IQC-BASED GAIN-SCHEDULED CONTROL OF LINEAR PERIODIC SYSTEMS USING OUTPUT FEEDBACK

by

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ABSTRACT

IQC-BASED GAIN-SCHEDULED CONTROL OF LINEAR PERIODIC SYSTEMS USING OUTPUT FEEDBACK

In this thesis, output feedback control of linear periodic systems is studied. The main approach is usign a gain-scheduled controller, where the synthesis of the controller is based on linear parameter-varying control theory and an integral quadratic constraint (IQC) result for linear periodic systems. Existence conditions for a stabilizing controller are given in terms of finite-dimensional linear matrix inequalities (LMI's).

For illustration of the application of the theory, the tracking control problem for a two-link arm system with a periodically excited base is studied. This problem is turned into an L2-gain minimization problem, which, using the aforementioned theories, can be cast as a minimization problem over a convex set defined by finite-dimensional LMI's. Using weighting functions, different controllers are designed for tracking different reference signals, in particular a unit step and a sinusoid. Nonlinear simulations are made for the closed-loop systems. The results show that the IQC result can be successfully applied to the output feedback control of linear periodic systems.

ÖZET

DOĞRUSAL PERİODİK SİSTEMLERİN ÇIKIŞ GERİ BESLEMESİ KULLANILARAK İKK-TEMELLİ KAZANÇ-AYARLAMALI DENETİMİ

Bu tezde doğrusal periodik sistemlerin çıktı geri beslemeli denetimi üzerine çalışılmıştır. Uygulanan yaklaşım kazanç-ayarlamalı denetimciler kullanmaktır. Denetimcinin sentezi, doğrusal parametre değişkenli denetim teorisine ve bir integral kuadratik kısıt (İKK) sonucuna dayanmaktadır. Sistemi kararlı duruma getiren bir denetimcinin varlık koşulları sonlu boyutlu doğrusal matris eşitsizlikleri olarak verilmiştir.

Teorinin uygulanışının gösterimi amacıyla, iki bağlantılı, temeli periodik tahrikli bir kolun izleme denetimi problemi üzerine çalışılmıştır. Bu problem sonlu sayıda doğrusal matris eşitsizliği ile ifade edilmiş bir L2-kazanç minimizasyon problemine dönüştürülebilmektedir. Ağırlık fonksiyonları kullanılarak, değişik referans sinyalleri için değişik denetimciler tasarlanmıştır. Kapalı devre sistem için doğrusal olmayan simulasyonlar yapılmıştır. Sonuçlar, mevcut integral kuadratik kısıt sonucunun başarıyla doğrusal periodik sistemlerin geri beslemeli denetimine uygulanabileceğini göstermektedir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS/ABBREVIATIONS

A, A_c	State matrices
B_c, B_p, B_{p_c}, B_u, B_w	Input matrices
$C(s)$	The nominal part of the controller
C_c, C_q, C_y, C_z	Output matrices
D_c, $D_{q_cp_c}$, D_{q_cy}, D_{up_c}	Direct feedthrough matrices of the controller
D_{qp}, D_{qu}, D_{qw}	Direct feedthrough matrices of the system
D_{yp}, D_{yw}	Direct feedthrough matrices of the system
D_{zp}, D_{zu}, D_{zw}	Direct feedthrough matrices of the system
$F(z)$	Discrete-time transfer function used in the dual IQC result
f_R	Periodic function of unit amplitude
G	The nominal part of the system
G_{cl}	The nominal part of the closed-loop system
$h(\tau)$	Time-varying effect of the base motion on the system
K	The matrix that consists of the controller parameters
k_N	Number of dominant harmonics
$\mathcal{L}_2$	The space of square integrable functions
M	Multiplier for the primal IQC
M_{cl}	Multiplier for the closed-loop system
N	Multiplier for the dual IQC
N_{cl}	Inverse of M_{cl}
q	The vector consisting of the angular positions q_1 and q_2
$R(t)$	Periodic function that describes the motion of the base
R_0	Amplitude of the periodic function $R(t)$
$\mathbb{R}$	The real line
$\mathbb{R}_e$	The extended real line, i.e., $\mathbb{R} \cup \{\infty\}$
t	Time
$u(\tau)$	The vector consisting of the control inputs u_1 and u_2
$\tilde{u}(\tau)$	The designed control input, i.e., $\tilde{u}(\tau) = u(\tau) + h(\tau)$
W_e	The weighting function for the error

W_I	The overall weighting function for the generalized disturbance
W_O	The overall weighting function for the controlled output
$W_{\dot{q}}$	The weighting function for the angular velocities
W_r	The weighting function for the reference signal
W_u	The weighting function for the input values
W_w	The weighting function for the disturbance input
X_{cl}	Inverse of Y_{cl}
Y_{cl}	Closed-loop Lyapunov matrix
α	Non-dimensional parameter
γ	$\mathcal{L}_2$-gain
Δ	Perturbation block of the system
$\boldsymbol{\Delta}$	A set of perturbation blocks defined by an IQC
$\tilde{\Delta}$	Alternative characterization of Δ in the primal IQC
$\Delta_A(\tau)$	The part of the state matrix due to unmodeled harmonics
Δ_c	Scheduling block of the controller
δ	Non-dimensional parameter
ϵ	Non-dimensional length
μ	Size of the output of Δ
μ_c	Size of the output of Δ_c
ρ	Size of the input of Δ
ρ_c	Size of the input of Δ_c
τ	Non-dimensionalized time
ψ	Non-dimensional frequency of f_R
ω	Frequency of the periodic function f_R
IQC	Integral Quadratic Constraint
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LTI	Linear Time-Invariant

1. INTRODUCTION

In this thesis, we study the output feedback control of linear periodic systems. Our approach is using a gain-scheduled controller, where the synthesis of the controller is based on linear parameter-varying (LPV) control theory [1] and an integral quadratic constraint (IQC) result for linear periodic systems [2, 3]. For illustration of the application of the theory, the tracking control problem for a parametrically excited rotating system (a two-link arm with a periodically excited base [4]) is studied.

Linear periodic systems have an important place in engineering applications. The two-link arm system studied in this thesis can be used as a preliminary model for the control of helicopter blades [5]. Another important area where linear periodic models are used is the attitude control of satellites [6]. We can also count unbalanced rotors and wind turbines as examples of linear periodic systems [2].

Control of linear periodic systems is a challenging research area. Some of the research on this subject has been concentrated on solutions of the periodic Riccati equation [7]. Some other approach is taken in [8]. In this paper, Floquet theory is used for constructing a Lyapunov-Floquet transformation that can be used in transforming linear periodic systems into linear time-invariant (LTI) systems. Then, using standard techniques developed for LTI systems, stabilizing controllers can be designed. The disadvantage of this approach is that the asymptotic stability is not guaranteed and the design process requires application of a trial-and-error procedure. Recently, an attempt has been made on guaranteeing the asymptotic stability by using the Lyapunov-Floquet transformation, the backstepping technique and the Floquet theory in a repeated procedure [9].

Stabilization of the two-link arm system with a base making simple harmonic motion is studied in [4]. Using Lyapunov-Floquet transformation, both full-state feedback and observer-based controllers are designed. In this thesis, we study the more challenging problem of reference tracking. In addition to the simple harmonic motion of the base, we also work on a more complex periodic motion form which can be expressed as an infinite sum of harmonic functions using Fourier series expansion.

The framework in which we study expresses a system as a feedback interconnection of an LTI system, G, and a perturbation block, Δ. The main principle in gain-scheduling is measuring Δ online and scheduling the controller, C, based on this measurement using a scheduling function $\Delta_c(\Delta)$. So, the controller has the same form with the system: it is represented as the feedback interconnection of C and Δ_c. In [1], existence conditions for such a controller are presented in terms linear matrix inequalities (LMIs) and a construction procedure is given. These results constitute the theoretical foundation on which the application in this thesis is based.

Synthesis of a gain-scheduled controller requires the definition of a set $\boldsymbol{\Delta}$ to which the perturbation block Δ belongs. Integral quadratic constraints are mathematical tools that are used for defining this set. In [10], they were presented to form a general framework for system analysis. Since then, the main focus has been on finding better IQC's for different types of perturbation blocks. In [2], an IQC for linear periodic systems was presented. In [3] this IQC result is used for full-state feedback control. The main motivation behind this thesis is to apply this IQC result to output feedback control of linear periodic systems.

The thesis outline is as follows: in Chapter 2, we introduce the main concepts and notations used throughout the thesis. Chapter 3 presents fundamental analysis results related with IQC's and the Kalman-Yakubovich-Popov (KYP) lemma which has a central role in control theory. In Chapter 4, we give the synthesis theorem that we use in designing gain-scheduled controllers. We also present a proof of the theorem for completeness reasons. Chapter 5 concentrates on linear periodic systems and how the aforementioned results can be applied to them. This is where we present the IQC result of [2] and [3]. In Chapter 6, we present the two-link arm system on which we study and apply our results. The tracking problem is introduced in Chapter 7, where the designs and simulation results are also presented. Finally in Chapter 8, we give a summary of the overall study made and we present our conclusions.

2. NOTATION AND DEFINITONS

We denote the space of square integrable functions that map $(-\infty, \infty)$ to $\mathbb{R}^n$ as $\mathcal{L}_2^n(-\infty, \infty)$, or shortly as $\mathcal{L}_2$. $\mathcal{L}_{2+}^n$ and $\mathcal{L}_{2-}^n$ denote the subspaces of $\mathcal{L}_2^n$ consisting of functions with the domain $[0, \infty)$ and $(-\infty, 0]$, respectively. The inner product $\langle x, y \rangle$ of two signals x and y in $\mathcal{L}_2^n$ is defined as

$$\langle x, y \rangle \triangleq \int_{-\infty}^{\infty} x(t)^T y(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(j\omega)^* \hat{y}(j\omega)\, d\omega,$$

where $\hat{x}$ and $\hat{y}$ denote the Fourier transforms of the signals x and y, respectively. Then, the $\mathcal{L}_2$-norm of a signal x in $\mathcal{L}_2^n$ can be defined as $||x||_2 \triangleq \langle x, x \rangle^{1/2}$.

The adjoint of an operator $\Delta\colon \mathcal{L}_2^\rho \to \mathcal{L}_2^\mu$ is defined as the operator $\Delta^*\colon \mathcal{L}_2^\mu \to \mathcal{L}_2^\rho$ that satisfy for all $q \in \mathcal{L}_2^\rho$ and $p \in \mathcal{L}_2^\mu$,

$$\langle \Delta q, p \rangle = \langle q, \Delta^* p \rangle .$$

Then, if Δ is a multiplication operator in time-domain, that is $\Delta(q)(t) = \Delta(t)q(t)$, Δ^* becomes simply the operator with the rule $(\Delta^* p)(t) = \Delta(t)^T p(t)$.

In order to quantify the "size" of a system G, we use the "induced 2-norm" ($\mathcal{L}_2$-gain) which is defined as

$$||G||_{i2} \triangleq \sup_{w \in \mathcal{L}_2,\, w \neq 0} \frac{||Gw||_2}{||w||_2}.$$

Then, the system G is said to be "bounded" if $||G||_{i2}$ is finite.

We say that an operator $\Pi : \mathcal{L}_2^\rho \to \mathcal{L}_2^\rho$ is "positive definite" if $\langle q, \Pi q \rangle > 0$ for all q in $\mathcal{L}_2$ and we show this condition as $\Pi > 0$. Similarly, "positive semi-definiteness" refers to the condition "$\langle q, \Pi q \rangle \geq 0$ for all q" and is shown as $\Pi \geq 0$. Definitions of "'negative definiteness" and "negative semi-definiteness" follow straightforwardly.

Now, we give some shorthand notations for matrices. We denote the number of positive, negative and zero eigenvalues of a real symmetric matrix M as n_+, n_- and n_0, respectively. Then, we define the inertia of M as $\text{in}(M) = (n_+, n_-, n_0)$. Finally, given a full column-rank matrix $M \in \mathbb{R}^{n\times m}$, $M_\perp \in \mathbb{R}^{n\times(n-m)}$ denotes the matrix whose image spans the kernel of M^T, that is, $M^T M_\perp = 0$ and $\begin{pmatrix} M & M_\perp \end{pmatrix}$ is invertible.

In certain places where the content of a matrix is obvious, we use a star $(\star)$ instead of the actual content for compactness reasons.

3. INTEGRAL QUADRATIC CONSTRAINTS AND ANALYSIS

Integral quadratic constraints form a general framework for system analysis. In this chapter, we define them and give a stability theorem that is based on IQC's, which is the main result of [10]. We also present a simplified version of this theorem which will be adequate to serve our purposes.

3.1. Integral Quadratic Constraints

In system analysis, an IQC is used to define a set $\boldsymbol{\Delta}$ to which the perturbation function Δ belongs. The set is characterized by the multiplier that defines the IQC.

Definition 3.1.1. *[10] The operator* $\Delta\colon \mathcal{L}_2^{\rho} \to \mathcal{L}_2^{\mu}$ *is said to satisfy the IQC defined by the multiplier* $\Pi\colon \mathcal{L}_2^{\rho+\mu} \to \mathcal{L}_2^{\rho+\mu}$ *if*

$$\left\langle \begin{pmatrix} I \\ \Delta \end{pmatrix} q, \Pi \begin{pmatrix} I \\ \Delta \end{pmatrix} q \right\rangle \geq 0$$

for all q *in* $\mathcal{L}_2$.

The main advantage of using an IQC is the possibility of better defining the set $\boldsymbol{\Delta}$, and thus reducing conservativity. For the same type of perturbation block, different IQC results making use of different characteristics of the perturbation block can be found.

3.2. Analysis via IQC's

Consider the basic feedback interconnection in Figure 3.1, where G is a causal linear time-invariant operator and Δ is a causal operator with bounded gain. The main result of [10] gives a stability theorem for this interconnection based on IQC's.

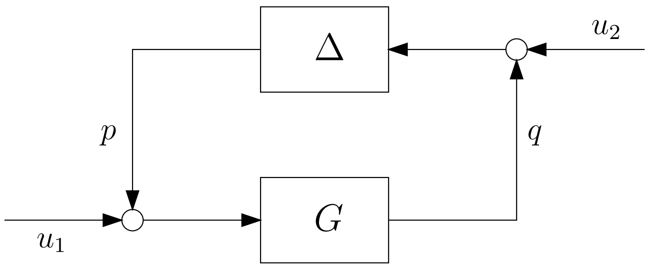

Figure 3.1. Basic feedback interconnection

Theorem 3.2.1. *[10] G - Δ feedback interconnection is stable if the followings hold:*

(i) G - τΔ feedback interconnection is well-posed for all $\tau \in [0, 1]$.

(ii) G is stable.

(iii) $\tau\Delta$ *satisfies the IQC defined by* Π *for all* $\tau \in [0, 1]$.

(iv) $\begin{pmatrix} G(j\omega) \\ I \end{pmatrix}^* \Pi(j\omega) \begin{pmatrix} G(j\omega) \\ I \end{pmatrix} < 0$ *for all* ω *in* $\mathbb{R}_e$.

Theorem 3.2.1 gives sufficient conditions for stability. In general, the multiplier Π is a real-rational transfer function that is bounded on the imaginary axis. In this thesis, we restrict ourselves to static multipliers $\Pi = \Pi^T \in \mathbb{R}^{(\rho+\mu)\times(\rho+\mu)}$ with the properties $\Pi_{11} > 0$, $\Pi_{22} < 0$, where Π is partitioned as $\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{pmatrix}$ conformable to $\begin{pmatrix} G_{cl}(j\omega) \\ I \end{pmatrix}$. Due to these properties, τ dependence can be eliminated. We also assume that the well-posedness condition is satisfied, which, in general, is the case for models of real physical systems. Definition of and information on the well-posedness condition can be found in [10]. Now, we give the so-called Kalman-Yakubovich-Popov (KYP) lemma, which is a fundamental result in control theory.

Lemma 3.2.2. *(Kalman-Yakubovich-Popov) [11] Let* $T(j\omega) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *and* $M = M^T$. *Assume that for all* ω *in* $\mathbb{R}$, $\det(j\omega I - A) \neq 0$. *Then, the following conditions are equivalent:*

(i) $T(j\omega)^* M T(j\omega) < 0$ *for all* ω *in* $\mathbb{R}_e$.

(ii) There exists a $P = P^T$ *such that*

$$\begin{pmatrix} A & B \\ C & D \\ I & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & P \\ 0 & M & 0 \\ P & 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \\ I & 0 \end{pmatrix} < 0.$$

Remark 3.2.3. If $C^T MC \geq 0$, then in the above lemma, A is Hurwitz stable if and only if $P = P^T > 0$.

A proof of the KYP lemma can be found in [11]. With the assumptions on M_{cl} and using the KYP lemma, we give a simplified version of Theorem 3.2.1.

Theorem 3.2.4. *Assume that the* G *-* Δ *feedback interconnection is well-posed, where* $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *and* Δ *is a multiplication operator in time-domain. Then, the interconnection is stable if the followings hold:*

(i) Δ *satisfies the IQC defined by* Π*, where* $\Pi_{11} > 0$ *and* $\Pi_{22} < 0$*.*

(ii) There exists a $P = P^T > 0$ *such that*

$$\left(\begin{array}{cc} A & B \\ \hline C & D \\ 0 & I \\ \hline I & 0 \end{array}\right)^T \begin{pmatrix} 0 & 0 & P \\ 0 & \Pi & 0 \\ P & 0 & 0 \end{pmatrix} \left(\begin{array}{cc} A & B \\ \hline C & D \\ 0 & I \\ \hline I & 0 \end{array}\right) < 0.$$

We will use this analysis result for designing stabilizing controllers. Now, we give a standard proof for it.

