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**SOME STUDIES ON THE PROPAGATION
OF WAVES IN A VISCOELASTIC MEDIUM**

by

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SUMMARY

In this study, the general theory of viscoelastic surface waves, the effects of initial stress on the reflection and refraction of plane waves, plane Lamb's problem in a viscoelastic medium under the influence of gravity and the propagation of plane waves in a viscoelastic plate under the influence of gravity are considered. Some special types of waves such as Love and Rayleigh waves are also included.

In each case, a viscoelastic medium of Voigt type which is often represented by a parallel spring - dashpot model is used.

Wave-velocity equations and the amplitude ratios of the reflected and the refracted waves to the incident wave are deduced. The imaginary part of the complex wave-velocity is neglected since it has a very small value when compared with the real part when the retardation time τ of the medium is less than 10 seconds.

Ö Z E T

Bu çalışmada viskoelastik yüzey dalgalarının genel teorisi, ön gerilmenin düzlem dalgaların yansıma ve kırılmasına etkileri, yerçekiminin etkisi altındaki viskoelastik bir ortamda Lamb problemi ve yerçekimi etkisi altındaki viskoelastik bir levhada düzlem dalgaların yayılması incelenmiştir. Aynı zamanda Love ve Rayleigh dalgaları gibi bazı özel dalgalar da ele alınmıştır.

Bütün problemlerde, paralel bağlı bir yay ve daspot sistemi ile belirlenen, Voigt tipi viskoelastik bir ortam kullanılmıştır.

Dalga hızı denklemleri ve yansıyan ve kırılan dalga genliklerinin gelen dalganın genliğine oranını veren denklemler elde edilmiştir. Ortamın gecikme süresi (retardation time) τ , 10 saniyenin altında olduğu zaman kompleks dalga hızının sanal kısmı, reel kısmına göre çok küçük olduğundan ihmal edilmiştir.

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INTRODUCTION

The theory of wave propagation in elastic solids have been developed during the last century by Stokes, Poisson, Rayleigh, Kelvin and others as an extension of the theory of elasticity applied to the problems of vibrating bodies. A detailed treatment on the subject can be found in the works of Love [23], Kolsky [20], Timoshenko [31], Ewing, Jardetsky and Press [17], Nowacki [27], Bullen [11], Achenbach [1] and Eringen [16].

During the first quarter of this century, the subject was neglected by physicists, partly because of new fields opened up by the discoveries of atomic physics and partly because the theory was in many ways ahead of the experimental work.

In recent years a remarkable interest has been observed in the subject. First reason for this change was that, as a result of the development of electronic techniques, elastic waves of high frequency can be produced and detected easily. Secondly, the development of new materials such as plastics has led to an increased interest in the theory of the mechanical behaviour of imperfectly elastic solids, where stress waves provide a powerful tool for studying the mechanical properties of such substances.

In the course of time, engineers and scientists have become increasingly conscious of the importance of anelastic behaviour of many materials. Thus mathematical formulations have been attempted and applied to practical problems. Among them are the theories of plastic and viscoelastic materials. Applied mathematicians and theoretical physicists have also studied the influence of other interacting fields upon the dynamic behaviour of solids. As a result of these studies, theories such as magneto-elasticity, magneto-visco-elasticity, magneto-thermo-viscoelasticity and theories for elasticity and viscoelasticity under initial stress have been developed. Biot [3] developed a new theory of elasticity under initial stress and with the help of this theory he had investigated basic problems of elastostatics and elasto-dynamics.

The reflection of plane waves at a free surface of a semi-infinite elastic solid medium has been dealt by Ewing, Jardetzky and Press [17]. The general cases of reflection and refraction of plane waves at a plane interface of two elastic solid media were considered by Jeffreys [19].

The effects of gravity, curvature and viscosity on wave propagation have been considered in the works of Ewing, Jardetzky and Press [17]. Bromwich [8] considered the influence of gravity on elastic waves and in particular on an elastic globe. Love [24] investigated the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of hydrostatic nature which is developed by a slow process of creep.

