BLACK HOLES IN AN EXPANDING UNIVERSE

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ABSTRACT

BLACK HOLES IN AN EXPANDING UNIVERSE

By performing transformations on the isotropic forms of different metrics, new cosmological black hole solutions were obtained by generalizing the static solution, the Schwarzchild solution, the McVittie and the Kerr solutions. We present cosmological black holes solutions that satisfy the known behaviours of the scale factor through the history of our Universe, the Radiation, Matter and Dark Energy dominated eras. In the static case we presented solutions satisfying all of the energy conditions, then a time generalization and analysis of the previous isotropic solution gave us the behaviour in time of the Schwarzschild black hole solution. By a choice of parameter, we obtained a non-singular equation of state parameter even though the energy density and pressure tend to infinity at the horizon. In that case the solution suggests that in the expanding universe, the equation of state parameter is negative and may eventually be satisfied by a dark energy dominated representation of the Universe. For the classical McVittie and Kerr solutions, an inductive reasoning led us to obtain all the solutions, Radiation, Matter and Dark energy dominated eras, by introducing two conformal factors and generalizing the metrics. Finally, the Weyl curvature and the volume expansion are calculated for the cosmological black holes and the effects, increases or decreases, of the volume expansions for all the cases are discussed. It is shown at the end that a multitude of black holes can impact the Universe as a whole by increasing or decreasing its volume.

ÖZET

GENİŞLEYEN EVRENDE KARA DELİKLER

Farklı metriklerin izotropik formlarına dönüşümler yaparak, yeni kozmolojik karadelik çözümleri elde edildi; statik, Schwarzchild, McVittie ve Kerr çözümlerinin genelleştirilmesiyle bulundu. Evrenimizin tarihi boyunca, ölçek faktörünün, radyasyon, madde ve kara enerjinin hakim olduğu dönemlerdeki bilinen davranışlarını sağlayan kozmolojik karadelik çözümleri sunduk. Statik durumda, enerji koşulların tümünü sağlayan çözümler sunduk. Sonra, önceki izotropik çözümün bir zaman genellemesini ve analizini yaparak, sonuç bize Schwarzschild kara delik çözümünün zamana bağlı davranışını verdi. Bir parametre seçim ile, enerji yoğunluğunun ve basıncın ufukta sonsuza eğilimli olsalar bile, hal denklemi parametresi tekil olmayan bir sonuç elde ettik. Bu durumda çözüm, genişleyen evrende, hal denklemi parametresinin negatif olması gerektiğini ve sonunda Evrenin karanlık enerji dönemi ile temsili olabileceğini gördük. Klasik McVittie ve Kerr çözümleri için, bir endüktif muhakeme yaparak, iki konformal faktör tanımlayarak ve metrikleri genelleştirerek, radyasyon, madde ve karanlık enerjinin hakim olduğu dönemlerdeki tüm çözümler elde edildi. Sonunda, Weyl eğriliği ve hacim genleşmesi kozmolojik kara delikler için hesaplandı. Etkileri, artış ya da azalmalar, tüm vakalar için hesaplandı. Bu karadelikler, çok sayıda olduklarında, hacimsel olarak, bir bütün olarak Evreni etkileyebilicekleri gösterildi.

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LIST OF SYMBOLS

$a(t)$	Scale Factor
$\dot{a}$	Time derivative of $a(t)$
$b(t)$	Conformal Factor
$\dot{b}$	Time derivative of $b(t)$
c	Speed of light in vacuum
$C^{\mu}_{\nu\sigma\tau}$	Weyl Curvature Tensor
$g_{\mu\nu}$	Metric Tensor
$\tilde{g}_{\mu\nu}$	Conformally Transformed Metric Tensor
$G_{\mu\nu}$	Einstein Tensor
G_N, G	Newtonian Gravitational Constant
$H, H(t)$	Hubble Parameter
K	Curvature of Space
k	Killing Vector Field
k^{α}	Arbitrary future-directed null vector
$\mathcal{L}_X$	Lie Derivative w.r.t Vector Field X
$\mathcal{M}, \mathcal{N}$	Topological Manifolds
p	Pressure
$R_{\mu\nu}$	Ricci Tensor
R	Ricci Curvature Scalar
$R^{\mu}_{\nu\sigma\tau}$	Riemann Tensor
S	Entropy
S^2	2-Sphere
S^3	3-Sphere
$T_{\mu,\sigma}$	$\partial_\sigma T_\mu$
$T_{\mu\nu}$	Energy-Momentum Tensor
$T_p\mathcal{M}$	Tangent Space to $\mathcal{M}$ at p
T	Tensor Field
T	Temperature

T_∞	Temperature at $r \to \infty$
V, W	Vector Fields
V	Volume
v^α	arbitrary vector field
x^μ, y^α	Coordinate Chart Maps
X	Vector Field
$\eta_{\mu\nu}$	Minkowski Metric Tensor
Λ	Cosmological Constant
ϕ	Map
ϕ^*	Pullback Map
ϕ_*	Pushforward Map
ρ	Energy Density
ρ_c	Critical Density
Σ	Hypersurface
$\Omega(t)$	Density Parameter
$\Omega(x)$	Conformal Factor
∇_μ	Covariant Derivative
∂	Partial Derivative

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
3D	Three Dimensional
FRW	Friedmann–Robertson–Walker
FLRW	Friedmann–Lemaître–Robertson–Walker
SCU	Swiss Cheese Universe
LTB	Lemaître-Tolman-Bondi

1. INTRODUCTION

General Relativity is vital for understanding systems as diluted as the Universe when the deviation from flatness is significant either due to high mass concentration or when large scales are involved. Cosmological black holes are of interest either as examples of non-isolated, time-dependent black hole solutions or as inhomogeneous cosmological models that help with the interpretation of the effects of those inhomogeneities on the universe. The Universe is not homogeneous on all scales, still totally homogeneous archetypes are regularly used to represent them, either with the thought that the general evolution of a universe with these inhomogeneities will not vary from that of a completely smooth universe, or with the belief that the expansion of the universe is governed by a perfectly homogeneous spatial expansion regardless of local inhomogeneities in the matter density.

General Relativity's Exact Solutions [1–5] have some definitions issues. On one hand, an analysis on the solutions of the Einstein equations with a right hand side can be made by setting directly the form of the energy-momentum tensor. For example, a spherically symmetric solution of the field equations can represent a star. On the other hand, one can introduce some structure, merely geometrical to a spacetime and search for a matter source that could explain these attributes. From the first way, the energy-momentum tensor should emerge from a proper, justifiable matter distribution or non-gravitational field. We would like to have a criterion, preferably purely mathematical to apply to the pretended "energy-momentum tensor". Unfortunately, no such depiction exist. Alternatively, we check for the energy conditions. These are analogous to limiting the eigenvalues and eigenvectors of a linear operator. On first glance, they are soft, meaning that they allow "solutions" that are not physically tenable, well-founded. They are also constraining, e.g. the strong energy condition is violated in any cosmological inflationary scenario [6, 7]. Moreover these energy conditions are seemingly violated by the Casimir effect [8]. Einstein recognized that an exact solution needed additional structure, a Lorentzian smooth manifold.

The problem of finding black hole solutions of the Einstein field equations in an expanding universe is a very important one, and while much has been written about the question, a convincing answer is still lacking. The trouble is that any metric at all is a "solution" to the Einstein field equations, in the sense that one can take an arbitrary metric, compute the Einstein tensor, and simply declare that the resulting "energy-momentum tensor" is the source. For the result to be physically interesting, though, one must do much more. One must show that the "energy-momentum tensor" one obtains this way is an actual energy-momentum tensor of a physically realistic configuration of physically reasonable matter. What is needed is a careful analysis of the energy-momentum tensor implied by the Einstein tensor.

In this chapter, spacetime, symmetry and isotropy will be discussed to serve as a scenery for this thesis, and then after presenting the standard model of cosmology, and the static and non-static spherically symmetric solutions, the motivation for studying black hole solutions will be discussed in further detail. The specific background knowledge and previous work necessary to understanding the work in this thesis will appear in Chapter 2, and then cosmological black hole solutions will be reviewed in Chapter 3. In Chapter 4, a static cosmological black hole solution and in Chapter 5 non-static black holes in a matter-energy background will be presented. Finally, the effect of cosmological black holes on the expansion of the universe will be examined in Chapter 6.

1.1. Spacetime

Spacetime is a topological manifold $\mathcal{M}$ in four dimensions, equipped with a smooth atlas, a torsion free connection and a time-orientation, on which there is a defined Lorentzian metric $g_{\mu\nu}$. The curvature of $g_{\mu\nu}$ is related to the matter distribution in spacetime by Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

The sign convention used in this thesis is signature $(-+++)$ and geometrized units set as $(G_N = c = 1)$, where G_N is the Newtonian gravitational constant, c the speed of light in vacuum and Λ the cosmological constant taken to be zero unless stated otherwise. Einstein's equations [1, 3, 4] may be thought of as second-order differential equations for the metric tensor field $g_{\mu\nu}$. By symmetry, these are ten independent equations, for the ten unknown functions of the metric components.

Solutions to Einstein's Field Equations can either be found by working on a metric $g_{\mu\nu}$ and calculating the left hand side of the field equations which will determine then the energy-momentum tensor $T_{\mu\nu}$ or by starting with the latter and working backwards in order to obtain the metric that corresponds to it. By working on a metric we may not obtain a physical energy-momentum tensor as this tensor may not satisfy energy conditions. On the other hand, determining the spacetime corresponding to a particular mass-energy distribution, starting with the energy-momentum tensor, doesn't easily allow the metric to be calculated.

From the twice contracted Bianchi identity, we obtain $\nabla_\mu G^{\mu\nu} = 0$. We can rewrite this equation as:

$$\partial_0 G^{0\nu} = -\partial_i G^{i\nu} - \Gamma^{\mu}_{\mu\lambda} G^{\lambda\nu} - \Gamma^{\nu}_{\mu\lambda} G^{\mu\lambda} \tag{1.2}$$

At first sight, the right hand side discloses that there are no third-order time derivatives; consequently there cannot be any on the left hand side. Thus, although $G^{\mu\nu}$ as a whole involves second-order time derivatives of the metric, the specific components $G^{0\nu}$ contain at most first-order derivatives. So the four equations represented by $G^{0\nu}$ rather constrain the initial data and the remaining six equations are the dynamical evolution equations.

1.2. Symmetry

1.2.1. Mathematical Aspects

Definitions. [9–11] Let us consider two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ with coordinate charts $x^\mu : \mathcal{M} \to \mathbb{R}^m$ and $y^\alpha : \mathcal{N} \to \mathbb{R}^n$, a map $\phi : \mathcal{M} \to \mathcal{N}$ and a function $f : \mathcal{N} \to \mathbb{R}$.

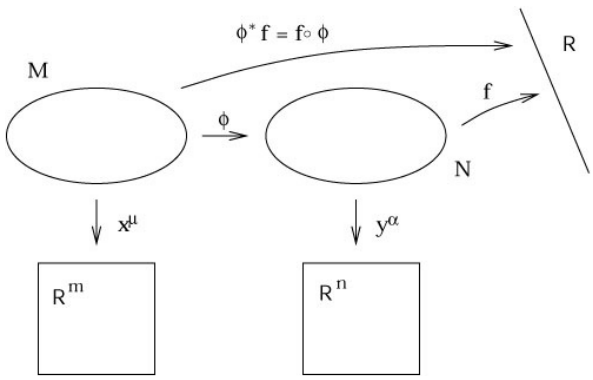

Figure 1.1. Pullback Map

We can compose ϕ with f to construct a map $(f o\, \phi) : \mathcal{M} \to \mathbb{R}$, which is simply a function on $\mathcal{M}$. We define the pullback of f by ϕ, denoted $\phi^* f$, by $\phi^* f = f o\, \phi$. If we have a function $g : \mathcal{M} \to \mathbb{R}$, there is no way we can compose g with ϕ to create a function on $\mathcal{N}$; but a vector can be thought of as a derivative operator that maps smooth functions to real numbers. If $V(p)$ is a vector at a point p on $\mathcal{M}$, we define the pushforward vector $\phi_* V$ at the point $\phi(p)$ on $\mathcal{N}$ by giving its action on functions on $\mathcal{N}$: $(\phi_* V)(f) = V(\phi_* f)$.

If ϕ is invertible and ϕ and ϕ^{-1} are smooth (which we always implicitly assume), then it defines a diffeomorphism between $\mathcal{M}$ and $\mathcal{N}$. In this case $\mathcal{M}$ and $\mathcal{N}$ are the same abstract manifold. The beauty of diffeomorphisms is that we can use both ϕ and ϕ^{-1} to move tensors from $\mathcal{M}$ to $\mathcal{N}$; that will allow us to define pushforward and pullback of arbitrary tensors. This provides another way of comparing tensors at different points on a manifold.

We say that a diffeomorphism ϕ is a symmetry of some tensor T if the tensor is invariant after being pulled back under ϕ, $\phi^*T = T$. Although symmetries may be discrete, it is more common to have a one-parameter family of symmetries ϕ_t. If the family is generated by a vector field V then the latter equation becomes $\mathcal{L}_V T = 0$ where $\mathcal{L}_X T_{\mu\nu}$ is the Lie derivative of the tensor $T_{\mu\nu}$ with respect to the vector field X.

The Lie derivative of the tensor $T_{\mu\nu}$ with respect to the vector field X is explicitly:

$$\mathcal{L}_X T_{\mu\nu} = T_{\mu\sigma} X^{\sigma}_{,\nu} + T_{\sigma\nu} X^{\nu}_{,\mu} + T_{\mu\nu,\sigma} X^{\sigma}$$

The most important symmetries are those of the metric, for which $\phi^* g_{\mu\nu} = g_{\mu\nu}$. A diffeomorphism of this type is called an isometry.

1.2.2. Spacetime symmetries

Killing Symmetry. A Killing vector field is one of the most important types of symmetries and is defined to be a smooth vector field that preserves the metric tensor: $\mathcal{L}_X g_{\mu\nu} = 0$, usually written in the form $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$.

Homothetic Symmetry. A homothetic vector field satisfies: $\mathcal{L}_X g_{\mu\nu} = 2c g_{\mu\nu}$ where c is a real constant.

Affine Symmetry. An affine vector field is one that satisfies: $\nabla_c(\mathcal{L}_X g_{\mu\nu}) = 0$. An affine vector field preserves geodesics and preserves the affine parameter. The above three vector field types are special cases of projective vector fields which preserve geodesics without necessarily preserving the affine parameter.

Conformal Symmetry. A conformal vector field satisfies: $\mathcal{L}_X g_{\mu\nu} = \phi \, g_{\mu\nu}$ where ϕ is a smooth real valued function on $\mathcal{M}$.