Proof. We have the state-space equation

$$\begin{aligned} \dot{x} &= Ax + Bp, \\ q &= Cx + Dp, \end{aligned}$$

and $p = \Delta q$. For stability, we want to show the existence of a Lyapunov function V which is positive definite and decreasing over all state trajectories. Let V be defined as

$$V(x) = x^T P x$$

for some $P = P^T > 0$. Then, we need

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + Bp)^T P x + x^T P (Ax + Bp) \\ &= \begin{pmatrix} x \\ p \end{pmatrix}^T \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} < 0.\end{aligned}$$

Due to the assumption that Δ is a multiplication operator, Condition (i) implies

$$\begin{pmatrix} I \\ \Delta(t) \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta(t) \end{pmatrix} > 0$$

for all t in $(-\infty, \infty)$. Then, for all q,

$$q^T \begin{pmatrix} I \\ \Delta \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta \end{pmatrix} q > 0$$

which is equivalent to

$$\begin{pmatrix} q \\ p \end{pmatrix}^T \Pi \begin{pmatrix} q \\ p \end{pmatrix} > 0.$$

We can write above inequality as

$$\begin{pmatrix} x \\ p \end{pmatrix}^T \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^T \Pi \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} > 0.$$

By adding this positive quantity to $\dot{V}(x)$ and requiring negativity of the sum, we can ensure negativity of $\dot{V}(x)$:

$$\begin{pmatrix} x \\ p \end{pmatrix}^T \left\{ \begin{pmatrix} \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \begin{pmatrix} \star \\ \star \end{pmatrix}^T \Pi \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \right\} \begin{pmatrix} x \\ p \end{pmatrix} < 0$$

In order to ensure that above inequlity is satisfied for all $\begin{pmatrix} x \\ p \end{pmatrix}$, we impose

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^T \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^T \Pi \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} < 0$$

which can be equivalently written as

$$\left(\begin{array}{cc} A & B \\ \hline C & D \\ 0 & I \\ \hline I & 0 \end{array}\right)^T \begin{pmatrix} 0 & 0 & P \\ 0 & \Pi & 0 \\ P & 0 & 0 \end{pmatrix} \left(\begin{array}{cc} A & B \\ \hline C & D \\ 0 & I \\ \hline I & 0 \end{array}\right) < 0.$$

This completes the proof. □

4. SYNTHESIS

In this chapter, using the stability result presented in Theorem 3.2.4, given a G-Δ feedback interconnection, we want to reach at existence conditions for a gain-scheduled output feedback controller C-Δ_c that stabilizes the system (Figure 4.1). We

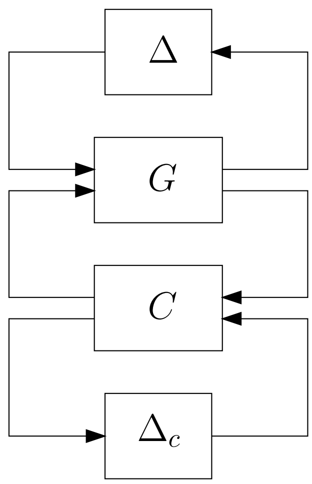

Figure 4.1. The system and the controller

can represent the closed-loop system as in Figure 4.2, where $\Delta_{cl} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_c \end{bmatrix}$.

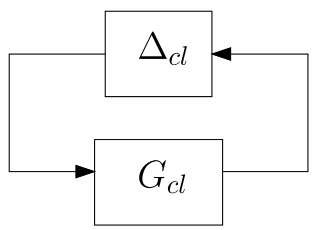

Figure 4.2. The closed loop system

4.1. IQC-Synthesis Theorem

We consider a system of the form:

$$\begin{aligned} \dot{x} &= Ax + B_p p + B_u u \\ q &= C_q x + D_{qp} p + D_{qu} u \quad \text{and} \quad p = \Delta q, \\ y &= C_y x + D_{yp} p \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $p(t) \in \mathbb{R}^\mu$, $u(t) \in \mathbb{R}^m$, $q(t) \in \mathbb{R}^\rho$, $y(t) \in \mathbb{R}^\nu$ and Δ is a bounded multiplication operator. The controller we design will have the form:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_{p_c} p_c + B_c y \\ q_c &= C_{q_c} x_c + D_{q_c p_c} p_c + D_{q_c y} y \quad \text{and} \quad p_c = \Delta_c q_c, \\ u &= C_c x_c + D_{u p_c} p_c + D_c y \end{aligned} \tag{4.2}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, $p_c(t) \in \mathbb{R}^{\mu_c}$, $q_c(t) \in \mathbb{R}^{\rho_c}$ and Δ_c is a bounded multiplication operator. Following result of [1] gives existence conditions for such a controller and a procedure for its construction.

Theorem 4.1.1. *[1] Let the primal IQC, defined by $M = M^T \in \mathbb{R}^{(\rho+\mu)\times(\rho+\mu)}$,*

$$\left\langle \begin{pmatrix} v \\ \Delta(v) \end{pmatrix}, M \begin{pmatrix} v \\ \Delta(v) \end{pmatrix} \right\rangle \geq 0 \qquad \forall\, v \in \mathcal{L}_2^\rho, \tag{4.3}$$

and the dual IQC, defined by $N = N^T \in \mathbb{R}^{(\rho+\mu)\times(\rho+\mu)}$,

$$\left\langle \begin{pmatrix} -\Delta^*(w) \\ w \end{pmatrix}, N \begin{pmatrix} -\Delta^*(w) \\ w \end{pmatrix} \right\rangle \leq 0 \qquad \forall\, w \in \mathcal{L}_2^\mu, \tag{4.4}$$

be satisfied and let the multipliers M and N have the properties

$$\begin{pmatrix} M_{11} & 0 \\ 0 & N_{11} \end{pmatrix} > 0, \quad \begin{pmatrix} M_{22} & 0 \\ 0 & N_{22} \end{pmatrix} < 0. \tag{4.5}$$

Assume that there exist matrices $X = X^T \in \mathbb{R}^{n\times n}$ *and* $Y = Y^T \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} \star \\ \star \end{pmatrix}_\perp^T \begin{pmatrix} \star \\ \hline \star \\ \star \\ \hline \star \end{pmatrix}^T \begin{pmatrix} 0 & 0 & Y \\ 0 & M & 0 \\ Y & 0 & 0 \end{pmatrix} \begin{pmatrix} A & B_p \\ \hline C_q & D_{qp} \\ 0 & I \\ \hline I & 0 \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \end{pmatrix}_\perp < 0, \tag{4.6}$$

$$\begin{pmatrix} \star \\ \star \end{pmatrix}_\perp^T \begin{pmatrix} \star \\ \hline \star \\ \star \\ \hline \star \end{pmatrix}^T \begin{pmatrix} 0 & 0 & X \\ 0 & N & 0 \\ X & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \hline 0 & I \\ -B_p^T & -D_{qp}^T \\ \hline -A^T & -C_q^T \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \end{pmatrix}_\perp > 0, \tag{4.7}$$

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0. \tag{4.8}$$

Then, there exist

(i) a multiplier $M_{cl} = M_{cl}^T \in \mathbb{R}^{(\rho+\rho_c+\mu+\mu_c)\times(\rho+\rho_c+\mu+\mu_c)}$ *with* $\mathrm{in}\,(M_{cl}) = (\rho+\rho_c, \mu+\mu_c, 0)$ *such that*

$$M_{cl} = P^T \begin{pmatrix} M & \star \\ \star & \star \end{pmatrix} P, \quad M_{cl}^{-1} = P^T \begin{pmatrix} N & \star \\ \star & \star \end{pmatrix} P, \quad P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

(ii) a $\Delta_c(\Delta)$ *such that*

$$\left\langle \begin{pmatrix} v_{cl} \\ \Delta_{cl}(v_{cl}) \end{pmatrix}, M_{cl} \begin{pmatrix} v_{cl} \\ \Delta_{cl}(v_{cl}) \end{pmatrix} \right\rangle \geq 0 \quad \forall\, v_{cl} \in \mathcal{L}_2^{\rho+\rho_c},$$

where $\Delta_{cl} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_c \end{bmatrix}$,

(iii) a controller of the form (4.2) that renders the closed-loop system stable.

Due to its central role in this study, we give the proof of this theorem in detail.

Proof. The closed-loop system can be represented by $G_{cl} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$, where

$$A_{cl} \triangleq \tilde{A} + \tilde{B}_u K \tilde{C}_y, \quad B_{cl} \triangleq \tilde{B}_p + \tilde{B}_u K \tilde{D}_{yp},$$
$$C_{cl} \triangleq \tilde{C}_q + \tilde{D}_{qu} K \tilde{C}_y, \quad D_{cl} \triangleq \tilde{D}_{qp} + \tilde{D}_{qu} K \tilde{D}_{yp},$$

and

$$\tilde{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_u \triangleq \begin{bmatrix} B_u & 0 & 0 \\ 0 & I_{n_c} & 0 \end{bmatrix}, \quad \tilde{C}_y \triangleq \begin{bmatrix} C_y & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix},$$
$$\tilde{B}_p \triangleq \begin{bmatrix} B_p & 0 \\ 0 & 0_{n_c \times \mu_c} \end{bmatrix}, \quad \tilde{C}_q \triangleq \begin{bmatrix} C_q & 0 \\ 0 & 0_{\rho_c \times n_c} \end{bmatrix}, \quad \tilde{D}_{yp} \triangleq \begin{bmatrix} D_{yp} & 0 \\ 0 & 0 \\ 0 & I_{\mu_c} \end{bmatrix},$$
$$\tilde{D}_{qu} \triangleq \begin{bmatrix} D_{qu} & 0 & 0 \\ 0 & 0 & I_{\rho_c} \end{bmatrix}, \quad \tilde{D}_{qp} \triangleq \begin{bmatrix} D_{qp} & 0 \\ 0 & 0_{\rho_c \times \mu_c} \end{bmatrix}, \quad K \triangleq \begin{bmatrix} D_c & C_c & D_{up_c} \\ B_c & A_c & B_{p_c} \\ D_{q_c y} & C_{q_c} & D_{q_c p_c} \end{bmatrix}.$$

Now, we want to show that the hypothesis of the theorem implies the existence of a controller matrix K, a scheduling function $\Delta_c(\Delta)$ and a closed-loop multiplier M_{cl} with which the feedback interconnection G_{cl} - Δ_{cl} can be proven to be stable. We look for the existence of a static multiplier M_{cl} that satisfies

$$M_{cl_{11}} > 0, \quad M_{cl_{22}} < 0, \tag{4.9}$$

where M_{cl} is partitioned as

$$M_{cl} = \begin{pmatrix} M_{cl_{11}} & M_{cl_{12}} \\ M_{cl_{12}}^T & M_{cl_{22}} \end{pmatrix}$$

conformable to $\begin{pmatrix} I \\ \Delta_{cl} \end{pmatrix}$. Imposing the constraints (4.9) on M_{cl} brings

(i) availability of a matrix elimination lemma (Lemma A.2.1),

(ii) removal of the τ dependence in Theorem 3.2.1,

(iii) equivalence of Hurwitz stability of G_{cl} to the positive-definiteness of Y_{cl}, which will be introduced below, in the application of KYP lemma (see Remark 3.2.3).

By Theorem 3.2.4, this interconnection is stable if for some multiplier M_{cl} and $\Delta_c(\Delta)$ the IQC

$$\left\langle \begin{pmatrix} v_{cl} \\ \Delta_{cl}(v_{cl}) \end{pmatrix}, M_{cl} \begin{pmatrix} v_{cl} \\ \Delta_{cl}(v_{cl}) \end{pmatrix} \right\rangle \geq 0 \quad \forall\, v_{cl} \in \mathcal{L}_2^{\rho+\rho_c}, \tag{4.10}$$

is satisfied and there exists a Lyapunov matrix $Y_{cl} = Y_{cl}^T > 0$ that satisfy

$$\left(\begin{array}{cc} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \\ 0 & I \\ \hline I & 0 \end{array}\right)^T \begin{pmatrix} 0 & 0 & Y_{cl} \\ 0 & M_{cl} & 0 \\ Y_{cl} & 0 & 0 \end{pmatrix} \left(\begin{array}{cc} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \\ 0 & I \\ \hline I & 0 \end{array}\right) < 0. \tag{4.11}$$

First, we will concentrate on the Condition (4.11). By a simple permutation, this can be equivalently written as

$$\begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \\ I & 0 \\ 0 & I \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & 0 & Y_{cl} & 0 \\ 0 & M_{cl_{11}} & 0 & M_{cl_{12}} \\ Y_{cl} & 0 & 0 & 0 \\ 0 & M_{cl_{12}}^T & 0 & M_{cl_{22}} \end{pmatrix}}_{\triangleq \Pi(M_{cl}, Y_{cl})} \begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \\ I & 0 \\ 0 & I \end{pmatrix} < 0.$$

Let

$$\begin{aligned}\begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} &= \begin{pmatrix} \tilde{A} + \tilde{B}_u K \tilde{C}_y & \tilde{B}_p + \tilde{B}_u K \tilde{D}_{yp} \\ \tilde{C}_q + \tilde{D}_{qu} K \tilde{C}_y & \tilde{D}_{qp} + \tilde{D}_{qu} K \tilde{D}_{yp} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A} & \tilde{B}_p \\ \tilde{C}_q & \tilde{D}_{qp} \end{pmatrix} + \begin{pmatrix} \tilde{B}_u \\ \tilde{D}_{qu} \end{pmatrix} K \begin{pmatrix} \tilde{C}_y & \tilde{D}_{yp} \end{pmatrix} \\ &\triangleq U_A + U_B K U_C. \end{aligned}$$

Then, we have the equivalent condition

$$\begin{pmatrix} U_A + U_B K U_C \\ I \end{pmatrix}^T \Pi(M_{cl}, Y_{cl}) \begin{pmatrix} U_A + U_B K U_C \\ I \end{pmatrix} < 0. \tag{4.12}$$

Now, in order to use Lemma A.2.1, we first must show that

$$\text{in}(\Pi) = (\rho + \rho_c + n + n_c, \mu + \mu_c + n + n_c, 0)\,.$$

For this purpose, first note that

$$\begin{aligned}\text{in}\,(\Pi) &= \text{in} \begin{pmatrix} 0 & Y_{cl} & 0 \\ Y_{cl} & 0 & 0 \\ 0 & 0 & M_{cl} \end{pmatrix} \\ &= \text{in} \begin{pmatrix} 0 & Y_{cl} \\ Y_{cl} & 0 \end{pmatrix} + \text{in}\,(M_{cl}). \end{aligned}$$

Since $\begin{pmatrix} 0 & Y_{cl} \\ Y_{cl} & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & -Y_{cl} \\ -Y_{cl} & 0 \end{pmatrix}$, we have

$$\begin{aligned}\text{in} \begin{pmatrix} 0 & Y_{cl} \\ Y_{cl} & 0 \end{pmatrix} &= \text{in} \begin{pmatrix} 0 & -Y_{cl} \\ -Y_{cl} & 0 \end{pmatrix} \\ &= (n, n, 0). \end{aligned}$$

Due to the constraints (4.9), we have $\mathrm{in}\,(M_{cl}) = (\rho + \rho_c, \mu + \mu_c)$, which leads to $\mathrm{in}\,(\Pi) = (\rho + \rho_c + n + n_c, \mu + \mu_c + n + n_c, 0)$. Now, we can conclude by Lemma A.2.1 that there exist a multiplier M_{cl}, a Lyapunov function Y_{cl} and a controller matrix K that satisfy (4.12) if and only if there exist M_{cl}, Y_{cl} that satisfy

$$(U_C^T)_\perp^T \begin{pmatrix} U_A \\ I \end{pmatrix}^T \Pi(M_{cl}, Y_{cl}) \begin{pmatrix} U_A \\ I \end{pmatrix} (U_C^T)_\perp < 0 \tag{4.13}$$

and

$$(U_B)_\perp^T \begin{pmatrix} I \\ -U_A^T \end{pmatrix}^T \Pi(M_{cl}, Y_{cl})^{-1} \begin{pmatrix} I \\ -U_A^T \end{pmatrix} (U_B)_\perp > 0. \tag{4.14}$$

It can be shown that

$$(U_C^T)_\perp = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \end{pmatrix}_\perp .$$

Then,

$$\begin{pmatrix} U_A \\ I \end{pmatrix} (U_C^T)_\perp = \begin{pmatrix} A & 0 & B_p & 0 \\ 0 & 0 & 0 & 0 \\ C_q & 0 & D_{qp} & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \end{pmatrix}_\perp$$

$$= \begin{pmatrix} A & B_p \\ 0 & 0 \\ C_q & D_{qp} \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \end{pmatrix}_{\perp}. \tag{4.15}$$

Let the partitionings

$$M_{cl_{11}} = \begin{pmatrix} M_{cl_{11/11}} & M_{cl_{11/12}} \\ M_{cl_{11/12}}^T & M_{cl_{11/22}} \end{pmatrix}, \quad M_{cl_{12}} = \begin{pmatrix} M_{cl_{12/11}} & M_{cl_{12/12}} \\ M_{cl_{11/21}} & M_{cl_{12/22}} \end{pmatrix},$$

$$M_{cl_{22}} = \begin{pmatrix} M_{cl_{22/11}} & M_{cl_{22/12}} \\ M_{cl_{22/12}}^T & M_{cl_{22/22}} \end{pmatrix},$$

and

$$Y_{cl} = \begin{pmatrix} Y_{cl_{11}} & Y_{cl_{12}} \\ Y_{cl_{12}}^T & Y_{cl_{22}} \end{pmatrix}.$$

Then, due to the zero rows in (4.15), by simple multiplication, Inequality (4.13) becomes

$$\begin{pmatrix} \star \\ \star \end{pmatrix}_{\perp}^T \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & 0 & Y_{cl_{11}} & 0 \\ 0 & M_{cl_{11/11}} & 0 & M_{cl_{12/11}} \\ Y_{cl_{11}} & 0 & 0 & 0 \\ 0 & M_{cl_{12/11}}^T & 0 & M_{cl_{22/11}} \end{pmatrix} \begin{pmatrix} A & B_p \\ C_q & D_{qp} \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \end{pmatrix}_{\perp} < 0,$$

which is equivalent to the LMI (4.6) with the definitions

$$M \triangleq \begin{pmatrix} M_{cl_{11/11}} & M_{cl_{12/11}} \\ M_{cl_{12/11}}^T & M_{cl_{22/11}} \end{pmatrix}, \quad Y \triangleq Y_{cl_{11}},$$

and with a simple permutation of rows and columns. In order to show that the Inequality (4.14) is equivalent to the LMI (4.7), first note that

$$U_{B_\perp} = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \end{pmatrix}_\perp .$$

Then, we have

$$\begin{aligned} \begin{pmatrix} I \\ -U_A^T \end{pmatrix} U_{B_\perp} &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -A^T & 0 & -C_q^T & 0 \\ 0 & 0 & 0 & 0 \\ -B_p^T & 0 & -D_{qp}^T & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \end{pmatrix}_\perp \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ -A^T & -C_q^T \\ 0 & 0 \\ -B_p^T & -D_{qp}^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \end{pmatrix}_\perp . \end{aligned} \tag{4.16}$$

Define $N_{cl} = M_{cl}^{-1}$ and $X_{cl} = Y_{cl}^{-1}$. Then, Π^{-1} can be written as

$$\Pi^{-1} = \begin{pmatrix} 0 & 0 & X_{cl} & 0 \\ 0 & N_{cl_{11}} & 0 & N_{cl_{12}} \\ X_{cl} & 0 & 0 & 0 \\ 0 & N_{cl_{12}}^T & 0 & N_{cl_{22}} \end{pmatrix}.$$

Let

$$N_{cl_{11}} = \begin{pmatrix} N_{cl_{11/11}} & N_{cl_{11/12}} \\ N_{cl_{11/12}}^T & N_{cl_{11/22}} \end{pmatrix}, \quad N_{cl_{12}} = \begin{pmatrix} N_{cl_{12/11}} & N_{cl_{12/12}} \\ N_{cl_{11/21}} & N_{cl_{12/22}} \end{pmatrix},$$

$$N_{cl_{22}} = \begin{pmatrix} N_{cl_{22/11}} & N_{cl_{22/12}} \\ N_{cl_{22/12}}^T & N_{cl_{22/22}} \end{pmatrix},$$

and

$$X_{cl} = \begin{pmatrix} X_{cl_{11}} & X_{cl_{12}} \\ X_{cl_{12}}^T & X_{cl_{22}} \end{pmatrix}.$$

Then, by simple multiplication and a permutation, Inequality (4.14) becomes

$$\begin{pmatrix} \star \\ \star \end{pmatrix}_{\perp}^T \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & X_{cl_{11}} \\ 0 & N_{cl_{11/11}} & N_{cl_{12/11}} & 0 \\ 0 & N_{cl_{12/11}}^T & N_{cl_{22/11}} & 0 \\ X_{cl_{11}} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ -B_p^T & -D_{qp}^T \\ -A^T & -C_q^T \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \end{pmatrix}_{\perp} > 0.$$

We define $N \triangleq \begin{pmatrix} N_{cl_{11/11}} & N_{cl_{12/11}} \\ N_{cl_{12/11}}^T & N_{cl_{22/11}} \end{pmatrix}$, $X \triangleq X_{cl_{11}}$, and obtain the LMI (4.7).