Using infinitesimal theory of strain some problems of wave propagation in viscoelastic solids are considered in the present thesis. All the problems are investigated within the mathematical framework of the theory of viscoelasticity.

In the first chapter, the general theory of viscoelastic surface waves in viscoelastic solids is studied. Also, the joint effect of gravity and viscosity on elastic waves are considered. In each case, the general discussion is extended to special type of waves such as Rayleigh, Love and Stoneley waves.

The first objective of Chapter II is to study the effects of initial stress on the reflection of plane waves at a free surface of a semi-infinite viscoelastic medium of Voigt type. The second objective is to investigate the reflection and refraction of plane waves at a plane interface of two viscoelastic solid media under initial stress.

The last chapter comprises two sections. In both of the sections the effect of gravity is taken into account. The first section deals with plane Lamb's problem in a viscoelastic solid medium of Voigt type under the influence of gravity, while the second section is concerned with the propagation of plane waves in a viscoelastic plate of Voigt type under the influence of gravity.

I. VISCOELASTIC SURFACE WAVES

1.1 INTRODUCTION

The importance of surface waves in an isotropic elastic solid medium has been felt long before, due to its relevance to earthquake waves and geophysical problems. The types of surface waves such as Rayleigh waves and Love waves were investigated for an elastic solid medium by many scientists, considering the different possibilities such as the nature of the material, the geometry of the boundary surface and different layers of material lying over it. A thorough literature can be found in the works of Love [25], Ewing, Jardetzky and Press [17], Nowacki [27] and Bullen [10]. Recently, considering the interaction of electromagnetic fields with the motion of deformable solids, De and Sengupta [15] have investigated the theory of general surface waves and applied it to the particular cases such as Rayleigh, Love and Stoneley waves.

The problems of attenuation of waves in viscoelastic media have been considered by Mattice and Lieber [26]. Rotational waves in a viscoelastic medium due to a twist on the surface of a spherical cavity have been studied by Chakraborty [13].

A subsequent investigation of the influence of gravity on surface waves was presented by Love [23] in his standard work, in which he showed that the velocity of Rayleigh waves is increased to a significant extent when the wave-length is large. Biot [3] developed a theory of initial stress and used it to investigate the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. Following the same theory of initial stress and using the dynamical equations of motion for a homogeneous isotropic elastic solid medium under initial stress, De and Sengupta [15] have studied the effect of gravity on elastic waves and vibrations.

In the first section of this chapter, the general theory of viscoelastic surface waves in Voigt solids will be studied. This general discussion will then be extended to Rayleigh, Love and Stoneley waves.

The joint effect of gravity and viscosity on elastic waves will be considered in the last section. Then the special cases will be investigated. In all these cases the wave velocity equations will be deduced.

1.2. SURFACE WAVES IN VISCOELASTIC MEDIA

1.2.1 General Theory

Let M_1 and M_2 be two homogeneous semi-infinite viscoelastic media welded in contact with each other along a common surface which extends to infinity in both directions where the medium M_2 is above the medium M_1 . Consider an orthogonal Cartesian coordinate system

$Ox_1x_2x_3$, where the origin O is at any point on the plane boundary and Ox_3 is directed normally into the medium M_2 (Fig. 1).

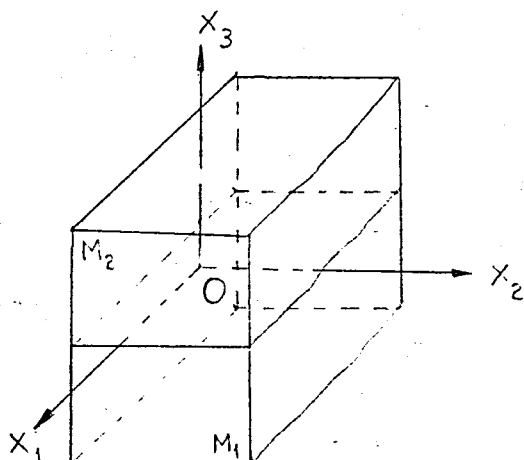


Figure 1

We consider a plane surface wave parallel to x_2 -axis moving in positive x_1 direction. The criterion for surface waves is that the displacement decays exponentially with distance from the common surface. Further, we assume that at any instant all the particles along any line parallel to the x_2 -axis have equal displacements. As a result of this assumption all the partial derivatives with respect to x_2 vanish. Let u_1 and u_3 represent the components of the displacement vector at any point (x_1, x_3) at an instant t . The dilatational and the rotational disturbances associated with the components u_1 and u_3 may be separated out by introducing the two scalar potential functions $\phi(x_1, x_3, t)$ and $\psi(x_1, x_3, t)$, called the displacement potentials given by the relations (1.1),