Curvature Symmetry. A curvature collineation is a vector field which preserves the Riemann tensor: $\mathcal{L}_X R^{\mu}_{\nu\sigma\tau} = 0$ where $R^{\mu}_{\nu\sigma\tau}$ are the components of the Riemann tensor.

Matter Symmetry. A less well-known form of symmetry concerns vector fields that preserve the energy-momentum tensor. These are variously referred to as matter collineations or matter symmetries and are defined by: $\mathcal{L}_X T_{\mu\nu} = 0$ where $T_{\mu\nu}$ are the energy-momentum tensor components.

Local and Global Symmetries. A local symmetry is symmetry of some physical quantity, which smoothly depends on the point of the base manifold. Such quantities can be for example an observable, a tensor or the Lagrangian of a theory. A global symmetry is a symmetry that holds at all points in the spacetime under consideration. Global symmetries require conservation laws, but not forces, in physics. General relativity has a local symmetry (general covariance, diffeomorphisms) which can be seen as generating the gravitational force. Special relativity only has a global symmetry (Lorentz symmetry or more generally Poincaré symmetry)

1.3. Isotropy

Qualitatively, isotropy refers to the absence of preferred directions in space.

Definition. A manifold $\mathcal{M}$ is isotropic around a point p if, for any two vectors V and W in $T_p\mathcal{M}$, there is an isometry of $\mathcal{M}$ such that the pushforward of W under the isometry is parallel with V (not pushed forward).

It is the isotropy of the Universe which is indicated by the observations of the cosmic microwave background.

Homogeneity is the statement that the metric is the same throughout the space. In other words, given any two points p and q in $\mathcal{M}$, there is an isometry which takes p into q. Note that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as $\mathbb{R} \times S^2$ in the usual metric), or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic everywhere then it is homogeneous.

Looking at distant galaxies, they appear to be receding from us; the universe is apparently not static, but changing with time. Therefore there is a tendency to use cosmological models with a universe that is homogeneous and isotropic in space, but not in time.

1.4. Friedmann–Lemaître–Robertson–Walker Cosmology

In cosmology a common approximation is that there is a slicing of spacetime into spacelike slices which are exactly homogeneous and isotropic. This means that there exists a coordinate system in which the $t =$ const hypersurfaces are homogeneous and isotropic. The proper time t which labels the hypersurfaces is called the cosmic time. The observations do not prove that the universe would be well described by a model which is exactly homogeneous and isotropic. But for the time being we will present the FLRW spacetime [12, 13], spatially homogeneous and isotropic, its curvature being the same at all points in space, but which can vary in time.

Noting K, the curvature of space, the FLRW metric is:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{1.3}$$

For $K = +1$, the space geometry at constant time is that of a 3-sphere, positively curved. The total volume of the universe is finite, though it grows in proportion to $a(t)^3$. For $K = 0$, the space geometry at constant time is Euclidean, also known as flat space. Space is infinite. For $K = -1$, the space geometry at constant time is that of a negatively curved, 3-dimensional "pseudo-sphere." Space is infinite. The non-vanishing components of the Einstein tensor are:

$$G^t_t = -3(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}) \tag{1.4}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}) \tag{1.5}$$

Let us rearrange these equations for the case of a perfect fluid form for the energy tensor as:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.6}$$

$$\text{with } T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p) \tag{1.7}$$

Isotropy implies that the fluid is at rest in the FLRW coordinates, so that $u_\mu = (1, 0, 0, 0)$ and (remember, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$). Homogeneity implies that $\rho = \rho(t)$ and $p = p(t)$. The Einstein equations become:

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G_N \rho}{3} \tag{1.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \tag{1.9}$$

These are the Friedmann equations. The general relativity version of energy and momentum conservation, energy-momentum continuity, follows from the Einstein equation. In the present case this becomes the energy continuity equation:

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \tag{1.10}$$

Since the fluid is at rest, there is no equation for the momentum.

We define $H = \frac{\dot{a}}{a}$. This quantity $H = H(t)$ gives the expansion rate of the universe, and it is called the Hubble parameter [14, 15]. Its present value H_0 is the Hubble constant. The dimension of H is 1/time (or velocity/distance). In time dt a distance gets stretched by a factor $1 + Hdt$ (a distance L grows with velocity HL). The Friedmann equation (1.7) connects the three quantities, the density ρ, the space curvature $\frac{K}{a^2}$, and the expansion rate H of the universe,

$$\rho = \frac{3}{8\pi G_N}(H^2 + \frac{K}{a^2}) = \rho_c + \frac{3K}{8\pi G_N a^2} \tag{1.11}$$

Here we have defined the critical density

$$\rho_c = \frac{3H^2}{8\pi G_N} \tag{1.12}$$

corresponding to a given value of the Hubble parameter. Defined this way, the critical density changes as the Hubble parameter evolves. Usually, by critical density we mean its present value, given by the value of the Hubble constant,

$$\rho_c = \rho_c(t_0) = \frac{3H_0^2}{8\pi G_N} \tag{1.13}$$

The nature of the curvature then depends on the density ρ:

$$\rho < \rho_c \implies K < 0 \tag{1.14}$$

$$\rho = \rho_c \implies K = 0 \tag{1.15}$$

$$\rho > \rho_c \implies K > 0 \tag{1.16}$$

The density parameter $\Omega(t)$ is defined as:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \tag{1.17}$$

Thus $\Omega = 1$ implies a flat, $\Omega < 1$ an open, and $\Omega > 1$ a closed universe.

The Friedmann equation can now be written as:

$$\Omega(t) = 1 + \frac{K}{Ha^2} \tag{1.18}$$

a very useful relation. Here K is a constant, and the other quantities are functions of time $\Omega(t)$, $H(t)$, and $a(t)$. To solve the Friedmann equations, we need the equation of state $p(\rho)$. The simplest cases are:

- "Matter" (called "matter" in cosmology, but "dust" in general relativity), meaning non-relativistic matter (particle velocities $v \ll 1$), for which $p \ll \rho$, so that we can forget the pressure, and approximate $p = 0$. From Eq. (1.9), $d(\rho a^3)/dt = 0$, or $\rho \propto a^{-3}$.

- "Radiation", meaning ultra-relativistic matter (where particle energies are $\gg$ their rest masses, which is always true for massless particles like photons), for which $p = \rho/3$. From Eq. (1.9), $d(\rho a^4)/dt = 0$, or $\rho \propto a^{-4}$.

- Vacuum energy (or the cosmological constant), for which $\rho = const$. From Eq. (1.9) follows the equation of state for vacuum energy: $p = -\rho$. Thus a positive vacuum energy corresponds to a negative vacuum pressure.

1.5. Static Spherically Symmetric Solutions

1.5.1. Time Independence

A spacetime $(\mathcal{M}, g)$ is stationary if it admits a timelike Killing vector field. Denoting k a timelike killing vector field and using adapted coordinates (t, x^i) so that $k = \frac{\partial}{\partial t}$, then $g_{\mu\nu}$ is independent of t:

$$ds^2 = g_{00}(x^l)dt^2 + 2g_{0i}(x^l)dtdx^i + g_{ij}(x^l)dx^i dx^j \text{ with } g_{00} < 0 \tag{1.19}$$

$(\mathcal{M}, g)$ is static if it admits a hypersurface-orthogonal timelike Killing vector field.

For a static spacetime, choose some surface Σ orthogonal to k and introduce adapted coordinates (t, x^i).

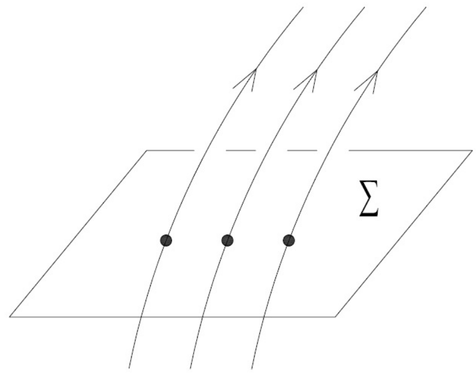

Figure 1.2. Hypersurface Σ

In these coordinates, Σ is the surface $t = 0$. Hence dt is normal to Σ, and k must be proportional to dt at $t = 0$. From the line element (1.19), we see that the covector dual to k has the form:

$$k = g_{00}(x^l)dt + g_{0i}(x^l)dx^i \tag{1.20}$$

We conclude that:

$$g_{0i}(x^l) = 0 \tag{1.21}$$

So in adapted coordinates (t, x^i), a static metric takes the form

$$ds^2 = g_{00}(x^l)dt^2 + g_{ij}(x^l)dx^i dx^j \text{ with } g_{00} < 0 \tag{1.22}$$

1.5.2. Spherical Symmetry

The standard metric on a unit two-sphere is:

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{1.23}$$

This has an $SO(3)$ isometry group. A spacetime is spherically symmetric if its isometry group has an $SO(3)$ sub-group, and the orbits of $SO(3)$ are two-spheres. The spacetime metric induces a metric on each two-sphere. By $SO(3)$ symmetry, it has to be proportional to (1.23). Let A be the area of a two-sphere and define a function r by

$$r = \sqrt{\frac{A}{4\pi}} \tag{1.24}$$

Then the metric on the two-sphere takes the form:

$$ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.25}$$

1.5.3. Solutions

Spherically symmetric spacetime is an important case in the study of general relativity for a number of reasons. Static solutions with spherically symmetric spacetimes could be used to describe the relativistic spheres in astrophysics. This is why different techniques and analysis are worked through to attain exact solutions. Stephani et al. [2] have emphasized the role of symmetries in classifying and categorizing exact solutions. Static spherically symmetric solutions may be found by examining the following metric and calculating the Einstein Field Equations [9]:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.26}$$

where $\alpha(r)$ and $\beta(r)$ are functions of the radial coordinate r.

We obtain the following non-vanishing components of the Einstein Tensor:

$$G^t_t = -\frac{e^{-2\beta}(2\beta_r r + e^{2\beta} - 1)}{r^2} \tag{1.27}$$

$$G^r_r = \frac{e^{-2\beta}(2\alpha_r r - e^{2\beta} + 1)}{r^2} \tag{1.28}$$

$$G^\theta_\theta = G^\phi_\phi = \frac{e^{-2\beta}(\alpha_r^2 r - \alpha_r\beta_r r + \alpha_{rr} r + \alpha_r - \beta_r)}{r} \tag{1.29}$$

where α_r,α_{rr} and β_r are partial 1st and 2nd derivatives of the functions $\alpha(r)$ and $\beta(r)$.

1.6. General Non-Static Spherically Symmetric Solutions

The study of general non-static spherically symmetric solutions is crucial to understand the behaviour of relativistic spheres in time. By working on the following metric:

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.30}$$

We obtain as before the non-vanishing components of the Einstein Tensor as:

$$G^t_t = -\frac{e^{-2\beta}(2\beta_r r + e^{2\beta} - 1)}{r^2} \tag{1.31}$$

$$G^r_t = e^{-2\beta}\frac{2\beta_t}{r} \tag{1.32}$$

$$G^r_r = \frac{e^{-2\beta}(2\alpha_r r - e^{2\beta} + 1)}{r^2} \tag{1.33}$$

$$G^\theta_\theta = G^\phi_\phi = e^{-2\beta}(\alpha_r^2 - \alpha_r\beta_r + \alpha_{rr} + \frac{1}{r}(\alpha_r - \beta_r)) + e^{-2\alpha}(\alpha_t\beta_t - \beta_t^2 - \beta_{tt}) \tag{1.34}$$

This is the best we can do for a general metric in a spherically symmetric spacetime. The next step is to actually solve Einstein's equations, which will allow us to determine explicitly the functions $\alpha(t, r)$ and $\beta(t, r)$.

2. EARLIER WORKS ON COSMOLOGICAL BLACK HOLES

2.1. Transformations

2.1.1. Conformal Transformations

Consider a spacetime $(\mathcal{M}, g_{\mu\nu})$, where $\mathcal{M}$ is a smooth $n-$dimensional manifold and $g_{\mu\nu}$ is a Lorentzian metric on $\mathcal{M}$. The following conformal transformation

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{2.1}$$

where Ω is a smooth, non-vanishing function of the spacetime point is a point-dependent rescaling of the metric and is called a conformal factor.

Conformal transformations [16] occur in many contexts in general relativity, in search of new solutions to the Einstein Field Equations and in particular in the definition of asymptotic flatness. In our case with a Lorentzian metric, we shall understand that v^μ is a timelike, null or spacelike vector with respect to $g_{\mu\nu}$ if and only if it satisfies the same property with respect to $\tilde{g}_{\mu\nu}$. Thus $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}, \tilde{g}_{\mu\nu})$ have identical causal structure. Conformal transformations preserve conformal curvature in the form of the Weyl tensor $C^{\mu}_{\nu\sigma\tau}$ (the relativistic equivalent of tidal forces), which corresponds to the trace-free part of the Riemann curvature tensor $R^{\mu}_{\nu\sigma\tau}$. We will not go through the necessary manipulations in detail, but the classical example of this conformal transformation can be understood by analyzing the Minkowski metric in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.2}$$

Nothing unusual will happen to the θ and ϕ coordinates, but we will want to keep careful track of the ranges of the other two coordinates.

Here we have $-\infty < t < +\infty$ and $0 \leq r < +\infty$. We perform a conformal transformation of the form

$$\tilde{g}_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \tag{2.3}$$

with $\Omega^2 = 4(1+v^2)^{-1}(1+u^2)^{-1}$ and where $u = t - r$ and $v = t + r$ are the advanced and retarded null coordinates with certain restrictions on their ranges. Then $\tilde{g}_{\mu\nu}$ is a smooth metric on the original Minkowski manifold and $(\mathbb{R}^4, \tilde{g}_{\mu\nu})$ can be smoothly extended to a larger spacetime such that the boundary of the Minkowski region in this larger spacetime gives us a more precise representation of infinity. Thus a definition of new coordinates T, R for Minkowski spacetime with new restrictions:

$$T = tan^{-1}v + tan^{-1}u \tag{2.4}$$

$$R = tan^{-1}v - tan^{-1}u \tag{2.5}$$

$$-\pi < T + R < \pi \tag{2.6}$$

$$-\pi < T - R < \pi \tag{2.7}$$

$$0 \leq R \tag{2.8}$$

give us the components of $\tilde{g}_{\mu\nu}$ in the coordinates T, R, θ, ϕ:

$$d\tilde{s}^2 = -dT^2 + dR^2 + sin^2R(d\theta^2 + sin^2\theta d\phi^2) \tag{2.9}$$

We note that this is the Lorentz metric on $\mathbb{R} \times S^3$ known as the Einstein Static Universe except that the coordinate ranges are restricted by the equations (2.6) to (2.8). From here other definitions such as conformal infinity and asymptotic flatness can be given but the point here is that by making conformal transformations we can obtain other solutions to the field equations. Another simple example is that by making a change from the cosmological time to a new a time coordinate for the FRW metric with the curvature of space $K = 0$ (flat), we may well obtain the Minkowski metric.

