# ON GENERALIZATIONS OF THE MCKAY CONJECTURE

by

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# ABSTRACT

## ON GENERALIZATIONS OF THE MCKAY CONJECTURE

McKay Conjecture asserts the existence of a one-to-one correspondence between sets of certain irreducible characters of any group and that of its local subgroups. In this thesis, we mainly study two generalizations of this conjecture, known as Isaacs-Navarro and Galois-McKay conjectures, along with their reductions to finite simple groups. The former considers the original conjecture via considering equivalence classes in $\bmod\, p$ whereas the other takes into account a particular action of a Galois group. Also, we construct some new $\mathcal{H}$-triples following [1].

# ÖZET

## MCKAY SANISININ GENELLEMELERİ ÜZERİNE

McKay sanısı herhangi bir grubun belli indirgenemez karakterlerinin kümesi ile bu grubun bazı lokal altgruplarının aynı şekilde kurulmuş kümeleri arasında birebir ve örten bir fonksiyon olduğunu söyler. Bu tezde, bu sanının Isaacs-Navarro ve Galois-McKay sanıları olarak adlandırılan iki genellemesini sonlu basit gruplara indirgenmesiyle birlikte çalışacağız. Bunlardan ilki özgün sanıyı mod $p$'deki denklik sınıflarını dikkate alırken diğeri belli bir Galois grubunun etkisiyle düşünerek geneller. Ayrıca [1] makalesini takip ederek bazı yeni $\mathcal{H}$-üçlüleri inşa edeceğiz.

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# LIST OF SYMBOLS

| | |
|---|---|
| $\mathrm{Aut}(G)$ | The set of automorphisms of $G$ |
| $\mathrm{Aut}(G)_H$ | The set of automorphisms of $G$ fixing $H$ setwise |
| $\mathbb{C}$ | Complex numbers |
| $\mathrm{cf}(G)$ | The set of class functions of $G$ |
| $c_g$ | Conjugation homomorphism by the element $g$ |
| $\mathrm{Cl}(G)$ | The set of conjugacy classes of $G$ |
| $\mathrm{Char}(G)$ | The set of characters of $G$ |
| $\mathrm{Char}(G \mid \theta)$ | The set of characters of $G$ whose irreducible constituents lie over $\theta$ |
| $C_G(H)$ | The set of fixed points of the action of $H$ on $G$ or the centralizer of $H$ in $G$ when $H \leq G$ |
| $G$, $H$, $K$, $M$, $N, ...$ | Finite groups |
| $\lvert G \rvert$ | Number of elements in the set $G$ |
| $\mathcal{G}$ | The Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ |
| $[G, N]$ | Commutator of $G$ and $N$ |
| $G_\theta$ | The stabilizer of the character $\theta$ in $G$ |
| $G_{\theta^{\mathcal{H}}}$ | The stabilizer of the set $\theta^{\mathcal{H}}$ in $G$ |
| $\mathrm{Gal}(F/K)$ | Galois group of $F$ over $K$ |
| $G' = [G, G]$ | The derived subgroup of $G$ |
| $G \rtimes H$ | Semidirect product of $G$ and $H$ |
| $\mathrm{GL}_n(F)$ | The set of $n \times n$ invertible matrices over the field $F$ |
| $(G, N, \theta)$ | A character triple |
| $(G, N, \theta)_{\mathcal{H}}$ | An $\mathcal{H}$-triple |
| $\mathcal{H}$ | The set of $\sigma \in \mathcal{G}$ such that $\sigma(\xi) = \xi^{p^k}$ for all roots of unity $\xi$ of order not divisible by $p$, where $p$ is a prime and $k$ is a fixed integer |
| $I_n$ | $n \times n$ identity matrix |
| $\mathrm{Irr}(G)$ | The set of irreducible characters of $G$ |
| $\mathrm{Irr}(G \mid \theta)$ | The set of irreducible characters of $G$ that lies over $\theta$ |

| | |
|---|---|
| $\mathrm{Irr}(G \mid \theta^{\mathcal{H}})$ | The set of $\chi \in \mathrm{Irr}(G)$ that lies over a character in the $\mathcal{H}$-orbit of $\theta$ |
| $\mathrm{Irr}(\chi)$ | The set of irreducible constituents of $\chi$ |
| $\mathrm{Irr}_H(G)$ | The set of $H$-invariant irreducible characters of $G$ |
| $\mathrm{Irr}_{p'}(G)$ | The set of irreducible characters of $G$ of degree not divisible by $p$ |
| $\mathrm{Irr}_{p'}^k(G)$ | The set of $\chi \in \mathrm{Irr}_{p'}(G)$ for which $\chi(1) \equiv \pm k \pmod p$ |
| $\ker \chi$ | Kernel of the character $\chi$ |
| $\mathrm{Lin}(G)$ | The set of linear characters of $G$ |
| $N \subseteq G$ and $G \supseteq N$ | $N$ is a subset of $G$ |
| $N_G(P)$ | Normalizer of $P$ in $G$ |
| $p$ | A prime number |
| $\mathcal{P}^G$ | Induction of the projective representation $\mathcal{P}$ to $G$ |
| $\mathcal{P}_N$ | Restriction of the projective representation $\mathcal{P}$ to $N$ |
| $\mathrm{Proj}(G)$ | The set of projective representations of $G$ |
| $\mathrm{Proj}(G \mid \alpha)$ | The set of projective representations of $G$ with factor set $\alpha$ |
| $\mathrm{Proj}_{\mathbb{Q}^{ab}}(G)$ | The set of projective representations of $G$ with entries in $\mathbb{Q}^{ab}$ |
| $\mathbb{Q}^{ab}$ | Smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all roots of unity |
| $\mathrm{Rep}(G)$ | The set of representations of $G$ |
| $\mathrm{Rep}(G \mid \theta)$ | The set of representations of $G$ affording a character in $\mathrm{Char}(G \mid \theta)$ |
| $\mathrm{Rep}_{\mathbb{Q}^{ab}}(G)$ | The set of representations of $G$ over $\mathbb{Q}^{ab}$ |
| $S$ | A simple group |
| $\mathrm{Syl}_p(G)$ | The set of Sylow $p$-subgroups of $G$, where $p \big\| |G|$ |
| $\mathrm{tr}$ | Trace |
| | |
| $\chi^G$ | Induction of the character $\chi$ to $G$ |
| $\chi_H$ | Restriction of the character $\chi$ to $H$ |
| $[\chi, \psi]$ | Inner product of characters $\chi$ and $\psi$ |
| $\theta^{\mathcal{H}}$ | $\mathcal{H}$-orbit of $\theta$ |

| | |
|---|---|
| $\bigsqcup$ | Disjoint union |
| $\iff$ | If and only if |
| $\lvert\ :\ \rvert$ | Index of groups |
| $\cong$ | Isomorphic to |
| $\otimes$ | Kronecker product |
| $\sim$ | Similar |
| $<$ | Proper subgroup |
| $\leq$ | Subgroup |
| $\trianglelefteq$ | Normal subgroup |

# LIST OF ACRONYMS/ABBREVIATIONS

| | |
|---|---|
| CFSG | Classification of finite simple groups |
| iGM$p$ | Inductive Galois-McKay condition for $p$ |
| iIN$p$ | Inductive Isaacs-Navarro condition for $p$ |

# 1. INTRODUCTION

In representation theory, it is of great importance to determine some information about the characters of a group from its local subgroups. These studies lead to Global-Local counting conjectures. Among them, John McKay conjectured, in 1972, that there exists a bijection between the set of irreducible characters of $2'$-degree of any group $G$ and that of some local subgroups of $G$. Indeed, the conjecture states that for a finite group $G$ of even order and $P \in \mathrm{Syl}_2(G)$, the number of irreducible characters of $G$ of degree not divisible by 2 is the same as that of $N_G(P)$. (See [2], for instance.) Since the announcement of this conjecture, some generalizations are made. Firstly, the same setting is claimed to hold for every prime and it is still named the McKay conjecture. In fact, letting $\mathrm{Irr}_{p'}(G)$ to denote the set of irreducible characters of $G$ of degree not divisible by some prime $p$, the McKay conjecture states the existence of a bijection

$$\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(N_G(P)), \tag{1.1}$$

where $P$ is any Sylow $p$-subgroup of $G$ and $N_G(P)$ is the normalizer subgroup of $P$ in $G$. Since then, some partial cases are checked to satisfy (1.1). As it is mentioned in [3], whenever $G$ is a solvable or $p$-solvable group, then the assertion is true. In the beginnings of this century, a reduction of the McKay conjecture to non-abelian finite simple groups is published in [4]. Now, by appealing to CFSG, certain conditions for all non-abelian finite simple groups need to be checked. Also, it is proven in [1] that these conditions on simple groups imply more than the McKay conjecture. As of today, the cases $p = 2$ and $p = 3$ are completed. (See [5, 6].) Therefore an affirmative answer to the conjecture is given and the original assertion made by John McKay is accomplished.

This conjecture has some other refinements appeared in [7]. For instance, it is asserted that the bijection (1.1) satisfies

$$\Omega(\theta)(1) \equiv \pm\theta(1) \pmod{p} \tag{1.2}$$

for every $\theta \in \mathrm{Irr}_{p'}(G)$. Moreover, considering the Galois actions on characters, the map in (1.1) does not commute with an action of $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, as any automorphism in this group fixes rational-valued characters and the number of rational-valued characters in $\mathrm{Irr}_{p'}(G)$ and in $\mathrm{Irr}_{p'}(N_G(P))$ are not always the same. However, if $\mathcal{H}$ is the subset of $\mathcal{G}$ containing the automorphisms $\sigma$ in $\mathcal{G}$ such that $\sigma(\xi) = \xi^{p^k}$ for some fixed $k$, for every root of unity $\xi$ of order not divisible by $p$, then it is asserted that the bijection in (1.1) is $\mathcal{H}$-equivariant. Recently, a reduction of this is given in [8]. Also, by using this reduction, it is checked in [9] that this conjecture is true for $p = 2$. In this thesis, we mainly study these two generalizations and their reduction theorems.

In Chapter 2, we review the basics of ordinary complex representations and character theory, and we give the essential results which we need later. Also we discuss group and Galois actions on characters. In Chapter 3, we study projective representations and character triples. Then we construct projective representations having entries in $\mathbb{Q}^{ab}$ that are associated with characters. In Chapter 4, we work on $\mathcal{H}$-triples that are constructed in [8]. Then we build new such triples analogous to the character triples in [1]. In Chapter 5, we give a proof of a reduction of the Isaacs-Navarro conjecture, by following [3]. Note that this is also done in [10]. Finally, we mention the reduction of the Galois-McKay conjecture.

Throughout the thesis, every group is finite, by assumption. Also $G$ always denotes a finite group.

# 2. ORDINARY REPRESENTATIONS

In this chapter, we follow [3], except otherwise noted.

**Definition 2.1.** *A map* $\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ *is called a (complex) representation of* $G$ *if* $\rho$ *is a group homomorphism. Then the complex valued map*

$$\begin{aligned} \chi\colon G &\longrightarrow \mathbb{C} \\ g &\longmapsto \mathrm{tr}(\rho(g)) \end{aligned} \tag{2.1}$$

*is said to be a (complex) character of* $G$*, afforded by* $\rho$*, where* $\mathrm{tr}(\rho(g))$ *denotes the trace of* $\rho(g)$*. The dimension of the representation equals* $n = \chi(1)$ *and it is called the degree of the character.*

Note that the homomorphism $\rho : G \longrightarrow \mathrm{GL}_1(\mathbb{C})$, given by $g \longmapsto 1$ defines a representation, affording the trivial character $1_G(g) = 1$, for all $g \in G$.

**Remark 2.2.** *Characters are constant on conjugacy classes, since for every* $x$, $g \in G$*, we have*

$$\mathrm{tr}\left(\rho(x^{-1}gx)\right) = \mathrm{tr}\left(\rho(x)^{-1}\rho(g)\rho(x))\right) = \mathrm{tr}\left(\rho(g)\right). \tag{2.2}$$

**Definition 2.3.** *Let* $\rho$ *and* $\rho'$ *be two representation of* $G$ *with equal dimensions. If there exists an invertible matrix* $A$ *for which*

$$A^{-1}\rho(g)A = \rho'(g) \tag{2.3}$$

*for every* $g \in G$*, then* $\rho$ *and* $\rho'$ *are called similar representations. Moreover if* $\rho$ *is similar to a representation of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{2.4}$$

*then* $\rho$ *is called reducible; otherwise, it's called irreducible. Then the irreducibility of a character is defined to be the irreducibility of a representation, which affords the character.*

By the following result, we determine the matrices commuting with representations.

**Lemma 2.4** (Schur's Lemma)**.** *Assume $\rho$ is an irreducible representation of $G$ of dimension $n$. If there exists an $n \times n$ matrix $A$ with entries in $\mathbb{C}$ such that for every $g \in G$, we have*

$$A\rho(g) = \rho(g)A, \tag{2.5}$$

*then $A$ is scalar.*

**Notation 2.5.** *We denote the set of*

- *(i) representations of $G$, by $\mathrm{Rep}(G)$;*
- *(ii) characters of $G$, by $\mathrm{Char}(G)$;*
- *(iii) irreducible characters of $G$, by $\mathrm{Irr}(G)$;*
- *(iv) irreducible characters of $G$ with degree not divisible by $p$, by $\mathrm{Irr}_{p'}(G)$;*
- *(v) conjugacy classes in $G$, by $\mathrm{Cl}(G)$;*
- *(vi) complex-valued functions that are constant on conjugacy classes of $G$, which are called class functions, by $\mathrm{cf}(G)$.*

**Remark 2.6.** *(i) $\mathrm{Irr}(G)$ and the set of characteristic functions of $\mathrm{Cl}(G)$ constitute bases for the vector space $\mathrm{cf}(G)$, so that $|\mathrm{Irr}(G)| = |\mathrm{Cl}(G)|$. In particular, $|\mathrm{Irr}(G)|$ is always finite. Also, if $G$ is abelian, then we have $|G| = |\mathrm{Irr}(G)|$.*

*(ii) Let $\chi \in \mathrm{Char}(G)$. Set*

$$\overline{\chi}(g) := \overline{\chi(g)}, \tag{2.6}$$

*where $\overline{\chi(g)}$ is the complex conjugate of $\chi(g)$. Then $\overline{\chi}$ is also a character of $G$, because if $\rho$ affords $\chi$, then $\overline{\rho}$ is a representation, affording $\overline{\chi}$, where $\overline{\rho}(g)$ is defined to be the matrix obtained by taking complex conjugate of each entry of the matrix $\rho(g)$, for all $g \in G$.*

*(iii) If $\rho$ is a representation, then $\rho(g)$ is similar to a diagonal matrix with roots of unity values in the diagonal. Therefore for any $\chi \in \mathrm{Char}(G)$, we have that $\chi(g)$ is a sum of roots of unity, for every $g \in G$. In particular, $\overline{\chi}(g) = \chi(g^{-1})$.*

*(iv) By using algebraic integers (which we omit) degrees of irreducible characters divide the order of the group. Moreover, we have the following generalization.*

**Theorem 2.7.** *Let $N \trianglelefteq G$ and assume $N$ is abelian. Then for every $\chi \in \mathrm{Irr}(G)$, we have $\chi(1)$ divides $|G : N|$.*

**Definition 2.8.** *Let $\chi, \psi \in \mathrm{Char}(G)$. Set*

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \tag{2.7}$$

*Then $[-, -]$ defines a Hermitian inner product on the vector space $\mathrm{cf}(G)$.*

There are two important orthogonality relations on this inner product, due to Schur.

**Theorem 2.9** (Schur's Orthogonality Relations)**.** *Let $\chi, \psi \in \mathrm{Irr}(G)$ and $g, h \in G$. Then we have*

$$[\chi, \psi] = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise}, \end{cases} \tag{2.8}$$

*and*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.9}$$

*Moreover, if $\chi \in \mathrm{Char}(G)$ with $[\chi, \chi] = 1$, then $\chi$ is irreducible.*

**Remark 2.10.** *(i) By taking $g = h = 1$ above, we have $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$.*

*(ii) For every $\chi, \psi \in \mathrm{Char}(G)$, $[\chi, \psi]$ is always a non-negative integer.*

*(iii) If $\chi \in \mathrm{Char}(G)$, then $\chi = \sum_{\psi \in \mathrm{Irr}(G)} [\chi, \psi]\, \psi$, as $\mathrm{Irr}(G)$ is a basis for $\mathrm{cf}(G)$ and $\mathrm{Char}(G) \subseteq \mathrm{cf}(G)$.*

**Proposition 2.11.** *Let $\chi, \psi \in \mathrm{Char}(G)$ be afforded by $\rho, \rho' \in \mathrm{Rep}(G)$, respectively. Write $m = \chi(1)$ and $n = \psi(1)$. Set*

$$\begin{aligned} \rho \otimes \rho' : G &\longrightarrow GL_{mn}(\mathbb{C}) \\ g &\longmapsto \rho(g) \otimes \rho'(g) \end{aligned} \tag{2.10}$$

*where $\rho(g) \otimes \rho'(g)$ is the usual Kronecker product of $\rho(g)$ and $\rho'(g)$. Then $\rho \otimes \rho'$ is a representation of $G$ affording $\chi\psi$. In particular, the product of two characters is still a character.*

**Definition 2.12** ( [11]). *For a group $G$, set $G' = [G, G] = \langle x^{-1}y^{-1}xy \,:\, x, y \in G\rangle$. Then $G'$ is called the derived subgroup of $G$.*

Note that $G' \trianglelefteq G$ and in fact, it is the minimal normal subgroup of $G$ with abelian quotient.

**Definition 2.13.** *Let $\mu \in \mathrm{Irr}(G)$. If $\mu(1) = 1$, then $\mu$ is called a linear character. Also, we set $\mathrm{Lin}(G) = \{\mu \in \mathrm{Irr}(G) \,:\, \mu(1) = 1\}$.*

**Theorem 2.14.** *For any finite group $G$, $\mathrm{Lin}(G) \cong G/G'$.*

The product of an irreducible character with a linear character is always irreducible, because

$$[\chi\mu, \chi\mu] = [\chi, \chi] = 1, \tag{2.11}$$

for all $\chi \in \mathrm{Irr}(G)$ and $\mu \in \mathrm{Lin}(G)$. However, the product of two non-linear irreducible characters need not be irreducible. For example, the non-linear irreducible character $\psi$ of the dihedral group of order 8 has degree 2. Then $\psi\psi$ has degree 4 and is not an irreducible character of this group.

When $G$ can be written as a direct product of $n$ groups, we describe the irreducible characters of $G$ as follows:

**Theorem 2.15.** *Let $G$ be the direct product of its subgroups $K_1, K_2, \ldots, K_n$. Then there is a bijection*

$$\begin{aligned} \mathrm{Irr}(K_1) \times \mathrm{Irr}(K_2) \times \cdots \times \mathrm{Irr}(K_n) &\longrightarrow \mathrm{Irr}(G) \\ (\psi_1, \psi_2, \ldots, \psi_n) &\longmapsto \psi_1 \times \psi_2 \times \cdots \times \psi_n, \end{aligned} \tag{2.12}$$

*where $\big(\psi_1 \times \psi_2 \times \cdots \times \psi_n\big)(x_1x_2\cdots x_n) = \psi_1(x_1)\,\psi_2(x_2)\,\cdots\,\psi_n(x_n)$, for every $x_i \in K_i$.*

Note that since $G$ is the direct product of $K_i$'s, we have $K_j \cap (\prod_{i\neq j} K_i) = 1$ for every $1 \leq j \leq n$, and therefore the definition $\psi_1 \times \psi_2 \times \cdots \times \psi_n$ makes sense. After defining central products of groups, we give a quite similar construction for irreducible characters of central products of groups.

## 2.1. Operations on Characters and Correspondences

**Definition 2.16.** *Let* $\chi \in \mathrm{Char}(G)$. *Then the kernel of* $\chi$ *is defined to be the set*

$$\ker\chi = \{g \in G : \ \chi(g) = \chi(1)\}. \tag{2.13}$$

**Definition 2.17.** *(i) (Inflation) Let* $N \trianglelefteq G$ *and let* $\tilde{\chi} \in \mathrm{Char}(G/N)$. *Define*

$$\begin{aligned} \chi\colon G &\longrightarrow \mathbb{C} \\ g &\longmapsto \tilde{\chi}(Ng) \end{aligned} \tag{2.14}$$

*Then* $\chi$ *is a character of* $G$*, called the inflation/lift of* $\tilde{\chi}$ *to* $G$.

*(ii) (Restriction) Let* $H \leq G$ *and* $\chi \in \mathrm{Char}(G)$. *Then*

$$\begin{aligned} \chi_H\colon H &\longrightarrow \mathbb{C} \\ h &\longmapsto \chi(h) \end{aligned} \tag{2.15}$$

*is a character of* $H$*, called the restriction of* $\chi$ *to* $H$.

*(iii) (Induction) Let* $H \leq G$ *and* $\theta \in \mathrm{Char}(H)$. *For any* $x \in G$*, set*

$$\dot{\theta}(x) := \begin{cases} \theta(x) & \text{if } x \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{2.16}$$

*and define*

$$\theta^G(x) = \frac{1}{|H|}\sum_{g\in G}\dot{\theta}(gxg^{-1}). \tag{2.17}$$

*Then* $\theta^G$ *is a character of* $G$*, called the induction of* $\theta$.

Note that inflation of characters preserve irreducibility. However, restriction and induction of characters need not preserve irreducibility. In the setting of Part (i) of the above definition, observe that $N \subseteq \ker\chi$. Moreover, a bijection between $\mathrm{Irr}(G/N)$ and the set of irreducible characters of $G$ containing $N$ in its kernel can be constructed, easily. Therefore, from now on, we'll not distinguish between these two sets and sometimes view $\mathrm{Irr}(G/N) = \{\chi \in \mathrm{Irr}(G) \,:\, N \subseteq \ker\chi\}$. In the setting of Part (iii), we can write

$$\theta^G(x) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \theta(gxg^{-1}) = \sum_{\substack{1 \leq i \leq n \\ t_i x t_i^{-1} \in H}} \theta(t_i x t_i^{-1}) \tag{2.18}$$

where $\{t_1, \ldots, t_n\}$ is a right transversal (that is, a complete set of right coset representatives) of $H$ in $G$.

**Remark 2.18.** *Let $H$ be a normal subgroup of $G$ and let $\chi \in \mathrm{Char}(G)$. For any $x \in G$, we have*

$$\chi^G(x) = \begin{cases} \dfrac{|G|}{|H|}\chi(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

*In particular,* $\chi^G(1) = \dfrac{|G|}{|H|}\chi(1)$.

**Definition 2.19.** *Let $\chi \in \mathrm{Char}(G)$. The characters $\psi \in \mathrm{Irr}(G)$ for which $[\chi, \psi] \neq 0$ are said to be the irreducible constituents of $\chi$, and the set of such irreducible characters is denoted by $\mathrm{Irr}(\chi)$. Also if $\theta \in \mathrm{Irr}(\psi_N)$ for some $\psi \in \mathrm{Irr}(G)$, set*

$$\mathrm{Irr}(G \,|\, \theta) = \{\chi \in \mathrm{Irr}(G) \,:\, [\chi_N, \theta] \neq 0\}, \tag{2.20}$$

*and write* $\mathrm{Char}(G \,|\, \theta) = \Big\{ \sum_{\chi \in \mathrm{Irr}(G \,|\, \theta)} c\chi \,:\, c \in \mathbb{Z}_{\geq 0} \Big\}$. *We'll also write* $\mathrm{Rep}(G \,|\, \theta)$ *for representations of $G$ having characters in* $\mathrm{Char}(G \,|\, \theta)$.