We can force the relation $Y_{cl}^{-1} = X_{cl}$ by imposing the LMI (4.8), since if LMI (4.8) is satisfied, we can always construct a $Y_{cl} = \begin{pmatrix} Y & Y_{cl_{12}} \\ Y_{cl_{12}}^T & Y_{cl_{22}} \end{pmatrix}$ with $Y_{cl}^{-1} = \begin{pmatrix} X & \star \\ \star & \star \end{pmatrix}$. In this case, any $Y_{cl_{12}}$ and $Y_{cl_{22}}$ work as long as

$$Y_{cl_{12}} Y_{cl_{22}}^{-1} Y_{cl_{12}}^T = Y - X^{-1}.$$

One can take $Y_{cl_{12}} = I$ and $Y_{cl_{22}} = (Y - X^{-1})^{-1}$.

We have shown that feasibility of the linear matrix inequalities (4.6), (4.7) and (4.8) with the relation

$$P^T \begin{pmatrix} M & \star \\ \star & \star \end{pmatrix}^{-1} P = P^T \begin{pmatrix} N & \star \\ \star & \star \end{pmatrix} P \tag{4.17}$$

and the constraints (4.5) is equivalent to the existence of a multiplier M_{cl} and a controller matrix K with which the Condition (4.11) is satisfied. Having shown that there are some possible choices of M_{cl}'s, the question is if it is possible to satisfy the IQC (4.10) defined by one such M_{cl} with some $\Delta_c(\Delta)$. We will show that it is, by actually constructing such an M_{cl} and Δ_c using the procedure presented in [1].

Construction of the Extended Multiplier $\mathbf{M_{cl}}$: [1] We want to find an extended multiplier that satisfies

$$M_{cl} = P^T \begin{pmatrix} M & \star \\ \star & \star \end{pmatrix} P, \quad M_{cl}^{-1} = P^T \begin{pmatrix} N & \star \\ \star & \star \end{pmatrix} P \tag{4.18}$$

and

$$M_{cl_{11}} > 0, \quad M_{cl_{22}} < 0. \tag{4.19}$$

We parameterize M_{cl} by a non-singular matrix T and a symmetric matrix E as

$$M_{cl} = P^T \begin{pmatrix} M & T \\ T^T & T^T E T \end{pmatrix} P.$$

If we choose E as $E = \left(M - N^{-1}\right)^{-1}$, then any non-singular T satisfies the coupling constraint (4.18). We want to find the suitable T matrix such that inequalities (4.19) are also satisfied. Let

$$Z_1 \triangleq \begin{pmatrix} I_\rho \\ 0 \end{pmatrix}, \quad Z_2 \triangleq \begin{pmatrix} I_{\rho_c} \\ 0 \end{pmatrix}, \quad Z_{1_\perp} \triangleq \begin{pmatrix} 0 \\ I_\mu \end{pmatrix}, \quad Z_{2_\perp} \triangleq \begin{pmatrix} 0 \\ I_{\mu_c} \end{pmatrix},$$

where the values of ρ_c and μ_c are not determined yet, but they are assumed to satisfy $\rho_c + \mu_c = \rho + \mu$. Then, Condition (4.19) is satisfied if and only if

$$\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}^T \begin{pmatrix} M & T \\ T^T & T^T E T \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} > 0$$

and

$$\begin{pmatrix} Z_{1_\perp} & 0 \\ 0 & Z_{2_\perp} \end{pmatrix}^T \begin{pmatrix} M & T \\ T^T & T^T E T \end{pmatrix} \begin{pmatrix} Z_{1_\perp} & 0 \\ 0 & Z_{2_\perp} \end{pmatrix} < 0.$$

Since $Z_1^T M Z_1 > 0$ and $Z_{1_\perp}^T M Z_{1_\perp} < 0$, above conditions are satisfied if and only if

$$T_1^T \left[E - Z_1 \left(Z_1^T M Z_1\right)^{-1} Z_1^T\right] T_1 > 0$$

and

$$T_2^T \left[E - Z_{1_\perp} \left(Z_{1_\perp}^T M Z_{1_\perp}\right)^{-1} Z_{1_\perp}^T\right] T_2 < 0,$$

where $\begin{bmatrix} T_1 & T_2 \end{bmatrix} \triangleq \begin{bmatrix} T Z_2 & T Z_{2_\perp} \end{bmatrix}$. We can construct such T_1 and T_2 by choosing T_1 as a basis for the positive eigenspace of $\left[E - Z_1 \left(Z_1^T M Z_1\right)^{-1} Z_1^T\right]$ and T_2 as a basis for the negative eigenspace of $\left[E - Z_{1_\perp}\left(Z_{1_\perp}^T M Z_{1_\perp}\right)^{-1} Z_{1_\perp}^T\right]$, but we have to prove that ρ_c (column number of T_1) and μ_c (column number of T_2) satisfy $\rho_c + \mu_c = \rho + \mu$. First, note that

$$\rho_c = n_+\left(E - Z_1\left(Z_1^T M Z_1\right)^{-1} Z_1^T\right)$$

and

$$\mu_c = n_-\left(E - Z_{1_\perp}\left(Z_{1_\perp}^T M Z_{1_\perp}\right)^{-1} Z_{1_\perp}^T\right).$$

In order to find about $n_+\left(E - Z_1\left(Z_1^T M Z_1\right)^{-1} Z_1^T\right)$, which will give us information about possible choices of T_1, we look at

$$n_+\left(\begin{bmatrix} Z_1^T M Z_1 & Z_1^T \\ Z_1 & E \end{bmatrix}\right)$$

which equals

$$n_+\left(E - Z_1\left(Z_1^T M Z_1\right)^{-1} Z_1^T\right) + n_+\left(Z_1^T M Z_1\right) \quad \text{and} \quad n_+\left(Z_1^T N^{-1} Z_1\right) + n_+(E).$$

Since we have $Z_{1_\perp}^T N Z_{1_\perp} < 0$, by duality we have $Z_1^T N^{-1} Z_1 > 0$. Noting that $Z_1^T M Z_1 > 0$, this leads to

$$n_+\left(E - Z_1\left(Z_1^T M Z_1\right)^{-1} Z_1^T\right) = n_+(E).$$

With a similar argument we can also conclude that

$$n_-\left(E - Z_{1_\perp}\left(Z_{1_\perp}^T M Z_{1_\perp}\right)^{-1} Z_{1_\perp}^T\right) = n_-(E).$$

Since $n_+(E) + n_-(E) = \rho + \mu$, we have proved that our assumption $\rho_c + \mu_c = \rho + \mu$ is valid. Then, we can take $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$.

Construction of $\boldsymbol{\Delta_c(\Delta)}$: [1] By Lemma A.2.1, conditions (4.3) and (4.4) imply the existence of a $\Delta_c(\Delta)$ with which the IQC (4.10) is satisfied. It is shown in [1] that one possible choice of Δ_c is

$$\Delta_c(\Delta) = -W_{22} + \begin{pmatrix} W_{21} & V_{21} \end{pmatrix} \begin{pmatrix} U_{11} & \star \\ W_{11}+\Delta & V_{11} \end{pmatrix}^{-1} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix},$$

where $U \triangleq M_{cl_{11}} - M_{cl_{12}} M_{cl_{22}}^{-1} M_{cl_{12}}^T$, $V \triangleq -M_{cl_{22}}^{-1}$ and $W \triangleq M_{cl_{22}}^{-1} M_{cl_{12}}^T$.

Construction of the Controller: [3] Inequality (4.12) can be written as

$$\begin{pmatrix} I \\ KU_C \end{pmatrix}^T \underbrace{\begin{pmatrix} U_A & U_B \\ I & 0 \end{pmatrix}^T \Pi \begin{pmatrix} U_A & U_B \\ I & 0 \end{pmatrix}}_{\Phi} \begin{pmatrix} I \\ KU_C \end{pmatrix} < 0.$$

By Lemma A.3.3, we can conclude that $n_-(\Phi) \geq n+n_c+\mu+\mu_c$. Assuming that U_B is full column-rank, by Lemma A.3.2 we have $n_-(\Phi) \leq n_-(\Pi) = n+n_c+\mu+\mu_c$, which leads to

$$n_-(\Phi) = n+n_c+\mu+\mu_c.$$

Since Φ is nonsingular, we can conclude that $\mathrm{in}\,(\Phi) = (m+n_c+\rho_c, n+n_c+\mu+\mu_c, 0)$. Then, using Corollary A.1.2, we can conclude that

$$\begin{pmatrix} -U_C^T K^T \\ I \end{pmatrix}^T \Phi^{-1} \begin{pmatrix} -U_C^T K^T \\ I \end{pmatrix} > 0$$

which is equivalent to

$$\begin{pmatrix} K^T \\ I \end{pmatrix}^T \Gamma \begin{pmatrix} K^T \\ I \end{pmatrix} > 0$$

where

$$\Gamma \triangleq \begin{pmatrix} -U_C^T & 0 \\ 0 & I \end{pmatrix}^T \Phi^{-1} \begin{pmatrix} -U_C^T & 0 \\ 0 & I \end{pmatrix}.$$

Then, if we take $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ as a basis for the positive eigenspace of Γ, we can write

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^T \Gamma \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} > 0.$$

Pre- and post-multiplication by V_2^{-T} and V_2^{-1} gives

$$\begin{pmatrix} V_1 V_2^{-1} \\ I \end{pmatrix}^T \Gamma \begin{pmatrix} V_1 V_2^{-1} \\ I \end{pmatrix} > 0.$$

This shows that

$$K = V_2^{-T} V_1^T$$

is one possible choice for the controller matrix K.

$\square$

4.2. $\mathcal{L}_2$-Gain Performance Design

By a slight modification, we can obtain a formulation for stabilizing a system

$$\begin{aligned} \dot{x} &= Ax + B_p p + B_w w + B_u u, \\ q &= C_q x + D_{qp} p + D_{qw} w + D_{qu} u \quad \text{and} \quad p = \Delta q, \\ z &= C_z x + D_{zp} p + D_{zw} w + D_{zu} u, \\ y &= C_y x + D_{yp} p + D_{yw} w, \end{aligned} \tag{4.20}$$

by a controller of the form (4.2) while satisfying $\mathcal{L}_2$-gain γ on the performance channel $w \to z$. The following theorem is again a result of [1].

Theorem 4.2.1. *[1] Let the primal IQC (4.3), defined by $M = M^T \in \mathbb{R}^{(\rho+\mu)\times(\rho+\mu)}$, and the dual IQC (4.4), defined by $N = N^T \in \mathbb{R}^{(\rho+\mu)\times(\rho+\mu)}$, be satisfied and let M and N have the properties (4.5). Assume that there exist $X = X^T \in \mathbb{R}^{n\times n}$, $Y = Y^T \in \mathbb{R}^{n\times n}$ and a real number $\gamma > 0$ such that*

$$\begin{pmatrix} \left(\star\right)_\perp^T \left(\star\right)^T \begin{pmatrix} 0 & 0 & Y & 0 \\ 0 & M & 0 & 0 \\ Y & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma I \end{pmatrix} \left(\begin{array}{ccc} A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ 0 & I & 0 \\ \hline I & 0 & 0 \\ \hline 0 & 0 & I \end{array}\right) \begin{pmatrix} C_y^T \\ D_{yp}^T \\ D_{yw}^T \end{pmatrix}_\perp & \star \\ \begin{pmatrix} C_z & D_{zp} & D_{zw} \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \\ D_{yw}^T \end{pmatrix}_\perp & -\gamma I \end{pmatrix} < 0 \tag{4.21}$$

$$\begin{pmatrix} \left(\star\right)_\perp^T \left(\star\right)^T \begin{pmatrix} 0 & 0 & X & 0 \\ 0 & N & 0 & 0 \\ X & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma I \end{pmatrix} \left(\begin{array}{ccc} I & 0 & 0 \\ \hline 0 & I & 0 \\ -B_p^T & -D_{qp}^T & -D_{zp}^T \\ \hline -A^T & -C_q^T & -C_z^T \\ \hline 0 & 0 & I \end{array}\right) \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix}_\perp & \star \\ \begin{pmatrix} B_w^T & D_{qw}^T & D_{zw}^T \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix}_\perp & \gamma I \end{pmatrix} > 0 \tag{4.22}$$

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0. \tag{4.23}$$

Then, there exists a controller interconnection C-Δ_c with which the closed-loop interconnection G_{cl}-Δ_{cl} is internally stable with $\mathcal{L}_2$-gain γ.

Proof. The proof is basically the same as the proof of Theorem 4.1.1 and therefore it is omitted. We just note that we modify the extended multiplier as

$$\tilde{M}_{cl} = \begin{pmatrix} M_{cl_{11}} & 0 & M_{cl_{12}} & 0 \\ 0 & \gamma^{-1} I & 0 & 0 \\ M_{cl_{12}}^T & 0 & M_{cl_{22}} & 0 \\ 0 & 0 & 0 & -\gamma I \end{pmatrix}$$

and use some Schur complement arguments in order to obtain a convex formulation in γ. □

5. APPLICATION TO LINEAR PERIODIC SYSTEMS

Linear periodic systems are linear time-varying systems with periodically changing system matrices. In order to work with them, we first put them into G-Δ feedback interconnection form.

5.1. Forming the G-Δ Feedback Interconnection

We are interested in the systems of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_w(t)w(t) + B_u(t)u(t) \\ z(t) &= C_z(t)x(t) + D_{zw}(t)w(t) + D_{zu}(t)u(t) \\ y(t) &= C_y(t)x(t) + D_{yw}(t)w(t) \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{m_w}$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^{\rho_z}$ and $y(t) \in \mathbb{R}^\nu$. We assume that

$$A(t) = A_0 + \sum_{k \in \{k_1,\ldots,k_N\}} \left(A_k^c \cos k\omega_0 t + A_k^s \sin k\omega_0 t\right),$$

and similarly for $B_w(t)$, $B_u(t)$, etc. We want to represent this system as a feedback interconnection of a linear time-invariant part, G, and a time-varying multiplication operator, Δ (Figure 5.1).

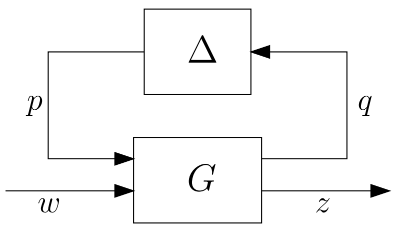

Figure 5.1. G-Δ feedback interconnection

With the guidance of [3], we define $\Delta\colon \mathcal{L}_2^{2k_N(n+\rho_z)} \to \mathcal{L}_2^{(n+\rho_z)}$ as

$$\Delta(q)(t) = \begin{bmatrix} \cos(k_1\omega_0 t)I & \sin(k_1\omega_0 t)I & \dots & \cos(k_N\omega_0 t)I & \sin(k_N\omega_0 t)I \end{bmatrix} q(t)$$

and we take G as:

$$\begin{pmatrix} \dot{x} \\ q \\ z \\ y \end{pmatrix} = \underbrace{\left[\begin{array}{c|ccc} A_0 & B_p & B_{w_0} & B_{u_0} \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_{z_0} & D_{zp} & D_{zw_0} & D_{zu_0} \\ C_{y_0} & D_{yp} & D_{yw_0} & 0 \end{array}\right]}_{G} \begin{pmatrix} x \\ p \\ w \\ u \end{pmatrix},$$

where

$$\begin{bmatrix} C_q & D_{qw} & D_{qu} \end{bmatrix} = \begin{bmatrix} A^c_{k_1} & B^c_{w_{k_1}} & B^c_{u_{k_1}} \\ C^c_{z_{k_1}} & D^c_{zw_{k_1}} & D^c_{zu_{k_1}} \\ C^c_{y_{k_1}} & D^c_{yw_{k_1}} & 0 \\ A^s_{k_1} & B^s_{w_{k_1}} & B^s_{u_{k_1}} \\ C^s_{z_{k_1}} & D^s_{zw_{k_1}} & D^s_{zu_{k_1}} \\ C^s_{y_{k_1}} & D^s_{yw_{k_1}} & 0 \\ \vdots & \vdots & \vdots \\ A^c_{k_N} & B^c_{w_{k_N}} & B^c_{u_{k_N}} \\ C^c_{z_{k_N}} & D^c_{zw_{k_N}} & D^c_{zu_{k_N}} \\ C^c_{y_{k_N}} & D^c_{yw_{k_N}} & 0 \\ A^s_{k_N} & B^s_{w_{k_N}} & B^s_{u_{k_N}} \\ C^s_{z_{k_N}} & D^s_{zw_{k_N}} & D^s_{zu_{k_N}} \\ C^s_{y_{k_N}} & D^s_{yw_{k_N}} & 0 \end{bmatrix}$$

and $B_p = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$, $D_{qp} = 0$, $D_{zp} = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, $D_{yp} = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$. Even though D_{qp} is identical to the zero matrix, we leave it in the generic formulation in the remainder of the thesis.

