [7].

The swap is another financial derivative, an agreement to exchange payment or cash flow between two parties. In the agreement, the payment date of cash flow and the factors in the calculation are specified. To illustrate, interest rate and exchange rate in the future have an impact on cash flow [8].

In addition, the options, which are crucial financial derivatives, are widely used in financial markets, especially for hedging operations. They give investors the right to buy or sell depending on the option's type. Firstly, the call options provide the right to buy the underlying asset on a specific date for a certain price determined previously. On the other hand, put options give holders the right to sell the underlying asset. This particular price is called strike price in terminology. Also, the exchange date that is defined in the contract is known as maturity date [9].

Furthermore, the options are divided mainly into two groups according to whether they are American or European type. The holders who have American options can exercise the options at any moment until the maturity date. On the other hand, the expiration date is the only day to exercise the European option for its holders. Also, options can be defined by their underlying assets. The bonds, futures, index, stocks, commodity, and swap are an example of underlying assets for options [9]. Moreover, option pricing is an important topic in financial mathematics. Many pieces of research have been conducted so far. However, the most famous studies were realized by Black&Scholes [10] and Cox-Ross-Rubinstein [11]. Both of these studies assume that the interest rate is constant throughout the option's life. Their studies are explained in the following chapter in detail.

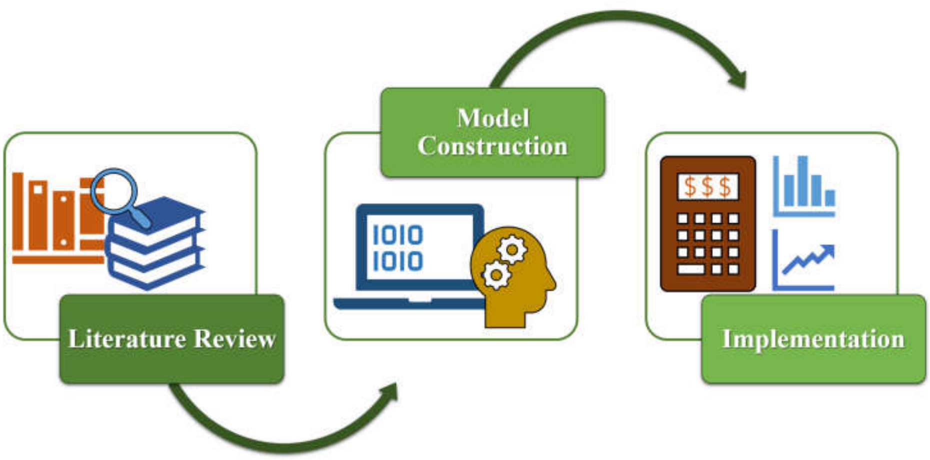

Figure 1.2. Thesis Methodology Chart.

In Figure 1.2, the research methodology of the thesis is represented. In chapter 2, literature related to the topics of option pricing and the interest rate models are reviewed. Also, the background is created before the model description. After that, in chapter 3, a new alternative model for price European type options under a stochastic interest rate environment is developed. This model enables us to take into consideration of interest rate behaviors in option pricing. Then, in chapter 4, the new model is used in the implementation with the real-market data. Also, the results are compared with the CRR model based on actual market prices. The accuracies of both models are examined for different strike prices and particular maturities. Finally, the conclusion of the pricing with the newly generated model exists in chapter 5.

2. RELATED LITERATURE

In this section of the thesis, some important models for both option pricing and interest rates are introduced as background in our model. Firstly, option pricing approaches under constant interest rate situations are described. After that, the models for the future evolution of interest rates are explained. Finally, kinds of literature about option pricing under stochastic interest rates are demonstrated.

2.1. Option Pricing Approaches Under Constant Interest Rate

First of all, Black and Scholes derived a closed-form equation for the prices of European-type options. The followings are assumed while deriving the formula in their paper. Initially, the interest rate is constant in the life of options. Also, there are no dividend and transaction costs. Additionally, the holder can exercise the option at maturity; in other words, the option is European type [10].

Table 2.1. Black and Scholes formula notations.

S	The stock price at time zero
K	Strike price of the option at maturity
r	Interest Rate
σ	The volatility of the stock price
T	Time to maturity
C	Call option price
P	Put option price

The equations

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2), \tag{2.1}$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1), \tag{2.2}$$

show that the Black and Scholes formula for the prices of call and put options, where

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \tag{2.3}$$

$$d_2 = \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{2.4}$$

Here, the expected stock price at maturity can be shown by $S_0N(d_1)e^{rT}$ and the term $N(d_2)$ represents the exercise probability of a call option [9].

Besides, this standard Black and Scholes formula was extended by Garman and Kohlagen [12] to develop a pricing model for foreign currency options since foreign currency options regard more than one interest rate. This phenomenon distinguishes foreign currency options from the Black and Scholes model. Also, in Garman and Kohlagen model, foreign and domestic interest rates are assumed constant like in the Black Scholes model. Garman and Kohlagen show the equation

$$C = e^{-r_F T} SN(d + \sigma\sqrt{T}) - e^{-r_D T} KN(d), \tag{2.5}$$

where

$$d = \frac{ln(\frac{S}{K}) + (r_D - r_F - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{2.6}$$

In the formulas above, the terms r_F and r_D show the foreign and domestic interest rate, respectively.

Furthermore, Cox et al. [11] introduced an option pricing model for discrete-time span using a binomial tree. In their model, option prices in each node are calculated by considering risk-neutral probability to satisfy the arbitrage-free condition. Also, delta hedging and replicating portfolio techniques can be handled. Moreover, they assumed that interest rates are constant and transaction costs and taxes do not exist in the model . Additionally, the stock price (S), the strike price (K), interest rate (r), and change rate of stocks (u, d) between consecutive periods are required to build their model. The framework of the model is explained below.

Let S, K, and r be the stock price, the strike price at the maturity, and the interest rate, respectively. Also, $(u - 1)$ and $(d - 1)$ show the possible rate of stock return in a period. Therefore, the stock price will be uS or dS at the end of 1^{st} period with the probabilities p and $(1-p)$. The Figure 2.1 illustrates that.

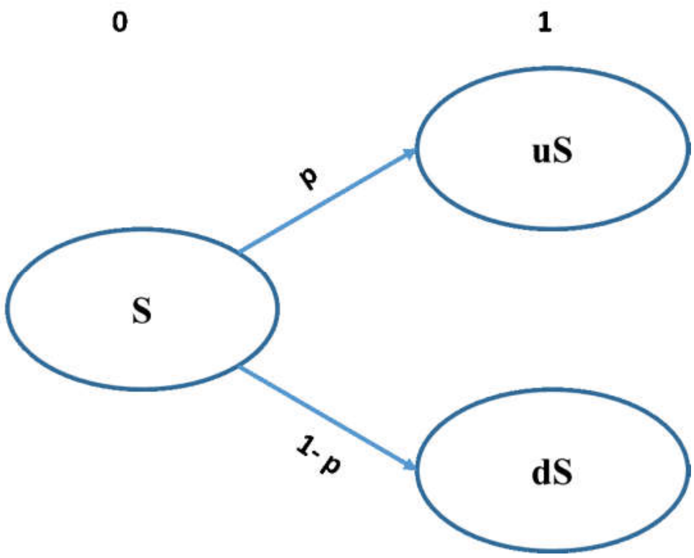

Figure 2.1. One Period Stock Binomial Tree.

The call option's value can take two possible values C_u and C_d, if the option is exercised at the end of 1^{st} period. Then, the value of C_d and C_u are equal to $max[0, dS–K]$ and $max[0, uS–K]$, respectively.

Moreover, option price at time 0 is found by replicating portfolio technique to satisfy the arbitrage-free condition. Let assume that B is the dollar amount in the portfolio at the beginning of time 0. If Δ shares of stock (S) are bought, the portfolio value equals $\Delta S+(B-\Delta S)$ at time 0. Here, $(B-\Delta S)$ can be either negative or positive. Its sign shows the borrowing or lending with the interest rate r. After that, the portfolio value at time end of period will be $\Delta uS+(B-\Delta S)(1+r)$ or $\Delta dS+(B-\Delta S)(1+r)$. By replicating derivatives, the value of Δ and B can be found in equations

$$\Delta uS + (B - \Delta S)(1 + r) = C_u, \tag{2.7}$$

$$\Delta dS + (B - \Delta S)(1 + r) = C_d. \tag{2.8}$$

Equations (2.7) and (2.8) can be written respectively as

$$B + \Delta(\frac{uS}{1+r} - S) = \frac{C_u}{1+r}, \quad (2.9)$$

$$B + \Delta(\frac{dS}{1+r} - S) = \frac{C_d}{1+r}. \quad (2.10)$$

As mentioned above, the probabilities of stock movements are p and $(1-p)$. So, multiplying the Equations (2.9) and (2.10) by these probabilities and summing them gives equation

$$B + \Delta(\frac{1}{1+r}[puS + (1-p)dS] - S) = \frac{1}{1+r}[pC_u + (1-p)C_d]. \quad (2.11)$$

Also, if the probability p is chosen so that $\frac{1}{(1+r)}[puS + (1-p)dS] - S = 0$, the equation above can be written as

$$B = \frac{1}{1+r}[pC_u + (1-p)C_d]. \quad (2.12)$$

Additionally, the value of probability can be easily found as $p = \frac{(1+r-d)}{(u-d)}$. Also, by solving Equations (2.9) and (2.10), the value of Δ can be obtained as

$$\Delta = \frac{C_u - C_d}{uS - dS}. \quad (2.13)$$

From the Equation (2.12), the option value at time 0 equals $C = \frac{1}{1+r}[pC_u + (1-p)C_d]$ after replicating the derivative; moreover, the arbitrage-free condition is satisfied with this method [13]. Moreover, this replicating portfolio technique can be extended for two periods case if the life of the option is two periods. In Figure 2.2, option prices for two periods case are shown. The possible prices at the end of 2^{nd} period are $C_{dd} = max[0, d^2S{-}K]$, $C_{du} = max[0, duS{-}K]$ and $C_{uu} = max[0, u^2S{-}K]$. By applying the previous methodology, C_u and C_d can be found as $C_u = \frac{pC_{uu}+(1-p)C_{ud}}{(1+r)}$ and $C_d = \frac{pC_{ud}+(1-p)C_{dd}}{(1+r)}$. Finally, the option price at time 0 equals $C = \frac{pC_u+(1-p)C_d}{(1+r)}$ according to the Cox-Ross-Rubinstein model. [11].

2.2. Interest Rate Models

Interest rate models represent the evolution of interest rates in the future. These models are mainly used to price interest rate derivatives in finance. Similar to option pricing models, there are continuous and discrete-time spans models. These are described in this section.

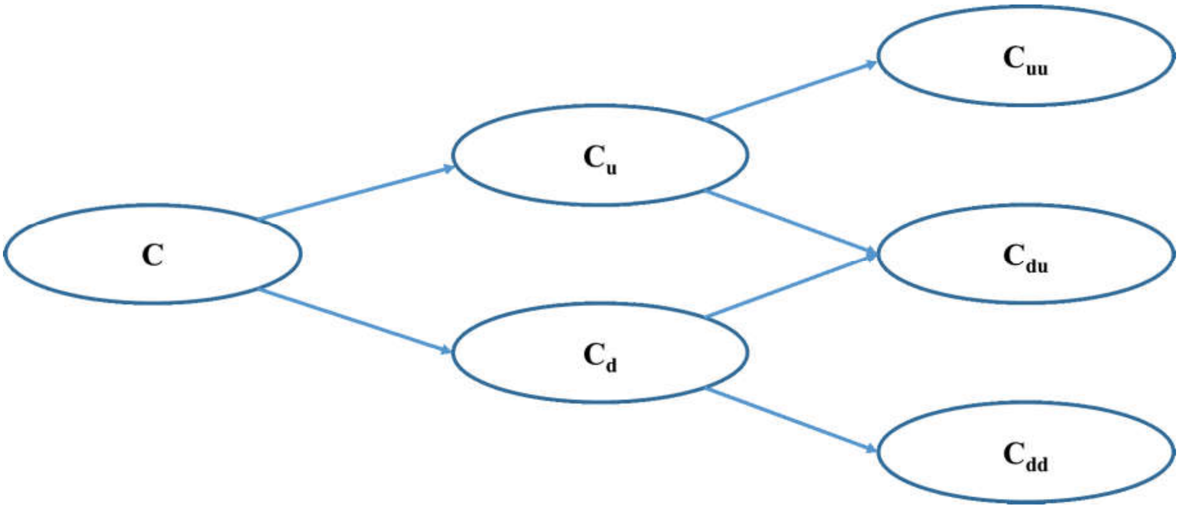

Figure 2.2. Two Periods CRR Binomial Tree.

2.2.1. Vasicek Interest Rate Model

The Vasicek model [14] assumes that change in the interest rate follows a stochastic differential equation, specifically Ornstein–Uhlenbeck process or mean reversion. The stochastic differential equation

$$dr_t = \alpha(\beta - r_t)d_t + \sigma dW_t \tag{2.14}$$

is served as the basis of the model. The drift factor in the equation $\alpha(\beta - r_t)d_t$ represents the expected change in the rate. Also, the term $(\beta - r_t)$ shows the mean-reverting property of the model. The terms in the drift factor part of the equation are described as follows:

- β is the mean level of interest in the long term. According to the Vasicek model, the interest rate in the long term converges to this level.
- α is the velocity in the mean-reversion process. Also, the term a is a non-negative constant in the model.
- r_t represents the instantaneous interest rate at time t.

To illustrate, if the instantaneous interest rate is less than the long-term mean level (β), the drift part of the equation becomes positive. Then, the interest rate will rise to a long-term mean level with the speed of (α). On the other hand, the drift part becomes negative if the current interest is greater than the long-term mean level. In that case, the interest rate will decrease to long-term man level with the speed of (α). In addition to the drift factor, the stochastic part of the equation is represented by σdW_t. The Wiener process dW_t represents the market risk with volatility σ.

Finally, the closed-form property and constant terms are the main advantages of the Vasicek Interest Rate model. Thanks to these characteristics, the implementation of the model can be simple. In the model, nonetheless, it is assumed that the interest rate is normally distributed, and this situation allows for a negative interest which is a significant drawback of the model.

2.2.2. Cox–Ingersoll–Ross Interest Rate Model

Another critical interest rate model is the Cox-Ingersoll-Ross model [15]. The model is the improved version of the Vasicek model. The main drawback of the former is the negative interest rate possible, as mentioned before. However, the Cox-Ingersoll-Ross interest rate model prevents the potential negative interest rate by adjusting the stochastic part of the equation. The stochastic differential equation

$$dr_t = \alpha(\beta - r_t)d_t + \sigma\sqrt{r_t}dW_t \tag{2.15}$$

constitutes the model. The drift terms of the equation are the same as the drift part of the Vasicek Model. Notations α, β, and r_t represent the speed of reversion, the long-term mean interest rate level, and instantaneous interest rate at time t, respectively.

The difference between these two models stems from the stochastic part of equations. Like in the Vasicek model, there are volatility term σ and the Wiener process dW_t in the Cox-Ingersoll-Ross model. However, the term $\sqrt{r_t}$ is added to the equation in the Vasicek interest rate model, additionally. This adjustment solves the problem of negative interest rate possibility in the Vasicek model. In addition, the drift factor of the equation dominates the change in interest rate when the rate is very close to zero since the stochastic part of the equation becomes very close to zero as well.

To sum up, the Cox-Ingersoll-Ross model both prevents the negative interest rate and enables mean-reversion in the long term like Vasicek Model. Also, the instantaneous interest rate at time t has an impact on the incremental variance.

2.2.3. Ho-Lee Interest Rate Model

In the paper of Ho-Lee [16], the interest rate model is created to price fixed-income derivatives such as bond pricing mainly. In the model, the arbitrage-free condition is taken into consideration while developing the model for pricing securities . Although binomial lattice is used in the paper while modeling interest rates movements, to observe characteristic features of the model, the change in the interest rate can be written as

$$dr_t = \theta_t d_t + \sigma dW_t, \tag{2.16}$$

where parameter θ_t is a function of time (t), and it shows the expected movements of interest rate. In other words, the parameter represents interest rate movement in the binomial lattice, and the amount of the change of this parameter in every step is calibrated from the market prices. In the Figure 2.3, the Ho-Lee model for interest rates is shown [17].

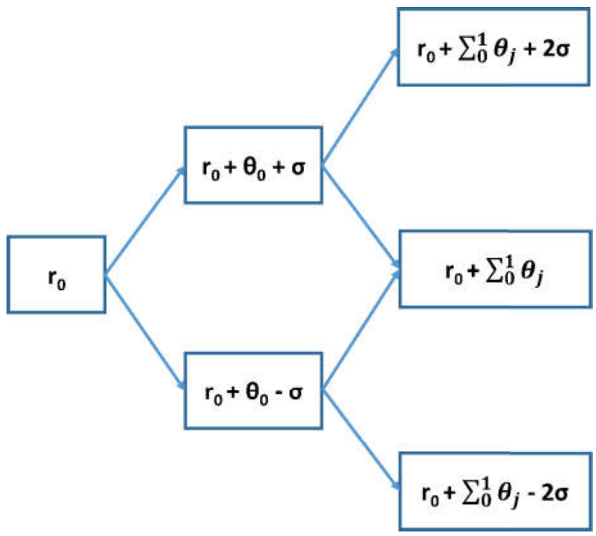

Figure 2.3. Ho-Lee Model Interest Rate Tree.