**Definition 2.20.** *Let $N \leq G$ and $\theta \in \mathrm{Irr}(N)$. Then $\theta$ is said to extend to $G$ if there is $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$.*

**Theorem 2.21** (Frobenius Reciprocity Principle)**.** *Let $N \leq G$, $\theta \in \mathrm{cf}(N)$ and $\chi \in \mathrm{cf}(G)$. Then*

$$[\chi_N, \theta] = [\chi, \theta^G]. \tag{2.21}$$

**Remark 2.22.** *By using Frobenius Reciprocity Principle,* $\mathrm{Irr}(G \,|\, \theta) = \mathrm{Irr}(\theta^G)$.

**Theorem 2.23.** *Let $N \trianglelefteq G$, $\theta \in \mathrm{cf}(N)$ and $g \in G$. Define*

$$\begin{aligned} \theta^g : N &\longrightarrow \mathbb{C} \\ n &\longmapsto \theta(gng^{-1}). \end{aligned} \tag{2.22}$$

*Then $\theta^g \in \mathrm{cf}(N)$. Moreover, $\theta \in \mathrm{Char}(G) \iff \theta^g \in \mathrm{Char}(G)$ and $[\theta, \theta] = [\theta^g, \theta^g]$.*

**Definition 2.24.** *Let $N \trianglelefteq G$, $\theta \in \mathrm{Char}(N)$ and $g \in G$. Then $\theta^g$ is called a $G$-conjugate of $\theta$. If $\theta^g(n) = \theta(n)$ for all $n \in N$, then $\theta$ is said to be fixed/stabilized by $g$. If every $g \in G$ stabilizes $\theta$, we call $\theta$ to be $G$-invariant. Set $G_\theta = \{g \in G : \theta^g = \theta\}$, called the stabilizer of $\theta$ in $G$.*

If $N = G$ in the above definition, then $G_\theta = G$, as characters are class functions. Notice that $\theta^n = \theta$ for all $n \in N$, so that $N \subseteq G_\theta$. Also if $x, y \in G_\theta$, then for every $n \in N$, $\theta^x(n) = \theta(n)$, therefore $\theta(n) = \theta^{(x^{-1})}(n)$, so that $x^{-1} \in G_\theta$ and also $\theta^{xy}(n) = \theta^y(n) = \theta(n)$, so that $xy \in G_\theta$. Thus $G_\theta$ is a subgroup of $G$.

**Theorem 2.25** (Clifford)**.** *Let $N \trianglelefteq G$, $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \,|\, \theta)$. Then*

$$\chi_N = [\chi_N, \theta] \sum_{t \in T} \theta^t, \tag{2.23}$$

*where $T \subseteq G$ is such that $\{\theta^t : t \in T\}$ is a complete list of different $G$-conjugates of $\theta$.*

By this theorem, $\chi(1) = [\chi_N, \theta]|T|\theta(1)$, because we have $\theta^t(1) = \theta(1)$ for all $t \in T$. Thus the degree of $\chi$ is divisible by $\theta(1)$. Also, whenever $\theta$ is $G$-invariant, we have

$$\chi_N = [\chi_N, \theta]\, \theta. \tag{2.24}$$

**Theorem 2.26** (Mackey's Formula)**.** *Let $H, N \leq G$ and $\theta \in \mathrm{cf}(H)$. Let $T$ be a subset of $G$ for which $G = \bigsqcup_{t \in T} HtN$. Then*

$$(\theta^G)_N = \sum_{t \in T} ((\theta^t)_{H^t \cap N})^N. \tag{2.25}$$

*In particular, if $N \trianglelefteq G$ and $G = NH$ (that is, $|T| = 1$), then*

$$(\theta^G)_N = (\theta_{H \cap N})^N. \tag{2.26}$$

Next, we record some significant character correspondences which we'll need later.

**Theorem 2.27** (The Clifford Correspondence)**.** *Let* $N \leq G$ *and* $\theta \in \mathrm{Irr}(N)$*. Then there exists a well defined bijection*

$$\begin{aligned} \mathrm{Irr}(G_\theta \,|\, \theta) &\longrightarrow \mathrm{Irr}(G \,|\, \theta) \\ \chi &\longmapsto \chi^G \end{aligned} \tag{2.27}$$

Note that if $p$ is a prime with $p \nmid |G : G_\theta|$, then this bijection maps $\mathrm{Irr}_{p'}(G_\theta \,|\, \theta)$ onto $\mathrm{Irr}_{p'}(G \,|\, \theta)$, as $\chi^G(1) = |G : G_\theta|\chi(1)$.

**Theorem 2.28** (The Gallagher Correspondence)**.** *Let* $N \trianglelefteq G$ *and assume that* $\theta \in \mathrm{Irr}(N)$ *extends to* $\chi \in \mathrm{Irr}(G)$*. Then there is a bijection*

$$\begin{aligned} \mathrm{Irr}(G/N) &\longrightarrow \mathrm{Irr}(G \,|\, \theta) \\ \psi &\longmapsto \chi\psi \end{aligned} \tag{2.28}$$

Notice that if the order of $G/N$ is a power of a prime $p$, then we have

$$\mathrm{Irr}_{p'}(G \,|\, \theta) = \{\chi\lambda \,:\, \lambda \in \mathrm{Lin}(G/N)\}. \tag{2.29}$$

**Theorem 2.29.** *Let* $N \trianglelefteq G$*,* $H \leq G$ *and suppose that* $G = NH$ *and* $\theta \in \mathrm{Irr}(N)$ *is* $G$*-invariant. Then the map* $\mathrm{Irr}(G \,|\, \theta) \longrightarrow \mathrm{Irr}(H \,|\, \theta_{N\cap H})$ *given by* $\chi \longrightarrow \chi_H$ *is a bijection.*

## 2.2. Groups Acting on $\mathrm{Char}(G)$

In this section, we first recall group actions and then review groups acting on characters.

**Definition 2.30.** *Let* $H$ *and* $G$ *be groups. We say that* $H$ *acts on* $G$ *if there is a map*

$$\begin{aligned} f \,:\, H \times G &\longrightarrow G \\ (h,\, x) &\longmapsto x^h \end{aligned} \tag{2.30}$$

*satisfying* $1^h = h$ *and* $f(h',\, (h,\, x)) = (x^h)^{h'} = x^{hh'} = f(h'h,\, x)$ *for every* $h,\, h' \in H$ *and* $x \in G$*. Additionally, if* $x^h y^h = (xy)^h$ *for all* $h \in H$ *and* $x,\, y \in G$*, then the action is* *said to be by automorphisms. Also*

$$C_G(H) = \{g \in G \,:\, g^h = g \text{ for every } h \in H\} \tag{2.31}$$

*is a subgroup of $G$, called the set of fixed points of this action.*

Set

$$\begin{aligned} \sigma : H &\longrightarrow \mathrm{Aut}(G) \\ h &\longmapsto \quad \sigma_h : \quad G \longrightarrow G \\ & \qquad\qquad\quad x \longmapsto x^h \end{aligned} \tag{2.32}$$

Then $H$ acts on $G$ by automorphisms if and only if $\sigma$ is a group homomorphism. Now, assume that $H$ acts on $G$ by automorphisms, then we construct $G \rtimes H = \{(x, h) : x \in G, h \in H\}$ and define an operation on this set by

$$(x, h) \cdot (x', h') = (x(x')^{h^{-1}}, hh'). \tag{2.33}$$

Then $G \rtimes H$ becomes a group, called the semi-direct product of $G$ and $H$, and $G \trianglelefteq G \rtimes H$.

In the above setting, if $\theta \in \mathrm{Irr}(G)$, then $\theta^h(x^h) := \theta(x)$ defines another irreducible character of $G$. Extending this linearly to $\mathrm{Char}(G)$, $H$ acts on $\mathrm{Char}(G)$ as

$$\begin{aligned} H \times \mathrm{Char}(G) &\longrightarrow \mathrm{Char}(G) \\ (h, \chi) &\longmapsto \chi^h \end{aligned} \tag{2.34}$$

Whenever $\chi^h = \chi \in \mathrm{Irr}(G)$ for all $h \in H$, we shall write $\chi \in \mathrm{Irr}_H(G)$.

By using the next theorem, we'll no longer distinguish between the characters of isomorphic groups.

**Theorem 2.31.** *Let $f : G \longrightarrow \tilde{G}$ be a group isomorphism. Then there exists a bijection*

$$\begin{aligned} \mathrm{Irr}(G) &\longrightarrow \mathrm{Irr}(\tilde{G}) \\ \psi &\longmapsto \psi^f \end{aligned} \tag{2.35}$$

*where $\psi^f(x) = \psi(x^{f^{-1}}) = \psi(f^{-1}(x))$ for every $x \in \tilde{G}$. Also this bijection can be extended linearly to a bijection between $\mathrm{Char}(G)$ and $\mathrm{Char}(\tilde{G})$.*

When a $p$-group $P$ acts on a $p'$-group $K$, by automorphisms, there is a nice correspondence between $\mathrm{Irr}_P(K)$ and $\mathrm{Irr}(C_K(P))$, given below.

**Theorem 2.32** (Glauberman Correspondence)**.** *Assume that $P$ is a $p$-group acting on a $p'$-group $K$, by automorphisms. If $\chi \in \mathrm{Irr}_P(K)$, then $\chi_{C_K(P)}$ has a unique irreducible constituent $\chi'$ with $p'$-multiplicity. Moreover we have that $[\chi_{C_K(P)}, \chi'] \equiv \pm 1 \pmod{p}$. In fact, the map*

$$\begin{aligned} {}' : \mathrm{Irr}_P(K) &\longrightarrow \mathrm{Irr}(C_K(P)) \\ \chi \quad &\longmapsto \quad \chi' \end{aligned} \tag{2.36}$$

*defines a bijection. In this construction, $\chi'$ is called the $P$-Glauberman correspondent of $\chi$.*

**Definition 2.33.** *Suppose a group $H$ acts on the sets $S$ and $T$. If there is a subset $R$ of $S$ such that $r^x \in R$ for all $x \in H$ and $r \in R$, then $R$ is said to be $H$-invariant or that $H$ normalizes $R$. If there is a bijection $f : S \to T$ commuting with the action of $H$, then $f$ is said to be $H$-equivariant.*

**Theorem 2.34.** *Consider the setting of the previous theorem. If a group $N$ acts by automorphisms on $K \rtimes P$ with normalizing $P$, then $N$ acts on $\mathrm{Irr}_P(K)$ and the action of $N$ commutes with the $P$-Glauberman correspondence.*

## 2.3. Galois Action on Char($G$)

In this section we follow Chapter 3 of [3] together with [8]. Let $\mathbb{Q}^{ab}$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all roots of unity. Also let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ be the set of automorphisms of $\mathbb{Q}^{ab}$ fixing $\mathbb{Q}$ element-wise and let $\theta \in \mathrm{Char}(H)$. If $\sigma \in \mathcal{G}$, define

$$\begin{aligned} \theta^\sigma \ : H &\longrightarrow \mathbb{Q}^{ab} \\ g &\longmapsto \theta(g)^\sigma = \sigma\left(\theta(g)\right) \end{aligned} \tag{2.37}$$

Then the map

$$\begin{aligned} f : \mathcal{G} \times \mathrm{Char}(H) &\longrightarrow \mathrm{Char}(H) \\ (\sigma, \theta) &\longmapsto \theta^\sigma \end{aligned} \tag{2.38}$$

defines an action of $\mathcal{G}$ on $\mathrm{Char}(H)$. Moreover $\theta$ is irreducible $\iff$ $\theta^\sigma$ is irreducible, because we have $[\theta^\sigma, \theta^\sigma] = [\theta, \theta]^\sigma = [\theta, \theta]$.

**Theorem 2.35.** *This action commutes with inflation, restriction and induction of characters.*

*Proof.* Let $\sigma \in \mathcal{G}$. It is obvious that if $K \leq H$, $(\theta_K)^\sigma = (\theta^\sigma)_K$ for every $\theta \in H$ and $\sigma \in \mathcal{G}$. If $K \trianglelefteq H$ and $\tilde{\theta} \in \mathrm{Irr}(H/K)$ lifts to $\chi$ in $H$, $\chi^\sigma(x) = \tilde{\theta}^\sigma(Kx)$ for all $x \in H$. Also whenever $H \leq G$,

$$\left(\theta^G\right)^\sigma(x) = \left(\frac{1}{|H|}\sum_{\substack{g\in G\\ gxg^{-1}\in H}} \theta(gxg^{-1})\right)^\sigma = \frac{1}{|H|}\sum_{\substack{g\in G\\ gxg^{-1}\in H}} \theta^\sigma(gxg^{-1}) = \left(\theta^\sigma\right)^G(x). \quad (2.39)$$

□

Finally, by combining group and Galois actions on characters, we have that if $\theta \in \mathrm{Char}(H)$, $\sigma \in \mathcal{G}$ and $K$ is a group acting $H$ by automorphisms, then set

$$\theta^{g\sigma}(x^g) := \theta(x)^\sigma \quad (2.40)$$

for any $g \in K$, $x \in H$, where by $\theta^{g\sigma}$, we mean $(\theta^g)^\sigma$. This defines an action on $\mathrm{Char}(H)$ and

$$\theta \in \mathrm{Irr}(H) \iff \theta^{g\sigma} \in \mathrm{Irr}(H). \quad (2.41)$$

# 3. PROJECTIVE REPRESENTATIONS AND CHARACTER TRIPLES

In this chapter, first, we study the basics of projective representations. After defining character triples, we define some ordering relations on them. Finally, we give the construction of projective representations over $\mathbb{Q}^{ab}$ extending characters. Throughout this chapter, we mainly follow [3], except otherwise stated.

## 3.1. Projective Representations

**Definition 3.1.** *A map $\mathcal{P} : G \to \mathrm{GL}_n(\mathbb{C})$ is called a projective representation of $G$ if there is a function $\alpha : G \times G \to \mathbb{C}^\times$ such that*

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x, y)\mathcal{P}(xy) \tag{3.1}$$

*for every $x, y \in G$. Also, the degree of $\mathcal{P}$ is defined to be $n$ (in [12] for instance).*

If $\mathcal{P}$ is a projective representation of $G$ with factor set $\alpha$, then as the multiplication in $GL_n(\mathbb{C})$ is associative, we obtain

$$\alpha(x, yz)\alpha(y, z) = \alpha(xy, z)\alpha(x, y), \tag{3.2}$$

for every $x, y, z \in G$.

**Definition 3.2.** *The function $\alpha$ given above is called a factor set of $\mathcal{P}$. Also any function $\alpha : G \times G \to \mathbb{C}^\times$ satisfying (3.2) is called a 2-cocycle.*

**Notation 3.3.** *We write* $\mathrm{Proj}(G)$ *to denote the set of projective representations of $G$.*

The following theorem can be found in [12].

**Theorem 3.4.** *If $\alpha$ is a 2-cocycle of $G$, then $G$ has a projective representation with factor set $\alpha$.*

**Notation 3.5.** *We'll write* $\mathrm{Proj}(G \mid \alpha)$ *to denote the set of projective representations of $G$ with factor set $\alpha$.*

**Definition 3.6.** *Let* $\mathcal{P} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ *and* $\mathcal{P}' : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ *be projective representations of* $G$.

*(i) They are called similar if there exists* $A \in \mathrm{GL}_n(\mathbb{C})$ *for which*

$$\mathcal{P}'(x) = A^{-1}\mathcal{P}(x)A, \tag{3.3}$$

*for every* $x \in G$*. In that case, we write* $\mathcal{P} \sim \mathcal{P}'$.

*(ii) If* $\mathcal{P}$ *is similar to a projective representation of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{3.4}$$

*then* $\mathcal{P}$ *is said to be reducible. Otherwise,* $\mathcal{P}$ *is irreducible.*

**Remark 3.7.**

- *Let* $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ *be of degree* $n$*. If* $\mathcal{P}' \in \mathrm{Proj}(G)$ *is similar to* $\mathcal{P}$*, then the factor set of* $\mathcal{P}'$ *is also* $\alpha$*. If* $\nu : G \longrightarrow \mathbb{C}^{\times}$ *is any function, then*

$$\begin{aligned} \nu\mathcal{P} : G &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ x &\longmapsto \nu(x)\mathcal{P}(x) \end{aligned} \tag{3.5}$$

*defines a projective representation of* $G$ *with factor set* $\nu\alpha$*, where*

$$\nu\alpha(x,\, y) = \frac{\nu(x)\nu(y)\alpha(x,\, y)}{\nu(xy)}, \tag{3.6}$$

*for every* $x,\, y \in G$.

- *Suppose that a group* $H$ *acts on* $G$*, by automorphisms. Then* $H$ *acts on* $\mathrm{Proj}(G)$ *by setting*

$$\begin{aligned} H \times \mathrm{Proj}(G) &\longrightarrow \mathrm{Proj}(G) \\ (h,\, \mathcal{P}) &\longmapsto \mathcal{P}^h \end{aligned} \tag{3.7}$$

*where* $\mathcal{P}^h(x^h) := \mathcal{P}(x)$*, for all* $x \in G$.

By the following result, we may assume that isomorphic groups have the same projective representations.

**Theorem 3.8.** *Assume* $G$ *and* $\tilde{G}$ *are finite groups and* $f : G \longrightarrow \tilde{G}$ *is a group isomorphism. If* $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ *is of degree* $n$*, define*

$$\begin{aligned} \mathcal{P}^f : \tilde{G} &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ x &\longmapsto \mathcal{P}(x^{f^{-1}}). \end{aligned} \tag{3.8}$$

*Then* $\mathcal{P}^f \in \mathrm{Proj}(\tilde{G}\,|\,\alpha^f)$*, where* $\alpha^f(x, y) = \alpha(x^{f^{-1}}, y^{f^{-1}})$*, for all* $x, y \in \tilde{G}$*. Moreover, there exists a bijection*

$$\begin{aligned} \mathrm{Proj}(G\,|\,\alpha) &\longrightarrow \mathrm{Proj}(\tilde{G}\,|\,\alpha^f) \\ \mathcal{P} &\longmapsto \mathcal{P}^f. \end{aligned} \tag{3.9}$$

*that preserves similarity and irreducibility.*

For any $g \in G$, the map $c_g : G \longrightarrow G$ given by conjugation is an automorphism of $G$. Then we write $\mathcal{P}^g$ in place of $\mathcal{P}^{c_g}$ to denote the projective representation obtained in the above theorem.

## 3.2. Character Triples

**Definition 3.9.** *Let* $N \trianglelefteq G$ *and* $\theta \in \mathrm{Irr}(N)$*. If* $\theta$ *is* $G$*-invariant, then* $(G, N, \theta)$ *is called a character triple.*

**Definition 3.10** (Character Triple Isomorphism)**.** *Let* $(G, N, \theta)$ *and* $(\tilde{G}, \tilde{N}, \tilde{\theta})$ *be character triples. Suppose that*

*(i) there is an isomorphism* $\tilde{}\, : G/N \longrightarrow \tilde{G}/\tilde{N}$*. Let* $\tilde{N} \leq \tilde{K} \leq \tilde{G}$ *be such that* $\widetilde{K/N} = \tilde{K}/\tilde{N}$ *and set* $\tilde{\psi} \in \mathrm{Irr}(\tilde{K}/\tilde{N})$ *by the image of* $\psi \in \mathrm{Irr}(K/N)$ *under the isomorphism,*

*(ii) for every* $N \leq K \leq G$*, there is a bijection* $f : \mathrm{Irr}(K\,|\,\theta) \longrightarrow \mathrm{Irr}(\tilde{K}\,|\,\tilde{\theta})$ *such that for any* $\chi \in \mathrm{Irr}(K\,|\,\theta)$

- $f$ *commutes with restriction, that is,* $(\chi_L)^f = (\chi^f)_L$ *for every* $N \leq L \leq K$*,*
- $(\chi\psi)^f = \chi^f\tilde{\psi}$ *for every* $\psi \in \mathrm{Irr}(K/N)$*.*

*Then the character triples* $(G, N, \theta)$ *and* $(\tilde{G}, \tilde{N}, \tilde{\theta})$ *are said to be isomorphic.*

**Remark 3.11.** *Suppose* $(G, N, \theta)$ *and* $(\tilde{G}, \tilde{N}, \tilde{\theta})$ *are isomorphic character triples.*

*(i) Isomorphism of character triples define an equivalence relation on the set of character triples.*

*(ii) There exists a bijection* $\mathrm{Irr}(G\,|\,\theta) \longrightarrow \mathrm{Irr}(\tilde{G}\,|\,\tilde{\theta})$.

*(iii) Let* $f$, $K$, $\chi$ *be as in Definition 3.10. Then by using Clifford's Theorem, we have*

$$\frac{\chi^f(1)}{\tilde{\theta}(1)} = \frac{\chi(1)}{\theta(1)}. \tag{3.10}$$

*(iv) Induction of characters commute with* $f$.

By the following theorem, if $(G,\, N,\, \theta)$ is a character triple, we may assume that $N \leq Z(G)$ and that $\theta$ is linear.

**Theorem 3.12.** *Let* $(G,\, N,\, \theta)$ *be a character triple. Then there is a character triple* $(\tilde{G},\, \tilde{N},\, \tilde{\theta})$ *isomorphic to* $(G,\, N,\, \theta)$ *such that* $\tilde{N} \leq Z(\tilde{G})$, $\ker \tilde{\theta} = \{1\}$, *and* $\tilde{\theta}(1) = 1$.

The following Proposition is Theorem 5.12 in [3].

**Proposition 3.13.** *Let* $K \trianglelefteq G$, $\theta \in \mathrm{Irr}(K)$ *and* $\chi \in \mathrm{Irr}(G\,|\,\theta)$. *Then* $\dfrac{\chi(1)}{\theta(1)}$ *divides the index of* $K$ *in* $G$.

*Proof.* By hypothesis, $(G_\theta,\, K,\, \theta)$ is a character triple. By Theorem 2.27, there exists $\varphi \in \mathrm{Irr}(G_\theta\,|\,\theta)$ for which $\varphi^G = \chi$, so that $\varphi(1) = \frac{\chi(1)}{|G\,:\,G_\theta|}$. Then by Theorem 3.12, we can assume that $K$ is central in $G_\theta$, so it is abelian, and that $\theta$ is linear. By Theorem 2.7, $\varphi(1) = \dfrac{\varphi(1)}{\theta(1)}$ divides $|G_\theta \,:\, K|$. Then we obtain $\chi(1)\big||G \,:\, K|$. □

### 3.3. Projective Representations Associated with Characters

**Definition 3.14.** *Let* $\mathcal{P} \in \mathrm{Proj}(G\,|\,\alpha)$, $N \trianglelefteq G$ *and* $\theta \in \mathrm{Irr}(N)$. *Suppose that*

*(i)* $\mathcal{P}_N$ *is an ordinary representation of* $N$ *which affords* $\theta$,

*(ii)* $\mathcal{P}(xy) = \mathcal{P}(x)\mathcal{P}(y)$ *and* $\mathcal{P}(yx) = \mathcal{P}(y)\mathcal{P}(x)$ *for every* $x \in G$, $y \in N$.

*Then* $\mathcal{P}$ *is said to be associated with* $\theta$.

Note that (ii) in the definition is equivalent to state that the factor set $\alpha$ of $\mathcal{P}$ satisfies

$$\alpha(x,\, y) = \alpha(y,\, x) = 1 \tag{3.11}$$

for every $x \in G,\, y \in N$.

In Section 3.6, we give a similar construction to the next theorem, by taking the field $\mathbb{Q}^{ab}$ instead of $\mathbb{C}$.