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3} \quad ; \quad u_3 = \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1} \quad (1.1)$$

These relations lead to:

$$\nabla^2 \phi = \Delta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}$$

$$\nabla^2 \psi = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

The equations of motion for a Voigt type solid may be written in the following form: (See App. A)

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \{(\lambda + 2\mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_j} + (\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_j$$

$$j = 1, 3 \quad (1.2)$$

where ρ is the density of the material, λ, μ Lamé constants and λ', μ' , the constants representing the effect of viscosity.

Following equations may be written after some mathematical manipulation over the Eq. (1.2) by using by using the potential functions ϕ and ψ .

$$\rho_i \frac{\partial^2 \phi_i}{\partial t^2} = (\lambda_i + 2\mu_i) \nabla^2 \phi_i + (\lambda'_i + 2\mu'_i) \nabla^2 \frac{\partial \phi_i}{\partial t} \quad (1.3)$$

$$\rho_i \frac{\partial^2 \psi_i}{\partial t^2} = \mu_i \nabla^2 \psi_i + \mu'_i \nabla^2 \frac{\partial \psi_i}{\partial t}, \quad i = 1, 2 \quad (1.4)$$

where the suffixes $i = 1, 2$ have been used to designate the relevant quantities for the media M_1 and M_2 respectively.

The Eqs. (1.3) and (1.4) will be solved for the medium M_1 . The corresponding solutions for the medium M_2 can be obtained following a similar manner. To solve the above equations in the medium M_1 we let

$$\begin{aligned}\phi_1 &= f(x_3)\exp[i\xi(x_1 - ct)] \\ \psi_1 &= g(x_3)\exp[i\xi(x_1 - ct)]\end{aligned}\tag{1.5}$$

This will lead to a particular solution corresponding to a group of simple harmonic waves of wavelength $2\pi/\xi$ travelling in the positive x_1 direction with a velocity of " c ".

One gets upon substituting the expression for ϕ_1 given by Eq. (1.5) into Eq. (1.3)

$$\frac{d^2f}{dx_3^2} + \xi^2 \left[\frac{\rho_1 c^2}{(\lambda_1 + 2\mu_1) - i\xi c(\lambda'_1 + 2\mu'_1)} - 1 \right] f = 0$$

The solution for the above equation can be written in the form of

$$f(x_3) = A_1 e^{-i\xi\gamma_1 x_3} + D_1 e^{i\xi\gamma_1 x_3}\tag{1.6}$$

where A_1 and D_1 are two unknown constants and

$$\gamma_1 = \left[\frac{\rho_1 c^2}{(\lambda_1 + 2\mu_1) - i\xi c(\lambda'_1 + 2\mu'_1)} - 1 \right]^{1/2}$$

In order to assume that we have the disturbance mostly confined to a small region close to the common surface, $f(x_3)$ must tend to zero as the distance from the boundary tends to infinity. This requirement is fulfilled if the real part of the argument of the exponential function is positive for the medium M_1 which in turn demands that the imaginary part of the complex quantity γ_1 should be positive. In view of this condition, the constant D_1 must vanish for the medium M_1 . Thus we have

$$\phi_1 = A_1 \exp[i\xi(-\gamma_1 x_3 + x_1 - ct)]$$

Substitution of the expression for ψ_1 from Eq. (1.5) into Eq. (1.4) yields

$$\psi_1 = B_1 \exp[i\xi(-\Delta_1 x_3 + x_1 - ct)]$$

where

$$\Delta_1 = \left[\frac{\rho_1 c^2}{\mu_1 - i\xi c \mu'_1} - 1 \right]^{1/2}$$

Note that the imaginary part of Δ_1 must be positive, in order to have a disturbance confined to the surface, that is a surface wave.