Moreover, unlike to Vasicek and Cox-Ingersoll-Ross model, mean-reversion is not incorporated in the Ho-Lee model. Also, there is no adjustment in order to remove the possibility of negative interest rates in the model like in the Vasicek model.

2.2.4. Hull-White Interest Rate Model

Hull-White [18] proposed another model by extending Vasicek and Cox-Ingersoll-Ross models to value interest rate derivatives. The motivation behind the extension is to reflect time dependency. For this reason, the term $\theta(t)$, which is a time-dependent function, is added to the drift part of stochastic differential equation. The difference between the original and extended versions is described in the equations below. Vasicek model equation is shown as

$$dr_t = \alpha(\beta - r_t)d_t + \sigma dW_t \tag{2.17}$$

and extented Vasicek model is

$$dr_t = [\theta(t) + \alpha(t)(\beta - r_t)]d_t + \sigma(t)dW_t \tag{2.18}$$

that can also be written as

$$dr_t = \alpha(t)[\frac{\theta(t)}{\alpha(t)} + (\beta - r_t)]d_t + \sigma(t)dW_t. \tag{2.19}$$

Mean reverting level in the extended Vasicek model, which Hull-White develops, is a function of time (t), as seen in Equation (2.19). While long term mean level in the Vasicek Model is constant β, in the extended version, it becomes $[\frac{\theta(t)}{\alpha(t)} + (\beta - r_t)]$.

Moreover, for the Cox-Ingersoll-Ross model, the same extension was applied. As mentioned before, the drift part of the CIR model is the same as the factor in Vasicek; however, the term $\sqrt{r}$, which resolves negative interest rates, exists in the stochastic part of the CIR model. The differential equation in the CIR model is represented as

$$dr_t = \alpha(\beta - r_t)d_t + \sigma\sqrt{r}dW_t \tag{2.20}$$

also, extented CIR model is

$$dr_t = [\theta(t) + \alpha(t)(\beta - r_t)]d_t + \sqrt{r}\sigma(t)dW_t. \tag{2.21}$$

Then, the Equation (2.21) can be written as

$$dr_t = \alpha(t)[\frac{\theta(t)}{\alpha(t)} + (\beta - r_t)]d_t + \sqrt{r}\sigma(t)dW_t. \tag{2.22}$$

Like in the extended Vasicek model, long term mean level in the extended version of CIR model converted to $[\frac{\theta(t)}{\alpha(t)} + (\beta - r_t)]$ [18].

2.2.5. Black-Derman-Toy Interest Rate Model

The Black-Derman-Toy model [19] was developed to price financial securities which have interest rate dependency, such as bonds. The model's inputs are mainly zero-coupon rates on bonds rate and volatilities of these same bonds. The principal methodology of the model is to match future rates with inputs introduced above. Also, long term mean-reverting level is not constant in this model due to the changes in the volatility of bonds with different maturities. Additionally, the vital point in the model which makes the model popular is the assumption that interest rates have log-normal distribution at any period. Thanks to this property of the model, the negative interest rate problem is eliminated. Although the Ho-Lee model uses the binomial tree for the interest rate model as well, the difference between these two models is that the Black-Derman-Toy model has log-normal property; but, Ho-Lee does not.

Besides, the model in the original paper is constructed via binomial lattice by using several bond rates and volatilities. While generating the tree, the arbitrage-free condition must be taken into consideration. For this reason, the risk-neutral probability should be utilized while calibrating the model. In their study, the risk-neutral probability is assumed as 0.50 for up and down market conditions. In Figure 2.4, a binomial tree with three periods is shown for the interest rates. At time 0, the interest rate equals r_0, which is the same as the return rate of zero-coupon rate with 1-year maturity. For the rates in the following periods, calibration with zero-coupon rates and volatilities should be implemented. The model uses log-normal property while defining the relation or ratio between possible interest rates in the same period. The relation between up (u) and down (d) market condition in the normal binomial tree can be shown like in the equations

$$u = e^{\sigma\sqrt{\Delta T}}, \tag{2.23}$$

$$d = e^{-\sigma\sqrt{\Delta T}}. \tag{2.24}$$

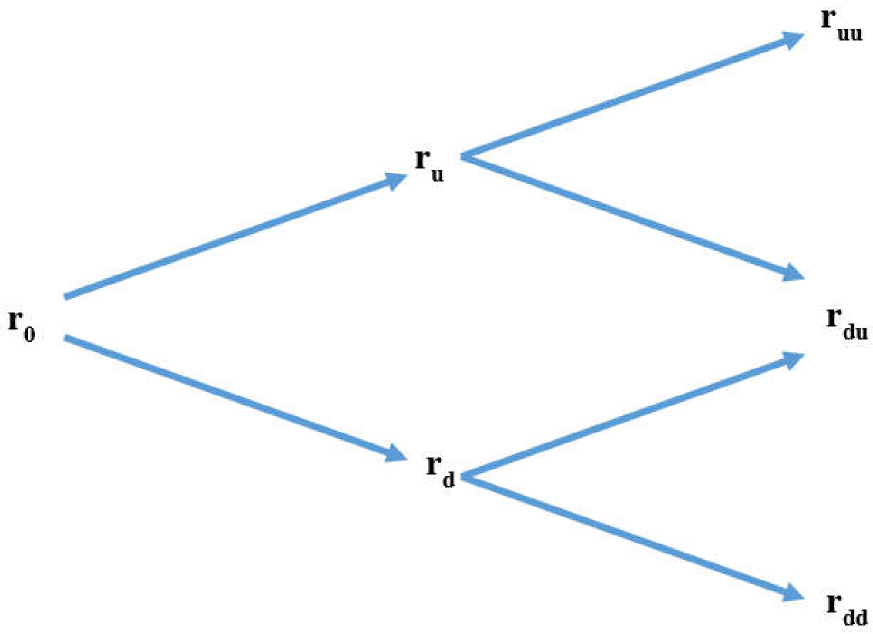

Figure 2.4. Two Periods Interest Rate Tree.

From the Equations above, the relation between conditions can be written as

$$\frac{u}{d} = e^{2\sigma\sqrt{\Delta T}}, \tag{2.25}$$

$$log(\frac{u}{d}) = 2\sigma\sqrt{\Delta T}. \tag{2.26}$$

Also, the Equation (2.26) will become $log(\frac{u}{d}) = 2\sigma$ if ΔT is taken 1. Owing to the log-normal property of the Black-Derman-Toy model, relation between two possible interest rates in the same period can be written as

$$log(\frac{r_u}{r_d}) = 2\sigma. \tag{2.27}$$

Hence, the ratio between these possible interest rates in the 1^{st} period is equal to $r_u = r_d e^{2\sigma}$. Also, this same methodology can be employed for 2^{nd} period. Then, the binomial tree is converted to Figure 2.5.

Even though the BDT model was introduced using a binomial tree in the original paper, the stochastic differential equation

$$dln(r) = [\theta_t + \frac{\sigma(t)'}{\sigma(t)} ln(r)]d_t + \sigma(t)dW_t \tag{2.28}$$

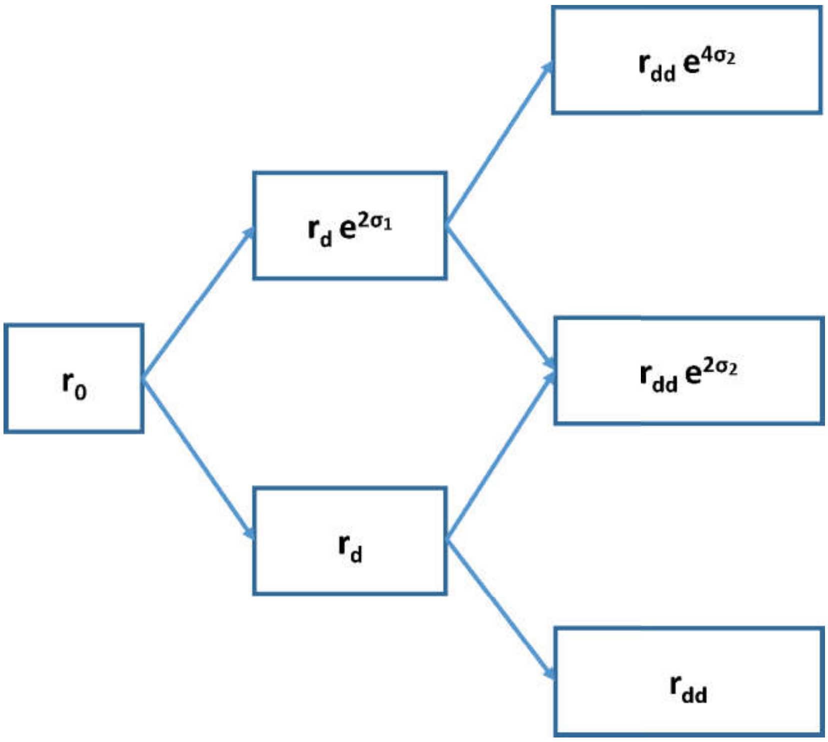

Figure 2.5. BDT Model Interest Rate Tree.

is written to observe the characteristics feature of the BDT model. Also, if the volatility is constant over time, the formula can be converted as

$$dln(r) = \theta_t d_t + \sigma(t) dW_t. \tag{2.29}$$

Moreover, according to Gerald et al. [20], the decrease in the volatility of term structure for local point, mean reversion property can be shown $\sigma(t)' < 0$. On the other hand, an increase in the volatility does not cause to show mean reversion .

Nonetheless, the model has no closed-form solution to price bonds [21]. Also, this situation was remarked by Magnou [22], who argued that closed-form with analytical approach could not be implemented because of log-normal property; hence, a numerical method like lattice is needed.

2.2.6. Black-Karasinski Interest Rate Model

Black-Karasinski [23] developed an interest rate model. They assumed that interest rates have log-normal distribution; hence, they cannot be negative in the long or short term. Moreover, the mean-reverting level changes over time in addition to local volatility. Thus, volatility and mean reversion are function of time in the model. Also, the stochastic differential equation of the model can be written as

$$dln(r) = [\theta_t - \alpha_t lnr_t]dt + \sigma_t dW_t, \tag{2.30}$$

where

- r_t shows the local interest rate,
- θ_t is the mean-reverting level, but it is a function of time,
- α_t is the adjusting speed while tending the long-term mean level,
- σ_t shows the local volatility.

The model enables to match cap, volatility curve, as well as the yield curve . However, an analytical solution for the interest rate options cannot be handled owing to log-normal property. For this reason, a numerical approach should be implemented like in the Black-Derman-Toy model [24].

2.3. Option Pricing Approaches Under Stochastic Interest Rate

In this part, the studies on option pricing under a stochastic interest rate environment are mentioned chronologically. Firstly, Bailey and Stulz [25] examined the biases which stem from using the Black-Scholes formula that assumes volatility and interest rate are constant while pricing options. They handled stock index options for this empirical study. Also, the investigation for stock index options pricing was conducted under the general equilibrium model in which interest rate and index volatility have a negative relationship. For this reason, it is indicated that the impact of stochastic volatility of the index on option pricing cannot be observed unless the interest rate is changed stochastically. Finally, in their study, they found that option prices were higher than the prices calculated with the Black-Scholes model for the deep-in-the-money cases.

Rabinovitch [26] derived a closed-form solution for call options on bonds and stocks. He also used Vasicek's interest model to embed it in the Black-Scholes formula. Also, the correlation between the underlying asset and interest rate exists in his approach. The difference between Rabinovitch's and Black-Scholes' formula results is remarkable if the correlation parameter and interest rate variance are enormous. Nonetheless, similar results are obtained from these two models when the correlation and variance are not relatively sizeable.

Besides, Amin and Jarrow [27] extended the model's scope, which was developed by Heath et al. [28] by adding other risky assets. Heath-Jarrow-Morton model was generated to price interest rate options. Also, in addition to European type call option pricing, they provided a methodology of forward and future contracts valuations.

Additionally, Goldstein and Zapatero [29] derived a closed-form solution to price a call option. Their model assumes that an exchange economy contains risk-free security such as bonds and stocks. Additionally, Vasicek's interest rate model was used to reflect the stochastic behavior of interest rates. Moreover, the security stock gives the owner a chance to get income from two sources. The first one is a dividend, and the other source is an increase in the stock. They described these incomes as a single factor which is the gain process. Also, the mean-reverting process is valid for dividend yield and gain factor in the model. Their model especially is an advanced version of Rabinovitch's model. In his model, dividend payment was not taken into consideration.

Moreover, Miltersen and Schwartz [30] developed a model for option pricing on the commodity futures. While generating the model, they assumed that not only interest rates change stochastically, but also convenience yield has stochastic behavior. Also, the Black-Scholes-Merton equation was generalized in their paper.

Kim and Kunitomo [31] derived a closed-form solution for option pricing in the economy in which interest rate shows stochastic behavior. They extended the Black-Scholes formula to asymptotic expansion under the assumption that interest rate volatility is small. Instead of the Gaussian process for interest rate, they relaxed that approach, and the Cox-Ingersoll-Ross interest rate model was used in their study in order to extend the Black-Scholes model.

Furthermore, Van Haastrecht and Pelsser [32] introduced the effects of other structures on option pricing. In addition to stochastic interest rate occasion, they consider the correlation, inflation, foreign and domestic rates. They emphasized the importance of these parameters on option pricing in a complex manner. Moreover, a pricing formula was derived for the securities such as forward options, swaps, and vanilla options .

Wilhelm [33] derived a European-type option pricing formula by incorporating the Black-Scholes model and stochastic interest rate structures with a Gaussian distribution. Specifically, a kind of Ho-Lee interest rate model was used in his formula. Also, he noticed that stock price movement and interest rates are dependent on each other.

Additionally, Fang [34] developed a model for pricing European type options under stochastic interest rate conditions. Like other researchers, he used Vasicek's interest rate model in his model. Also, the martingale property was implemented in the model.

Finally, Abudy and Izhakian [35] derived a new closed-form solution on the option pricing of stocks. They used Vasicek's interest rate model and the Black-Scholes option pricing model. Also, the correlation between interest rates and stocks was embedded in the formula. After that, it is found that their model gives more accurate results than the Black-Scholes model gives for the case of around-the-money, especially .

On the other hand, De Simone [36] proposed another approach for option pricing under stochastic interest rate condition in his doctoral thesis. In that thesis, joint dynamics of stock prices and interest rates are illustrated with binomial lattice. However, while calculating the stock option in a node, the interest rate that belongs to the wrong period is used to calculate the discount rate on page 35. Also, on page 39, replicating portfolio is miswritten since while constituting the replicating portfolio equations, constant Δ and constant amount of zero-coupon bond should be used to satisfy the arbitrage-free condition. However, these values are not constant in all four equations in his thesis written for the same period. For this reason, replicating portfolio techniques is formed erroneously. Consequently, this approach cannot satisfy the arbitrage-free situation .

What we contribute to option pricing is that how European options can be priced at different maturities in a stochastic interest rate environment. The developed model is based on the CRR, and the BDT models since having lognormal properties and working on binomial trees are attractive features of using these two models. In addition, satisfying the arbitrage-free condition is another significant factor for these models.

3. THE MODEL OF OPTION PRICING WİTH RANDOM INTEREST RATE AND A NUMERICAL EXAMPLE

In this thesis, the binomial tree is used while generating a model that handles the effect of stochastic behavior of interest rate on option pricing. These models are utilized for two separated objectives: option pricing and determination of interest rates. Previous studies of Cox-Ross-Rubinstein [11], and Black-Derman-Toy [19], that introduced option pricing model and interest rate model respectively, are combined to constitute an option pricing model under stochastic interest rate environment in this work. In addition to being easy to implement, accordance with log-normal distribution and mean-reversion are incentive properties of these models to utilize in our model.

At the beginning of this study, the joint probabilities of stock and interest rate movements explained in CRR, and BDT models are calculated in each step of the binomial tree. Each model has two possible directions in every step of its binomial tree; hence, joint probability calculation gives four possibilities in the next step. However, this approach for creating a binomial tree cannot satisfy arbitrage-free condition with replicating portfolio technique. Even though De Simone [36] tried to use the joint probability for the option pricing model under stochastic interest rate, he created replicating portfolio formulas insufficiently, as mentioned in the previous section.

The joint probability approach gives four possible option prices for the first period in the binomial trees larger than one step. However, in that situation, at least two possible options in the first period should be equal in the replicating portfolio technique. As expected, such equality cannot happen as either stock prices or interest rates between nodes are different when there are two dynamics: stock price and interest rates. To illustrate, possible option prices are shown with $C_{1,1}$, $C_{1,2}$, $C_{1,3}$ and $C_{1,4}$ in Figure 3.1.