**Theorem 3.15.** *If $(G,\, N,\, \theta)$ is a character triple, then there exists $\mathcal{P} \in \mathrm{Proj}(G)$ associated with $\theta$.*

**Remark 3.16.** *If $(G,\, N,\, \theta)$ is a character triple and $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ is associated with $\theta$, then for every $c \in C_G(N)$ and $n \in N$, we have*

$$\mathcal{P}(c)\mathcal{P}(n) = \mathcal{P}(cn) = \mathcal{P}(nc) = \mathcal{P}(n)\mathcal{P}(c). \tag{3.12}$$

*Then by Schur's Lemma, $\mathcal{P}(c)$ is a scalar matrix, so $\mathcal{P}(c) = \mu_c I_{\theta(1)}$, for some non-zero constant $\mu_c$. This defines a function*

$$\begin{aligned} \mu \,:\, C_G(N) &\longrightarrow \mathbb{C}^\times \\ c &\longmapsto \mu_c \end{aligned} \tag{3.13}$$

**Lemma 3.17.** *Assume $(G,\, N,\, \theta)$ is a character triple. Let $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ be associated with $\theta$. Then*

$$\alpha(gn,\, g'n') = \alpha(g,\, g') \tag{3.14}$$

*for every $g,\, g' \in G,\, n,\, n' \in N$.*

Under the setting of the above lemma, we note that $\alpha(n,\, n') = 1$ for all $n,\, n' \in N$.

**Lemma 3.18.** *Let $(G,\, N,\, \theta)$ be a character triple and $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ be associated with $\theta$.*

*(i)* $\mathcal{P}(xnx^{-1}) = \mathcal{P}(x)\mathcal{P}(n)\mathcal{P}(x)^{-1}$ *for every $x \in G,\, n \in N$.*

*(ii)* $\mathcal{Q} \in \mathrm{Proj}(G)$ *is associated with $\theta$ if and only if $\mathcal{Q} \sim \nu\mathcal{P}$ for some unique function $\nu \,:\, G \longrightarrow \mathbb{C}^\times$ constant on $N$-cosets and $\nu(1) = 1$.*

*Proof.* These follow from Lemma 10.10 in [3] and Remark 1.3 in [8]. □

### 3.4. Construction of Projective Representations

As for ordinary representations, we describe the product of projective representations. Let $\mathcal{P} \in \mathrm{Proj}(G \mid \alpha)$ and $\mathcal{Q} \in \mathrm{Proj}(G \mid \beta)$ be of degrees $m$ and $n$, respectively. Define

$$\begin{aligned} \mathcal{P} \otimes \mathcal{Q} \,:\, G &\longrightarrow GL_{mn}(\mathbb{C}) \\ x &\longmapsto \mathcal{P}(x) \otimes \mathcal{Q}(x) \end{aligned} \tag{3.15}$$

Then for any $x,\, y \in G$,

$$\begin{aligned} (\mathcal{P} \otimes \mathcal{Q})(x) \cdot (\mathcal{P} \otimes \mathcal{Q})(y) &= \big(\mathcal{P}(x) \otimes \mathcal{Q}(x)\big)\big(\mathcal{P}(y) \otimes \mathcal{Q}(y)\big) \\ &= \mathcal{P}(x)\mathcal{P}(y) \otimes \mathcal{Q}(x)\mathcal{Q}(y) \\ &= \alpha(x,\, y)\beta(x,\, y)\big(\mathcal{P}(xy) \otimes \mathcal{Q}(xy)\big). \end{aligned} \tag{3.16}$$

Thus $\mathcal{P} \otimes \mathcal{Q} \in \mathrm{Proj}(G \mid \alpha\beta)$, where

$$\alpha\beta(x,\, y) = \alpha(x,\, y)\beta(x,\, y). \tag{3.17}$$

Also, notice that if $\mathcal{Q} \in \mathrm{Proj}(G \mid \alpha^{-1})$, where $\alpha^{-1}(x,\, y) = \alpha(x,\, y)^{-1}$, for all $x,\, y \in G$, then $\mathcal{P} \otimes \mathcal{Q} \in \mathrm{Rep}(G)$.

**Theorem 3.19.** *Let $(G,\, N,\, \theta)$ be a character triple and assume that $\mathcal{P} \in \mathrm{Proj}(G \mid \alpha)$ is associated with $\theta$. Then there exists an injective function*

$$\begin{aligned} \phi \,:\, \mathrm{Proj}(G/N \mid \alpha^{-1}) &\longrightarrow \mathrm{Rep}(G \mid \theta), \\ \mathcal{Q} &\longmapsto \mathcal{Q} \otimes \mathcal{P} \end{aligned} \tag{3.18}$$

*preserving similarity and irreducibility. Moreover, if $\psi \in \mathrm{Char}(G \mid \theta)$, then there exists $\mathcal{Q} \in \mathrm{Proj}(G/N \mid \alpha^{-1})$ such that $\phi(\mathcal{Q})$ affords $\psi$.*

Now, we describe restriction, inflation and induction of projective representations. In order to obtain a suitable result, in the setting of induction, we consider character triples.

**Definition 3.20.** *(i) Let $\mathcal{P} \in \mathrm{Proj}(G \mid \alpha)$ and $H \leq G$. Then $\mathcal{P}_H(h) := \mathcal{P}(h)$ defines a projective representation of $H$ with factor set $\alpha_{H \times H}$, called the restriction of $\mathcal{P}$ to $H$.*

*(ii) Let $N \trianglelefteq G$ and $\overline{\mathcal{P}} \in \mathrm{Proj}(G/N \,|\, \overline{\alpha})$. Then $\mathcal{P}(g) := \overline{\mathcal{P}}(Ng)$ gives a projective representation of $G$ with factor set $\alpha(g, g') = \overline{\alpha}(Ng, Ng')$, called the lift of $\overline{\mathcal{P}}$ to $G$.*

Considering the inflation of projective representations, a bijection between the set $\mathcal{S} = \{\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha) \,:\, \mathcal{P}(ng) = \mathcal{P}(g) \text{ for all } n \in N,\ g \in G\}$ and $\mathrm{Proj}(G/N \,|\, \overline{\alpha})$ can easily be constructed. Thus, from now on, we'll not distinguish between these two sets and write $\mathcal{P} \in \mathrm{Proj}(G/N \,|\, \alpha)$ to mean that $\mathcal{P} \in \mathcal{S}$. Moreover, when $(G, N, \theta)$ is a character triple, we have $\mathcal{S} = \{\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha) \,:\, \mathcal{P}(n) = \mathcal{P}(1) \text{ for all } n \in N\}$.

The following proposition can be found in p.712 of [13].

**Proposition 3.21.** *Let $N \trianglelefteq G$, $H \leq G$ such that $G = NH$ and let $M = N \cap H$. Assume $(G, N, \theta)$ and $(H, M, \varphi)$ are character triples with $\varphi^N = \theta$. Let $\mathcal{P} \in \mathrm{Proj}(H \,|\, \alpha)$ be associated with $\varphi$. Let $\mathcal{T} = \{t_1 = 1, t_2, \ldots, t_r\}$ be a right transversal of $H$ in $G$. For every $x \in G$, set*

$$\dot{\mathcal{P}}_{ij}(x) = \begin{cases} \mathcal{P}(t_i x t_j^{-1}) & \text{if } t_i x t_j^{-1} \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{3.19}$$

*and define*

$$\mathcal{P}^G(x) = \begin{pmatrix} \dot{\mathcal{P}}_{11}(x) & \dot{\mathcal{P}}_{12}(x) & \cdots & \dot{\mathcal{P}}_{1r}(x) \\ \dot{\mathcal{P}}_{21}(x) & \dot{\mathcal{P}}_{22}(x) & \cdots & \dot{\mathcal{P}}_{2r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}_{r1}(x) & \dot{\mathcal{P}}_{r2}(x) & \cdots & \dot{\mathcal{P}}_{rr}(x) \end{pmatrix}. \tag{3.20}$$

*Then $\mathcal{P}^G$ becomes a projective representation of $G$ associated with $\theta$, with factor set uniquely described by $\alpha$.*

*Proof.* By construction, $\mathcal{T}$ is also a right transversal of $M$ in $N$, so that $\mathcal{T} \subseteq N$. Note that for every $x \in G$, there exists precisely one non-zero block in each row and column in $\mathcal{P}^G(x)$. Since if $\dot{\mathcal{P}}_{ik}(x)$ and $\dot{\mathcal{P}}_{il}(x)$ are nonzero, then we have $t_i x t_k^{-1} \in H$ and $t_i x t_l^{-1} \in H$ and therefore $(t_i x t_k^{-1})^{-1}(t_i x t_l^{-1}) \in N \cap H = M$, so that $t_k = t_l$. Similarly, it is true for columns.

Define $\beta : G \times G \longrightarrow \mathbb{C}^\times$ by $\beta(x, y) = \alpha(t_i x t_j^{-1}, t_j y t_k^{-1})$, where $t_i, t_j, t_k \in \mathcal{T}$ satisfy $t_i x t_j^{-1} \in H$ and $t_j y t_k^{-1} \in H$. Actually, by the above argument, $t_i$ determines $t_j$ and $t_k$, uniquely. If $t_{i'} x t_{j'}^{-1}$ and $t_{j'} y t_{k'}^{-1}$ are also in $H$, then

$$\alpha(t_i x t_j^{-1}, t_j y t_k^{-1}) = \alpha(t_{i'} x t_{j'}^{-1}, t_{j'} y t_{k'}^{-1}), \tag{3.21}$$

because $(t_i x t_j^{-1})^{-1}(t_{i'} x t_{j'}^{-1}) \in N \cap H = M$ and $(t_j y t_k^{-1})^{-1}(t_j y t_k^{-1}) \in M$ and the result follows by Lemma 3.17. Thus $\beta$ is well defined. Then for any $x, y \in G$, the $ij$-th entry of

$$\mathcal{P}^G(x)\mathcal{P}^G(y) \tag{3.22}$$

is either 0 or is equal to $\mathcal{P}(t_i x t_k^{-1})\mathcal{P}(t_k y t_j^{-1}) = \alpha(t_i x t_k^{-1}, t_k y t_j^{-1})\mathcal{P}(t_i x y t_j^{-1})$ if there is a $k$ for which $t_i x t_k^{-1} \in H$ and $t_k y t_j^{-1} \in H$. Therefore

$$\mathcal{P}^G(x)\mathcal{P}^G(y) = \beta(x, y)\mathcal{P}^G(xy). \tag{3.23}$$

This shows that $\mathcal{P}^G \in \mathrm{Proj}(G \,|\, \beta)$.

Note that, if $x \in N$, then we have

$$\dot{\mathcal{P}}_{ij}(x) = \begin{cases} \mathcal{P}(t_i x t_j^{-1}) & \text{if } t_i x t_j^{-1} \in H \cap N \text{ (since } T \subseteq N), \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

Therefore $(\mathcal{P}^G)_N = (\mathcal{P}_M)^N$. As $\mathcal{P}$ is associated with $\varphi$ and $\mathcal{P}_M$ is a representation of $M$ affording $\varphi$, we have $(\mathcal{P}^G)_N$ affords $\theta$. Let $g \in G$ and $n \in N$. Then for some $t_i, t_j, t_k \in T$ with $t_i g t_k^{-1} \in H$ and $t_k n t_j^{-1} \in H$, we have $\beta(g, n) = \alpha(t_i g t_k^{-1}, t_k n t_j^{-1}) = 1$, by noting that $t_k n t_j^{-1} \in M$. Similarly, $\beta(n, g) = 1$. Then we obtain

$$\mathcal{P}^G(g)\mathcal{P}^G(n) = \mathcal{P}^G(gn) \qquad \text{and} \qquad \mathcal{P}^G(n)\mathcal{P}^G(g) = \mathcal{P}^G(ng) \tag{3.25}$$

for every $n \in N$, $g \in G$. Hence $\mathcal{P}^G$ is associated with $\theta$. $\square$

In the above construction, observe that $\beta_{H\times H} = \alpha$ and that if $x \in H \cap C_G(N)$ and $\mathcal{P}(x) = \lambda I_{\varphi(1)}$ for some constant $\lambda$, then $\mathcal{P}^G(x) = \lambda I_{\theta(1)}$.

**Definition 3.22.** *In the setting of Proposition 3.21, $P^G$ is called the induced projective representation.*

### 3.5. Orderings on Character Triples

First, we define an ordering on character triples as follows:

**Definition 3.23.** *Assume $(G, N, \theta)$ and $(H, M, \varphi)$ are character triples. If*

*(i)* $G = NH$, $M = N \cap H$, *and*
*(ii)* *there exist $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$, $\mathcal{P}' \in \mathrm{Proj}(H \,|\, \alpha')$ which are associated with $\theta$ and $\varphi$, respectively, and $\alpha_{H\times H} = \alpha'$,*

*then we shall write $(G, N, \theta) \geq (H, M, \varphi)$. In this case, we say that $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq (H, M, \varphi)$.*

**Lemma 3.24.** *If $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq (H, M, \varphi)$ and $N \leq K \leq G$, then $(\mathcal{P}_K, \mathcal{P}'_{K\cap H})$ is associated with $(K, N, \theta) \geq (K \cap H, M, \varphi)$.*

Therefore this ordering is compatible with restriction of projective representations.

Suppose now that $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq (H, M, \varphi)$ and let $N \leq K \leq G$ be an intermediate subgroup. Then $(\mathcal{P}_K, \mathcal{P}'_{K\cap H})$ is associated with $(K, N, \theta) \geq (K \cap H, M, \varphi)$. Let $\psi \in \mathrm{Char}(K \,|\, \theta)$. By Theorem 3.19, there exists some $\mathcal{Q} \in \mathrm{Proj}(K/N \,|\, \alpha^{-1}_{K\times K})$ such that $\mathcal{Q} \otimes \mathcal{P}_K$ affords $\psi$, because $\mathcal{P}_K$ is associated with $\theta$. Then as $\alpha_{(K\cap H)\times(K\cap H)} = \alpha'_{(K\cap H)\times(K\cap H)}$, we have that $\mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H}$ is a representation of $K \cap H$. By the isomorphism, $f : G/N \longrightarrow H/M$, $Nx \longmapsto Mx$, $x \in G$, and Theorem 3.8, $\mathcal{Q}^f \in \mathrm{Proj}(K \cap H/M \,|\, (\alpha^{-1}_{K\times K})^f)$. Considering the construction of $f$, $\mathcal{Q}^f = \mathcal{Q}_{K\cap H}$.

Then as $\mathcal{P}'_{K\cap H}$ is associated with $\varphi$, $\mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H} \in \operatorname{Rep}(K\cap H \,|\, \varphi)$. Define

$$\begin{aligned} \tau_K : \qquad & \operatorname{Char}(K \,|\, \theta) \longrightarrow \operatorname{Char}(K\cap H \,|\, \varphi). \\ \psi = \operatorname{tr}&(\mathcal{Q} \otimes \mathcal{P}_K) \longmapsto \operatorname{tr}(\mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H}) \end{aligned} \tag{3.26}$$

Now, let $\chi \in \operatorname{Char}(K \,|\, \theta)$ and write $\chi = \operatorname{tr}(\mathcal{R} \otimes \mathcal{P}_K)$ for some $\mathcal{R} \in \operatorname{Proj}(K/N \,|\, \alpha^{-1}_{K\times K})$. Then

$$\begin{aligned} \chi = \psi; &\Longleftrightarrow \mathcal{Q} \otimes \mathcal{P}_K \text{ and } \mathcal{R} \otimes \mathcal{P}_K \text{ are similar} \\ ; &\Longleftrightarrow \mathcal{Q} \text{ and } \mathcal{R} \text{ are similar} \\ ; &\Longleftrightarrow \mathcal{Q}^f = \mathcal{Q}_{K\cap H} \text{ and } \mathcal{R}^f = \mathcal{R}_{K\cap H} \text{ are similar} \\ ; &\Longleftrightarrow \mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H} \text{ and } \mathcal{R}_{K\cap \mathcal{H}} \otimes \mathcal{P}'_{K\cap H} \text{ are similar} \\ ; &\Longleftrightarrow \tau_K(\chi) = \tau_K(\psi), \end{aligned} \tag{3.27}$$

by using Theorem 3.8 and Theorem 3.19. Thus $\tau_K$ is well-defined and injective. We observe that

$$\begin{aligned} \psi \text{ is irreducible}; &\Longleftrightarrow \mathcal{Q} \otimes \mathcal{P}_K \in \operatorname{Rep}(K \,|\, \theta) \text{ is irreducible} \\ ; &\Longleftrightarrow \mathcal{Q} \in \operatorname{Proj}(K/N \,|\, \alpha^{-1}_{K\times K}) \text{ is irreducible} \\ ; &\Longleftrightarrow \mathcal{Q}^f = \mathcal{Q}_{K\cap H} \in \operatorname{Proj}(K/N \,|\, \alpha^{-1}_{(K\cap H)\times(K\cap H)}) \text{ is irreducible} \\ ; &\Longleftrightarrow \mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H} \in \operatorname{Rep}(K\cap H \,|\, \varphi) \text{ is irreducible} \\ ; &\Longleftrightarrow \tau_K(\psi) \text{ is irreducible.} \end{aligned} \tag{3.28}$$

Thus $\tau_K\big(\operatorname{Irr}(K \,|\, \theta)\big) \subseteq \operatorname{Irr}(K\cap H \,|\, \varphi)$. Next, if we let $\chi \in \operatorname{Char}(K\cap H \,|\, \varphi)$, then $\chi = \operatorname{tr}(\mathcal{R} \otimes \mathcal{P}'_{K\cap H})$ for some $\mathcal{R} \in \operatorname{Proj}\big((K\cap H)/M \,|\, (\alpha'^{-1}_{(K\cap H)\times(K\cap H)})\big)$. Then $\mathcal{R}^{f^{-1}} \in \operatorname{Proj}\big(K/N \,|\, (\alpha'^{-1}_{(K\cap H)\times(K\cap H)})^{f^{-1}}\big)$. By noting that $(\alpha'^{-1}_{(K\cap H)\times(K\cap H)})^{f^{-1}}(x, y) = \alpha^{-1}(x, y)$, for all $x, y \in K$, we have

$$\mathcal{R}^{f^{-1}} \otimes \mathcal{P}_K \in \operatorname{Rep}(G \,|\, \theta) \tag{3.29}$$

such that $\tau_K\big(\operatorname{tr}(\mathcal{R}^{f^{-1}} \otimes \mathcal{P}_K)\big) = \chi$. Thus $\tau_K$ is onto and hence it is a bijection sending irreducible characters onto irreducible characters.

Now, let $N \leq J \leq K$. Then

$$\tau_J(\psi_J) = \tau_J(\operatorname{tr}(\mathcal{Q}_J \otimes \mathcal{P}_J)) = \operatorname{tr}(\mathcal{Q}_{J\cap H} \otimes \mathcal{P}'_{J\cap H}) = (\tau_K(\psi))_{J\cap H}. \tag{3.30}$$

Moreover, if $\vartheta \in \mathrm{Irr}(K \mid \theta)$ and $\rho \in \mathrm{Rep}(K)$ affords $\vartheta$, then

$$\begin{aligned}\tau_K(\vartheta\psi) &;= \tau_K\big(\,\mathrm{tr}(\rho \otimes (\mathcal{Q} \otimes \mathcal{P}_K))\big)\ \tau_K\big(\,\mathrm{tr}((\rho \otimes \mathcal{Q}) \otimes \mathcal{P}_K)\big)\\ &;= \mathrm{tr}\,\big((\rho \otimes \mathcal{Q})_{K\cap H} \otimes \mathcal{P}'_{K\cap H}\big)\\ &;= \vartheta_{K\cap H}\tau_K(\psi).\end{aligned} \tag{3.31}$$

Also, for any $h \in H$, $\psi^h = \mathrm{tr}(\mathcal{Q}^h \otimes (\mathcal{P}^h)_{K^h}) \in \mathrm{Irr}(K^h \mid \theta)$. Then with some effort,

$$\tau_{K^h}(\psi^h) = \mathrm{tr}\,\big((\mathcal{Q}^h)_{K^h\cap H} \otimes ((\mathcal{P}')^h)_{K^h\cap H}\big) = \tau_K(\psi)^h. \tag{3.32}$$

Hence this proves:

**Theorem 3.25** ( [3] and [14]). *If $(G, N, \theta) \geq (H, M, \varphi)$, then the character triples $(G, N, \theta)$ and $(H, M, \varphi)$ are isomorphic by the function $\tau_K$ constructed above, in the sense of Definition 3.10. Moreover $\tau_{K^h}(\psi^h) = \tau_K(\psi)^h$ for all $h \in H$, $N \leq K \leq G$ and $\psi \in \mathrm{Irr}(K \mid \theta)$.* □

**Definition 3.26.** *Let $(G, N, \theta)$ and $(H, M, \varphi)$ be character triples. Suppose*

*(i) $C_G(N) \leq H$,*

*(ii) $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq (H, M, \varphi)$,*

*(iii) for every $c \in C_G(N)$, $\mathcal{P}(c) = \mu_c I_{\theta(1)}$ and $\mathcal{P}'(c) = \mu(c) I_{\varphi(1)}$ for some constant $\mu_c$.*

*Then we write $(G, N, \theta) \geq_c (H, M, \varphi)$ and say that $(\mathcal{P}, \mathcal{P}')$ is associated with this (central) ordering.*

Notice that restriction is also compatible with this ordering, as in Lemma 3.24. The following constructions will be needed in Chapter 5 and their proofs can be found in [3].

**Lemma 3.27.** *Assume $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq_c (H, M, \varphi)$. Then for any $N \leq K \leq G$ and $\psi \in \mathrm{Irr}(K \mid \theta)$, we have*

$$\mathrm{Irr}(\psi_{C_K(N)}) = \mathrm{Irr}(\tau_K(\psi)_{C_K(N)}), \tag{3.33}$$

*where $\tau_K$ is as given in Theorem 3.25.*

**Lemma 3.28.** *Suppose that $(G,\, N,\, \theta) \geq_c (H,\, M,\, \varphi)$ and let $K \subseteq \ker\theta \cap \ker\varphi$. If $C_{G/K}(N/K) = C_G(N)K/K$, then $(G/K,\, N/K, \overline{\theta}) \geq_c (H/K,\, M/K,\, \overline{\varphi})$, where $\overline{\theta}(Kn) = \theta(n)$ and $\overline{\varphi}(Km) = \varphi(m)$, for every $n \in N,\ m \in M$.*

### 3.6. Constructing Projective Representations over $\mathbb{Q}^{ab}$

First, we introduce new notations to ease writing.

**Notation 3.29.** *We shall denote the set of ordinary representations of a group $G$ over the field $\mathbb{Q}^{ab}$, by $\mathrm{Rep}_{\mathbb{Q}^{ab}}(G)$. Similarly, we write $\mathrm{Proj}_{\mathbb{Q}^{ab}}(G)$ for the set of projective representations of $G$ with entries in $\mathbb{Q}^{ab}$. Also $\mathrm{Proj}_{\mathbb{Q}^{ab}}(G \,|\, \alpha)$ can be defined analogously.*

By the following theorem from [8], we may assume that entries of projective representations associated with characters are in $\mathbb{Q}^{ab}$.

**Theorem 3.30.** *Suppose $(G,\, N,\, \theta)$ is a character triple and $\mathcal{P} \in \mathrm{Proj}(G \,|\, \alpha)$ is associated with $\theta$. Then there exists $\mathcal{Q} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G)$ similar to $\mathcal{P}$, associated with $\theta$, whose factor set only takes roots of unity values. In that case, there exists some positive integer $k$ for which $\mathcal{Q}(g)^k = I_{\theta(1)}$, for every $g \in G$.*

We'll give a construction of such a projective representation, by changing the field in Chapter 5 of [3]. Proofs of the results follow almost the same lines of that and we write some of them to illustrate the construction. We first need a lemma.