5.2. An IQC Result for the Dual IQC

Below, we give a result taken from [2] and [3] for the dual IQC. This result forms a convex set (defined in terms of LMI's) of multipliers N with which the dual IQC is satisfied.

Proposition 5.2.1. *[2, 3] Let N be defined as $N = Re(V^* L V)$, where*

$$
\begin{aligned}
V &\triangleq \begin{bmatrix} I & jI & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & jI & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & jI & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & I \end{bmatrix} \\
&\in \mathbb{C}^{(k_N+1)(n+\rho_z)\times(2k_N+1)(n+\rho_z)}
\end{aligned}
$$

and L satisfies

$$
\begin{bmatrix} A_F^T P A_F - P & A_F^T P B_F \\ B_F^T P A_F & B_F^T P B_F \end{bmatrix} + \begin{bmatrix} C_F^T \\ D_F^T \end{bmatrix} L \begin{bmatrix} C_F & D_F \end{bmatrix} \leq 0 \qquad (5.2)
$$

for some $P = P^T$, where A_F, B_F, C_F, D_F is any controllable realization of

$$
F(z) = \frac{1}{(z+a)^{k_N}} \begin{bmatrix} -z^{k_1} I \\ \vdots \\ -z^{k_N} I_{n+\rho_z} \\ I \end{bmatrix},
$$

for some real number $a \neq 1$. Then, the operator Δ satisfies the IQC

$$
\int_0^\infty \begin{pmatrix} -\Delta^T(w)(t) \\ w(t) \end{pmatrix}^T N \begin{pmatrix} -\Delta^T(w)(t) \\ w(t) \end{pmatrix} dt \leq 0 \quad \forall\, w \in \mathcal{L}_2. \qquad (5.3)
$$

Now, we present the proof following the steps of [3].

Proof. By KYP lemma, there exists a $P = P^T$ that satisfies Inequality (5.2) if and only $F(z)^* L F(z) \leq 0$ for all $|z| = 1$. This is equivalent to

$$\begin{pmatrix} -z^{k_1} I \\ \vdots \\ -z^{k_N} I \\ I \end{pmatrix}^* L \begin{pmatrix} -z^{k_1} I \\ \vdots \\ -z^{k_N} I \\ I \end{pmatrix} \leq 0 \quad \forall |z| = 1.$$

Now, if we parameterize complex numbers z on the unit circle by t, we can write the above inequality equivalently as

$$\begin{pmatrix} -e^{ik_1\omega_0 t} I \\ \vdots \\ -e^{ik_N\omega_0 t} I \\ I \end{pmatrix}^* L \begin{pmatrix} -e^{ik_1\omega_0 t} I \\ \vdots \\ -e^{ik_N\omega_0 t} I \\ I \end{pmatrix} \leq 0 \quad \forall t \in [0, \infty).$$

We can see that this is equivalent to

$$\begin{pmatrix} -\Delta^T(t) \\ I \end{pmatrix}^T V^* L V \begin{pmatrix} -\Delta^T(t) \\ I \end{pmatrix} \leq 0 \quad \forall t \tag{5.4}$$

since

$$\begin{aligned} V \begin{pmatrix} -\Delta^T(t) \\ I \end{pmatrix} &= \begin{pmatrix} -\left(\cos\left(k_1\omega_0 t\right) + j \sin\left(k_1\omega_0 t\right)\right) \\ \vdots \\ -\left(\cos\left(k_N\omega_0 t\right) + j \sin\left(k_N\omega_0 t\right)\right) \\ I \end{pmatrix} \\ &= \begin{pmatrix} -e^{ik_1\omega_0 t} I \\ \vdots \\ -e^{ik_N\omega_0 t} I \\ I \end{pmatrix}. \end{aligned}$$

Since $\operatorname{Im}(V^* L V)$ is skew symmetric, the corresponding part drops out in the quadratic constraint and (5.4) becomes equivalent to

$$\begin{pmatrix} -\Delta^T(t) \\ I \end{pmatrix}^T N \begin{pmatrix} -\Delta^T(t) \\ I \end{pmatrix} \leq 0 \quad \forall t \in [0, \infty).$$

Then, it is straightforward to conclude that (5.3) holds. □

5.3. The Primal IQC

Unfortunately, there is no IQC result in the literature for the primal IQC which uses the relation between the cosine and sine terms in Δ. One can use Lemma A.1.1 and obtain LMI's in M^{-1} using Proposition 5.2.1, but, this would violate convexity of the overall problem since M itself is also a matrix variable. For this reason, we choose to make a different approach that is presented in [1].

The primal IQC is satisfied if

$$\begin{pmatrix} I & & 0 \\ & \ddots & \\ 0 & & I \\ \delta_1 I & \dots & \delta_{2k_N} I \end{pmatrix}^T M \begin{pmatrix} I & & 0 \\ & \ddots & \\ 0 & & I \\ \delta_1 I & \dots & \delta_{2k_N} I \end{pmatrix} \geq 0 \tag{5.5}$$

for all δ_k in $[-1, 1]$, $k \in \{1, \dots, 2k_N\}$. If we let δ_k's vary between the two extremes -1 and 1, we obtain 2^{2k_N}-many Δ_i's that correspond to the vertices of the set $\boldsymbol{\Delta} \triangleq \operatorname{Co}\{\Delta_1, \dots, \Delta_{2^{2k_N}}\}$. Then, if we define $\tilde{\Delta} \triangleq \begin{bmatrix} \delta_1 I & \dots & \delta_{2k_N} I \end{bmatrix}$, $\tilde{\Delta}$ belongs to $\boldsymbol{\Delta}$ for all δ_k in $[-1, 1]$. Now, if we impose the set of LMI's

$$\begin{pmatrix} I \\ \Delta_i \end{pmatrix}^T M \begin{pmatrix} I \\ \Delta_i \end{pmatrix} > 0 \quad \forall i \in \{1, \dots, 2^{2k_N}\}, \tag{5.6}$$

due to a convexity argument (5.5) is satisfied for all $\tilde{\Delta} \in \boldsymbol{\Delta}$. In order to see this, first define

$$
\begin{aligned}
f_q(\tilde{\Delta}) &= q^T \begin{pmatrix} I \\ \tilde{\Delta} \end{pmatrix}^T M \begin{pmatrix} I \\ \tilde{\Delta} \end{pmatrix} q \\
&= q^T \left(M_{11} + M_{12}\tilde{\Delta} + \tilde{\Delta}^T M_{12}^T + \tilde{\Delta}^T M_{22}\tilde{\Delta} \right) q.
\end{aligned}
$$

Now, note that since $M_{22} < 0$, $f_q(\tilde{\Delta})$ is concave in $\tilde{\Delta}$. Due to concavity, $f_q(\tilde{\Delta})$ takes its minimum values at the extreme points, that is at Δ_i's. So, by imposing the LMI's (5.6), we guarantee that $f_q(\Delta_i)$ is positive definite for all Δ_i, which implies that $f_q(\tilde{\Delta})$ is positive definite for all $\tilde{\Delta} \in \mathbf{\Delta}$.

6. PARAMETRICALLY EXCITED ROTATING SYSTEM

The two-link arm system, taken from [4], is shown in Figure 6.1. The base of the arm makes periodic motion along line OA, which itself rotates with a constant angular velocity Ω in counter-clockwise direction. The whole system is lying on a horizontal plane so that there is no effect of gravity. The links connecting the masses with the base and with each other have length L. The joints located at the base and the first mass have stiffness constant k and damping coefficient c. T_1 and T_2 denote the net control torques applied at the two joints. The two masses are assumed to be equal, that is, $m_1 = m_2 = m$.

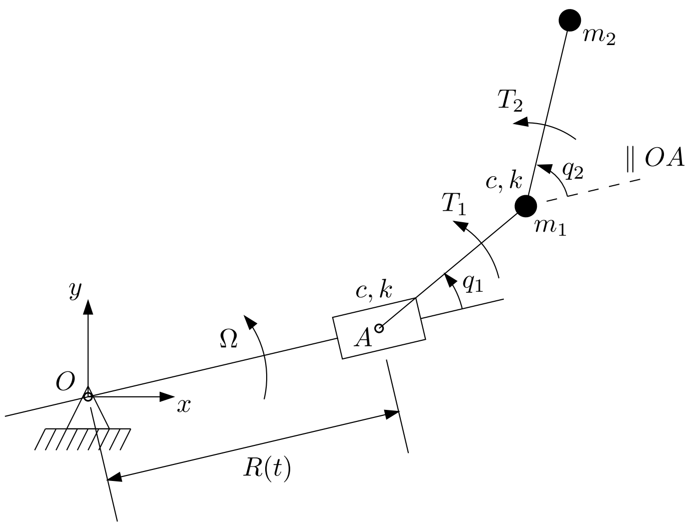

Figure 6.1. The two-link arm system with a periodically excited base

6.1. The Equations of Motion

Using Lagrange's equations, the equations of motion of the system are found to be

$$\begin{aligned}
&2mL^2\ddot{q}_1 + mL^2\cos(q_2 - q_1)\ddot{q}_2 \\
&+ 2c\dot{q}_1 - mL^2\sin(q_2 - q_1)\dot{q}_2^2 - \left[c + 2mL^2\Omega\sin(q_2 - q_1)\right]\dot{q}_2 \\
&+ 4mL\Omega\dot{R}\cos q_1 + 2mL\left(\Omega^2 R - \ddot{R}\right)\sin q_1 + 2kq_1 \\
&- kq_2 - mL^2\Omega^2\sin(q_2 - q_1) = T_1,
\end{aligned} \tag{6.1}$$

and

$$
\begin{aligned}
& mL^2 \cos{(q_2 - q_1)}\ddot{q}_1 + mL^2\ddot{q}_2 \\
& + mL^2 \sin{(q_2 - q_1)}\dot{q}_1^2 + \left[2mL^2\Omega \sin{(q_2 - q_1)} - c\right] \dot{q}_1 + c\dot{q}_2 \\
& - kq_1 + 2mL\Omega\dot{R}\cos q_2 + mL\left(\Omega^2 R - \ddot{R}\right)\sin q_2 + kq_2 \\
& + mL^2\Omega^2 \sin{(q_2 - q_1)} = T_2.
\end{aligned} \tag{6.2}
$$

In order to work with non-dimensional parameters, we define the following non-dimensional quantities with the guidance of [4]:

- non-dimensional time: $\tau = t\left(\frac{k}{mL^2}\right)^{\frac{1}{2}}$,
- $\delta = \left(\frac{c^2}{mL^2k}\right)^{\frac{1}{2}}$,
- $\alpha = \frac{mL^2\Omega^2}{k}$,
- $\varepsilon = \frac{R_0}{L}$,
- $\psi = \omega\left(\frac{mL^2}{k}\right)^{\frac{1}{2}}$,
- $u = \begin{pmatrix} \frac{T_1}{k} \\ \frac{T_2}{k} \end{pmatrix}$,

where R_0 is the amplitude of the periodic function $R(t) \triangleq R_0 f_R(t)$ with frequency ω. Then, we obtain the non-dimensional equations of motion

$$
\begin{aligned}
& 2q_1'' + \cos{(q_2 - q_1)}q_2'' + 2\delta q_1' - \sin{(q_2 - q_1)}{q_2'}^2 - \left[\delta + 2\sqrt{\alpha}\sin{(q_2 - q_1)}\right] q_2' \\
& + 4\varepsilon\sqrt{\alpha}f_R'\cos q_1 + 2\varepsilon\left(\alpha f_R - f_R''\right)\sin q_1 + 2q_1 - q_2 - \alpha\sin{(q_2 - q_1)} = u_1,
\end{aligned} \tag{6.3}
$$

and

$$
\begin{aligned}
& \cos{(q_2 - q_1)}q_1'' + q_2'' + \sin{(q_2 - q_1)}{q_1'}^2 + \left[2\sqrt{\alpha}\sin{(q_2 - q_1)} - \delta\right] q_1' + \delta q_2' \\
& - q_1 + 2\varepsilon\sqrt{\alpha}f_R'\cos q_2 + \varepsilon\left(\alpha f_R - f_R''\right)\sin q_2 + q_2 + \alpha\sin{(q_2 - q_1)} = u_2.
\end{aligned} \tag{6.4}
$$

where "prime" ($'$) denotes differentiation with respect to τ. The system can be linearized by assuming small angular displacements q_1, q_2 and small angular velocities q_1',

q_2'. The resulting linear system becomes

$$Mq''(\tau) + Cq'(\tau) + [K_1 + K_2 (\alpha f_R(\tau) - f_R''(\tau))] \, q(\tau) = u(\tau) + h(\tau),$$

where

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \delta \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$
$$K_1 = \begin{pmatrix} 2+\alpha & -1-\alpha \\ -1-\alpha & 1+\alpha \end{pmatrix}, \quad K_2 = \varepsilon \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$h(\tau) = -\varepsilon\sqrt{\alpha} \begin{pmatrix} 4 \\ 2 \end{pmatrix} f_R'(\tau).$$

Let $x \triangleq \begin{pmatrix} q_1 \\ q_2 \\ q_1' \\ q_2' \end{pmatrix}$. Then, we can represent the system dynamics in state-space as

$$x' = \begin{bmatrix} 0 & I \\ -M^{-1}\left[K_1 + (\alpha f_R - f_R'')K_2\right] & -M^{-1}C \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tilde{u}, \tag{6.5}$$

where $\tilde{u}(\tau) \triangleq u(\tau) + h(\tau)$. Since we know $h(\tau)$, if we can design some control law that gives the required $\tilde{u}$ for some specific purpose, we can find the actual control input by simple subtraction as $u(\tau) = \tilde{u}(\tau) - h(\tau)$. We assume that only the angular positions are available for feedback:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{6.6}$$

We study this system for two different cases. The first one is the case that is also studied in [4] where $f_R(t) = \sin \omega t$. In the second case, we give a more complex periodic wave form that can be expressed as a combination of sines and cosines using the Fourier series expansion.

6.2. Case I: Simple Harmonic Motion of the Base

In this case, the base is assumed to make simple sinusoidal motion of frequency ω, that is, $f_R(t) = \sin \omega t$, or for the non-dimensional system, $f_R(\tau) = \sin \psi\tau$. Then, the state-space equations become

$$\begin{aligned} x' &= A(\tau)x + B_u\tilde{u}, \\ y &= C_y x, \end{aligned}$$

where

$$A(\tau) = \begin{bmatrix} 0 & I \\ -M^{-1}\left[K_1 + (\alpha+\psi^2)K_2 \sin\psi\tau\right] & -M^{-1}C \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix},$$

and $C_y = \begin{bmatrix} I_2 & 0 \end{bmatrix}$. Since $A(\tau)$ depends periodically on τ, we can use the procedure presented in Section 5.1 and put this system into a G-Δ feedback interconnection form. $A(\tau)$ can be written as

$$A(\tau) = A_0 + A_1^c \cos\psi\tau + A_1^s \sin\psi\tau,$$

where

$$A_0 = \begin{bmatrix} 0 & 0 \\ -M^{-1}K_1 & M^{-1}C \end{bmatrix}, \quad A_1^c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_1^s = \begin{bmatrix} 0 & 0 \\ -(\alpha+\psi^2)M^{-1}K_2 & 0 \end{bmatrix}.$$

Then, Δ can be defined as

$$\Delta(\tau) = \begin{bmatrix} \cos\psi\tau\, I_4 & \sin\psi\tau\, I_4 \end{bmatrix},$$

and we take $C_q = \begin{bmatrix} A_1^c \\ A_1^s \end{bmatrix}$ with $B_p = I_4$. The system becomes

$$\begin{aligned} x' &= A_0 x + B_p p + B_u u, \\ q &= C_q x, \\ y &= C_y x, \end{aligned}$$

and $p = \Delta q$.

