GENERALIZED COMPLEX GEOMETRY AND NILMANIFOLDS

by

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ABSTRACT

GENERALIZED COMPLEX GEOMETRY AND NILMANIFOLDS

The generalized complex geometry is a relatively new and highly popular subject which also has applications in theoretical physics. It studies the geometric structures on $TM \oplus T^*M$. That is to say, this new geometry develops a new language that treats tangent and contangent bundles of a manifold simultaneously. In this thesis, we will first study essential properties of generalized complex geometry. On the way we will see, how this approach gives a new way to study complex and symplectic structures. This observation directs us to investigate the generalized complex and generalized Kähler structures on nilmanifolds since these spaces contain interesting examples of complex and symplectic geometries. Although it is known that some six dimensional nilmanifolds do no admit neither symplectic nor complex structures, they all admit generalized complex structure. In order to understand these structures in details, we explicitly construct a generalized complex structure on a nilmanifold. Moreover, discussing the Hodge theory and the formality property of generalized complex structures, we will show that if a nilmanifold admits a generalized Kähler structure then it has a trivial nilpotent Lie algebra.

ÖZET

GENELLEŞTİRİLMİŞ KOMPLEKS GEOMETRİ VE NİL-UZAYLARI

Genelleştirilmiş kompleks geometri farklı alanlardan birçok araştırmacının ilgisini çeken, göreceli olarak yeni sayılabilecek bir çalışma sahasıdır. Bu yeni konunun üzerinde çalıştığı kavramlar, bir çokkatlının ne teğet uzayından ne de eş-teğet uzayından gelirler. Bunların yerine, iki farklı kavram birleştirilip, bizim genelleştirilmiş teğet uzayı diye isimlendireceğimiz uzay üzerinde çalışılır. Bu tezin amacı, bu yeni konunun karakteristik özelliklerini anlamak olacaktır. Bunu yaparken, kompleks ve simplektik geometrilerin yeni kurgu içerisinde, farklı bir şekilde anlaşılabileceğini göreceğiz. Böylece konunun daha net anlaşılabilmesi için nil-çokkatlılarını çalışacağız. Bu uzaylar, tarihsel olarak birçok ilginç örnek ve karşı örneğe doğal kaynak oluşturmuşlardır. Yine göreceğiz ki, genel teorinin aksine, tüm altı boyutlu nil-çokkatlılarının üzerine genelleştirilmiş kompleks yapı tanımlanabilir. Ne var ki, Kähler çokkatlıları için durum genel durumla örtüşüyor. Yani genelleştirilmiş Kähler yapısına sahip olabilecek tek nil-çokkatlısı sınıfı, bu sınıfa karşılık gelen Lie algebrasının bariz olması durumudur.

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LIST OF SYMBOLS

A^*	The adjoint of a linear map A
$CL(V \oplus V^*)$	The Clifford algebra generated by $V \oplus V^*$ and $< \cdot, \cdot >$
E^0	Annihilator of the subspace E
$End(V)$	Linear maps from V to V
$graph(\epsilon, E)$	The set given by the graph of the map ϵ defined on E
J	Complex structure on a manifold M
$\mathcal{J}$	Generalized Complex structure
$\mathbb{I}$	Identity map on the fibres of generalized tangent bundle
I	Identity map on the fibres of tangent bundle
L	The maximal isotropic subspace corresponding to GCS $\mathcal{J}$
$\mathfrak{so}(m, m)$	The Lie algebra of $SO(m, m)$
$SO(m, m)$	Special orthogonal group with signature (m, m) inner product
TM	The tangent bundle of M
T^*M	The cotangent bundle of M
$T \oplus T^*$	Generalized tangent bundle: $TM \oplus T^*M$
$\mathbb{V}$	The generalized tangent space $V \oplus V^*$

$\epsilon(X, \cdot)$	The one form constructed by a two form ϵ
$\xi\|_E$	Restriction of a form ξ defined on V to a subspace E
ω	Symplectic structure on a manifold M
$\pi\|_V L$	Projection of the subspace L onto V

$<, >$	Natural pairing
$\{e^i\}$	A basis for vector spaces
$\{e_i\}$	A dual basis corresponding to $\{e^i\}$
$\bigwedge^k V^*$	Degree k forms defined on V
$(,)_{top}$	The m-th degree part of an differential form

LIST OF ACRONYMS/ABBREVIATIONS

AGCS	Almost Generalized Complex Structure
GCG	Generalized Complex Geometry
GCS	Generalized Complex Structure
GKS	Generalized Kähler Structure

1. INTRODUCTION

Generalized geometry is a relatively new and intensely studied branch of differential geometry in which, cotangent and tangent bundles are treated together. In other words, it unifies and extends the notions developed separately for tangent and cotangent spaces to generalized tangent bundle $TM \oplus T^*M$. This subject was introduced by N.Hitchin [1] in 2003. In the introduction of his paper, N. Hitchin states: "we introduce in this paper a geometrical structure on a manifold which generalizes both the concept of a Calabi-Yau manifold and that of a symplectic manifold " which explains the origins of this new field.

Following Hitchin's work, this subject initially developed further by his Ph.D. students M. Gualtieri [2], and G. Cavalcanti [3]. In fact, M. Gualtieri's Ph.D. thesis [2] is commonly accepted as the main source on this field. Although Hitchin's initial motivation did not stem from physics, it was soon recognized that this subject has very deep relations with one of the major study areas in theoretical physics, namely the String Theory. M. Gualtieri introduced the notion of Generalized Kähler Structure and showed that this is equivalent to bi-Hermitian geometry [2] which was developed years ago by physicists, S. Gates, C. Hull and M. Rocek [4] as a general solution to the $(2, 2)$ supersymmetric sigma model. Developing further this relation, C. Hull and B. Zweibach introduced "Double Field Theory" [5] which provides a framework for geometrical understanding of the so called T-duality in the String Theory. (For further discussion of T-duality from generalized complex geometry (GCG) point of view, see [6]. For more detailed discussion of applications of this subject in physics, see [7] and [8].) Thanks to these connections generalized geometry receives a big amount of interest from physicists.

Besides its importance in physics, generalized geometry is also a very natural and interesting subject to study for mathematicians. In this new setting, complex, symplectic and Poisson geometries appear as special cases of generalized complex geometry. Thus, the tools developed for these geometries work in harmony. For example, a generalized version of Darboux theorem and the decomposition of differential operator are some results which we can get their generalized analogous in GCG [2]. In addition to its unification property, it enables to study some important mathematical objects, like Dirac geometry and Courant algebroids introduced by J. Courant [9], from different perspectives. For more motivation and further applications of generalized geometry, see [10] and [11].

One of the our main goals in this thesis is to apply the theory of GCS on nilmanifolds. Nilmanifolds are homogeneous spaces that are given as the quotients of nilpotent Lie groups by co-compact discrete subgroups. Since their appearance in the article of Thurston [12] as the first example of a symplectic manifold which do not admit a Kähler Structure, they are studied intensively as a source of interesting examples for geometrical structures. The classification of nilmanifolds with dimension less than seven was done by L. Magnin in [13]. and in [14] it is shown that, up to isomorphisms there are 34 different equivalence classes of six dimensional nilmanifolds . Moreover, due to results in [15] and [16], we know that the five of 34 different classes of nilmanifolds does not admit a symplectic or a complex structure. Since in the above we stated that GCG is closely related to complex and symplectic structures, it is natural to ask whether these five classes of nilmanifolds do admit generalized complex structures (GCS). In their articles [14], M. Gualtieri and G. Cavalcanti ask this question and show that every six dimensional nilmanifold admit a GCS. Taking one step further, motivated by the Thurston's original paper we can wonder whether nilmanifolds admits generalized Kähler structures (GKS). This question is answered in Cavalcanti's article [17].

This thesis is organized as follows. In the second chapter, we will discuss the generalized complex structure in the linear algebra setting. In that chapter, we will introduce basic concepts, definitions and theorems without taking into account the integrability issues of the structure. In the third chapter, we extend our definitions to manifolds and state the integrability conditions on these structures. Then we will discuss GKS and metric structures on generalized geometry. Finally, in chapter four, we will focus on the constructions of GCS and GKS on nilmanifolds.

Finally, we assume basic knowledge on differential geometry, Riemannian geometry and complex geometry. For comprehensive introduction to these subjects, see for example [18] and [19].

2. LINEAR ALGEBRA SETTING

Throughout this chapter, we will work on $\mathbb{V} \doteq V \oplus V^*$ where V is an m-dimensional vector space over $\mathbb{R}$ and V^* its dual space. In fact, this setting is an essential ingredient of generalized geometry since pointwise we will work on such a vector space in general. After introducing the basic tools, we will define linear generalized complex structures on $\mathbb{V}$. In Sections 2.3 and 2.4 we will try to understand GCS in terms of maximal isotropic subspaces and pure spinors. The maximal isotropic description of a GCS will be particularly important since the integrability conditions will be mainly discussed in terms of closedness of these subspaces under a suitably defined bracket called Courant bracket. On the other hand, pure spinor description will be useful when we study GCS on nilmanifolds. Moreover, the pure spinors enable us to extend some well-known results like Dolbeault decomposition theorem on complex manifolds and closedness condition on symplectic manifolds. In this chapter we mainly follow the articles [2] and [20].

In this chapter, we mainly follow the articles [2] and [20] and our conventions are as follows : An element of the space $\mathbb{V}$ will always be denoted in the form $X + \xi$. In this notation, capital letters from Latin alphabet, e.g. X, Y, represent the elements from V and the letters from Greek alphabet, e.g. ξ, η, represent the elements from V^*. Also $\{e^1, \cdots e^m\}$ stands for a basis for V, and $\{e_1, \cdots e_m\}$ stands for a basis for V^*.

2.1. Linear Algebra of $\mathbb{V}$

In this section, we will develop some basic tools on $\mathbb{V}$. The first one is a symmetric, signature (m,m) bilinear form $<,>$, which is defined in the following way:

$$< X + \xi, Y + \eta > \; = \; \frac{1}{2}(\eta(X) + \xi(Y)) \tag{2.1}$$

This bilinear form will be called the natural pairing. Notice that, if we choose

$$\{e^1 + e_1, ..., e^m + e_m, e^1 - e_1, ..., e^m - e_m\}$$

as a basis for $\mathbb{V}$ then the matrix representation of the natural pairing becomes

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

which proves that this bilinear form has signature (m, m). If we study in the basis $\{e^1, ..., e^m, e_1, ..., e_m\}$, then the matrix representation of the natural pairing $<,>$ is $\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Also, in this case if we write a linear map $M = \begin{pmatrix} A & \beta \\ B & C \end{pmatrix}$ from $\mathbb{V}$ to itself, then the adjoint of M with respect to the natural pairing will be $M^* = \begin{pmatrix} C^* & \beta^* \\ B^* & A^* \end{pmatrix}$. Now we will investigate the action of elements of $\mathfrak{so}(m, m)$. An arbitrary element of $\mathfrak{so}(m, m)$ can be written in the form $\begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}$ where $A \in End(V)$, $\beta \in \bigwedge^2 V$ and $B \in \bigwedge^2 V^*$. If we consider B as a map from V to V^*, its action which is extended to $\mathbb{V}$ can be written as :

$$B \doteq \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ B(X, \cdot) \end{pmatrix} \tag{2.2}$$

We will denote this map again by B. It is easy the see that the map B is in $\mathfrak{so}(m, m)$:

$$< B(u), v > + < u, B(v) > = \frac{1}{2}(B(X, Y) + B(Y, X) = 0)$$

where $u = X + \xi$ and $v = Y + \eta$ are any two elements of $\mathbb{V}$. Similarly define the mappings

$$A \doteq \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} A(X) \\ -A^*(\xi) \end{pmatrix}, \quad \beta \doteq \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} \beta(\xi) \\ 0 \end{pmatrix} \tag{2.3}$$

Similar to the above argument, again one can show that all these mappings are the elements of $\mathfrak{so}(m,m)$. Also, let us write the $SO(m,m)$ action of these elements. We know that the exponential map takes an element of the Lie algebra and maps it into the corresponding Lie group. So, the exponentials of the elements of $\mathfrak{so}(m,m)$ given in Equations 2.2 and 2.3 are given by :

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad e^A = \begin{pmatrix} e^A & 0 \\ 0 & (e^{A^*})^{-1} \end{pmatrix}, \qquad e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

Also their corresponding $SO(m,m)$ action on an element of $\mathbb{V}$ is given by :

$$\begin{aligned} e^B(X+\xi) &= X+\xi+B(X,\cdot) \\ e^A(X+\xi) &= e^A(X)+(e^{A^*})^{-1}(\xi) \\ e^\beta(X+\xi) &= X+\beta(\xi,\cdot)+\xi \end{aligned} \tag{2.5}$$

2.2. Linear Generalized Complex Structure

In this section, we will give the definition of the GCS on $\mathbb{V}$ and then discuss some properties of these structures. Throughout this thesis we will use the symbol $\mathcal{J}$ for a GCS on $\mathbb{V}$. Also, the symbol $\mathbb{I}$ is used for the identity map on $\mathbb{V}$ and I denotes the identity map of V.

Definition 2.1. *A generalized complex structure GCS on a vector space* $\mathbb{V}$ *is an endomorphism* $\mathcal{J}$ *of* $\mathbb{V}$ *such that* $\mathcal{J}^2 = -\mathbb{I}$ *and* $\mathcal{J}^* = -\mathcal{J}$.

Remark 2.2. *Notice that the definition of a GCS generalizes the complex structure condition* $J^2 = -I$ *and the symplectic structure condition* $\omega^* = -\omega$.

Proposition 2.3. $\mathcal{J}$ *is a generalized complex structure on* $\mathbb{V}$ *if and only if* $\mathcal{J}^2 = -\mathbb{I}$ *and it is orthogonal with respect to the natural pairing.*

Proof. Assume that $\mathcal{J}$ is a GCS on $\mathbb{V}$. Then, it is clearly a complex structure on $\mathbb{V}$ and also $\mathcal{J}^*\mathcal{J} = -\mathcal{J}\mathcal{J} = -(-\mathbb{I}) = \mathbb{I}$, which proves the first part of the claim. Conversely, assume that $\mathcal{J}^2 = -\mathbb{I}$ and $\mathcal{J}^*\mathcal{J} = \mathbb{I}$ then $(\mathcal{J}^*\mathcal{J})\mathcal{J} = \mathcal{J}\mathbb{I}$ which implies $\mathcal{J}^* = -\mathcal{J}$. □

We will now observe that usual complex and symplectic structures are naturally embedded in the notion of generalized complex structure.