**Lemma 3.31** ( [15]). *Let $F$ be a field of characteristic 0. Two representations with entries in $F$ are similar if and only if they afford the same character.*

**Theorem 3.32.** *Suppose $(G,\, N,\, \theta)$ is a character triple with $G/N \cong \langle Nx \rangle$ for some $x \in G$. If $\rho \in \mathrm{Rep}_{\mathbb{Q}^{ab}}(N)$ affords $\theta$, then there exists $\tau \in \mathrm{Rep}_{\mathbb{Q}^{ab}}(G)$ such that $\tau_N = \rho$.*

*Proof.* Construct a representation $\rho^x : N \longrightarrow \mathrm{GL}_{\theta(1)}(\mathbb{Q}^{ab})$ of $N$ by setting $\rho^x(n) = \rho(xnx^{-1})$. As $x$ stabilizes $\theta$, $\mathrm{tr}(\rho^x) = \theta$ and therefore $\rho^x = A\rho A^{-1}$, for some $A \in$

$\mathrm{GL}_{\theta(1)}(\mathbb{Q}^{ab})$, by Lemma 3.31. Write $r = |G : N|$. Since $x^r \in N$, we have

$$A^r \rho(n) A^{-r} = \rho(x^r n x^{-r}) = \rho^{x^r}(n) = \rho(x^r)\rho(n)\rho(x^r)^{-1}. \tag{3.34}$$

Then by Schur's Lemma,

$$\rho(x^r) = \lambda(x^r) A^r \tag{3.35}$$

for some $\mathbb{Q}^{ab}$-valued function $\lambda$. Set

$$\tau(nx^k) = \lambda(x^r)^{\frac{k}{r}} \rho(n) A^k, \tag{3.36}$$

for every $n \in N$, $0 \leq k < r$. This defines a representation of $G$ whose entries are in $\mathbb{Q}^{ab}$, extending $\rho$. □

**Theorem 3.33.** *Let $(G, N, \theta)$ be a character triple and assume $\rho \in \mathrm{Rep}_{\mathbb{Q}^{ab}}(N)$ affords $\theta$. For all $x \in G$, set*

$$\mathcal{R}(x) := \tau_x(x), \tag{3.37}$$

*where $\tau_x$ is an extension of $\rho$ to $\langle N, x\rangle$ given by Theorem 3.32. Then $\mathcal{R} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G)$ and it's associated with $\theta$. Moreover, the factor set of $\mathcal{R}$ only takes roots of unity values.*

*Proof.* This follows from Theorem 5.5 of [3], by using the preceding result. □

In the proof of the next theorem, we follow the construction given in the proof of Theorem 8.16 in [16].

**Theorem 3.34.** *Suppose $(G, N, \theta)$ is a character triple and $\mathcal{P} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G \,|\, \alpha)$ is associated with $\theta$. Let $N \leq K \leq G$ and $\psi \in \mathrm{Irr}(K \,|\, \theta)$. Considering Theorem 3.19, there exists $\mathcal{Q} \in \mathrm{Proj}(K/N \,|\, \alpha^{-1}_{K \times K})$ such that $\mathcal{Q} \otimes \mathcal{P}_K$ is an ordinary representation of $K$ that affords $\psi$. As the map $\phi$ in Theorem 3.19 preserves similarity, we can assume that the entries of $\mathcal{Q}$, and so of $\mathcal{Q} \otimes \mathcal{P}_K$, are in $\mathbb{Q}^{ab}$, by using Theorem 3.30. Assume $K \trianglelefteq G_\psi$. Then there exists $\mathcal{R} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_\psi)$ such that $\mathcal{R}_K = \mathcal{Q} \otimes \mathcal{P}_K$ and $\mathcal{R} = \mathcal{S} \otimes \mathcal{P}$ for some $\mathcal{S} \in \mathrm{Proj}(G_\psi)$ with $\mathcal{S}_K = \mathcal{Q}$.*

*Proof.* As $(G_\psi, K, \psi)$ is a character triple, there exists $\mathcal{R} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_\psi)$ such that

$\mathcal{R}_K = \mathcal{Q} \otimes \mathcal{P}_K$. Since $\theta$ is $G$-invariant, by Clifford's Theorem, $\psi_N = m\theta$ with $m = \dfrac{\psi(1)}{\theta(1)}$. Note that $\mathcal{R}(n) = I_{\frac{\psi(1)}{\theta(1)}} \otimes \mathcal{P}(n)$, for all $n \in N$ and write

$$\mathcal{R}(x) = \begin{pmatrix} \mathcal{R}_{11}(x) & \cdots & \mathcal{R}_{1m}(x) \\ \vdots & \ddots & \vdots \\ \mathcal{R}_{m1}(x) & \cdots & \mathcal{R}_{mm}(x) \end{pmatrix} \tag{3.38}$$

for all $x \in G_\psi$. As $\mathcal{P}^x(n) = \mathcal{P}(x)\mathcal{P}(n)\mathcal{P}(x)^{-1}$ and $(I \otimes \mathcal{P}^x(n))\mathcal{R}(x) = \mathcal{R}(x)(I \otimes \mathcal{P}(n))$, for every $x \in G_\psi$, $n \in N$, where $I = I_{\frac{\psi(1)}{\theta(1)}}$ is the identity matrix, we obtain

$$\mathcal{P}(x)^{-1}\mathcal{R}_{i\,j}(x)\mathcal{P}(n) = \mathcal{P}(n)\mathcal{P}(x)^{-1}\mathcal{R}_{i\,j}(x). \tag{3.39}$$

By Schur's lemma, as $\mathcal{P}_N$ is an ordinary irreducible representation of $N$,

$$\mathcal{R}_{i\,j}(x) = \lambda_{i\,j}(x)\mathcal{P}(x) \tag{3.40}$$

for some $\lambda_{i\,j}(x) \in \mathbb{Q}^{ab}$ (as the entries of $\mathcal{R}$ and $\mathcal{P}$ are in $\mathbb{Q}^{ab}$). Set

$$\mathcal{S}(x) = \begin{pmatrix} \lambda_{11}(x) & \cdots & \lambda_{1m}(x) \\ \vdots & \ddots & \vdots \\ \lambda_{m1}(x) & \cdots & \lambda_{mm}(x) \end{pmatrix}. \tag{3.41}$$

Then $\mathcal{S}$ is a projective representation of $G_\psi$ and $\mathcal{R} = \mathcal{S} \otimes \mathcal{P}$. Since $\mathcal{R}_K = \mathcal{S}_K \otimes \mathcal{P}_K = \mathcal{Q} \otimes \mathcal{P}_K$, we have $\mathcal{S}_K = \mathcal{Q}$. $\square$

# 4. $\mathcal{H}$-TRIPLES

Until Section 4.2, we mainly follow [8], except otherwise stated. Recall that by $\mathcal{G}$, we denote the group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, where $\mathbb{Q}^{ab}$ is the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all roots of unity. For any group $G$, we write $g\sigma \in G \times \mathcal{G}$ to mean that $g \in G$ and $\sigma \in \mathcal{G}$.

**Theorem 4.1.** *Let $\sigma \in \mathcal{G}$ and suppose that $\mathcal{P} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G \mid \alpha)$. Then*

$$\mathcal{P}^{\sigma}(x) := \mathcal{P}(x)^{\sigma} = \sigma(\mathcal{P}(x)), \tag{4.1}$$

*where $\sigma$ is applied to each entry, defines a projective representation of $G$ with entries in $\mathbb{Q}^{ab}$ and with factor set $\alpha^{\sigma}(x, y) := \alpha(x, y)^{\sigma} = \sigma(\alpha(x, y))$ for all $x, y \in G$. In particular, $\mathcal{G}$ acts on $\mathrm{Proj}(G)$.*

**Definition 4.2.** *Fix a prime $p$. Let $\mathcal{H} \subseteq \mathcal{G}$ be the set of automorphisms in $\mathcal{G}$ that maps every root of unity $\xi$ of order not divisible by $p$ to $\xi^{p^k}$ for some fixed but arbitrary integer $k$. If $\theta \in \mathrm{Irr}(G)$, we denote by $\theta^{\mathcal{H}} = \{\theta^{\sigma} : \sigma \in \mathcal{H}\}$, the $\mathcal{H}$ orbit of $\theta$. Let $N \leq G$ and $\theta \in \mathrm{Irr}(N)$. Set $G_{\theta^{\mathcal{H}}} := \{g \in G : \theta^g = \theta^{\sigma}$ for some $\sigma \in \mathcal{H}\}$. Then this is the set of stabilizer of $\theta^{\mathcal{H}}$ in $G$.*

**Definition 4.3.** *Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. If $G_{\theta^{\mathcal{H}}} = G$, then $(G, N, \theta)_{\mathcal{H}}$ is said to be an $\mathcal{H}$-triple.*

**Remark 4.4.** 
*(i) Every character triple is also an $\mathcal{H}$-triple, as $G_\theta \leq G_{\theta^{\mathcal{H}}} \leq G$. However, the converse is not always true.*
*(ii) $G_\theta$ is a normal subgroup of $G_{\theta^{\mathcal{H}}}$, because $G_\theta = G_{\theta^\sigma}$ for every $\sigma \in \mathcal{G}$.*
*(iii) If $(G, N, \theta)_{\mathcal{H}}$ is an $\mathcal{H}$-triple, then so is $(G, N, \theta^{\sigma})_{\mathcal{H}}$ for any $\sigma \in \mathcal{H}$.*

**Proposition 4.5.** *Let $K \trianglelefteq G$, $\varphi \in \mathrm{Irr}(K)$, $g\sigma \in (G \times \mathcal{G})_{\varphi}$. Assume that $\mathcal{P} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_{\varphi} \mid \alpha)$ is associated with $\varphi$, then*

$$\mathcal{P}^{g\sigma}(x) := \mathcal{P}(gxg^{-1})^{\sigma} = \sigma\big(\mathcal{P}(gxg^{-1})\big) \tag{4.2}$$

*is a projective representation of $G_{\varphi}$ with entries in $\mathbb{Q}^{ab}$, where $\sigma$ is applied entry-wise, with factor set $\beta(x, y) := \alpha(gxg^{-1}, gyg^{-1})^{\sigma}$ (for all $x, y \in G_{\varphi}$), and $\mathcal{P}^{g\sigma}$ is also associated with $\varphi$.*

*Proof.* For any $x,\, y \in G_\varphi$, we have

$$\begin{aligned} \mathcal{P}^{g\sigma}(x)\mathcal{P}^{g\sigma}(y) &= \mathcal{P}(gxg^{-1})^\sigma\, \mathcal{P}(gyg^{-1})^\sigma = \big(\mathcal{P}(gxg^{-1})\mathcal{P}(gyg^{-1})\big)^\sigma & (4.3)\\ &= \big(\alpha(gxg^{-1},\, gyg^{-1})\mathcal{P}(gxg^{-1}gyg^{-1})\big)^\sigma = \beta(x,\, y)\, \mathcal{P}^{g\sigma}(xy). & (4.4)\end{aligned}$$

Therefore $\mathcal{P}^{g\sigma} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_\varphi \,|\, \beta)$. Since $K \trianglelefteq G$ and $\mathcal{P}_K \in \mathrm{Rep}(K)$ affords $\varphi$, we have that $(\mathcal{P}^{g\sigma})_K = (\mathcal{P}_K)^{g\sigma}$ is a representation of $K$, affording $\varphi^{g\sigma} = \varphi$. Also $\beta(k,\, x) = \alpha(gkg^{-1},\, gxg^{-1})^\sigma = 1^\sigma = 1$ and $\beta(x,\, k) = 1$ for every $k \in K$, $x \in G_\varphi$. Thus $\mathcal{P}^{g\sigma}$ is associated with $\varphi$. □

**Lemma 4.6.** *Assume the hypothesis of the preceding Proposition. Then there is a unique function $\nu_{g\sigma} : G_\varphi \longrightarrow \mathbb{C}$ such that $\mathcal{P}^{g\sigma} \sim \nu_{g\sigma}\mathcal{P}$ with $\nu_{g\sigma}$ being constant on $K$-cosets and $\nu_{g\sigma}(1) = 1$.*

*Proof.* The result follows from Lemma 3.18, as $\mathcal{P}^{g\sigma}$ and $\mathcal{P}$ are associated with $\theta$. □

### 4.1. An Order Relation on $\mathcal{H}$-triples

**Definition 4.7.** *Assume $(G,\, N,\, \theta)_\mathcal{H}$ and $(H,\, M,\, \varphi)_\mathcal{H}$ are $\mathcal{H}$-triples. We shall write $(G,\, N,\, \theta)_\mathcal{H} \geq_c (H,\, M,\, \varphi)_\mathcal{H}$ if the following group theoretical and structural conditions hold:*

*(i)* $G = NH$, $M = N \cap H$ *and* $C_G(N) \leq H$,
*(ii)* $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$,
*(iii)* *there exist* $\mathcal{P} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_\theta \,|\, \alpha)$ *and* $\mathcal{P}' \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(H_\varphi \,|\, \alpha')$ *such that*
- $\mathcal{P}$ *is associated with* $\theta$ *and* $\mathcal{P}'$ *is associated with* $\varphi$,
- $\alpha$ *and* $\alpha'$ *only take roots of unity values and* $\alpha|_{H_\varphi \times H_\varphi} = \alpha'$,
- *for every* $c \in C_G(N)$, $\mathcal{P}(c) = \mu(c)I_{\theta(1)}$ *and* $\mathcal{P}'(c) = \mu(c)I_{\varphi(1)}$ *for some function* $\mu : C_G(N) \longrightarrow \mathbb{C}^\times$,

*(iv)* *for any* $h\sigma \in (H \times \mathcal{H})_\theta$, *consider the functions* $\nu_{h\sigma}$ *and* $\nu'_{h\sigma}$ *satisfying* $\mathcal{P}^{h\sigma} \sim \nu_{h\sigma}\mathcal{P}$

*and* $(\mathcal{P}')^{h\sigma} \sim \nu'_{h\sigma}\mathcal{P}'$ *as in Lemma 4.6, then*

$$\nu_{h\sigma}|_{H_\varphi} = \nu'_{h\sigma}. \tag{4.5}$$

*We also say that* $(\mathcal{P}, \mathcal{P}')$ *is associated with* $(G, N, \theta)_{\mathcal{H}} \geq_c (H, M, \varphi)_{\mathcal{H}}$.

Notice that if $(H\times\mathcal{H})_\theta = (H\times\mathcal{H})_\varphi$, then $H_\theta = H_\varphi$. Since $C_{G_\theta}(N) = C_G(N)$, Part 3 of Definition 4.7 is equivalent to state that $(\mathcal{P}, \mathcal{P}')$ is associated with $(G_\theta, N, \theta) \geq_c (H_\varphi, M, \varphi)$, with entries in $\mathbb{Q}^{ab}$ and with factor sets only taking roots of unity values.

The following lemma will be quite useful in the proof of Proposition 4.10.

**Lemma 4.8.** *It suffices to check (iv) for a transversal of* $H_\theta$ *in* $(H \times \mathcal{H})_\theta$.

**Lemma 4.9.** *Suppose* $(G, N, \theta)_{\mathcal{H}} \geq_c (H, M, \varphi)_{\mathcal{H}}$. *If* $N \leq K \leq G_\theta$, *then consider the map* $\tau_K : \mathrm{Irr}(K \,|\, \theta) \longrightarrow \mathrm{Irr}(K \cap H \,|\, \varphi)$ *in Theorem 3.25. Then* $\tau_K$ *is* $(N_H(K) \times \mathcal{H})_\theta$*-equivariant.*

## 4.2. Construction of New $\mathcal{H}$-triples

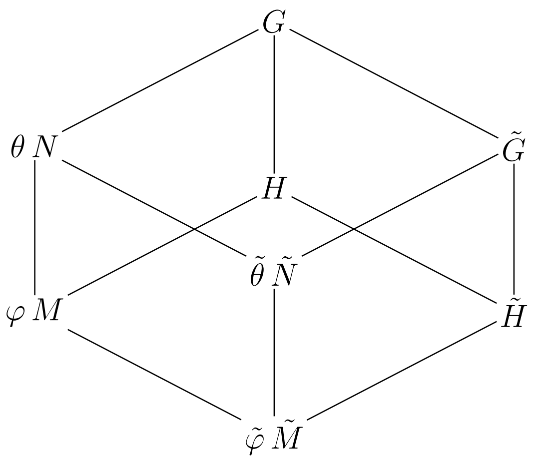
In this section, we prove some new results on $\mathcal{H}$-triples that are refinements from character triples. See [1] and [13] for the following facts on character triples. We actually implement the proofs in these articles to $\mathcal{H}$-triples.

**Proposition 4.10.** *Let* $G$ *be a finite group,* $N \trianglelefteq G$, $H \leq G$, $\tilde{G} \leq G$, $M = N \cap H$, $\tilde{L} = L \cap \tilde{G}$ *for any* $L \leq G$ *and that* $\tilde{\theta} \in \mathrm{Irr}(\tilde{N})$ *and* $\tilde{\varphi} \in \mathrm{Irr}(\tilde{M})$ *such that* $\theta = \tilde{\theta}^N$ *and* $\varphi = \tilde{\varphi}^M$ *are irreducible characters of* $N$ *and* $M$, *respectively. Suppose that* $G = NH$, $H = \tilde{H}M$, $C_G(N) \leq H$, *that* $(\tilde{G}, \tilde{N}, \tilde{\theta})_{\mathcal{H}} \geq_c (\tilde{H}, \tilde{M}, \tilde{\varphi})_{\mathcal{H}}$ *and that there are bijections*

$$\begin{array}{ccc} \mathrm{Irr}(\tilde{K} \,|\, \tilde{\theta}^{\mathcal{H}}) \longrightarrow \mathrm{Irr}(K \,|\, \theta^{\mathcal{H}}) & \qquad & \mathrm{Irr}(\tilde{K} \cap H \,|\, \tilde{\varphi}^{\mathcal{H}}) \longrightarrow \mathrm{Irr}(K \cap H \,|\, \varphi^{\mathcal{H}}) \\ \psi \longmapsto \psi^K & & \vartheta \longmapsto \vartheta^{K\cap H} \end{array} \tag{4.6}$$

*for all* $N \leq K \leq G$. *Then we have*

$$(G, N, \theta)_{\mathcal{H}} \geq_c (H, M, \varphi)_{\mathcal{H}}. \tag{4.7}$$




*Proof.* First note that $G = NH = NM\tilde{H} = N\tilde{H}$.

*Step 1: $(G, N, \theta)_{\mathcal{H}}$ and $(H, M, \varphi)_{\mathcal{H}}$ are $\mathcal{H}$-triples.*

By hypothesis, we have that $\tilde{G}_{\tilde{\theta}^{\mathcal{H}}} = \tilde{G}$ and $\tilde{H}_{\tilde{\varphi}^{\mathcal{H}}} = \tilde{H}$, and it suffices to check $G_{\theta^{\mathcal{H}}} = G$ and $H_{\varphi^{\mathcal{H}}} = H$. Let $g \in G$ and write $g = nx$ for some $n \in N$ and $x \in \tilde{H}$. Since $\tilde{G}_{\tilde{\theta}^{\mathcal{H}}} = \tilde{G}$, there exists some $\sigma \in \tilde{H}$ for which $\tilde{\theta}^x = \tilde{\theta}^\sigma$. Then

$$\theta^g = (\tilde{\theta}^N)^{nx} = (\tilde{\theta}^N)^x = (\tilde{\theta}^x)^N = (\tilde{\theta}^\sigma)^N = (\tilde{\theta}^N)^\sigma = \theta^\sigma, \tag{4.8}$$

by noting that $\sigma$ commutes with induction of characters and that $\tilde{N} \trianglelefteq \tilde{G}$. Therefore $g \in G_{\theta^{\mathcal{H}}}$. In a similar manner, we obtain $\tilde{H}_{\tilde{\varphi}^{\mathcal{H}}} = \tilde{H}$.

*Step 2: $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$.*

We have $(\tilde{H} \times \mathcal{H})_{\tilde{\theta}} = (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$ and induction of characters gives a bijection between $\{\tilde{\theta}^\sigma : \sigma \in \mathcal{H}\}$ and $\{\theta^\sigma : \sigma \in \mathcal{H}\}$, by assumption. Now, let $h\sigma \in (H \times \mathcal{H})_\theta$ and write $h = mx$ for some $m \in M$ and $x \in \tilde{H}$. Then

$$\tilde{\theta}^N = \theta = \theta^{h\sigma} = \theta^{x\sigma} = (\tilde{\theta}^N)^{x\sigma} = (\tilde{\theta}^{x\sigma})^N, \tag{4.9}$$

because $\tilde{N} \trianglelefteq \tilde{G}$, by using the above bijection. Since $\tilde{G}_{\tilde{\theta}^{\mathcal{H}}} = \tilde{G}$, there exists $\tau \in \mathcal{H}$ for which $\tilde{\theta}^x = \tilde{\theta}^\tau$, so that $\tilde{\theta}^{\tau\sigma} = \tilde{\theta}^{x\sigma} = \tilde{\theta}$. Therefore $x\sigma \in (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$. Then

$$\varphi^{h\sigma} = (\tilde{\varphi}^M)^{mx\sigma} = (\tilde{\varphi}^M)^{x\sigma} = (\tilde{\varphi}^{x\sigma})^M = \varphi, \tag{4.10}$$

because $\tilde{M} \trianglelefteq \tilde{H}$. Therefore $h\sigma \in (H \times \mathcal{H})_{\varphi}$. Conversely, by using the facts $\tilde{H} = \tilde{H}_{\tilde{\varphi}^{\mathcal{H}}}$ and the bijection between $\{\tilde{\varphi}^{\sigma} : \sigma \in \mathcal{H}\}$ and $\{\varphi^{\sigma} : \sigma \in \mathcal{H}\}$ given by induction (in the hypothesis), we obtain that $(H \times \mathcal{H})_{\varphi} \subseteq (H \times \mathcal{H})_{\theta}$.

Now, suppose that there are projective representations $\mathcal{P} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(\tilde{G}_{\tilde{\theta}} \,|\, \alpha)$ and $\mathcal{P}' \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(\tilde{H}_{\tilde{\varphi}} \,|\, \alpha')$ such that $(\mathcal{P}, \mathcal{P}')$ is associated with $(\tilde{G}, \tilde{N}, \tilde{\theta})_{\mathcal{H}} \geq_c (\tilde{H}, \tilde{M}, \tilde{\varphi})_{\mathcal{H}}$.

*Step 3:* $\left(\mathcal{P}^{G_\theta}, (\mathcal{P}')^{H_\varphi}\right)$ *is associated with* $(G, N, \theta)_{\mathcal{H}} \geq_c (H, M, \varphi)_{\mathcal{H}}$.

We first check some group theoretical identities in order to use Proposition 3.21.

- $G_\theta = N\tilde{G}_{\tilde{\theta}}$ : If $x \in \tilde{G}_{\tilde{\theta}}$, then $\theta^x = (\tilde{\theta}^N)^x = (\tilde{\theta}^x)^N = \theta$, as $\tilde{N} \trianglelefteq \tilde{G}$, so $x \in G_\theta$. If $x \in G_\theta$, write $x = ny$ for some $n \in N$ and $y \in \tilde{H}$, then $\tilde{\theta}^N = \theta = \theta^{ny} = (\tilde{\theta}^N)^y = (\tilde{\theta}^y)^N$. Since $\tilde{G}_{\tilde{\theta}^{\mathcal{H}}} = \tilde{G}$, there exists $\sigma \in \mathcal{H}$ for which $\tilde{\theta}^y = \tilde{\theta}^\sigma$. Since there is a bijection from $\{\tilde{\theta}^\sigma : \sigma \in \mathcal{H}\}$ onto $\{\theta^\sigma : \sigma \in \mathcal{H}\}$, given by induction, we have $\tilde{\theta}^y = \tilde{\theta}$, so that $y \in \tilde{G}_{\tilde{\theta}}$. Therefore $x \in N\tilde{G}_{\tilde{\theta}}$, as desired.
- $N \cap \tilde{G}_{\tilde{\theta}} = \tilde{N}$ : Since $N \cap \tilde{G}_{\tilde{\theta}} \leq N \cap \tilde{G} = \tilde{N}$ and $\tilde{\theta} \in \mathrm{cf}(\tilde{N})$, the equality follows.