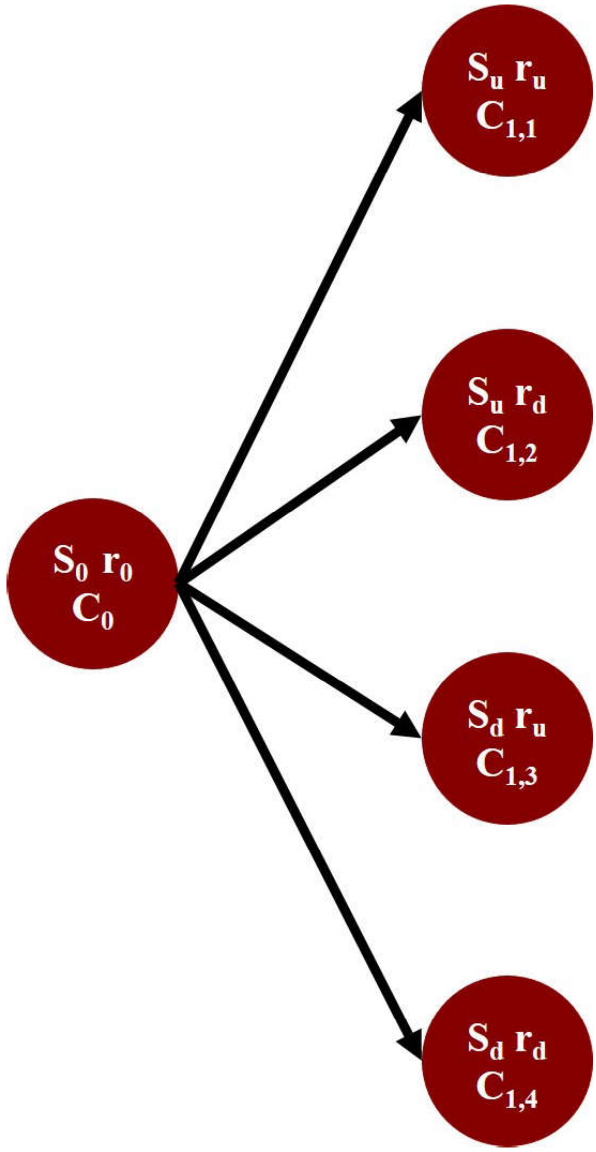

Figure 3.1. Joint Probability - Possible Nodes.

Also, the possible interest rates and stock prices can be seen in these four nodes.

After that, for those for nodes, the replicating portfolio equations are written as

$$C_0 - \Delta S_0 = \frac{C_{1,1}}{1+r_0} - \frac{\Delta S_u}{1+r_0}, \tag{3.1}$$

$$C_0 - \Delta S_0 = \frac{C_{1,2}}{1+r_0} - \frac{\Delta S_u}{1+r_0}, \tag{3.2}$$

$$C_0 - \Delta S_0 = \frac{C_{1,3}}{1+r_0} - \frac{\Delta S_d}{1+r_0}, \tag{3.3}$$

$$C_0 - \Delta S_0 = \frac{C_{1,4}}{1+r_0} - \frac{\Delta S_d}{1+r_0}. \tag{3.4}$$

In the equation system, option prices pairs $C_{1,1}$ - $C_{1,2}$ and $C_{1,3}$ - $C_{1,4}$ should be equal to each other to satisfy all equations. However, it is impossible since the options in pairs are priced with different interest rates, as illustrated in Figure 3.1; hence, the option prices must be different from each other. For this reason, the arbitrage-free condition cannot be satisfied by using the joint probability approach. In our model, instead of joint probability, the same probability space for interest rate and stock movements is used to ensure arbitrage-free condition in pricing options. If there is one common probability space, two possible movements proceeded from each node in the binomial tree, like in the CRR and BDT models. Thus, the same probability space technique enables to create option pricing model under a stochastic interest rate with the arbitrage-free condition.

Nonetheless, two issues have arisen in the model generation. The first one is how risk-neutral probabilities are evaluated. In the BDT model, there is no restriction on the probability of interest rate movements. The BDT model assumes that the up movement in interest rate has a 0.5 probability. Thanks to this flexible characteristic of the BDT model, risk-neutral probabilities can be calculated like in the CRR model by adopting the volatility of a stock.

The second issue is how the direction of stock prices and interest rates movements are determined since neutrally, there are four possibilities for stock and interest rate in each step. On the other hand, using the same probability spaces gives two possible movements. Therefore, a lack of movements for correspondings probability space emerges. This problem is illustrated in Table 3.1 below.

Table 3.1. Model probability table.

Joint Probability			**Same Probability Space**		
CRR/BDT	**Up**	**Down**	**CRR/BDT**	**Up**	**Down**
Up	p * q	p * (1-q)	Up	p	-
Down	(1-p) * q	(1-p) * (1-q)	Down	-	(1-p)

Our model has an assumption to overcome this second issue from an economic perspective. The up and down movements of stocks and interest rates are assumed to be the same in the model. Luintel and Paudyal [37] found that price and stock index in the economy has a positive relationship in the long term since the revenue of companies raise at the same rate of increase in the price of goods and services. Hence, stock investment is still a good way to hedge rising inflation. Also, Boudoukh and Richardson [38] supported the relationship between inflation and return of stocks in a positive way by conducting an empirical study on stock markets in the U.K, and U.S. After that, higher inflation brings about a rise in the interest rate, ultimately [39]. Therefore, the direction of movements is assumed to be the same for the model's stocks and interest rates, as seen in Figure 3.2.

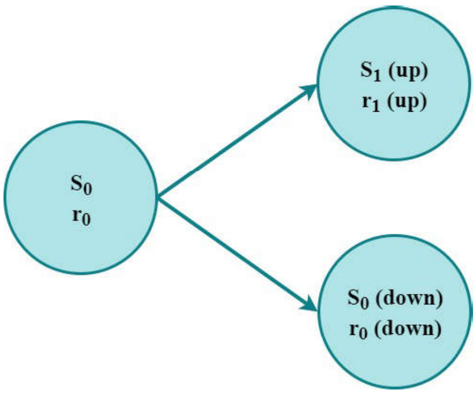

Figure 3.2. One Period Stock and Interest Rate Tree.

Furthermore, both interest rate possibilities in the same step can be an up movement or down movement compared to the previous step. It is not required that there should be at least one up and down movement in the same step. For instance, the interest rate can increase for both up and down movements in a step with a difference hike because the direction and size of the interest rate change mainly stem from bond prices and volatilities. To illustrate, when the zero-coupon rate for a long-term bond is much higher, the interest rate can tend to increase in two possible movements for that step with different hikes.

Although we assume a positive correlation between stock and interest rates, our model can handle the negative correlation perfectly. For instance, interest rates can rise in a period while stock prices are declining. Additionally, there can be an unclear pattern in movements between periods. The binomial tree can include both negative and positive correlations in different steps. In our model, the crucial point is that the same probability space methodology should be used.

Moreover, in our model, stock and interest rate movements affect each other step by step. To illustrate, the interest rate impacts risk-neutral probability with stock price volatility, demonstrated as

$$p = \frac{e^{r\Delta T} - d}{u - d}, u = e^{\sigma\sqrt{\Delta T}}, d = e^{-\sigma\sqrt{\Delta T}}. \tag{3.5}$$

Consequently, CRR and BDT models constitute a cycle, as seen in Figure 3.3. As a result, via this interactive relation in each step, the effect of interest rate on the option pricing can be considered. Additionally, in Table 3.2, notations used in the model are shown. In the following sections, before explaining the generation of the model, CRR and BDT models are briefly described again.

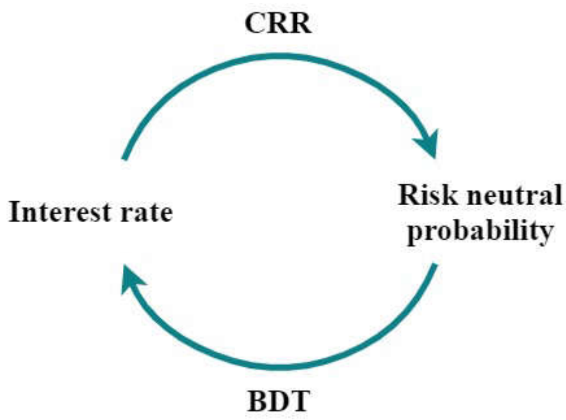

Figure 3.3. Interaction between CRR and BDT Models.

Table 3.2. Model notations.

$\boldsymbol{S_i}$	Stock price at time i
$\boldsymbol{r_{i,j}}$	Interest rate at time i and in movement j
$\boldsymbol{\sigma}$	Stock price volatility
$\boldsymbol{p_{i,j}}$	Risk neutral probability at time i and in movement j
Pi	Price of Zero-Coupon Bond for i years maturity
$\boldsymbol{\sigma_i}$	Volatility of Zero-Coupon Bond for i years maturity
$\boldsymbol{\Delta T}$	Time difference between two consecutive steps
$\boldsymbol{F_i}$	Face value of bond
K	Strike price of option at maturity
$\boldsymbol{C_{i,j}}$	Option price at time i and in movement j

3.1. CRR MODEL

According to CRR model, stock price movements can be shown in a binomial tree. Let S_0 be stock price at time 0. Then, the price S_1 of stock in the 1^{st} period can take two possible values: uS_0 and dS_0 with the probability of p and (1-p), respectively. After that, the price S_2 in the 2^{nd} period will be uuS_0, udS_0, and ddS_0 according to the model. The stock price will be uuS_0 in the 2^{nd} period with the probability of p^2 in the tree. On the other hand, the probability of ddS_0 equals $(1-p)^2$ according to Cox-Ross-Rubinstein Model.

3.2. BDT MODEL

As explained before, Black-Derman-Toy introduced a single-factor interest rate model by developing a binomial tree. In the model, possible interest rates in the future are calibrated from bonds that have different maturities. For instance, let r be the current interest rate; it is also the rate of return for a 1-year zero-coupon bond. Then, the price of the bond now equals $P_1 = \frac{100}{(1+r)}$ if the face value of a bond is assumed \$100. In addition to a 1-year bond, a 2-year bond exists with a different rate of return. If r_2 is the rate of return for that 2-year bond, the current price of this bond equals $P_2 = \frac{100}{(1+r_2)^2}$, explicitly. Also, the volatilities of these bonds are σ_1 and σ_2 like the rate of return. After that, possible interest rates in the next year can be calculated using the information explained above. First of all, a 1-year bond and current interest rate have the same rate of return (r).

Besides, the 2-years bond's price will change in the next year concerning the interest rate at the beginning of next year. The bond price will be recalculated with the interest rate of next year. The change in the price of that bond is shown in Figure 3.4. In the following year, the price of a 2-years bond will be $P_2(up)$ or $P_2(down)$. Additionally, possible interest rates in the next year are r_u and r_d, as seen in Figure 3.5.

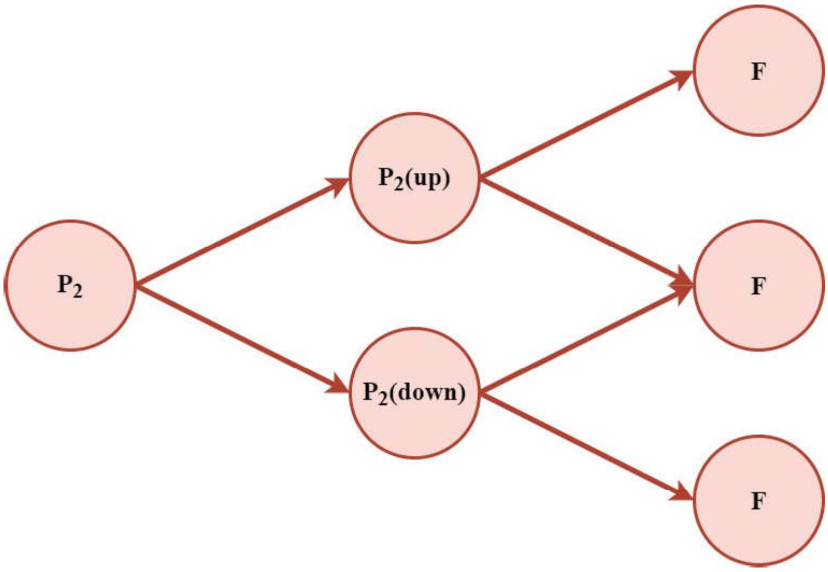

Figure 3.4. Bond Price Tree.

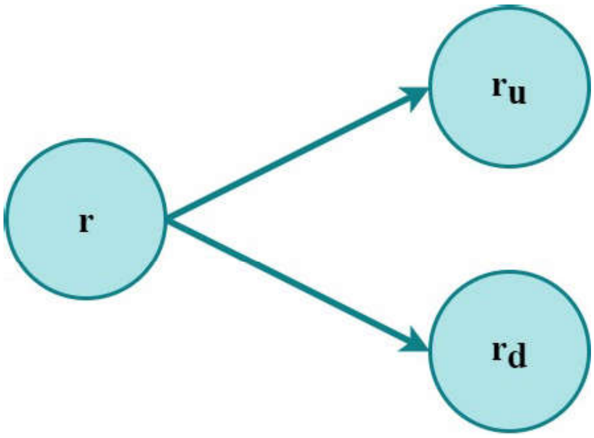

Figure 3.5. Interest Rate Tree.

To find possible interest rates in the next year, the price P_2 should be matched with the current interest rate and possible interest rates in the next year as follows:

$$P_2 = \frac{0.5\frac{100}{1+r_d} + 0.5\frac{100}{1+r_u}}{1+r}. \tag{3.6}$$

In the Equation (3.6), the principal methodology briefly takes the bond to its maturity step by step. Moreover, the formulas $u = e^{\sigma\sqrt{\Delta T}}$ and $d = e^{-\sigma\sqrt{\Delta T}}$ are increase and decrease rate in the binomial tree as mentioned before. Then, volatility can be shown as

$$\sigma = \frac{1}{2\sqrt{\Delta T}} log(\frac{u}{d}). \tag{3.7}$$

Moreover, in the BDT model, interest rates are log-normally distributed, and if ΔT is taken 1-year, the equation can be converted to

$$\sigma = \frac{1}{2} log(\frac{r_u}{r_d}). \tag{3.8}$$

Also, the volatility of the 2-years zero-coupon bond is σ_2. Therefore, the relation between possible interest rates in the next year can be written as

$$e^{2\sigma_2} * r_d = r_u. \tag{3.9}$$

Finally, Equation (3.6) can be easily solved by typing r_u in r_d, and these possible interest rates can be found.

3.3. CRR-BDT MODEL

First of all, stock prices can be calculated before the interest rate in each step since the input of stock price calculation is only stock volatility regardless of interest rate. In Table 3.3, the parameters and risk-neutral probabilities are written to create a one-period binomial tree according to the CRR model, and a stock price can be Su or Sd in the first step, like in Figure 3.6. Also, possible interest rates in the next period can be found by adopting the risk-neutral probability in BDT calibration. Besides, it should be noticed that the rule

$$0 < d < 1 + r < u \tag{3.10}$$

should be assumed to satisfy the arbitrage-free condition in our model, like in the CRR model. In any different case, anyone can take advantage of either borrowing money and buying stock or shorting stock and lending money to increase the wealth.

Table 3.3. Parameters and risk neutral probabilities of the CRR model.

CRR	**Multiplier**	**Risk Neutral Probability**
Up	u	p
Down	d	(1-p)

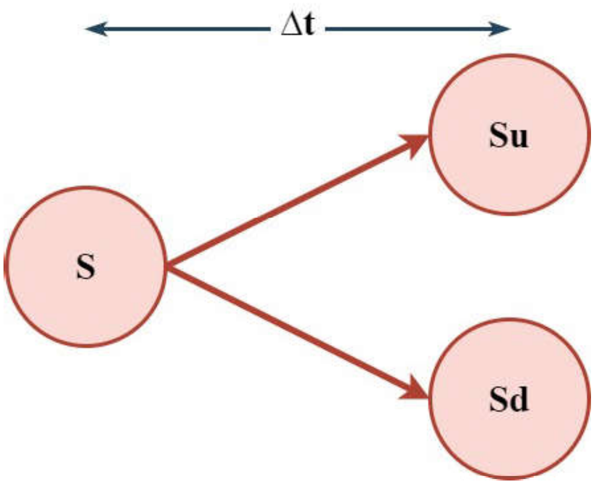

Figure 3.6. Stock Price Tree.

3.3.1. 1^{st} Period Calculations

At the beginning of the tree (Period 0), stock price, interest rate, and stock price volatility are represented by S_0, r_0, and σ, respectively. In order to calculate values in the 1^{st} period, the following implementations should be applied step by step.

<u>3.3.1.1. 1^{st} Period Stock Change Calculations.</u> In the CRR model, the change rates of the stock price for one period are calculated by using stock price volatility like in the followings: $u = e^{\sigma}$ and $d = e^{-\sigma}$. Then, the risk neutral probability between periods 0 and 1 is calculated as $p_{0,0} = \frac{e^{r_{0,0}} - d}{u - d}$. Thus, the stock price in the 1^{st} period (S_1) can be $S_0 u$ with the probability $p_{0,0}$ or $S_0 d$ with the probability $(1 - p_{0,0})$.

<u>3.3.1.2. 1^{st} Period Intrest Rate Calculation.</u> BDT model calibration for 1^{st} period requires risk-neutral probability, zero-coupon bond price (P_2), and zero-coupon bond volatility (σ_2) with years to maturity 2 in order to calculate possible interest rates in 1^{st} period. Rather than assuming risk-neutral probability as 0.50 like in the original Black Derman Toy Model, in our model, risk-neutral probability equals the value found in section 3.3.1.1 calculation part, as explained before. To illustrate, for 1^{st} period, the risk-neutral probability of up movement condition equals $p_{0,0}$. In equation

$$P_2 = \frac{p_{0,0} * \frac{F_2}{1+r_{(1,1)}} + (1 - p_{0,0}) * \frac{F_2}{1+r_{(1,-1)}}}{1 + r_{(0,0)}}, \tag{3.11}$$

except $r_{(1,1)}$ and $r_{(1,-1)}$ which are possible interest rates in the 1^{st} period, all terms are known. Also, in the BDT model, the relation between these two interest rates is $\frac{r_{(1,1)}}{r_{(1,-1)}} = e^{2\sigma_2}$, as mentioned before. Therefore, Equation (3.11) above can be shaped as

$$P_2 = \frac{p_{0,0} * \frac{F_2}{1+e^{2\sigma_2} * r_{(1,-1)}} + (1 - p_{0,0}) * \frac{F_2}{1+r_{(1,-1)}}}{1 + r_{(0,0)}}. \tag{3.12}$$

The adjusted equation can be easily solved, and possible interest rates in the 1^{st} period can be found. After calculating stock prices and interest rates in the 1^{st} period, a binomial tree can be generated for one step, as illustrated in Figure 3.7. By using the values found in the 1^{st} period, such as interest rates and stock prices, the same methodology can be applied for 2^{nd} period calculations. Now, calculated interest rates affect the risk-neutral probabilities, as mentioned before. However, since the BDT model has no closed-form solution, a numerical approach should be used instead of analytical formula. For this reason, the methodology should be used implemented gradually with the numerical method.