The corresponding expressions for ϕ_2 and ψ_2 in the medium M_2 may be determined by following the same method outlined above.

Thus the solutions for ϕ and ψ in the media M_1 and M_2 can be written as

$$\phi_1 = A_1 \exp[i\xi(-\gamma_1 x_3 + x_1 - ct)] \quad (1.7)$$

$$\psi_1 = B_1 \exp[i\xi(-\Delta_1 x_3 + x_1 - ct)]$$

$$\phi_2 = A_2 \exp[i\xi(\gamma_2 x_3 + x_1 - ct)] \quad (1.8)$$

$$\psi_2 = B_2 \exp[i\xi(\Delta_2 x_3 + x_1 - ct)]$$

$$\gamma_j = \left[\frac{\rho_j c^2}{(\lambda_j + 2\mu_j) - i\xi c(\lambda'_j + 2\mu'_j)} - 1 \right]^{1/2} \quad j = 1, 2 \quad (1.9)$$

$$\Delta_j = \left[-\frac{\rho_j c^2}{\mu_j - i\xi c \mu'_j} - 1 \right]^{1/2}$$

Note that, the unknown constants A_j and B_j ($j = 1, 2$) are to be determined using the boundary conditions:

1. The displacements over the bounding surface between the media, M_1 and M_2 must be continuous at all times.

$$\begin{aligned} (u_1)_1 &= (u_1)_2 \\ (u_3)_1 &= (u_3)_2 \end{aligned} \quad \text{at } x_3 = 0$$

2. The stress components σ_{31} , σ_{32} , σ_{33} [See App. A] must also be continuous across the bounding surface

$$\begin{aligned} (\sigma_{31})_1 &= (\sigma_{31})_2 \\ (\sigma_{33})_1 &= (\sigma_{33})_2 \end{aligned} \quad \text{at } x_3 = 0$$

Condition on the continuity of the displacement field across the boundary leads to,

$$A_1 - \Delta_1 B_1 = A_2 + \Delta_2 B_2 \quad (1.10)$$

$$\gamma_1 A_1 + B_1 = -\gamma_2 A_2 + B_2 \quad (1.11)$$

The components of the stress field for a viscoelastic body of Voigt type are given by [See App. A]

$$(\sigma_{31})_j = (\mu_j + \mu_j' \frac{\partial}{\partial t}) (2 \frac{\partial^2 \phi_j}{\partial x_3 \partial x_1} - \frac{\partial^2 \psi_j}{\partial x_1^2} + \frac{\partial^2 \psi_j}{\partial x_3^2}) \quad (1.12)$$

$$(\sigma_{33})_j = (\lambda_j + \lambda_j' \frac{\partial}{\partial t}) \nabla^2 \phi_j + 2(\mu_j + \mu_j' \frac{\partial}{\partial t}) (\frac{\partial^2 \phi_j}{\partial x_3^2} - \frac{\partial^2 \psi_j}{\partial x_3 \partial x_1}) \quad (1.13)$$

Thus, applying the second boundary conditions we get

$$(\mu_1 - i\xi c \mu_1') (2\gamma_1 A_1 + B_1 - \Delta_1^2 B_1) = (\mu_2 - i\xi c \mu_2') (-2\gamma_2 A_2 + B_2 - \Delta_2^2 B_2) \quad (1.14)$$

$$\begin{aligned}
 & -(1 + \gamma_1^2)A_1(\lambda_1 - i\xi c\lambda_1') - 2(\mu_1 - i\xi c\mu_1')(A_1\gamma_1^2 + B_1\Delta_1) = \\
 & -(1 + \gamma_2^2)A_2(\lambda_2 - i\xi c\lambda_2') - 2(\mu_2 - i\xi c\mu_2')(A_2\gamma_2^2 - B_2\Delta_2) \quad (1.15)
 \end{aligned}$$