Similarly, we also obtain $H_\varphi = M\tilde{H}_{\tilde{\varphi}}$ and $M \cap \tilde{H}_{\tilde{\varphi}} = \tilde{M}$.

Define $\mathcal{Q} = \mathcal{P}^{G_\theta}$ and $\mathcal{Q}' = (\mathcal{P}')^{H_\varphi}$ as in Proposition 3.21. Write $\beta$ and $\beta'$ for their factor sets, respectively. Then $\beta$ and $\beta'$ only take roots of unity values and the entries of $\mathcal{Q}$ and $\mathcal{Q}'$ are in $\mathbb{Q}^{ab}$, by construction. Letting $x = ax' \in H_\varphi$ and $y = by' \in H_\varphi$ for some $x', y' \in \tilde{H}_{\tilde{\varphi}}$ and $a, b \in M$, we have

$$\beta(x, y) = \beta(x', y') = \alpha(x', y') = \alpha'(x', y') = \beta'(x', y') = \beta'(x, y), \tag{4.11}$$

since $\mathcal{Q}$ is associated with $\theta$ and $\alpha_{\tilde{H}_{\tilde{\varphi}} \times \tilde{H}_{\tilde{\varphi}}} = \alpha'$. Therefore $\beta_{H_\theta \times H_\theta} = \beta'$, as $H_\theta = H_\varphi$ and $\tilde{H}_{\tilde{\theta}} = \tilde{H}_{\tilde{\varphi}}$. We claim that $C_G(N) \subseteq C_{\tilde{G}}(\tilde{N})$. Let $c \in C_G(N)$. Then by Theorem 3.32, $\theta = \tilde{\theta}^N$ extends to some $\chi \in \mathrm{Irr}(\langle N, c\rangle \,|\, \theta)$. Since $c \in Z(\langle N, c\rangle)$, $\chi(c) \neq 0$. By hypothesis, induction of characters constructs a bijection $\mathrm{Irr}(\langle N, c\rangle \cap \tilde{G} \,|\, \tilde{\theta}^{\mathcal{H}}) \longrightarrow \mathrm{Irr}(\langle N, c\rangle \,|\, \theta^{\mathcal{H}})$, so there exists some $\eta \in \mathrm{Irr}(\langle N, c\rangle \cap \tilde{G} \,|\, \tilde{\theta}^{\mathcal{H}})$ such that $\chi = \eta^{\langle N, c\rangle}$. By the definition of character induction, there must exist some $x \in \langle N, c\rangle$ such that

$xcx^{-1} \in \langle N, c\rangle \cap \tilde{G}$ and $\eta(xcx^{-1}) \neq 0$, as $\chi(c) \neq 0$. However, since $c \in Z(\langle N, c\rangle)$, $xcx^{-1} = c \in \langle N, c\rangle \cap \tilde{G}$, for all $x \in \langle N, c\rangle$ so $c \in \tilde{G}$. Thus $c \in C_{\tilde{G}}(\tilde{N})$. Finally, if $c \in C_G(N)$, $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar matrices, associated to the same scalar, by hypothesis. Then by the remark below Proposition 3.21, $P^{G_\theta}(c)$ and $(P')^{H_\varphi}(c)$ are also associated to the same scalar.

Now, let $h\sigma \in (H \times \mathcal{H})_\theta$ and write $\mathcal{Q}^{h\sigma} \sim \nu_{h\sigma}\mathcal{Q}$ and $(\mathcal{Q}')^{h\sigma} \sim \nu'_{h\sigma}\mathcal{Q}'$ for some unique functions $\nu_{h\sigma}\colon G_\theta \longrightarrow \mathbb{C}^\times$, $\nu'_{h\sigma}\colon H_\varphi \longrightarrow \mathbb{C}^\times$, as in Lemma 4.6. Then as the final step, we need to check that

$$\nu_{h\sigma}|_{H_\varphi} = \nu'_{h\sigma}|_{H_\varphi} \tag{4.12}$$

for every $h\sigma \in (H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$.

*Step 4: It suffices to check $\nu_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}} = \nu'_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}}$ for every $r\sigma \in (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$, in order to obtain (4.12).*

Let $h\sigma \in (H \times \mathcal{H})_\theta$. By Lemma 4.6, $\nu_{h\sigma}$ is constant on $N$-cosets, $\nu'_{h\sigma}$ is constant on $M$-cosets and $\nu_{h\sigma}(1) = \nu'_{h\sigma}(1) = 1$. By Lemma 4.8, it suffices to check (4.12) for a transversal of $H_\varphi$ in $(H \times \mathcal{H})_\varphi$. Let $R$ be a complete set of coset representatives of $H_\varphi$ in $H$, then $R$ is also a complete set of coset representatives of $\tilde{H}_{\tilde{\varphi}}$ in $\tilde{H}$, because

$$H/H_\varphi = M\tilde{H}/M\tilde{H}_{\tilde{\varphi}} \cong \tilde{H}/(M\tilde{H}_{\tilde{\varphi}} \cap \tilde{H}) = \tilde{H}/\tilde{H}_{\tilde{\varphi}}, \tag{4.13}$$

where $\tilde{H}_{\tilde{\varphi}} = M\tilde{H}_{\tilde{\varphi}} \cap \tilde{H}$, because if $x \in M\tilde{H}_{\tilde{\varphi}} \cap \tilde{H}$, then $x = my$ for some $m \in M$, $y \in \tilde{H}_{\tilde{\varphi}}$, and as $y \in \tilde{H}$, we have $m \in \tilde{H} \cap M = \tilde{M}$. Notice that if $h\sigma = xr\sigma \in (H \times \mathcal{H})_\varphi$, for some $x \in H_\varphi$ and $r \in R$, then $\varphi^{h\sigma} = \varphi^{r\sigma}$, so that $r\sigma \in (R \times \mathcal{H})_\varphi$. Also if $r\sigma \in (R \times \mathcal{H})_\varphi$, then $xr\sigma \in (H \times \mathcal{H})_\varphi$, for every $x \in H_\varphi$. Therefore a transversal of $H_\varphi$ in $(H \times \mathcal{H})_\varphi$ can be viewed as a subset of $(R \times \mathcal{H})_\varphi$. Now, if $r\sigma \in (R \times \mathcal{H})_\varphi$, then $r \in \tilde{H}$ as $R \subseteq \tilde{H}$. Then

$$\tilde{\varphi}^M = \varphi = \varphi^{r\sigma} = (\tilde{\varphi}^M)^{r\sigma} = (\tilde{\varphi}^{r\sigma})^M, \tag{4.14}$$

since $\tilde{M} \trianglelefteq \tilde{H}$. As induction defines a bijection $\mathrm{Irr}(\tilde{M} \mid \tilde{\varphi}^{\mathcal{H}}) \longrightarrow \mathrm{Irr}(M \mid \varphi^{\mathcal{H}})$, we have that $\tilde{\varphi}^{r\sigma} = \tilde{\varphi}$, because $\tilde{\varphi}^r = \tilde{\varphi}^\tau$ for some $\tau \in \mathcal{H}$ by using the equality $\tilde{H} = \tilde{H}_{\tilde{\varphi}}$. Therefore $r\sigma \in (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$. Finally, since $\nu_{r\sigma}$ and $\nu'_{r\sigma}$ are constant on $M$-cosets, the result follows.

*Step 5: For all* $r\sigma \in (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$*, we have* $\nu_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}} = \nu'_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}}$.

Let $r\sigma \in (\tilde{H} \times \mathcal{H})_{\tilde{\varphi}}$. Since $(\mathcal{P}, \mathcal{P}')$ gives $(\tilde{G}, \tilde{N}, \tilde{\theta})_{\mathcal{H}} \geq_c (\tilde{H}, \tilde{M}, \tilde{\varphi})_{\mathcal{H}}$, there exist unique functions $\mu_{r\sigma} : \tilde{G}_{\tilde{\theta}} \longrightarrow \mathbb{C}^{\times}$ and $\mu'_{r\sigma} : \tilde{H}_{\tilde{\varphi}} \longrightarrow \mathbb{C}^{\times}$ constant on $\tilde{N}$ and $\tilde{M}$ cosets, respectively and $\mu_{r\sigma}(1) = \mu'_{r\sigma}(1) = 1$ such that $\mathcal{P}^{r\sigma} \sim \mu_{r\sigma}\mathcal{P}$, $(\mathcal{P}')^{r\sigma} \sim \mu'_{r\sigma}\mathcal{P}'$ and $\mu_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}} = \mu'_{r\sigma}|_{\tilde{H}_{\tilde{\varphi}}}$. Set $\mathcal{T} = \{t_1 = 1, t_2, ..., t_n\}$ to be a right transversal of $\tilde{N}$ in $N$. Then $t_i r = r(r^{-1}t_i r) = r\tilde{n}_i t_{i'}$ for some $\tilde{n}_i \in \tilde{N}$, $t_{i'} \in \mathcal{T}$. For any $x \in \tilde{H}_{\tilde{\varphi}}$,

$$\mathcal{Q}^{r\sigma}(x) = \mathcal{P}^{G_\theta}(rxr^{-1})^{\sigma} = \begin{pmatrix} \dot{\mathcal{P}}_{11}(rxr^{-1})^{\sigma} & \dot{\mathcal{P}}_{12}(rxr^{-1})^{\sigma} & \cdots & \dot{\mathcal{P}}_{1n}(rxr^{-1})^{\sigma} \\ \dot{\mathcal{P}}_{21}(rxr^{-1})^{\sigma} & \dot{\mathcal{P}}_{22}(rxr^{-1})^{\sigma} & \cdots & \dot{\mathcal{P}}_{2n}(rxr^{-1})^{\sigma} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}_{n1}(rxr^{-1})^{\sigma} & \dot{\mathcal{P}}_{n2}(rxr^{-1})^{\sigma} & \cdots & \dot{\mathcal{P}}_{nn}(rxr^{-1})^{\sigma} \end{pmatrix}, \tag{4.15}$$

where

$$\begin{aligned} \dot{\mathcal{P}}_{ij}(rxr^{-1}) &= \begin{cases} \mathcal{P}(t_i rxr^{-1}t_j^{-1})^{\sigma} & \text{if } t_i rxr^{-1}t_j^{-1} \in \tilde{G}_{\tilde{\theta}}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \mathcal{P}^{r\sigma}(\tilde{n}_i t_{i'} x t_{j'}^{-1} \tilde{n}_j^{-1}) & \text{if } t_i rxr^{-1}t_j^{-1} = r\tilde{n}_i t_{i'} x t_{j'}^{-1}\tilde{n}_j^{-1} r^{-1} \in \tilde{G}_{\tilde{\theta}}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{4.16}$$

Then as $\mathcal{P}^{r\sigma} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(\tilde{G}_{\tilde{\theta}})$ is associated with $\tilde{\theta}$,

$$\mathcal{P}^{r\sigma}(\tilde{n}_i t_{i'} x t_{j'}^{-1}\tilde{n}_j^{-1}) = \mathcal{P}^{r\sigma}(\tilde{n}_i)\mathcal{P}^{r\sigma}(t_{i'} x t_{j'}^{-1})\mathcal{P}^{r\sigma}(\tilde{n}_j)^{-1}, \tag{4.17}$$

whenever $\tilde{n}_i t_{i'} x t_{j'}^{-1}\tilde{n}_j^{-1} \in \tilde{G}_{\tilde{\theta}}$, by noting that $\mathcal{P}^{r\sigma}(\tilde{n}_j)^{-1} = \mathcal{P}^{r\sigma}(\tilde{n}_j^{-1})$, as $\mathcal{P}^{r\sigma}|_{\tilde{N}}$ is an ordinary representation of $\tilde{N}$. If we write

$$A := \begin{pmatrix} \mathcal{P}^{r\sigma}(\tilde{n}_1) & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{P}^{r\sigma}(\tilde{n}_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{P}^{r\sigma}(\tilde{n}_r) \end{pmatrix}, \tag{4.18}$$

then

$$\mathcal{Q}^{r\sigma}(x) = A \begin{pmatrix} \dot{\mathcal{P}}^{r\sigma}(t_{1'}xt_{1'}^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_{1'}xt_{2'}^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_{1'}xt_{r'}^{-1}) \\ \dot{\mathcal{P}}^{r\sigma}(t_{2'}xt_{1'}^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_{2'}xt_{2'}^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_{2'}xt_{r'}^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}^{r\sigma}(t_{r'}xt_{1'}^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_{r'}xt_{2'}^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_{r'}xt_{r'}^{-1}) \end{pmatrix} A^{-1}, \tag{4.19}$$

where

$$\dot{\mathcal{P}}^{r\sigma}(t_{i'}xt_{j'}^{-1}) = \begin{cases} \mathcal{P}^{r\sigma}(t_{i'}xt_{j'}^{-1}) & \text{if } rt_{i'}xt_{j'}^{-1}r^{-1} \in \tilde{G}_{\tilde{\theta}}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Notice that $A$ is independent of $x$ and only depends on $r$ and also that the conjugated matrix in (4.19) is similar to

$$\begin{pmatrix} \dot{\mathcal{P}}^{r\sigma}(t_1xt_1^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_1xt_2^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_1xt_r^{-1}) \\ \dot{\mathcal{P}}^{r\sigma}(t_2xt_1^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_2xt_2^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_2xt_r^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}^{r\sigma}(t_rxt_1^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_rxt_2^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_rxt_r^{-1}) \end{pmatrix}. \tag{4.21}$$

Now, if $t_ixt_j^{-1}$ and $t_kxt_l^{-1}$ are elements of $\tilde{G}_{\tilde{\theta}}$, then $(t_ixt_j^{-1})(t_kxt_l^{-1}) \in N \cap \tilde{G}_{\tilde{\theta}} = \tilde{N}$. Since $\mu_{r\sigma}$ is constant on $\tilde{N}$-cosets, we obtain $\mu_{r\sigma}(t_kxt_l^{-1}) = \mu_{r\sigma}(x)$, for any $t_k$, $t_l \in \mathcal{T}$ with $t_kxt_l^{-1} \in \tilde{G}_{\tilde{\theta}}$. As $\dot{\mathcal{P}}^{r\sigma}(t_ixt_j^{-1}) \sim \mu_{r\sigma}(t_ixt_j^{-1})\dot{\mathcal{P}}(t_ixt_j^{-1})$, we have

$$\begin{aligned} \mathcal{Q}^{r\sigma}(x) &\sim \begin{pmatrix} \dot{\mathcal{P}}^{r\sigma}(t_1xt_1^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_1xt_2^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_1xt_r^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}^{r\sigma}(t_rxt_1^{-1}) & \dot{\mathcal{P}}^{r\sigma}(t_rxt_2^{-1}) & \cdots & \dot{\mathcal{P}}^{r\sigma}(t_rxt_r^{-1}) \end{pmatrix} \\ &\sim \mu_{r\sigma}(x) \begin{pmatrix} \dot{\mathcal{P}}(t_1xt_1^{-1}) & \dot{\mathcal{P}}(t_1xt_2^{-1}) & \cdots & \dot{\mathcal{P}}(t_1xt_r^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\mathcal{P}}(t_rxt_1^{-1}) & \dot{\mathcal{P}}(t_rxt_2^{-1}) & \cdots & \dot{\mathcal{P}}(t_rxt_r^{-1}) \end{pmatrix} \\ &\sim \mu_{r\sigma}(x)\mathcal{Q}(x). \end{aligned} \tag{4.22}$$

Similarly, $(\mathcal{Q}')^{r\sigma}(x) \sim \mu'_{r\sigma}(x)\mathcal{Q}'(x)$. Then by the uniqueness of $\nu_{r\sigma}$ and $\nu'_{r\sigma}$, we get $\nu_{r\sigma}(x) = \mu_{r\sigma}(x) = \mu'_{r\sigma}(x) = \nu'_{r\sigma}(x)$, as desired. □

**Proposition 4.11.** *Let $N \trianglelefteq G$, $H \le G$, $M = N \cap H$ and assume $G = NH$. Let $K \le M$ be a normal subgroup of $G$ and let $\eta \in \mathrm{Irr}_{p'}(N)$ be $(H \times \mathcal{H})$-invariant with $\eta_K \in \mathrm{Irr}(K)$. Suppose $(G/K, N/K, \overline{\theta})_{\mathcal{H}} \ge_c (H/K, M/K, \overline{\varphi})_{\mathcal{H}}$. If $\theta$ and $\varphi$ are the* *lifts of* $\overline{\theta}$ *and* $\overline{\varphi}$ *to* $N$ *and* $M$*, respectively, then we have*

$$(G,\, N,\, \theta\eta)_{\mathcal{H}} \geq_c (H,\, M,\, \varphi\eta_M)_{\mathcal{H}}. \tag{4.23}$$

*Proof.* As $G = NH$, we have that $\eta$ is $G \times \mathcal{H}$-invariant. For every $g \in G$, there exists $\sigma \in \mathcal{H}$ for which $\theta^g(n) = \theta(gng^{-1}) = \overline{\theta}(Kgng^{-1}) = \overline{\theta}^{Kg}(Kn) = \overline{\theta}(Kn)^\sigma = \theta(n)^\sigma$ for every $n \in N$ and $(\theta\eta)^g = \theta^g\eta^g = \theta^\sigma\eta = \theta^\sigma\eta^\sigma = (\theta\eta)^\sigma$, since $(G/K)_{\mathcal{H}} = G/K$ and $\eta$ is $G \times \mathcal{H}$-invariant. Therefore $(G,\, N,\, \theta\eta)_{\mathcal{H}}$ is an $\mathcal{H}$-triple. Similarly, $(H,\, M,\, \varphi\eta_M)_{\mathcal{H}}$ is also an $\mathcal{H}$-triple. As $C_G(N)K/K \leq C_{G/K}(N/K) \leq H/K$, we obtain $C_G(N) \leq H$.

*Claim:* $(H \times \mathcal{H})_{\theta\eta} = (H \times \mathcal{H})_{\varphi\eta_M}$

Let $h\sigma \in (H \times \mathcal{H})_{\theta\eta}$ and write $\overline{h} = Kh$. As Gallagher's Theorem gives a bijection $\mathrm{Irr}(N/K) \longrightarrow \mathrm{Irr}(N \,|\, \eta_K)$, $\overline{\theta} \longmapsto \theta\eta$, we obtain $\overline{\theta}^{\overline{h}\sigma} = \overline{\theta}$. Then $\overline{\varphi}^{\overline{h}\sigma} = \overline{\varphi}$, since we have $(\overline{H} \times \mathcal{H})_{\overline{\theta}} = (\overline{H} \times \mathcal{H})_{\overline{\varphi}}$, by hypothesis. Again by Gallagher Correspondence, there is a bijection $\mathrm{Irr}(M/K) \longrightarrow \mathrm{Irr}(M \,|\, \eta_K)$, $\overline{\varphi} \longmapsto \varphi\eta_M$. Therefore $h\sigma \in (H \times \mathcal{H})_{\varphi\eta_M}$. Similarly, the converse follows.

By hypothesis, there exists a pair of projective representations $(\overline{\mathcal{P}},\, \overline{\mathcal{P}}')$ associated with $(G/K,\, N/K,\, \overline{\theta})_{\mathcal{H}} \geq_c (H/K,\, M/K,\, \overline{\varphi})_{\mathcal{H}}$. Suppose that the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ are $\overline{\alpha}$ and $\overline{\alpha'}$, respectively. Note that since $(G_\theta,\, N,\, \eta)$ is a character triple, there exists $\mathcal{Q} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(G_\theta \,|\, \beta)$, for some factor set $\beta$ only taking roots of unity values, associated with $\eta$.

*Claim:* $(\mathcal{P} \otimes \mathcal{Q},\, \mathcal{P}' \otimes \mathcal{Q}_{H_\varphi})$ *is associated with* $(G,\, N,\, \theta\eta)_{\mathcal{H}} \geq_c (H,\, M,\, \varphi\eta_M)_{\mathcal{H}}$*, where* $\mathcal{P}$ *and* $\mathcal{P}'$ *are the lifts of* $\overline{\mathcal{P}}$ *and* $\overline{\mathcal{P}'}$*, respectively.* .

Write $\alpha$ and $\alpha'$ for the factor sets of $\mathcal{P}$ and $\mathcal{P}'$, respectively. By Gallagher Correspondence, there exists a bijection $\mathrm{Irr}(N/K) \longrightarrow \mathrm{Irr}(N \,|\, \eta_K)$, $\overline{\theta} \longmapsto \theta\eta$ and $g \in G_\theta$ if and only if $\theta\eta = \theta^g\eta = (\theta\eta)^g$ . Therefore $G_{\theta\eta} = G_\theta$ and similarly, $H_{\varphi\eta_M} = H_\varphi$. Since $\mathcal{P}$ and $\mathcal{Q}$ are projective representations of $G_\theta$, we have $\mathcal{P} \otimes \mathcal{Q} \in \mathrm{Proj}(G_\theta \,|\, \alpha\beta)$, also the entries of $\mathcal{P} \otimes \mathcal{Q}$ are in $Q^{ab}$. Then similarly $\mathcal{P}' \otimes \mathcal{Q}_{H_\varphi} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(H_\varphi \,|\, \alpha'\beta_{H_\varphi \times H_\varphi})$.

Since $(\mathcal{P}\otimes\mathcal{Q})_N = (\mathcal{P}_N \otimes \mathcal{Q}_N)$ and $(\mathcal{P}'\otimes\mathcal{Q}_{H_\varphi})_M = \mathcal{P}'_M \otimes \mathcal{Q}_M$, they afford $\theta\eta$ and $\varphi\eta_M$, respectively. For any $n \in N$, $g \in G_\theta$,

$$\big(\alpha\beta\big)(g,\,n) = \alpha(g,\,n)\beta(g,\,n) = 1 \tag{4.24}$$

and $\big(\beta\alpha\big)(n,\,g) = 1$, since $\mathcal{P}$ lifts $\overline{\mathcal{P}}$ and $\mathcal{Q}$ is associated with $\eta$. Also

$$(\alpha\beta)_{H_{\theta\eta}\times H_{\theta\eta}} = \alpha_{H_\theta\times H_\theta}\beta_{H_\varphi\times H_\varphi} = \alpha'\beta_{H_\varphi\times H_\varphi}. \tag{4.25}$$

For any $c \in C_G(N)$, $Kc \in C_{G/K}(N/K)$, and since $\overline{\mathcal{P}}(Kc)$, $\overline{\mathcal{P}}'(Kc)$, $(\mathcal{P}\otimes\mathcal{Q})(c) = \overline{\mathcal{P}}(Kc)\otimes\mathcal{Q}(c)$ and $(\mathcal{P}'\otimes\mathcal{Q}_{H_\varphi})(c) = \overline{\mathcal{P}}'(Kc)\otimes\mathcal{Q}(c)$ are scalar matrices, so is $\mathcal{Q}(c)$. Since $\overline{\mathcal{P}}(Kc)$, $\overline{\mathcal{P}}'(Kc)$ are associated to the same scalar, by hypothesis, it follows that $(\mathcal{P}\otimes\mathcal{Q})(c)$ and $(\mathcal{P}'\otimes\mathcal{Q}_{H_\varphi})(c)$ are also associated to the same scalar.