6.3. Case II

The periodic function $f_R(t)$ is shown in Figure 6.2, where $T = 2\pi/\omega$. Using

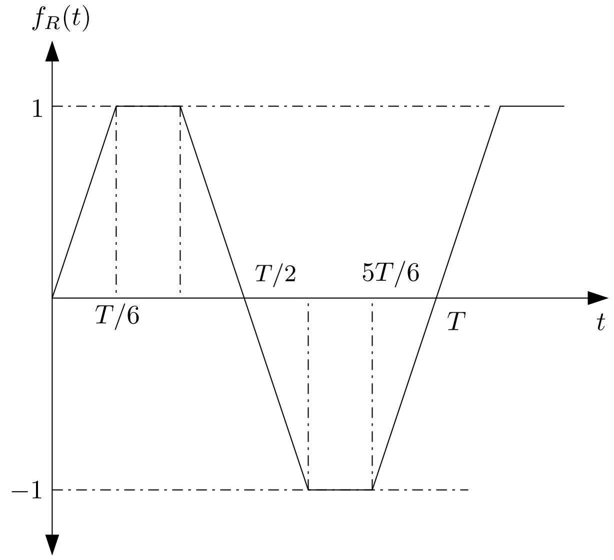

Figure 6.2. The periodic motion of the base in Case II

Fourier series expansion formula, f_R can be expressed in the non-dimensional time τ

as

$$f_R(\tau) = \sum_{k=1}^{\infty} \left(a_k \cos k\psi\tau + b_k \sin k\psi\tau\right), \tag{6.7}$$

where

$$\begin{aligned} a_k &= \frac{3}{k^2\pi^2}\left(\cos\frac{k\pi}{3} + \cos\frac{2k\pi}{3} - \cos\frac{4k\pi}{3} - \cos\frac{5k\pi}{3}\right), \\ b_k &= \frac{3}{k^2\pi^2}\left(\sin\frac{k\pi}{3} + \sin\frac{2k\pi}{3} - \sin\frac{4k\pi}{3} - \sin\frac{5k\pi}{3}\right). \end{aligned}$$

Then, we can express f_R' and f_R'' as

$$\begin{aligned} f_R'(\tau) &= \sum_{k=1}^{\infty} k\psi\left(-a_k \sin k\psi\tau + b_k \cos k\psi\tau\right), \\ f_R''(\tau) &= -\sum_{k=1}^{\infty} k^2\psi^2\left(a_k \cos k\psi\tau + b_k \sin k\psi\tau\right). \end{aligned}$$

Then, the state-space equations become

$$\begin{aligned} x' &= A(\tau)x + B_u\tilde{u}, \\ y &= C_y x, \end{aligned}$$

where

$$A(\tau) = \begin{bmatrix} 0 & I \\ -M^{-1}\left[K_1 + K_2 \sum_{k=1}^{\infty}(\alpha + k^2\psi^2)\left(a_k \cos k\psi\tau + b_k \sin k\psi\tau\right)\right] & -M^{-1}C \end{bmatrix},$$

and $B_u = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$, $C_y = \begin{bmatrix} I_2 & 0 \end{bmatrix}$. Then, we choose the first k_N-many frequencies as the dominant harmonics of the system and express $A(\tau)$ as

$$A(\tau) = A_0 + \sum_{k=1}^{k_N} \left(A_k^c \cos k\psi\tau + A_k^s \sin k\psi\tau\right) + \Delta_A(\tau),$$

where

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & I \\ -M^{-1}K_1 & -M^{-1}C \end{bmatrix}, \\
A_k^c &= \begin{bmatrix} 0 & 0 \\ -(\alpha^2 + k^2\psi^2)a_k M^{-1}K_2 & 0 \end{bmatrix}, \\
A_k^s &= \begin{bmatrix} 0 & 0 \\ -(\alpha^2 + k^2\psi^2)b_k M^{-1}K_2 & 0 \end{bmatrix}
\end{aligned}
$$

and

$$
\Delta_A(\tau) = \sum_{k=k_N+1}^{\infty} \begin{bmatrix} 0 & 0 \\ -(\alpha^2 + k^2\psi^2)M^{-1}K_2 & 0 \end{bmatrix} (a_k \cos k\psi\tau + b_k \sin k\psi\tau)\,.
$$

If we define

$$
\Delta(\tau) = \begin{bmatrix} \cos\psi\tau\, I_4 & \sin\psi\tau\, I_4 & \dots & \cos k_N\psi\tau\, I_4 & \sin k_N\psi\tau\, I_4 \end{bmatrix},
$$

and $C_q = \begin{bmatrix} A_1^c \\ A_1^s \\ \vdots \\ A_{k_N}^c \\ A_{k_N}^s \end{bmatrix}$ with $B_p = I_4$, we can express the system as

$$
\begin{aligned}
x' &= A_0 x + \Delta_A(\tau)x + B_p p + B_u \tilde{u}, \\
q &= C_q x, \\
y &= C_y x, \quad \text{and} \quad p = \Delta q.
\end{aligned}
$$

Now, we can treat the term $\Delta_A(\tau)x$ as disturbance on the system and substitute it with $B_w w$, where $B_w = \begin{bmatrix} 0 \\ -M^{-1}K_2 \end{bmatrix}$ and w is the disturbance input. If we choose the controlled output as $z = x$, the system becomes

$$\begin{aligned} x' &= A_0 x + B_p p + B_w w + B_u \tilde{u}, \\ q &= C_q x, \\ z &= x, \\ y &= C_y x, \end{aligned}$$

and $p = \Delta q$.

7. REFERENCE TRACKING DESIGN AND SIMULATIONS

In this chapter, we study the reference tracking problem for the two-link arm system introduced in the previous chapter. We begin with shortly introducing the tracking problem.

7.1. The Tracking Problem

In the tracking problem, a controller is synthesized such that given a reference command, $r(t)$, to the system, some chosen output, $z(t)$ of the system will follow that reference command. This can be achieved by designing a controller that minimizes the $\mathcal{L}_2$-gain of the closed-loop interconnection along the channel $r(t) \to e(t)$.

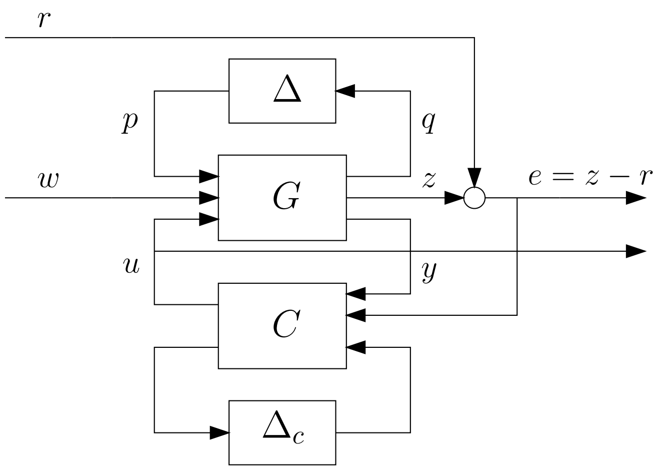

Figure 7.1. The tracking problem

In our general formulation, the system G is described by the following state-space equations:

$$\begin{aligned}
\dot{x} &= Ax + B_p p + B_w w + B_u u,\\
q &= C_q x + D_{qp} p + D_{qw} w + D_{qu} u,\\
z &= C_z x + D_{zp} p + D_{zw} w + D_{zu} u,\\
y &= C_y x + D_{yp} p + D_{yw} w.
\end{aligned} \tag{7.1}$$

In this formulation, we want $z(t)$ to track $r(t)$. We also want attenuation of the disturbance w, so we define the generalized disturbance $\tilde{w} = \begin{pmatrix} w \\ r \end{pmatrix}$. We define the controlled output as $\tilde{z} = \begin{pmatrix} e \\ u \end{pmatrix}$, where we include u so that we can push down the control input levels as we desire. So, we can state our problem as an $\mathcal{L}_2$-gain minimization problem of a $\tilde{G}$-Δ feedback interconnection with the performance channel $\tilde{w} \to \tilde{z}$. The state-space equations of the new interconnection become

$$\begin{aligned}
\dot{x} &= Ax + B_p p + \begin{bmatrix} B_w & 0 \end{bmatrix} \tilde{w} + B_u u,\\
q &= C_q x + D_{qp} p + \begin{bmatrix} D_{qw} & 0 \end{bmatrix} \tilde{w} + D_{qu} u,\\
\tilde{z} &= \begin{bmatrix} Cz \\ 0 \end{bmatrix} + \begin{bmatrix} D_{zp} \\ 0 \end{bmatrix} p + \begin{bmatrix} D_{zw} & -I \\ 0 & 0 \end{bmatrix} \tilde{w} + \begin{bmatrix} D_{zu} \\ I \end{bmatrix} u,\\
\tilde{y} &= \begin{bmatrix} Cy \\ Cz \end{bmatrix} + \begin{bmatrix} D_{yp} \\ D_{zp} \end{bmatrix} p + \begin{bmatrix} D_{yw} & 0 \\ D_{zw} & -I \end{bmatrix} \tilde{w} + \begin{bmatrix} 0 \\ D_{zu} \end{bmatrix} u.
\end{aligned} \tag{7.2}$$

with $p = \Delta q$.

7.2. Adding Weighting Functions to the System

Generally, in order to achieve the desired performance, one has to add weighting functions to the performance channel. For this reason, we let $\tilde{w} = W_I \hat{w}$ and $\hat{z} = W_O \tilde{z}$,

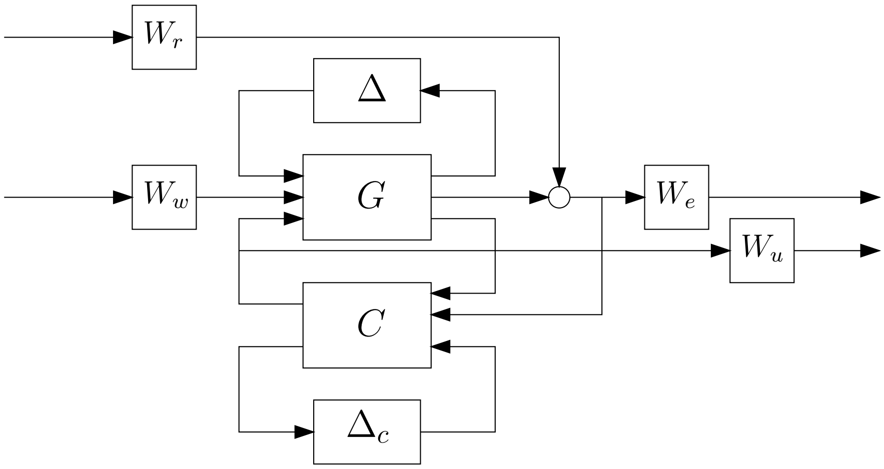

Figure 7.2. The tracking problem with weighting functions

where

$$W_I = \begin{pmatrix} W_w I_{m_w} & 0 \\ 0 & W_r I_{\rho_z} \end{pmatrix}, \quad \text{and} \quad W_O = \begin{pmatrix} W_e I_{\rho_z} & 0 \\ 0 & W_u I_m \end{pmatrix},$$

where W_w, W_r, W_e and W_u are weighting functions corresponding to w, r, e and u, respectively. The design problem becomes $\mathcal{L}_2$-gain minimization of the performance channel $\hat{w} \to \hat{z}$. Let $\tilde{G}$ be defined as

$$\begin{pmatrix} \dot{x} \\ q \\ \tilde{z} \\ \tilde{y} \end{pmatrix} = \underbrace{\left[\begin{array}{c|ccc} A & B_p & B_{\tilde{w}} & B_u \\ \hline C_q & D_{qp} & D_{q\tilde{w}} & D_{qu} \\ C_{\tilde{z}} & D_{\tilde{z}p} & D_{\tilde{z}\tilde{w}} & D_{\tilde{z}u} \\ C_{\tilde{y}} & D_{\tilde{y}p} & D_{\tilde{y}\tilde{w}} & 0 \end{array}\right]}_{\tilde{G}} \begin{pmatrix} x \\ p \\ \tilde{w} \\ u \end{pmatrix},$$

and $W_I = \left[\begin{array}{c|c} A_I & B_I \\ \hline C_I & D_I \end{array}\right]$, $W_O = \left[\begin{array}{c|c} A_O & B_O \\ \hline C_O & D_O \end{array}\right]$. Then, the 'augmented' system that maps $\hat{w}$ to $\hat{z}$ becomes

$$
\begin{aligned}
\dot{x}_a &= \begin{bmatrix} A & B_{\tilde{w}}C_I & 0 \\ 0 & A_I & 0 \\ B_O C_{\tilde{z}} & B_O D_{\tilde{z}\tilde{w}} C_I & A_O \end{bmatrix} x_a + \begin{bmatrix} B_p \\ 0 \\ B_O D_{\tilde{z}p} \end{bmatrix} p + \begin{bmatrix} B_{\tilde{w}} D_I \\ B_I \\ B_O D_{\tilde{z}\tilde{w}} D_I \end{bmatrix} \hat{w} \\
&+ \begin{bmatrix} B_u \\ 0 \\ B_O D_{\tilde{z}u} \end{bmatrix} u, \qquad (7.3) \\
q &= \begin{bmatrix} C_q & D_{q\tilde{w}} C_I & 0 \end{bmatrix} x_a + D_{qp} p + D_{q\tilde{w}} D_I \hat{w} + D_{qu} u, \\
\hat{z} &= \begin{bmatrix} D_O C_{\tilde{z}} & D_O D_{\tilde{z}\tilde{w}} C_I & C_O \end{bmatrix} x_a + D_O D_{\tilde{z}p} p + D_O D_{\tilde{z}\tilde{w}} D_I \hat{w} + D_O D_{\tilde{z}u} u, \\
\tilde{y} &= \begin{bmatrix} C_{\tilde{y}} & D_{\tilde{y}\tilde{w}} C_I & 0 \end{bmatrix} x_a + D_{\tilde{y}p} p + D_{\tilde{y}\tilde{w}} D_I \hat{w},
\end{aligned}
$$

where $x_a \triangleq \begin{pmatrix} x \\ x_I \\ x_O \end{pmatrix}$.

7.3. The Controller Design and Simulations

If we represent the augmented system (7.3) by $\hat{G}$, the tracking problem becomes the problem of $\mathcal{L}_2$-gain minimization of the feedback interconnection $\hat{G}$-Δ by designing a controller interconnection C-Δ_c. We use the MATLAB toolbox YALMIP and solver SeDuMi for implementing the LMI's that come from Theorem 4.2.1 and finding a γ-minimizing controller. We design controllers for Case I and Case II, and in each case we make designs for four different sets of parameters (ε, δ, α, ψ) as in [4], which are

- Parameter Set I: $\varepsilon = 0.4$, $\delta = 0$, $\alpha = 0$, $\psi = 2.417$.
- Parameter Set II: $\varepsilon = 0.4$, $\delta = 0$, $\alpha = 0.666$, $\psi = 3.025$.
- Parameter Set III: $\varepsilon = 0.4$, $\delta = 0.0163$, $\alpha = 0$, $\psi = 2.417$.
- Parameter Set IV: $\varepsilon = 0.4$, $\delta = 0.0163$, $\alpha = 0.666$, $\psi = 3.025$.

Note that based on the analysis made in [5], ψ is chosen in each case such that it is equal to the second natural frequency of the system. This increases the unstability of the system. We make designs for two different purposes: tracking a unit step and tracking a sinusoid. The nonlinear equations of motion are simulated in MATLAB Simulink, and the results are plotted.

7.3.1. Step Tracking for Case I

For all parameter sets, we used the same weighting functions which are

$$W_r(s) = \frac{100}{s+100} I_2, \quad W_e(s) = \begin{pmatrix} \frac{1100}{s+1} & 0 \\ 0 & \frac{1000}{s+1} \end{pmatrix}, \quad \text{and} \quad W_u = \frac{s}{s+1} I_2.$$

We made the designs for a γ value of 240. In the simulations, we used a transfer function $\frac{5}{s+5}$ in order to soften the step function while applying it as the reference signal. Simulation results show that the system tracks the reference input successfully.

7.3.2. Tracking a Sinusoid: Case I

The weights chosen in the design procedure are

$$W_r(s) = \frac{6}{s+6} I_2, \quad W_e(s) = \begin{pmatrix} \frac{1200}{s+4.5} & 0 \\ 0 & \frac{1100}{s+4.5} \end{pmatrix}, \quad \text{and} \quad W_u = \frac{0.25s}{s+5} I_2.$$

In the figures, red curves denote the reference signal. We track a sinusoid of amplitude one and frequency π rad/s. For the parameter sets I and III, the designs were made for $\gamma = 65$, while for the sets II and IV, $\gamma = 95$. The state variables q_1 and q_2 follow the reference input with a slight latency. We also make simulations for the reference signals with different amplitudes, that is,

$$r_1(\tau) = 0.75 \sin(\pi\tau), \quad \text{and} \quad r_2(\tau) = 1.25 \sin(\pi\tau).$$

This increases the nonlinearity effects on the system due to the terms $\cos(q_2 - q_1)$ and $\sin(q_2 - q_1)$ in the equations of motion. The simulation results show that the reference signals are tracked successfully with a slight latency.

7.3.3. Step Tracking for Case II

In this design process, we include also the angular velocities in the controlled output with a corresponding weighting function $W_{\dot{q}}$. The weights chosen in the design procedure are

$$W_w(s) = \frac{0.1}{s+0.1} I_2, \quad W_r(s) = \frac{0.1}{s+0.1} I_2,$$
$$W_e(s) = \begin{pmatrix} \frac{1200}{s+1} & 0 \\ 0 & \frac{1100}{s+1} \end{pmatrix}, \quad W_{\dot{q}}(s) = \frac{1}{s+1}$$

and $W_u = \frac{0.1s}{s+10} I_2$. For the first parameter set, we made a design for $\gamma = 40$. For the other sets, the designs are made for a γ value of 50. We chose the number of harmonics as $k_N = 3$. For the dual IQC, we choose the arbitrary number a in Proposition 5.2.1 as $a = 2$. In the simulation results, q_1 and q_2 reach and stay at the reference input with slight variations. The slight variations are observed to be due to the vertices of $f_R(t)$ (sudden movements of the base). These vertices are the most difficult points to approximate by the finite sum of sines and cosines. The variations are observed to be taken care of by the controller quickly.

7.3.4. Tracking a Sinusoid: Case II

The designs made for unit step tracking are also successful in tracking a sinusoid. The simulation results are presented below. Again, the system is simulated also for sinusoids of different amplitudes. The system is observed to track the reference signals successfully without latency.