Since Eqs. (1.10), (1.11), (1.14) and (1.15) are all homogeneous, the constants A_1 , B_1 , A_2 and B_2 will have a non-trivial solution of the following transcendental equation, obtained by taking the determinant of the coefficient matrix.

$$\begin{vmatrix}
 1 & -\Delta_1 & -1 & -\Delta_2 \\
 \gamma_1 & 1 & \gamma_2 & -1 \\
 2\gamma_1(\mu_1 - i\xi c\mu_1') & (1 - \Delta_1^2)(\mu_1 - i\xi c\mu_1') & 2\gamma_2(\mu_2 - i\xi c\mu_2') & -(1 - \Delta_2^2)(\mu_2 - i\xi c\mu_2') \\
 -(1 + \gamma_1^2)(\lambda_1 - i\xi c\lambda_1') & -2\Delta_1(\mu_1 - i\xi c\mu_1') & (1 + \gamma_2^2)(\lambda_2 - i\xi c\lambda_2') & -2\Delta_2(\mu_2 - i\xi c\mu_2') \\
 -2\gamma_1^2(\mu_1 - i\xi c\mu_1') & & +2\gamma_2^2(\mu_2 - i\xi c\mu_2') &
 \end{vmatrix} = 0 \quad (1.16)$$

The roots of the Eq. (1.16) determine the velocity "c" of the surface waves propagating along the common boundary of two semi-infinite viscoelastic solids of Voigt type.

Since the velocity "c" is complex, we can separate out the real and the imaginary parts. Thus,

$$c = c_1 + ic_2$$

When the retardation time τ [See App.B] is between 0.1 to 1 seconds, c_2 can be neglected when compared by c_1 . Real part of the solution of the Eq. (1.16) for different types of viscoelastic materials is given in "Numerical Results" section in Chapter 4.

1.2.2 Rayleigh Waves

The possibility of a wave travelling along the free surface of an elastic half-space such that the disturbance is largely confined to the neighbourhood of the boundary was first considered by Lord Rayleigh (1887), who showed that its velocity of propagation is smaller than that of body waves.

Now, we shall consider the propagation of Rayleigh waves in a semi-infinite viscoelastic solid medium of Voigt type. In this case, the medium M_2 is replaced by vacuum, so that the plane boundary now becomes the free surface. As a result, the constants A_2 and B_2 related to medium M_2 vanish which leads to the following modification of the Eqs. (1.14) and (1.15)

$$2\gamma_1 A_1 + B_1(1 - \Delta_1^2) = 0 \quad (1.17)$$

$$[(1 + \gamma_1^2)(\lambda_1 - i\xi c\lambda_1') + 2\gamma_1^2(\mu_1 - i\xi c\mu_1')]A_1 + 2\Delta_1(\mu_1 - i\xi c\mu_1')B_1 = 0 \quad (1.18)$$

The wave velocity equation for "c" can be obtained by eliminating A_1 and B_1 from the Eqs. (1.17) and (1.18):

$$4\gamma_1\Delta_1(\mu_1 - i\xi c\mu_1') - (1 - \Delta_1^2)[(1 + \gamma_1^2)(\lambda_1 - i\xi c\lambda_1') + 2\gamma_1^2(\mu_1 - i\xi c\mu_1')] = 0 \quad (1.19)$$

Substituting the values of γ_1 and Δ_1 from Eq. (1.9) into Eq. (1.19) and using the notations

$$\alpha_o^2 = [(\lambda_1 + 2\mu_1) - i\xi c(\lambda_1' + 2\mu_1')]/\rho_1$$

$$\beta_o^2 = (\mu_1 - i\xi c\mu_1')/\rho_1$$

we get

$$\frac{4\rho_1\beta_0}{\alpha_0}(c^2 - \alpha_0^2)^{\frac{1}{2}}(c^2 - \beta_0^2)^{\frac{1}{2}} - \frac{\rho_1(2\beta_0^2 - c^2)}{\alpha_0^2\beta_0^2}[c^2(\alpha_0^2 - 2\beta_0^2) + 2\beta_0^2(c^2 - \alpha_0^2)] = 0 \quad (1.20)$$

which on simplification becomes

$$1 - \frac{8\beta_0^2}{c^2}(24 - 16\frac{\beta_0^2}{\alpha_0^2})\frac{\beta_0^4}{c^4} - 16(1 - \frac{\beta_0^2}{\alpha_0^2})\frac{\beta_0^6}{c^6} = 0 \quad (1.21)$$

Equation (1.21), with slight changes in notations, is the same equation given in the discussion on the propagation of plane Rayleigh waves in a Voigt solid half-space in the standard work of Ewing, Jardetzky and Press [17].