Now, let $h\sigma \in (H\times\mathcal{H})_{\theta\eta} = (H\times\mathcal{H})_{\varphi\eta_M}$ and $x \in H_\varphi = H_\theta$. By Lemma 4.6 and hypothesis, there exists unique functions $\nu_{h\sigma}\colon G_\theta \longrightarrow \mathbb{C}^\times$ and $\nu'_{h\sigma}\colon H_\varphi \longrightarrow \mathbb{C}^\times$ such that $\mathcal{P}^{h\sigma} \sim \nu_{h\sigma}\mathcal{P}$ and $(\mathcal{P}')^{h\sigma} \sim \nu'_{h\sigma}\mathcal{P}'$ satisfying $\nu_{h\sigma}|_{H\varphi} = \nu'_{h\sigma}|_{H_\varphi}$. Also, there exists a unique function $\mu_{h\sigma} : G_\theta \longrightarrow \mathbb{C}^\times$ for which $\mathcal{Q}^{h\sigma} \sim \mu_{h\sigma}\mathcal{Q}$. Then

$$(\mathcal{P}\otimes\mathcal{Q})^{h\sigma} = \mathcal{P}^{h\sigma}\otimes\mathcal{Q}^{h\sigma} \sim \nu_{h\sigma}\,\mu_{h\sigma}(\mathcal{P}\otimes\mathcal{Q}) \tag{4.26}$$

and

$$(\mathcal{P}'\otimes\mathcal{Q}_{H_\varphi})^{h\sigma} = (\mathcal{P}')^{h\sigma}\otimes(\mathcal{Q}_{H_\varphi})^{h\sigma} \sim \nu'_{h\sigma}\mu_{h\sigma}|_{H_\varphi}(\mathcal{P}'\otimes\mathcal{Q}_{H_\varphi}), \tag{4.27}$$

by noting that $H_\varphi \trianglelefteq H$. Since $\nu_{h\sigma}|_{H\varphi} = \nu'_{h\sigma}|_{H_\varphi}$, we obtain

$$(\nu_{h\sigma}\,\mu_{h\sigma})|_{H_\varphi} = \nu'_{h\sigma}\mu_{h\sigma}|_{H_\varphi}. \tag{4.28}$$

Hence, this completes the proof of $(G,\,N,\,\theta\eta)_{\mathcal{H}} \geq_c (H,\,M,\,\varphi\eta_M)_{\mathcal{H}}$. □

Notice that, by taking $\eta = 1_N$ in the above theorem, we get $(G, N,\,\theta)_{\mathcal{H}} \geq_c (H,\,M,\,\varphi)$, under the same setting.

Except in the last part of the proof of the next result, we do not need the assumption that $\psi$ extends $\theta$ and until there, we give a general proof.

**Proposition 4.12.** *Suppose* $(G,\,N,\,\theta)_{\mathcal{H}} \geq (H,\,M,\,\varphi)_{\mathcal{H}}$. *For any* $N \leq K \leq G_\theta$, *let* $\tau_K :\ \mathrm{Irr}(K\,|\,\theta) \longrightarrow \mathrm{Irr}(K\cap H\,|\,\varphi)$ *be the* $(N_H(K)\times\mathcal{H})_\theta$*-equivariant bijection given by*

*Lemma 4.9. Assume* $\psi \in \mathrm{Irr}(K \mid \theta)$ *extends* $\theta$*, then*

$$(N_G(K)_{\psi^{\mathcal{H}}},\, K,\, \psi)_{\mathcal{H}} \geq_c (N_H(K)_{\psi^{\mathcal{H}}},\, K \cap H,\, \tau_K(\psi))_{\mathcal{H}}. \tag{4.29}$$

*Proof.* By assumption, we have $G = NH$, $M = N \cap H$ and $C_G(N) \leq H$. We first check that $N_G(K)_{\psi^{\mathcal{H}}} = KN_H(K)_{\psi^{\mathcal{H}}}$, that $K \cap N_H(K)_{\psi^{\mathcal{H}}} = K \cap H$ and that $C_{N_G(K)_{\psi^{\mathcal{H}}}}(K) \leq N_H(K)_{\psi^{\mathcal{H}}}$. If $g \in N_G(K)_{\psi^{\mathcal{H}}}$, then $g = nh$ for some $n \in N$ and $h \in H$. By noting that $K = K^{nh} = K^h$ and that $= \psi^h = \psi^{nh} = \psi^\sigma$ for some $\sigma \in \mathcal{H}$, we have $nh \in KN_H(K)_{\psi^{\mathcal{H}}}$. On the other hand, as $\psi \in \mathrm{cf}(K)$, we have $K \subseteq N_G(K)_{\psi^{\mathcal{H}}}$ and $N_H(K)_{\psi^{\mathcal{H}}} \subseteq N_G(K)_{\psi^{\mathcal{H}}}$. Therefore $N_G(K)_{\psi^{\mathcal{H}}} = KN_H(K)_{\psi^{\mathcal{H}}}$. Naturally $K \cap N_H(K)_{\psi^{\mathcal{H}}} \subseteq K \cap H$. Conversely, $K \cap H \subseteq N_H(K)$ and if $x \in K \cap H$, then $\psi^x = \psi$, as $\psi \in \mathrm{cf}(K)$. Thus $K \cap N_H(K)_{\psi^{\mathcal{H}}} = K \cap H$. As $C_G(N) \leq H$,

$$C_{N_G(K)_{\psi^{\mathcal{H}}}}(K) \leq C_{N_G(K)_{\psi^{\mathcal{H}}}}(N) \leq H \cap N_G(K) = N_H(K). \tag{4.30}$$

Next, we check the equality $(N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_\psi = (N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_{\tau_K(\psi)}$. Let $h\sigma \in (N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_\psi$. By Clifford's Theorem, we have $\theta^{h\sigma} = \theta$, because $h\sigma$ fixes $\psi$ and the unique irreducible constituent of $\psi_N$ is $\theta$. Therefore $h\sigma \in (N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_\theta$. Since $\tau_K$ is $(N_H(K) \times \mathcal{H})_\theta$-equivariant, $\tau_K(\psi)^{h\sigma} = \tau_K(\psi^{h\sigma}) = \tau_K(\psi)$. Thus $h\sigma \in (N_H(K) \times \mathcal{H})_{\tau_K(\psi)}$. On the other hand, if $h\sigma \in (N_H(K) \times \mathcal{H})_{\tau_K(\psi)}$, then since $H_\theta = H_\varphi$ by assumption, we have $\varphi$ is $K \cap H$-invariant. Then by an application of Clifford's Theorem, $\varphi^{h\sigma} = \varphi$, and since $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$, we get $h\sigma \in (N_H(K) \times \mathcal{H})_\theta$. As $\tau_K$ is $(N_H(K) \times \mathcal{H})_\theta$-equivariant, it follows that $h\sigma \in (N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_\psi$.

Note that $N_G(K)_\psi = N_{G_\theta}(K)_\psi$, because if $x \in N_G(K)_\psi$, then by Clifford's Theorem, $x$ stabilizes $\theta$. In particular, $N_H(K)_\psi = N_{H_\theta}(K)_\psi$.

Suppose now that $(\mathcal{P}, \mathcal{P}')$ gives $(G, N, \theta)_{\mathcal{H}} \geq (H, M, \varphi)_{\mathcal{H}}$, and write $\alpha$, $\alpha'$ for their factor sets, respectively. Then there exists $\mathcal{Q} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(K/N \mid \alpha^{-1}_{K \times K})$ for which $\mathcal{Q} \otimes \mathcal{P}_K$ affords $\psi$ and $\mathcal{Q}_{K \cap H} \otimes \mathcal{P}_{K \cap H}$ affords $\tau_K(\psi)$. By Theorem 3.34, there is $\mathcal{R} \in \mathrm{Proj}_{\mathbb{Q}^{ab}}(N_G(K)_\psi \mid \beta)$ such that $\mathcal{R}_K = \mathcal{Q} \otimes \mathcal{P}_K$ and $\mathcal{R} = \mathcal{S} \otimes \mathcal{P}_{N_G(K)_\psi}$ for some $\mathcal{S} \in \mathrm{Proj}(N_G(K)_\psi \mid \gamma)$, where $\beta = \gamma\, \alpha^{-1}_{N_G(K)_\psi \times N_G(K)_\psi}$. Set $\mathcal{R}' = \mathcal{S}_{N_H(K)_\psi} \otimes \mathcal{P}'_{N_H(K)_\psi}$. Then

$\mathcal{R}' \in \mathrm{Proj}(N_H(K)_\psi \,|\, \beta')$, where $\beta' = \gamma_{N_H(K)_\psi \times N_H(K)_\psi}\,(\alpha')^{-1}_{N_H(K)_\psi \times N_H(K)_\psi}$. We claim that $(\mathcal{R},\, \mathcal{R}')$ gives

$$(N_G(K)_{\psi^{\mathcal{H}}},\, K,\, \psi)_{\mathcal{H}} \geq_c (N_H(K)_{\psi^{\mathcal{H}}},\, K \cap H,\, \tau_K(\psi))_{\mathcal{H}}. \tag{4.31}$$

As the entries of $\mathcal{R}$, $\mathcal{P}$ and $\mathcal{P}'$ are in $\mathbb{Q}^{ab}$, so are the entries of $\mathcal{S}$ and $\mathcal{R}'$. Also as $\alpha$, $\alpha'$ and $\beta$ only take roots of unity values, so do $\gamma$ and $\beta'$. Since $\mathcal{R}$ is associated with $\psi$, we note that $\mathcal{R}' = \mathcal{S}_{K\cap H} \otimes \mathcal{P}'_{K\cap H} = \mathcal{Q}_{K\cap H} \otimes \mathcal{P}'_{K\cap H}$ affords $\tau_K(\psi)$ and that

$$\beta'(h,\, k) = \gamma(h,\, k)\alpha'(h,\, k)^{-1} = \gamma(h,\, k)\alpha(h,\, k)^{-1} = \beta(h,\, k) = 1, \tag{4.32}$$

for every $h \in N_H(K)_\psi$, $k \in K \cap H$, similarly, $\beta'(k,\, h) = 1$. Therefore $\mathcal{R}'$ is associated with $\tau_K(\psi)$. By noting $N_H(K)_\psi \subseteq H_\theta$, we have

$$\beta(x,\, y) = \gamma(x,\, y)\alpha(x,\, y)^{-1} = \gamma(x,\, y)\alpha'(x,\, y)^{-1} = \beta'(x,\, y), \tag{4.33}$$

for every $x,\, y \in N_H(K)_\psi$. That is, $\beta_{N_H(K)_\psi \times N_H(K)_\psi} = \beta'$. Next, let $c \in C_{N_G(K)_{\psi^{\mathcal{H}}}}(K) \subseteq C_G(N)$. By hypothesis, $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar matrices and $\mathcal{R}(c)$ and $\mathcal{R}'(c)$ are also scalar. Thus $\mathcal{S}(c)$ is scalar. As $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated to the same scalar, by hypothesis, so are $\mathcal{R}(c)$ and $\mathcal{R}'(c)$.

Now, as $\psi(1) = \theta(1)$, by assumption, $\mathcal{S}$ can be seen as a function $N_G(K)_\psi \to \mathbb{Q}^{ab}$. Let $h\sigma \in (N_H(K)_{\psi^{\mathcal{H}}} \times \mathcal{H})_\psi$. Since $h\sigma$ stabilizes $\theta$, by Clifford's theorem, we have $\mathcal{P}^{h\sigma} \sim \nu_{h\sigma}\mathcal{P}$ as well as $(\mathcal{P}')^{h\sigma} \sim \nu'_{h\sigma}\mathcal{P}'$, for some $\mathbb{Q}^{ab}$-valued functions $\nu$ and $\nu'$. Also $\nu_{h\sigma}|_{H_\theta} = \nu'_{h\sigma}|_{H_\theta}$. Then

$$\begin{aligned} \mathcal{R}^{h\sigma}(x) = \mathcal{S}^{h\sigma}(x) \otimes \mathcal{P}^{h\sigma}(x) \sim \eta_{h\sigma}(x)\,(\mathcal{S}(x) \otimes \mathcal{P}(x)) \\ (\mathcal{R}')^{h\sigma}(x) = \mathcal{S}^{h\sigma}(x) \otimes (\mathcal{P}')^{h\sigma}(x) \sim \eta'_{h\sigma}(x)\,(\mathcal{S}(x) \otimes \mathcal{P}'(x)) \end{aligned} \tag{4.34}$$

where $\eta_{h\sigma}(x) = \dfrac{\nu_{h\sigma}(x)}{\mathcal{S}^{h\sigma}(x)}$ and $\eta'_{h\sigma}(x) = \dfrac{\nu'_{h\sigma}(x)}{\mathcal{S}^{h\sigma}(x)}$. Then we have

$$\eta_{h\sigma}|_{N_H(K)_\psi} = \eta'_{h\sigma}|_{N_H(K)_\psi}, \tag{4.35}$$

since $\nu_{h\sigma}|_{H_\theta} = \nu'_{h\sigma}|_{H_\theta}$ and $N_H(K)_\psi \subseteq H_\theta$.

□

# 5. GENERALIZATIONS OF THE MCKAY CONJECTURE

In this chapter, we first review some group theoretical facts. Then we implement the proof of the reduction of the McKay conjecture given in [3] to the Isaacs-Navarro conjecture. Note that this is actually established in [10], without using character triples. We'll use character triples to ease the notation and make some small changes in arguments. Finally, we discuss the Galois-McKay Conjecture.

## 5.1. Auxiliary Results for Reduction Theorems

First, we record a well known theorem, called the Frattini argument.

**Theorem 5.1** (Frattini argument, [11])**.** *Let $N \trianglelefteq G$ and let $p$ be a prime dividing $|N|$. If $P \in \mathrm{Syl}_p(N)$, then $G = NN_G(P)$.*

**Definition 5.2** ( [3])**.** *A group $G$ with order divisible by a prime $p$ is said to be $p$-solvable if there is a sequence*

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{5.1}$$

*such that $G_{i+1}/G_i$ has $p'$ or $p$-power order, for all $0 \leq i < n$.*

**Lemma 5.3.** *Let $N \trianglelefteq G$ with $p \nmid |N|$ and assume that $NP \trianglelefteq G$ for some $P \in \mathrm{Syl}_p(G)$. Then $G$ is $p$-solvable.*

*Proof.* Note that $\{1\} \trianglelefteq N \trianglelefteq NP \trianglelefteq G$ is a sequence with desired properties. □

All the following results in this section are taken from [3], except otherwise noted.

### 5.1.1. Central Products of Groups

In this subsection we define central products of groups and describe irreducible characters of them as the product of irreducible characters of these groups.

**Definition 5.4.** *Let $K_1,\, K_2,\, \ldots, K_n$ be a collection of subgroups of $G$ with $[K_i,\, K_j] = 1$ for every $1 \leq i \neq j \leq n$ and $n \geq 2$. Suppose that $G = K_1K_2 \cdots K_n$ and that $K_j \cap \left( \prod_{i \neq j} K_i \right) = \bigcap_{i=1}^{n} K_t$ for all $j$. Then $G$ is said to be the central product of $K_i$'s.*

Write $Z = \bigcap_{i=1}^{n} K_t$, then for any $z \in Z$ and $g = x_1 \cdots x_n \in G$, where $x_i \in K_i$, we have $g^{-1}zg = z$, as $[K_i,\, K_j] = 1$ and $z \in K_i$ for every $1 \leq i \leq n$. Next, we describe $\mathrm{Irr}(G \,|\, \vartheta)$ for any $\vartheta \in \mathrm{Irr}(Z)$.

**Theorem 5.5.** *Let $\vartheta \in \mathrm{Irr}(Z)$ and assume that $G = K_1 \cdots K_n$ is the central product of $K_i$'s. Then there exists a bijection*

$$\begin{aligned} \mathrm{Irr}(K_1 \,|\, \vartheta) \times \mathrm{Irr}(K_2 \,|\, \vartheta) \times \cdots \times \mathrm{Irr}(K_n \,|\, \vartheta) &\longrightarrow \mathrm{Irr}(G \,|\, \vartheta) \\ (\psi_1,\, \psi_2,\, \ldots,\, \psi_n) &\longmapsto \psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n, \end{aligned} \tag{5.2}$$

*where $\psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n$ is given by*

$$\big(\psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n\big)(x_1x_2 \cdots x_n) = \psi_1(x_1)\psi_2(x_2) \cdots \psi_n(x_n)$$

*for every $x_i \in K_i$. Moreover, $\psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n$ is the unique irreducible character of $G$ for which*

$$[(\psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n)_{K_i},\, \psi_i] \neq 0,$$

*for every $1 \leq i \leq n$.*

The following corollary discusses $n = 2$ case of the above theorem, with one subgroup being central.

**Corollary 5.6.** *Let $N \leq Z(G)$, $H \leq G$ such that $G = NH$, and let $\theta \in \mathrm{Irr}(N)$. Set $M = N \cap H$, then the map*

$$\begin{aligned} \mathrm{Irr}(G \,|\, \theta) &\longrightarrow \mathrm{Irr}(H \,|\, \theta_M) \\ \chi &\longmapsto \chi_H \end{aligned} \tag{5.3}$$

*is a bijection. Also we can write any $\chi \in \mathrm{Irr}(G \,|\, \theta)$ as $\chi = \theta \cdot \chi_H$.*

### 5.1.2. Simple Groups

**Definition 5.7.** *Let $H$ be a finite group.*

*(i) If $H = H'$, then $H$ is called perfect.*

*(ii) If $H$ is perfect and $H/Z(H)$ is simple, then $H$ is called quasisimple.*

**Theorem 5.8** (CFSG)**.** *Any finite simple group belongs to precisely one of the following sets:*

- *Cyclic groups of prime order,*
- *Alternating groups, $A_n$, with $n \geq 5$,*
- *Sporadic simple groups (there are 26 of them),*
- *Simple groups of Lie type.*

**Theorem 5.9** ( [11])**.** *Assume $H$ is a minimal normal subgroup of a finite group $G$, then $H \cong S^n$ for some simple group $S$ and $n \geq 1$.*

**Definition 5.10.** *(i) A pair $(H, \pi)$ is said to be a central extension of a group $S$ if $\pi : H \longrightarrow S$ is a surjection and $\ker \pi \subseteq Z(H)$.*

*(ii) A map $f : (H, \pi) \longrightarrow (K, \pi')$ is called a morphism of central extensions if $f : H \to K$ is a group homomorphism, $(H, \pi)$ and $(K, \pi')$ are central extensions of a group $S$ and $\pi' \circ f = \pi$.*

*(iii) Let $(H, \pi)$ be a central extension of $S$. Suppose that for any central extension $(K, \pi')$ of $S$, there is a unique morphism $f : (H, \pi) \longrightarrow (K, \pi')$. Then $(H, \pi)$ is called a universal (central) extension of $S$. In that case, $H$ is said to be a universal covering (group) of $S$.*

**Theorem 5.11.** *Let $S$ be a finite perfect group with a universal covering group $H$. Then $H$ is also perfect. If $S$ is non-abelian simple, then $H$ is quasisimple.*

**Notation 5.12.** *If $H \leq G$, then $\mathrm{Aut}(G)_H$ denotes the set of automorphisms of $G$ fixing $H$ setwise.*

**Theorem 5.13.** *Let $N$ be a perfect group and $N/Z(N) \cong S^n$ for some finite non-abelian simple group $S$ and $n \geq 1$. If $H$ is a universal covering of $S$, then $H^n$ is a* *universal covering of $N$.*

### 5.1.3. Action of $S_n$ on Product of Groups

If $1 \neq G$ is a finite group, then $S_n$ acts on $G^n$ by setting

$$\begin{aligned} S_n \times G^n &\longrightarrow G^n \\ (\sigma, (g_1, \ldots, g_n)) &\longmapsto (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}). \end{aligned} \tag{5.4}$$

**Lemma 5.14.** *Let $G$ be a finite quasisimple group, $H \leq G$ and $n \geq 1$.*

*(i)* $\mathrm{Aut}(G^n) = \mathrm{Aut}(G)^n \rtimes S_n$.
*(ii)* $\mathrm{Aut}(G^n)_{H^n} = (\mathrm{Aut}(G)_H)^n \rtimes S_n$.
*(iii) Whenever $G^n \leq N \leq G^n \rtimes \mathrm{Aut}(G^n)$ and $Z \subseteq Z(G^n)$ is $N$-invariant, then $C_{N/Z}(G^n/Z) = Z(G^n)/Z$.*

## 5.2. The Isaacs-Navarro Conjecture

In this section, first, we study some of the proved cases of the Isaacs-Navarro Conjecture. A reduction of this conjecture to non-abelian simple groups is actually established in [10], without using character triple orderings. We shall use the ideas in [3], in order to write a similar proof of that, with using character triple orderings.

**Notation 5.15.** *For a prime $p$ diving $|G|$ with $P \in \mathrm{Syl}_p(G)$, we write*

$$\mathrm{Irr}^k_{p'}(G) = \{\chi \in \mathrm{Irr}_{p'}(G) : \ \chi(1) \equiv \pm k \,(\mathrm{mod} p)\}, \tag{5.5}$$

*for any $1 \leq k < p$. Also if $H$ acts by automorphisms on $G$, then we denote by $\mathrm{Irr}^k_{p',H}(G)$, the set $\mathrm{Irr}^k_{p'}(G) \cap \mathrm{Irr}_H(G)$.*

**Conjecture 5.16** (The Isaacs-Navarro Conjecture, [3])**.** *Let $p$ be a prime dividing $|G|$ and let $P \in \mathrm{Syl}_p(G)$. Then*

$$|\,\mathrm{Irr}^k_{p'}(G)| = |\,\mathrm{Irr}^k_{p'}(N_G(P))|, \tag{5.6}$$

*for every $1 \leq k < p$.*

Whenever $G$ has a normal $p$-complement (that is, there exist $N \trianglelefteq G$, $p \nmid |N|$ and $P \in \mathrm{Syl}_p(G)$ such that $G = NP$) or $G$ is $p$-solvable, for some prime $p$ dividing $|G|$, this conjecture is true. We'll prove the case that $G$ has a normal $p$-complement, by following the ideas in [3]. First, we need a lemma.

**Lemma 5.17** (Corollary 6.3 in [3])**.** *Suppose $N \trianglelefteq G$ with $(|N|, |G : N|) = 1$, then every $G$-invariant irreducible character of $N$ extends to $G$.*

**Theorem 5.18.** *Let $p$ be a prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$. Suppose $N \trianglelefteq G$ with $p \nmid |N|$ and $G = NP$. Then $|\operatorname{Irr}^k_{p'}(G)| = |\operatorname{Irr}^k_{p'}(N_G(P))|$, for every $1 \leq k < p$.*

*Proof.* Fix some $k$ and $\psi \in \operatorname{Irr}^k_{p'}(G)$. By Proposition 3.13, if $\theta \in \operatorname{Irr}(\psi_N)$, then $\dfrac{\psi(1)}{\theta(1)}$ divides $|P|$, so that $\psi_N = \theta \in \operatorname{Irr}^k_{p'}(N)$. Also $\theta$ is $P$-invariant, as so is $\psi$. Then by Gallagher Correspondence, there is a bijection $\operatorname{Irr}(G/N) \longrightarrow \operatorname{Irr}(G \,|\, \theta)$, $\lambda \longmapsto \lambda\psi$. Therefore $\operatorname{Irr}^k_{p'}(G \,|\, \theta) = \{\lambda\psi \,:\, \lambda \in \operatorname{Lin}(G/N)\}$. As $\psi_N$ is irreducible, it follows that

$$\operatorname{Irr}^k_{p'}(G) = \bigsqcup_{\theta \in \operatorname{Irr}^k_{p',\,P}(N)} \operatorname{Irr}^k_{p'}(G \,|\, \theta). \tag{5.7}$$

Also $\operatorname{Irr}^k_{p'}(G \,|\, \theta)$ is non-empty for any $\theta \in \operatorname{Irr}^k_{p',\,P}(N)$, by Lemma 5.17, and in fact, such a $\theta$ extends to $G$. Thus

$$\begin{aligned} |\operatorname{Irr}^k_{p'}(G)| &= \textstyle\sum_{\theta \in \operatorname{Irr}^k_{p',\,P}(N)} |\{\lambda\psi_\theta \,:\, \lambda \in \operatorname{Lin}(G/N)\}| \\ &= |\operatorname{Irr}^k_{p',\,P}(N)| \cdot |\operatorname{Lin}(P)| \end{aligned} \tag{5.8}$$

where $\psi_\theta$ is an extension of $\theta$ to $G$ and we use $G/N \cong P$. Similarly,

$$|\operatorname{Irr}^k_{p'}(N_G(P))| = |\operatorname{Irr}^k_{p'}(C_N(P))| \cdot |\operatorname{Lin}(P)|, \tag{5.9}$$

because $N_G(P) \cong C_N(P) \times P$ in this setting. As $P$-Glauberman correspondence maps $\operatorname{Irr}^k_{p',\,P}(N)$ onto $\operatorname{Irr}^k_{p'}(C_N(P))$, the result follows. $\square$

The following theorem from [17] proves the Isaacs-Navarro Conjecture in $p$-solvable case. This result will be very useful in the proof of Theorem 5.29.