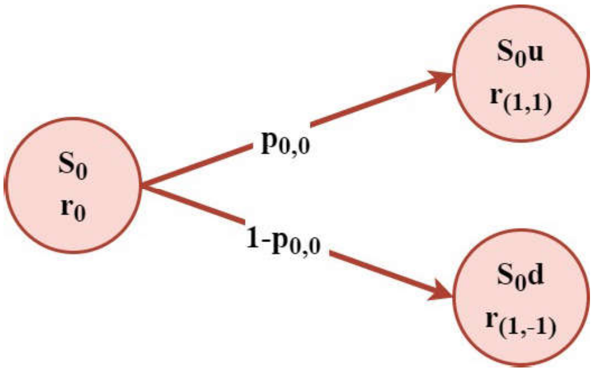

Figure 3.7. 1^{st} Period - Stock and Interest Rate Tree.

3.3.2. 2^{nd} Period Calculations

After calculating values in the 1^{st} period, stock prices and interest rates in step 2 of the tree can be found with the same methodology.

3.3.2.1. 2^{nd} Period Stock Change Calculation. Change rates of stock are the same as rates in the 1^{st} period calculation owing to the constant volatility of the stock. However, there should be two risk neutral probabilities between 1^{st} and 2^{nd} periods since there are two possible interest rates for the 1^{st} period. The first one is $p_{1,1}$ which represents the 1^{st} period and up movement condition in this period. Another probability is $p_{1,-1}$ and it shows the down movement condition in the same period. Therefore, the risk neutral probabilities between 1^{st} and 2^{nd} period are evaluated as $p_{1,1} = \frac{e^{r(1,1)}-d}{u-d}$ and $p_{1,-1} = \frac{e^{r(1,-1)}-d}{u-d}$.

After calculating stock price and risk-neutral probabilities, a binomial tree of stock prices for the 2^{nd} period is generated like in Figure 3.8.

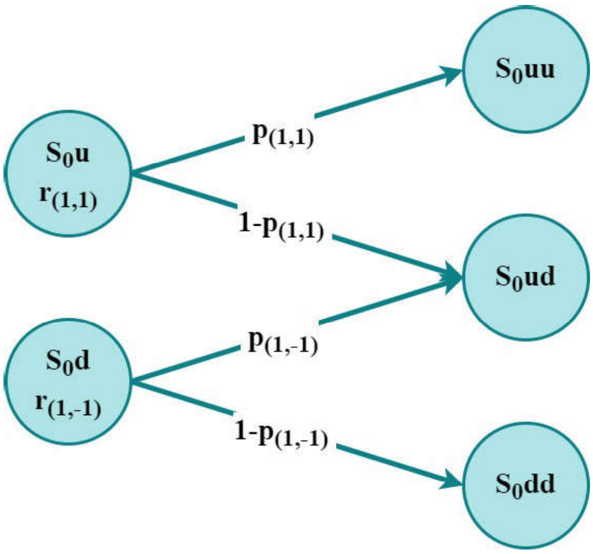

Figure 3.8. 2^{nd} Period - Stock and Interest Rate Tree.

3.3.2.2. 2^{nd} Period Interest Rate Calculation. The possible interest rates in the 2^{nd} period according to the BDT model are calculated by solving the equation

$$P_3 = \frac{p_{(0,0)} * \frac{p_{(1,1)} * \frac{F_3}{1+r_{(2,1)}} + (1-p_{(1,1)}) * \frac{F_3}{1+r_{(2,0)}}}{1+r_{(1,1)}} + (1-p_{(0,0)}) * \frac{p_{(1,-1)} * \frac{F_3}{1+r_{(2,0)}} + (1-p_{(1,-1)}) * \frac{F_3}{1+r_{(2,-1)}}}{1+r_{(1,-1)}}}{1+r_{(0,0)}} \tag{3.13}$$

where, except for $r_{2,1}$, $r_{2,0}$ and $r_{2,-1}$ all terms are known. Like in 1^{st} period, there is a relation between these possible interest rates as

$$\frac{r_{(2,0)}}{r_{(2,-1)}} = \frac{r_{(2,1)}}{r_{(2,0)}} = e^{2\sigma_3}. \tag{3.14}$$

Therefore, $r_{(2,1)}$ and $r_{(2,0)}$ can be written as $r_{(2,-1)}e^{4\sigma_3}$ and $r_{(2,-1)}e^{2\sigma_3}$, respectively. After replacing the terms, only one unknown term is left. Finally, the equation can be solved via this simplification. In Figure 3.9, the 2-periods binomial tree is illustrated by calculating stock price and interest rates.

3.3.3. Long Term Periods

The formula for interest rate expands upwards in the following steps due to the BDT model structure; however, the methodology is the same in all steps. In brief,

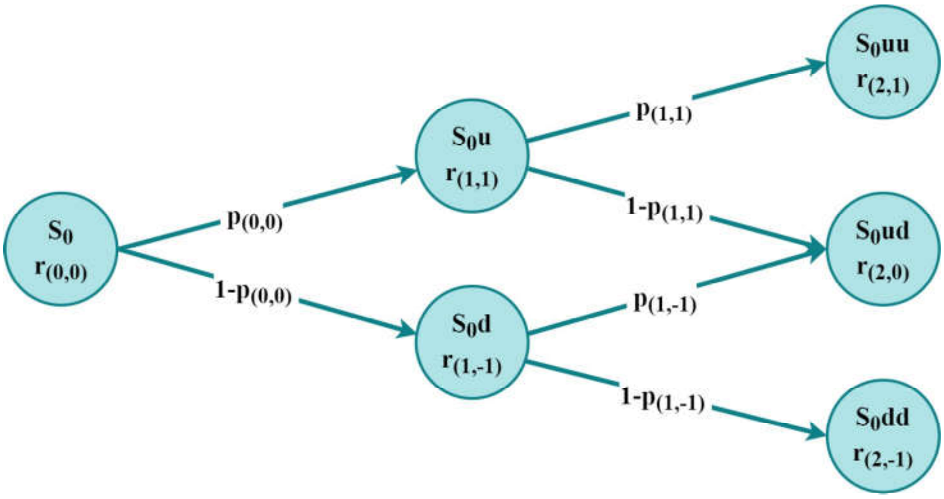

Figure 3.9. Two Periods Binomial Tree.

interest rate and risk-neutral probability affect each other in a sequence that creates a continuous and iterative cycle. In Figure 3.10, stock prices, risk-neutral possibilities, and possible interest rates are shown with a binomial tree for four periods.

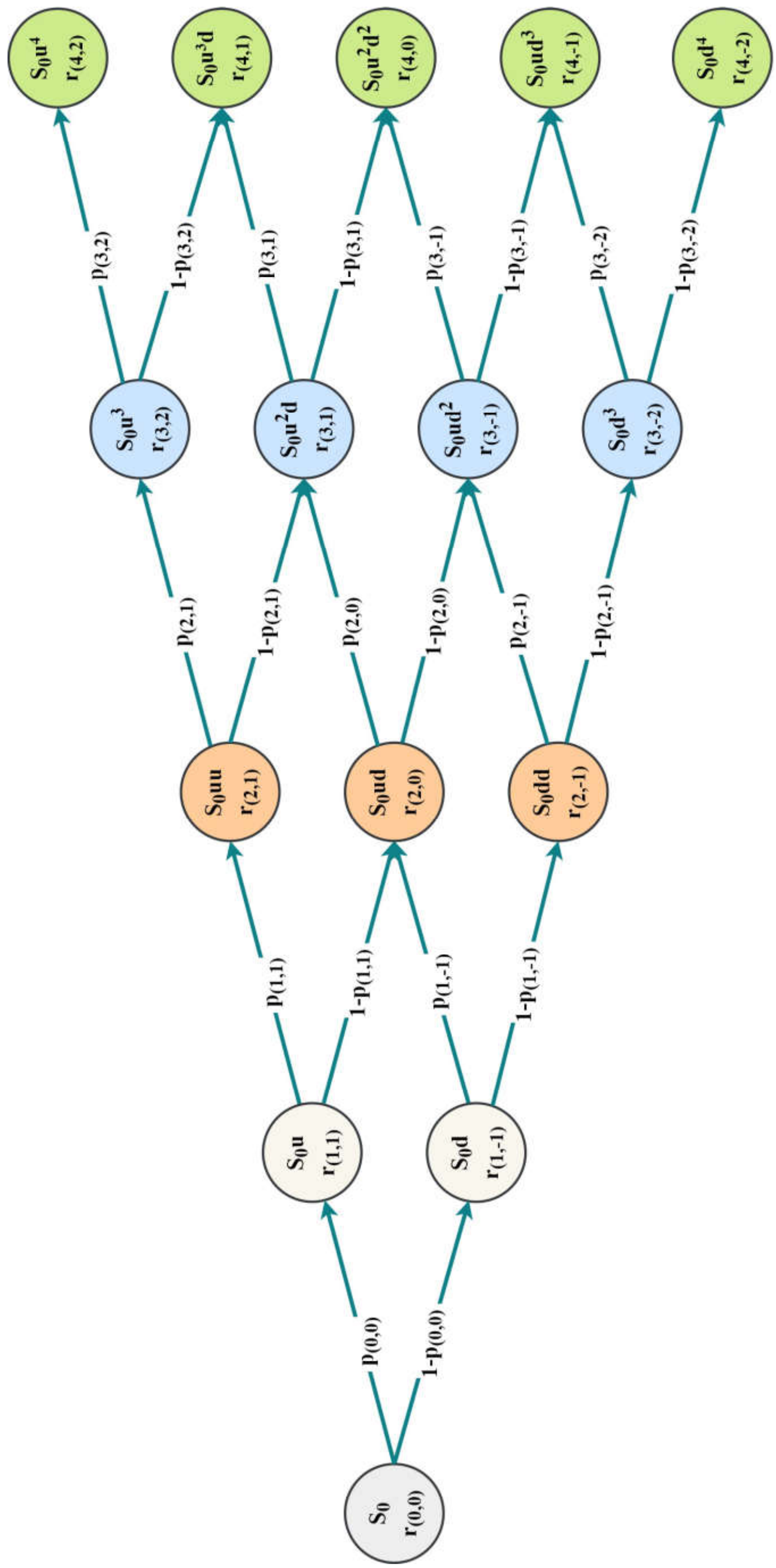

Figure 3.10. Four Periods Binomial Tree.

Finally, the algorithm of the CRR-BDT model generation is described as shown in Figure 3.11.

Inputs: Index volatility (σ), bond price list ($BondPrice[0,..,N]$), bond volatility list ($BondVolatility[0,..,N]$)

Outputs: Risk-neutral probabilities ($p[0,..,N]$) and possible interest rates ($r[0,..,N]$) of CRR-BDT model tree for N periods

for $i = 1$ to N **do**

  Calculate risk-neutral probabilities between periods i and $(i+1)$;

  $p[i] = risk\_neutral\_prob\_calculation\ (\sigma,\ r[i])$;

  Compute possible interest rates for the periods $i \to (i+1)$;

  $r[i{+}1] = BDT\_calibration\ (p[0,..,i],\ BondPrice[0,..,i],\ BondVolatility[0,..,i])$

end for

Figure 3.11. CRR-BDT Model Algorithm.

3.3.4. Option Pricing

The price of an option can be easily calculated by using replicating portfolio technique after creating a binomial tree with the stock price, risk-neutral probabilities, and possible interest rates. Using risk-neutral probabilities obtained above enables us to price options by satisfying arbitrage-free condition. Below, the procedure of option pricing is explained with the notations defined previously.

At the maturity T, the price of a call option equals $max(0, S_T - K)$. Additionally, the option price in an earlier period can be attained using risk-neutral probabilities and interest rates. Risk neutral probabilities are needed to find expected values of prices. Also, interest rates are required to obtain the discounted value of these expected values. In Figure 3.12, call option prices in every step of the tree with maturity ($T = 2$) are pointed.

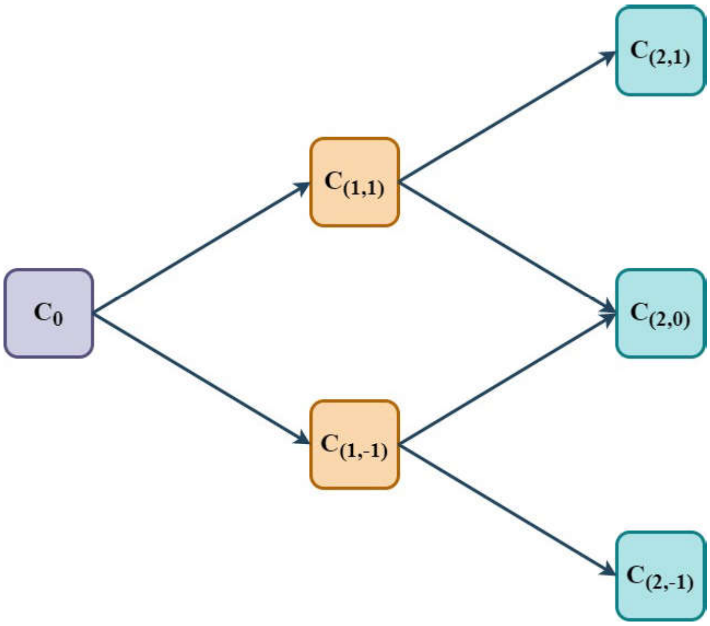

Figure 3.12. Two Periods Option Price Tree.

The option price C_0 at $T = 0$ is reached by calculating prices from the last nodes to a beginning node of the tree step by step. Prices of these options can be appraised like in following equations

$$C_{2,1} = max(u^2 S_0 - K, 0), \tag{3.15}$$

$$C_{2,0} = max(udS_0 - K, 0), \tag{3.16}$$

$$C_{2,-1} = max(d^2 S_0 - K, 0), \tag{3.17}$$

$$C_{1,1} = \frac{p_{1,1} * C_{2,1} + (1 - p_{1,1}) * C_{2,0}}{e^{r_{1,1}}}, \tag{3.18}$$

$$C_{1,-1} = \frac{p_{1,-1} * C_{2,0} + (1 - p_{1,-1}) * C_{2,-1}}{e^{r_{1,-1}}}, \tag{3.19}$$

$$C_0 = \frac{p_{0,0} * C_{1,1} + (1 - p_{0,0}) * C_{1,-1}}{e^{r_{0,0}}}. \tag{3.20}$$

Furthermore, the following equations are satisfied for replicating portfolio technique and arbitrage-free condition. Equations

$$\Delta = \frac{C_{2,1} - C_{2,0}}{S_0u^2 - S_0ud}, \tag{3.21}$$

$$C_{1,1} + \Delta(\frac{S_0u^2}{e^{r_{1,1}}} - S_0u) = \frac{C_{2,1}}{e^{r_{1,1}}}, \tag{3.22}$$

$$C_{1,1} + \Delta(\frac{S_0ud}{e^{r_{1,1}}} - S_0u) = \frac{C_{2,0}}{e^{r_{1,1}}} \tag{3.23}$$

are used for 1^{st} period-up movement condition. Also, formulas

$$\Delta = \frac{C_{2,0} - C_{2,-1}}{S_0ud - S_0d^2}, \tag{3.24}$$

$$C_{1,-1} + \Delta(\frac{S_0ud}{e^{r_{(1,-1)}}} - S_0d) = \frac{C_{2,0}}{e^{r_{(1,-1)}}}, \tag{3.25}$$

$$C_{1,-1} + \Delta(\frac{S_0d^2}{e^{r_{(1,-1)}}} - S_0d) = \frac{C_{2,-1}}{e^{r_{(1,-1)}}} \tag{3.26}$$

are handled for 1^{st} period - down movement condition. Finally, equations

$$\Delta = \frac{C_{1,1} - C_{1,-1}}{S_0u - S_0d}, \tag{3.27}$$

$$C_0 + \Delta(\frac{S_0u}{e^{r_{0,0}}} - S_0) = \frac{C_{1,1}}{e^{r_{0,0}}}, \tag{3.28}$$

$$C_0 + \Delta(\frac{S_0d}{e^{r_{0,0}}} - S_0) = \frac{C_{1,-1}}{e^{r_{0,0}}} \tag{3.29}$$

are applied for the period 0.

3.4. Numerical Example

A numerical example is figured out to clarify the model in this section. For instance, let the stock price at time 0 (S_0) be \$100 with volatility (σ) of 20% per annum. Also, zero-coupon bonds used in BDT calibration with different maturities are shown in Table 3.4.

Table 3.4. Example of zero-coupon rates and volatilities.

Years to Maturity	Zero-Coupon Rates (%)	Zero-Coupon Volatilities (%)	Zero-Bond Prices
1	6.0	20.0	94.339
2	7.0	18.0	87.343
3	8.0	16.0	79.383

3.4.1. 1^{st} Period Calculations

First of all, stock price movement is generated, and risk-neutral probability is calculated by using the current interest rate. Then, BDT calibration employs this probability. Also, the time between each node is assumed annually.