Example 2.4. *Let J be a complex structure on V. Then*

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

defines a GCS on $\mathbb{V}$. This is because

$$\mathcal{J}^2 = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} = \begin{pmatrix} J^2 & 0 \\ 0 & (J^2)^* \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} = -\mathbb{I}$$

Also we have

$$\mathcal{J}^* = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}^* = \begin{pmatrix} (J^*)^* & 0 \\ 0 & (-J)^* \end{pmatrix} = -\mathcal{J}.$$

Example 2.5. *Let ω be a symplectic structure on V. Then*

$$\begin{aligned} \mathcal{J}_\omega &= \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \\ (\mathcal{J}_\omega)^* &= \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & (-\omega^{-1})^* \\ (\omega)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} = -\mathcal{J}_\omega \end{aligned}$$

So ω defines a GCS on $\mathbb{V}$.

2.3. Maximal Isotropic Subspaces

As we know, we can determine a complex structure J by specifying its i-eigenspace. Here we will see that a similar property is valid for a GCS. However, first we need to define maximal isotropic subspaces and investigate some properties of these subspaces.

Definition 2.6. *A subspace L is called an isotropic subspace of a vector space $\mathbb{V}$ if $< u, v >= 0$ for any elements $u, v \in L$. Moreover, L is called maximal isotropic subspace if we cannot find a subspace $L^{'}$ such that $L < L^{'} < \mathbb{V}$.*

Proposition 2.7. *If L is an isotropic subspace of $\mathbb{V}$ with $dim(L) = m$, then L is maximal isotropic.*

Proof. Assume that L is an isotropic subspace and has dimension m. Suppose we have a subspace $L^{'}$ such that $dim(L^{'}) = m + 1$. Consider a basis

$$\left\{e^1 + e_1, ..., e^m + e_m, e^1 - e_1, ..., e^m - e_m\right\}$$

of $\mathbb{V}$. The subspaces $V_{\pm} = span\,\{e^1 \pm e_1, ..., e^m \pm e_m\}$ defined in this way are positive definite and negative definite subspaces of $\mathbb{V}$. Hence $L^{'} \cap V_+ = \{0\}$ (similarly $L^{'} \cap V_- = \{0\}$). However, this implies $dim(L^{'} \oplus V_+) \geq 2m + 1$ which is a contradiction. □

We will give now an important example of a maximal isotropic subspace. Then we will show that this actually is the generic example of maximal isotropic subspaces.

Example 2.8. *Let us define the following subspace of $\mathbb{V}$*

$$L(E, \epsilon) = \{X + \xi \in E \oplus V^* : \xi|_E = \epsilon(X, \cdot)\} \tag{2.6}$$

where $\epsilon \in \wedge^2 E^*$. First of all, this space is isotropic because ϵ is chosen to be an anti-symmetric bilinear map. Moreover, this is a maximal isotropic subspace, since we will see in next Lemma 2.9 that dimension of this space is m. So, Proposition 2.7 shows that $L(E, \epsilon)$ is a maximal isotropic.

Lemma 2.9. *Every maximal isotropic subspace L of $\mathbb{V}$ is in the form of $L(E,\epsilon)$. More explicitly, it is the direct sum of the annihilator E^0 of the subspace E of V with the graph of a two form ϵ defined on a subspace E i.e. $L = graph(\epsilon) \oplus E^0 = \left\{X + \epsilon(X,\cdot) \in \mathbb{V} \mid X \in E \,, \epsilon \in \bigwedge^2 V^*\right\} \oplus E^0$*

Proof. Let us define E in the following way: $E \doteq \pi|_V L$, where $\pi|_V$ is the projection onto V. Observe that E^0 is a subspace of L and also the kernel of the linear map $\pi|_V$. So, we can write $L = L' \oplus E^0$ where L' is the complementary subspace of E^0 in L. Since we take out the kernel of the mapping $\pi|_L$ we know that L' is isomorphic to its image E. That is to say we can find an isomorphism A such that, $L' = \{X + AX \mid X \in V,\ A : E \longrightarrow \pi\,|_{V^*}(L^{'})\}$. Finally, we need to show that $A \in \bigwedge^2 E^*$. We know that L' is an isotropic subspace so $0 = \langle X + AX, Y + AY\rangle = A(X,Y) + A(Y,X) = 0$. So, define $\epsilon(X,Y) \doteq A(X)(Y)$ □

Let us note that, all these results can be extended to complex vector spaces by extending our natural pairing to complex bilinear pairing.

Theorem 2.10. *All maximal isotropic subspaces of $\mathbb{V}\otimes\mathbb{C}$ can be written as $e^{\sigma}(E\oplus E^0)$ where $\sigma = B + i\omega$ is an element of $\bigwedge^2(V^* \otimes \mathbb{C})$ and e^{σ} acts on $X + \xi \in \mathbb{V}\otimes\mathbb{C}$ as follow*

$$e^{\sigma}(X + \xi) = X + \xi + \sigma(X,\cdot) \tag{2.7}$$

Proof. In Lemma 2.9, we have already showed that all maximal isotropic subspaces are in the form $L(E,\epsilon)$. Now, assume that every maximal isotropic is of the form $L(E,\epsilon)$. Then, we can choose σ as an element of $\bigwedge^2(V^* \otimes \mathbb{C})$ such that $\sigma(X,Y) = \epsilon(X,Y)$ for $X, Y \in E$. □

Corollary 2.11. *If L is a maximal isotropic subspace, then $dim(L) = m$.*

Proof. Assume that L is a maximal isotropic subspace of $\mathbb{V}$. Due to Lemma 2.9 , $L = L(E,\epsilon) = L' \oplus E^0$. A basis for L' can be given by $\{u^1 + Au^1, u^2 + Au^2 ... u^k + Au^k\}$

where $\{u^i\}$ is a basis for E and A is the map given in the proof of Lemma 2.9. Moreover, we know E^0 has dimension $m-k$ which implies that the dimension of L must be m. $\square$

Now we are ready to relate generalized complex structures on $\mathbb{V}$ and the maximal isotropic subspaces of $\mathbb{V} \otimes \mathbb{C}$.

Proposition 2.12. *Defining a generalized complex structure on* $\mathbb{V}$ *is equivalent to the specification of a maximal isotropic subspace* $L < \mathbb{V} \otimes \mathbb{C}$ *such that* $L \cap \bar{L} = \{0\}$.

Proof. First, if we have a generalized complex structure, its i-eigenspace L satisfies properties which are mentioned in this proposition. That is,

$$< u, v >=< \mathcal{J}(u), \mathcal{J}(v) >=< iu, iv >= - < u, v >$$

which implies $< u, v >= 0$. Also its dimension equals to 2m where m is the real dimension of V. Finally, it is clear that $L \cap \bar{L} = 0$ since $\bar{L}$ is the $(-i)$-eigenspace of $\mathcal{J}$. For the converse, if we are given a maximal isotropic subspace L such that $L \cap \bar{L} = 0$ then construct an endomorphism $\mathcal{J}$ such that L is the i-eigenspace of L and $\bar{L}$ is the $(-i)$-eigenspace of $\mathcal{J}$. $\square$

Definition 2.13. *Let* L *be a maximal isotropic subspace of* $\mathbb{V} \otimes \mathbb{C}$ *and* $(L \cap \bar{L}) = K \otimes \mathbb{C}$ *Then,*

$$dim_{\mathbb{C}}(L \cap \bar{L}) = dim_{\mathbb{R}}(K) = r$$

is called the real index of L.

Remark 2.14. *Thanks to Proposition 2.12 and Definition 2.13, we see that studying generalized complex structures on a vector space* $\mathbb{V}$ *is equivalent to studying maximal isotropic subspaces of* $\mathbb{V} \otimes \mathbb{C}$ *with real index zero. Therefore, it will be useful for us to study maximal isotropics to get a deeper understanding of these structures.*

Proposition 2.15. *(Proposition 4.4 of [2]) The maximal isotropic subspace* $L(E, \epsilon)$

has real index zero if and only if $E+\bar{E}=V\otimes\mathbb{C}$ and ϵ is such that the real skew 2-form $\omega_\Delta = i(\epsilon|_{E\cap\bar{E}} - \bar{\epsilon}|_{E\cap\bar{E}})$ is nondegenerate on Δ where $\Delta\otimes\mathbb{C} = E\cap\bar{E}$

Proof. First assume that $L(E,\epsilon)$ is a maximal isotropic subspace with real index zero. So $L\cap\bar{L}=\{0\}$ and $L\cup\bar{L}=E\otimes\mathbb{C}$. This implies $E\cup\bar{E}=V\otimes\mathbb{C}$. To prove the second part of the claim, suppose that there is a nonzero vector $X\in\Delta$ such that $(\epsilon-\bar{\epsilon})(X,\cdot)=0$. This condition implies that on $E\cap\bar{E}$, $\epsilon(X,\cdot)=\bar{\epsilon}(X,\cdot)$. So, we can define $\xi\in\mathbb{V}^*\otimes\mathbb{C}$ such that $\xi|_E=\epsilon(X,\cdot)$ and $\xi|_{\bar{E}}=\bar{\epsilon}(X,\cdot)$. On the intersection, this one form is well-defined since we have $\epsilon(X,\cdot)=\bar{\epsilon}(X,\cdot)$. Therefore, we conclude that a nonzero element $X+\xi$ is in $L\cap\bar{L}$ which is a contradiction. Conversely, assume that $E+\bar{E}=V\otimes\mathbb{C}$ and $\omega=i(\epsilon|_{E\cap\bar{E}}-\bar{\epsilon}|_{E\cap\bar{E}})$ is non-degenerate on $\Delta\otimes\mathbb{C}$. Suppose $X+\xi\in L\cap\bar{L}$. Then, the action of ξ on E given by $\epsilon(X,\cdot)$ and the action of ξ on $\bar{E}$ is given by $\bar{\epsilon}(X,\cdot)$. Also, on the intersection these must be equal to each other. So, on Δ we would have $\epsilon(X,\cdot)=\bar{\epsilon}(X,\cdot)$. Therefore, $\omega X,\cdot=0$ on Δ. By assumption of nondegeneracy of ω we get $X=0$. Recall that $\xi|_E=\epsilon(X,\cdot)=0$ and $\xi|_{\bar{E}}=\bar{\epsilon}(X,\cdot)=0$. Since, we assume that $E\oplus\bar{E}=V\otimes\mathbb{C}$ we conclude that $\xi=0$. Hence, we see that if $X+\xi\in L\cap L'$ then $X+\xi=0$. □

Definition 2.16. *As we proved in Proposition 2.12, we can uniquely determine the generalized complex structure $\mathcal{J}$ by its corresponding maximal isotropic subspace $L(E,\epsilon)$. The dimension of the space E^0 will be called the type of this generalized complex structure.*

2.4. Clifford Algebra of $\mathbb{V}$

In this section we will discuss the Clifford algebra generated by $(\mathbb{V},<,>)$. This algebra will be denoted by $CL(\mathbb{V})$. In this context the exterior algebra $\bigwedge V^*$ of V can be seen as the spinor representation of $CL(\mathbb{V})$. Understanding spinor description will be very useful in our later discussions. In particular, when we discuss the GCS on nilmanifolds, we will mostly define the structure by just stating its corresponding pure spinor, which will be defined in the next section. Let us note that, detailed information about Clifford algebras and pure spinors can be found in [21].

Definition 2.17. *The Clifford algebra of a vector space W, with a symmetric, bilinear form $< \cdot, \cdot >$ on it, is the vector space $\bigwedge W$ with an associative product called the Clifford product. This new algebra will be denoted by $CL(W)$ and the Clifford product is defined by the following relations*

$$\begin{aligned} e^i e^j + e^j e^i &= 2 < e^i, e^j > \\ \omega 1 &= 1\omega = \omega \end{aligned} \tag{2.8}$$

where e^i is an element of the oriented orthonormal basis for W, ω is any element in $\bigwedge W$ and $1 \in \bigwedge^0 W$.

Definition 2.18. *A left ideal $\mathcal{L}$ of the algebra $\mathcal{A}$ is called a minimal left ideal if it does not contain any other nontrivial left ideal. The spinor representation of a Clifford algebra $CL(W)$ is the regular representation on a minimal left ideal $S \subset CL(W)$. An element of the spinor representation is called a spinor.*

Let us now take $W = \mathbb{V}$. Since V and V^* are maximal isotropic subspaces, the $CL(V)$ is a subalgebra of $CL(\mathbb{V})$ and this subalgebra is isomorphic to $\bigwedge V$. Similarly, $\bigwedge V^* \simeq CL(V^*) < CL(\mathbb{V})$. Moreover, $\bigwedge^m V$ generates a left ideal since

$$(e^i + e_i)\sigma = (e^i + e_i)(ce^1 e^2 \cdots e^m) = ce_i(e^1 e^2 \cdots e^m) = e_i \sigma \tag{2.9}$$

Where $\sigma \in \bigwedge^m V$. Therefore, any element of this left ideal can be written as $\varphi\sigma$ where $\varphi \in \bigwedge V^*$. Due to this ideal we get a natural action on $\wedge V^*$:

$$(X + \xi) \cdot \phi = i_X \phi + \xi \wedge \phi \tag{2.10}$$

Notice that this is a Clifford action, that is :

$$
\begin{aligned}
(X+\xi)^2\cdot\phi &= i_X(i_X\phi+\xi\wedge\phi)+\xi\wedge(i_X\phi+\xi\wedge\phi) \\
&= i_X(i_X\phi)+(i_X\xi)\wedge\phi-\xi\wedge i_X\phi+\xi\wedge i_X\phi+\xi\wedge\xi\wedge\phi \\
&= (i_X\xi)\phi \\
&= <X+\xi,X+\xi>\phi \qquad (2.11)
\end{aligned}
$$

So, it is natural to choose $\bigwedge V^*$ as a spinor representation of $CL(\mathbb{V})$. Treating forms as spinors for $CL(\mathbb{V})$ will give a different and easy way to understand GCS in our later discussions. We will also define a bilinear form on the space of spinors which will be called the Mukai pairing of forms. Define the main anti-automorphism α of the $CL(\mathbb{V})$ as, $\alpha(v_1\cdots v_k)=v_k\cdots v_1$ where $v_i\in CL(\mathbb{V})$. Then, the Mukai pairing is a map $(\cdot,\cdot):S\otimes S\longmapsto det(V^*)$ such that $(s,t)=(\alpha(s)\wedge t)_{top}$ where $(,)_{top}$ means taking the m-th degree part of the form.