**Theorem 5.19.** *Let $p$ be a prime dividing $|G|$. If $G$ $p$-solvable and $P \in \mathrm{Syl}_p(G)$, then there exist bijections $\operatorname{Irr}^k_{p'}(G) \to \operatorname{Irr}^k_{p'}(N_G(P))$ for every $1 \leq k < p$.*

The following theorem is just a refinement of Theorem 10.17 in [3].

**Theorem 5.20.** *Let $p$ be a prime, $Z \subseteq Z(G)$, $\zeta \in \mathrm{Irr}(Z)$ and assume that $(G, N, \theta) \geq_c (H, M, \varphi)$. Suppose that $p$ does not divide $\theta(1)$ and $\varphi(1)$, and that $\theta$ and $\varphi$ lie over $\zeta_{Z\cap M}$. Then for every $1 \leq k < p$, there exists a bijection*

$$\mathrm{Irr}^k_{p'}(G \,|\, \zeta \cdot \theta) \longrightarrow \mathrm{Irr}^l_{p'}(H \,|\, \zeta \cdot \varphi), \tag{5.10}$$

*where* $l \equiv k\dfrac{\varphi(1)}{\theta(1)} \,(\mathrm{mod}\, p)$.

*Proof.* By Theorem 10.17 in [3], there is a bijection $\mathrm{Irr}_{p'}(G \,|\, \zeta \cdot \theta) \longrightarrow \mathrm{Irr}_{p'}(H \,|\, \zeta \cdot \varphi)$, given by $\psi \longmapsto \tau_G(\psi)$, where $\tau_G$ is as in Theorem 3.25, and $\dfrac{\psi(1)}{\theta(1)} = \dfrac{\tau_G(\psi)(1)}{\varphi(1)}$. Then $\psi(1) \equiv \pm k \,(\mathrm{mod}\, p)$ if and only if $\tau_G(\psi)(1) \equiv \pm k\dfrac{\varphi(1)}{\theta(1)} \,(\mathrm{mod}\, p)$. □

**Theorem 5.21** ( [3] and [14])**.** *Suppose that $(G, N, \theta) \geq_c (H, M, \varphi)$ and let $c_g : N \longrightarrow N$, $n \longmapsto g^{-1}ng$, where $g \in G$. Also let $\hat{G}$ be a finite group such that $N \trianglelefteq \hat{G}$. Define $\epsilon : G \longrightarrow \mathrm{Aut}(N)$ and $\hat{\epsilon} : \hat{G} \longrightarrow \mathrm{Aut}(N)$, by $g \longmapsto c_g$. If $\epsilon(G) = \hat{\epsilon}(\hat{G})$, then*

$$(\hat{G}, N, \theta) \geq_c (\hat{\epsilon}^{-1}(\epsilon(H)), M, \varphi). \tag{5.11}$$

The following definition is a generalization of Definition 10.23 in [3]. Also this can be found in [10].

**Definition 5.22.** *Let $p$ be a prime and let $S$ be a non-abelian simple group. Also let $P$ be a Sylow $p$-subgroup of a universal covering group $G$ of $S$. We say that $S$ satisfies iINp if*

*(i)* *there exists an $\mathrm{Aut}(G)_P$-invariant subgroup $N_G(P) \leq M < G$,*

*(ii)* *there exist $\mathrm{Aut}(G)_P$-equivariant bijections*

$$\Omega_k : \mathrm{Irr}^k_{p'}(G) \longrightarrow \mathrm{Irr}^k_{p'}(M) \tag{5.12}$$

*for every $1 \leq k < p$, and*

*(iii)* *$(G \rtimes (\mathrm{Aut}(G)_P)_\theta, G, \theta) \geq_c (M \rtimes (\mathrm{Aut}(G)_P)_\theta, M, \Omega_k(\theta))$ for every $\theta \in \mathrm{Irr}^k_{p'}(G)$ and $1 \leq k < p$.*

Note that whenever $P \trianglelefteq G$, $M$ can be taken to be $G$. However, in this case, the result is trivial and we omit it. Also we disregard this situation in the case of the Galois-McKay Conjecture. Also Part (ii) of this definition is equivalent to say that there is an $\mathrm{Aut}(G)_P$-equivariant bijection $\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(M)$ such that $\Omega(\theta)(1) \equiv \pm\theta(1)(\mathrm{mod} p)$ for all $\theta \in \mathrm{Irr}_{p'}(G)$. Also, in (iii), we'll consider $\Omega$ instead of $\Omega_k$.

**Theorem 5.23** ( [3]). *Let $H \leq G$ and $A \leq \mathrm{Aut}(G)_H$. Suppose that $\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(H)$ is an $A$-equivariant bijection and that $(G \rtimes A_\theta, G, \theta) \geq_c (H \rtimes A_\theta, H, \Omega(\theta))$ for every $\theta \in \mathrm{Irr}_{p'}(G)$. Let $B = A^n \rtimes S_n$. Then there exists a $B$-equivariant bijection*

$$\begin{aligned} \Omega^n &: \mathrm{Irr}_{p'}(G^n) \longrightarrow \mathrm{Irr}_{p'}(H^n) \\ \chi = \chi_1 \times \cdots \times \chi_n &\longmapsto \Omega(\chi_1) \times \cdots \times \Omega(\chi_n) \end{aligned} \tag{5.13}$$

*and $(G^n \rtimes B_\chi, G^n, \chi) \geq_c (H^n \rtimes B_\chi, H^n, \Omega^n(\chi))$ for every $\chi \in \mathrm{Irr}_{p'}(G)$.*

Notice that if we also assume $\Omega(\theta)(1) \equiv \pm\theta(1)(\mathrm{mod} p)$ for every $\theta \in \mathrm{Irr}_{p'}(G)$, then $\Omega^n(\chi)(1) \equiv \pm\chi(1)(\mathrm{mod} p)$ for every $\chi \in \mathrm{Irr}_{p'}(G^n)$, by construction.

### 5.3. A Reduction Theorem for the Isaacs-Navarro Conjecture

For reducing the conjecture to a question on non-abelian simple groups, we first investigate the case where $G$ is perfect. The results below are just refinements of the proofs of Theorem 10.25 and Theorem 10.26 of [3] and we give their proofs for completeness. We first need a lemma.

**Lemma 5.24.** *Fix a prime $p$. Let $G$ be a perfect group with $p \big| |G|$ and $G/Z(G) \cong S^n$ for some non-abelian simple group $S$ and $n \geq 1$. For a universal covering $H$ of $S$, let $Q \in \mathrm{Syl}_p(H)$. As $H^n$ is a universal covering group of $G$ (by using Theorem 5.13), there exists a surjective map $\pi : H^n \longrightarrow G$ with $K = \ker \pi \subseteq Z(H^n)$. Suppose that $S$ satisfies iINp, with the Malle intermediate subgroup $N_H(Q) \subseteq N < H$. Then there exists an $\mathrm{Aut}(H^n)_{Q^n, K}$-equivariant bijection*

$$\Lambda : \mathrm{Irr}_{p'}(H^n/K) \longrightarrow \mathrm{Irr}_{p'}(N^n/K) \tag{5.14}$$

*such that* $\Lambda(\theta)(1) \equiv \pm\theta(1) \pmod p$ *and*

$$(H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, H^n/K,\, \theta) \geq_c (N^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, N^n/K,\, \Lambda(\theta)) \tag{5.15}$$

*for every* $\theta \in \mathrm{Irr}_{p'}(H^n/K)$*, where* $Aut(H^n)_{Q^n,K}$ *is the set containing automorphisms in* $Aut(H^n)_{Q^n}$ *fixing* $K$ *setwise.*

*Proof.* As $S$ satisfies iIN$p$, there exists an $\mathrm{Aut}(H)_Q$-equivariant bijection

$$\Omega\,:\,\mathrm{Irr}_{p'}(H) \longrightarrow \mathrm{Irr}_{p'}(N) \tag{5.16}$$

such that $\Omega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and

$$(H \rtimes (\mathrm{Aut}(H)_Q)_\theta,\, H,\, \theta) \geq_c (N \rtimes (\mathrm{Aut}(H)_Q)_\theta,\, N,\, \Omega(\theta)) \tag{5.17}$$

for every $\theta \in \mathrm{Irr}_{p'}(H)$. Since $N$ is $\mathrm{Aut}(H)_Q$-invariant, by hypothesis, we have that $\mathrm{Aut}(H)_Q \subseteq \mathrm{Aut}(H)_N$. Then by Theorem 5.23, there is a one-to-one correspondence

$$\Omega^n\,:\,\mathrm{Irr}_{p'}(H^n) \longrightarrow \mathrm{Irr}_{p'}(N^n) \tag{5.18}$$

which is $(\mathrm{Aut}(H)_Q)^n \rtimes S_n = \mathrm{Aut}(H^n)_{Q^n}$-equivariant (since $H$ is quasisimple) and $\Omega^n(\theta)(1) \equiv \pm\theta(1) \pmod p$ and

$$(H^n \rtimes (\mathrm{Aut}(H^n)_{Q^n})_\theta,\, H^n,\, \theta) \geq_c (N^n \rtimes (\mathrm{Aut}(H^n)_{Q^n})_\theta,\, N^n,\, \Omega^n(\theta)) \tag{5.19}$$

for every $\theta \in \mathrm{Irr}_{p'}(H^n)$. In particular, as $(\mathrm{Aut}(H^n)_{Q^n,K})_\theta \leq (\mathrm{Aut}(H^n)_{Q^n})_\theta$,

$$(H^n \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, H^n,\, \theta) \geq_c (N^n \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, N^n,\, \Omega^n(\theta)) \tag{5.20}$$

for every $\theta \in \mathrm{Irr}_{p'}(H^n)$. Then since $\mathrm{Irr}(\theta_{Z(H^n)}) = \mathrm{Irr}(\Omega^n(\theta)_{Z(H^n)})$, by using Lemma 3.27 and the relation $K \subseteq Z(H^n)$, we obtain that $K \subseteq \ker\theta$ if and only if $K \subseteq \ker\Omega^n(\theta)$. Thus $\Omega^n$ induces a bijection

$$\begin{aligned}\Lambda\,:\mathrm{Irr}_{p'}(H^n/K) &\longrightarrow \mathrm{Irr}_{p'}(N^n/K),\\ \theta &\longmapsto \Omega^n(\theta)\end{aligned} \tag{5.21}$$

by viewing $\theta \in \mathrm{Irr}_{p'}(H^n/K)$ as $\theta \in \mathrm{Irr}_{p'}(H^n)$ with $K \subseteq \ker\theta$. Notice that $\Lambda(\theta)(1) \equiv \pm\theta(1) \pmod p$. Finally, since $K \subseteq \ker\theta \cap \ker\Lambda(\theta)$ for every $\theta \in \mathrm{Irr}_{p'}(H^n/K)$ and $C_{H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta}(H^n/K) = Z(H^n)/K = C_{H^n \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta}(H^n)/K$, we obtain

$$(H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, H^n/K,\, \theta) \geq_c (N^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, N^n/K,\, \Lambda(\theta)), \tag{5.22}$$

for every $\theta \in \mathrm{Irr}_{p'}(H^n/K)$, by applying Lemma 3.28 to (5.20). Note that we identify $(H^n \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta)/K$ by $H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta$, as $K \cong K \rtimes 1$. □

**Theorem 5.25.** *Fix a prime $p$. Let $G \trianglelefteq A$ be finite groups with $p \big| |G|$, $G = G'$ and $G/Z(G) \cong S^n$ for some non-abelian simple group $S$ and $n \geq 1$. Let $R \in \mathrm{Syl}_p(G)$. Suppose that $S$ satisfies iINp. Then there exists an $N_A(R)$-invariant subgroup $N_G(R) \subseteq M < G$ and an $N_A(R)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(M) \tag{5.23}$$

*such that $\Omega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and*

$$(A_\theta,\, G,\, \theta) \geq_c (N_A(M)_{\Omega(\theta)},\, M,\, \Omega(\theta)) \tag{5.24}$$

*for all $\theta \in \mathrm{Irr}_{p'}(G)$.*

*Proof.* Let $H$ be a universal covering group of $S$ and let $(H^n, \pi)$ be a universal central extension of $G$, by using Theorem 5.13. Set $K = \ker \pi$. This gives an isomorphism

$$\nu : G \longrightarrow H^n/K. \tag{5.25}$$

Since $R \in \mathrm{Syl}_p(G)$, we have $\nu(R) \in \mathrm{Syl}_p(H^n/K)$. Write $\nu(R) = Q^nK/K$ for some $Q \in \mathrm{Syl}_p(H)$. As $S$ satisfies iIN$p$, there exists an $\mathrm{Aut}(H)_Q$-invariant subgroup $N_H(Q) \subseteq N < H$. Then by using Lemma 5.24, there exists an $\mathrm{Aut}(H^n)_{Q^n,K}$-equivariant bijection

$$\Lambda : \mathrm{Irr}_{p'}(H^n/K) \longrightarrow \mathrm{Irr}_{p'}(N^n/K) \tag{5.26}$$

such that for every $\theta \in \mathrm{Irr}_{p'}(H^n/K)$, we have $\Lambda(\theta)(1) \equiv \pm\theta(1) \pmod p$ and

$$(H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, H^n/K,\, \theta) \geq_c (N^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta,\, N^n/K,\, \Lambda(\theta)). \tag{5.27}$$

Note that $\mathrm{Aut}(H^n)_{Q^n,K}$ and $\mathrm{Aut}(H^n/K)_{Q^nK/K}$ are isomorphic by the map $\sigma \longmapsto \tilde{\sigma}$, with $\tilde{\sigma}(Kx) := K\sigma(x)$ and that both of these groups act on any $\mathrm{Aut}(H^n)_{Q^n,K}$-invariant subgroup of $H^n/K$ in the same way. As a result, $H^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta \cong H^n/K \rtimes (\mathrm{Aut}(H^n/K)_{Q^nK/K})_\theta$ and $N^n/K \rtimes (\mathrm{Aut}(H^n)_{Q^n,K})_\theta \cong N^n/K \rtimes (\mathrm{Aut}(H^n/K)_{Q^nK/K})_\theta$ and (5.27) turns out to be

$$(H^n/K \rtimes \Gamma_\theta,\, H^n/K,\, \theta) \geq_c (N^n/K \rtimes \Gamma_\theta,\, N^n/K,\, \Lambda(\theta)), \tag{5.28}$$

where $\Gamma = \mathrm{Aut}(H^n/K)_{Q^nK/K}$.

Notice that as $N$ is $\mathrm{Aut}(H)_Q$-invariant, $N^n$ is $(\mathrm{Aut}(H)_Q)^n \rtimes S_n = \mathrm{Aut}(H^n)_{Q^n}$-invariant (the equality holds since $H$ is quasisimple) and therefore $N^n/K$ is invariant under the action of $\mathrm{Aut}(H^n)_{Q^n,K}$. Thus $M := \nu^{-1}(N^n/K)$ is $\mathrm{Aut}(G)_R$-equivariant. Also since $K \subseteq Z(H^n)$, we have

$$N_{H^n/K}(Q^nK/K) = N_H(Q)^n/K \subseteq N^n/K, \tag{5.29}$$

and $\nu(N_G(R)) \subseteq N^n/K$, so that $M \supseteq N_G(R)$. Then considering the preimages of $\nu$ and the bijection $\Lambda$, there is an $\mathrm{Aut}(G)_R$-equivariant bijection

$$\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(M) \tag{5.30}$$

such that for every $\theta \in \mathrm{Irr}_{p'}(G)$, we have $\Omega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and

$$(G \rtimes (\mathrm{Aut}(G)_R)_\theta,\, G,\, \theta) \geq_c (M \rtimes (\mathrm{Aut}(G)_R)_\theta,\, M,\, \Omega(\theta)). \tag{5.31}$$

Set $\hat{A} := G \rtimes \mathrm{Aut}(G)_R$. Then $N_{\hat{A}}(M) = N_G(M) \rtimes \mathrm{Aut}(G)_R = M \rtimes \mathrm{Aut}(G)_R$, by the Frattini argument. Then (5.31) can be re-written as

$$(\hat{A}_\theta,\, G,\, \theta) \geq_c (N_{\hat{A}}(M)_\theta,\, M,\, \Omega(\theta)). \tag{5.32}$$

Define homomorphisms

$$\begin{array}{rl} \epsilon : A \longrightarrow \mathrm{Aut}(G), & \hat{\epsilon} : \hat{A} \longrightarrow \mathrm{Aut}(G) \\ x \longmapsto c_x & x \longmapsto c_x \end{array}, \tag{5.33}$$

where $c_x$ is the conjugation map. Then $\epsilon(G) = \hat{\epsilon}(G)$, so that $\epsilon(A) = \epsilon(GN_A(R)) \subseteq \hat{\epsilon}(\hat{A})$. Finally, the result follows, by applying the argument in the last part of the proof of Theorem 10.25 in [3]. Also see the proof of Theorem 5.38 for a similar construction. □

**Lemma 5.26** (Lemma 9.3 in [3]). *Let $p$ be a prime dividing $|G|$. If $P \in \mathrm{Syl}_p(G)$, $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}_{p'}(G)$, then $\chi_N$ has a $P$-invariant irreducible constituent. Moreover, if $\theta$ and $\psi$ are such constituents, then there exists $x \in N_G(P)$ for which $\theta^x = \psi$.*

**Lemma 5.27.** *Assume $N = HZ \trianglelefteq G$ with $H \trianglelefteq G$ and $Z \leq Z(G)$. Let $\mu \in \mathrm{Irr}(Z)$ and $P \in \mathrm{Syl}_p(G)$. If $\mathcal{S}$ is a transversal of the action of $N_G(P)$ on $\mathrm{Irr}_{p',P}(H \mid \mu_{H\cap Z})$, then*

$$\mathrm{Irr}^k_{p'}(G \mid \mu) = \bigsqcup_{\theta \in \mathcal{S}} \mathrm{Irr}^k_{p'}(G \mid \mu \cdot \theta), \tag{5.34}$$

*for every $1 \leq k \leq p-1$.*

*Proof.* Note that, by Corollary 5.6, we have $\mathrm{Irr}(N \mid \mu) = \{\mu \cdot \theta \,:\, \theta \in \mathrm{Irr}(H \mid \mu_{H\cap Z})\}$. Let $\chi \in \mathrm{Irr}_{p'}(G \mid \mu)$, then $\chi_N$ has a $P$-invariant irreducible constituent and any two such constituents are $N_G(P)$-conjugate, by Lemma 5.26. Also, as $\mu$ is $G$-invariant, all the irreducible constituents of $\chi_N$ lie over $\mu$. Therefore

$$\chi_N = \mu \cdot \theta_1 + \cdots + \mu \cdot \theta_n \tag{5.35}$$

for some $\theta_i \in \mathrm{Irr}_{p'}(H \mid \mu_{H\cap Z})$, as all $\theta_i$'s have the same degree. Since $\mu$ is $G$-invariant and $(\mu \cdot \theta)^x = \mu^x \cdot \theta^x$, for any $x \in G$, we have that all $\theta_i$'s are $N_G(P)$-conjugate, by Lemma 5.26. Since some $\theta_i$ is $P$-invariant, if $\theta_j = \theta_i^x$ for some $x \in N_G(P)$, then for any $y \in P$, $\theta_j^y = \theta_i^{xp} = \theta_i^{(xpx^{-1}x)} = \theta_i^x = \theta_j$. Thus it follows that all $\theta_i$'s are $P$-invariant.

On the other hand, if $\chi \in \mathrm{Irr}_{p'}(G \mid \mu \cdot \theta)$, then $\chi \in \mathrm{Irr}_{p'}(G \mid \mu)$, by noting $\mu \cdot \theta$ lies over $\mu$. Moreover, if $\chi \in \mathrm{Irr}_{p'}(G \mid \mu \cdot \theta) \cap \mathrm{Irr}_{p'}(G \mid \mu \cdot \psi)$ for some $\theta$, $\psi \in \mathcal{S}$, then by the above argument, $\theta$ is $N_G(P)$-conjugate to $\psi$ and therefore $\theta = \psi$. Then the result follows. □

**Definition 5.28** ( [3])**.** *Suppose that $S$ is a simple group with $S \cong H/N$ for some $N \trianglelefteq H \leq G$. Then $S$ is said to be involved in $G$.*

**Theorem 5.29.** *Let $p$ be a prime dividing $|G|$, $Z \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$. Assume that $\mu \in \mathrm{Irr}_{p',P}(Z)$ and that every non-abelian simple group $S$ involved in $G/Z$ satisfies the iINp. Then*

$$|\,\mathrm{Irr}^k_{p'}(G \mid \mu)| = |\,\mathrm{Irr}^k_{p'}(N_G(P)Z \mid \mu)| \tag{5.36}$$

*for every $1 \leq k < p$.*

*Proof.* We'll proceed by induction on the index of $Z$ in $G$.

*Step 1: We can assume $G = G_\mu$, $Z \leq N_G(P)$ and $\mu$ is linear.*

As $P \subseteq (N_G(P)Z)_\mu \subseteq G_\mu$, by Clifford correspondence, we obtain

$$|\,\mathrm{Irr}^k_{p'}(G_\mu \mid \mu)| = |\,\mathrm{Irr}^l_{p'}(G \mid \mu)| \quad \text{and} \quad |\,\mathrm{Irr}^k_{p'}((N_G(P)Z)_\mu \mid \mu)| = |\,\mathrm{Irr}^m_{p'}(N_G(P)Z \mid \mu)|, \tag{5.37}$$

where $l \equiv k|G : G_\mu| \pmod p$ and $m \equiv k|(N_G(P)Z) : (N_G(P)Z)_\mu| \pmod p$. Note that

$$\frac{|N_G(P)Z|}{|(N_G(P)Z)_\mu|} = \frac{|N_G(P) : N_{G_\mu}(P)|}{|N_G(P) \cap Z : N_{G_\mu}(P) \cap Z|} = |N_G(P) : N_{G_\mu}(P)|, \tag{5.38}$$

since if $x \in N_G(P) \cap Z$, then $x \in G_\mu$, as $\mu$ is a class function of $Z$. Also we have

$$|G : N_G(P)| \equiv |G_\mu : N_{G_\mu}(P)| \equiv 1 \pmod p, \tag{5.39}$$

so that $l \equiv m \pmod p$. Therefore we may assume that $\mu$ is $G$-invariant.

By Theorem 3.12, there exists a character triple $(\tilde{G}, \tilde{Z}, \tilde{\mu})$ isomorphic to $(G, Z, \mu)$ such that $\tilde{Z} \leq Z(\tilde{G})$ and $\tilde{\mu}(1) = 1$. Let $\tilde{} : G/Z \longrightarrow \tilde{G}/\tilde{Z}$ be the map as in Definition 3.10. Then $\widetilde{N_G(P)Z/Z} = N_{\tilde{G}}(\tilde{P})/\tilde{Z}$ and there exist bijections

$$\mathrm{Irr}^k_{p'}(G \,|\, \mu) \longrightarrow \mathrm{Irr}^{\frac{k}{\mu(1)} \bmod p}_{p'}(\tilde{G} \,|\, \tilde{\mu}) \tag{5.40}$$

and

$$\mathrm{Irr}^k_{p'}((N_G(P)Z) \,|\, \mu) \longrightarrow \mathrm{Irr}^{\frac{k}{\mu(1)} \bmod p}_{p'}(N_{\tilde{G}}(\tilde{P}) \,|\, \tilde{\mu}), \tag{5.41}$$

by noting that $p \nmid \mu(1)$ and $\tilde{\mu}$ is linear. Thus we can assume that $Z \leq Z(G) \leq N_G(P)$ and that $\mu$ is linear.