To get cosmological models with Weyl curvature, conformal transformations would only be useful for taking models that already have Weyl curvature and transforming them to introduce expansion to obtain models that exist as part of cosmological models.

2.1.2. Kerr-Schild Transformations

The classical Kerr-Schild Ansatz [17], in which the metrics are as $\tilde{g} = \eta + 2Hl \otimes l$, where η is the Minkowski metric and l is a null 1-form, was very rewarding in finding solutions of the vacuum Einstein field equations. A 1-form is defined as a linear scalar function of a vector. That is, a 1-form takes a vector as input and outputs a scalar. The Kerr metric was in fact originally presented in its Kerr-Schild form [18], and the general Kerr-Schild vacuum solution was explicitly found [19–22]. The solution was also prosperous when it was implemented to the Einstein-Maxwell equations [20, 23] and to the case of null radiation. The Kerr-Schild Ansatz was soon generalized to the case where the background metric was not flat [24–26]. Consequently, two metrics $\tilde{g}$ and g are linked by a generalized Kerr-Schild relation if there exist a function H and a null 1-form l such that:

$$\tilde{g} = g + 2Hl \otimes l \tag{2.10}$$

Again, a lot of exact solutions have been found using the generalized Kerr-Schild relation. Kerr-Schild transformations are on the same footing as isometries (which leave the metric invariant, $\tilde{g} = g$), or conformal transformations ($\tilde{g} = \Omega^2 g$). As in the latter cases, in many situations the interesting point is not the existence of a discrete transformation, but the existence of a continuous group of such transformations admitted by the given metric.

2.2. Swiss Cheese Black Holes

The elementary cosmological black holes are Swiss Cheese Black Holes [5].

The Swiss Cheese Universe (SCU) [27] is the spacetime including a spherically symmetric black hole in an expanding universe, and obtained by connecting a Schwarzschild spacetime with a dust-dominated Friedmann-Robertson-Walker (FRW) spacetime at a given spherically symmetric timelike hypersurface, Σ, by the Israel junction condition with no energy density confined on the junction surface [28, 29]. Lemaître [13],Tolman [30],Bondi [31] presented general spherically symmetric spacetimes. It is understable that this toy model is not so realistic because of its high degree of symmetry but nevertheless it helps as a testing ground for the effects of inhomogeneities when fitting the cosmological data without dark energy. The LTB metric is used also to describe a local underdense bubble in FRW universe, for which there is some evidence both from supernova [32] and galaxy data [33]. In some way, the study of the Swiss Cheese Black Holes is somewhat impractical since the dense and overdense regions compensate each other and the external FRW universe is completely uninfluenced by them, which makes them inefficient because we need to study our Universe that is with inhomogeneities.

Let us consider a spherically symmetric dust universe. The line element of the LTB metric reads:

$$ds^2 = -dt^2 + X(t,r)^2 dr^2 + A(t,r)^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.11}$$

where $X(t,r)$ and $A(t,r)$ are both functions of t and r. By calculating the non-vanishing components of the Einstein Tensor we obtain:

$$G^t_t = -\frac{\dot{A}^2}{A^2} - 2\frac{\dot{A}\dot{X}}{AX} - \frac{1}{A^2} + 2\frac{A''}{AX^2} + \frac{A'^2}{A^2X^2} - 2\frac{X'A'}{AX^3} \tag{2.12}$$

$$G^t_r = 2\frac{\dot{A}'}{A} - 2\frac{A'\dot{X}}{AX} \tag{2.13}$$

$$G^r_r = -2\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} - \frac{1}{A^2} + \frac{A'^2}{A^2X^2} \tag{2.14}$$

$$G^\theta_\theta = G^\phi_\phi = -\frac{\ddot{A}}{A} - \frac{\ddot{X}}{X} - \frac{\dot{A}\dot{X}}{AX} + \frac{A''}{AX^2} - \frac{X'A'}{AX^3} \tag{2.15}$$

where $A'(t,r) = \frac{\partial A}{\partial r}$ and $X'(t,r) = \frac{\partial X}{\partial r}$.

The energy-momentum tensor of the spherically symmetric dust solution implies $G^t_r = 0$ and $G^r_r = G^\theta_\theta = G^\phi_\phi = 0$. We have therefore:

$$\frac{\dot{A}'}{A'} = \frac{\dot{X}}{X} \tag{2.16}$$

Solving (2.16) we obtain:

$$X(t,r) = B(r)A'(t,r) \tag{2.17}$$

where $B(r)$ is a function that depends only the radial coordinate r. By redefining $B(r) = \frac{1}{\sqrt{1-k(r)}}$ with $k(r) < 1$ we can rewrite the LTB metric in its usual form:

$$ds^2 = -dt^2 + \frac{A'(t,r)^2}{1-k(r)}dr^2 + A(t,r)^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.18}$$

$k(r)$ is a function associated with the curvature of $t =$ const hypersurfaces. The FRW metric is the limit $A(t,r) \to a(t)r$ and $k(r) \to Kr^2$. Further analysis can be worked through the dust approximation $G^r_r = G^\theta_\theta = G^\phi_\phi = 0$.

2.3. Kerr-Schild Cosmological Black Holes

In the original article [18], the Kerr metric was also presented in terms of coordinates (t, x, y, z):

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2mr^3}{r^4 + a^2z^2}\left[dt + \frac{r(xdx + ydy)}{a^2 + r^2} + \frac{a(ydx - xdy)}{a^2 + r^2} + \frac{z}{r}dz\right]^2 \tag{2.19}$$

with $a = \frac{J}{m}$, J and m being the angular momentum and mass of the Kerr black hole, and where $r = r(x, y, z)$ which is a function determined implicitly by:

$$x^2 + y^2 + z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right] \tag{2.20}$$

The limit $m \to 0$ is manifestly Minkowski space:

$$ds^2 \to -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.21}$$

The $a \to 0$ limit is:

$$ds^2 \to -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r}\left[dt + \frac{(xdx + ydy + zdz)}{r}\right]^2 \tag{2.22}$$

with $r = \sqrt{x^2 + y^2 + z^2}$ After a change of coordinates this can also be written as:

$$ds^2 \to -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) + \frac{2m}{r}[dt + dr]^2 \tag{2.23}$$

which is recognized as the Schwarzschild spacetime in advanced Eddington-Finkelstein coordinates. The full $m \neq 0$ and $a \neq 0$ metric is of the Kerr-Schild form:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2mr^3}{r^4 + a^2z^2} l_\mu l_\nu \tag{2.24}$$

where we have $l_\mu = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right)$ And vice versa, this transformation can be used to obtain the Kerr metric from the Minkowski space with:

$$2H = \frac{2mr^3}{r^4 + a^2z^2} \tag{2.25}$$

$$l_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right) \tag{2.26}$$

which also means that the Schwarzschild metric ($a = 0$) is also a Kerr-Schild transformation of Minkowski space with:

$$2H = \frac{2m}{r} \tag{2.27}$$

$$l_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{2.28}$$

We remarkably note that Kerr-Schild transformations can be used to generate new metrics by taking a known metric and adding a component based on a scalar field H and null 1-form l_μ.

Since a black hole is a Kerr-Schild transformation of Minkowski space, and FRW is a conformal transformation of Minkowski space, it is clear one can start with Minkowski space, do a Kerr-Schild transformation to get a black hole, and then perform a conformal transformation to get a black hole in an FRW background. Alternatively, one can start with Minkowski space, do a conformal transformation to get an FRW universe, and then perform a Kerr-Schild transformation to get a black hole in an FRW background. Thus, both approaches are similar, the only difference being whether the Kerr-Schild part of the metric contains the conformal factor or not. The same solution can be obtained either way if the scalar field H contains the conformal factor when performing the Kerr-Schild transformation after the conformal transformation.

2.4. Isotropic Cosmological Black Holes

McVittie [34, 35] was the first to reach on a cosmological black hole solution. It was basically the Schwarzschild metric with a conformal transformation, but in an isotropic form. In his model, McVittie emphasized that the mass of the black hole is a function of time, but as the universe was expanding, the mass decreased by the scale factor. The McVittie model provides a homogeneous energy density, an isotropic pressure and the fact that the black hole does not expand with the Universe. A detailed review of the mass-particle of McVittie's solution will be presented in the next chapter and new work concerning McVittie's black hole will be given as an extension on the model in Chapter 5.

3. ISOTROPIC COSMOLOGICAL BLACK HOLES

3.1. Schwarzschild Black Holes

The vacuum solution which describes a spherical symmetric spacetime was first obtained by K. Schwarzschild [36]. The Schwarzschild solution describes the gravitational field outside a spherical non-rotating mass, without charge, and the cosmological constant set to zero.

Derivation of the Schwarzschild Black Hole. Our source is static, spherically symmetric, non-rotating and without charge. So we can express the metric as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.1}$$

Before even calculating connection coefficients and Einstein's equations we can make a simplification: Let us a define a new coordinate:

$$\tilde{r} = e^{\gamma}r \tag{3.2}$$

with

$$d\tilde{r} = e^{\gamma}dr + e^{\gamma}rd\gamma = \big(1 + r\frac{d\gamma}{dr}\big)e^{\gamma}dr \tag{3.3}$$

Then the metric (3.1) becomes:

$$ds^2 = -e^{2\alpha(r)}dt^2 + \big(1 + r\frac{d\gamma}{dr}\big)^{-2}e^{2\beta(r)-2\gamma(r)}d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.4}$$

where all the functions depending on r, depend on $\tilde{r}$ obviously and we can relabel:

$$\tilde{r} \to r \tag{3.5}$$

$$\big(1 + r\frac{d\gamma}{dr}\big)^{-2} e^{2\beta(r)-2\gamma(r)} \to e^{\beta(r)} \tag{3.6}$$

and the metric is simply:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.7}$$

Since we are in vacuum, and it implies that the Einstein Field Equations give us $R_{\mu\nu} = 0$, we will be calculating for the following, the non-vanishing components of the Ricci Tensor and then we will set them to zero to obtain the solution.

$$R_{tt} = e^{2(\alpha-\beta)}(\alpha_r^2 - \alpha_r\beta_r + \alpha_{rr} + \frac{2}{r}\alpha_r) \tag{3.8}$$

$$R_{rr} = -\alpha_r^2 - \alpha_r\beta_r + \alpha_{rr} + \frac{2}{r}\beta_r \tag{3.9}$$

$$R_{\theta\theta} = e^{-2\beta}[r(\beta_r - \alpha_r) - 1] + 1 \tag{3.10}$$

$$R_{\phi\phi} = R_{\theta\theta} sin^2\theta \tag{3.11}$$

Now we will set the Ricci Tensor to zero. Since both R_{tt} and R_{rr} vanish independently we have:

$$e^{2(\beta-\alpha)}R_{tt} + R_{rr} = 0 \tag{3.12}$$

which gives us:

$$\frac{2}{r}(\alpha_r + \beta_r) = 0 \tag{3.13}$$

which implies $\alpha(r) = -\beta(r) + c$, where c is a constant. By making a time coordinate change $t \to e^{-ct}$ we can set the constant c to zero and get $\alpha(r) = -\beta(r)$.

Continuing with $R_{\theta\theta} = 0$, we need to work on:

$$e^{2\alpha}(2r\alpha_r + 1 - e^{-2\alpha}) = 0 \tag{3.14}$$

$$i.e.\ \ e^{2\alpha}(2r\alpha_r + 1) = 1 \tag{3.15}$$

which is equivalent to:

$$\partial_r(re^{2\alpha}) = 1 \tag{3.16}$$

Solving this by integrating we get:

$$e^{2\alpha} = 1 - \frac{R_S}{r} \tag{3.17}$$

And the metric becomes:

$$ds^2 = -(1 - \frac{R_S}{r})dt^2 + \frac{1}{1 - \frac{R_S}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.18}$$

where R_S is an undetermined constant, usually denoted as the Schwarzschild Radius. Further considerations of the weak field limit can be taken into account to interpret the value of R_S as $2m$ where m is the mass of the black hole. Replacing it in the metric we obtain the famous form:

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.19}$$

Isotropic Form. Let's relabel the previous metric as:

$$ds^2 = -(1 - \frac{2m}{\bar{r}})dt^2 + \frac{1}{1 - \frac{2m}{\bar{r}}}d\bar{r}^2 + \bar{r}^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.20}$$

By making a coordinate transformation on the radial coordinate $\bar{r}$ we can write the Schwarzschild metric in its isotropic form:

$$\bar{r} = \frac{m}{2}\left(\sqrt{\frac{2r}{m}} + \sqrt{\frac{m}{2r}}\right)^2 \tag{3.21}$$

This transformation is invariant under $\frac{2r}{m} \to \frac{m}{2r}$ i.e. $r \to \frac{m^2}{4r}$. The region $\bar{r} \leq 2m$ i.e. inside the black hole is symmetric, therefore we only need to investigate $\bar{r} \geq 2m$. As one can check, all the physical quantities, obtained by generalizing the isotropic metric, have the same invariance and the solution should be considered only for the outside of the black hole. We obtain the isotropic form of the Schwarzschild black hole:

$$ds^2 = -\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}dt^2 + (1+\frac{m}{2r})^4(dr^2 + r^2 d\Omega^2) \text{ where } d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{3.22}$$

The event horizon $\bar{r} = 2m$ is now $r = \frac{m}{2}$. The cosmological form of this metric is obtained by a conformal transformation as:

$$ds^2 = a(t)^2[-\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}dt^2 + (1+\frac{m}{2r})^4(dr^2 + r^2 d\Omega^2)] \tag{3.23}$$

where we denote $a(t)$ as the scale factor. Now we perform a coordinate change as $d\tau = a(t)dt$ to derive:

$$ds^2 = -\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}d\tau^2 + a(t)^2(1+\frac{m}{2r})^4(dr^2 + r^2 d\Omega^2) \tag{3.24}$$

where τ is the cosmological time coordinate. The event horizon remains at $r = \frac{m}{2}$, since the conformal transformation preserves the causal structure of the original isotropic black hole spacetime. We observe that the $r \to +\infty$ limit is the flat FLRW metric with only matter in it, which is the Einstein-de Sitter Universe model.