Proposition 2.19. *Let $v\in\mathbb{V}$ be any element and $s,\omega\in\bigwedge V^*$ then*

$$(v\cdot s,v\cdot\omega)=<v,v>(s,\omega) \qquad (2.12)$$

Proof. To prove this claim, we assume without loss of generality that $s\in\bigwedge^k V^*$ and $\omega\in\bigwedge^{m-k}V^*$. Let $v=X+\xi$. The action of this element to the spinors is given by $v\cdot s=i_xs+\xi\wedge s$. So, we compute:

$$
\begin{aligned}
(v\cdot s,v\cdot\omega) &= (\alpha(i_Xs+\xi\wedge s)\wedge(i_X\omega+\xi\wedge\omega))_{top} \\
&= (-1)^{k-1}(i_X\alpha(s)\wedge\xi\wedge\omega)+\alpha(s)\wedge\xi\wedge i_X\omega \\
&= (-1)^{k-1}(i_X\alpha(s)\wedge\xi\wedge\omega)+\alpha(s)\wedge\xi\wedge i_X\omega-\xi(X)\alpha(s)\wedge\omega \\
&+ \xi(X)\alpha(s)\wedge\omega \\
&= <v,v>(s,t) \qquad (2.13)
\end{aligned}
$$

Here, in the last step we used the fact that first three terms of (2.13) equals to $i_X(\alpha(s)\wedge$

$\xi \wedge \omega$) which is zero. □

It is a well-known fact that the spin group is a double cover of the special orthogonal group. So, it has an induced action on the vectors of $\mathbb{V}$. This action is defined in the following way:

$$\begin{aligned} \rho \quad &: \quad Spin(m,m) \longrightarrow SO(m,m) \\ &\quad \rho(g)(v) = gvg^{-1}, \qquad g \in Spin(m,m), \quad v \in \mathbb{V} \end{aligned} \tag{2.14}$$

Here in the right hand side, multiplications are Clifford multiplication and in the left hand side we have the usual $SO(m,m)$ action on vectors. This Lie group homomorphism induces a Lie algebra isomorphism $\mathfrak{spin}(m,m) \simeq \mathfrak{so}(m,m)$ and a Lie algebra action :

$$\begin{aligned} d\rho : \mathfrak{spin}(m,m) \quad &\longrightarrow \quad \mathfrak{so}(m,m) \\ d\rho_g(v) \quad &= \quad gv - vg \end{aligned} \tag{2.15}$$

Therefore, for $g \in \mathfrak{spin}(m,m)$ the action given by $gv - vg$ must be equal to usual $\mathfrak{so}(m,m)$ action of $d\rho_g$ on $\mathbb{V}$. Recall that we have mentioned the $\mathfrak{so}(m,m)$ action on $\mathbb{V}$, in Equations 2.2), (2.3) . Now, we will define spinorial action of the elements of $\mathfrak{so}(m,m)$.

Example 2.20. *(Example 2.10 of [2])(B-action) Equation (2.2) defines the $\mathfrak{so}(m,m)$ action of an element $B \in \bigwedge V^*$. The corresponding element is in the Lie algebra $\mathfrak{spin}(m,m)$ whose action is similar to $B = b_{ij}e^i \wedge e^j$ is $b_{ij}e^j e^i$ since*

$$\begin{aligned} e^i \wedge e^j \quad &: \quad e_k \longmapsto \delta^i_k e^j - \delta^j_k e^i \\ e^j e^i \quad &: \quad e_k \longmapsto e^j e^i e_k - e_k e^j e^i \\ &= \quad e^j(\delta^i_k - e_k e^i) - (\delta^j_k - e^j e_k)e^i \\ &= \quad \delta^i_k e^j - \delta^j_k e^i \end{aligned} \tag{2.16}$$

notice that the action of this element to an element of the spinor space is given by

$$\begin{aligned}(e^j e^i)\varphi &= e^j \wedge (e^i \wedge \varphi) \\ &= -B \wedge \varphi \end{aligned} \tag{2.17}$$

Therefore the action of the Spin(m, m) can be obtained by exponentiating (2.17)

$$e^{-B}\varphi = (1 - B \wedge \frac{1}{2} B \wedge B - \frac{1}{3!} B \wedge B \wedge B \cdots) \wedge \varphi \tag{2.18}$$

Example 2.21. *Similar to the above argument we can discuss the spin(m, m) correspondence of the action of an element $A \in End(V)$. The action of $A = A_i^j e^i \otimes e_j$ as an element of $\mathfrak{so}(m, m)$ is given in the Equation 2.3 as $A(X + \xi) = A(X) - A^*(\xi)$. The corresponding element in spin(m, m) is given by $\frac{1}{2} A_i^j (e_j e^i - e^i e_j)$.*

$$(e^\lambda \otimes e_j)(e_k + e^i) = \delta_k^\lambda e_j - \delta_j^i e^\lambda \tag{2.19}$$

$$\begin{aligned} &\frac{1}{2}(e_j e^\lambda - e^\lambda e_j)(e_k + e^i) - \frac{1}{2}(e_k + e^i)(e_j e^\lambda - e^\lambda e_j) \\ &= \frac{1}{2}\left[e_j e^\lambda e_k - e^\lambda e_j e_k + e_j e^\lambda e^i - e^\lambda e_j e^i\right] \\ &\quad - \frac{1}{2}\left[e_k e_j e^\lambda - e_k e^\lambda e_j + e^i e_j e^\lambda - e^i e^\lambda e_j\right] \\ &= \frac{1}{2}\left[e_j(\delta_k^\lambda - e_k e^\lambda) - (e^\lambda e_j e^k) + (e_j e^\lambda e^i) - e^\lambda(\delta_j^i - e^i e_j)\right] \\ &\quad - \frac{1}{2}\left[e_k e_j e^\lambda - (e^\lambda e_k - \delta_k^\lambda) e_j + (\delta_j^i - e_j e^i) e^\lambda - e^i e^\lambda e_j\right] \\ &= \delta_k^\lambda e_j - \delta_j^i e^\lambda \end{aligned} \tag{2.20}$$

In the last line, we make use of the fact that $e^i e^j = -e^j e^i$ and $e_i e_j = -e_j e_i$. Now, we *try to understand the action of this element on a spinor element φ:*

$$\begin{aligned}\frac{1}{2}A_i^j(e_je^i - e^ie_j)\varphi &= \frac{1}{2}A_i^j i_{e_j}(e^i \wedge \varphi) - \frac{1}{2}A_i^j(e^i \wedge i_{e_j}\varphi) \\ &= \frac{1}{2}A_i^j\delta_j^i\varphi - \frac{1}{2}A_i^j(e^i \wedge i_{e_j}\varphi) - \frac{1}{2}A_i^j(e^i \wedge i_{e_j}\varphi) \\ &= \frac{1}{2}A_i^i\varphi - A_i^j(e^i \wedge i_{e_j}\varphi) \\ &= \frac{1}{2}Tr(A)\varphi - A^*\varphi \end{aligned} \tag{2.21}$$

The exponentiation of this element gives the spinorial action of A which is $g\varphi = \sqrt{detg}(g^*)^{-1}\varphi$ where $g \in GL(V)^+$ where $GL(V)^+$ is the connected component of identity element of $GL(m, V)$.

2.5. Pure Spinors

Definition 2.22. *Let φ be any nonzero spinor. Define the subspace L_φ of $\mathbb{V}$:*

$$L_\varphi = \{v \in \mathbb{V} : v \cdot \varphi = 0\} \tag{2.22}$$

This space is called the null-space of φ.

In fact, we can show that any null-space is isotropic with respect to our natural inner product $<,>$ on $\mathbb{V}$.

$$2 < v, w > \varphi = (vw + wv) \cdot \varphi = 0 \quad \text{for} \;\; v, w \in L_\varphi \tag{2.23}$$

Which implies $< v, w >= 0$. Now we are ready to define pure spinors:

Definition 2.23. *A spinor φ is called pure if L_φ is maximal isotropic subspace. Let us remark that any constant multiple of a pure spinor is also a pure spinor corresponding to the same maximal isotropic subspace. This line of pure spinors will be called the canonical line of pure spinors.(When we study on manifolds, we will use the term the canonical line bundle for the bundle of pure spinors.)*

Notice that we have defined the null-spaces in terms of Clifford action of the elements of $\mathbb{V}$ on the differential forms. Now, using the relation between $SO(m,m)$ and $Spin(\mathbb{V})$ we will get an useful relation:

Proposition 2.24. *Let L be a maximal isotropic subspace corresponding to the pure spinor φ. Also let g be an element of $Spin(\mathbb{V})$. Then the relation between the action given in Equation 2.10 and the $SO(m,m)$ action of the element given by the image of g under the homomorphism ρ which is defined in Equation 2.14 is given by :*

$$\rho(g^{-1})L_{\varphi} = L_{g\cdot\varphi} \tag{2.24}$$

Proof. Note that

$$\begin{aligned} L_{g\cdot\varphi} &= \{v \in \mathbb{V} :\ v \cdot (g \cdot \varphi) = 0\} \\ &= \left\{v \in \mathbb{V} :\ (g^{-1}v) \cdot (g \cdot \varphi) = 0\right\} \\ &= \left\{v \in \mathbb{V} :\ \rho(g^{-1})(v \cdot \varphi) = 0\right\} \\ &= \rho(g^{-1}) \cdot \{v \in \mathbb{V} :\ v \cdot \varphi = 0\} \end{aligned} \tag{2.25}$$

□

Example 2.25. *Consider $1 \in \wedge V^*$.*

$$L_1 = \{X + \xi : i_X 1 + \xi \wedge 1 = 0\} \tag{2.26}$$

However $\xi \wedge 1 = 0 \Leftrightarrow \xi = 0$ and $i_X 1 = 0, \quad \forall X \in V$. Therefore we see that $L_1 = V = L(V, 0)$. We know that V is maximal isotropic subspace of $\mathbb{V}$ so 1 is a pure spinor. Using Equation 2.24 we can find another maximal null-subspaces. Apply spinor action $e^B \cdot 1 = e^B \wedge 1 = e^B$. From this action we get $L_{e^B} = e^{-B}L(V, 0) = L(V, -B) = \{X - i_X B :\ X \in V\}$.

We have a dual of Definition 2.23. Every maximal isotropic can be identified by a pure spinor line bundle. Now we will try to make this clear. (We will follow arguments given in [20])

Proposition 2.26. *$L_\varphi = L(E,0) = E \oplus E^0$ if and only if $\varphi = c(\theta_1 \wedge ... \wedge \theta_k)$ where $\{\theta_i\}$ is a basis for E^0 and c is any nonzero constant.*

Proof. Let us assume that $L_\varphi = E \oplus E^0$ then we must have $(X+\xi)\cdot\varphi = 0$ if and only if $i_X\varphi = 0$ and $\xi \wedge \varphi = 0$. These equations must hold $\forall X \in E$ and $\forall \xi \in E^0$. Therefore these equations are valid if and only if $\varphi = c(\theta_1 \wedge ... \wedge \theta_k)$. □

Let $\mathcal{J}$ be a GCS on $\mathbb{V}$ and L be its i-eigenspace. Then, we get $CL(\mathbb{V}\otimes\mathbb{C}) \cong CL(L \oplus \bar{L})$. The action of $CL(L\oplus\bar{L})$ on pure spinor φ corresponding to the L gives a new $\mathbb{Z}$ grading of forms on V. This decomposition will be given by the following construction

$$\begin{aligned} U_k &= \wedge^{nm-k}\bar{L}\cdot\rho \\ \wedge(V^*\otimes\mathbb{C}) &= U_{-n}\oplus\cdots\oplus U_n. \end{aligned} \tag{2.27}$$

Proposition 2.27. *(Proposition 2.3 of [3]) In the above decomposition each U_k can be defined as the ki-eigenspace of the GCS $\mathcal{J}$ in the Lie algebra action.*

Proof. We know that L and $\bar{L}$ are maximal isotropic subspaces of $\mathbb{V}\otimes\mathbb{C}$ so we have the identification $CL(\mathbb{V}\otimes\mathbb{C}) \cong CL(L\oplus\bar{L})$. Following the argument given in Section 2.4, we try to find the spinorial action of $\mathcal{J}$ to the space $U_k = (\bigwedge^{n-k}\bar{L})\cdot\rho$, where ρ is a pure spinor of L. Notice that under this setting one can see $\mathcal{J}$ as

$$\mathcal{J} = \begin{pmatrix} iI_{m\times m} & 0 \\ 0 & -iI_{m\times m} \end{pmatrix} \tag{2.28}$$

Since, if we choose a basis $e_1,\cdots,e_{2n}$ for the i- eigensubspace L of $\mathcal{J}$ then we can write $iI = ie^i \otimes e_i$. It is clear that $iI\cdot\rho = 0$ since $e_i\cdot\rho = 0$ for any $e_i \in L$. So, using (2.3) we can see that the action of $\mathcal{J}$ on $\varphi \in (\bigwedge^{n-k} \bar{L}) \cdot \rho$

$$\begin{aligned} \mathcal{J} \cdot \varphi &= \frac{1}{2}Tr(iI)\varphi - (iI) \cdot \varphi \\ &= in\varphi - (i(n-k))\varphi \\ &= ik\varphi \end{aligned} \tag{2.29}$$

□

Proposition 2.28. *Let $L(E, \epsilon)$ be any maximal isotropic. Then, the defining pure spinor line bundle U_L is given by*

$$\varphi_L = c(e^{-B}\theta_1 \wedge ... \wedge \theta_k) \tag{2.30}$$

such that $\iota^* B = \epsilon$ and θ_i's are basis for E^0.