3.4.1.1. 1^{st} Period Stock Change Calculations. Initially, the rates u and d derived from the stock volatility should be calculated to create a one-step binomial tree and find risk-neutral probability. These rates are found as

$$u = e^{0.20\sqrt{1}} = 1.221, \tag{3.30}$$

$$d = e^{-0.20\sqrt{1}} = 0.819. \tag{3.31}$$

Therefore, possible stock prices are

$$S_1(up) = 1.221 * 100 = 122.10, \tag{3.32}$$

$$S_1(down) = 0.819 * 100 = 81.9. \tag{3.33}$$

Moreover, risk-neutral probability between period 0 and 1^{st} period can be obtained with the equation

$$p_{(0,0)} = \frac{e^{r\Delta T} - d}{u - d} = \frac{e^{0.06*1} - 0.819}{1.221 - 0.819} = 0.604. \tag{3.34}$$

After finding risk-neutral probability, BDT calibration can be implemented to obtain possible interest rates at the beginning of the 1^{st} period.

3.4.1.2. 1^{st} Period Interest Rate Calculations. According to Table 3.4, 2-years zero-coupon bond rate equals 7.0% with a volatility of 20%. Also, the face values of bonds are 100$ with a price of 87.343$. By appealing risk-neutral probability $p_{(0,0)}$ found above, possible interest rates for the 1^{st} period can be attained by solving equations

$$P_2 = \frac{p_{(0,0)} * \frac{F_2}{1+r_u} + (1 - p_{(0,0)}) * \frac{F_2}{1+r_d}}{1 + r_0}, \tag{3.35}$$

$$87.343 = \frac{0.604 * \frac{100}{1+r_u} + (1 - 0.604) * \frac{100}{1+r_d}}{1 + 0.06}. \tag{3.36}$$

Also, $r_u = r_d * e^{2\sigma\sqrt{\Delta T}}$ and the volatility of the 2-years zero-coupon bond is 0.18 from Table 3.4. Hence, r_u equals $r_d e^{2*0.18\sqrt{1}}$. Then Equation (3.36) can be written as

$$87.343 = \frac{0.604 * \frac{100}{1+r_d e^{0.36}} + (1 - 0.604) * \frac{100}{1+r_d}}{1 + 0.06}. \tag{3.37}$$

Now, the equation can be easily solved, and r_d is 6.36%. Thus, r_u equals 9.11% due to the two possible interest rates ratio. Finally, all of the variables in the 1^{st} period are determined, and the first step of the binomial tree is constituted, as demonstrated in Figure 3.13.

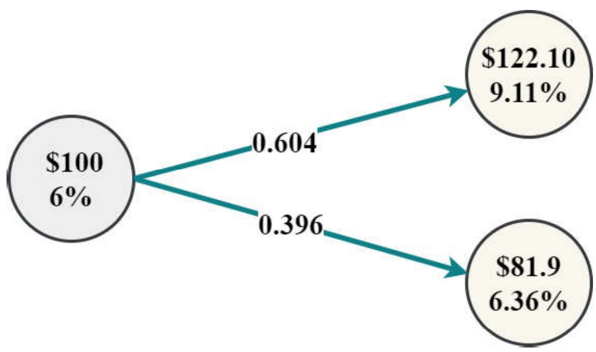

Figure 3.13. Binomial Tree for 1^{st} Step of Example.

3.4.2. 2^{nd} Period Calculations

After finding variables in the 1^{st} period, the same implementations should be followed for the 2^{nd} period calculations. Firstly, risk-neutral probabilities should be found. Then, BDT calibration should be applied by using these risk-neutral probabilities to detect possible interest rates.

3.4.2.1. 2^{nd} Period Stock Change Calculations. The rates u and d remain the same since they are affected only by stock volatility regardless of interest rate. Therefore, the same rates are used as specified in the 1^{st} period calculation, and possible stock prices in 2^{nd} are shown in the equations

$$S_2(up, up) = 1.221 * 122.1 = 149.0841, \tag{3.38}$$

$$S_2(up, down) = S_2(down, up) = 81.90 * 1.221 = 100, \tag{3.39}$$

$$S_2(down, down) = 81.90 * 0.819 = 67.0761. \tag{3.40}$$

Furthermore, the risk-neutral probability for the 2^{nd} period changes now because possible interest rates have consequences on risk-neutral probability. Since there are now two nodes at the end of the binomial tree, each node should be 2 different risk-neutral probability calculations. These are $p_{(1,1)}$ and $p_{(1,-1)}$. Second notations (1) and (-1) show the movement condition in the tree. Firstly, the probability for down movement condition, in which the possible interest rate equals 6.36%, is found as

$$p_{(1,-1)} = \frac{e^{r\Delta T} - d}{u - d} = \frac{e^{0.0636*1} - 0.819}{1.221 - 0.819} = 0.613. \tag{3.41}$$

After that, this probability for up movement, where possible interest is 9.11%, is calculated as

$$p_{(1,1)} = \frac{e^{r\Delta T} - d}{u - d} = \frac{e^{0.0911*1} - 0.819}{1.221 - 0.819} = 0.687. \tag{3.42}$$

By using these risk-neutral probabilities, possible interest rates at the beginning of the next period can be obtained with BDT calibration.

3.4.2.2. 2^{nd} Period Interest Rate Calculations. According to Table 3.4, 3-years zero-coupon bond rate equals 8.0% with volatility 16% and price 79.383$. By using the same methodology for the 1^{st} period calculation, possible interest rates are obtained by solving equations

$$P_3 = \frac{p_{(0,0)} * \frac{p_{(1,1)} * \frac{F_3}{1+r_{(2,1)}} + (1-p_{(1,1)}) * \frac{F_3}{1+r_{(2,0)}}}{1+r_{(1,1)}} + (1 - p_{(0,0)}) * \frac{p_{(1,-1)} * \frac{F_3}{1+r_{(2,0)}} + (1-p_{(1,-1)}) * \frac{F_3}{1+r_{(2,-1)}}}{1+r_{(1,-1)}}}{1 + r_{(0,0)}}, \tag{3.43}$$

$$79.383 = \frac{0.604 * \frac{0.687 * \frac{100}{1+r_{(2,1)}} + (1-0.687) * \frac{100}{1+r_{(2,0)}}}{1+0.0911} + (1 - 0.604) * \frac{0.613 * \frac{100}{1+r_{(2,0)}} + (1-0.613) * \frac{100}{1+r_{(2,-1)}}}{1+0.0636}}{1 + 0.07}. \tag{3.44}$$

Besides, $r_{(2,1)} = r_{uu} = r_{dd} * e^{4\sigma\sqrt{\Delta T}} = r_{(2,-1)} * e^{4\sigma\sqrt{\Delta T}}$ and $r_{(2,0)} = r_{ud} = r_{dd} * e^{2\sigma\sqrt{\Delta T}} = r_{(2,-1)} * e^{2\sigma\sqrt{\Delta T}}$ as mentioned before. In addition, the volatility of the 3-years zero-coupon bond equals 0.16 according to Table 3.4. Hence, $r_{(2,1)}$ and $r_{(2,0)}$ equals to $r_{(2,-1)} * e^{4*0.16\sqrt{1}}$ and $r_{(2,-1)} * e^{2*0.16\sqrt{1}}$, respectively. Then, Equation (3.39) can be written in terms of $r_{(2,-1)}$. After that, there is only one unknown variable in the equation; thus, the equation can be figured out with new adjusting. In the new equation, $r_{(2,-1)}$ is calculated as 5.90% and $r_{(2,0)}$ and $r_{(2,1)}$ equal to 8.12% and 11.19 % ,respectively. Finally, all of the variables in the 2^{nd} period are found. The binomial tree with these new findings can be generated like in Figure 3.14.

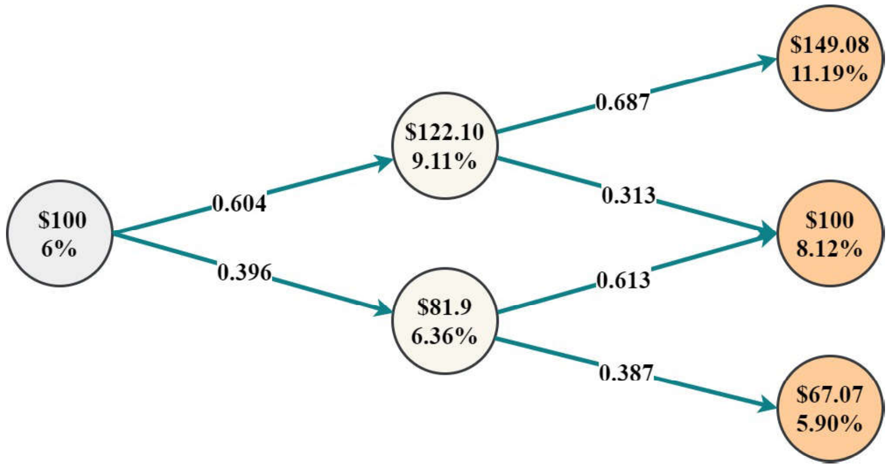

Figure 3.14. Final Binomial Tree of Example.

3.4.3. Option Pricing Example

After constituting the binomial tree, option pricing can be implemented conveniently. To illustrate, there is a call option with strike price $K = \$80$ and maturity $T = 2$. The price of this option at time 0 (C_0) is calculated by solving the following equations step by step.

Prices in 2^{nd} period are calculated as

$$C_{2,1} = max(u^2 S_0 - K, 0) = max(149.0841 - 80, 0) = 69.0841, \tag{3.45}$$

$$C_{2,0} = max(udS_0 - K, 0) = max(100 - 80, 0) = 20, \tag{3.46}$$

$$C_{2,-1} = max(d^2 S_0 - K, 0) = max(67.0761 - 80, 0) = 0. \tag{3.47}$$

Additionally, option prices in 1^{st} period are found with equations

$$C_{1,1} = \frac{p_{1,1} * C_{2,1} + (1 - p_{1,1}) * C_{2,0}}{e^{r_{1,1}}}, \tag{3.48}$$

$$C_{1,1} = \frac{0.687 * 69.0841 + (1 - 0.687) * 20}{e^{0.0911}}, \tag{3.49}$$

$$C_{1,1} = 49.043, \tag{3.50}$$

$$C_{1,-1} = \frac{p_{(1,-1)} * C_{2,0} + (1 - p_{(1,-1)}) * C_{(2,-1)}}{e^{r_{(1,-1)}}}, \tag{3.51}$$

$$C_{1,-1} = \frac{0.613 * 20 + (1 - 0.613) * 0}{e^{0.0636}}, \tag{3.52}$$

$$C_{1,-1} = 11.504. \tag{3.53}$$

Finally, option price at time 0 can be found as

$$C_0 = \frac{p_{0,0} * C_{1,1} + (1 - p_{0,0}) * C_{1,-1}}{e^{r_{0,0}}}, \tag{3.54}$$

$$C_0 = \frac{p_{0,0} * C_{1,1} + (1 - p_{0,0}) * C_{1,-1}}{e^{r_{0,0}}}, \tag{3.55}$$

$$C_0 = \frac{0.604 * 49.043 + (1 - 0.604) * 11.504}{e^{0.06}}. \tag{3.56}$$

$$C_0 = 32.187. \tag{3.57}$$

The option price equals \$32.187 at time 0. To illustrate, the tree of these option prices is shown in Figure 3.15.

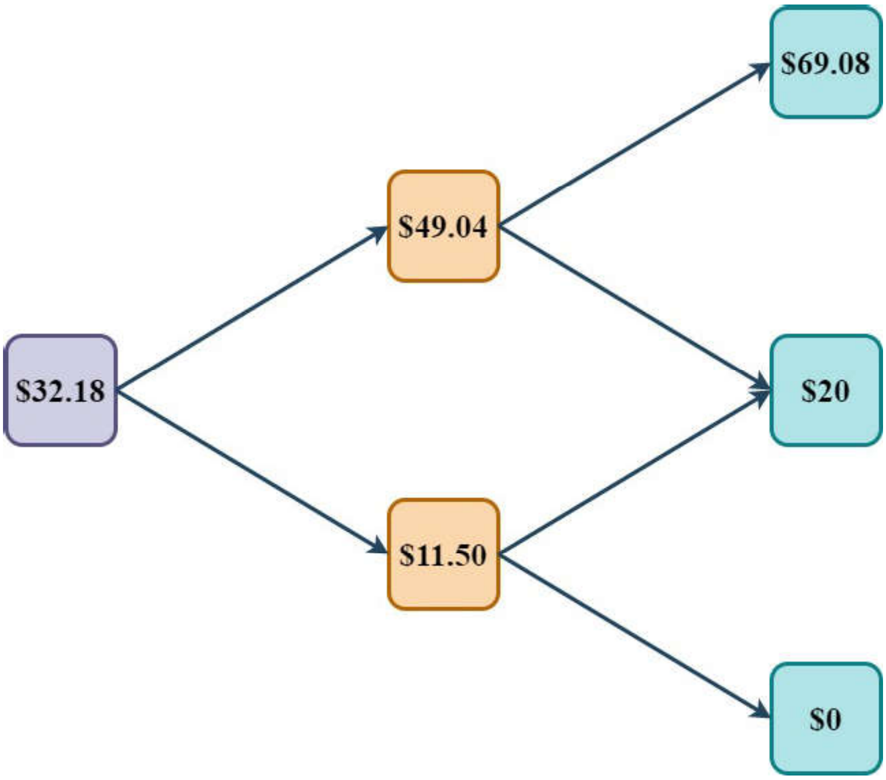

Figure 3.15. Option Price Tree of Example.

4. IMPLEMENTATION AND SIMULATION

The CRR-BDT model explained in the previous chapter is an extension version of the CRR model. In this chapter, the implementation of the CRR-BDT model is demonstrated with actual market data. Also, a simulation under generic numerical instances is shown in the last part of the chapter. Furthermore, the market prices of options and the CRR model's estimation are observed to compare the models' accuracies according to actual market prices.

4.1. Market Data Implementation

This section mainly consists of two parts. Firstly, the options whose maturities are at least one year are priced. Then, these estimations are compared with the real market prices and the prices derived from the CRR model. After pricing options with longer maturities, that implementation is made for less than 1-year maturity options. Finally, the comparison with the CRR model and market values is realized again. The purpose of pricing options with different maturities is to observe which maturities the model works more effectively.

4.1.1. Long Term Pricing

In this part, by using the CRR-BDT and CRR models, we price 147 options in the market. Then, the predictions of both models are compared with the actual market prices. Moreover, maturities of the options vary between 1 and 4-years, as well as different strike prices. The part begins with bootstrapping technique that is a method for obtaining zero-coupon rates from coupon bonds. Then, S&P500 index volatility is calculated to define index change rates. After calculating volatility and obtaining zero-coupon rates, the CRR-BDT model is generated. Finally, the options instances are valued, and the accuracies of the models are compared.

4.1.1.1. Bootstrapping. As explained previously, the BDT tree is generated from zero-coupon bond rates and volatilitilies. For this reason, the zero-coupon bond should be obtained. Before describing this obtaining mechanism, the terms zero-coupon bonds and coupon-bonds are defined in this section. First of all, zero-coupon bonds, also called discount bonds, pay the face value of the bond to its holder only on the maturity date. In other words, another coupon payment is not realized until the maturity [40]. For instance, there is a zero-coupon bond maturing on time T with the face value of F. Also, its yield-to-maturity or the bond rate at time t is equal to $r(t,T)$. If $T - t = n$, the price of this bond at time t, (P_t), is calculated by using the formula

$$P_t = \frac{F}{(1 + r(t,T))^n}. \tag{4.1}$$

On the other hand, a coupon bond make other payments periodically until maturity date in addition to face value [41]. If annual coupon rate equals to c, then this coupon payment amount C is calculated as

$$C = c * F. \tag{4.2}$$

Additionally, if the number of payments equals N during the lifetime of the bond and the yield to maturity is shown as $r(t,T)$, the price of this security at time t, P_t, can be found by using the formula

$$P_t = \sum_{i=1}^{N} \frac{C}{(1 + r(t,T))^i} + \frac{F}{(1 + r(t,T))^N}. \tag{4.3}$$

Also, the Figure 4.1 and 4.2 represent the difference between these two type of bonds.

Figure 4.1. Coupon Bond Cash Flow.

Furthermore, Treasury notes and bonds of the U.S. government make coupon

Figure 4.2. Zero-Coupon Bond Cash Flow.

payments every six months until maturity. Also, zero-coupon bonds are released for up to only one year by Treasury. For the bonds which maturities greater than 1-year, coupon payment is realized semi-annually [21]. In order to work with actual market data, the bond rates of the U.S. government should be converted to zero-coupon bond rates for BDT tree generation. At this point, the bootstrapping technique enables us to obtain zero-coupon rates from coupon bonds. This methodology is figured out with the following example. For instance, there are four different bonds with separate maturities. These are six months, 1-year, 1.5-years, and 2-years bonds with the bond prices $100. Table 4.1 illustrates the bond yields and maturities.

Table 4.1. Bootstrapping example-bond yields and maturities.

Time to Maturity (Year)	**Yield**	**Price**
0.5	5%	$100
1	5.5%	$100
1.5	6%	$100
2	6.5%	$100

First of all, the 6-months and 1-year bonds do not make a payment until maturity date as explained before. For this reason, these securities are zero-coupon bonds and zero-coupon rates for these bonds are the same as the yields to maturities in the table. However, several calculation steps are needed to obtain zero-coupon bond rates for the maturities 1.5 and 2-years.