The Einstein Tensor. Relabeling τ by t and calculating the non-vanishing components of the Einstein Tensor we obtain the following:

$$G^t_t = -\frac{3\dot{a}^2}{a^2}\frac{(m+2r)^2}{(m-2r)^2} \tag{3.25}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\frac{(m+2r)^2}{(m-2r)^2} \tag{3.26}$$

$$G^t_r = 8m\frac{\dot{a}}{a}\frac{(m+2r)}{(m-2r)^3} \tag{3.27}$$

Comments. By posing ρ and p respectively as the energy density and the pressure and comparing them with previous results that we have found:

$$8\pi(\rho - 3p) = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\frac{(m+2r)^2}{(m-2r)^2} \tag{3.28}$$

We can write the equation of state as:

$$p = \frac{\rho}{3} - \frac{2}{8\pi}(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\frac{(m+2r)^2}{(m-2r)^2} \tag{3.29}$$

By checking that $p = G^r_r = G^\theta_\theta = G^\phi_\phi$ we can obtain p and ρ:

$$\rho = \frac{1}{8\pi}\frac{3\dot{a}^2}{a^2}\frac{(m+2r)^2}{(m-2r)^2} \tag{3.30}$$

$$p = -\frac{1}{8\pi}(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\frac{(m+2r)^2}{(m-2r)^2} \tag{3.31}$$

It is noteworthy to add that $\frac{p}{\rho}$ is independent of the position.

Comments. For a radiation dominated universe, the behaviour of $a(t)$ is as $t^{1/2}$, and we obtain:

$$\rho = \frac{1}{8\pi}\frac{3}{4t^2}\frac{(m+2r)^2}{(m-2r)^2} \tag{3.32}$$

$$p = \frac{1}{8\pi}\frac{1}{4t^2}\frac{(m+2r)^2}{(m-2r)^2} \tag{3.33}$$

The pressure is one third of the energy density, the energy density is always positive as it is in a radiation filled FLRW universe. Also the energy conditions are satisfied. As $r \to 0$ or as $r \to +\infty$, the energy density and pressure are those of the standard FRW universe. As $r \to \frac{m}{2}$ the event horizon, the energy density and pressure tend to infinity.

For a matter dominated universe, $a(t)$ goes like $t^{2/3}$ and we get:

$$\rho = \frac{1}{8\pi}\frac{4}{3t^2}\frac{(m+2r)^2}{(m-2r)^2} \tag{3.34}$$

$$p = 0 \tag{3.35}$$

The energy density is positive everywhere. As $r \to 0$ or as $r \to +\infty$ the energy density is that of a matter-dominated FLRW universe. When $r \to \frac{m}{2}$, at the event horizon, the energy density is infinite. The pressure is zero everywhere, as the FLRW matter-dominated model. Here the energy conditions are also satisfied.

3.2. McVittie's Point Mass in an expanding universe

With a time dependent model McVittie [34, 35] incorporated two solutions to depict a spherically symmetric metric that describes a point mass embedded in an expanding spatially-flat universe. Very similar to the previous section, the McVittie metric differs from the Schwarzschild metric by a factor $\frac{1}{a(t)}$. So in some sense, the mass is scaled down by the expansion of the Universe.

The McVittie metric is:

$$ds^2 = -\frac{(1-\frac{m}{2ra(t)})^2}{(1+\frac{m}{2ra(t)})^2}dt^2 + a(t)^2(1+\frac{m}{2ra(t)})^4(dr^2 + r^2 d\Omega^2) \tag{3.36}$$

Looking at the Einstein tensor as in the previous section, the only non-zero components are:

$$G_t^t = -3\frac{\dot{a}^2}{a^2} \tag{3.37}$$

$$G_r^r = G_\theta^\theta = G_\phi^\phi = -\frac{(1-\frac{5m}{2ra})\frac{\dot{a}^2}{a^2} + (1+\frac{m}{2ra})\frac{2\ddot{a}}{a}}{(1-\frac{m}{2ra})} \tag{3.38}$$

Again for ρ and p:

$$\rho = 3\frac{\dot{a}^2}{a^2} \tag{3.39}$$

$$p = -\frac{(1-\frac{5m}{2ra})\frac{\dot{a}^2}{a^2} + (1+\frac{m}{2ra})\frac{2\ddot{a}}{a}}{(1-\frac{m}{2ra})} \tag{3.40}$$

The energy density is spatially homogeneous, but we notice that at $r = \frac{m}{2a}$ the pressure is infinite. As $r \to +\infty$ the pressure takes the value of the FLRW Universe.

Comments. For a radiation dominated universe, the behaviour of $a(t)$ is as $t^{1/2}$, and we obtain:

$$\rho = \frac{1}{8\pi}\frac{3}{4t^2} \tag{3.41}$$

$$p = \frac{1}{8\pi}\frac{(1+\frac{7m}{2ra})}{4t^2(1-\frac{m}{2ra})} \tag{3.42}$$

Outside, the pressure falls of from positive infinity at $r \to (\frac{m}{2a})^+$ to $p = \frac{1}{8\pi}\frac{7}{4t^2}$ as $r \to +\infty$. Outside $r = \frac{m}{2a}$, the magnitude of the pressure is greater than that of the energy density for $r \leq \frac{5m}{2a}$, so the dominant energy condition is violated there.

For a matter dominated universe, $a(t)$ goes like $t^{2/3}$ and we get:

$$\rho = \frac{1}{8\pi}\frac{4}{3t^2} \tag{3.43}$$

$$p = \frac{\frac{m}{2ra}}{3\pi t^2(1 - \frac{m}{2ra})} \tag{3.44}$$

We remark that outside, the pressure falls of from positive infinity at $r \to (\frac{m}{2a})^+$ as $\frac{1}{r}$ to zero as $r \to +\infty$. Outside $r = (\frac{m}{2a})^+$, the magnitude of the pressure is greater than that of the energy density for $r \leq 3\frac{m}{2a}$, so the dominant energy condition is violated in that region.

4. STATIC COSMOLOGICAL BLACK HOLE SOLUTION

In Einstein's theory of general relativity, static solutions with spherically symmetric spacetimes could be used to describe the relativistic spheres in astrophysics. This is why different techniques and analysis are worked through to attain exact solutions. There are models [2, 31, 37] for static, spherically symmetric solutions which provide leading paths. John *et al.* [38] derived an exact isotropic solution which reduces to a recurrence equation with variable, rational coefficients of order three. Models of "black hole interiors" which satisfy the weak energy condition and where the matter content is specified by an equation of state of the elastic type were proposed [39]. Careful examinations of spaces with some rotational symmetries which provided a rich spectrum of different phases of black objects with distinct topologies for horizons were presented thoroughly [40] and Pons *et al.* [41] considered Einstein-Gauss-Bonnet black holes, a new aspect is that there can occur a non-central naked singularity, that can be averted by imposing a range for black hole mass. Also, Davidson *et al.* [42] have proven that any static metric with a Killing horizon in the presence of a perfect fluid, is necessarily a Schwarzschild solution with a vanishing proper energy density and a vanishing proper pressure. Solutions describing black holes embedded in gradually increasing perfect fluid and the phenomenon for $p = \nu\rho$ were demonstrated.

The motivation that drove us was the possibility if a solution satisfying only the isotropy condition existed in a static spacetime. In the following section we will propose a two parameter solution of the Field Equations. One of the parameters is related to the mass of the black hole and the other to the energy density in the universe.

4.1. Isotropic Solution

Let us write the metric in a static spacetime with a function $f(r)$ [43].

$$ds^2 = -\frac{(1-f(r))^2}{(1+f(r))^2}dt^2 + (1+f(r))^4(dr^2 + r^2 d\Omega^2) \text{ where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (4.1)$$

The motivation for this ansatz was to investigate the singularity at $r = 0$. Instead of putting a solution like $\frac{1}{r}$ directly we searched for a more general form for a solution. The calculations lead us to the following non-vanishing components of the Einstein tensor G^t_t, G^r_r and $G^\theta_\theta = G^\phi_\phi$.

$$G^t_t = 4\frac{f_{rr} + \frac{2f_r}{r}}{r^2(1+f)^5} \tag{4.2}$$

$$G^r_r = 4\frac{f_r(f_r + \frac{f}{r})}{r^2(-1+f)(1+f)^5} \tag{4.3}$$

$$G^\theta_\theta = G^\phi_\phi = 2\frac{f_{rr}f - f_r^2 + \frac{f_r f}{r}}{r^2(-1+f)(1+f)^5} \tag{4.4}$$

where $f_r = \frac{\partial f(r)}{\partial r}$ and $f_{rr} = \frac{\partial^2 f(r)}{\partial r^2}$.

The only condition that we impose is the isotropy condition:

$$G^r_r = G^\theta_\theta = G^\phi_\phi \tag{4.5}$$

What we obtain is rather intriguing, the unique solution that satisfies (4.5) is:

$$f(r) = \frac{1}{\sqrt{a^2r^2 + 2b}} = (a^2r^2 + 2b)^{-1/2} \tag{4.6}$$

where a and b are constants.

We obtain the following non-vanishing components of the Einstein tensor:

$$G^t_t = -\frac{24a^2b}{(\sqrt{a^2r^2 + 2b} + 1)^5} \tag{4.7}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = \frac{8a^2b}{(\sqrt{a^2r^2 + 2b} - 1)(\sqrt{a^2r^2 + 2b} + 1)^5} \tag{4.8}$$

These can also be written as:

$$G^t_t = -\frac{24a^2b}{(1+\frac{1}{f})^5} \tag{4.9}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = \frac{8a^2b}{(-1+\frac{1}{f})(1+\frac{1}{f})^5} \tag{4.10}$$

Remembering that $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$ with ρ and p representing respectively the density and pressure, we can rewrite the latter equations as:

$$\rho = \frac{24a^2b}{(1+\frac{1}{f})^5} \tag{4.11}$$

$$p = \frac{8a^2b}{(-1+\frac{1}{f})(1+\frac{1}{f})^5} \tag{4.12}$$

Denoting ν, the equation of state parameter:

$$\nu = \frac{1}{3(\sqrt{a^2r^2+2b}-1)} = \frac{1}{3(\frac{1}{f(r)}-1)} \tag{4.13}$$

With (4.6), $a = \frac{2}{GM}$ and $b = \frac{8}{3}G^3M^2\rho_H$

$$\rho = \frac{32\rho_H}{(\frac{1}{f(r)}+1)^5} \tag{4.14}$$

$$p = \nu\rho \tag{4.15}$$

ρ_H being the energy density at the horizon.

Also the non-vanishing components of the Weyl Tensor are:

$$C_{trtr} = -2C_{t\theta t\theta} = -2C_{t\phi t\phi} = 2C_{r\phi r\phi} = 2C_{\theta\phi\theta\phi} = \frac{4a^4r^2}{f(r)(\frac{1}{f(r)}+1)^6} \tag{4.16}$$

Singularity. We immediately observe the horizon singularity at $f(r) = 1$, we note that the pressure $p \to \infty$ whereas the density and the Weyl Tensor components are finite at the horizon. On the other hand for $r \to \infty$, we see that the equation of state paramter ν goes to zero and the components of the Weyl tensor signal a massive object of mass M. Far from the black hole, matter is approximated as stationary dust particles which produce no pressure. Analysis on $f(r)$ implies that we can circumvent this singularity for $\sqrt{a^2r^2 + 2b} > 1$ i.e. for $b > \frac{1}{2}$ for all r.

Energy Conditions. [44]

Table 4.1. Energy Conditions

Name	Statement	Conditions
Weak	$T_{\alpha\beta}v^\alpha v^\beta \geq 0$	$\rho \geq 0, \quad \rho + p_i > 0$
Null	$T_{\alpha\beta}k^\alpha k^\beta \geq 0$	$\rho + p_i \geq 0$
Strong	$\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)v^\alpha v^\beta \geq 0$	$\rho + \sum_i p_i \geq 0, \quad \rho + p_i \geq 0$
Dominant	$-T^\alpha_{\ \beta}v^\beta$ future directed	$\rho \geq 0, \quad \rho \geq \lvert p_i \rvert$

Some of the energy conditions are formulated in terms of a normalized, future-directed, but otherwise arbitrary vector field v^α. This represents the four-velocity of an arbitrary observer in spacetime, and k^α an arbitrary future-directed null vector.

We note that as $r \to 0$ all the energy conditions are satisfied for $b \geq \frac{8}{9}$. If we choose $b = 2$, we have as $r \to 0$, $p = \frac{\rho}{3}$ which is the radiation equation of state and as $r \to \infty$, $p = 0$ the dust equation of state.

Temperature. In the expanding Universe [45], the second law of thermodynamics, as applied to a comoving volume element of unit coordinate volume and physical volume V, implies that:

$$TdS = d(\rho V) + pdV = d[(\rho + p)V] - Vdp, \tag{4.17}$$

where ρ and p are the equilibrium density and pressure. Moreover the integrability condition,

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \tag{4.18}$$

relates the energy density and pressure:

$$T\frac{dp}{dT} = \rho + p \tag{4.19}$$

or equivalently

$$dp = \frac{\rho + p}{T} dT \tag{4.20}$$

The latter equation can also be written as:

$$\frac{d}{dT}(\frac{\rho + p}{T}) = \frac{1}{T}\frac{d\rho}{dT} \tag{4.21}$$

Now, the interesting aspect of this metric with $f(r)$, the density and the pressure formulations that we have derived previously is that we can calculate via the latter equation the temperature as a function of r.

Doing the necessary calculations we have:

$$\frac{T}{T_\infty} = \frac{1 + f(r)}{1 - f(r)} \tag{4.22}$$

where T_∞ is the temperature at $r \to \infty$. We see that the change of temperature is quite mild unless there is a horizon in which case the temperature is infinite at the horizon. We note that the equation of state parameter is:

$$\nu = \frac{1}{6}(\frac{T}{T_\infty} - 1) \tag{4.23}$$

and observe that ν has a linear dependence on temperature.

5. NON-STATIC BLACK HOLES IN MATTER-ENERGY BACKGROUND

In Einstein's theory of general relativity, two exact solutions were well known and studied throughout the years. One is the Schwarzschild solution that describes the gravitational field outside a spherical non-rotating mass, without charge, and the cosmological constant set to zero. This was practical to model spacetime outside a star, a planet or a black hole. The isotropic form with a conformal time transformation as we have presented as the cosmological Schwarzschild solution in chapter 3. Concerning the impacts in an expanding universe, there are works on the cosmological effects of expansion on local systems [46,47], which explore the local attraction in a gravitationally bound system and analyse the solution of general relativity representing a black hole embedded in a special cosmological background. The other model is the McVittie metric [34], [35] , which incorporated two solutions to depict a spherically symmetric metric that describes a point mass embedded in an expanding spatially-flat universe. In order to describe relativistic spheres in general relativity, one must consider time dependent solutions.