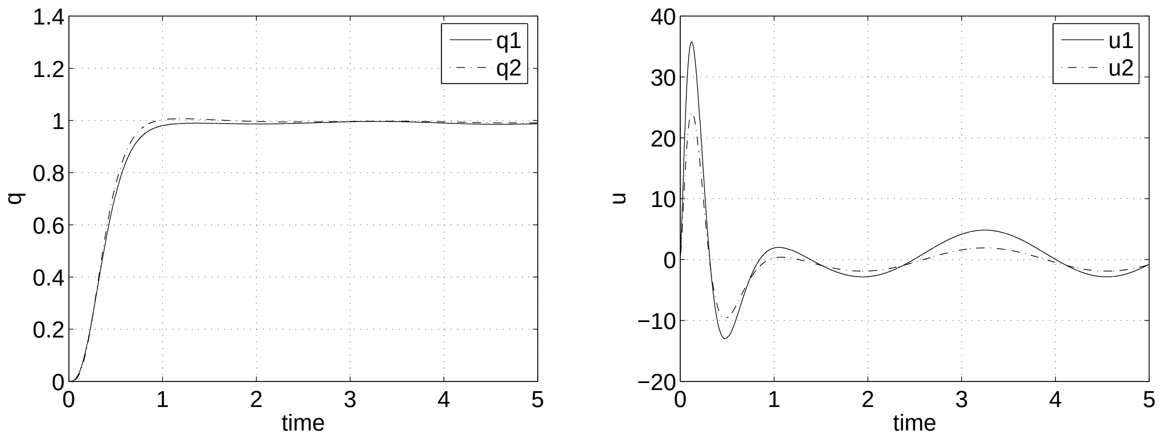

Figure 7.3. Step tracking for Case I with the Parameter Set I

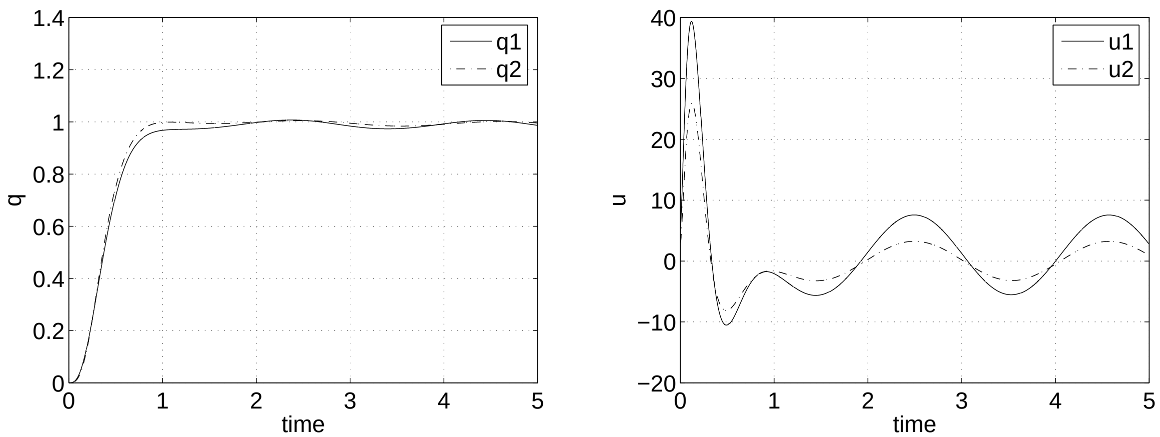

Figure 7.4. Step tracking for Case I with the Parameter Set II

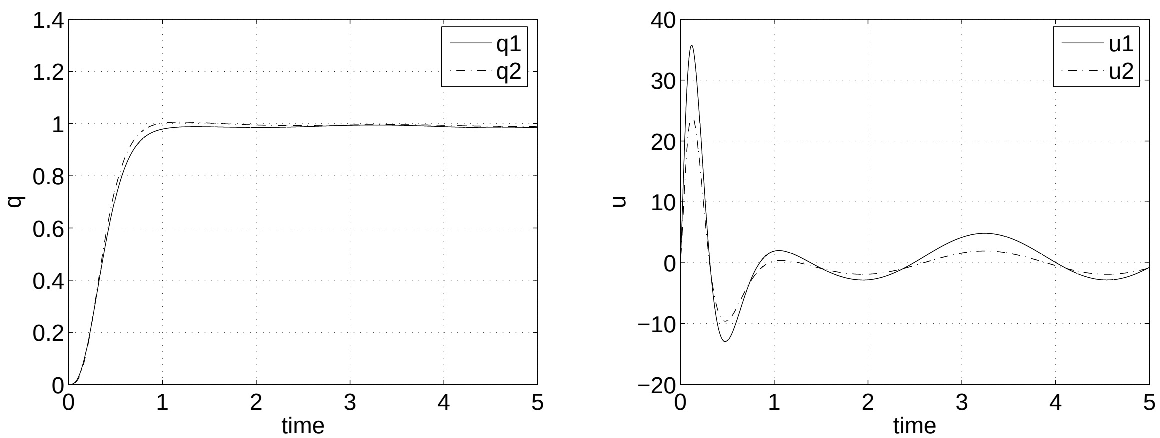

Figure 7.5. Step tracking for Case I with the Parameter Set III

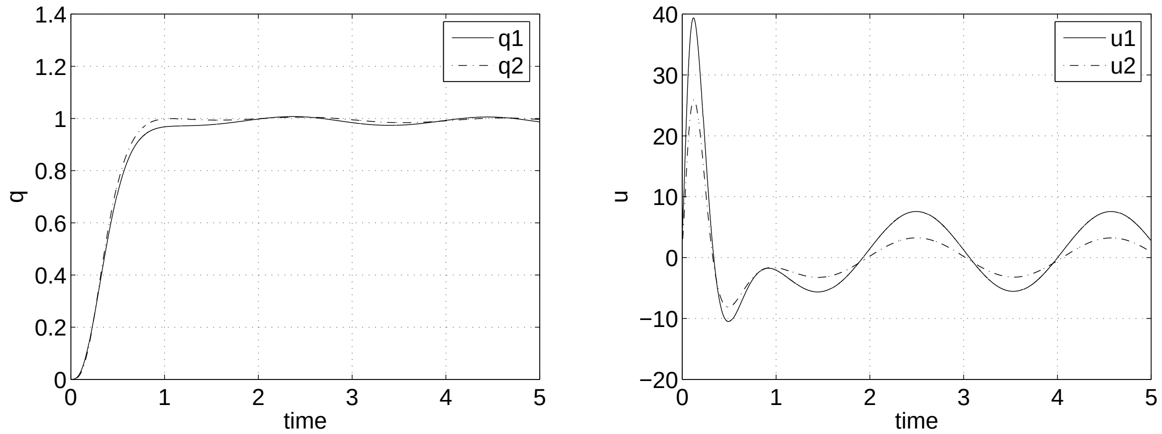

Figure 7.6. Step tracking for Case I with the Parameter Set IV

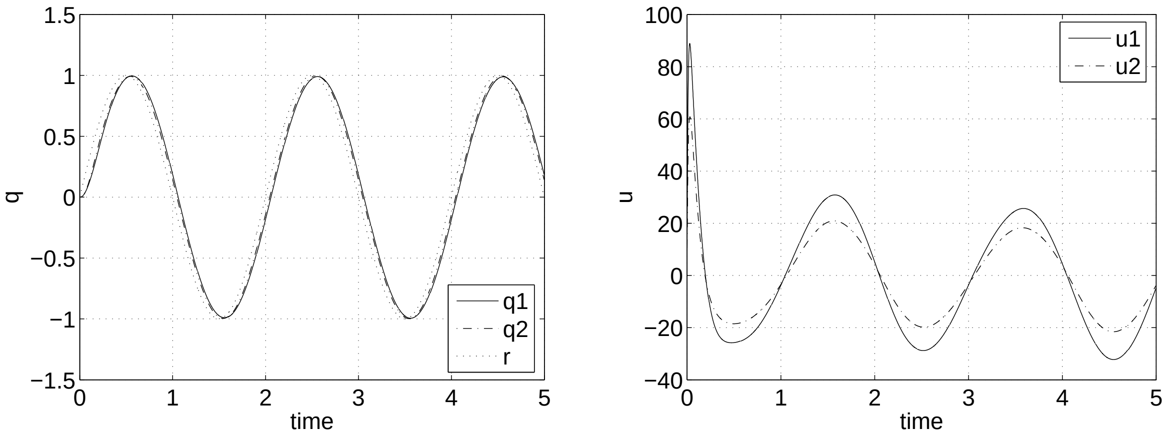

Figure 7.7. Tracking a sinusoid for Case I with the Parameter Set I

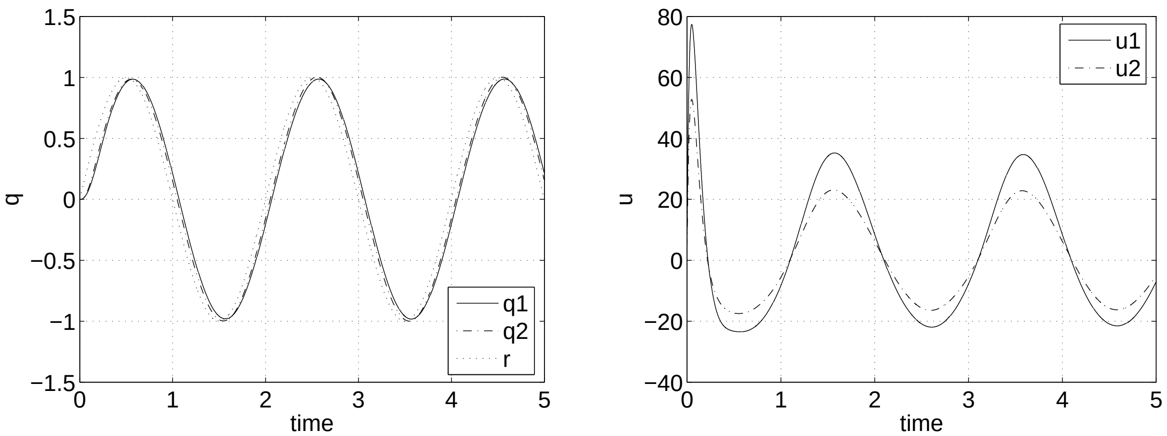

Figure 7.8. Tracking a sinusoid for Case I with the Parameter Set II

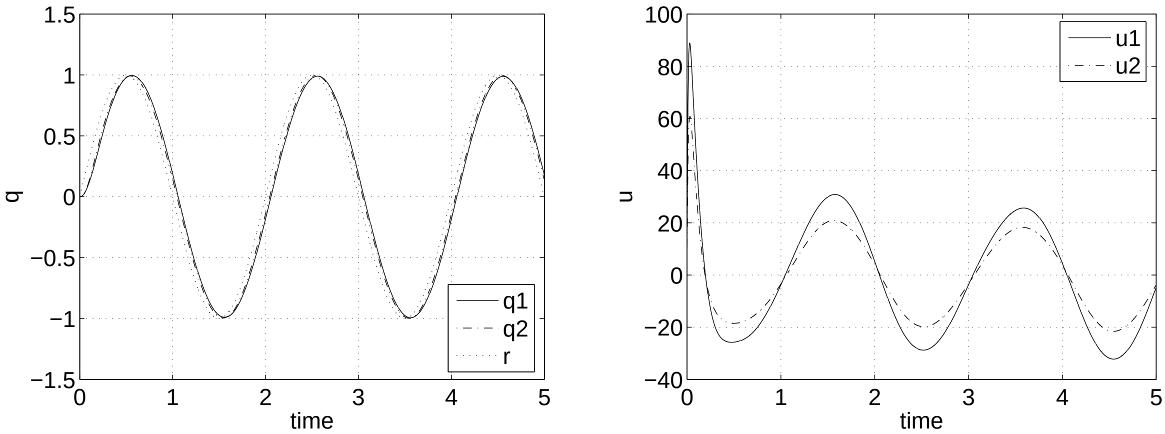

Figure 7.9. Tracking a sinusoid for Case I with the Parameter Set III

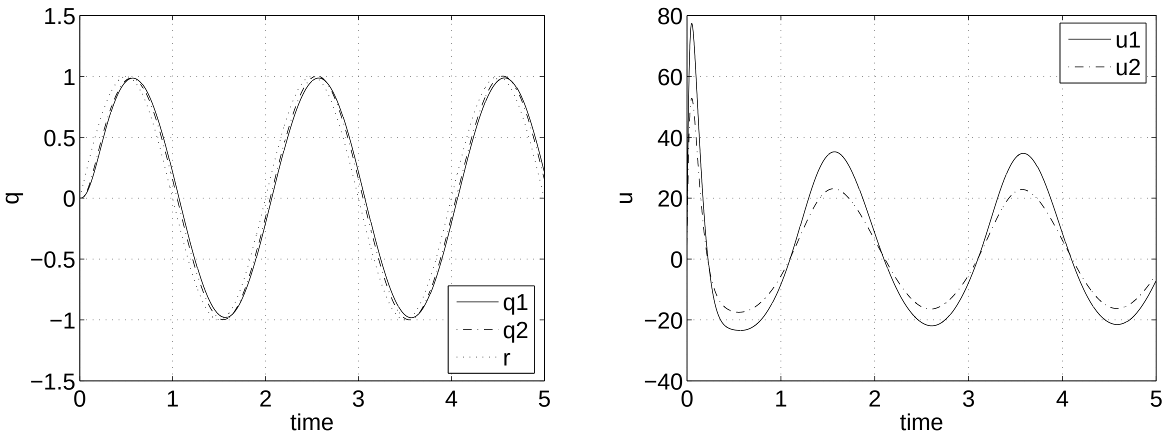

Figure 7.10. Tracking a sinusoid for Case I with the Parameter Set IV

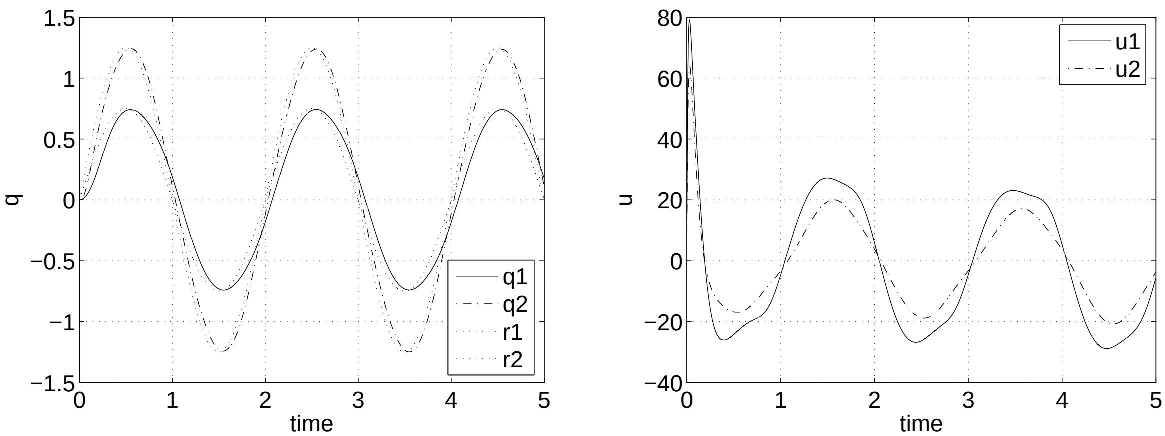

Figure 7.11. Tracking sinusoids of different amplitudes: Case I with Parameter Set I

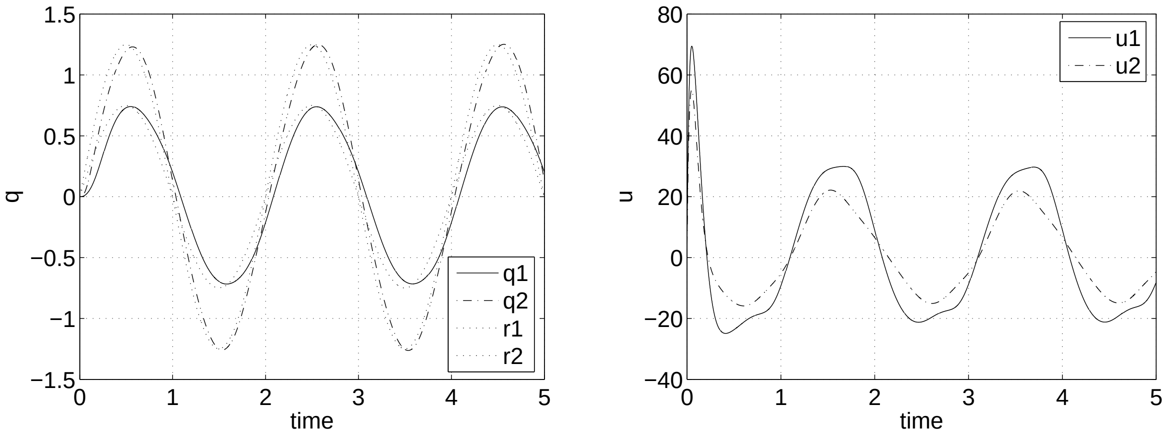

Figure 7.12. Tracking sinusoids of different amplitudes: Case I with Parameter Set II

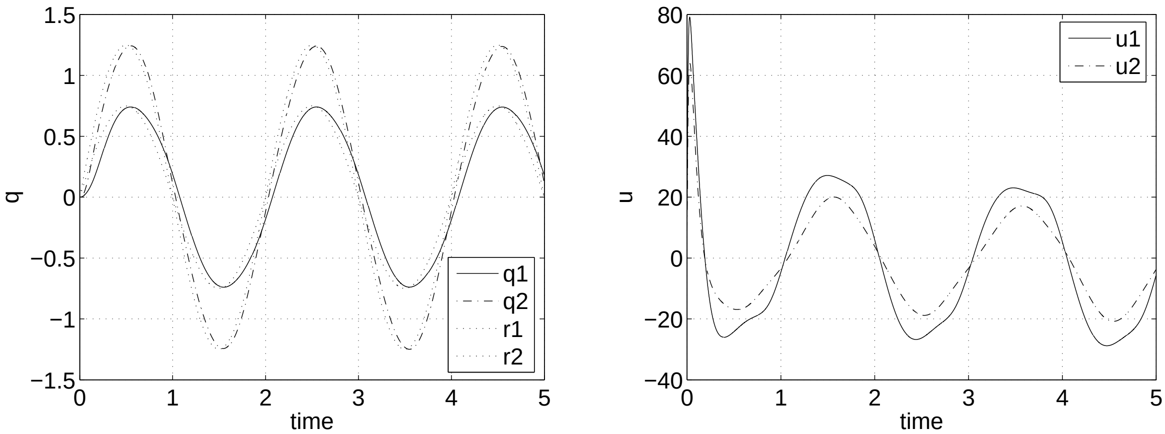

Figure 7.13. Tracking sinusoids of different amplitudes: Case I with Parameter Set III

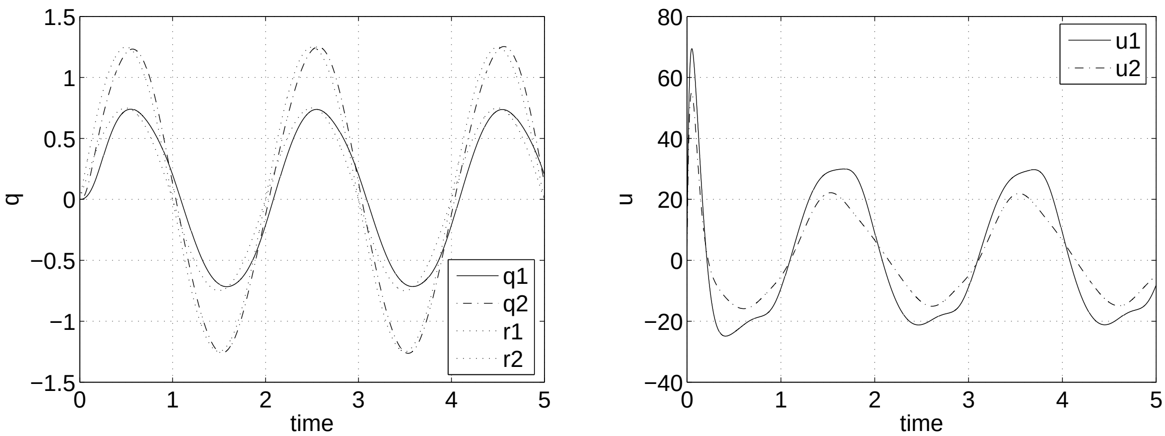

Figure 7.14. Tracking sinusoids of different amplitudes: Case I with Parameter Set IV