There are 3 coupon payments for the 3^{rd} bond. Since the yield of this bond is 6%

annually, the amount of \$3 is paid in every 6 months to holder as a coupon payment. The coupon payment amount calculation for 1.5-years bond is shown as

$$C_{1.5} = \$100 * \frac{6\%}{2} = \$3. \tag{4.4}$$

Also, on the maturity date, the holder gains \$103 by summing face value and coupon payment. All of these payments should be discounted with the respective discount or zero-coupon rates to obtain a 1.5-year zero-coupon rate. Because the 0.5 and 1-year discount rates have already been known, only the 1.5-year zero-coupon bond rates ($d_{1.5}$) is an unknown variable given by

$$\$100 = \frac{\$3}{1 + \frac{5\%}{2}} + \frac{\$3}{(1 + \frac{5.5\%}{2})^2} + \frac{\$103}{(1 + \frac{d_{1.5}}{2})^3}. \tag{4.5}$$

Solving the equation above gives the 6.02% as 1.5-years zero-coupon rate.

Besides, with the same method, 2-years zero-coupon rate can be found. Additionally, the values found above as 1.5 years zero coupon rate is used to discount cash flow. Also, the yield for 2-years bond is 6.5%; hence, coupon payment equals \$3.25 per semi-annual. In order to find 2-years zero-coupon rate, the equation

$$\$100 = \frac{\$3.25}{1 + \frac{5\%}{2}} + \frac{\$3.25}{(1 + \frac{5.5\%}{2})^2} + \frac{\$3.25}{(1 + \frac{6.02\%}{2})^3} + \frac{\$103.25}{(1 + \frac{d_2}{2})^4} \tag{4.6}$$

can be written. In the equation above, 2-years zero-coupon bond rate is found as 6.54%. Finally, by using the bootstrapping technique, zero-coupon bond rates for longer than 1-year maturity can be obtained from coupon-bond yields. In Table 4.2, the coupon bond yields and the zero-coupon bond rates are demonstrated.

Table 4.2. Bootstrapping example-bond yields and zero-coupon rates.

Time to Maturity (Year)	**Yield**	**Zero-Coupon Rates**	**Price**
0.5	5%	5%	\$100
1	5.5%	5.5%	\$100
1.5	6%	6.02%	\$100
2	6.5%	6.54%	\$100

Besides, the daily treasury yield curve rates of bonds can be displayed on the U.S. Department of the Treasury website [42]. In Table 4.3, the example data is shown.

Table 4.3. Daily treasury yield curve rates.

Date	**6 Mo**	**1 Yr**	**2 Yr**	**3 Yr**	**5 Yr**	**7 Yr**	**10 Yr**	**20 Yr**	**30 Yr**
11.03.2021	0.07	0.17	0.47	0.77	1.19	1.46	1.6	2.01	2.00
11.04.2021	0.07	0.14	0.41	0.69	1.10	1.37	1.53	1.96	1.96
11.05.2021	0.07	0.14	0.39	0.66	1.04	1.30	1.45	1.88	1.87
11.08.2021	0.07	0.16	0.45	0.75	1.13	1.38	1.51	1.91	1.89
11.09.2021	0.06	0.14	0.41	0.71	1.08	1.32	1.46	1.86	1.83
11.10.2021	0.07	0.17	0.51	0.83	1.23	1.45	1.56	1.96	1.92
11.12.2021	0.07	0.17	0.53	0.85	1.24	1.47	1.58	1.99	1.95
11.15.2021	0.06	0.18	0.53	0.87	1.26	1.51	1.63	2.05	2.01
11.16.2021	0.07	0.17	0.54	0.87	1.27	1.52	1.63	2.06	2.02
11.17.2021	0.06	0.18	0.52	0.85	1.24	1.49	1.60	2.04	2.00
11.18.2021	0.06	0.18	0.52	0.84	1.22	1.47	1.59	2.01	1.97
11.19.2021	0.06	0.18	0.52	0.86	1.22	1.45	1.54	1.95	1.91

Even though some maturities, such as 4-years bond yield rate, do not exist in the table, the yield curve rates for this non-displayed maturity can be settled by interpolating existing bonds rates. To illustrate, the yield to maturity rate of a 4-years maturity bond can be defined as interpolation of 3 and 5-years maturity bonds rates. After adjusting the data in the table, bootstrapping should be made for each day in the table since the volatility of the rates of the zero-coupon bond is also required for BDT calibration. Hence, the zero-coupon rates for multiple days are needed. After finding the zero-coupon volatilities of different maturities for the chosen data window, the result can be scaled with the number of trading days to define annual volatility. In this thesis, calendar or trading days are assumed 252 days. The formula of the scaling is written in the equation

$$\sigma_{annually} = \sigma_{daily} * \sqrt{252}. \tag{4.7}$$

After applying the interpolation and the bootstrapping method, zero-coupon rates for specific maturities can be obtained. In the thesis, option pricing is made for securities up to four years of maturity. For this reason, 1-year, 2-years, 3-years, and 4-years zero-coupon bond rates are found from daily Treasury yield curve rates. In Table 4.4, zero-coupon bond rates for these maturities are shown.

4.1.1.2. Index Volatility Calculation. As explained previous chapter, the stock or index change rates between consecutive steps in CRR model and our model is equal to $u = e^{\sigma\sqrt{\Delta T}}$ and $d = e^{-\sigma\sqrt{\Delta T}}$. Also, the option pricing for S&P 500 index is handled in this thesis. Thus, the volatility of the S&P500 index should be used to obtain the rates of u and d in order to create a binomial tree. On the date 11.20.2021, historical closing prices in the last year are taken [43]. Then, the index volatility is calculated for this historical data in this thesis. In Table 4.5, the annual volatility and average price of the index are shown.

Table 4.4. Daily zero-coupon bond rates.

Date	**1 Yr**	**2 Yr**	**3 Yr**	**4 Yr**	**5 Yr**
11.03.2021	0.170	0.471	0.773	0.986	1.202
11.04.2021	0.140	0.411	0.693	0.901	1.110
11.05.2021	0.140	0.391	0.663	0.855	1.049
11.08.2021	0.160	0.451	0.753	0.946	1.140
11.09.2021	0.140	0.411	0.713	0.900	1.089
11.10.2021	0.170	0.511	0.834	1.037	1.242
11.12.2021	0.170	0.531	0.854	1.052	1.252
11.15.2021	0.180	0.531	0.874	1.072	1.272
11.16.2021	0.170	0.541	0.874	1.077	1.282
11.17.2021	0.180	0.521	0.854	1.052	1.252
11.18.2021	0.180	0.521	0.844	1.037	1.231
11.19.2021	0.180	0.521	0.864	1.047	1.231

Furthermore, the index change rates u, and d can be obtained by using this found annual volatility like in the equations

$$\mathbf{u} = e^{12.06\%\sqrt{1}} = 1.128, \tag{4.8}$$

$$\mathbf{d} = e^{-12.06\%\sqrt{1}} = 0.886. \tag{4.9}$$

Moreover, the S&P 500 index's binomial tree can be constituted with these obtained rates above. Figure 4.3 illustrates this 4-years binomial tree created on the date 11.20.2021 with the closing price $4697.

Table 4.5. S&P 500 Index Volatility and Average Price.

Annual Volatiltiy	12.06%
Average Price	$4163.15
Last Price	$4697.96

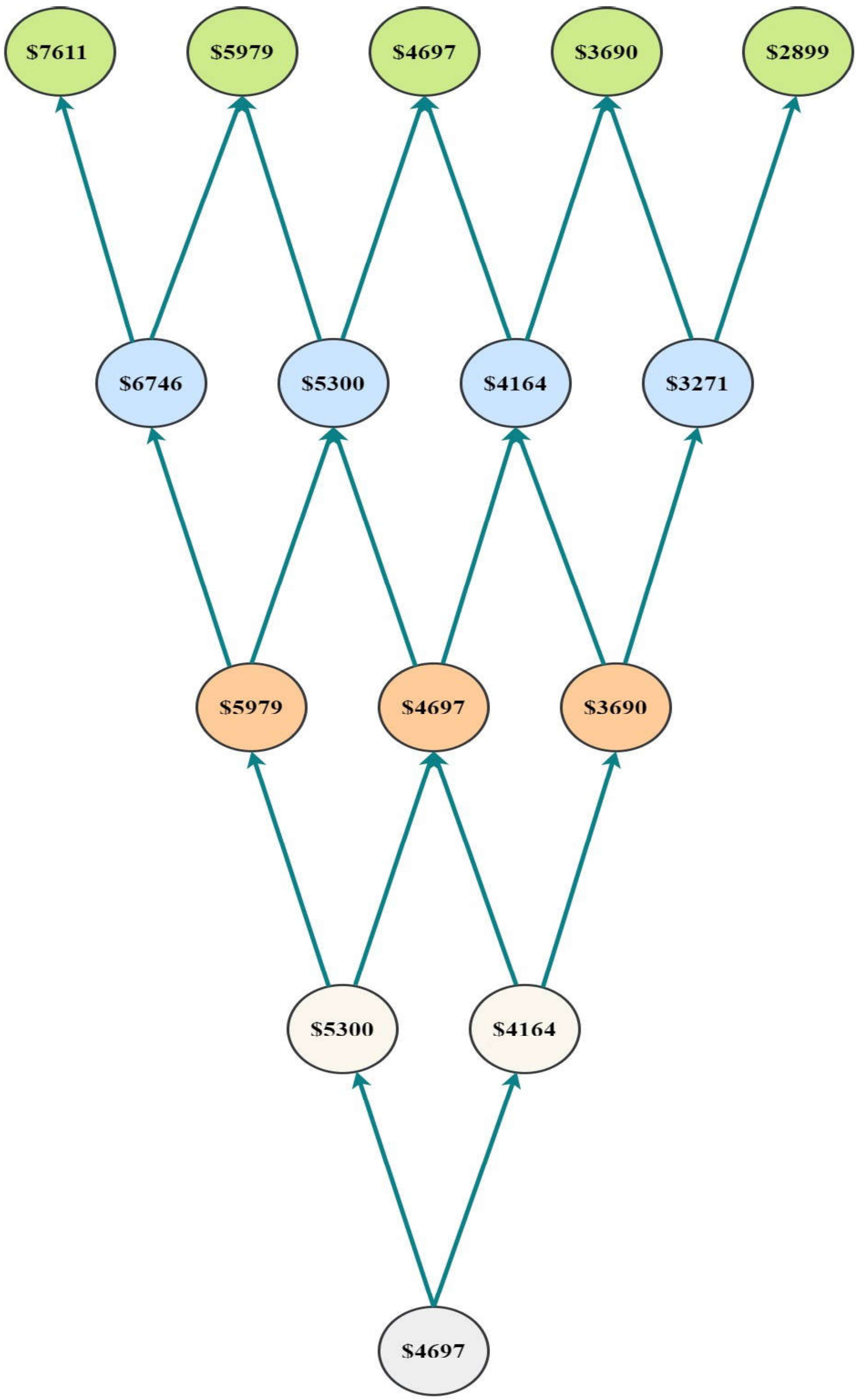

Figure 4.3. S&P 500 Index Binomial Tree.

4.1.1.3. Generating the CRR-BDT Model. After bootstrapping and index volatility calculation, zero-coupon rates, zero-coupon volatilities, and index change rates are obtained. In other words, all essential inputs of our model are available. As defined in the previous section, the binomial tree should be created step by step in the model. In a step, firstly, the risk-neutral probability should be calculated by using the index change rates and the interest rate in that specific period. After that, this estimated probability should be used in BDT calibration to define possible interest rates in the following step. This cycle will continue until maturity, as explained in Model Section. While implementing this methodology, the binomial tree is also formed step by step. Then, option pricing can be made for different maturities with this generated tree.

In this part of the implementation, risk-neutral probabilities and possible interest rates in the following periods are found via a binomial tree constituted by the BDT-CRR model. The parameters risk-neutral probabilities and possible interest rates are calculated with a code. The code exists in the Appendix section of the thesis. Moreover, these obtained parameters are added to the binomial tree illustrated in Figure 4.3. The new adjusted tree is shown in Figure 4.4; also, each step shows one year. For instance, possible interest rates between 1^{st} and 2^{nd} steps are 0.85% and 0.88%. Also, risk-neutral probabilities between the beginning step and 1^{st} step are calculated as 0.477 and 0.523. Additionally, the highest possible interest rate in the last period is found at 2.40%.

Finally, all required parameters and binomial tree are ready to make the option pricing. In the following part, an option pricing for the specific strike price and the maturity is figured out using these obtained parameters.

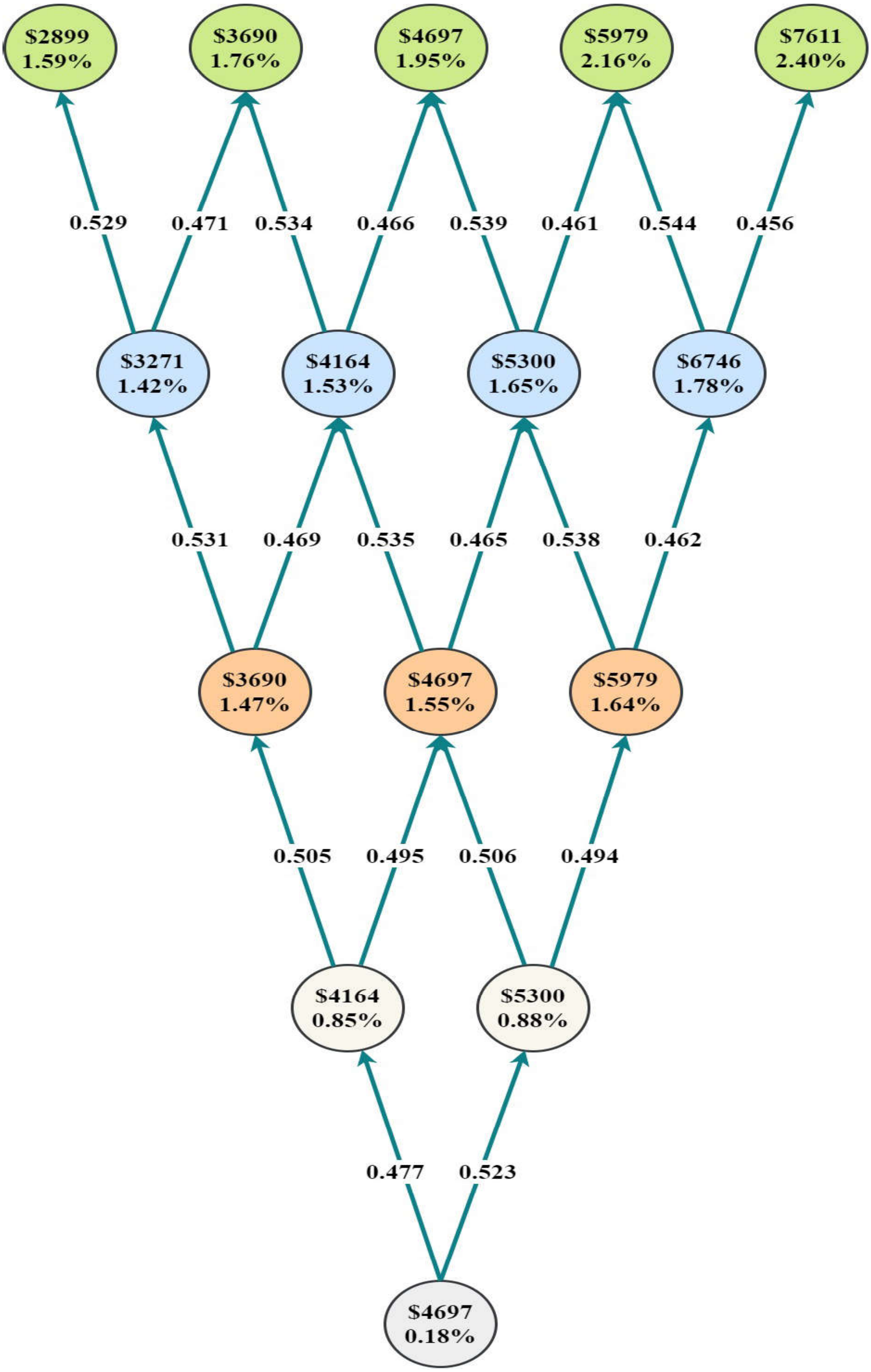

Figure 4.4. S&P 500 Index and Interest Rate Tree.

4.1.1.4. Option Pricing. After creating the binomial tree in the previous section, call option prices for 1,2,3 and 4-years maturities are estimated. Actually, each maturity shows the one step in the binomial tree indicated in Figure 4.4. When the strike price is equal to K, the call option with 1-year maturity (1^{st} step) is priced as

$$C_{(1,-1)} = max[4164 - K, 0], \tag{4.10}$$

$$C_{(1,1)} = max[5300 - K, 0], \tag{4.11}$$

$$C_0 = \frac{p_{(0,0)} * C_{(1,1)} + (1 - p_{(0,0)}) * C_{(1,-1)}}{e^{r_{0,0} * \Delta T}}, \tag{4.12}$$

$$C_0 = \frac{0.523 * C_{(1,1)} + 0.477 * C_{(1,-1)}}{e^{0.18\%}}. \tag{4.13}$$

Hence, if the strike price (K) is \$5000, call option price C_0 can be estimated by substituting the value of strike price. Then, these formulas are converted to

$$C_{(1,-1)} = max[4164 - 5000, 0] \quad =0, \tag{4.14}$$

$$C_{(1,1)} = max[5300 - 5000, 0] \quad =300, \tag{4.15}$$

$$C_0 = \frac{0.523 * 300 + 0.477 * 0}{e^{0.18\%}} =156.617. \tag{4.16}$$

As indicated above, the price of the call option with 1-year maturity and \$5000 strike price is estimated as \$156.617.