5.1. Conformal Extension of the Schwarzschild Metric

We assume a form of the metric motivated by the Schwarzschild black hole in isotropic coordinates by replacing the mass parameter by an arbitrary function of r and t. Then we will impose the equality of the spatial diagonal elements of the energy-momentum tensor. Let us describe this by considering the following metric in a non-static spacetime generalized by a function $f(r,t)$ and a cosmological scale factor $a(t)$.

$$ds^2 = -\frac{(1-f(r,t))^2}{(1+f(r,t))^2}dt^2 + a(t)^2(1+f(r,t))^4(dr^2 + r^2 d\Omega^2) \text{ where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{5.1}$$

The calculations lead us to the following non-vanishing components of the Einstein tensor G^t_t, G^t_r, G^r_r and $G^\theta_\theta = G^\phi_\phi$. The non-vanishing components of the Einstein tensor are presented in the Appendix A. The only condition that we impose is the analogous condition of the static case:

$$G^r_r = G^\theta_\theta = G^\phi_\phi \tag{5.2}$$

The unique solution that satisfies (5.2) is:

$$f(r,t) = \frac{1}{\sqrt{m(t)^2 r^2 + 2n(t)}} \tag{5.3}$$

where $m(t)$ and $n(t)$ are functions that depend on t. We notice that at $r = r_H$ we have $f(r_H, t) = 1$, where r_H is the horizon. We denote p and ρ respectively the pressure and the energy density. As $r \to r_H$, we set $\dot{f}(r,t) \to 0$ so that $\frac{p}{\rho}$ is non-singular at the horizon, this gives:

$$f(r,t) = \frac{1}{\sqrt{m(t)^2 (r^2 - r_H^2) + 1}} \tag{5.4}$$

With (5.1) and (5.4) the non-vanishing components of the Einstein tensor are presented in the Appendix B. We note that as $r \to \infty$ we have:

$$G^t_t = -3\frac{\dot{a}^2}{a^2} \tag{5.5}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \tag{5.6}$$

And as $r \to r_H$:

$$G^t_t = -\frac{12}{a^2(1-f)^2}(\dot{f}a + \dot{a})^2 \tag{5.7}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -\frac{8}{a^2(1-f)^3} a\dot{f}(\dot{f}a + \dot{a}) \tag{5.8}$$

We restate that $T^{\mu}_{\nu} = \text{diag}(-\rho, p, p, p)$, and for further purposes we denote the equation of state parameter:

$$\nu = \frac{p}{\rho} = -\frac{2}{3(1-f)} \frac{a\dot{f}}{\dot{f}a + \dot{a}} \tag{5.9}$$

What we observe for the equation of state parameter and the function $m(t)$ as $r \rightarrow r_H$ is:

$$m^2(t) = -\frac{3\nu}{2r_H^2} ln(a(t)). \tag{5.10}$$

We can see that for this solution to exist for an expanding universe we need $\nu \leq 0$. e.g. the dark energy solution $p = -\rho$.

5.2. McVittie Extension

Lake *et al.* [48] studied the McVittie solution that contains a black hole in an expanding universe, worked on specific solutions that asypmtote ΛCDM cosmology and da Silva *et al.* [49] worked on the main characteristics of the McVittie solution for different choices of the scale factor. Nandra *et al.* [50] present a tetrad based procedure to solve Einstein's field equations to derive metrics describing a point mass in an expanding universe and Nolan [51] proves the existence of solutions representing more general spherical objects embedded in a Robertson Walker universe.

This leads us to a very naïve question; why should the conformal factor (scale factor) $a(t)$ influence the same way the temporal part and spatial part of the metric. Can a more general expression of the metric be presented? The motivation that drove us also to this, was the question whether or not the event horizon was increasing in an expanding universe. Thus we wanted to see if a time dependent conformal factor can influence, the event horizon of this black hole and the dynamics that it creates with time.

Let us write the metric with two conformal factors $a(t)$, the scale factor and $b(t)$, a conformal factor [52]. From now on, we will call “the McVittie Extension”, the following metric in isotropic form with $a(t)$ and $b(t)$ which reads as:

$$ds^2 = -\frac{(1-\frac{mb(t)}{2r})^2}{(1+\frac{mb(t)}{2r})^2}dt^2 + a(t)^2(1+\frac{mb(t)}{2r})^4(dr^2 + r^2 d\Omega^2) \tag{5.11}$$

The calculations lead us to the following non-vanishing components of the Einstein tensor:

$$G^t_t = -3\,\frac{\dot{a}^2}{a^2}\,\frac{1}{(1-\frac{mb}{2r})^2}\,\big(1+\frac{m}{2r}(b+2\dot{b}\,\frac{a}{\dot{a}})\big)^2 \tag{5.12}$$

$$G^t_r = -8m\frac{(b\dot{a}+\dot{b}a)}{a}\frac{(1+\frac{mb}{2r})}{(1-\frac{mb}{2r})^3} \tag{5.13}$$

$$\begin{aligned} G^r_r = G^\theta_\theta = G^\phi_\phi = & -\frac{1}{(1-\frac{mb}{2r})^3}\big(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2} \\ & +\frac{1}{a^2}(\frac{m}{2r})(b\dot{a}^2 + 16a\dot{a}\dot{b} + 2a\ddot{a}b + 4a^2\ddot{b}) \\ & +\frac{1}{a^2}(\frac{m}{2r})^2(-b^2\dot{a}^{+}4ab\dot{a}\dot{b} - 2ab^2\ddot{a} + 16a^2\dot{b}^2) \\ & +\frac{1}{a^2}(\frac{m}{2r})^3(-b^3\dot{a}^2 - 12ab^2\dot{a}\dot{b} - 2ab^3\ddot{a} - 8a^2b\dot{b}^2 - 4a^2b^2\ddot{b})\big) \end{aligned} \tag{5.14}$$

5.2.1. Radiation

Let us model this by assuming that the energy density resides primarily in light particles having relativistic velocities. The Universe is radiation dominated; The equation of state is $p = \frac{\rho}{3}$. By equating the powers of $\frac{m}{2r}$ to zero, we obtain the following system of differential equations:

$$\dot{a}^2 + a\ddot{a} = 0 \tag{5.15}$$

$$-4b\dot{a}^2 - 4ba\ddot{a} - 8a^2\ddot{b} - 40a\dot{a}\dot{b} = 0 \tag{5.16}$$

$$2b^2\dot{a}^2 + 2b^2a\ddot{a} - 4ba\dot{a}\dot{b} - 20\dot{b}^2a^2 = 0 \tag{5.17}$$

$$b^3\dot{a}^2 + b^3a\ddot{a} + 8b^2a\dot{a}\dot{b} + 2b^2a^2\ddot{b} + 6ba^2\dot{b}^2 = 0 \tag{5.18}$$

The unique solution that satisfies this system is:

$$a(t) = Ct^{1/2} \text{ and } b(t) = 1 \tag{5.19}$$

where C is a constant. This the known behaviour of the scale factor in the radiation dominated era of the Universe. In addition we have:

$$G^t_t = -\frac{3}{4t^2}\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.20}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = \frac{1}{4t^2}\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.21}$$

$$G^t_r = -\frac{4m}{t}\frac{(1+\frac{m}{2r})}{(1-\frac{m}{2r})^3} \tag{5.22}$$

This means that for an energy density ρ and pressure p, we have $p = \frac{\rho}{3} = \frac{1}{4t^2}\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2}$. G^t_r is a momentum component, we notice that matter infalls for $r \geq \frac{m}{2}$.

5.2.2. Dust approximation

Matter dominated Universe is modeled by dust approximation. The matter is approximated as stationary dust particles which produce no pressure. For $p = 0$, by equating the powers of $\frac{m}{2r}$ to zero, we obtain the following system of differential equations:

$$G^r_r = G^\theta_\theta = G^\phi_\phi = 0 \tag{5.23}$$

We obtain a system of differential equations:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 0 \tag{5.24}$$

$$b\dot{a}^2 + 16a\dot{a}\dot{b} + 2a\ddot{a}b + 4a^2\ddot{b} = 0 \tag{5.25}$$

$$-b^2\dot{a}^2 + 4ab\dot{a}\dot{b} - 2ab^2\ddot{a} + 16a^2\dot{b}^2 = 0 \tag{5.26}$$

$$-b^3\dot{a}^2 - 12ab^2\dot{a}\dot{b} - 2ab^3\ddot{a} - 8a^2b\dot{b}^2 - 4a^2b^2\ddot{b} = 0 \tag{5.27}$$

The unique solution that satisfies this system of equations is

$$a(t) = Ct^{2/3} \text{ and } b(t) = 1 \tag{5.28}$$

where C is a constant. We have found here, from our generalized metric (5.8), the well known behaviour of the scale factor $a(t)$. In addition for

$$a(t) = Ct^{2/3} \text{ and } b(t) = 1 \tag{5.29}$$

we have:

$$G_t^t = -\frac{4}{3t^2}\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.30}$$

$$G_r^t = -\frac{16}{3t}m\frac{(1+\frac{m}{2r})}{(1-\frac{m}{2r})^3} \tag{5.31}$$

This means that for an energy density ρ, we have $\rho = \frac{4}{3t^2}\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2}$ and the behaviour of G_r^t is as $\frac{1}{t}$. Since G_r^t is a momentum component, we notice that matter infalls for $r \geq \frac{m}{2}$.

From this we note that the energy due to dust calculated for any finite volume including the black hole is infinite. In his latest talk [53], S.W.Hawking states that "black holes are not black", this can be a manifestation of what we have stated previously. Another fact that we need to state is that, as we propose it, our event horizon is not static because of $b(t)$. Our result for dust solution is $b(t) = 1$, which implies a static event horizon. This has been pointed also by A.Davidson and S.Rubin [54]: an evolving universe can host locally a static event horizon.

5.2.3. Dark Energy

For a dark energy dominated Universe we have the equation of state $p = -\rho$.

By equating the powers of r to zero, we obtain the following system of differential equations:

$$-\dot{a}^2 + a\ddot{a} = 0 \tag{5.32}$$

$$-4b\dot{a}^2 + 4ba\ddot{a} + 8a^2\ddot{b} + 8a\dot{a}\dot{b} = 0 \tag{5.33}$$

$$2b^2\dot{a}^2 - 2b^2a\ddot{a} + 4ba\dot{a}\dot{b} + 4\dot{b}^2a^2 = 0 \tag{5.34}$$

$$b^3\dot{a}^2 - b^3a\ddot{a} - 2b^2a^2\ddot{b} + 2ba^2\dot{b}^2 = 0 \tag{5.35}$$

The solutions that satify these are:

$$a(t) = Ce^{\sqrt{\frac{\Lambda}{3}}t} \text{ and } b(t) = 1 \tag{5.36}$$

and

$$a(t) = Ce^{\sqrt{\frac{\Lambda}{3}}t} \text{ and } b(t) = \frac{1}{a(t)} \tag{5.37}$$

where c is a constant.

Dark Energy Solution 1. For $a(t) = Ce^{\sqrt{\frac{\Lambda}{3}}t}$ and $b(t) = 1$, we obtain:

$$G^t_t = G^r_r = G^\theta_\theta = G^\phi_\phi = -\Lambda\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.38}$$

$$G^t_r = -8m\sqrt{\frac{\Lambda}{3}}\frac{(1+\frac{m}{2r})}{(1-\frac{m}{2r})^3} \tag{5.39}$$

This means that for an energy density ρ and pressure p, we have:

$$\rho = \Lambda\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.40}$$

$$p = -\Lambda\frac{(1+\frac{m}{2r})^2}{(1-\frac{m}{2r})^2} \tag{5.41}$$

Dark Energy Solution 2. For $a(t) = Ce^{\sqrt{\frac{\Lambda}{3}}t}$ and $b(t) = \frac{1}{a(t)}$, we obtain simply

$$G^t_r = 0 \tag{5.42}$$

$$G^t_t = G^r_r = G^\theta_\theta = G^\phi_\phi = -\Lambda \tag{5.43}$$

i.e. $\rho = \Lambda$ and $p = -\Lambda$.

In an empty or a dark energy dominated Universe the horizon is a coordinate singularity, but with other matter-energy forms the horizon must be a real singularity as mentioned in the talk [53].

Diagonal Einstein Tensor. Another approach may be to consider only a diagonal Einstein tensor, an energy momentum tensor without any cross terms, i.e. by considering directly only $G^t_r = 0$ and getting by this $b(t) = \frac{1}{a(t)}$ which is a special case of the latter paragraph. Replacing the value of $b(t)$ in (5.12-5.13-5.14) we have:

$$G^t_t = -3\,\frac{\dot{a}^2}{a^2} \tag{5.44}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -\frac{1}{(1-\frac{m}{2ra})}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \left(\frac{m}{2ra}(2\frac{\ddot{a}}{a} - 5\frac{\dot{a}^2}{a^2})\right)\right) \tag{5.45}$$

We notice that this is the solution produced again by our generalized metric (5.9). They are identical to the equations of the McVittie metric. By inserting the Hubble parameter:

$$H = \frac{\dot{a}}{a} \tag{5.46}$$

Equations (5.44 and 5.45) become:

$$G^t_t = -3H^2 \tag{5.47}$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = -\frac{1}{(1-\frac{m}{2ra})}\big(2\dot{H} + 3H^2 + \frac{m}{2ra}(2\dot{H} - 3H^2)\big) \tag{5.48}$$

With the energy density ρ and pressure p:

$$\rho = 3H^2 \tag{5.49}$$

$$p = -\frac{1}{(1-\frac{m}{2ra})}\big(2\dot{H} + 3H^2 + \frac{m}{2ra}(2\dot{H} - 3H^2)\big) \tag{5.50}$$

For $H = \text{const}$, $p = -\rho = -3H^2$ which is independent of r. Equivalently, it can be shown that if G^r_r, G^θ_θ, G^ϕ_ϕ are independent of r, H is constant. We find that for $b(t) = \frac{1}{a(t)}$ and $H = \text{const}$, $\rho = -p = 3H^2$ which is constant. This is the vacuum (dark) energy for dark energy dominated universe.

5.3. Kerr Metric

It is well known that the gravitational field of a rotating black hole is described by the Kerr metric [18], [55]. Vaidya *et al.* [56] have obtained an exact solution of the Einstein's equations which describes the field of a radiating Kerr particle embedded in Einstein static universe. Vaidya [57] has also given a very general form of the Kerr-Schild metric satisfying the Einstein's field equations. Tzounis *et al.* [58] explored the properties of the ergoregion and the location of the curvature singularities for the Kerr black hole distorted by the gravitational field of external sources and they also studied the scalar curvature invariants of the horizon and compared their behaviour with the case of the isolated Kerr black hole. Lake *et al.* [59] examined the global structure of the family of Kerr-de Sitter spacetimes. Abdelqader *et al.* [60] presented an invariant characterization of the physical properties of the Kerr spacetime and introduced two dimensionless invariants, constructed out of some known curvature invariants, that act as detectors for the event horizon and ergosurface of the Kerr black hole. Gibbons *et al.* [61] discussed the global structure of the metrics, and obtain formulae for the surface gravities and areas of the event horizons. There is a considerable interest in generalizing the Kerr metric to describe the non-static field of a rotating star, that is why the main aim is to derive an exact solution of Einstein's equations which describes the field of a Kerr source embedded in a rotating expanding universe.