The wave velocity equation derived above is studied for different values of viscoelastic parameters when $\tau = 0.1$ and 1 seconds. The imaginary part of the complex "c" is neglected as in the general case. Numerical solutions for the real part is given in the "Numerical Results" section in Chapter 4.

When the material is perfectly elastic, we put $\lambda_1' = \mu_1' = 0$ in Eq. (1.20) which then becomes

$$4(1 - \frac{c^2}{\alpha_1^2})^{\frac{1}{2}}(1 - \frac{c^2}{\beta_1^2})^{\frac{1}{2}} - (2 - \frac{c^2}{\beta_1^2})^2 = 0$$

where

$$\alpha_1^2 = (\lambda_1 + 2\mu_1)/\rho_1 \quad ; \quad \beta_1^2 = \mu_1/\rho_1$$

This result is in agreement with the corresponding classical result [11].

1.2.3 Love Waves

For Rayleigh waves the direction of vibration of the horizontal components should be parallel to the direction of propagation, whereas horizontal components parallel to the wave front are often found. Love (1911) has suggested that these waves can be accounted for by assuming that the elasticity and density of the outer layer of the earth differs from that in the interior. He showed that transverse waves can be propagated through such an outer layer without penetrating into the interior. Waves of this type are known as "Love waves".

We now consider the case where the half-space M_1 is covered by a medium M_2 of finite thickness H (Fig. 2). Assume that the two media are welded to each other over the common surface.

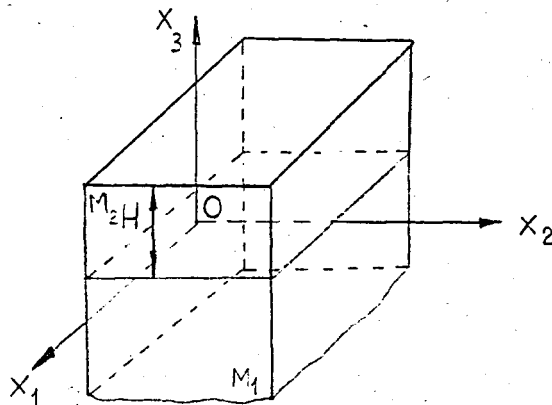


Figure 2.

Since this is the case of surface waves with displacements perpendicular to the plane of propagation we only consider the component of displacement u_2 which is in the direction of x_2 -axis. For the medium M_1 , we proceed in the same manner as in the case of two half-spaces welded to each other. Thus, with the previous

notations Eq. (1.2) can be written for $j = 2$, in the following form

$$\rho_i \frac{\partial^2 (u_2)_i}{\partial t^2} = \mu_i \nabla^2 (u_2)_i + \mu_i' \nabla^2 \frac{\partial}{\partial t} (u_2)_i \quad (1.22)$$

Let

$$(u_2)_1 = g(x_3) \exp[i\xi(x_1 - ct)] \quad (1.23)$$

Substituting the above equation into Eq. (1.22) leads to the solution

$$(u_2)_1 = C_1 \exp[i\xi(-\Delta_1 x_3 + x_1 - ct)] \quad (1.24)$$

Note that the imaginary part of Δ_1 must be positive, in order to have a disturbance confined to the surface.

For the medium M_2 we have to retain the full solution as the displacement no longer diminishes with the increasing distance from the common boundary. Thus

$$(u_2)_2 = A' \exp[i\xi(\Delta_2 x_3 + x_1 - ct)] + B' \exp[i\xi(-\Delta_2 x_3 + x_1 - ct)] \quad (1.25)$$

where the imaginary part of the complex quantity Δ_2 need not to be positive as in the previous case.