Proof. We know that

$$\begin{aligned} L_{e^B \cdot \varphi} &= e^{-B}L_\varphi = e^{-B}L(E, \epsilon) \\ &= e^{-B}\{X + \xi : \xi|_E = \epsilon(X, \cdot)\} \\ &= \{X + \xi - i_X B : \xi|_E = \epsilon(X, \cdot)\} \end{aligned} \tag{2.31}$$

If we choose $\epsilon = \iota^* B$ we will get (2.31) equals to $L(E, 0)$. We have shown in Proposition 2.26 that the pure spinor of the subspace $L(E, 0)$ is $\Omega = c(\theta_1 \wedge ... \wedge \theta_k)$. Therefore we get that $e^B \cdot \varphi = \Omega \Longrightarrow \varphi = e^{-B}\Omega$. □

Proposition 2.29. *(Proposition 5.14 of [20]) Let L and L' be two maximal isotropic subspace of $\mathbb{V}$ and let φ and φ' their corresponding pure spinor lines. Then $L_\varphi \cap L_{\varphi'} = \{0\}$ if and only if $(\varphi, \varphi') \neq 0$ where $(,)$ is the Mukai pairing defined on spinors.*

Proof. Here without loss of generality we can assume that $L_\varphi = L(E, 0)$(since Mukai pairing is invariant under B- field transformations, we can fix it by multiplying suitable

B-transformation). Also let $L_{\varphi'} = L(E', \epsilon)$ again we can find a B-transformation such that $L(E', \epsilon) = e^B L(E', 0)$. Let $\varphi' = e^{-B}\Omega'$ and $\varphi = \Omega$.

$$\begin{aligned} (\varphi, \varphi') &= [\alpha(\Omega) \wedge e^{-B}\Omega']_{top} \\ &= \pm[e^B \wedge \Omega \wedge \Omega']_{top} \end{aligned} \tag{2.32}$$

if $E + E' \neq V \otimes \mathbb{C}$ then we can easily see that $Ann(E) \cap Ann(E') \neq \{0\}$. So we get $\Omega \wedge \Omega' = 0$. However, in this case we would get $L_\varphi \cap L_{\varphi'} \neq \{0\}$. Since $(E')^0 \subset L'$ and $E^0 \subset L$. As the second case, assume that $E + E' = V \otimes \mathbb{C}$. Let us write $L_\varphi = \tilde{L} + E^0$ and $L_{\varphi'} = \tilde{L}' + E'^0$ where $\tilde{L} = \{X + B(X, \cdot) |\ X \in E\}$ (similar definition for $\tilde{L}'$). Also consider bases $\{u_1, \cdots, u_k\}$ for $E \cap E'$, let $\{v_1, \cdots, v_l\}$ be a basis for the complement of $E \cap E'$ in E. finally $\{f_1, \cdots, f_r\}$ for the complement of $E \cap E'$ in E'. The duals of these vectors are represented with $\{u_1^*, \cdots, u_k^*\}, \{v_1^*, \cdots, v_l^*\}, \{f_1^*, \cdots, f_r^*\}$ With these vectors we can write:

$$\begin{aligned} L_\varphi &= span\{u_1, \cdots, u_k, v_1, \cdots, v_l, f_1^* \cdots f_r^*\} \\ L_{\varphi'} &= span\{u_1 + B(u_1, \cdot), \cdots, u_k + B(u_k, \cdot), \\ &\quad f_1 + B(f_1, \cdot), \cdots, f_r + B(f_r, \cdot), v_1^*, \cdots, v_l^*\} \end{aligned} \tag{2.33}$$

Now if we have $L_\varphi \cap L_{\varphi'} \neq \{0\}$, then we must have a nonzero volume element from wedge products of basis elements of these two spaces :

$$\begin{aligned} &(f_1 + Bf_1) \wedge ... \wedge (f_r + Bf_r) \wedge (u_1 + Bu_1) \wedge \cdots \wedge (u_k + Bu_k) \\ &\wedge v_1^* \wedge \cdots \wedge v_l^* \wedge u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_l \wedge f_1^* \wedge \cdots \wedge f_r^* \\ = \ &\pm c f_1 \wedge \cdots \wedge f_r \wedge u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_l \wedge \\ &Bu_1 \wedge \cdots \wedge Bu_k \wedge \Omega \wedge \Omega' \end{aligned} \tag{2.34}$$

Hence, (2.34) implies that $L_\varphi \cap L_{\varphi'} \neq \{0\}$ if and only if

$$Bu_1 \wedge \cdots \wedge Bu_k \wedge \Omega \wedge \Omega' \neq 0 \tag{2.35}$$

Write $B = C + A$ where A, C are two forms such that A contains only the basis elements $\{u_1^* \cdots u_r^*\}$ and C contains all the other terms in B. So, (2.35) becomes:

$$Au_1 \wedge \cdots \wedge Au_k \wedge \Omega \wedge \Omega' \neq 0 \tag{2.36}$$

Notice that A maps $\{u_1 \cdots u_r\}$ to $\{u_1^* \cdots u_r^*\}$ and $Au_1 \wedge \cdots \wedge Au_k$ is the determinant of this map. We see that $L_\varphi \cap L_{\varphi'} \neq \{0\}$ if and only if A is an isomorphism. Hence, A^r is a volume form for E. Therefore $[e^B \wedge \Omega \wedge \Omega']_{top} = [e^A \wedge \Omega \wedge \Omega']_{top} = A^r \wedge \Omega \wedge \Omega' \neq 0$. □

Corollary 2.30. *Let φ be a pure spinor corresponding to maximal isotropic L. L has real index zero if and only if $\varphi = exp(B + i\omega)\Omega$ satisfies the following property: $\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$.*

Proof. $\varphi = exp(B + i\omega)\Omega$ then $\bar{\varphi} = exp(B - i\omega)\bar{\Omega}$ which is the pure spinor corresponding to maximal isotropic subspace $\bar{L}$. Applying $B-$field transformation e^{-B} and then using Proposition 2.29, we see that L has real index zero if and only if

$$(\varphi, \bar{\varphi}) = [e^{-i2\omega} \wedge \Omega \wedge \bar{\Omega}] \neq 0 \tag{2.37}$$

which implies that $\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$. □

3. GENERALIZED COMPLEX STRUCTURES ON MANIFOLDS

In the previous chapter, we have defined the generalized complex structures on a vector space. Now we will extend this definition to the fibres of the $TM \oplus T^*M$ and try to understand the integrability conditions. From now on, the letter T denotes the tangent bundle of the manifold M, T^* denotes the cotangent bundle of M and $T \oplus T^*$ denotes the generalized tangent bundle $T \oplus T^*$. Also, the structures defined on the vector spaces, like the natural pairing and the Mukai pairing are extended to $T \oplus T^*$ by defining them fibrewise on $T \oplus T^*$.

Notice that in the usual complex geometry we define the integrability of a complex structure J in terms of closedness of its i-eigenspace in terms of a Lie bracket. However, we have not yet defined a suitable candidate for the Lie bracket on the sections of $T \oplus T^*$. So, our first aim is to establish this. In Sections 3.2 and 3.3, we discuss the notion of generalized metrics which will lead us to define generalized Hodge star operation. Using this operation we can define adjoint and Laplacian of the differential operator d. In Section 3.4, we will briefly study the generalized Kähler manifolds. Finally, in Sections 3.5 and 3.6, we will discuss the results similar to the $\partial\bar{\partial}$-lemma in complex geometry and formality in differential graded algebras. These results will be used to put a restriction on manifolds admitting a GKS. In this chapter, we mainly follow the related chapters of the following articles: [20], [2], [17] and [22].

3.1. Generalized Complex Manifolds

Definition 3.1. *Let M be a differentiable manifold. At each fibre of $T \oplus T^*$, an almost generalized complex structure (AGCS) on M is a map $\mathcal{J}$ from this fibre to itself such that $\mathcal{J}^2 = -\mathbb{I}$ and $\mathcal{J}^* = -\mathcal{J}$.*

As we have already mentioned in the introduction of this chapter, the integrability of an almost complex structure J is equivalent to the closedness of its i-eigenbundle under the Lie bracket. On the other hand, Newlander- Nirenberg theorem states the equivalence of this condition to vanishing of the Nijenhuis tensor which is defined as $N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$. To generalize these theorems and definitions, first we need to define a bracket in the sections of $T \oplus T^*$. To achieve this goal, we will take advantage of the duality between Lie bracket and the exterior derivative. That is to say, using the exterior and interior derivatives we can naturally define the Lie bracket by the following formula:

$$[\mathcal{L}_X, i_Y]\alpha = i_{[X,Y]}\alpha.$$

Extending this formula by using the spinor action instead of interior derivative, we define

$$[u,v]_D \cdot \alpha \doteq [\mathcal{L}_u, v] \cdot \alpha$$

where u, v are sections of $T \oplus T^*$ and $\alpha \in \bigwedge T^*$. The definition of the Lie derivative is given by the Cartan's formula

$$\mathcal{L}_u \alpha = d(u \cdot \alpha) + u \cdot d\alpha.$$

More explicitly :

$$[u,v]_D \cdot \alpha \quad = \quad d(u \cdot (v \cdot \alpha)) + u \cdot (d(v \cdot \alpha)) - v \cdot (d(u \cdot \alpha)) - v \cdot (u \cdot d\alpha)$$

$$
\begin{aligned}
&= di_X i_Y\alpha + d\eta(X)\wedge\alpha + \eta(X)d\alpha - d\eta\wedge i_X\alpha + \eta\wedge di_X\alpha \\
&\quad + d\xi\wedge i_Y\alpha - \xi\wedge di_Y\alpha + d\xi\wedge\eta\wedge\alpha - \xi\wedge d\eta\wedge\alpha + \xi\wedge\eta\wedge d\alpha \\
&\quad + i_X di_Y\alpha + i_X d\eta\wedge\alpha + d\eta\wedge i_X\alpha - \eta(X)d\alpha + \eta\wedge i_X d\alpha \\
&\quad + \xi\wedge di_Y\alpha + \xi\wedge d\eta\wedge\alpha - \xi\wedge\eta\wedge d\alpha - i_Y di_X\alpha - i_Y d\xi\wedge\alpha \\
&\quad - d\xi\wedge i_Y\alpha + \xi(Y)d\alpha - \xi\wedge i_Y d\alpha - \eta\wedge di_X\alpha - \eta\wedge d\xi\wedge\alpha \\
&\quad + \eta\wedge\xi\wedge d\alpha - i_Y i_X d\alpha - \xi(Y)d\alpha + \xi\wedge i_Y d\alpha - \eta\wedge i_X d\alpha - \eta\wedge\xi\wedge d\alpha \\
&= di_X i_Y\alpha + i_X di_Y\alpha - i_Y i_X d\alpha - i_Y di_X\alpha + d\eta(X)\wedge\alpha + i_X d\eta\wedge\alpha \\
&\quad - i_Y d\xi\wedge\alpha \\
&= i_{[X,Y]}\alpha + di_X\eta\alpha + i_X d\eta\wedge\alpha - i_Y d\xi\wedge\alpha \\
&= ([X,Y] + \mathcal{L}_X\eta - i_Y d\xi)\cdot\alpha \qquad (3.1)
\end{aligned}
$$

The last Equation 3.1 gives an explicit definition of the bracket $[\cdot,\cdot]_D$ which is called the Dorfman bracket. Unlike the Lie bracket, the Dorfman bracket is not skew-symmetric. In fact, its skew-symmetrization gives what we want.

Definition 3.2. *Let $X+\xi$ and $Y+\eta$ in $TM \oplus T^*M$ the Courant bracket $[\cdot,\cdot]_{co}$ is defined in the following way*

$$[u,v]_{co} \doteq \frac{1}{2}\left([u,v]_D - [v,u]_D\right) \qquad (3.2)$$

If we take $u = X+\xi$ and $v = Y+\eta$ then the Courant bracket becomes

$$[X+\xi, Y+\eta]_{co} = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \qquad (3.3)$$

Now we are ready to define the integrability of $\mathcal{J}$ in terms of closedness of its i-eigenspace with respect to the Courant bracket.

Definition 3.3. *An AGCS $\mathcal{J}$ is said to be integrable or a generalized complex structure (GCS) if its i-eigenspace L is closed under the Courant bracket $[,]_{co}$.*

Example 3.4. *Note that, in Example 2.4 we have produced a generalized complex structure from a complex structure defined on a vector space. Now, suppose we have* *an almost complex structure J on M. Then almost generalized complex structure* $\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$ *is integrable if and only if J is integrable. To prove this claim, first note that $L = T_{0,1} \oplus T^*_{1,0}$ where $T_{0,1}$ denotes the anti-holomorphic vectors and $T^*_{1,0}$ denotes the holomorphic 1-forms on M. Let $X+\xi$ and $Y+\eta$ be two sections of L. Then, their Courant bracket*

$$[X+\xi, Y+\eta]_{co} = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \tag{3.4}$$

is closed if $[X,Y]$ is closed under Lie bracket, i.e. J is integrable. For the converse of the statement assume that J is integrable. Then, the Courant bracket gives

$$\begin{aligned} [X+\xi, Y+\eta]_{co} &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \\ &= [X,Y] + d(i_X\eta) + i_X(d\eta) - d(i_Y\xi) - i_Y(d\xi) \\ &= [X,Y] + i_X(\bar{\partial}\eta) - i_Y(\bar{\partial}\xi) \end{aligned} \tag{3.5}$$

Notice that this is a section of $T_{0,1} \oplus T_{1,0}$. So $\mathcal{J}$ is an GCS on M.

Now, we define the Nijenhuis operator for $\mathcal{J}$. Then, we will claim that the integrability of an AGCS is equivalent to vanishing of its Nijenhuis operator.

Definition 3.5. *Let $(M, \mathcal{J})$ be a manifold with an AGCS $\mathcal{J}$ on it. The Nijenhuis operator of this AGCS is defined as*

$$N_{\mathcal{J}}(u,v) = [u,v]_{co} + \mathcal{J}[\mathcal{J}u, v]_{co} + \mathcal{J}[u, \mathcal{J}v]_{co} - [\mathcal{J}u, \mathcal{J}v]_{co} \tag{3.6}$$

where u, v are the sections of $T \oplus T^*$.

Proposition 3.6. *Let $(M, \mathcal{J})$ be a manifold with an AGCS $\mathcal{J}$. $\mathcal{J}$ is integrable to a GCS if and only if $N_{\mathcal{J}}(u,v) = 0$ for any $u, v \in T \oplus T^*$.*

Proof. The proof is exactly the same with the case when we have an almost complex structure. Details can be found in [18]. □

In Section 2.2, we have defined linear generalized complex structures on vector spaces and then we characterized them with maximal isotropic subspaces. Also in Lemma 2.9, we gave explicit form of the maximal isotropic subspaces with real index zero. The following proposition give, a criteria on integrability of these subbundles .