The Kerr metric describes the geometry of spacetime in the vicinity of a mass M rotating with angular momentum J. We add to this metric in Boyer-Lindquist coordinates a scale factor that we denote $b(t)$. The metric can be written as follows [62]:

$$\begin{aligned} ds^2 = &-(1-\frac{2Mr}{\rho^2})dt^2 - b(t)\big(\frac{4Mr\alpha sin^2\theta}{\rho^2}\big)d\phi dt + b(t)^2\big(\frac{\rho^2}{\Delta}\big)dr^2 \\ &+ b(t)^2(\rho^2)d\theta^2 + b(t)^2(r^2+\alpha^2+\frac{2Mr\alpha^2 sin^2\theta}{\rho^2})\sin^2\theta d\phi^2 \end{aligned} \tag{5.51}$$

where $\alpha = \frac{J}{M}$, $\rho^2 = r^2+\alpha^2 cos^2\theta$ and $\Delta = r^2-2Mr+\alpha^2$. The non-vanishing components of the Einstein Tensor are: G^t_t, $G^r_r = G^\theta_\theta = G^\phi_\phi$, G^t_r,G^t_θ, G^t_ϕ, G^r_ϕ and G^θ_ϕ.

5.3.1. Radiation

Let us model this by assuming that the energy density resides primarily in light particles having relativistic velocities. The Universe is radiation dominated; The equation of state is $p = \frac{\rho}{3}$, and the unique solution that satisfies this is:

$$b(t) = b_1 t^{1/2} \tag{5.52}$$

where b_1 is a constant. The non-vanishing components of the Einstein Tensor are:

$$G^t_t = -\frac{3}{4t^2}\,\frac{[a^2cos^2\theta(\Delta)+r^4+2Mra^2+a^2r^2]}{\rho^2\Delta} \tag{5.53}$$

$$G^r_r = \frac{1}{4t^2}\,\frac{[a^2cos^2\theta(\Delta)+r^4+2Mra^2+a^2r^2]}{\rho^2\Delta} \tag{5.54}$$

$$G^\theta_\theta = G^\phi_\phi = \frac{1}{4t^2}\,\frac{[a^2cos^2\theta(\Delta)+r^4+2Mra^2+a^2r^2]}{\rho^2\Delta} \tag{5.55}$$

$$G^t_r = -\frac{M}{t}\,\frac{[r^6+2a^2r^4-4a^2Mr^3+a^4r^2-(r^4-4Mr^3+2a^2r^2+a^4)a^2cos^2\theta]}{\Delta^2\rho^4} \tag{5.56}$$

$$G^t_\theta = \frac{M}{t}\,\frac{(\alpha^2+r^2)r\alpha^2 sin2\theta}{\Delta\rho^4} \tag{5.57}$$

$$G^t_\phi = 0 \tag{5.58}$$

$$G^r_\phi = -\frac{1}{b_1 t^{3/2}} \frac{\alpha M sin^2\theta(3r^4 + \alpha^2 r^2 + \alpha^2 r^2 cos^2\theta - \alpha^4 cos^2\theta)}{\rho^6} \tag{5.59}$$

$$G^\theta_\phi = \frac{1}{b_1 t^{3/2}} \frac{\alpha^3 M r(sin2\theta)(sin^2\theta)}{\rho^6} \tag{5.60}$$

5.3.2. Dust approximation

Matter dominated Universe is modeled by dust approximation. The matter is approximated as stationary dust particles which produce no pressure. For $p = 0$,

$$G^r_r = G^\theta_\theta = G^\phi_\phi = 0 \tag{5.61}$$

The unique solution that satisfies this system of equations is

$$b(t) = b_1 t^{2/3} \tag{5.62}$$

where b_1 is a constant. Now for the non-vanishing components of the Einstein Tensor we get:

$$G^t_t = -\frac{4}{3t^2} \frac{[a^2 cos^2\theta(\Delta) + r^4 + 2Mra^2 + a^2 r^2]}{\rho^2 \Delta} \tag{5.63}$$

$$G^t_r = -\frac{4}{3t} \frac{M[r^6 + 2a^2 r^4 - 4a^2 M r^3 + a^4 r^2 - (r^4 - 4Mr^3 + 2a^2 r^2 + a^4)a^2 cos^2\theta]}{\Delta^2 \rho^4} \tag{5.64}$$

$$G^t_\theta = \frac{4}{3t} \frac{Mr\alpha^2 sin2\theta(\alpha^2 + r^2)}{\Delta \rho^4} \tag{5.65}$$

$$G^t_\phi = 0 \tag{5.66}$$

$$G^r_\phi = -\frac{4}{3b_1 t^{5/3}} \frac{\alpha M sin^2\theta(3r^4 + \alpha^2 r^2 + \alpha^2 r^2 cos^2\theta - \alpha^4 cos^2\theta)}{\rho^6} \tag{5.67}$$

$$G^\theta_\phi = \frac{4}{3b_1 t^{5/3}} \frac{\alpha^3 M r(sin2\theta)(sin^2\theta)}{\rho^6} \tag{5.68}$$

5.3.3. Dark Energy

For a dark energy dominated universe let us solve the equation: $p = -\rho$ i.e. $G^t_t = -\rho$ and $p = G^r_r = G^\theta_\theta = G^\phi_\phi$. we obtain the solution:

$$b(t) = b_1 e^{\sqrt{\frac{\Lambda}{3}}\ t} \tag{5.69}$$

where b_1 is a constant. The components of the Einstein Tensor are:

$$G^t_t = -\frac{\Lambda\big((\Delta)(\alpha^2 cos^2\theta) + r^4 + 2Mr\alpha^2 + \alpha^2 r^2\big)}{\rho^2\ \ \Delta} \tag{5.70}$$

$$G^r_r = -\frac{\Lambda\big((\Delta)(\alpha^2 cos^2\theta) + r^4 + 2Mr\alpha^2 + \alpha^2 r^2\big)}{\rho^2\ \ \Delta} \tag{5.71}$$

$$G^\theta_\theta == -\frac{\Lambda\big((\Delta)(\alpha^2 cos^2\theta) + r^4 + 2Mr\alpha^2 + \alpha^2 r^2\big)}{\rho^2\ \ \Delta} \tag{5.72}$$

$$G^\phi_\phi = -\frac{\Lambda\big((\Delta)(\alpha^2 cos^2\theta) + r^4 + 2Mr\alpha^2 + \alpha^2 r^2\big)}{\rho^2\ \ \Delta} \tag{5.73}$$

$$G^t_r = -\frac{2M\sqrt{3\Lambda}}{3}\frac{[r^6 + 2a^2r^4 - 4a^2mr^3 + a^4r^2 - (r^4 - 4mr^3 + 2a^2r^2 + a^4)a^2cos^2\theta]}{\Delta^2\rho^4} \tag{5.74}$$

$$G^t_\theta = \frac{2}{3}\frac{(Mr\alpha^2 sin2\theta)\sqrt{3\Lambda}}{\Delta\rho^4} \tag{5.75}$$

$$G^t_\phi = 0 \tag{5.76}$$

$$G^r_\phi = -\frac{2}{3b_1}\frac{\alpha M sin^2\theta(3r^4 + \alpha^2r^2 + \alpha^2r^2cos^2\theta - \alpha^4cos^2\theta)(\sqrt{3\Lambda}e^{-\sqrt{\frac{\Lambda}{3}}\ t})}{\rho^6} \tag{5.77}$$

$$G^\theta_\phi = \frac{2}{3b_1}\frac{\alpha^3 Mr(sin2\theta)(sin^2\theta)(\sqrt{3\Lambda}e^{-\sqrt{\frac{\Lambda}{3}}\ t})}{\rho^6} \tag{5.78}$$

6. INFLUENCE OF BLACK HOLES ON THE UNIVERSE

In this chapter the Weyl curvature will be calculated for the cosmological black holes, since Weyl curvature should lead to shear that could influence the volume expansion of the universe. The volume expansion will then be calculated for co-ordinate volumes.

6.1. Weyl Curvature

Since Weyl curvature is the relativistic equivalent of tidal force, then the Weyl curvature in the cosmological black hole spacetimes should be expected to introduce shear in the velocity field. Thus, the Weyl curvature of the cosmological black holes will be calculated in this section.

6.1.1. Static Solution

For the generalized metric presented in Chapter 4,

$$ds^2 = -\frac{(1-f(r))^2}{(1+f(r))^2}dt^2 + (1+f(r))^4(dr^2 + r^2 d\Omega^2) \text{ where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (6.1)$$

we obtain the following Weyl Curvature Scalar:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{16}{3}\frac{(f_{rr}f^2r - 3f_r^2fr - f_{rr}fr - f_rf^2 + 9f_r^2r - 2rf_{rr} + f_rf + 2f_r)^2}{(1-f)^2(1+f)^{12}r^2} \quad (6.2)$$

Introducing the form obtained (4.4):

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{192r^4a^8f^{10}}{(1+f)^{12}} \quad (6.3)$$

remembering that a is a constant here.

6.1.2. Schwarzschild and Conformally Extended Schwarzschild Metric

The Weyl scalar for an isolated black hole with the metric

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{1}{(1 - \frac{2m}{r})}dr^2 + r^2(d\theta^2 + sin\theta^2 d\phi^2) \tag{6.4}$$

can be written as:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \tag{6.5}$$

Using the isotropic form

$$ds^2 = -\frac{(1 - \frac{m}{2r})^2}{(1 + \frac{m}{2r})^2}dt^2 + (1 + \frac{m}{2r})^4(dr^2 + r^2 d\Omega^2) \tag{6.6}$$

we have for the Weyl scalar:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6(1 + \frac{m}{2r})^{12}} \tag{6.7}$$

The Weyl scalar for the constant-mass isotropic cosmological black holes is:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6 a(t)^4(1 + \frac{m}{2r})^{12}} \tag{6.8}$$

This differs from the Weyl curvature for a Schwarzschild black hole, by the presence of the $(1 + \frac{m}{2r})^{12}$. As $r \to 0$, the Weyl curvature goes to zero at the origin. This makes sense because the region inside the event horizon in the isotropic black hole spacetime is a remapping of the region outside the event horizon in the standard Schwarzschild spacetime so there should be zero Weyl curvature at the origin. The Weyl curvature is finite at the event horizon and is infinite only at the Big Bang.

For the generalized metric presented in Chapter 5,

$$ds^2 = -\frac{(1-f(r,t))^2}{(1+f(r,t))^2}dt^2 + a(t)^2(1+f(r,t))^4(dr^2+r^2d\Omega^2) \text{ where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{6.9}$$

we obtain the following Weyl Curvature Scalar:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{16}{3}\frac{(f_{rr}f^2r - 3f_r^2fr - f_{rr}fr - f_rf^2 + 9f_r^2r - 2rf_{rr} + f_rf + 2f_r)^2}{(1-f)^2(1+f)^{12}r^2a(t)^4} \tag{6.10}$$

Introducing the form obtained (5.4):

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{192r^4m(t)^8f^{10}}{(1+f)^{12}a(t)^4} \tag{6.11}$$

6.1.3. McVittie Extension

The Weyl scalar for the McVittie Metric is:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6a(t)^6(1+\frac{m}{2ra(t)})^{12}} \tag{6.12}$$

analogous to the constant-mass isotropic black holes. As with the case of the constant-mass black holes, the Weyl curvature is zero at the origin, but unlike the constant-mass black holes, the Weyl curvature is zero at the Big Bang.

With our extension on the McVittie metric, using

$$ds^2 = -\frac{(1-\frac{mb(t)}{2r})^2}{(1+\frac{mb(t)}{2r})^2}dt^2 + a(t)^2(1+\frac{mb(t)}{2r})^4(dr^2+r^2d\Omega^2) \tag{6.13}$$

we get the following Weyl scalar:

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{48m^2b(t)^2}{r^6a(t)^4(1+\frac{mb(t)}{2r})^{12}} \tag{6.14}$$

This differs from the original McVittie Weyl curvature scalar by a factor of $b(t)^2$ in the nominator and $a(t)^2 b(t)^{12}$ in the denominator, the values of which are solved in Chapter 5 and give the expected results in a general form.

6.2. Volume Expansion

The volume in some comoving region Ω is given by

$$V = \int_{\Omega} \sqrt{h} d^3 x \tag{6.15}$$

where h is the determinant of the spatial part of the metric.

6.2.1. Static Solution

The volume integrated over a given radius range r_a to r_b for the static solution is:

$$V = \int_{r_a}^{r_b} \int_0^{\pi} \int_0^{2\pi} \sqrt{(1+f(r))^{12} r^4 sin^2\theta} dr d\theta d\phi \tag{6.16}$$

which simplifies to

$$V = 4\pi \int_{r_a}^{r_b} \sqrt{(1+f(r))^{12} r^4} dr \tag{6.17}$$

Since this solution does not evolve in time, we cannot talk of a volume expansion.

6.2.2. FLRW

The volume integrated over a given radius range r_a to r_b for the FRW metric is:

$$V = \int_{r_a}^{r_b} \int_0^{\pi} \int_0^{2\pi} \sqrt{a(t)^6 r^4 sin^2\theta} dr d\theta d\phi \tag{6.18}$$

which simplifies to

$$V = 4\pi a(t)^3 \int_{r_a}^{r_b} r^2 dr \tag{6.19}$$

Taking the time derivative in this case gives us:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} \tag{6.20}$$

so the volume goes as the cube of the scale factor, and the volume expansion is unaffected in FLRW Universe.

6.2.3. Schwarzschild and Conformally Extended Schwarzschild Metric

The volume integrated over a given radius range r_a to r_b for the Schwarzschild solution is:

$$V = \int_{r_a}^{r_b} \int_0^{\pi} \int_0^{2\pi} \sqrt{a(t)^6(1+f(r,t))^{12} r^4 sin^2\theta} dr d\theta d\phi \tag{6.21}$$

which simplifies to

$$V = 4\pi a(t)^3 \int_{r_a}^{r_b} (1+f(r,t))^6 r^2 dr \tag{6.22}$$

Taking the time derivative in this case gives us:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} + 6\frac{\int_{r_a}^{r_b}(1+f)^5 f_t r^2 dr}{\int_{r_a}^{r_b}(1+f)^6 r^2 dr} \tag{6.23}$$

Now plugging the solution that we have in (5.4):

$$f(r,t) = \frac{1}{\sqrt{m(t)^2(r^2 - r_H^2) + 1}} \tag{6.24}$$

we get integrating from $r_a = r_H$ to $r_b = r$,

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} - 6\dot{m}m(t)\frac{\int_{r_H}^{r} \frac{(\sqrt{m(t)^2(r^2-r_H^2)+1}+1)^5 r^2(r^2-r_H^2)dr}{(m(t)^2(r^2-r_H^2)+1)^4}}{\int_{r_H}^{r} \frac{(\sqrt{m(t)^2(r^2-r_H^2)+1}+1)^6 r^2 dr}{(m(t)^2(r^2-r_H^2)+1)^3}} \tag{6.25}$$

Further analysis can be made by specifying the function $m(t)$, by also considering a thin shell of infinitesimal thickness Δr and evaluating the volume expansion.