Furthermore, in the empirical study, the pricing implementation of the call option is conducted with different strike prices and different maturities. The years from 2022 to 2025 are settled as maturities. Also, the S&P 500 Index price is \$4697 on the calculation day, as mentioned before. In Table 4.6, summary statistics of the data are shown. There are 147 data points in our calculation. Also, the table provides the number of years until maturity date (ΔT) and strike price ranges (\$$K$). For instance, 105 different options for the 2-years maturity are priced in total.

Moreover, the empirical estimation results with the data points summarized above are presented in the following table for the 1-year maturity. Table 4.7 includes strike prices for each data point. In addition to calculated prices by the CRR-BDT model,

Table 4.6. Summary statistics of option data sample.

Strike Price	**Number of Years Until Maturity**				**Total**
	1	**2**	**3**	**4**	
K $\leq$ $4500	16	98	1	5	120
$4500 <K $\leq$ $5000	7	7	2	2	18
$5000 <K	-	-	6	3	9
Total	23	105	9	10	**147**

market prices for each option are given in the table. For instance, in Table 4.7, our model calculates the call option price, which has strike price $1400 and 1-year maturity, as $3300.48; while the market price is $3264. Finally, the statistical results for different maturities and comparison with the CRR model in which the interest rate is constant throughout the option's life are demonstrated in the following section.

Table 4.7. 1-year maturity option prices.

Strike Price	**CRR-BDT Model**	**Market Price**
1400	3300.48	3264.00
2600	2102.64	1940.80
3550	1154.34	1228.64
3600	1104.43	1080.20
3975	730.11	868.26
4000	705.15	859.75
4250	500.43	516.08
4275	488.51	469.17
4300	476.60	613.94
4350	452.78	563.30
4375	440.87	565.24
4400	428.96	557.00
4425	417.04	506.50
4450	405.13	487.19
4475	393.22	476.44
4500	381.31	416.96
4525	369.40	438.12
4550	357.49	443.32
4575	345.57	304.31
4600	333.66	406.21
4625	321.75	352.18
4650	309.84	354.50
4675	297.93	365.57

4.1.1.5. Comparision with CRR Model. As a reminder, the CRR model assumes that the interest rate is constant during the life of an option. In this section, the CRR and CRR-BDT model are compared to observe the impact of the interest rate on option pricing. This comparison measures the deviation of the model estimation from the option prices in the actual market data. In other words, mispricing of the models according to actual market prices is examined in this section.

The accuracy of the models based on the actual prices in the market can be found by using statistical measures the Mean Square Error (MSE) and the Mean Absolute Deviation (MAPE). It can be assumed that a model with lower MSE or MAPE gives better estimation than other models. The measurements MSE and MAPE can be defined as

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(C_i - M_i)^2, \tag{4.17}$$

$$MAPE = \frac{1}{N}\sum_{i=1}^{N}|\frac{(C_i - M_i)}{M_i}|, \tag{4.18}$$

where N is the number of options, C_i is the i^{th} option price calculated by the model, and M_i shows the actual market price of the i^{th} option.

The comparison between these two models is summarized in Table 4.8. The table includes MSE and MAPE criteria separately for each maturity. Also, the overall results for both models are demonstrated in Table 4.8. According to Table 4.8, the CRR-BDT model gives slightly better estimations than the CRR model in the overall MAPE comparison. On the other hand, the CRR model makes less mispricing for the MSE criterion overall. Furthermore, the CRR-BDT model has a more accurate output, especially for three and 4-years maturities. In particular, the difference between MSE and MAPE values is remarkable for this maturity. This situation mainly stems from an expected increase in the interest rate in the market for these years.

Additionally, market prices for some options are unreasonable due to trading volatility. For instance, the call option expired in December 2023 with the strike price

Table 4.8. Model comparision in yearly analysis.

Model	**Criterions**	**1**	**2**	**3**	**4**	**Overall**
CRR-BDT	**MAPE**	0.14	0.18	0.34	0.21	0.18
	MSE	8246.01	68624.75	15318.08	17563.21	52440.49
CRR	**MAPE**	0.14	0.17	0.42	0.28	0.19
	MSE	8246.01	64883.74	24398.74	31434.83	51267.94

$1300, is priced $2793.61 in the market. At that time, another call option with the same maturity date and $1400 strike price has $3104.2. market price. Reasonably, the price of the second option should be lower. However, some options are priced irrational because of trading in the market. In Table 4.9, this situation is illustrated. The options with strike price $1300 and $1100 break the price increment significantly.

Table 4.9. Unreasonable market prices.

Strike Price ($)	**Option Price in the Market ($)**
1500	3010.1
1400	3104.2
1300	**2793.61**
1200	3250.9
1100	**3169.8**
1000	3596

Adversely, this mispricing in the market decreases the accuracies of both models. Then, the models' estimations for these mispriced options have more significant absolute percentage error. In Figure 4.5, the distribution of these deviations is shown. According to the histogram in Figure 4.5, the major part of pricing has a lower than 0.25 absolute percentage error. Moreover, some variations are greater than the error 0.5. Even though the portion of these high deviations in the pricing is low, they impact the accuracy of the models overall.

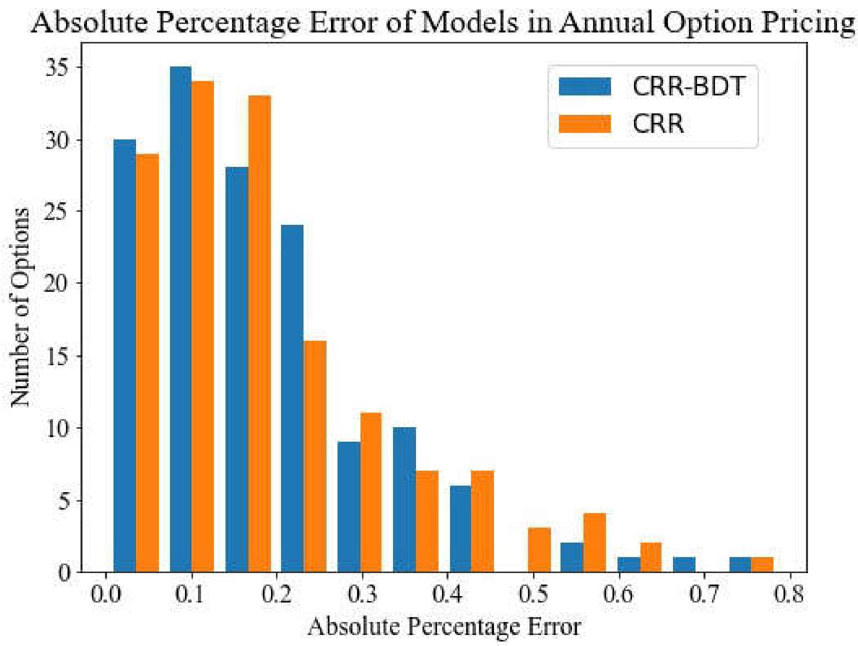

Figure 4.5. Distribution of Absolute Percentage Error of Models.

Moreover, the calculated and market prices are illustrated in Figure 4.6 to observe the deviation in the option pricing and the minor diversion in the market prices. By analyzing Figure 4.6, it can be argued that market prices and model estimations are close to each other in the 1-year maturity case. Also, the models overprice options that are approximately less than $3000 strike price. After this threshold, market prices are above model prices. Furthermore, in the 2-years maturity cases, values of the options in the market are below model estimation until almost $3300 strike price. Additionally, both models make predictions close to each other in the option pricing with 2-years maturity. Also, the mentioned market's erratic pricing can be easily seen with the sharp declines in that period.

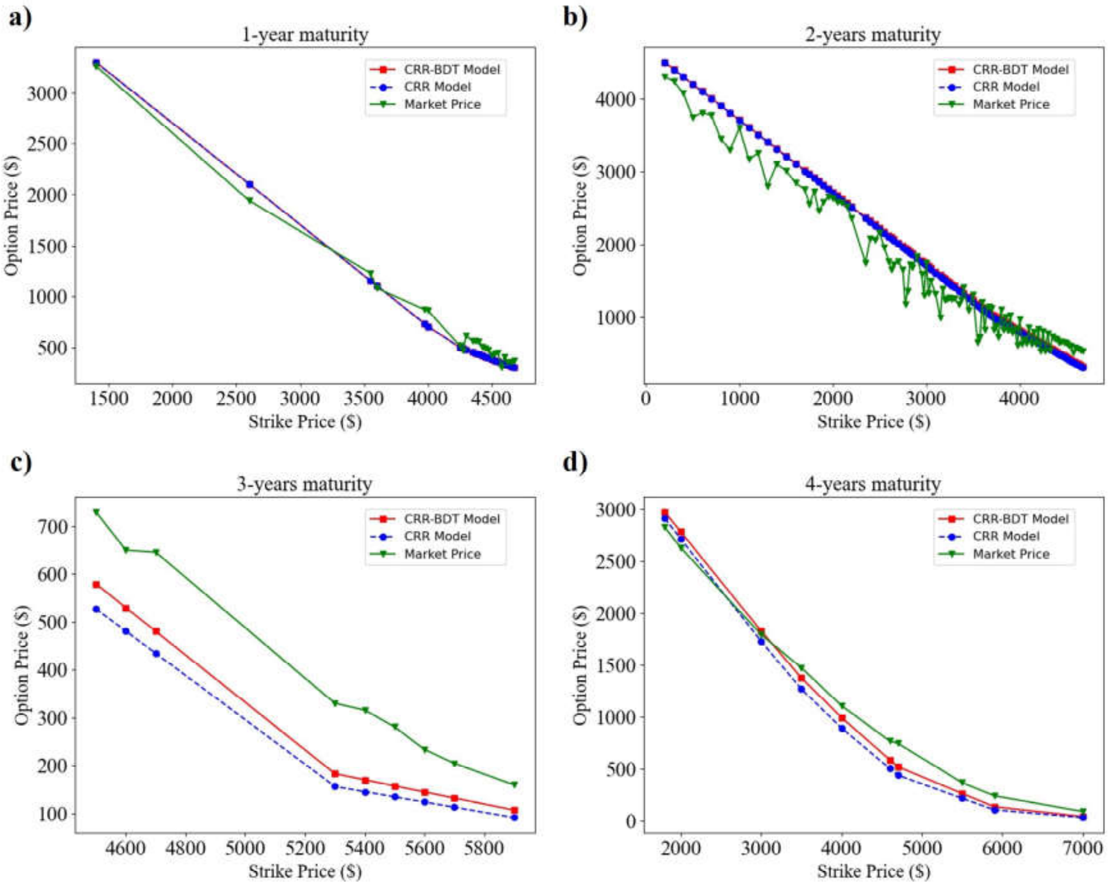

Figure 4.6. Models and Market Prices for Different Maturities.

Moreover, there is no remarkable, unusual market pricing for the options with three and 4-years maturities. On the other hand, the deviation between model prices and market prices for the 3-years maturity instances is considerable compared to different maturities. However, the CRR-BDT model gives slightly better results than the CRR model for the 3-years maturity with all strike prices. Finally, model and market prices remain close for 3-year options; also, the CRR-BDT model gives a closer estimation of market prices in most cases. As explained before, the CRR-BDT model estimates closer results in 3 and 4-years maturities because the model considers the expected increase in the future interest rate.

4.1.2. Option Pricing for Monthly Terms

In addition to option pricing for annual terms, the values of options whose maturities are less than one year are estimated in the thesis. Unlike annual pricing, bootstrapping is not required for monthly pricing. Because the yield curve rates until 1-year maturity do not give any coupon payment to holders, as mentioned before. For this reason, yield curve rates and zero-coupon rates are the same. Hence, the bootstrapping method can be skipped in monthly pricing. To compare the monthly pricing performance of models according to market data, 587 options with different strike prices and maturity are estimated.

4.1.2.1. Yield Curve Rates. Zero-coupon rates can be taken directly from the website of the U.S Treasury Department. The rates on the day of this study are shown in Table 4.10. The 4, 5, 7, 8, 9, 10, and 11-months yield curve rates not included in the table can be derived from interpolation of other rates, as applied before. For example, 6-months zero-coupon rate on the day 11.30.2021 is equal to 0.10%. Additionally, the binomial tree includes 12-periods is generated in implementation to define interest rates, risk-neutral probabilities, and possible stock movements.

Table 4.10. Monthly yield curve rates.

Date	**1 Mo**	**2 Mo**	**3 Mo**	**6 Mo**	**1 Yr**
11.01.2021	0.05	0.09	0.05	0.06	0.15
11.02.2021	0.05	0.06	0.05	0.07	0.15
11.03.2021	0.05	0.07	0.05	0.07	0.17
11.04.2021	0.05	0.05	0.04	0.07	0.14
11.05.2021	0.05	0.06	0.05	0.07	0.14
11.08.2021	0.04	0.06	0.06	0.07	0.16
11.09.2021	0.04	0.05	0.04	0.06	0.14
11.10.2021	0.06	0.06	0.05	0.07	0.17
11.12.2021	0.05	0.05	0.05	0.07	0.17
11.15.2021	0.06	0.06	0.05	0.06	0.18
11.16.2021	0.06	0.06	0.05	0.07	0.17
11.17.2021	0.06	0.05	0.05	0.06	0.18
11.18.2021	0.12	0.05	0.05	0.06	0.18
11.19.2021	0.11	0.04	0.05	0.06	0.18
11.22.2021	0.07	0.03	0.05	0.07	0.20
11.23.2021	0.06	0.04	0.06	0.07	0.21
11.24.2021	0.14	0.05	0.06	0.10	0.24
11.26.2021	0.11	0.04	0.06	0.10	0.20
11.29.2021	0.07	0.04	0.06	0.10	0.21
11.30.2021	0.11	0.05	0.05	0.10	0.24

4.1.2.2. Monthly Index Change Rates. The index change rates (u, d) are standardized for the monthly process. For this reason, daily volatility should be scaled in order to find monthly volatility. Additionally, it assumed that there are 21 trading days in a month. Hence, the scaling formula can be written as

$$\sigma_{monthly} = \sigma_{daily} * \sqrt{21}. \tag{4.19}$$

After that, the u and d rates for the one month can be obtained as

$$u_{monthly} = e^{\sigma_{monthly}}, \tag{4.20}$$

$$d_{monthly} = e^{-\sigma_{monthly}}. \tag{4.21}$$

Also, these change rates are the same for both models since these are affected by only the stock volatility regardless of interest rates.

4.1.2.3. Estimation Results and Comparision of the Models. Finally, the options from the real-market data are estimated with CRR and CRR-BDT model after specifying the parameters. Again the same criteria are utilized to compare the models. Table 4.11 compares CRR-BDT and CRR model accuracy in the option pricing for a monthly term.

Table 4.11. Model comparision in monthly term.

Criteria	**CRR-BDT**	**CRR**
MAPE	0.2448214	0.243985
MSE	23399.115	23325.44
RMSE	152.96769	152.7267

According to the results in the table, there is no significant difference between the monthly pricing models. The Mean Absolute Percentage Error (MAPE) values of the CRR-BDT and CRR models are 0.244 and 0.243, respectively. Also, Mean Square Error (MSE) is slightly less in the CRR model. However, this difference is not significant. Hence, both models estimate the value of options very close to each other. Furthermore, Figure 4.7 presents the market price and model estimation in the chart to observe predictions closeness of the model.

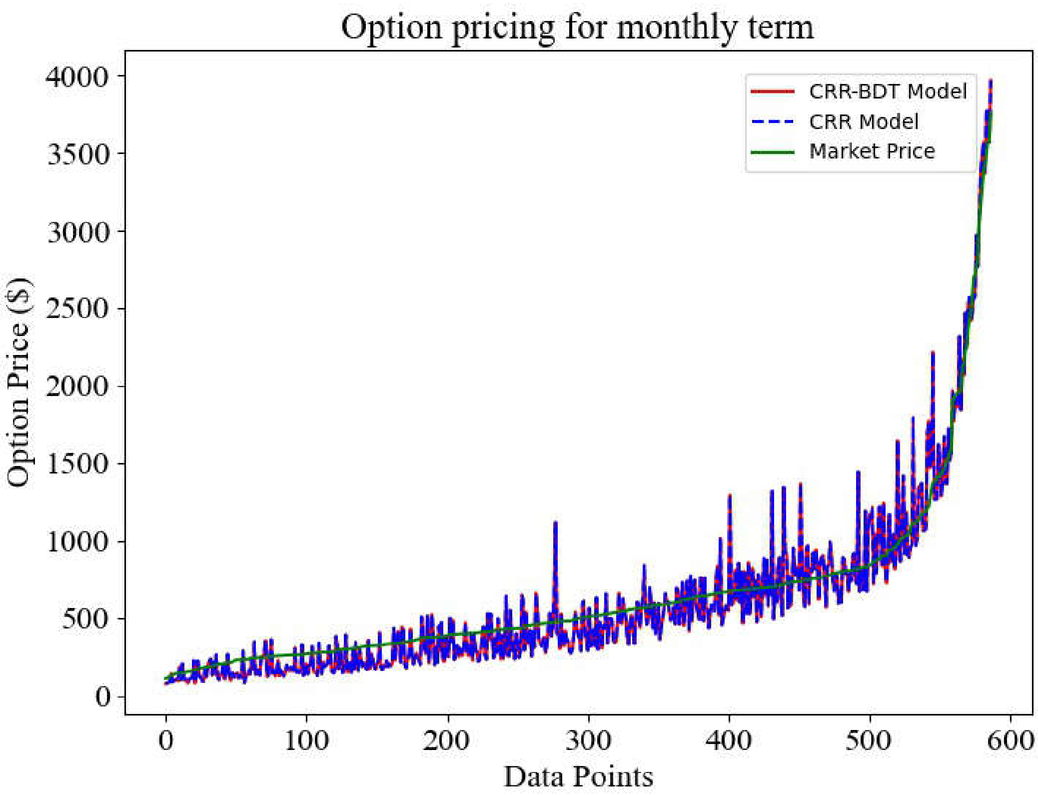

Figure 4.7. Models and Market Valuation for Monthly Pricing.