Now, the displacement component u_2 and the stress component σ_{32} must be continuous over the plane of contact. Hence

$$\begin{aligned} (u_2)_1 &= (u_2)_2 \\ (\sigma_{32})_1 &= (\sigma_{32})_2 \end{aligned} \quad \text{at } x_3 = 0 \quad (1.26)$$

The stress component σ_{32} is given by (See Appendix A)

$$(\sigma_{32})_j = (\mu_j + \mu_j' \frac{\partial}{\partial t}) \frac{\partial (u_2)_j}{\partial x_3} \quad (1.27)$$

Applying the boundary conditions we get

$$C_1 = A' + B' \quad (1.28)$$

$$-C_1 \Delta_1 (\mu_1 - i\xi c \mu_1') = (A' - B') (\mu_2 - i\xi c \mu_2') \Delta_2 \quad (1.29)$$

Elimination of C_1 , using Eqs. (1.28) and (1.29) leads to

$$\begin{aligned} A' [\Delta_2 (\mu_2 - i\xi c \mu_2') + \Delta_1 (\mu_1 - i\xi c \mu_1')] = \\ B' [\Delta_2 (\mu_2 - i\xi c \mu_2') - \Delta_1 (\mu_1 - i\xi c \mu_1')] \end{aligned} \quad (1.30)$$

One must also use the boundary condition that there is no stress over the free surface. Thus

$$(\sigma_{32})_2 = 0 \quad \text{at } x_3 = H \quad (1.31)$$

Applying the last boundary condition we get

$$A' e^{i\xi \Delta_2 H} = B' e^{-i\xi \Delta_2 H} \quad (1.32)$$

Eliminating A' and B' using Eqs. (1.30) and (1.32), we obtain

$$\Delta_2 (\mu_2 - i\xi c \mu_2') \tan(\xi \Delta_2 H) + i \Delta_1 (\mu_1 - i\xi c \mu_1') = 0 \quad (1.33)$$

which is the characteristic equation for the wave velocity "c".

In case the material is perfectly elastic, we set $\mu_1' = \mu_2' = 0$ and the resulting equation is in agreement with the corresponding classical result [11].

1.2.4 Stoneley Waves

In the classical theory, Stoneley waves are generalized form of Rayleigh waves propagating along the common boundary of two half-spaces.

From the discussion on the general theory, we observe that Stoneley waves propagating along the common surface between two semi-infinite viscoelastic solid media of Voigt type are determined by the roots of the Eq. (1.16).

We also note that when M_1 and M_2 are perfectly elastic, λ'_1 , μ'_1 , λ'_2 , and μ'_2 all vanish and Eq. (1.16) becomes

$$\begin{vmatrix} 1 & -\Delta_1 & -1 & -\Delta_2 \\ \gamma_1 & 1 & \gamma_2 & -1 \\ 2\rho_1\beta_1^2\gamma_1 & \rho_1\beta_1^2(1-\Delta_1^2) & 2\rho_2\beta_2^2\gamma_2 & -\rho_2\beta_2^2(1-\Delta_2^2) \\ \rho_1\{\alpha_1^2(1+\gamma_1^2)-2\beta_1^2\} & 2\rho_1\beta_1^2\Delta_1 & -\rho_2\{\alpha_2^2(1+\gamma_2^2)-2\beta_2^2\} & 2\rho_2\beta_2^2\Delta_2 \end{vmatrix} = 0$$

where

$$\alpha_j^2 = \frac{\lambda_j + 2\mu_j}{\rho_j} \quad ; \quad \beta_j^2 = \frac{\mu_j}{\rho_j} \quad , \quad j = 1, 2$$

The expanded form of the above determinant can be written as

$$\begin{aligned} & c^4[(\rho_1 - \rho_2)^2 + (\rho_1\gamma_2 + \rho_2\gamma_1)(\rho_1\Delta_2 + \rho_2\Delta_1)] \\ & + 2Qc^2[\gamma_1\Delta_1\rho_2 - \gamma_2\Delta_2\rho_1 - \rho_1 + \rho_2] + Q^2(\gamma_1\Delta_1 + 1)(\gamma_2\Delta_2 + 1) = 0 \end{aligned}$$

where

$$Q = 2(\rho_1 \beta_1^2 - \rho_2 \beta_2^2)$$

The last equation is in agreement with the corresponding classical result of general surface waves for perfectly elastic isotropic solid media [11].