6.2.4. McVittie

The volume integrated over a given radius range r_a to r_b for the McVittie metric is:

$$V = \int_{r_a}^{r_b} \int_0^{\pi} \int_0^{2\pi} \sqrt{a(t)^6(1 + \frac{m}{2ra(t)})^{12} r^4 sin^2\theta dr d\theta d\phi} \tag{6.26}$$

which simplifies to

$$V = 4\pi a(t)^3 \int_{r_a}^{r_b} (1 + \frac{m}{2ra(t)})^6 r^2 dr \tag{6.27}$$

Taking the time derivative in this case gives us:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} - \frac{3m\dot{a}}{a^2}\frac{\int_{r_a}^{r_b}(1 + \frac{m}{2ra})^5 r dr}{\int_{r_a}^{r_b}(1 + \frac{m}{2ra})^6 r^2 dr} \tag{6.28}$$

For informative purposes the integrand will not be presented here. The volume expansion can't be evaluated starting from $r_a = 0$, so evaluating from $r_a = \frac{m}{2a(t)}$ to $r_b = r$ yields for large values of r and with m and $a(t)$ both finite,

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a}(1 - \frac{3m}{2ra(t)}) \tag{6.29}$$

which for $r \to \infty$, $\frac{\dot{V}}{V} \to \frac{3\dot{a}}{a}$. The volume expansion only asymptotically approaches that of FLRW as $r \to \infty$.

Considering a thin shell of infinitesimal thickness Δr, based on (6.22) the volume expansion as a function of the radius is given by:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} - \frac{3m\dot{a}}{a^2}\frac{(1+\frac{m}{2ra})^5 r\Delta r}{(1+\frac{m}{2ra})^6 r^2\Delta r} = 3\frac{\dot{a}}{a} - \frac{3m\dot{a}}{a^2}\frac{1}{(1+\frac{m}{2ra})r} \tag{6.30}$$

which in the limit of small r goes as $\frac{\dot{V}}{V} = -\frac{3\dot{a}}{a}$ and for large r as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a}(1 - \frac{m}{ra(t)})$, which for $r \to \infty$ goes as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a}$. The decrease in the volume expansion of the shell is slightly less than that of the essentially spherical volume of radius r (6.23), which makes sense if the volume expansion is being affected more near the black hole than at the boundary of the sphere.

6.2.5. McVittie Extension

The volume integrated over a given radius range r_a to r_b for the metric is:

$$V = \int_{r_a}^{r_b}\int_0^{\pi}\int_0^{2\pi}\sqrt{a(t)^6(1+\frac{mb(t)}{2r})^{12} r^4 sin^2\theta drd\theta d\phi} \tag{6.31}$$

which simplifies to

$$V = 4\pi a(t)^3\int_{r_a}^{r_b}(1+\frac{mb(t)}{2r})^6 r^2 dr \tag{6.32}$$

Taking the time derivative in this case gives us:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} + 3m\dot{b}\frac{\int_{r_a}^{r_b}(1+\frac{mb(t)}{2r})^5 r dr}{\int_{r_a}^{r_b}(1+\frac{mb(t)}{2r})^6 r^2 dr} \tag{6.33}$$

The volume expansion can't be evaluated starting from $r_a = 0$, so evaluating from $r_a = \frac{mb(t)}{2}$ to $r_b = r$ yields for large values of r and with m finite,

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} + \frac{9m\dot{b}}{2r} \tag{6.34}$$

and with $b(t)$ finite we obtain the same solution as the McVittie case $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a}$. Considering a thin shell of infinitesimal thickness Δr, based on (6.27) the volume expansion as a function of the radius is given by:

$$\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} + 3m\dot{b}\frac{(1+\frac{mb(t)}{2r})^5 r\Delta r}{(1+\frac{mb(t)}{2r})^6 r^2\Delta r} = 3\frac{\dot{a}}{a} + 3m\dot{b}\frac{1}{(1+\frac{mb(t)}{2r})r} \tag{6.35}$$

which in the limit of small r goes as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a} + \frac{6\dot{b}}{b}$ and for large r as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a} + \frac{3m\dot{b}}{r}$, which for $r \to \infty$ goes as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a}$.

Comparing with the FLRW solution, the decrease in the volume expansion of the McVittie cosmological black holes and not in the non-expanding, static constant-mass isotropic cosmological black holes suggest that there may be shear; although a careful specific study of the factors $m(t)$ and $b(t)$ may enlighten on the behaviour of these "expansions". These calculations only show what happens to a volume defined by a constant value of the coordinate r.

7. SUMMARY AND DISCUSSION

7.1. Static Solution

The static solution suggests that the vacuum black hole solution attains a singular horizon even if a small amount of isotropic matter-energy is introduced. In this case one expects a non-static solution with isotropic matter-energy falling into the black hole. In case of a large amount of isotropic matter-energy, the horizon disappears. We physically expect that again, matter falling into the black hole eventually causes a singular horizon to appear, with the horizon getting bigger as matter-energy keeps falling into the black hole. Thus investigation of such non-static solutions is relevant.

The fact that the equation of state parameter, $\nu > 1$, near the singular horizon is probably due to the fact that interactions other than gravity are neglected. It is a well know fact that the increase of pressure as real matter falls into the black hole causes thermonuclear fusion to occur.

From cosmological solutions, we know that matter-energy in a spatially flat universe affects the universe to expand. This may be relevant for our solution which is static and the tendancy of matter-energy to expand is balanced by the pull of the black hole.

For large values of the dimensionless parameter b, the equation of state remains close to the dust equation of state for all r. For values of $b > \frac{1}{2}$, there is no horizon and the maximum value of energy density ρ_0 is attained for $r = 0$. For $b >> \frac{1}{2}$, the relation between b and ρ_0 is

$$b = (\frac{9}{2\pi^2 G^6 M^4 \rho_0^2})^{\frac{1}{3}} \tag{7.1}$$

More generally we can say that our solution gives us also regular objects (here notably small mass objects), beside the static black hole solution.

7.2. Non-Static Solutions

7.2.1. Conformal Extension of the Schwarzschild Solution

For the Schwarzschild black hole solution that we have presented, a diagonal energy-momentum tensor satisfying $G^r_r = G^\theta_\theta = G^\phi_\phi$ corresponds to an isotropic and homogeneous fluid. An energy momentum tensor satisfying $G^r_r = G^\theta_\theta = G^\phi_\phi$ with the only non-diagonal non-vanishing term G^t_r means that we can still talk of a pressure but there is a non-vanishing radial momentum density corresponding to the infalling matter.

We have studied the effects of a time dependent isotropic metric with an energy momentum tensor whose diagonal part is isotropic and contains a radial momentum. Consequently, field equations were presented for various cases to illustrate our point of view. We have derived the energy, pressure and the equation of state parameter ν that satisfied the condition $G^r_r = G^\theta_\theta = G^\phi_\phi$ and obtained a non-singular expression for $\nu = \frac{p}{\rho}$ that binds the scale factor, r_H, ν and the function figuring in the metric proposed for the latter condition.

7.2.2. McVittie Extension

For the McVittie Extension presented, we have investigated the effects of conformal factors $a(t)$ and $b(t)$ in the background of a homogeneous and isotropic expanding universe. Consequently, field equations were presented for various cases to illustrate our point of view. From a more generalized metric that we have proposed, we have shown that, depending on given conditions, we can obtain the characteristics of $a(t)$ and $b(t)$ for the radiation dominated universe $p = \frac{\rho}{3}$, the matter dominated universe ($p = 0$, matter dominated dust approximation) and the vacuum (dark) energy dominated universe $p = -\rho$ with the McVittie metric. Systems of differential equations were solved to obtain from a new, generalized form of the metric. The non-vanishing components of Einstein tensors were explicitly given in every case.

For all the cases, G^t_r, a momentum component, made us notice that matter infalls for $r \geq \frac{m}{2}$. The particular classic McVittie solution was found to be a special solution of the Dark energy solutions that we have presented.

Concerning the event horizon of the black hole, we need to make some remarks. In our terms, the event horizon is $r = 2Gmb(t)$. For the expanding universe in the radiation dominated era, with $b(t) = 1$ we note that the event horizon is $a(t)r = 2Gma(t)$. The black hole event horizon is expanding as $t^{\frac{1}{2}}$. For the expanding universe, with the dust approximation, $b(t) = 1$ we note that the event horizon is $a(t)r = 2Gma(t)$. This means that the black hole event horizon increases with the expansion as $t^{\frac{2}{3}}$. In the case fo the dark energy dominated part, we have in the first case that $b(t) = 1$ and the event horizon goes as $a(t)r = 2Gma(t)$, i.e. as $e^{\sqrt{\frac{\Lambda}{3}}t}$ and for the second case, the McVittie case, we have $b(t) = \frac{1}{a(t)}$ and $a(t)r = 2Gm$ constant; the black hole event horizon is constant with the expansion.

In standard cosmology, for a universe with matter, $p = 0$, and the energy density which is proportional to the square of the Hubble parameter varies as a^{-3} and for purely dark energy with $p = -\rho$ the Hubble parameter is constant. ΛCDM fit to today's expanding universe gives:

$$(\frac{H}{H_0})^2 = 0,75 + 0,25(\frac{a_0}{a})^3 \tag{7.2}$$

where H_0 and a_0 are respectively the current values of the Hubble parameter and the scale factor. Hence today's universe is 75% dark energy dominated for which the event horizon is constant. But due to the 25% matter we expect the event horizon to increase but not at a slower rate than $t^{\frac{2}{3}}$.

7.2.3. Kerr Generalization

For the Kerr black hole, it is desirable to introduce a mass M rotating with angular momentum J where both J and M are time dependent but in this case the equations do not yield any solution.

By calculating the energy-momentum tensor, we have obtained some off-diagonal terms $G^t_r, G^t_\theta, G^t_\phi$ which are momentum density components, and some shear stress like G^r_ϕ and G^θ_ϕ.

On the other hand, it is well-known that introducing a cosmological constant is equivalent to adding a part to the energy-momentum tensor satisfying with our sign conventions $G^t_t = -\rho = p = G^r_r = G^\theta_\theta = G^\phi_\phi$. The solutions that we get by solving the Einstein equations with a cosmological constant are the same; only the interpretation of the energy-momentum tensor is different. In our presentation we choose a language where there is no cosmological constant, but a term satisfying
$G^t_t = -\rho = p = G^r_r = G^\theta_\theta = G^\phi_\phi$ in the energy-momentum tensor.

In the recent observations that have demonstrated the existence of a binary stellar-mass black hole system [63] and the first observation of a binary black hole merger, it has been pointed out that the angular momentum observed was very high. This may support a detail of our work concerning the angular momentum components that we have noted in calculating the Einstein tensor. Since we believe that there is a black hole at the center of each galaxy and that the rotation axes of galaxies are not isotropically distributed in space, an isotropic and homogeneous cosmology does not seem to be realistic.

From a more generalized Kerr metric in Boyer-Lindquist coordinates that we have proposed in this thesis, we have shown that, depending on given conditions, we can obtain the characteristics of the scale factor $b(t)$ for the radiation dominated universe $p = \frac{\rho}{3}$, for the matter dominated universe ($p = 0$, matter dominated dust approximation) and the vacuum (dark) energy dominated universe ($p = -\rho$). We notice the known results that for $\alpha \to 0$ the Boyer-Lindquist line element reproduces the Schwarzschild line element in an expanding universe and for $M \to 0$ the flat expanding Minkowski space.

8. CONCLUSION

In this thesis, we have obtained new cosmological black hole solutions by generalizing the static solution, the Schwarzchild solution, the McVittie Extension and the Kerr solution by performing transformations on the isotropic forms of the different metrics to obtain cosmological black holes that have physical spacetimes. The Kerr black holes and the McVittie metric black holes, with and without their generalization with a conformal factor $b(t)$ were studied to see where they are physical and where they are solutions to the Einstein Field Equations.

For the static solution that has been presented, we have found that, by a careful choice of parameters, solutions satisfying all of the energy conditions can be derived. Moreover our model has some significant consequences. Even if the volume expansion does not give us any facts since the solution is static, the Weyl components, being non-singular, signal a massive object. Studies on the equation of state parameter show that near the black hole, the energy density resides primarily in light particles having relativistic velocities and we have $p = \frac{\rho}{3}$. Far from the black hole, the matter behaves as dust particles with $p = 0$. In the case of a static solution we observe that the pressure is singular at the horizon, it is infinite, whereas the density is finite. The model is expected to converge to a state of infinite pressure at the horizon since the particles will be accumulated towards there with a given energy-density. An examination made on one of the parameters that we use to describe our new isotropic formed metric may give us a non-singular horizon but we suspect that when we would introduce more energy-density, particles to the system and study the variation of these through time a singular horizon would appear. By inspecting that parameter we have also derived that the energy density of the universe prevents small mass objects from ever attaining a horizon. Therefore we needed to perform an analysis in time.

An immediate time generalization and analysis of the previous isotropic solution gave us the behaviour in time of the Schwarzschild black hole solution. Beside the energy density and pressure, we obtain a radial momentum component showing the infalling matter in time through the event horizon. A delicacy on the singularity of these quantities is noted and we can, by a choice of parameter, have a non-singular equation of state parameter even though the energy density and pressure tend to infinity. In which case the solution suggests that in the expanding universe, the equation of state parameter is negative and may eventually be satisfied by a dark energy dominated representation of the Universe. Although the energy density and pressure become infinite at the event horizon, the energy conditions are not violated, and this may be possible and pragmatic, since the Universe would have begun that way and would simply need to preserve the original density and pressure at the event horizon.