Proposition 3.7. *(Proposition 4.19 of [2]) Let $E \subset T \otimes \mathbb{C}$ be a subbundle and $\varepsilon \in \bigwedge^2 E^*$. Then the maximal isotropic $L(E, \varepsilon)$ defines an integrable generalized complex structure if and only if E is involutive and $d_E \varepsilon = 0$. Here $d_E \varepsilon$ is defined in the following way : Consider the inclusion map $i : E \longrightarrow T$. Its pull-back is denoted by $i^* : \bigwedge T^* \longrightarrow \bigwedge E^*$. Then for $\omega \in \bigwedge T^*$*

$$d_E(i^* \omega) = i^*(d\omega) \tag{3.7}$$

Proof. (We will sketch the proof given in [2].) Let us assume that $L(E, \varepsilon)$ is integrable. That is to say, if $X + \xi$ and $Y + \eta$ are two elements of $L(E, \varepsilon)$ then their Courant bracket $Z + \zeta = [X + \xi, Y + \eta]_{co}$ is also an element of the subbundle. If we write this condition explicitly using the definition of the Courant bracket given in (3.3), we will get the following conditions

$$Z = [X, Y] \in E \qquad \text{and} \qquad \zeta|_E - i_Z \varepsilon = i_Y i_X d_E \varepsilon = 0.$$

The first one of these equations implies the involutivity of E and the second one implies the closedness of ε. For the converse, just reverse the same arguments. □

Example 3.8. *In Example 2.5, we observed that if we have a symplectic structure ω on V then $\mathcal{J}_\omega = \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix}$ defines a GCS on V. Now, assume that we have a manifold M together with a symplectic structure ω on it. Then $\mathcal{J}_\omega$ is a GCS if and only if $d\omega = 0$. To prove this claim, just use Proposition 3.7 and note that in this case $\varepsilon = \omega$ and $d_E = d$.*

Lemma 3.9. *Let $(M, \mathcal{J})$ be an almost generalized complex structure. Let ρ be a element in the canonical line bundle. Then $\mathcal{J}$ is integrable if and only if $d\rho = (X + \xi) \cdot \rho$*

Proof. Let us state the following identity ; (whose proof can be found in lemma 4.24 of [2].)

$$A \cdot B \cdot d\rho = d(B \cdot A \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + [A, B]_{co} \cdot \rho - d < A, B > \wedge \rho \quad (3.8)$$

This equation holds for any sections A, B of $T \oplus T^*$. So, if we choose $A, B \in L$ then Equation (3.8) becomes

$$A \cdot B \cdot d\rho = [A, B]_{co} \cdot \rho \quad (3.9)$$

Since L is isotropic and $A \cdot \rho = 0$ for any $A \in L$. Let us now assume that $d\rho = (X + \xi) \cdot \rho$ then $A \cdot B \cdot ((X + \xi) \cdot \rho) = 0$ which implies $[A, B]_{co} \cdot = 0$. Therefore, L is closed under the Courant bracket. Conversely, assume that L is closed under the Courant bracket. Then (3.8) becomes $A \cdot B \cdot \rho = 0$. This implies that $d\rho \subset U_{-n} \oplus U_{-n+1}$. However, d changes the parity of a differential form. that is if ρ is an even form then $d\rho$ and $(X + \xi) \cdot = i_X \rho + \xi \wedge \rho$ are odd forms. Therefore, we get $d\rho \in U_{-n+1}$. □

We have already seen how generalized complex geometry includes the integrability condition of symplectic geometry, see Examples 3.4 and 3.8. Now, by using the lemma above, we will obtain the complex geometry analogue of the integrability condition.

Theorem 3.10. *(Theorem 4.23 of [2])*
Let $\mathcal{J}$ be a generalized almost complex structure, and define

$$\begin{aligned} \bar{\partial} &= \pi_{k+1} \circ d : C^\infty(U_k) \longrightarrow C^\infty(U_{k+1}), \\ \partial &= \pi_{k-1} \circ d : C^\infty(U_k) \longrightarrow C^\infty(U_{k-1}) \end{aligned} \quad (3.10)$$

where π_k is the projection onto U_k and U_k is defined as in Equation 2.27. Then $\mathcal{J}$ is integrable if and only if $d = \partial + \bar{\partial}$.

Proof. (We will sketch the arguments given in the article [2].) We have already proved that $d(U_{-n}) \in U_{-n+1}$ if and only if L is integrable. For the rest of the proof, just use the method of induction on n for Equation (3.8) and the fact that multiplication by $A \in L$ decreases the degree by one and multiplication by $B \in \bar{L}$ increases the degree by one. $\square$

3.2. Generalized Metric

Now, let us choose an m dimensional, positive definite subbundle C_+ of $T \oplus T^*$ with respect to natural pairing on it. Then, the orthogonal complement of this space $C^\perp$ will be negative definite and denoted by C_-. Since $C_+ \cap T = \{0\}$ the choice of the C_+ is equivalent to the choice of a linear bundle map $A : T \longrightarrow T^*$ such that the graph is C_+.

Proposition 3.11. *The subbundle C_+ is given as $C_+ = graph(g + b)(T)$ where g is a positive definite metric on M and b is an element of $\bigwedge^2 T^*$*

Proof. Let us choose an orthonormal basis $\{a_1, \cdots, a_n\}$ for C_+ and for an orthonormal basis for C_- for $\{a^1, \cdots, a^n\}$. We know that under such these basis $<,>$ will be in the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Since we have $C_+ \cap T = \{0\}$ and $C_+ \cap T^* = \{0\}$ then $a_i|_T = e_i$ $a^i|_{T^*} = e^i$ must be a basis for the corresponding subbundles. Therefore, we can define an isomorphism $A : T \longrightarrow T^*$ such that $e_i + A(e_i) = a_i$. That is to say, C_+ is the graph of T under the isomorphism A. The symmetric part will be g and the antisymmetric part of A will be denoted by b. Similarly, the space C_- can be expressed as the graph of $g - b$ in this setting. $\square$

Definition 3.12. *The choice of the subbundle C_+ naturally defines a metric $\mathcal{G}$, which is called the generalized metric on $T \oplus T^*$. The metric is explicitly given as:*

$$\mathcal{G} \doteq <,> |_{C_+} - <,> |_{C_-} \tag{3.11}$$

Using the natural pairing and the generalized metric we can define a map G such that $\mathcal{G}(u, v) =< Gu, v >$. In fact, due to natural pairing we can consider $\mathcal{G}$ as a map from $T \oplus T^*$ to itself. Under this identification, clearly, $\mathcal{G}$ will be equal to G. So we will interchangeably call them as generalized metric structures. Also note that G has the following properties: $G^2 = \mathbb{I}$ and $G^* = G$.

Let us understand the action of G on the sections $u = X + \xi$ of $T \oplus T^*$. Assume that $C_+^0 = graph(g, T)$, i.e. $b = 0$. If we decompose u into orthogonal components in $C_\pm$ we get

$$X + \xi = \frac{1}{2}(X + g^{-1}(\xi, \cdot) + g(X+, \cdot) + \xi) + \frac{1}{2}(X - g^{-1}(\xi, \cdot) - g(X+, \cdot) + \xi)$$

If we define $u_\pm = \frac{1}{2}[X \pm g^{-1}(\xi, \cdot) \pm g(X+, \cdot) + \xi]$ then

$$G(X + \xi) = G(u) = G(u_+ + u_-) = u_+ - u_- = g^{-1}(\xi, \cdot) + g(X, \cdot) \qquad (3.12)$$

Here, we have used the fact that $G = \mathbb{I}$ on C_+ and $G = -\mathbb{I}$ on C_- . In the general case, we have $C_+ = e^b C_+^0$, that is C_+ is the $B-$ field action of C_+^0. So in matrix represention, we can write the general form of G as

$$G = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} \qquad (3.13)$$

3.3. The Born-Infeld Metric

Recall that in the usual Riemannian case, we can define a Hodge star operation on differential forms and using this operator, we define Hodge dual of d-operator and finally we arrive at “The Hodge Decomposition Theorem” . Now, we will discuss an analogous operator and try to get a similar result. (In this section, we will mostly follow section 4.3.2 of [20] and section 2.1 of [23].)

We have seen that the generalized metric can be written as:

$$G = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} \qquad (3.14)$$

The term $g - bg^{-1}b$ defines a new Riemannian metric on T and the invariant volume form induced from this metric is given by $vol_G = \dfrac{det(g+b)}{\sqrt{(detg)}}$, [22]. Using this volume form we will define a metric on the spinors. Now consider the case $b = 0$. Let C_+^0 be the graph of g only. Then if we let $\{e^1, \cdots, e^n\}$ be an oriented basis for the fibres of T then $\{e^1 + e_1, \cdots, e^n + e_n\}$ will be an oriented basis of the subbundle C_+^0. If we relabel the elements of this basis as $a_i \doteq e^i + e_i$, then the product $*_0 = a_1 \cdots a_n$ acts on forms by Clifford multiplication. So, we have

$$\alpha(*_0)\beta = \alpha(\star_g \beta) \tag{3.15}$$

where α is the main anti-automorphism mentioned in Section 2.4 and $\star_g$ is the usual Hodge star operator defined by the metric g. Furthermore, we find

$$\begin{aligned} *_0(*_0\beta) &= (a_1 \cdots a_n)(a_1 \cdots a_n)\beta \\ &= (a_1 \cdots a_n \cdot a_1 \cdots a_n)\beta \\ &= (-1)^{n-1}(a_2 \cdots a_n \cdot a_2 \cdots a_n)\beta \\ &= (-1)^{n-1}(-1)^{n-2} \cdots (-1)\beta \\ &= (-1)^{\frac{n(n-1)}{2}}\beta \end{aligned} \tag{3.16}$$

So $(*_0)^2 = (-1)^{\frac{n(n-1)}{2}}$. Using these properties, one can show that

$$g(\alpha, \beta)dvol_g =< \alpha, *_0\tilde{\beta} > \tag{3.17}$$

Where $\tilde{\beta} = \star_g(\star_g\beta)$ and $g(\alpha, \beta)$ is the Hodge metric defined on the forms. Let us now consider the general case that is when $b \neq 0$. In this case, we observed that $C_+ = e^b C_+^0$. If $\{e^1 + e_1, \cdots, e^n + e_n\}$ is an oriented basis of the subbundle C_+^0, then

$\left\{e^b(e^i+e_i)\right\}_{i=1}^n$ will be an oriented basis for C_+

$$
\begin{aligned}
(e^b u)(e^{-b}\omega) &= (X+\xi+b(X,\cdot))\cdot(\omega - b\wedge\omega + \frac{1}{2!}b^2\wedge\omega - \frac{1}{3!}b^3\wedge\omega + \cdots) \\
&= \omega(X,\cdot) + \xi\wedge\omega + b(X,\cdot)\wedge\omega - b(X,\cdot)\wedge\omega - b\wedge\omega(X,\cdot) \\
&\quad -\xi\wedge b\wedge\omega - b(X,\cdot)\wedge b\wedge\omega + 2\frac{1}{2!}b(X,\cdot)\wedge b\wedge\omega + \frac{1}{2!}b^2\wedge\omega(X,\cdot) \\
&\quad +\frac{1}{2!}\xi\wedge b^2\wedge\omega + \frac{1}{2!}b(X,\cdot)\wedge b^2\wedge\omega - 3\frac{1}{3!}b(X,\cdot)\wedge b^2\wedge\omega \\
&\quad -\frac{1}{3!}b^3\wedge\omega(X,\cdot) - \frac{1}{3!}\xi\wedge b^3\wedge\omega - \frac{1}{3!}b(X,\cdot)\wedge b^3\wedge\omega\cdots \\
&= \omega(X,\cdot) + \xi\wedge\omega - b\wedge\omega(X,\cdot) - b\wedge\xi\wedge\omega + \frac{1}{2!}b\wedge\omega(X,\cdot) \\
&\quad +\frac{1}{2!}b^2\wedge\xi\wedge\omega - \frac{1}{3!}b^3\wedge\omega(X,\cdot) - \frac{1}{3!}b^3\wedge\xi\wedge\omega \\
&= e^{-b}(u\cdot\omega)
\end{aligned}
\tag{3.18}
$$

where ω is a spinor and u is a section of $T\oplus T^*$ and $b^i = b\wedge\cdots\wedge b$ wedge product of b, i times. Therefore, we get

$$
\begin{aligned}
*\,\omega &= (e^b a_1)(e^b a_2)\cdots(e^b a_n)\cdot\omega \\
&= (e^b a_1)(e^b a_2)\cdots(e^b a_n)\cdot e^{-b}(e^b\omega) \\
&= (e^b a_1)(e^b a_2)\cdots(e^{-b} a_n)\cdot e^b\omega) \\
&= e^{-b} *_0 e^b\omega
\end{aligned}
$$

Thus, $* = e^{-b} *_0 e^b$. If we define $G(\alpha,\beta)\ dvol_G \doteq <\alpha, *\tilde{\beta}>$, then

$$
\begin{aligned}
G(\alpha,\beta) &= <\alpha, *\tilde{\beta}> \\
&= <\alpha, e^{-b} *_0 e^b\tilde{\beta}> \\
&= <e^b\alpha, *_0 e^b\tilde{\beta}> \\
&= <e^b\alpha, *_0\widetilde{e^b\beta}> \\
&= g(e^b\alpha, e^b\beta>\ \ dvol_g
\end{aligned}
\tag{3.19}
$$

which implies the symmetry of $G(\alpha,\beta)$ and positive definiteness of $G(\cdot,\cdot)$. We are now ready to define a positive definite, Hermitian inner product, which is called the

Born-Infeld metric on differential forms.

Definition 3.13. *[22] Let M be a compact manifold. The Born-Infeld metric is the metric h on differential forms given by*

$$h(\alpha, \beta) = \int G(\alpha, \bar{\beta})\, dvol_G \tag{3.20}$$

Definition 3.14. *The generalized Hodge star operation $\star_G$ is defined as: $\star_G \beta = *\tilde{\beta}$. Notice that, in this setting we have $< \alpha, \star_G \beta >= G(\alpha, \beta)$*

3.4. Generalized Kähler Manifolds

Now, we will discuss one of the most important subset of generalized complex manifolds, namely generalized Kähler manifolds. Similar to complex case, GKS are a generalization of the usual the Kähler structure. In this section, definition of these structures will be given and then an important decomposition of differential forms will be constructed. These results will be used in the next section when we try to get formality property on generalized Kähler manifolds. We know that this property is satisfied by differential graded algebras of Kähler manifolds and due to this property we can check whether some manifolds admit Kähler structures or not [17].