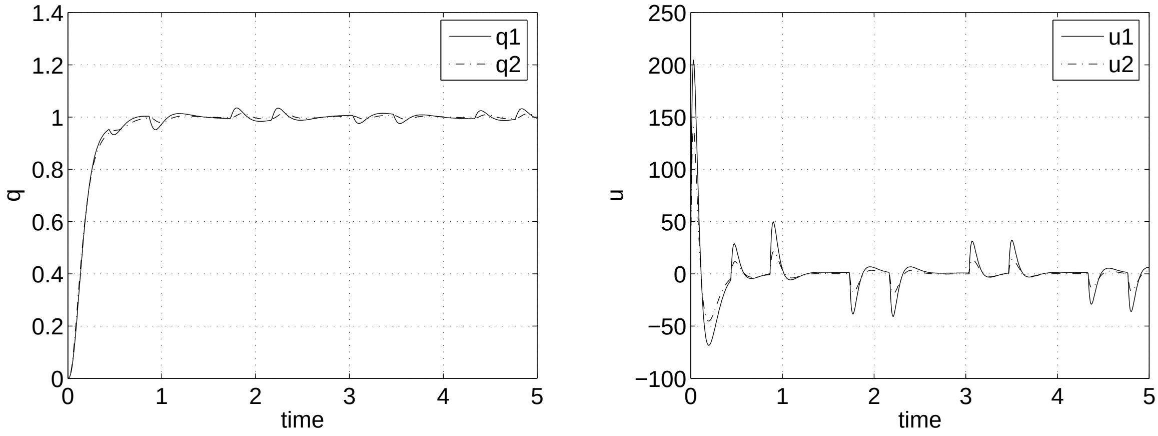

Figure 7.15. Step tracking for Case II with the Parameter Set I

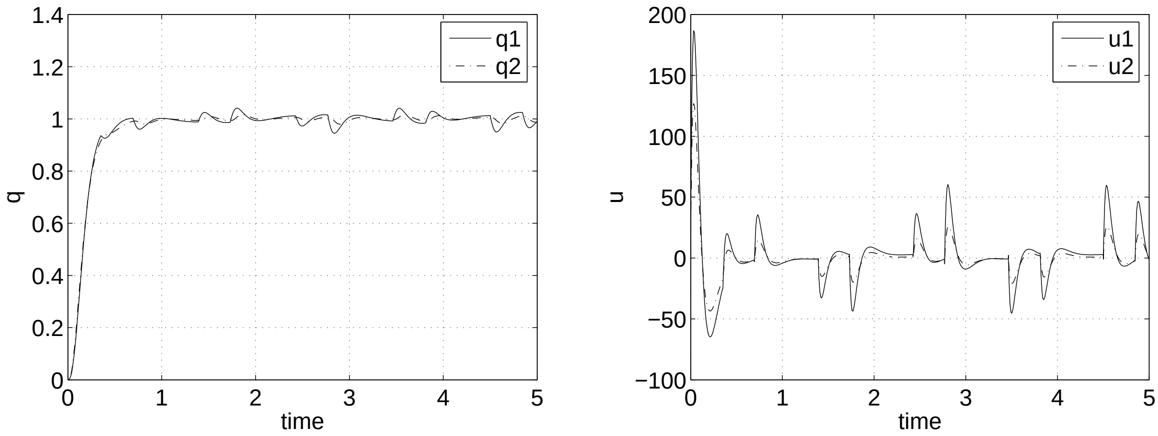

Figure 7.16. Step tracking for Case II with the Parameter Set II

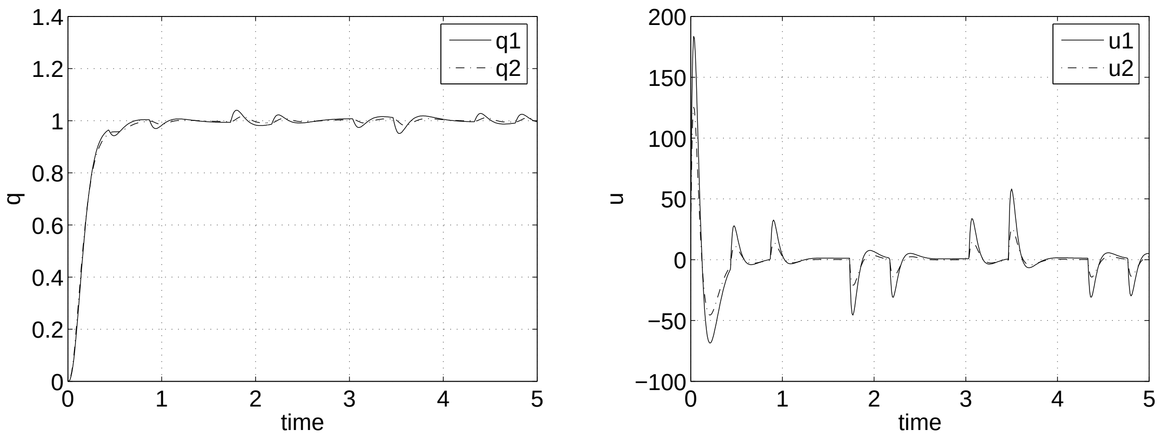

Figure 7.17. Step tracking for Case II with the Parameter Set III

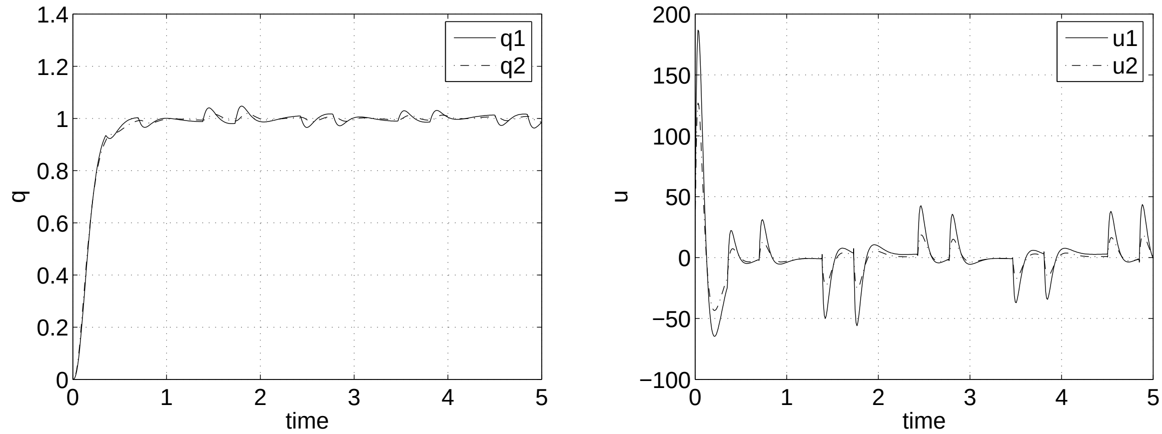

Figure 7.18. Step tracking for Case II with the Parameter Set IV

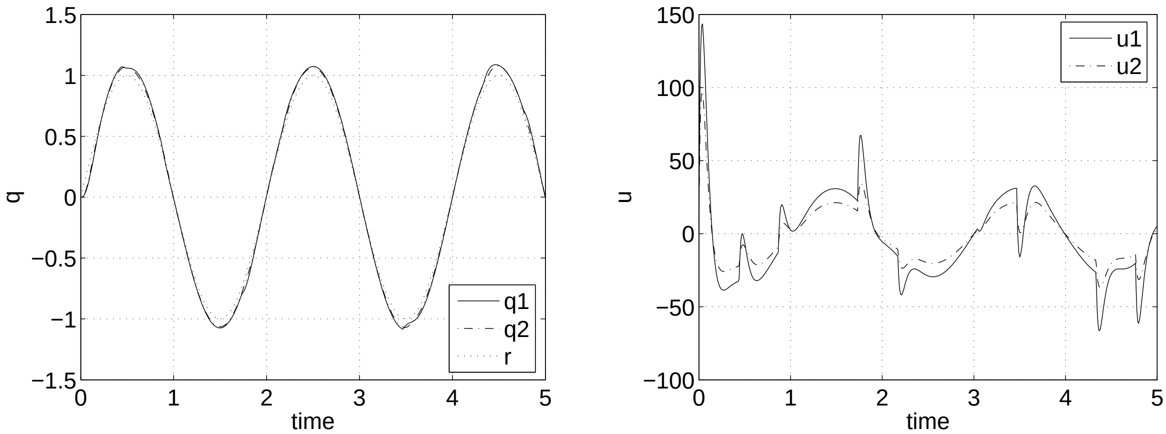

Figure 7.19. Tracking a sinusoid for Case II with the Parameter Set I

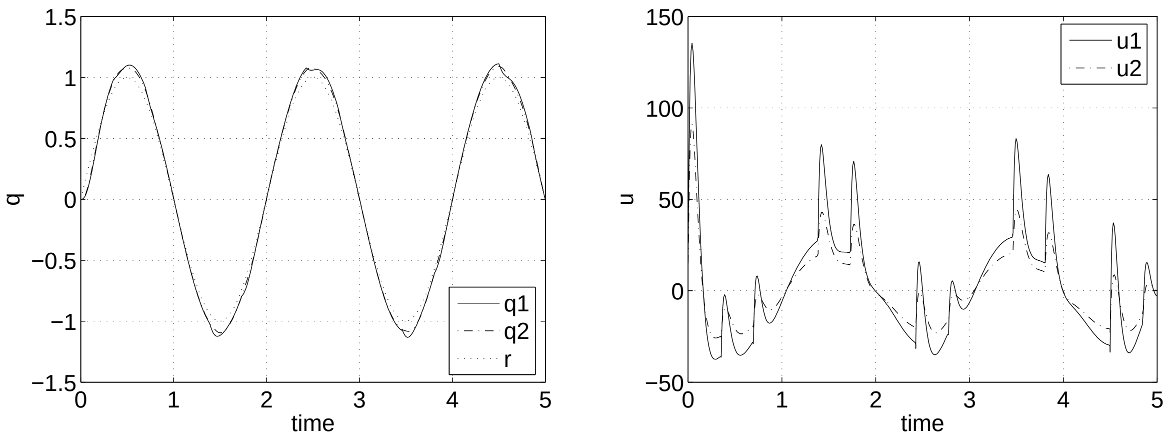

Figure 7.20. Tracking a sinusoid for Case II with the Parameter Set II

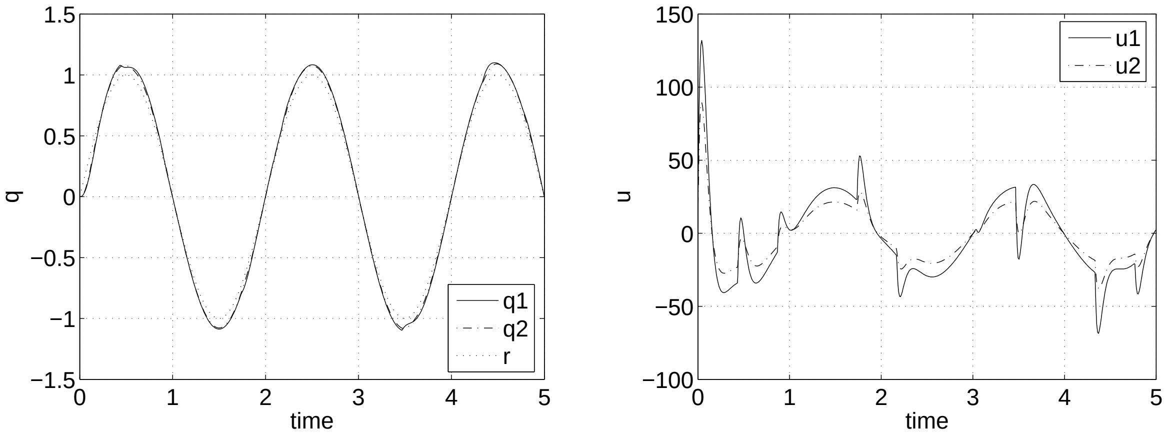

Figure 7.21. Tracking a sinusoid for Case II with the Parameter Set III

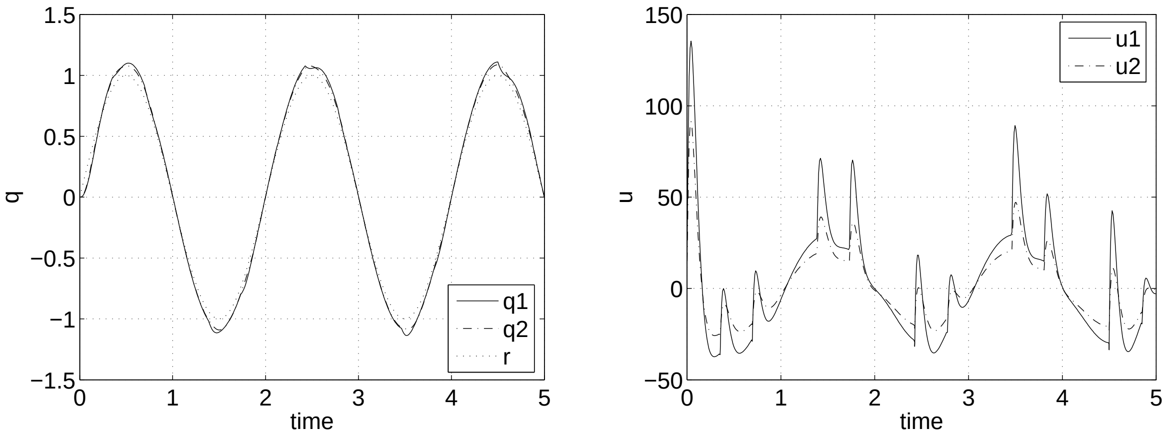

Figure 7.22. Tracking a sinusoid for Case II with the Parameter Set IV

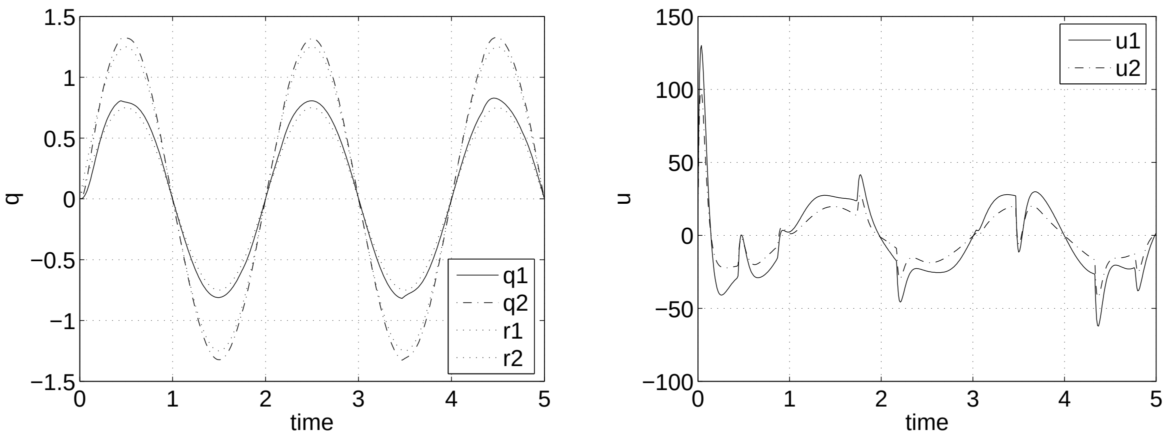

Figure 7.23. Tracking sinusoids of different amplitudes: Case II, the Parameter Set I

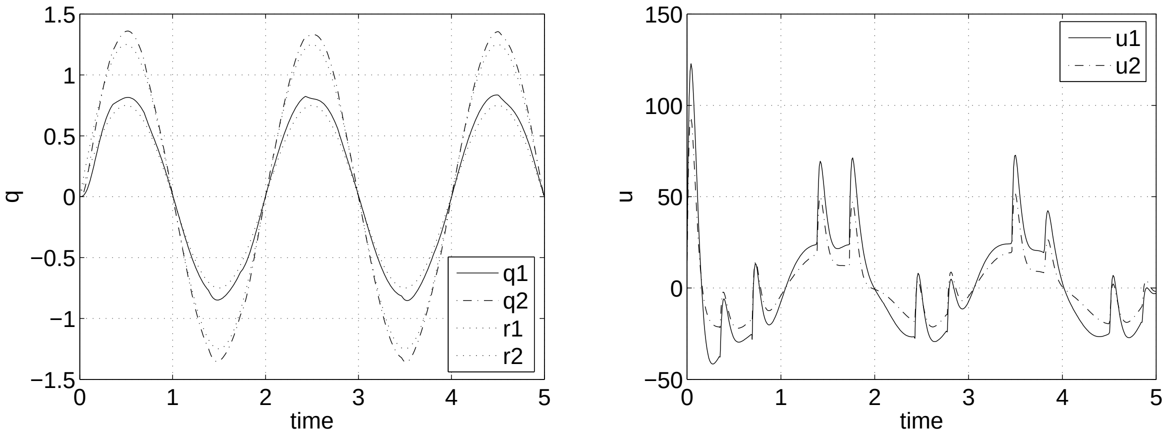

Figure 7.24. Tracking sinusoids of different amplitudes: Case II, the Parameter Set II

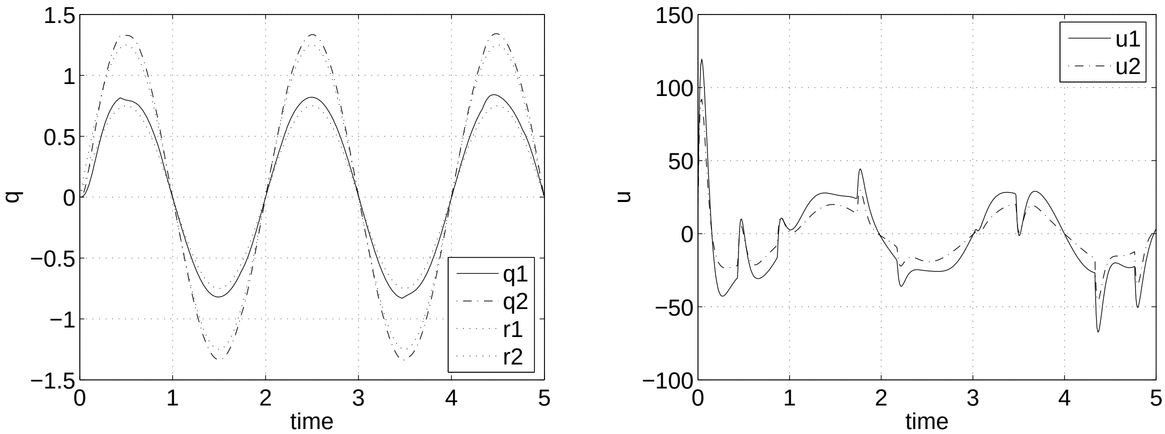

Figure 7.25. Tracking sinusoids of different amplitudes: Case II, the Parameter Set III