By observing the graph, it can be clearly figured out that model predictions are close to each other. Also, the estimation of the models in some points deviates a lot from the market price, which is shown as the green line. The trading in the market can be a reason for these deviations.

To sum up, the limited interest rate change probability in the short term causes the models to give close results. On the other hand, the high probability of interest rate change in the long term causes the CRR-BDT model to work more effectively. For this reason, the CRR-BDT model gives more accurate results than the CRR model.

4.2. Simulation with a Synthetic Data Example

The simulation aims to observe how the prices of options valued with two different models change as time progresses. First of all, there are two call options with the same strike price and maturity: A and B. The first one is calculated by the CRR model, while the other is calculated using the CRR-BDT model. Also, the underlying asset of these securities is a stock, and the stock price is equal to \$100 at time t. The stock movement rates (u, d) are assumed as 1.16 and 0.862, respectively. The stock tree is generated with these change rates, and the tree is the same for both models.

Besides, the possible interest rates in the following periods are found by using the BDT tree in the simulation. The zero-coupon rates used in BDT calibration are demonstrated in Table 4.12.

Table 4.12. Synthetic zero-coupon rates and volatilities.

Maturity	**Zero-Coupon Rates**	**Zero-Coupon Volatilities**	**Zero-Bond Prices ($)**
t+1	2.00%	0.18	98.039216
t+2	3.00%	0.16	94.259591
t+3	3.50%	0.14	90.194271
t+4	4.00%	0.12	85.480419
t+5	4.50%	0.10	80.245105

Additionally, the options are priced again in each step to observe price changes. The new stock price and new interest rate are required to valuation due to the expected changes in the interest rate and stock prices in market conditions. At this point, stock price and interest rate are assumed as the average of possible stock prices and possible interest rates in each specific period, respectively. In addition, it should be noticed that the stock tree is generated after each step because the expected stock price becomes the beginning node of the tree, and the tree should be calculated again with this new node.

Moreover, for option, A priced with CRR model, only $\bar{r}$ which is calculated by taking an average of possible interest rate in the step is handled. On the other hand, stock volatility and zero coupon bonds are used to generate a BDT tree for option B. Hence, while pricing option B, the change in the interest rate in the future is considered. After each step, the BDT tree is re-generated because the new current interest rate and time progress cause changes in the bond prices. Finally, option pricing is realized regarding risk-neutral probabilities and interest rates. Also, the strike price (K) and maturity date are equal to \$50 and $t + 5$, respectively.

4.2.1. Stock Price and Interest Rate Movements

Firstly, the binomial tree of stock movement is generated with the rates (u, d) as defined previously. For example, the possible stock prices in step $t + 1$ are found as

$$S_{up}(t + 1) = uS = 1.160 * 100 = 116, \tag{4.22}$$

$$S_{down}(t + 1) = dS = 0.862 * 100 = 86.2. \tag{4.23}$$

Then, the expected stock price in the $(t + 1)$ is calculated by taking average of these possible stock values. Thus, S_{t+1} equals $\frac{116+86.2}{2}$ = \$101.10. Furthermore, Figure 4.8 illustrates the stock binomial tree generated at the time $(t + 1)$. The tree begins with the price \$101.10 as calculated. Finally, the expected stock prices in all periods are shown in Figure 4.9. While pricing options in each step, it is based on these expected stock prices.

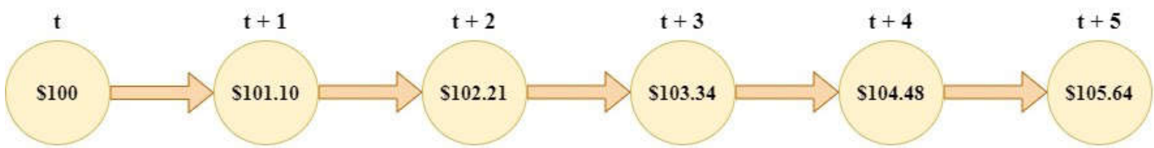

Figure 4.9. Expected Stock Prices.

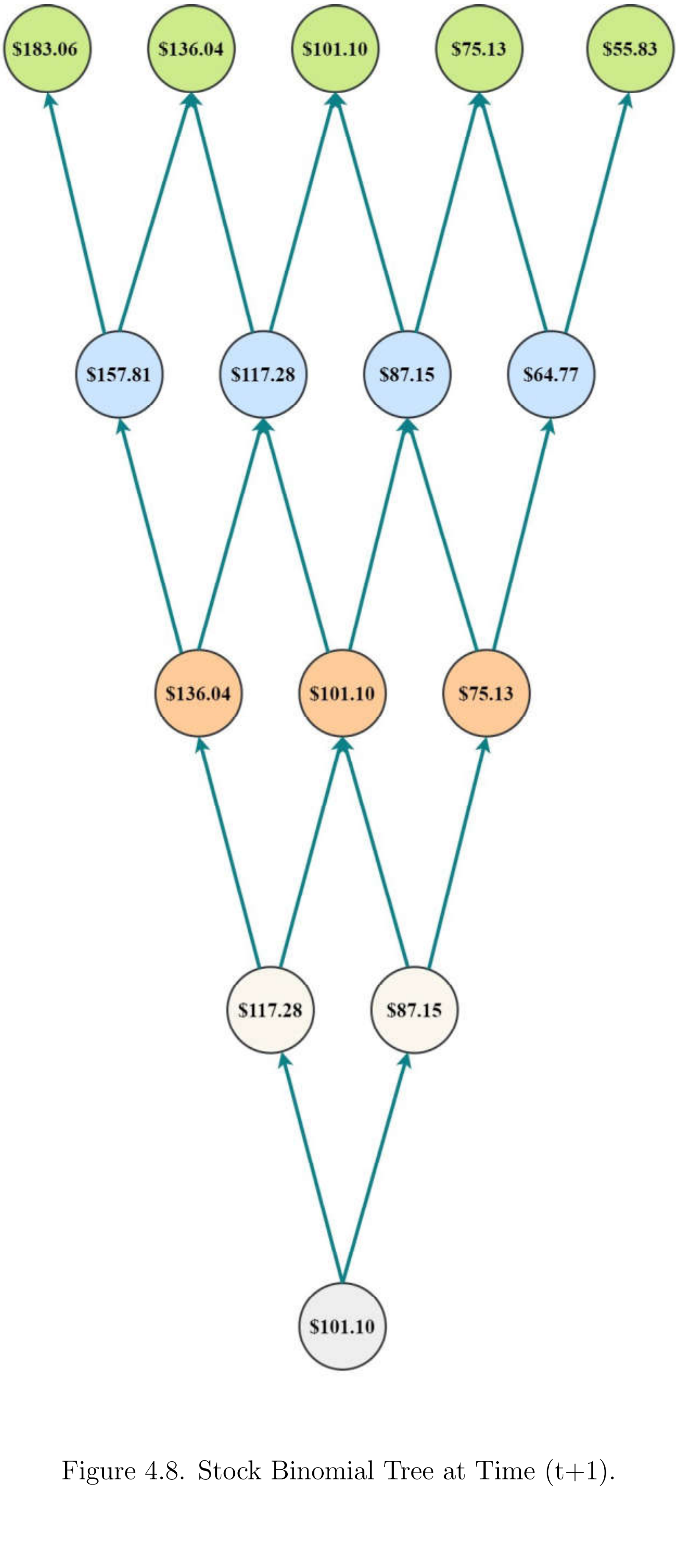

Figure 4.8. Stock Binomial Tree at Time (t+1).

Furthermore, $\bar{r}$ is calculated in every step again. The expected interest rate in a specific step is used to find discount values. Figure 4.10 shows a part of the BDT tree created at time t.

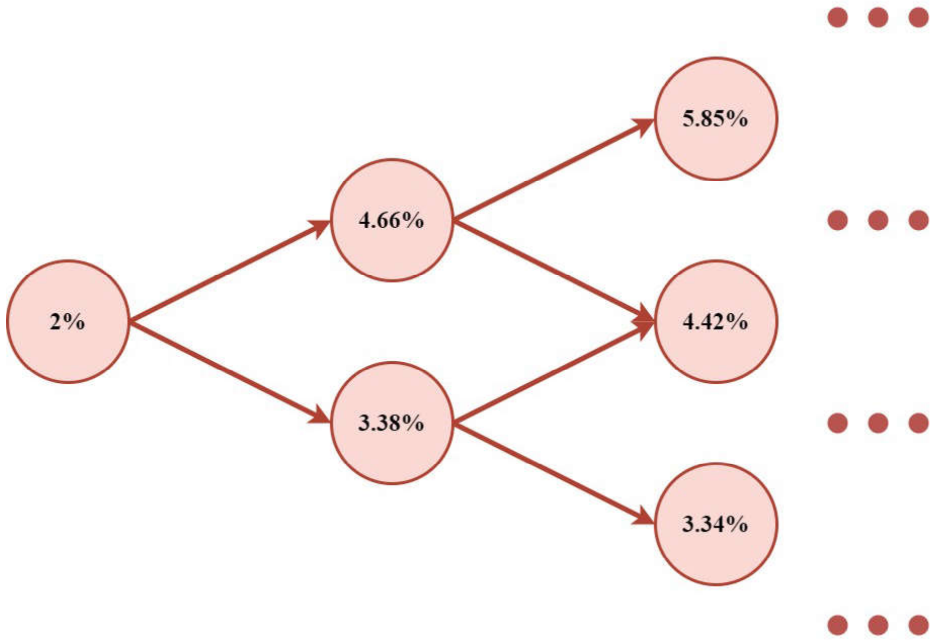

Figure 4.10. BDT Tree at Time (t).

The figure shows that possible interest rates in period 1 are 3.38% and 4.66%. Thus, the expected interest rate $\bar{r}_{(t+1)}$ in the period t+1 is found as 0.0402 by calculating the mean value of these possible interest rates. When the time reaches $t + 1$ in the simulation, the BDT tree is generated again with the new beginning node, and this implementation repeats until maturity in the simulation. Finally, Figure 4.11 shows $\bar{r}$ in all periods. These rates are used in the option pricing realized by the CRR model.

4.2.2. Option Pricing and Results

After finding parameters, option prices with both models are estimated. In Table 4.13, the estimation of the models is included. Following the results, the prices in the time series calculated by the CRR-BDT model decrease regularly. On the other hand, Option A's prices face more remarkable changes in comparison with option B as time

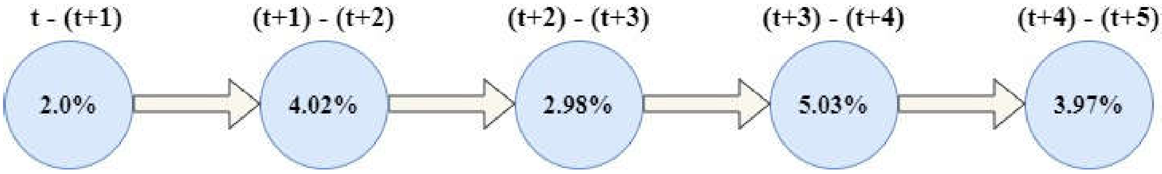

Figure 4.11. $\bar{r}$ in all Periods.

progresses.

Table 4.13. Prices of option A and B.

Option Price ($)	**t**	**t+1**	**t+2**	**t+3**	**t+4**	**t+5**
Option A (CRR)	54,80	58,537	56,495	58,135	56,433	55,640
Option B (CRR-BDT)	60,13	59,353	58,542	57,651	56,398	55,640

Furthermore, the price-time graph for both models' predictions is shown in Figure 4.12. In the graph, the degree of slope between two consecutive prices is higher in Option A in terms of logarithmic change. Moreover, the price graph of Option B is a monotonically decreasing function.

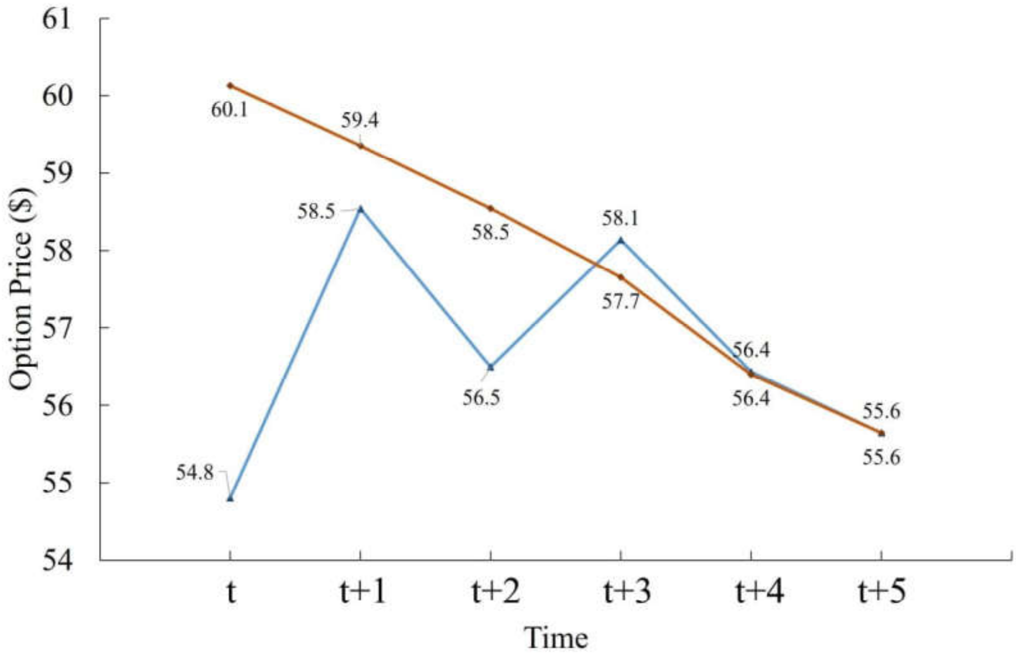

Figure 4.12. Prices of Option A and B.

Table 4.14. Changes in the prices of option A and B.

Change (log)	**Option A**	**Option B**
(t) → (t+1)	0.065969	-0.01301
(t+1) → (t+2)	-0.03551	-0.01376
(t+2) → (t+3)	0.028616	-0.01534
(t+3) → (t+4)	-0.02971	-0.02197
(t+4) → (t+5)	-0.01479	-0.01352
Abs. Avg.	0.034919	0.01552
St. Dev.	0.043274	0.003711

Additionally, Table 4.14 shows the logarithmic changes in the prices. The absolute average of the logarithmic change in the prices are 4.327% and 1.55% for options A and B. Finally, the standard deviation of the logarithmic change in the prices equals 4.32% for option A, and this value is 0.37% for option B. Hence, a pricing model like option B can be better than the option A type model to prevent sharp price changes.

5. CONCLUSION

The important securities options in financial markets are widely used in financial operations by actors. As defined before, many studies for option pricing have been conducted until now, and it is still an outstanding research topic in financial mathematics. Primarily, studies are carried out for the stochastic interest rate environment and different underlying assets.

Besides, the option in finance is an important topic since it enables investors to cost-efficiency. Also, options can give greater potential returns, and they are used in business strategies such as hedging. Due to its benefit and area of usage, option pricing is gaining importance.

We have developed a numerical option pricing model under a stochastic interest environment in this work. While generating the model, the Cox-Ross-Rubinstein option pricing model and the Black-Derman-Toy short-rate model have unified. Also, the constructed model satisfies the arbitrage-free condition. Moreover, the model is compared with the Cox-Ross-Rubinstein model that assumes the interest rate is constant throughout the option's life. Also, this comparison was based on actual option prices in the market.

To sum up, both models give similar accuracies for the maturities 2-years and less. However, the generated model estimates option prices for three and 4-years maturities slightly better. This difference proceeds from the expected interest rate changes in the longer maturities since the developed model takes into consideration the possible interest rate in the following periods. Consequently, the model can be a better alternative for the longer maturities.

Finally, what we contribute can be summarized as follows in this study:

- The model satisfies the arbitrage-free condition. Anyone cannot guarantee to make money with the option prices derived from our model.
- The model consolidates the two widely-used models: CRR and BDT. Both can work on a binomial tree; hence, our model can generate a binomial tree.
- The model provides a numerical solution by using binomial lattice. Also, it is easy to implement.
- The model regards possible interest rate changes in the future. Thus, it considers the expected interest rates in the market while pricing options.

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APPENDIX A: Python Code for Model Implementation

The code for the bootstrapping technique is in the following link.

`https://github.com/mrt4748/Master_Thesis/blob/main/bootstrapping.ipynb`

The code for the generating CRR-BDT model is in the link below.

`https://github.com/mrt4748/Master_Thesis/blob/main/CRR_BDT_Model.ipynb`

The option valuation code exists in following link.

`https://github.com/mrt4748/Master_Thesis/blob/main/option_pricing.ipynb`