Definition 3.15. *Let $\mathcal{J}$ be a GCS on $T \oplus T^*$ and $\mathcal{G}$ be a generalized metric defined as in Equation 3.11. If these two structures are compatible (i.e. $\mathcal{G}(\mathcal{J}(v), \mathcal{J}(u)) = \mathcal{G}(u, v)$) then we will call $\mathcal{G}$ as a Hermitian structure on $T \oplus T^*$. The compatibility gives $\mathcal{J}^* G \mathcal{J} = G$ which implies $G\mathcal{J} = \mathcal{J}G$ since $\mathcal{J}^* = -\mathcal{J}$.*

Remark 3.16. *Notice that when we have a Hermitian structure then we have one more AGCS on the space namely $\mathcal{J}_2 \doteq G\mathcal{J}$ which is another AGCS since $(G\mathcal{J})^2 = (G\mathcal{J})(G\mathcal{J}) = (G^2)(\mathcal{J}^2) = -\mathbb{I}$. On the other hand, this AGCS $\mathcal{J}_2$ is not integrable to a GCS in general. This extra condition will lead us to GKS.*

Definition 3.17. *A generalized Kähler structure on $T \oplus T^*$ is given by two commuting GCS $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $G = -\mathcal{J}_1 \mathcal{J}_2$ gives a generalized metric.*

Remark 3.18. *In the case of generalized Kähler structures since* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *commute* $\mathcal{J}_2$ *preserves the* i*-eigenspace* L_1 *of the GCS* $\mathcal{J}_1$*. So* L_1 *can be decomposed into two parts* $L_1 = L_1^+ \oplus L_1^-$*. In this decomposition, we have defined* $L^+ = L_1 \cap L_2$ *and* $L^- = L_1 \cap \bar{L}_2$*.*

Note that in the case of GCS we have defined a decomposition of forms in (2.27) and in Proposition 2.27, we proved that U_k is the ik-eigenspace of $\mathcal{J}$. Since in GKS case we have two GCS $\mathcal{J}_1$ and $\mathcal{J}_2$, we get the following decomposition of forms

$$(\bigwedge T^* \otimes \mathbb{C}) = \bigoplus_{p,q} U_{p,q} \tag{3.21}$$

where $U_{p,q} = U_p \cap U_q$. In Proposition 3.10 we have seen that if GCS is integrable then the exterior derivative decomposes in to two part, namely, $d = \partial + \bar{\partial}$ such that $\partial : U_k \longrightarrow U_{k-1}$. Let us now consider the GKS case. The exterior derivative d will take an element β of $U_{p,q}$. As an element of U_p, $d\beta \in U_{p-1} \oplus U_{p+1}$ and as an element of U_q, $d\beta = U_{q-1} \oplus U_{q+1}$. So, we have the following proposition :

Proposition 3.19. *Let* $(M, \mathcal{J}_1, \mathcal{J}_2)$ *be a GKS. The the exterior derivative decomposes in four components*

$$\begin{aligned} \delta_+ &: U_{p,q} \longrightarrow U_{p+1,q+1} \\ \delta_- &: U_{p,q} \longrightarrow U_{p+1,q-1} \\ \bar{\delta}_+ &: U_{p,q} \longrightarrow U_{p-1,q-1} \\ \bar{\delta}_- &: U_{p,q} \longrightarrow U_{p-1,q+1} \end{aligned}$$

Let us note that with this decomposition in fact we get the following decomposition $\partial_{\mathcal{J}_1} = \delta_+ + \delta_-$ and $\partial_{\mathcal{J}_2} = \bar{\delta}_+ + \bar{\delta}_-$

We have seen that $\mathcal{J}_2$ is an complex structure on L_1. Also it is integrable with respect to the Courant Bracket. So, the differential operator defined on L_1 can be decomposed into two parts. In other words, we can write $d_{L_1} = \partial_{L_1}^+ + \partial_{L_1}^-$ where

$\partial^+_{L_1}$, $\partial^-_{L_1}$ is defined like this

$$\begin{aligned} \wedge^{p,q} L_1^* &= \wedge^p (L_1^+)^* \otimes \wedge^q (L_1^-)^* \\ \partial^+_{L_1} &= \pi_{p+1,q} \circ d_{L_1} : C^\infty(\wedge^{p,q} L_1^*) \longrightarrow C^\infty(\wedge^{p+1,q} L_1^*) \\ \partial^-_{L_1} &= \pi_{p,q+1} \circ d_{L_1} : C^\infty(\wedge^{p,q} L_1^*) \longrightarrow C^\infty(\wedge^{p,q+1} L_1^*) \end{aligned} \tag{3.22}$$

Using the identity

$$\partial(\alpha \cdot \rho) = (d_{L_1}\alpha) \cdot \rho + (-1)^k \alpha \cdot d\rho \tag{3.23}$$

given in Cavalcanti's paper [17] and degree matching arguments, for any element α in $\bigwedge^k(\bar{L}_1)$, we can also derive the following relation:

$$\delta_+(\alpha \cdot \rho) = (\partial^+_{L_1}\alpha) \cdot \rho + (-1)^k \alpha \cdot d\rho \tag{3.24}$$

which will be useful when we try to construct GKS on nilmanifolds.

3.5. Hodge Decomposition

In Section 3.3, we have constructed a Hermitian metric on forms. We know that in the usual complex geometry similar constructions leads to ∂- harmonic and $\bar{\partial}$-harmonic forms. Finally, these give the complex version of Hodge's theorem. In this section, we will show that in generalized geometry we have similar results for δ and $\bar{\delta}$ operators defined in Proposition 3.19.

Lemma 3.20. *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a compact generalized Kähler manifold without boundary. δ_+ be the operator defined in Proposition 3.19 and let $h(\cdot,\cdot)$ be the Born-Infeld inner product defined on forms. Then the adjoint of δ_+ with respect to this inner product is given by $\delta_+^* = -\bar{\star}_G \delta_+ \bar{\star}_G^{-1}$ where $\bar{\star}_G$ is defined in the following way $\bar{\star}_G \sigma = \star_G \bar{\sigma}$ where $\sigma \in \bigwedge(T^* \otimes \mathbb{C})$.*

Proof. Without loss of the generality, let us choose $\sigma \in U_{p,q}$ and $\beta \in U_{-p,-q-1}$. Using the identity

$$< \delta_+\sigma, \beta > + < \sigma, \delta_+\beta >= d(\alpha(\sigma) \wedge \beta). \tag{3.25}$$

given in the lemma 2.5 of [23], we will get :

$$\begin{aligned} h(\delta_+\sigma, \beta) &= \int_M < \delta_+\sigma, \star_G\bar{\beta} > \\ &= -\int_M < \sigma, \delta_+ \star_G \bar{\beta} > \\ &= -\int_M < \sigma, \bar{\star}_G\bar{\star}_G^{-1}\delta_+ \star_G \bar{\beta} > \\ &= h(\sigma, -\bar{\star}_G^{-1}\delta_+\bar{\star}_G\beta). \end{aligned} \tag{3.26}$$

Equation 3.26 implies $\delta_+^* = -\bar{\star}_G\delta_+\bar{\star}_G^{-1}$. □

The following lemma will enable us to understand the action of generalized Hodge star operation of forms. The proof can be found in [24].

Lemma 3.21. *(Lemma 2.6 of [24]) If G is a generalized metric structure compatible with a GCS $\mathcal{J}_1$, then the generalized Hodge star operation assumes $U_{p,q}$ as its i^{p+q} eigenspace. More explicitly, for an element $\sigma \in U_{p,q}$ the action of generalized hodge star is given by $\star_G\sigma = i^{p+q}\sigma$*

Note that in the usual Hodge theory, after we construct the adjoint differential operators, we define the Laplacian operator $\Delta = dd^* + d^*d$. Similarly we will define the Laplacian operators for the differential operators we have constructed so far. The relation between them will be stated in the next theorem.

Theorem 3.22. *(Theorem 2.1 of [23]) In a compact generalized Kähler manifold without boundary, the operators defined in Equation 3.19 and their adjoints satisfy*

$$\delta_+^* = \bar{\delta}_+ \quad \textit{and} \quad \delta_-^* = -\bar{\delta}_- \tag{3.27}$$

Moreover, the Laplacians of these operators satisfy the relations

$$\begin{aligned} 4\Delta_d &= 2\Delta_{\partial_{\mathcal{J}_1}} = 2\Delta_{\bar{\partial}_{\mathcal{J}_1}} = 2\Delta_{\partial_{\mathcal{J}_2}} = 2\Delta_{\bar{\partial}_{\mathcal{J}_2}} \\ &= \Delta_{\delta_+} = \Delta_{\delta_-} = \Delta_{\bar{\delta}_+} = \Delta_{\bar{\delta}_-} \end{aligned} \tag{3.28}$$

Proof. (We will sketch the proof given in the article [23].) We have stated that in a compact generalized Kähler manifold $\delta_+^* = -\bar{\star}_G \delta_+ \bar{\star}_G^{-1}$ by, Lemma 3.20. Using Lemma 3.21, for $\sigma \in U_{p,q}$ and noting that $\bar{\sigma} \in U_{-p,-q}$ we have :

$$\begin{aligned} \delta_+^* \sigma &= -\bar{\star}_G \delta_+ \bar{\star}_G^{-1} \sigma \\ &= -i^{p+q} \bar{\star}_G \delta_+ \bar{\sigma} \\ &= -i^{p+q} i^{(-p+1-q+1)} \bar{\delta}_+ \sigma \\ &= \bar{\delta}_+ \sigma \end{aligned} \tag{3.29}$$

Also for the relation $\delta_-^* = -\bar{\delta}_-$ we have

$$\begin{aligned} \delta_-^* \sigma &= -\bar{\star}_G \delta_- \bar{\star}_G^{-1} \sigma \\ &= -i^{p+q} \bar{\star}_G \delta_- \bar{\sigma} \\ &= -i^{(-p+1-q-1} i^{p+q)} \bar{\delta}_- \sigma \\ &= -\bar{\delta}_- \sigma. \end{aligned} \tag{3.30}$$

So, we have proved the first part of the theorem. For the second part of the claim, let us note that,

$$d = \partial_{\mathcal{J}_1} + \bar{\partial}_{\mathcal{J}_2} = \delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-$$

By using the fact that $d^2 = (\partial\bar{\partial} + \bar{\partial}\partial) = 0$, we can derive the following

$$\begin{aligned} d^2 &= \delta_+\bar{\delta}_+ + \bar{\delta}_+\delta_+ + \delta_-\bar{\delta}_- + \bar{\delta}_-\delta_- = 0 \\ -\delta_-\bar{\delta}_- - \bar{\delta}_-\delta_- &= \delta_+\bar{\delta}_+ + \bar{\delta}_+\delta_+ \\ \delta_-\delta_-^* + \delta_-^*\delta_- &= \delta_+\delta_+^* + \delta_+^*\delta_+ \\ \Delta_{\delta_-} &= \Delta_{\delta_+} \end{aligned} \tag{3.31}$$

For the other identities

$$\begin{aligned} \Delta_d &= (dd^* + d^*d) \\ &= (\delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-)(\delta_+^* + \delta_-^* + \bar{\delta}_+^* + \bar{\delta}_-^*) \\ &\quad +(\delta_+^* + \delta_-^* + \bar{\delta}_+^* + \bar{\delta}_-^*)(\delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-) \\ &= 2(\delta_+\bar{\delta}_+ + \bar{\delta}_+\delta_+) + 2(\delta_-\bar{\delta}_- + \bar{\delta}_-\delta_-) \\ &= 2\Delta_{\delta_+} + 2\Delta_{\delta_-} = 4\Delta_{\delta_+} \end{aligned} \tag{3.32}$$

□

Following the usual arguments in Kähler geometry we can define analogues of Hodge decomposition theorem and $\partial\bar{\partial}-$lemma. These theorems in Kähler geometry put restrictions on manifolds admitting a Kähler structures. For more detailed arguments on generalized Hodge decomposition and $\partial\bar{\partial}-$ lemma, see [3] and [22].

Theorem 3.23. *($\delta_+\delta_-$-lemma of [22])*
In a compact generalized Kähler manifold

$$Im\delta_+ \cap Ker\delta_- = Im\delta_- \cap Ker\delta_+ = Im(\delta_+\delta_-) \tag{3.33}$$

3.6. Formality in Generalized Kähler Geometry

In this section by following the arguments given in article [17], we will discuss an important property of generalized Kähler geometry, namely formality. This property provides a criteria for the existence of a Kähler structure on a manifolds. Similarly, one can show that nilmanifolds with non-trivial nilpotent Lie algebras cannot have Kähler structures since they are not formal. So, we will use an analogous argument to show that they do not admit GKS.
In order to achieve our aim, we need some definitions related to differential graded algebras. Let us note that, by a differential graded algebra we mean a graded algebra endowed with a differential operator which must be a derivation.

Definition 3.24. *A differential graded algebra is called minimal if it is free as a differential graded algebra and has generators* $e_1, \cdots, e_n, \cdots$ *such that*
i)The degree of the generators form a weakly increasing sequence,
ii) There are finitely many generators,
iii)The differential operator satisfies $de_i \in$ *span* $\{e_1, \cdots, e_{i-1}\}$

Definition 3.25. *A minimal model* $\mathcal{M}$*, for a differential graded algebra* $\mathcal{A}$ *is a minimal differential graded algebra together with a quasi-isomorphism* $\psi : \mathcal{M} \longrightarrow \mathcal{A}$*.*

Definition 3.26. *A differential graded algebra* $\mathcal{A}$ *is called formal if it has the same minimal model as its cohomology when we consider the cohomology with trivial differential operator.*

Example 3.27. *(Nilpotent Lie Algebra)(Example 1 of [17]) One can show that if g is a nilpotent Lie algebra then,* $(\bigwedge g^*, d)$ *is minimal differential graded algebra. However, if the bracket is non-trivial it is not formal. A proof of this statement will be given in Proposition 4.1.*

Theorem 3.28. *(Theorem 2.15 of [23])*

If $\mathcal{J}_1, \mathcal{J}_2$ *is a GKS on a compact manifold and* $\mathcal{J}_1$ *has a trivial canonical bundle, then* $(\Omega(\bar{L}_1), d_{L_1})$ *is a formal differential graded algebra.*

Proof. (We will sketch the proof given in the article [23].) Let us assume that $\mathcal{J}_1$ has canonical bundle satisfying $d\rho = 0$. In this case Equation 3.23 becomes

$$\delta_+(\alpha \cdot \rho) = (\partial^+_{L_1}\alpha) \cdot \rho \tag{3.34}$$

and using this relation between δ_+ and $\partial^+_{L_1}$, Theorem 3.23 implies

$$Im(\partial^+_{L_1}) \cap Ker(\partial^-_{L_1}) = Im(\partial^-_{L_1}) \cap Ker(\partial^+_{L_1}) = Im(\partial^+_{L_1}\partial^-_{L_1}) \tag{3.35}$$

Let us prove this claim. Let $\alpha \in Im(\partial^+_{L_1}) \cap Ker(\partial^-_{L_1})$. Then $\alpha = \partial^+_{L_1}\beta$ and $\partial^-_{L_1}\alpha = 0$. If we define $\tilde{\alpha} = \alpha \cdot \rho$ and $\tilde{\beta} = \beta \cdot \rho$

$$\tilde{\alpha} = \alpha \cdot \rho = (\partial^+_{L_1}\beta) \cdot \rho = \delta_+(\beta \cdot \rho). \tag{3.36}$$

So, $\tilde{\alpha} \in Im(\delta_+)$. Similarly,

$$\delta_-(\alpha \cdot \rho) = (\partial^-_{L_1}\alpha) \cdot \rho = 0. \tag{3.37}$$

So, $\tilde{\alpha} \in Ker(\delta_-)$. Thanks to the $(\delta_+\delta_-)$-lemma 3.23 , we have $\tilde{\alpha} \in Ker(\delta_+) \cap Im(\delta_-)$. This implies,

$$\begin{aligned} 0 &= \delta_+(\alpha \cdot \rho) = (\partial^+_{L_1}\alpha) \cdot \rho \\ \alpha \cdot \rho &= \tilde{\alpha} = \delta_-(\gamma \cdot \rho) = (\partial^-_{L_1}\gamma) \cdot \rho. \end{aligned} \tag{3.38}$$