Instead of only scaling down the mass parameter by the expansion of the Universe, we foresee a solution with conformal factors that are different. An inductive reasoning led us to obtain all the solutions, Radiation, Matter and Dark energy dominated eras, with two conformal factors. Consequently the McVittie solution is found to be a special solution of the Dark energy model. It is worthwhile to review the studies [64], where the Kerr metric is presented in the Boyer-Lindquist form and a conformal transformation is applied, and [65] where a conformal transformation of the Schwarzschild metric was made on the Eddington-Finkelstein coordinates. These two studies obtained the same result without plainly saying that the pressure and energy density would have been found negative inside the event horizon. So we must always remember try to assess if the solution is physical or not before noticing that they satisfy the Einstein Field Equations. These spacetimes are not just conformal transformations of black holes [66], so they do not have to maintain the causal structure at the event horizon. Other matter energy type generalizations may infer different properties concerning their singularities, energy density and pressure, as S.W.Hawking said in his “black holes are not black” talk, and may well carry a real singularity at their event horizon. It is noteworthy to say that the generalization of the Kerr metric presented in Chapter 5 has completely reproduced the behaviour of the scale factor with their corresponding eras.

An analysis on the universe's volume expansion as a result of the influence of black holes was presented. Results show that some of the cosmological black holes are decreasing the universe's volume expansion. This may be due to the excess matter-energy that is related with the black holes, introducing shear to the model. Others suggest that the volume expansion will depend on the behaviour of the parameters that we have posed in time, that is to say that the volume may be deformed either way for the Schwarschild metric generalization depending on a function $m(t)$, its derivative $\dot{m}(t)$ and also their respective signs in time. For the McVittie extension, the same goes for $b(t)$, its derivative $\dot{b}(t)$ and their signs. We need to imply that with a single black hole in an infinite universe, the volume expansion may not have a direct effect, but a lattice of black holes with different volume influences, increasing or decreasing, can impact the Universe as a whole and could well have formed our Universe.

To conclude, previous solutions have been more thoroughly analyzed. New cosmological black hole and independently static isotropic black hole solutions have been found, their energy-momentum tensors discussed. Cases were presented concerning their physicality, representations, singularities and also the energy conditions that they satisfy. The behaviour of the scale and conformal factors were found to satisfy the known radiation, matter and dark energy dominated universe models. We remarkably note that the Universe is composed of overdensities and underdensities, we also believe that there is a black hole at the center of each galaxy and that the rotation axes of galaxies are not isotropically distributed in space, while the existence of absolute rotation is generally assumed, it was found not to be valid [63]. Therefore an isotropic and homogeneous cosmology does not seem to be realistic. This is why the search for black hole solutions of the Einstein field equations in an expanding universe is very important and crucial to understand our Universe.

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APPENDIX A: CONFORMALLY EXTENDED SCHWARZSCHILD WITH $f(r,t)$

$$
\begin{aligned}
G^t_t = &-\frac{1}{(1+f)^5(1-f)^2}\Bigg(3\frac{\dot a^2}{a^2}(f^7) + (12\frac{\dot a}{a}\dot f + 21\frac{\dot a^2}{a^2})(f^6)\\
&+ (12\dot f^2 + 72\frac{\dot a}{a}\dot f + 63\frac{\dot a^2}{a^2})(f^5)\\
&+ (60\dot f^2 + 180\frac{\dot a}{a}\dot f + 105\frac{\dot a^2}{a^2})(f^4) + (120\dot f^2 + 240\frac{\dot a}{a}\dot f + 105\frac{\dot a^2}{a^2})(f^3)\\
&+ (120\dot f^2 + 180\frac{\dot a}{a}\dot f + 63\frac{\dot a^2}{a^2} - 4\frac{f_{r,r}}{a^2} - 8\frac{f_r}{ra^2})(f^2)\\
&+ (60\dot f^2 + 72\frac{\dot a}{a}\dot f + 21\frac{\dot a^2}{a^2} + 8\frac{f_{r,r}}{a^2} + 16\frac{f_r}{ra^2})(f)\\
&+ (12\dot f^2 + 12\frac{\dot a}{a}\dot f + 3\frac{\dot a^2}{a^2} - 4\frac{f_{r,r}}{a^2} - 8\frac{f_r}{ra^2})\Bigg)\\
G^t_r = &-\frac{4(ff_{r,t} - f_rf_t - f_{r,t} - \frac{\dot a}{a}f_r)(1+f)}{(1-f)^3}\\
G^r_r = &-\frac{1}{(1+f)^5(1-f)^3}\Bigg((2\frac{\ddot a}{a} + \frac{\dot a^2}{a^2})(f^8)\\
&+ (4\ddot f + 12\frac{\dot a}{a}\dot f + 12\frac{\ddot a}{a} + 6\frac{\dot a^2}{a^2})(f^7)\\
&+ (8\dot f^2 + 20\ddot f + 56\frac{\dot a}{a}\dot f + 28\frac{\ddot a}{a} + 14\frac{\dot a^2}{a^2})(f^6)\\
&+ (24\dot f^2 + 36\ddot f + 84\frac{\dot a}{a}\dot f + 28\frac{\ddot a}{a} + 14\frac{\dot a^2}{a^2})(f^5)\\
&+ (20\ddot f)(f^4) + (-80\dot f^2 - 20\ddot f - 140\frac{\dot a}{a}\dot f - 28\frac{\ddot a}{a} - 14\frac{\dot a^2}{a^2} - 4\frac{f_r}{ra^2})(f^3)\\
&+ (-120\dot f^2 - 36\ddot f - 168\frac{\dot a}{a}\dot f - 28\frac{\ddot a}{a} - 14\frac{\dot a^2}{a^2} + 8\frac{f_r}{ra^2} - 4\frac{f_r^2}{a^2})(f^2)\\
&+ (-72\dot f^2 - 20\ddot f - 84\frac{\dot a}{a}\dot f - 12\frac{\ddot a}{a} - 6\frac{\dot a^2}{a^2} - 4\frac{f_r}{ra^2} + 8\frac{f_r^2}{a^2})(f)\\
&+ (-16\dot f^2 - 4\ddot f - 16\frac{\dot a}{a}\dot f - 2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - 4\frac{f_r^2}{a^2})\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
G^\theta_\theta = G^\phi_\phi = -\frac{1}{(1+f)^5(1-f)^3}\Bigg(& (2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})(f^8) \\
& + (4\ddot{f} + 12\frac{\dot{a}}{a}\dot{f} + 12\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2})(f^7) \\
& + (8\dot{f}^2 + 20\ddot{f} + 56\frac{\dot{a}}{a}\dot{f} + 28\frac{\ddot{a}}{a} + 14\frac{\dot{a}^2}{a^2})(f^6) \\
& + (24\dot{f}^2 + 36\ddot{f} + 84\frac{\dot{a}}{a}\dot{f} + 28\frac{\ddot{a}}{a} + 14\frac{\dot{a}^2}{a^2})(f^5) + (20\ddot{f})(f^4) \\
& + (-80\dot{f}^2 - 20\ddot{f} - 140\frac{\dot{a}}{a}\dot{f} - 28\frac{\ddot{a}}{a} - 14\frac{\dot{a}^2}{a^2} - 2\frac{f_r}{ra^2} - 2\frac{f_{rr}}{a^2})(f^3) \\
& + (-120\dot{f}^2 - 36\ddot{f} - 168\frac{\dot{a}}{a}\dot{f} - 28\frac{\ddot{a}}{a} - 14\frac{\dot{a}^2}{a^2} + 4\frac{f_r}{ra^2} + 2\frac{f_r^2}{a^2} + 4\frac{f_{rr}}{a^2})(f^2) \\
& + (-72\dot{f}^2 - 20\ddot{f} - 84\frac{\dot{a}}{a}\dot{f} - 12\frac{\ddot{a}}{a} - 6\frac{\dot{a}^2}{a^2} - 2\frac{f_r}{ra^2} - 4\frac{f_r^2}{a^2} - 2\frac{f_{rr}}{a^2})(f) \\
& + (-16\dot{f}^2 - 4\ddot{f} - 16\frac{\dot{a}}{a}\dot{f} - 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + 2\frac{f_r^2}{a^2})\Bigg)
\end{aligned}
$$

APPENDIX B: CONFORMALLY EXTENDED SCHWARZSCHILD WITH $f(r,t) = \frac{1}{\sqrt{m(t)^2(r^2-r_H^2)+1}}$

$$
\begin{aligned}
G^t_r &= -4m^2 r(\frac{\dot{a}}{a} - \frac{\dot{m}}{m}) f^3 \frac{(1+f)}{(1-f)^3} \\
G^t_t &= -\frac{1}{(1+f)^5(1-f)^2}\Bigg(3\frac{\dot{a}^2}{a^2} + (21\frac{\dot{a}^2}{a^2})f \\
&+ (63\frac{\dot{a}^2}{a^2})f^2 \\
&+ (105\frac{\dot{a}^2}{a^2} - 12\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) + 12\frac{m^2}{a^2})f^3 \\
&+ (105\frac{\dot{a}^2}{a^2} - 72\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) - 24\frac{m^2}{a^2})f^4 \\
&+ (63\frac{\dot{a}^2}{a^2} - 180\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) + 12\frac{m^2}{a^2} - 12\frac{r^2m^4}{a^2})f^5 \\
&+ (21\frac{\dot{a}^2}{a^2} - 240\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) + 12m^2\dot{m}^2(r^4+r_H^4) - 24r^2m^2\dot{m}^2r_H^2 + 24\frac{r^2m^4}{a^2})f^6 \\
&+ (3\frac{\dot{a}^2}{a^2} - 180\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) + 60m^2\dot{m}^2(r^4+r_H^4) - 120r^2m^2\dot{m}^2r_H^2 - 12\frac{r^2m^4}{a^2})f^7 \\
&+ (-72\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) - 240r^2m^2\dot{m}^2r_H^2 + 120m^2\dot{m}^2(r^4+r_H^4))f^8 \\
&+ (-12\frac{\dot{a}}{a}m\dot{m}(r^2-r_H^2) - 240r^2m^2\dot{m}^2r_H^2 + 120m^2\dot{m}^2(r^4+r_H^4))f^9 \\
&+ (-120r^2m^2\dot{m}^2r_H^2 + 60m^2\dot{m}^2(r^4+r_H^4))f^{10} \\
&+ (-24r^2m^2\dot{m}^2r_H^2 + 12m^2\dot{m}^2(r^4+r_H^4))f^{11}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
G^r_r = G^\theta_\theta = G^\phi_\phi = -\frac{1}{(1+f)^5(1-f)^3}\Bigg(& 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 6(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})f \\
& + 14(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})f^2 \\
& + (14(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) - 4(r^2 - r_H^2)(\ddot{m}m + \dot{m}^2 + 4m\dot{m}\frac{\dot{a}}{a}))f^3 \\
& - 4((r^2 - r_H^2)(5\ddot{m}m + 5\dot{m}^2 + 21m\dot{m}\frac{\dot{a}}{a}) + \frac{m^2}{a^2})f^4 \\
& + (-14(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) - 4(r^2 - r_H^2)(9\ddot{m}m + 9\dot{m}^2 + 42m\dot{m}\frac{\dot{a}}{a}) \\
& + 12m^2\dot{m}^2(r^4 + r_H^4) - 24m^2\dot{m}^2r^2r_H^2 + 8\frac{m^2}{a^2})f^5 \\
& + (-14(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) - 20(r^2 - r_H^2)(\ddot{m}m + \dot{m}^2 + 7m\dot{m}\frac{\dot{a}}{a}) \\
& + 76m^2\dot{m}^2(r^4 + r_H^4) - 152m^2\dot{m}^2r^2r_H^2 + 4\frac{m^2}{a^2}(m^2r^2 + 1))f^6 \\
& + (-6(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) - 20(r^2 - r_H^2)(\ddot{m}m + \dot{m}^2) + 180m^2\dot{m}^2(r^4 + r_H^4) \\
& - 360m^2\dot{m}^2r^2r_H^2 - 8\frac{r^2m^4}{a^2})f^7 \\
& + (-(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) + 4(r^2 - r_H^2)(9\ddot{m}m + 9\dot{m}^2 + 21m\dot{m}\frac{\dot{a}}{a}) \\
& + 180m^2\dot{m}^2(r^4 + r_H^4) - 360m^2\dot{m}^2r^2r_H^2 + 4\frac{r^2m^4}{a^2})f^8 \\
& + (4(r^2 - r_H^2)(5\ddot{m}m + 5\dot{m}^2 + 14m\dot{m}\frac{\dot{a}}{a}) + 20m^2\dot{m}^2(r^4 + r_H^4) \\
& - 40m^2\dot{m}^2r^2r_H^2)f^9 \\
& + (4(r^2 - r_H^2)(\ddot{m}m + \dot{m}^2 + 3m\dot{m}\frac{\dot{a}}{a}) - 108m^2\dot{m}^2(r^4 + r_H^4) \\
& - 216m^2\dot{m}^2r^2r_H^2)f^{10} \\
& + (168m^2\dot{m}^2r^2r_H^2 - 84m^2\dot{m}^2(r^4 + r_H^4))f^{11} \\
& + (40m^2\dot{m}^2r^2r_H^2 - 20m^2\dot{m}^2(r^4 + r_H^4))f^{12}\Bigg)
\end{aligned}
$$

APPENDIX C: KERR METRIC WITH $b(t)$

$$G^t_t = -3\frac{\dot{b}[a^2cos^2\theta(\Delta) + r^4 + 2Mra^2 + a^2r^2]}{b^2\rho^2\Delta}$$

$$G^r_r = -\frac{[a^2cos^2\theta(\Delta) + r^4 + 2Mra^2 + a^2r^2](2\ddot{b}b + \dot{b})}{\Delta\rho^2b^2}$$

$$G^\theta_\theta = -\frac{[a^2cos^2\theta(\Delta) + r^4 + 2Mra^2 + a^2r^2](2\ddot{b}b + \dot{b})}{\Delta\rho^2b^2}$$

$$G^\phi_\phi = -\frac{[a^2cos^2\theta(\Delta) + r^4 + 2Mra^2 + a^2r^2](2\ddot{b}b + \dot{b})}{\Delta\rho^2b^2}$$

$$G^t_r = -\frac{-2M\dot{b}[r^6 + 2a^2r^4 - 4a^2Mr^3 + a^4r^2 - (r^4 - 4Mr^3 + 2a^2r^2 + a^4)a^2cos^2\theta]}{\Delta^2\rho^4b}$$

$$G^t_\theta = \frac{2\dot{b}(\alpha^2 + r^2)Mr\alpha^2sin2\theta}{\Delta\rho^4b}$$

$$G^t_\phi = 0$$

$$G^r_\phi = -\frac{2\dot{b}\alpha Msin^2\theta(3r^4 + \alpha^2r^2 + \alpha^2r^2cos^2\theta - \alpha^4cos^2\theta)}{b^2\rho^6}$$

$$G^\theta_\phi = \frac{2\dot{b}\alpha^3Mr(sin2\theta)(sin^2\theta)}{b^2\rho^6}$$