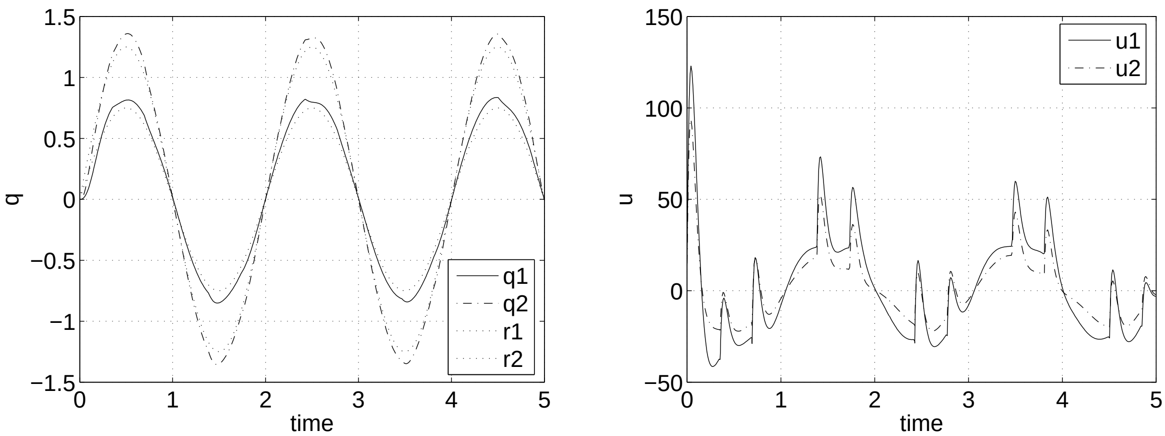

Figure 7.26. Tracking sinusoids of different amplitudes: Case II, the Parameter Set IV

8. SUMMARY AND CONCLUSIONS

Using LPV results of [1] and the IQC result of [2] and [3], we have provided a general procedure for the gain-scheduled control of linear periodic systems. We tried to give a more complete picture of the theory by presenting the proofs of the synthesis theorem and the IQC result.

For illustration of the application of the theory, we studied the tracking control problem of a two-link arm system with a periodically excited base. We modeled the system using Lagrange's equations and linearized it by making small angles assumption. The resulting system was a typical linear periodic system with a periodically changing state matrix in state-space representation. We studied this system for different parameters.

We presented a general formulation for the tracking problem. Our approach was to minimize the $\mathcal{L}_2$-gain of the controlled system (G_{cl}-Δ_{cl} feedback interconnection) from the reference input to the tracking error while stabilizing the system. After constructing the augmented system with weighting functions, we designed controllers by trying different weighting functions and comparing the simulation results. We made desings for different types of reference signals, in particular, the unit step input and sinusoidal signals.

The designed systems were found to be successful in tracking the reference signals. The most challenging task was step tracking for Case II, where the sudden movements of the base caused some variations around the reference signal, but the effect of these movements were observed to be taken care of quickly by the controller.

The simulation results show that the IQC result Proposition 5.3 can be successfully applied to the output feedback control of linear periodic systems. In the design procedure for the tracking problem, different choices of weighting functions enable us to make different designs considering the particular reference signal that is to be tracked.

In order to improve the designs made in this thesis, one can consider the nonlinearities that are ignored in the linearized model and try to make a design that is robust against these nonlinearities. Nevertheless, all the simulations made in this study were nonlinear simulations and the closed-loop system proved to be successful at least in the certain nonlinear regions it covered.

The design procedure using weighting functions is based on intuition and trial-and-error and a search for better weighting functions is always possible. Different designs for different criteria, e.g., different reference signals, tight input bounds, etc., can be practiced.

Finally, formulation of an IQC result for the primal IQC that uses more information about the structure of the certain perturbation block can be considered as a future work area.

APPENDIX A: AUXILIARY RESULTS

In this chapter, we present some auxiliary results that are used in the proof of Theorem 4.1.1.

A.1. Duality Results

Below, we give a duality result for static integral quadratic constraints.

Lemma A.1.1. *[3] Let* $M = M^T \in \mathbb{R}^{(p+q)\times(p+q)}$ *be such that* $\mathrm{in}(M) = \{p, q, 0\}$*. Let* $\Delta : \mathcal{L}_2^p \to \mathcal{L}_2^q$ *be a bounded multiplication operator. Then,*

$$\left\langle \begin{pmatrix} I \\ \Delta \end{pmatrix} v, M \begin{pmatrix} I \\ \Delta \end{pmatrix} v \right\rangle \geq 0 \quad \forall\, v \in \mathcal{L}_2^p$$

if and only if

$$\left\langle \begin{pmatrix} -\Delta^* \\ I \end{pmatrix} w, M^{-1} \begin{pmatrix} -\Delta^* \\ I \end{pmatrix} w \right\rangle \leq 0 \quad \forall\, w \in \mathcal{L}_2^q.$$

Corollary A.1.2. *Let* $\Phi = \Phi^T \in \mathbb{R}^{(p+m)\times(p+m)}$ *have* $\mathrm{in}(\Phi) = (p, m, 0)$ *and let* $K \in \mathbb{R}^{m\times p}$ *be given. Then,*

$$\begin{pmatrix} I_m \\ K \end{pmatrix}^T \Phi \begin{pmatrix} I_m \\ K \end{pmatrix} < 0$$

if and only if

$$\begin{pmatrix} -K^T \\ I_p \end{pmatrix}^T \Phi^{-1} \begin{pmatrix} -K^T \\ I_p \end{pmatrix} > 0.$$

A.2. Matrix Elimination Lemma

The following lemma has a central role in the proof of Theorem 4.1.1. It makes it possible to eliminate the controller parameters from the inequalities and obtain existence conditions for a stabilizing controller in terms of LMI's.

Lemma A.2.1. [**1**] *Let* $A \in \mathbb{R}^{k\times n}$, $B \in \mathbb{R}^{k\times m}$, $C \in \mathbb{R}^{p\times n}$ *and* $\Pi = \Pi^T \in \mathbb{R}^{(k+n)\times(k+n)}$ *be given. Assume* $\mathrm{in}(\Pi) = \{k, n, 0\}$. *Then, there exists a* $K \in \mathbb{R}^{m\times p}$ *such that*

$$\begin{pmatrix} A + BKC \\ I_n \end{pmatrix}^T \Pi \begin{pmatrix} A + BKC \\ I_n \end{pmatrix} < 0$$

if and only if

$$(C^T)_\perp^T \begin{pmatrix} A \\ I_n \end{pmatrix}^T \Pi \begin{pmatrix} A \\ I_n \end{pmatrix} (C^T)_\perp < 0,$$
$$B_\perp^T \begin{pmatrix} I_k \\ -A^T \end{pmatrix}^T \Pi^{-1} \begin{pmatrix} I_k \\ -A^T \end{pmatrix} B_\perp > 0.$$

A.3. Some Linear Algebra Results

The following lemmas are very basic results in linear algebra. They are mainly used in the construction procedure of the controller.

Lemma A.3.1. *[12] Let* $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times n}$. *Then, the followings are equivalent:*

(i) $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} > 0.$

(ii) $A > 0$ *and* $C - B^T A^{-1} B > 0.$

(iii) $A - BC^{-1}B^T > 0$ *and* $C > 0.$

Lemma A.3.2. *[3] Let* $M = M^T \in \mathbb{R}^{n\times n}$ *and let* $B \in \mathbb{R}^{n\times m}$ *be full column-rank. Then,* $n_-(M) \geq n_-(B^T M B)$.

Lemma A.3.3. *[12] Let* $M = M^T \in \mathbb{R}^{(n+p)\times(n+p)}$*. Then,*

$$n_-(M) \geq p$$

if and only if there exists a full column-rank $B \in \mathbb{R}^{(n+p)\times p}$ *such that*

$$B^T AB < 0.$$

APPENDIX B: DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion for the system can be obtained by Lagrange's method. Now, we shortly state Lagrange's formulation.

B.1. Lagrange's Equations

For a system of N particles, let $\underline{r}_j$'s be the position vectors and $\underline{F}_j^{nc}$ denote the non-conservative applied force on the j-th particle. If we define the Lagrangian L of the system as $L = T - V$, where T is the total kinetic energy of the system and V stands for the potential energy, then, system dynamics can be expressed by the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{nc} \quad \text{for} \quad i = 1, 2, ..., n_{dof}$$

where q_i's are the generalized independent coordinates, n_{dof} is the number of degrees of freedom of the system, and $Q_i^{nc} = \sum_{j=1}^{N} \underline{F}_j \cdot \frac{\partial \underline{r}_j}{\partial q_i}$ is the generalized non-conservative force associated with the generalized coordinate q_i.

B.2. The Equations of Motion of the Two-Link Arm System

We want to express the kinetic and potential energies of the system in terms of the generalized coordinates q_1, q_2 and their time-derivatives $\dot{q}_1$, $\dot{q}_2$. The kinetic energy, T, of the system can be expressed as the sum of the kinetic energies of the two masses:

$$T = \frac{1}{2}m(\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1) + \frac{1}{2}m(\dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2),$$

where $\dot{\underline{r}}_1$ and $\dot{\underline{r}}_2$ denote the velocities of the first and second masses, respectively. We can express the position vectors $\underline{r}_1$ and $\underline{r}_2$ as

$$\underline{r}_1 = R[\cos(\Omega t)\underline{i} + \sin(\Omega t)\underline{j}] + L\underline{e},$$
$$\underline{r}_2 = \underline{r}_1 + L\underline{u},$$

where the unit vectors $\underline{i}$, $\underline{j}$, $\underline{e}$, $\underline{e}_\perp$, $\underline{u}$ and $\underline{u}_\perp$ are taken as shown in Figure B.1. We have

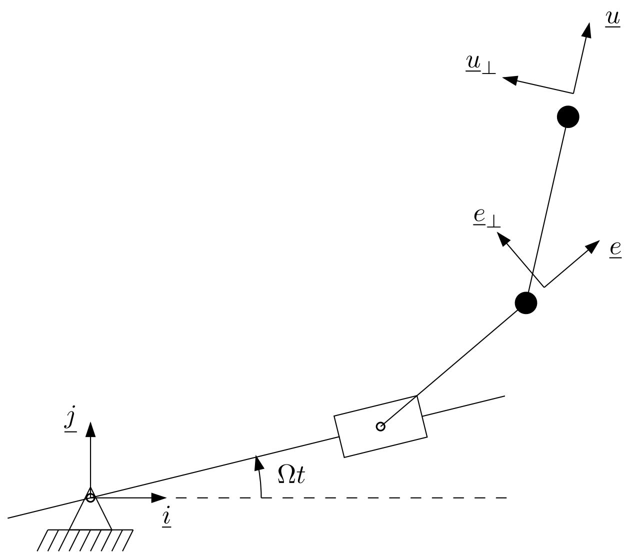

Figure B.1. The unit vectors

assumed that the base lies along the x-direction at $t = 0$, which leads to the following relationships between the rotating unit vectors $\underline{e}$, $\underline{e}_\perp$, $\underline{u}$, $\underline{u}_\perp$ and the stationary ones $\underline{i}$, $\underline{j}$:

$$\begin{aligned}
\underline{e} &= \cos(\Omega t + q_1)\underline{i} + \sin(\Omega t + q_1)\underline{j}, \\
\underline{e}_\perp &= -\sin(\Omega t + q_1)\underline{i} + \cos(\Omega t + q_1)\underline{j}, \\
\underline{u} &= \cos(\Omega t + q_2)\underline{i} + \sin(\Omega t + q_2)\underline{j}, \\
\underline{u}_\perp &= -\sin(\Omega t + q_2)\underline{i} + \cos(\Omega t + q_2)\underline{j}.
\end{aligned}$$

Taking time-derivative of $\underline{r}_1$, we obtain

$$\begin{aligned}\dot{\underline{r}}_1 &= \left[\dot{R}\cos(\Omega t) - R\Omega\sin(\Omega t) - L(\Omega+\dot{q}_1)\sin(\Omega t+q_1)\right]\underline{i} \\ &\quad + \left[\dot{R}\sin(\Omega t) + R\Omega\cos(\Omega t) + L(\Omega+\dot{q}_1)\cos(\Omega t+q_1)\right]\underline{j}.\end{aligned}$$

The velocity of the second mass, $\dot{\underline{r}}_2$, is

$$\begin{aligned}\dot{\underline{r}}_2 &= \dot{\underline{r}}_1 + L(\Omega+\dot{q}_2)\underline{u}_\perp \\ &= \dot{\underline{r}}_1 + L(\Omega+\dot{q}_2)\left[-\sin(\Omega t+q_2)\underline{i} + \cos(\Omega t+q_2)\underline{j}\right].\end{aligned}$$

Then, T can be calculated to be

$$\begin{aligned}T &= m\Big[\dot{R}^2 + R^2\Omega^2 + L^2(\Omega+\dot{q}_1)^2 - 2\dot{R}L(\Omega+\dot{q}_1)\sin q_1 \\ &\quad +2R\Omega L(\Omega+\dot{q}_1)\cos q_1 - \dot{R}L(\Omega+\dot{q}_2)\sin q_2 \\ &\quad +R\Omega L(\Omega+\dot{q}_2)\cos q_2 + L^2(\Omega+\dot{q}_1)(\Omega+\dot{q}_2)\cos(q_2-q_1) \\ &\quad +\frac{1}{2}L^2(\Omega+\dot{q}_2)^2\Big].\end{aligned}$$

The potential energy stored in the system is due to the torsional springs located at the joints at the base and the first mass, and can be written as

$$V = \frac{1}{2}kq_1^2 + \frac{1}{2}k(q_2-q_1)^2.$$

Then, the Lagrangian L becomes

$$\begin{aligned}L &= T - V \\ &= m\Big[\dot{R}^2 + R^2\Omega^2 + L^2(\Omega+\dot{q}_1)^2 - 2\dot{R}L(\Omega+\dot{q}_1)\sin q_1 \\ &\quad +2R\Omega L(\Omega+\dot{q}_1)\cos q_1 - \dot{R}L(\Omega+\dot{q}_2)\sin q_2 \\ &\quad +R\Omega L(\Omega+\dot{q}_2)\cos q_2 + L^2(\Omega+\dot{q}_1)(\Omega+\dot{q}_2)\cos(q_2-q_1) \\ &\quad +\frac{1}{2}L^2(\Omega+\dot{q}_2)^2\Big] - \frac{1}{2}kq_1^2 - \frac{1}{2}k(q_2-q_1)^2.\end{aligned}$$

Now, we can obtain two coupled differential equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{nc} \quad \text{for} \quad i = 1, 2.$$

Taking derivatives, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) &= m\Big[2L^2\ddot{q}_1 - 2L\ddot{R}\sin q_1 - 2L\dot{R}\cos q_1\dot{q}_1 + 2L\Omega\dot{R}\cos q_1 \\
&\quad -2L\Omega R\sin q_1\dot{q}_1 + L^2\cos(q_2 - q_1)\ddot{q}_2 \\
&\quad -L^2\sin(q_2 - q_1)(\Omega + \dot{q}_2)(\dot{q}_2 - \dot{q}_1)\Big], \\
\frac{\partial L}{\partial q_1} &= m\big[-2L\dot{R}(\Omega + \dot{q}_1)\cos q_1 - 2L\Omega R(\Omega + \dot{q}_1)\sin q_1 \\
&\quad +L^2(\Omega + \dot{q}_1)(\Omega + \dot{q}_2)\sin(q_2 - q_1)\big] - kq_1 + k(q_2 - q_1),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) &= m\big[-L\ddot{R}\sin q_2 - L\dot{R}\cos q_2\dot{q}_2 + L\Omega\dot{R}\cos q_2 - L\Omega R\sin q_2\dot{q}_2 \\
&\quad +L^2\cos(q_2 - q_1) - L^2(\Omega + \dot{q}_1)\sin(q_2 - q_1)(\dot{q}_2 - \dot{q}_1) + L^2\ddot{q}_2\big], \\
\frac{\partial L}{\partial q_2} &= m\big[-L\dot{R}(\Omega + \dot{q}_2)\cos q_2 - L\Omega R(\Omega + \dot{q}_2)\sin q_2 \\
&\quad -L^2(\Omega + \dot{q}_1)(\Omega + \dot{q}_2)\sin(q_2 - q_1)\big] - k(q_2 - q_1).
\end{aligned}$$

Then, the equations of motion become

$$\begin{aligned}
& m\left[2L^2\ddot{q}_1 + L^2\cos(q_2 - q_1)\ddot{q}_2 - L^2\sin(q_2 - q_1)\dot{q}_2^2 - 2L^2\Omega\sin(q_2 - q_1)\dot{q}_2\right. \\
& \left. -L^2\Omega^2\sin(q_2 - q_1) + 4L\Omega\dot{R}\cos q_1 + 2L\left(\Omega^2 R - \ddot{R}\right)\sin q_1\right] \\
& + 2kq_1 - kq_2 = T_1 - 2c\dot{q}_1 + c\dot{q}_2, \\
& m\left[L^2\cos(q_2 - q_1)\ddot{q}_1 + L^2\ddot{q}_2 + L^2\sin(q_2 - q_1)\dot{q}_1^2 + 2L^2\Omega\sin(q_2 - q_1)\dot{q}_1\right. \\
& \left. L^2\Omega^2\sin(q_2 - q_1) + L\left(\Omega^2 R - \ddot{R}\right)\sin q_2 + 2L\Omega\dot{R}\cos q_2\right] \\
& - kq_1 + kq_2 = T_2 + c\dot{q}_1 - c\dot{q}_2.
\end{aligned}$$

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