So, we get the first part of the our claim: $Im(\partial^+_{L_1}) \cap Ker(\partial^-_{L_1}) = Im(\partial^-_{L_1}) \cap Ker(\partial^+_{L_1})$. For the other part, just apply the same argument and $\alpha \in Im(\partial^-_{L_1}) \cap Ker(\partial^+_{L_1})$ in this case. Finally, we get the natural extension of (3.35):

$$Im(d_{L_1}) \cap Ker(d^c_{L_1}) = Im(d^c_{L_1}) \cap Ker(d_{L_1}) = Im(d_{L_1}d^c_{L_1}) \tag{3.39}$$

where $d^c_{L_1} = i(\partial^+_{L_1} - \partial^-_{L_1})$. So, consider the following differential graded algebras

$\Omega(\bar{L}_1, d_{L_1})$, algebra of $d^c_{L_1}$ closed forms $(\Omega_c(\bar{L}_1), d_{L_1})$ with the differential d_{L_1} and finally the cohomology $(H_c(\bar{L}_1), d_{L_1})$ of $\bar{L}_1$ with respect to $d^c_{L_1}$ and its differential is d_{L_1}. It is obvious that we have the following maps between these differential graded algebras

$$\begin{aligned} i &: (\Omega_c(\bar{L}_1), d_{L_1}) \longrightarrow (\Omega(\bar{L}_1), d_{L_1}) \\ \pi &: (\Omega_c(\bar{L}_1), d_{L_1}) \longrightarrow (H_c(\bar{L}_1), d_{L_1}). \end{aligned}$$

These are the injection and the projection mappings respectively. In fact, these mappings form isomorphisms between cohomologies of these algebras. Therefore, if we have a minimal model $\mathcal{M}$ for $(\Omega_c(\bar{L}_1), d_{L_1})$ and $\tilde{\varphi}$ is the morphism between them, then one can construct the following isomorphism between $\mathcal{M}$ and $H(\bar{L}_1)$.

$$\begin{aligned} i \circ \tilde{\varphi} &: \mathcal{M} \longrightarrow (\Omega(\bar{L}_1), d_{L_1}) \\ i^* \circ (\pi^*)^{-1} \circ \pi \circ \tilde{\varphi} &: \mathcal{M} \longrightarrow H(\bar{L}_1) \end{aligned} \tag{3.40}$$

This implies that $(\Omega(\bar{L}_1), d_{L_1})$ is formal for a compact generalized Kähler manifold whose canonical bundle satisfies the relation $d\rho = 0$. □

4. NILMANIFOLDS

In this chapter, we discuss the GCS on nilmanifolds. Historically, these spaces are the first examples of manifolds which admit symplectic structure but not the Kähler structures, see [12]. Also, it is known that on six dimensions five classes of nilmanifolds admit neither a symplectic nor a complex structure [14]. So, it is natural to ask whether these manifolds admit a GCS. In this section we will obtain following results: Every six dimensional nilmanifold admit a GCS. If also, a nilmanifold admits a Kähler structure then its corresponding Lie algebra must be trivial. Throughout this chapter we followed the articles [17] and [14].

4.1. Nilmanifolds and Nilpotent Lie Algebras

Definition 4.1. *Let G be connected, simply connected, nilpotent Lie group and $\mathcal{J}$ is a left invariant GCS on G. We can identify this structure on G with a GCS defined on $\mathfrak{g}$. Let Γ be a cocompact discrete subgroup of G. In this case, the quotient space $M = \Gamma \backslash G$ is called a nilmanifold. This manifold has an induced GCS from the structure defined on G. We will denote this GCS again by $\mathcal{J}$ and call this structure as left invariant GCS on M. Thanks to the left invariance of these structure it is enough to study on the Lie algebra $\mathfrak{g}$ of G*

In the remaining part of this chapter we will work on six dimensional nilmanifolds. Since the exterior derivative is dual to Lie bracket, in the sense that $d\alpha(X,Y) = -\alpha([X,Y])$ for an element $\alpha \in \mathfrak{g}^*$, we will use the exterior derivative to describe Lie algebra $\mathfrak{g}^*$. In addition to this, we will use the following notation to describe structure constants:

$$(0, 0, 0, 12, 13, 23 + 13)$$

which means that

$$de_1 = 0, de_2 = 0, de_3 = 0, de_4 = e_1 + e_2, de_5 = e_1 \wedge e_3, de_6 = e_2 \wedge e_3 + e_1 \wedge e_3$$

From now on we will use a short hand notation $e_i \wedge e_j = e_i e_j$. Since $\mathfrak{g}$ comes from a nilpotent Lie group its Lie algebra is also nilpotent in the sense that the following sequence of ideals of $\mathfrak{g}$

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^i = \left[\mathfrak{g}^{i-1}, \mathfrak{g}\right]$$

will end $\mathfrak{g}^s = 0$ for finite $s \in \mathbb{Z}$. Also we can construct annihilators of each $\mathfrak{g}^i$

$$V_i = \left\{\alpha \in \mathfrak{g}^* | \quad d\alpha \in \wedge^2 V_{i-1}\right\}, \quad V_0 = \{0\} \tag{4.1}$$

In fact, we can easily see by induction that each element of the V_i is an annihilator of $\mathfrak{g}^i$. For $i = 1$, we need to show that $\alpha(X) = 0$ for an arbitrary element $\alpha \in V_1$ and $X \in \mathfrak{g}^1$. Remember that being an element of $\mathfrak{g}^1$ implies that $X = [Y, Z]$ for elements $Y, Z \in \mathfrak{g}$. So, since $\alpha \in V_1$ implies $d\alpha = 0$ we get the following

$$d\alpha(Y, Z) = 0 = -\alpha([Y, Z]) = \alpha(X)$$

Assume that for $i = k - 1$ our claim holds, i.e. V_i is the annihilator of the $\mathfrak{g}^i$. In order to prove our claim we need to show that V_k is the annihilator of $\mathfrak{g}^k$. Let us choose X as an arbitrary element of $\mathfrak{g}^k$. That is to say, X is in the form $X = [Y, Z]$ where $Y \in \mathfrak{g}^{k-1}$ and $Z \in \mathfrak{g}$. If we calculate $\alpha(X)$

$$\alpha(X) = \alpha([Y, Z]) = d\alpha(Y, Z) = 0$$

since $d\alpha \in \wedge^2 V_{k-1}$ which is the annihilator of the $\mathfrak{g}^{k-1}$

Definition 4.2. *The nilpotent degree of a p-form α, which is denoted by $nil(\alpha)$, is the smallest i such that $\alpha \in \bigwedge^p V_i$.*

Now we can prove our claim stated in Example 3.27:

Proposition 4.3. *The nilpotent Lie algebras are minimal differential graded algebras. However, they are not formal unless they are trivial.*

Proof. Thanks to the property given in Equation 4.1 we can choose a basis inductively such that $de_i \in \wedge^2 < e_1, \cdots, e_{i-1} >$ where we use the notation $\wedge^2 < e_1, \cdots, e_{i-1} >$ to denote the two forms generated by the forms $e_1, \cdots, e_{i-1}$. It is clear that, this property implies that these algebras are minimal. Also note that, if the differential is not trivial then de_n must be nonzero (otherwise for any element β of $\mathfrak{g}^*$ we have $d\beta = 0$) and $e_1 \cdots e_{n-1}$ is an exact form. If the algebra $\mathfrak{g}^*$ is formal then there is an isomorphism between the differential graded algebras:

$$\psi : (\wedge \mathfrak{g}^*, d) \longrightarrow H(\mathfrak{g}) \tag{4.2}$$

However, this map cannot be an isomorphism since in this case we would have:

$$0 \neq \psi(e_1 \cdots e_n) = \psi(e_1 \cdots e_{n-1})\psi(e_n) = 0 \tag{4.3}$$

So, we conclude that a nilpotent Lie algebra is not formal unless it is trivial. □

4.2. Generalized Complex Structures on Nilmanifolds

In this section we will first show that any GCS on a nilmanifold satisfy the relation $d\rho = 0$ where ρ is any element in the canonical line bundle of the GCS, see Definition 2.23. Then, we will see the effect of the type on the construction of GCS. Finally, we will give an explicit GCS for a specific six dimensional nilmanifold.

Now, we will prove a theorem which is one of the most important results about GCS on nilmanifolds. Before that we need a technical lemma. The proof of this lemma can be found in [14] so we will omit the proof here.

Lemma 4.4. *(Lemma 3.3 of [14]) Let* $\rho = e^{B+i\omega}\Omega$ *be an element of the canonical line bundle of a GCS defined on a nilmanifold* M*. Recall that* Ω *is given in the form* $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ *(Proposition 2.28). In this case* $\{\theta_1, \cdots \theta_k\}$ *can be chosen so that* $nil(\theta_i) < nil(\theta_j)$ *whenever* $i < j$ *and also the set* $\{\theta_j,\ nil(\theta_j) > i\}$ *are linearly independent modulo* V_i*. Moreover, suppose that none of these* θ_i*'s satisfy the condition* $nil(\theta_i) = \lambda$ *but there is a* θ_l *such that* $nil(\theta_l) = \lambda + 1$*. Then,* V_{j+1}/V_j *must have dimension two or grater.*

Theorem 4.5. *(Theorem 3.1 of [14]) Any left invariant global trivialization* ρ *of the canonical bundle must be a closed differential form.*

Proof. Let M be a nilmanifold on which a left invariant GCS $\mathcal{J}$ is defined. We know that this structure corresponds to a canonical line bundle ρ which is given in the following form

$$\rho = e^{(B+i\omega)}\Omega, \qquad \Omega = \theta_1 \wedge \cdots \wedge \theta_k \tag{4.4}$$

Also from Lemma 3.9, we know that integrability implies the following: $d\rho = (X+\xi)\cdot\rho$ for some $X + \xi \in \mathbb{V}$. When we expand this condition

$$\begin{aligned} d\rho = (X+\xi)\cdot\rho &= i_X\rho + \xi \wedge \rho \\ d(B+i\omega)e^{(B+i\omega)}\Omega + e^{(B+i\omega)}d\Omega &= i_X(B+i\omega)e^{(B+i\omega)}\Omega + \\ & \quad e^{(B+i\omega)}i_X\Omega + e^{(B+i\omega)}\xi \wedge \Omega \\ d(B+i\omega)\wedge\Omega + d\Omega &= i_X(B+i\omega)\wedge\Omega + i_X\Omega + \xi\wedge\Omega \end{aligned} \tag{4.5}$$

from $k + 1$-part of this equation we get :

$$d\Omega = i_X(B+i\omega)\wedge\Omega + \xi\wedge\Omega. \tag{4.6}$$

Multiplying both sides of Equation 4.6 with θ_i, we find

$$\theta_i \wedge d\Omega = d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_k = 0. \tag{4.7}$$

As we stated in Lemma 4.4, we can choose θ_i's in such a way that the elements of the set $\{\theta_j : nil(\theta_j) > i\}$ are linearly independent modulo V_i [14]. This fact allows us to say that

$$(d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_j) \wedge \theta_{j+1} \wedge \cdots \wedge \theta_k = 0 \tag{4.8}$$

which implies

$$d\theta_i \wedge \theta_1 \wedge \cdots \wedge \theta_j = 0 \;, \qquad\qquad j < i. \tag{4.9}$$

However, this argument holds for all i so, we conclude that $d\Omega = 0$. Also, when we compare degree $k+3$ forms in Equation 4.5 we get

$$d(B + i\omega) \wedge \Omega = 0 \tag{4.10}$$

Finally, combining the result 4.10 with $d\Omega = 0$, we get $d\rho = 0$ which is what we wanted to show. □

Now by using the above machinery we will show how we can put some restrictions on the type of GCS of a six dimensional nilmanifolds.

Proposition 4.6. *[14] If a six dimensional nilmanifold M has nilpotent Lie algebra given by $(0, 0, 0, 12, 14, -)$ and has a nilpotent index four, then M does not admit a left invariant GCS of type 2.*

Proof. Suppose that M has a nilpotent Lie algebra given by $(0, 0, 0, 12, 14, -)$ and it admits a GCS of type two whose corresponding pure spinor is $\rho = e^{(B+i\omega)}\theta_1 \wedge \theta_2$. Here we choose $\{\theta_i\}$ as in Lemma 4.4. From Equation 4.9 we find that $nil(\theta_1) = 0$. So that $\theta_1 = z_1e_1 + z_2e_2 + z_3e_3$. Notice that since we suppose $\mathfrak{g}$ has nilpotent index four we have $dim(V_{i+1}/V_i) = 1$ for $i \geq 1$. So Lemma 4.4 implies $nil(\theta_2) \leq 2$, that is $\theta_2 = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4$. Now, we will apply the conditions $d\Omega = 0$ and

$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$ which imply :

$$\begin{aligned} \Omega &= \theta_1\theta_2 = (z_1w_2 - z_2w_1)e_1e_2 + (z_1w_3 - z_3w_1)e_1e_3 + \\ &\quad (z_1w_4)e_1e_4 + (z_2w_3 - z_3w_2)e_2e_3 + (z_2w_4)e_2e_4 + (z_3w_4)e_3e_4 \end{aligned}$$

Then taking the exterior derivative of Ω, we get :

$$d\Omega = -(z_3w_4)e_3e_1e_2 = 0 \tag{4.11}$$

The condition (4.11) implies that

$$z_3 = 0 \qquad \text{or} \qquad w_4 = 0 \tag{4.12}$$

$$\begin{aligned} \Omega \wedge \bar{\Omega} &= (z_1w_2 - z_2w_1)\overline{(z_3w_4)}e_1e_2e_3e_4 + (z_1w_3 - z_3w_1)\overline{(z_2w_4)}e_1e_3e_2e_4 + \\ &\quad (z_1w_4)\overline{(z_2w_3 - z_3w_2)}e_1e_4e_2e_3 + (z_2w_3 - z_3w_2)\overline{(z_1w_4)}e_2e_3e_1e_4 + \\ &\quad (z_2w_4)\overline{(z_1w_3 - z_3w_1)}e_2e_4e_1e_3 + (z_3w_4)\overline{(z_1w_2 - z_2w_1)}e_3e_4e_1e_2 \end{aligned} \tag{4.13}$$

From Equations 4.12 and 4.13 we deduce that $z_3 = 0$. So $\theta_1 = z_1e_1 + z_2e_2$ and $\Omega = (z_1w_2 - z_2w_1)e_1e_2 + (z_1w_3)e_1e_3 + (z_1w_4)e_1e_4 + (z_2w_3)e_2e_3 + (z_2w_4)e_2e_4$. Moreover the condition $\omega \wedge \Omega \wedge \bar{\Omega} = 0$ implies ω must be nondegenerate on $\{e^5, e^6\}$. Using this we find that $B + i\omega = (k_1e_1 + \cdots k_5e_5)e_6 + \alpha$ where $\alpha \in \bigwedge^2 \langle e_1, \cdots, e_5\rangle$ and $k_5 \neq 0$. Equation 4.5 implies

$$d\rho = d(B + i\omega) \wedge \Omega = 0 \tag{4.14}$$

In our case we get

$$\begin{aligned} d\rho &= (k_4e_1e_2e_6 + k_5e_1e_4e_6 - (k_1e_1 + \cdots k_5e_5)de_6 + d\alpha) \wedge \Omega \\ &= k_5z_2w_3e_1e_4e_6e_2e_3 - [(k_1e_1 + \cdots k_5e_5)de_6] \wedge [(z_2w_3)e_2e_3 + (z_2w_4)e_2e_4] \\ &\quad + d\alpha \wedge \Omega \end{aligned} \tag{4.15}$$

Notice that in Equation 4.15 the only term that contains e_6 is $k_5z_2w_3e_1e_4e_6e_2e_3$ (recall that $d\sigma \in V_{i-1}$ if $\sigma \in V_i$) and the coefficient of this term $k_5z_2w_3$ is nonzero. First note that, we have assume that k_5 is a nonzero constant. Also, if $z_2 = 0$ then $\theta_1\bar{\theta}_1 = 0$ which implies that $\Omega \wedge \bar{\Omega} = 0$ and finally $w_3 = 0$ implies $\Omega = (z_1w_2 - z_2w_1)e_1e_2 + (z_1w_4)e_1e_4 + (z_2w_4)e_2e_4$ and hence $\Omega \wedge \bar{\Omega} = 0$ again. Therefore we can conclude that $d\rho$ cannot be a closed pure spinor which gives a contradiction. □

Using similar arguments we can put some restrictions on the type of a left invariant GCS on nilmanifolds. Moreover, these restrictions in some cases help us to find GCS explicitly.