Now, let $N/Z$ be a minimal normal subgroup of $G/Z$. By Theorem 5.9, $N/Z \cong S^n$ for some simple group $S$ and $n \geq 1$.

*Step 2: We have $PN = NP \trianglelefteq G$.*

As $N \trianglelefteq G$, $N_G(P)$ acts on $\mathrm{Irr}_{p', P}(N \,|\, \mu)$, then set $\mathcal{T}$ to be a transversal of this action. If $\chi \in \mathrm{Irr}_{p'}(G \,|\, \mu)$, $\chi_N$ has an irreducible constituent $\theta \in \mathrm{Irr}_{p', P}(N \,|\, \mu)$, by Lemma 5.26 and Clifford's Theorem. Therefore $\chi \in \mathrm{Irr}_{p'}(G \,|\, \theta)$. Also again by Lemma 5.26, if $\psi \in \mathrm{Irr}_{p', P}(N \,|\, \mu) \cap \mathrm{Irr}(\chi_N)$, then $\psi = \theta^x$ for some $x \in N_G(P)$. On the other hand, if $\chi \in \mathrm{Irr}_{p'}(G \,|\, \theta)$ for some $\theta \in \mathcal{T}$, then we have $\chi \in \mathrm{Irr}_{p'}(G \,|\, \mu)$, as $\theta$ lies over $\mu$. Thus we obtain

$$\mathrm{Irr}^k_{p'}(G \,|\, \mu) = \bigsqcup_{\theta \in \mathcal{T}} \mathrm{Irr}^k_{p'}(G \,|\, \theta) \tag{5.42}$$

and similarly

$$\operatorname{Irr}_{p'}^{k}(N_G(P)N \mid \mu) = \bigsqcup_{\theta \in \mathcal{T}} \operatorname{Irr}_{p'}^{k}(N_G(P)N \mid \theta), \tag{5.43}$$

for every $1 \leq k < p$. Notice now that if $S$ is involved in $G/N$, that is there exist subgroups $A/N \trianglelefteq B/N \leq G/N$ for which $S \cong B/N \big/ A/N \cong B/A$, then $A/Z \trianglelefteq B/Z \leq G/Z$ and $S \cong B/Z \big/ A/Z$ and therefore $S$ is involved in $G/Z$. As $|G : N| < |G : Z|$, by induction, we have

$$|\operatorname{Irr}_{p'}^{k}(G \mid \theta)| = |\operatorname{Irr}_{p'}^{k}(N_G(P)N \mid \theta)|, \tag{5.44}$$

for every $1 \leq k \leq p-1$. Then combining the above equations, we get

$$|\operatorname{Irr}_{p'}^{k}(G \mid \mu)| = |\operatorname{Irr}_{p'}^{k}(N_G(P)N \mid \mu)|, \tag{5.45}$$

since the unions in (5.42) and (5.43) are disjoint. Now, if $S$ is involved in $N_G(P)N/Z$, then so is in $G/Z$. As $|N_G(P)N : Z| < |G : Z|$, by induction, we have

$$\left|\operatorname{Irr}_{p'}^{k}(N_G(P)N \mid \mu)\right| = \left|\operatorname{Irr}_{p'}^{k}(N_G(P) \mid \mu)\right| \tag{5.46}$$

for every $1 \leq k \leq p-1$, and the result establishes. Hence we can assume $N_G(P)N = G$, that is, $PN \trianglelefteq G$.

*Step 3: We can assume $N/Z$ is non-abelian and $p \nmid \left|N/Z\right|$.*

If $N/Z$ is abelian, not a $p$-group and $p \big| |N/Z|$, then $N/Z$ has a proper characteristic Sylow p-subgroup and it follows that $N/Z$ cannot be minimal. Thus we can assume $N/Z$ is a $p$-group. Then the sequence of subgroups

$$ZP/Z \leq NP/Z \leq G/Z \tag{5.47}$$

implies that $ZP = NP$, because $ZP/Z \in \operatorname{Syl}_p(G/Z)$. Therefore $N_G(P) = G$, as $P$ is a characteristic subgroup of $ZP$. If $p \nmid |N/Z|$, then $G$ becomes $p$-solvable, by Lemma 5.3, and this case is established in Theorem 5.19.

*Final Step: If $N/Z \cong S^n$ for some non-abelian simple group $S$ with $p \big| |S|$ and $n \geq 1$, then the result follows.*

As $N/Z \cong S^n$, we have that $N/Z$ is perfect. Therefore $N'Z/Z = N/Z$ and

$N'Z = N$. Write $H = N'$ and notice $H \trianglelefteq G$. As $Z \leq Z(G)$, $[H, Z] = 1$ and therefore $N$ is the central product of $H$ and $Z$. Since $Z(H)/H \cap Z$ is an abelian normal subgroup of $H/H \cap Z$ and $N/Z = HZ/Z \cong H/H \cap Z \cong S^n$, we have that $Z(H) = H \cap Z$ and that $H/Z(H)$ is also perfect. By repeating the same argument, $H'Z(H) = H$ and $H'Z = N$. Also $N/H'$ is abelian. Since $N'$ is the minimal normal subgroup of $N$ with abelian quotient, $H \leq H'$ and therefore $H$ is perfect. Note that $R := H \cap P \in \mathrm{Syl}_p(H)$. Then for any $x \in N_G(P)$, we have $x^{-1}Rx = x^{-1}Hx \cap x^{-1}Px = R$ as $H \trianglelefteq G$. Therefore $N_G(P) \subseteq N_G(R)$. By the Frattini argument, $G = HN_G(R) = HN_G(P)$.

If $\chi \in \mathrm{Irr}(N \,|\, \mu)$, then $\chi_H \in \mathrm{Irr}(H \,|\, \mu_{Z(H)})$ and $\chi = \mu \cdot \chi_H$, by using Corollary 5.6. Also conversely, if $\theta \in \mathrm{Irr}(H \,|\, \mu_{Z(H)})$, then $\mu \cdot \theta \in \mathrm{Irr}(N \,|\, \mu)$. Thus we have

$$\mathrm{Irr}(N \,|\, \mu) = \{\mu \cdot \theta \,:\, \theta \in \mathrm{Irr}(H \,|\, \mu_{Z(H)})\}. \tag{5.48}$$

Let $\mathcal{S}$ be a transversal of the action of $N_G(P)$ on $\mathrm{Irr}_{p',\,P}(H \,|\, \mu_{Z(H)})$. Then by appealing to Lemma 5.27, we have

$$\mathrm{Irr}^k_{p'}(G \,|\, \mu) = \bigsqcup_{\theta \in \mathcal{S}} \mathrm{Irr}^k_{p'}(G \,|\, \mu \cdot \theta) \tag{5.49}$$

for every $1 \leq k < p$. Then by Theorem 5.25, there is an $N_G(R)$-invariant subgroup $N_H(R) \leq M < H$ and an $N_G(R)$-equivariant bijection

$$\varOmega \,:\, \mathrm{Irr}_{p'}(H) \longrightarrow \mathrm{Irr}_{p'}(M) \tag{5.50}$$

such that $\varOmega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and

$$(G_\theta,\, H,\, \theta) \geq_c (N_G(M)_{\varOmega(\theta)},\, M,\, \varOmega(\theta)) \tag{5.51}$$

for every $\theta \in \mathrm{Irr}_{p'}(H)$. Then by Theorem 5.20, there are bijections

$$\mathrm{Irr}^k_{p'}(G_\theta \,|\, \mu \cdot \theta) \longrightarrow \mathrm{Irr}^{k\frac{\varOmega(\theta)(1)}{\theta(1)} \bmod p}_{p'}(N_G(M)_{\varOmega(\theta)} \,|\, \mu \cdot \varOmega(\theta)) \tag{5.52}$$

for every $\theta \in \mathrm{Irr}_{p'}(H \,|\, \mu_{Z(H)})$ and $1 \leq k < p$.

Because of $N_G(P) \subseteq N_G(R)$, $M$ is $N_G(P)$-invariant and $\varOmega$ is $N_G(P)$-equivariant. By the Frattini argument, $N_G(M) = MN_G(R) = MN_G(P)$. Then for any $\theta \in \mathrm{Irr}_{p'}(H)$ and $x = my \in N_G(M)_{\varOmega(\theta)}$ with $m \in M$, $y \in N_G(R)$, we have

$$\varOmega(\theta)^x = \varOmega(\theta)^y = \varOmega(\theta^y) = \varOmega(\theta^x), \tag{5.53}$$

since $\varOmega$ is $N_G(R)$-equivariant. Since $\varOmega$ is a bijection, we get $N_G(M)_{\varOmega(\theta)} = N_G(M)_\theta$.

Next, we claim that $\varOmega(\mathcal{S}) = \{\varOmega(\theta) : \theta \in \mathcal{S}\}$ is a transversal of the action of $N_G(P)$ on $\operatorname{Irr}_{p',P}(M \mid \mu_{M\cap Z})$. Note that by Lemma 3.27, $\operatorname{Irr}(\theta_{Z\cap H}) = \operatorname{Irr}(\varOmega(\theta)_{Z\cap H})$ and in particular we have $\operatorname{Irr}(\theta_{M\cap Z}) = \operatorname{Irr}(\varOmega(\theta)_{M\cap Z})$, for every $\theta \in \mathcal{S}$. Therefore

$$\theta \in \operatorname{Irr}_{p',P}(H \mid \mu_{M\cap Z}) \iff \varOmega(\theta) \in \operatorname{Irr}_{p',P}(M \mid \mu_{M\cap Z}),. \tag{5.54}$$

by recalling that $\varOmega$ is $N_G(P)$-equivariant. Notice then that $\varOmega(\theta_1) = \varOmega(\theta_2) \in \varOmega(\mathcal{S})$ if and only if $\theta_1 = \theta_2 \in \mathcal{S}$. Hence our claim follows, and then by Lemma 5.27, we obtain

$$\operatorname{Irr}^k_{p'}(N_G(M) \mid \mu) = \bigsqcup_{\theta\in\mathcal{S}} \operatorname{Irr}^k_{p'}(N_G(M) \mid \mu \cdot \varOmega(\theta)), \tag{5.55}$$

since $MZ \trianglelefteq N_G(M)$, $Z \subseteq Z(N_G(M))$ and $N_{N_G(M)}(P) = N_G(P)$.

Next, combining the above equalities and using the Clifford Correspondence,

$$\begin{aligned} |\operatorname{Irr}^k_{p'}(G \mid \mu)| &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^k_{p'}(G \mid \mu\cdot\theta)| \\ &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^{k_1}_{p'}(G_{\mu\cdot\theta} \mid \mu\cdot\theta)| \\ &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^{k_1}_{p'}(G_\theta \mid \mu\cdot\theta)| \\ &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^{k_2}_{p'}(N_G(M)_{\varOmega(\theta)} \mid \mu\cdot\varOmega(\theta))| \\ &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^{k_2}_{p'}(N_G(M)_{\mu\cdot\varOmega(\theta)} \mid \mu\cdot\varOmega(\theta))| \\ &= \textstyle\sum_{\theta\in\mathcal{S}} |\operatorname{Irr}^{k_3}_{p'}(N_G(M) \mid \mu\cdot\varOmega(\theta))| \\ &= |\operatorname{Irr}^{k_3}_{p'}(N_G(M) \mid \mu)| \end{aligned} \tag{5.56}$$

where

$$\begin{aligned} k_1 &\equiv \frac{k}{|G : G_\theta|} \pmod p \\ k_2 &\equiv k_1 \frac{\varOmega(\theta(1))}{\theta(1)} \pmod p \\ k_3 &\equiv k_2 |N_G(M) : N_G(M)_\theta| \pmod p \end{aligned} \tag{5.57}$$

and we note that $N_G(M)_{\mu\cdot\varOmega(\theta)} = N_G(M)_{\varOmega(\theta)} = N_G(M)_\theta$ and $G_{\mu\cdot\theta} = G_\theta$. Also notice that

$$k_3 \equiv k \frac{\varOmega(\theta)(1)}{\theta(1)} \cdot \frac{|G_\theta : N_{G_\theta}(M)|}{|G : N_G(M)|} \equiv k \pmod p, \tag{5.58}$$

since $\varOmega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and $|G : N_G(M)| = \frac{|G : N_G(P)|}{|N_G(M) : N_{N_G(M)}(P)|} \equiv 1 \pmod p$ and similarly $|G_\theta : N_{G_\theta}(M)| \equiv 1 \pmod p$. Thus it suffices to establish

$$|\operatorname{Irr}^k_{p'}(N_G(M) \mid \mu)| = |\operatorname{Irr}^k_{p'}(N_G(P) \mid \mu)|. \tag{5.59}$$

By noting that $M \cap N_G(P) \supseteq H \cap N_G(P)$, as $H \cap N_G(P) \leq N_H(R) \subseteq M < H$, we have $|N_G(M) : N_G(P)| < |HN_G(P) : N_G(P)| \leq |G : N_G(P)|$. Therefore $N_G(M) < G$. Also note that for any $S$ involved in $N_G(M)/Z$, $S$ is involved in $G/Z$. Finally, by using induction, as $|N_G(M) : Z| < |G : Z|$, we obtain the result. □

By using the main result in [1], this conjecture can be generalized as follows:

**Conjecture 5.30.** *Let $H \trianglelefteq G$ be finite groups and $R \in \mathrm{Syl}_p(H)$. There exists an $N_G(R)$-invariant subgroup $N_H(R) \subseteq M < H$ and an $N_G(R)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(H) \longrightarrow \mathrm{Irr}_{p'}(M) \tag{5.60}$$

*satisfying for every $\theta \in \mathrm{Irr}_{p'}(H)$, that $\Omega(\theta)(1) \equiv \pm\theta(1) \pmod p$ and that*

$$(G_\theta,\, H,\, \theta) \geq_c (N_G(M)_\theta,\, M,\, \Omega(\theta)). \tag{5.61}$$

The following Corollary follows from Theorem 5.21, Theorem 5.29 and the main result in [1].

**Corollary 5.31.** *If Conjecture 5.30 is true for every universal covering of simple groups, then it is also true for every finite group.*

### 5.4. A Reduction of the Galois-McKay Conjecture

**Conjecture 5.32** (The Galois-McKay Conjecture, [8]). *If $p$ is a prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$, then there is an $\mathcal{H}$-equivariant bijection $\mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(N_G(P))$.*

**Definition 5.33** ( [8]). *We shall write $\mathrm{Irr}^{\mathrm{rel}}_{p'}(G \,|\, \theta)$ for the set $\{\chi \in \mathrm{Irr}_{p'}(G \,|\, \theta) : p \nmid \frac{\chi(1)}{\theta(1)}\}$.*

**Definition 5.34** ( [8]). *For any non-abelian simple group $S$ with order divisible by a prime $p$, let $P$ be a Sylow $p$-subgroup of a universal covering group $G$ of $S$. Suppose that*

*(i) there exists an $\mathrm{Aut}(G)_P$-invariant subgroup $N_G(P) \leq M < G$,*

*(ii) there exists an* $\mathrm{Aut}(G)_P \times \mathcal{H}$*-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(M), \tag{5.62}$$

*and*

*(iii)* $(G \rtimes (\mathrm{Aut}(G)_P)_{\theta^{\mathcal{H}}}, G, \theta)_{\mathcal{H}} \geq_c (M \rtimes (\mathrm{Aut}(G)_P)_{\theta^{\mathcal{H}}}, M, \Omega(\theta))_{\mathcal{H}}$ *for every* $\theta \in \mathrm{Irr}_{p'}(G)$.

*Then S is said to satisfy iGMp.*

Note that taking $Z = 1$ in the next theorem, we recover the Galois-McKay Conjecture.

**Theorem 5.35** ( [8]). *Fix a prime p which divides* $|G|$. *Let* $P \in \mathrm{Syl}_p(G)$, $Z \trianglelefteq G$ *and* $\mu \in \mathrm{Irr}_P(Z)$. *Suppose that every non-abelian simple group S involved in* $G/Z$ *satisfies iGMp. Then there is a bijection*

$$\mathrm{Irr}^{\mathrm{rel}}_{p'}(G \,|\, \mu) \longrightarrow \mathrm{Irr}^{\mathrm{rel}}_{p'}(N_G(P)Z \,|\, \mu) \tag{5.63}$$

*which commutes with the action of* $\mathcal{H}$.

The proof of this theorem can be found in [8]. It follows a similar idea as that of Theorem 5.29. However, as character triple isomorphisms are not appropriate anymore, new constructions are used and the steps of the proof is adapted to the set $\mathrm{Irr}^{\mathrm{rel}}_{p'}(G \,|\, \mu)$. Therefore this conjecture reduces to checking the conditions in Definition 5.34 for every universal covering of simple groups. We generalize this conjecture, inspired by [1].

**Conjecture 5.36.** *Let* $H \trianglelefteq G$ *be finite groups,* $R \in \mathrm{Syl}_p(H)$. *There is an* $N_G(R)$*-invariant subgroup* $N_H(R) \subseteq M < H$ *and an* $N_G(R) \times \mathcal{H}$*-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(H) \longrightarrow \mathrm{Irr}_{p'}(M) \tag{5.64}$$

*satisfying for every* $\theta \in \mathrm{Irr}_{p'}(H)$ *that*

$$(G_{\theta^{\mathcal{H}}}, H, \theta)_{\mathcal{H}} \geq_c (N_G(M)_{\theta^{\mathcal{H}}}, M, \Omega(\theta))_{\mathcal{H}}. \tag{5.65}$$

Next lemma is Theorem 2.9 in [8] and it is rewritten by following its proof. This will be useful in the proof of Theorem 5.38.

**Lemma 5.37.** *Suppose* $(G, N, \theta)_{\mathcal{H}} \geq_c (H, M, \varphi)_{\mathcal{H}}$ *and let* $\epsilon$, $\hat{\epsilon}$ *and* $\hat{G}$ *be as in Theorem 5.21. If* $MC_{\hat{G}}(N) \subseteq \hat{H} \leq \hat{G}$ *and* $\hat{\epsilon}(\hat{H}) = \epsilon(H)$, *then*

$$(\hat{G}, N, \theta)_{\mathcal{H}} \geq_c (\hat{H}, M, \varphi)_{\mathcal{H}}. \tag{5.66}$$

A similar result to the next theorem is proven at the end of the proof of Theorem 3.3 in [8]. We give a complete proof for that.

**Theorem 5.38.** *Conjecture 5.36 is true for every* $H \trianglelefteq G$ *if and only if it is true for* $H \trianglelefteq H \rtimes \mathrm{Aut}(H)_R$.

*Proof.* Suppose the conjecture is true with respect to $H \trianglelefteq H \rtimes \mathrm{Aut}(H)_R$ and let $G$ be a group with $H \trianglelefteq G$. Define homomorphisms

$$\epsilon : H \rtimes \mathrm{Aut}(H)_R \longrightarrow \mathrm{Aut}(H), \qquad \hat{\epsilon} : G \longrightarrow \mathrm{Aut}(H) \tag{5.67}$$

by conjugation. By hypothesis, there exists an $N_{H\rtimes \mathrm{Aut}(H)_R}(R)$-invariant subgroup $M$ and an $N_{H\rtimes \mathrm{Aut}(H)_R}(R) \times \mathcal{H}$-equivariant bijection $\Omega$. Notice that $\epsilon(H) = \hat{\epsilon}(H)$ and

$$\hat{\epsilon}(G) = \hat{\epsilon}(HN_G(R)) \subseteq \epsilon(H \rtimes \mathrm{Aut}(H)_R), \tag{5.68}$$

as $\hat{\epsilon}(N_G(R)) \subseteq \mathrm{Aut}(H)_R$ and $\hat{\epsilon}$ is a homomorphism. Let $x \in N_G(R)$, then there exists some $y \in H \rtimes \mathrm{Aut}(H)_R$ for which

$$c_x = \hat{\epsilon}(x) = \epsilon(y) = c_y. \tag{5.69}$$

As $x$ and $y$ act in the same way on any subgroup of $H$ and on their characters, $M$ is $N_G(R)$-invariant and $\Omega$ is $N_G(R) \times \mathcal{H}$-equivariant. Note also that $x$ and $y$ act on $\theta^{\mathcal{H}}$ in the same manner, because if $\theta^x = \theta^\sigma$ for some $\sigma \in \mathcal{H}$, then naturally $\theta^y = \theta^\sigma$.

Now, suppose that for any $\theta \in \mathrm{Irr}_{p'}(H)$ we have

$$(H \rtimes (\mathrm{Aut}(H)_R)_{\theta^{\mathcal{H}}}, H, \theta)_{\mathcal{H}} \geq_c (N_{H\rtimes \mathrm{Aut}(H)_R}(M)_{\theta^{\mathcal{H}}}, M, \Omega(\theta))_{\mathcal{H}}. \tag{5.70}$$

Set $B = \{x \in H \rtimes \mathrm{Aut}(H)_R : \epsilon(x) = \hat{\epsilon}(y)$ for some $y \in G_{\theta^{\mathcal{H}}}\}$. Note that by the above argument and its construction, $\epsilon(B) = \hat{\epsilon}(G_{\theta^{\mathcal{H}}})$. As $B \subseteq (H \rtimes \mathrm{Aut}(H)_R)_{\theta^{\mathcal{H}}} =$

$H \rtimes (\mathrm{Aut}(H)_R)_{\theta^{\mathcal{H}}}$, we have

$$(B,\, H,\, \theta)_{\mathcal{H}} \geq_c (N_B(M)_{\theta^{\mathcal{H}}},\, M,\, \Omega(\theta))_{\mathcal{H}}. \tag{5.71}$$

Finally, observe that

$$\begin{aligned} \epsilon(x) \in \epsilon(N_B(M)_{\theta^{\mathcal{H}}}) &\iff \text{there exists } y \in G_{\theta^{\mathcal{H}}} \text{ normalizing } M \text{ with } \epsilon(x) = \hat{\epsilon}(y) \\ &\iff \hat{\epsilon}(y) \in \hat{\epsilon}(N_G(M)_{\theta^{\mathcal{H}}}) \end{aligned} \tag{5.72}$$

As $MC_{G_{\theta^{\mathcal{H}}}}(H) \subseteq N_G(M)_{\theta^{\mathcal{H}}}$, it follows by Lemma 5.37, that

$$(G_{\theta^{\mathcal{H}}},\, H,\, \theta)_{\mathcal{H}} \geq_c (N_G(M)_{\theta^{\mathcal{H}}},\, M,\, \Omega(\theta))_{\mathcal{H}}. \tag{5.73}$$

□

# 6. CONCLUSION

In this thesis, we studied Isaacs-Navarro and Galois-McKay conjectures, with their reductions to finite simple groups. We reproved a reduction theorem of the Isaacs-Navarro conjecture, by using character triples as in [3], with considering the necessary Malle intermediate subgroups. Then, inspired by [1], we generalized the concept of the Galois-McKay Conjecture to include a certain $\mathcal{H}$-triple ordering. We believe that the same conditions on simple groups imply this generalization. Also, by proceeding as in [1], we proved some $\mathcal{H}$-triple relations in Chapter 4. These will be necessary to construct a similar proof for the above claim. For a future study, we plan to work on a proof of this. The challenge in this construction is mainly studying actions on $\mathcal{H}$-orbits of characters and checking Condition (iv) in Definition 4.7.

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