Let M be a nilmanifold with nilpotent Lie algebra $(0, 0, 12, 13, 14 + 35)$. We will investigate type two GCS on this nilmanifolds. First of all, let us write V_i's explicitly.

$$V_1 =< e_1, e_2, e_3 >, V_2 =< e_1, e_2, e_3, e_4 >,\ V_3 = V \tag{4.16}$$

So $\theta_1 = z_1e_1 + z_2e_2 + z_3e_3$. Due to Lemma 4.4 and the fact that $dim(V_3/V_2) = 1$,we have $\theta_2 = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4 + w_5e_5$, and w_4 or w_5 are nonzero.

$$\begin{aligned}
\Omega &= \theta_1 \wedge \theta_2 = \theta_1\theta_2 = (z_1w_2 - z_2w_1)e_1e_2 + (z_1w_3 - z_3w_1)e_1e_3 + \\
&\quad (z_1w_4)e_1e_4 + (z_1w_5)e_1e_5 + (z_2w_3 - z_3w_2)e_2e_3 + (z_2w_4)e_2e_4 + (z_2w_5)e_2e_5 + \\
&\quad (z_3w_4)e_3e_4 + (z_3w_5)e_3e_5 \\
d\Omega &= -(z_2w_5)e_2e_1e_3 - (z_3w_4)e_3e_1e_2 \\
&= 0 \Longleftrightarrow z_2w_5 = z_3w_4 \qquad (4.17)
\end{aligned}$$

$$
\begin{aligned}
\Omega \wedge \bar{\Omega} \;=\; & \left[(z_1w_2 - z_2w_1)\overline{(z_3w_4)} + (z_3w_4)\overline{(z_1w_2 - z_2w_1)}\right] e_1e_2e_3e_4 + \\
& \left[(z_1w_2 - z_2w_1)\overline{(z_3w_5)} + (z_3w_5)\overline{(z_1w_2 - z_2w_1)}\right] e_1e_2e_3e_5 + \\
& \left[(z_1w_3 - z_3w_1)\overline{(z_2w_4)} + (z_2w_4)\overline{(z_1w_3 - z_3w_1)}\right] e_1e_3e_2e_4 + \\
& \left[(z_1w_3 - z_3w_1)\overline{(z_2w_5)} + (z_2w_5)\overline{(z_1w_3 - z_3w_1)}\right] e_1e_3e_2e_5 + \\
& \left[(z_1w_4)\overline{(z_2w_3 - z_3w_2)} + (z_2w_3 - z_3w_2)\overline{(z_1w_4)}\right] e_1e_4e_2e_3 + \\
& \left[(z_1w_5)\overline{(z_2w_3 - z_3w_2)} + (z_2w_3 - z_3w_2)\overline{(z_1w_5)}\right] e_1e_5e_2e_3 + \\
& \left[(z_1w_5)\overline{(z_2w_4)} + (z_2w_4)\overline{(z_1w_5)}\right] e_1e_5e_2e_4 + \\
& \left[(z_1w_5)\overline{(z_3w_4)} + (z_3w_4)\overline{(z_1w_5)}\right] e_1e_5e_3e_4 + \\
& \left[(z_2w_4)\overline{(z_3w_5)} + (z_3w_5)\overline{(z_2w_4)}\right] e_2e_4e_3e_5 + \\
& \left[(z_2w_5)\overline{(z_3w_4)} + (z_3w_4)\overline{(z_2w_5)}\right] e_2e_5e_3e_4 + \\
& \left[(z_1w_4)\overline{(z_2w_5)} + (z_2w_5)\overline{(z_1w_4)}\right] e_1e_4e_2e_5 + \\
& \left[(z_1w_4)\overline{(z_3w_5)} + (z_3w_5)\overline{(z_1w_4)}\right] e_1e_4e_3e_5 +
\end{aligned}
$$

Rearranging the terms we get the following coefficients.

The coefficients of $e_1e_2e_3e_4$:

$$
\begin{aligned}
a \;=\; & Re(z_1w_4(\overline{z_2w_3})) - Re(z_1w_3(\overline{z_2w_4})) + Re(z_1w_2(\overline{z_3w_4})) - \\
& Re(z_1w_4(\overline{z_3w_2})) + Re(z_3w_1(\overline{z_2w_4})) - Re(z_2w_1(\overline{z_3w_4}))
\end{aligned} \tag{4.18}
$$

The coefficients of $e_1e_2e_3e_5$:

$$
\begin{aligned}
b \;=\; & Re(z_1w_2(\overline{z_3w_5})) + Re(z_1w_3(\overline{z_2w_5})) - Re(z_2w_1(\overline{z_3w_5})) - \\
& Re(z_1w_5(\overline{z_1w_3})) - Re(z_3w_1(\overline{z_2w_5})) - Re(z_1w_5(\overline{z_3w_2}))
\end{aligned} \tag{4.19}
$$

The coefficients of $e_1e_2e_4e_5$:

$$
c \;=\; Re(z_1w_5(\overline{z_2w_4})) - Re(z_1w_4(\overline{z_2w_5})) \tag{4.20}
$$

The coefficients of $e_1e_3e_4e_5$:

$$d = Re(z_1w_5(\overline{z_3w_4})) - Re(z_1w_4(\overline{z_3w_5})) \tag{4.21}$$

The coefficients of $e_2e_3e_4e_5$:

$$f = Re(z_2w_5(\overline{z_3w_4})) - Re(z_2w_4(\overline{z_3w_5})) \tag{4.22}$$

Notice that e^6 is in the kernel of $\Omega \wedge \bar{\Omega}$. Suppose the other vector in the kernel is in the form $v_1e^1 + \cdots + v_5e^5$ then we get the following conditions

$$\begin{array}{lll} cv_1 - fv_3 = 0\,, & cv_2 + dv_3 = 0\,, & cv_5 - av_3 = 0 \\ cv_4 + bv_3 = 0\,, & dv_1 + fv_2 = 0\,, & dv_4 - bv_2 = 0 \\ cv_5 + av_2 = 0\,, & av_1 - fv_5 = 0\,, & ev_4 + bv_1 = 0 \\ & av_4 + bv_5 = 0 & \end{array} \tag{4.23}$$

Finally, we compute the condition $d\rho = 0$. As we have seen in the above computations, we can write $B + i\omega = (k_1e_1 + k_2e_2 + \cdots + k_5e_5)\, e_6 + \alpha$ such that at least one of the k_i's are nonzero and $\alpha \in \bigwedge^2 < e_1, \cdots, e_5 >$.

$$\begin{aligned} d(B+i\omega)\Omega \;=\; & (k_4z_3w_4)e_1e_2e_3e_4e_6 + (k_4z_3w_5)e_1e_2e_3e_5e_6 - \\ & (k_5z_2w_4)e_1e_2e_3e_4e_6 - (k_5z_2w_5)e_1e_2e_3e_5e_6 + \\ & (k_1z_2w_4)e_1e_2e_3e_4e_5 - (k_2z_3w_5)e_1e_2e_3e_4e_5 - \\ & (k_2z_1w_4)e_1e_2e_3e_4e_5 + (k_3z_2w_5)e_1e_2e_3e_4e_5 + \\ & (k_4z_1w_2)e_1e_2e_3e_4e_5 - (k_4z_2w_1)e_1e_2e_3e_4e_5 - \\ & (k_5z_2w_3)e_1e_2e_3e_4e_5 + (k_5z_3w_2)e_1e_2e_3e_4e_5 \end{aligned} \tag{4.24}$$

The Equation 4.24 must be equal to zero. So, in particular

$$w_4(k_4z_3 - z_2k_5) = 0$$
$$w_5(k_4z_3 - z_2k_5) = 0$$

Since w_4 and w_5 cannot be equal to zero simultaneously, we see that

$$k_4z_3 - z_2k_5 = 0 \tag{4.25}$$

Also the remaining terms in Equation 4.24 must satisfy :

$$w_4(k_1z_2 - z_1k_2) + w_5(k_3z_2 - z_3k_2) + k_4(z_1w_2 - w_1z_2) + k_5(z_3w_2 - w_3z_2) = 0 \tag{4.26}$$

To sum up:
The nilmanifold whose nilpotent Lie algebra given by $(0, 0, 0, 12, 13, 14 + 35)$ admits a type two left invariant GCS if and only if the conditions found in Equations 4.17, 4.23, 4.26, 4.25 must hold. Also a, b, c, d, f given in Equation 4.22 should not be zero simultaneously. These conditions allows us to construct type two GCS on $(0, 0, 0, 12, 13, 14+35)$.

Proposition 4.7. $\rho = e^{(e_1e_6+e_3e_6)+i(e_3e_6-e_1e_6)}(ie_1+e_2+ie_3)(e_4+ie_5)$ *gives a type two left invariant GCS on the nilmanifold with associated nilpotent Lie algebra* $(0, 0, 0, 12, 13, 14+35)$*. Also note that this GCS does not apper in the list given in [14] which says that these structures do not need to be unique.*

Proof. Just check the conditions 4.17, 4.23, 4.26, 4.25 and 4.22 for

$$\theta_1 = ie_1 + e_2 + ie_3 \qquad \text{and} \qquad \theta_2 = e_4 + ie_5$$
$$B + i\omega = (e_1e_6 + e_3e_6) + i(e_3e_6 - e_1e_6)$$

□

Remark 4.8. *As we already stated in the introduction, there are 34 different nilmanifold classes and only five of them do not admit a complex or symplectic structure. These classes are given in [14] as:*

$$(0, 0, 12, 13, 14, 34 + 52)$$
$$(0, 0, 0, 12, 13, 14 + 23, 34 + 52)$$
$$(0, 0, 0, 12, 13, 14 + 35)$$
$$(0, 0, 0, 12, 23, 14 + 35)$$
$$(0, 0, 0, 0, 12, 15 + 34)$$

Notice that we already find a GCS on the nilmanifold class $(0, 0, 0, 12, 13, 14 + 35)$*. Using similar arguments as we used to obtain the conditions in the proof of Proposition 4.7, one can show that all of these nilmanifolds admit a GCS [14].*

We know that except the trivial nilpotent Lie algebra, none of these algebras admit a Kähler structure. So it is natural to ask whether they admit generalized Kähler structures or not. The answer is:

Theorem 4.9. *[17] If a nilpotent Lie algebra* $\mathfrak{g}$ *admits a generalized Kähler structure then* $\mathfrak{g}$ *is Abelian.*

Proof. We have showed in Theorem 3.6 that if a manifold admits a generalized Kähler structure with trivial canonical bundle then its differential graded algebra $\Omega(\bar{L}_1, d_{L_1})$ must be formal. Moreover, Example (3.27) shows that the differential graded nilpotent algebras are not formal except they are trivial. Also note that $\bar{L}_1$ is nilpotent subalgebra of the nilpotent algebra $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathbb{C}$. So we can conclude that only Abelian nilpotent Lie algebras can admit GKS. □

5. CONCLUSION

In this thesis, we studied the generalized complex geometry which is a generalizations of the complex and symplectic structures. In fact, these two are just some special examples of GCG. Then, in order to demonstrate that the GCS is more than just the union of these structures, we considered GCS on six dimensional nilmanifolds.It is known that five classes of nilmanifolds admit neither a complex nor a symplectic structure. In the article [14] the nilpotent Lie algebra $(0, 0, 0, 12, 13, 14 + 35)$ is one of the classes admitting neither structures. Following the arguments given in this article, in Corollary 4.7 we explicitly constructed a generalized complex structure on these nilmanifolds. Proceeding similarly, we aimed to understand GKS on nilmanifolds. Following the article [17], we discussed the Hodge theory of GCS and formality property. These notions brought us to the result that if a nilmanifold admits a GKS then its corresponding Lie algebra must be trivial, (Theorem 4.9).

Our main purpose was to understand essential properties of the generalized complex structures and applications of these in the nilmanifolds. For this aim, we tried to study these structures on the most basic level. Therefore, we avoided going into the relations of the subject with Courant algebroids and gerbes which provide higher generalization of the subject. For these one can consult the articles [2], [9] and [25]. Also we did not mention $H-$ fluxes and generalized diffeomorphisms which have important counterparts in the physics literature, see [6] and [2]. In addition to these, recall that in Section 3.2, we have define a metric on the generalized tangent bundle. So, one of the natural study area in generalized geometry is to study the Riemannian geometric properties of these structures. Especially, understanding the properties of the generalized Riemannian curvature and generalized Ricci tensors is an interesting task for both mathematicians and physicists. For details see [26] and [27].

In the search of generalized Kähler manifolds in Section 3.4, we limited ourselves to a small portion of the subject. Historically, the relation with the generalized complex geometry and the string theory rooted in the equivalence of the generalized Kähler structures and Bi-Hermitian structures. Originated from this relation, now string theory and generalized complex geometry have very tight relations. Moreover, very large amount of researchers still try to make this relation more apparent, see [5], [8] and [7].

Let us also mention some possible further study topics in the field GCS on nilmanifolds. In Chapter 4.1 we focused on six dimensional nilmanifolds and showed that all of them do admit a GCS. The natural question in the sequence is, whether this fact is valid for all nilmanifolds or not. In fact, one can find eight dimensional nilmanifolds which does not admit a GCS. This result makes things much interesting since it implies that the construction of GCS on nilmanifolds is not a trivial task.

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