1.3 SURFACE WAVES IN VISCOELASTIC MEDIA UNDER THE INFLUENCE OF GRAVITY

1.3.1 General Theory

Let M_1 and M_2 be two homogeneous semi-infinite viscoelastic media welded in contact with each other along a common surface which extends to infinity in both directions where the medium M_2 is above the medium M_1 . Consider an orthogonal Cartesian coordinate system $Ox_1x_2x_3$, where the origin O is at any point on the plane boundary and Ox_3 is directed normally into the medium M_1 (Fig. 3).

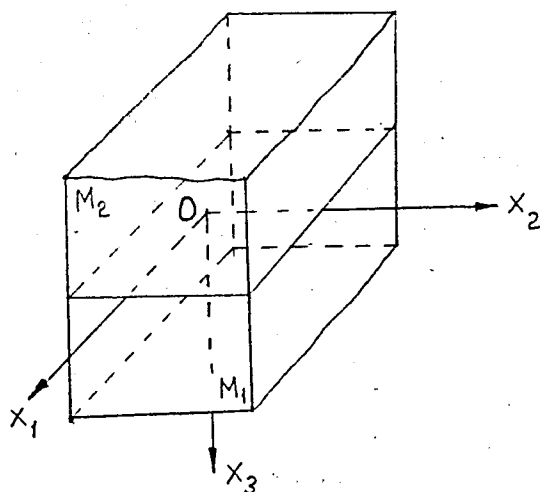


Figure 3.

We consider a plane surface wave parallel to x_2 -axis, moving in positive x_1 -direction. Further, we assume that at any instant all the particles along any line parallel to the x_2 -axis have equal displacements so that all the partial derivatives with respect to x_2 vanish.

Let u_1 and u_3 represent the components of the displacement vector at any point (x_1, x_3) at an instant t . The dilatational and rotational disturbances associated with the components u_1 and u_3 may be separated out by introducing the two scalar potential functions $\phi(x_1, x_3, t)$ called the displacement potentials given by the relations[1]

$$\begin{aligned} u_1 &= \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} \\ u_3 &= \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \end{aligned} \quad (1.34)$$

which lead to

$$\begin{aligned} \nabla^2 \phi &= \Delta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \\ \nabla^2 \psi &= \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \end{aligned}$$

The dynamical equations of motion of the three-dimensional problem under the influence of gravity are [3]

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho g \frac{\partial u_3}{\partial x_1} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho g \frac{\partial u_3}{\partial x_2} &= \rho \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} - \rho g \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (1.35)$$

where ρ is the density of the medium, g is the acceleration of gravity and σ_{11} , σ_{12} etc. are the stress components.

According to Voigt's definition, the stress-strain relations in a viscoelastic medium can be written as (See App. A)

$$\sigma_{ij} = \delta_{ij} \left(\lambda + \lambda' \frac{\partial}{\partial t} \right) + 2 \left(\mu + \mu' \frac{\partial}{\partial t} \right) e_{ij} \quad (1.36)$$

Substitution of Eq. (1.36) into Eqs. (1.35) yields

$$\begin{aligned} \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right] \frac{\partial \Delta}{\partial x_1} + (\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_1 + \rho g \frac{\partial u_3}{\partial x_1} \\ = \rho \frac{\partial^2 u_1}{\partial t^2} \end{aligned}$$

$$(\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_2 = \rho \frac{\partial^2 u_2}{\partial t^2} \quad (1.37)$$

$$\begin{aligned} \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right] \frac{\partial \Delta}{\partial x_3} + (\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_3 \\ - \rho g \frac{\partial u_1}{\partial x_1} = \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned}$$

where ρ is the density of the material, λ , μ Lamé constants and λ' , μ' the constants representing the effect of viscosity. Using the potential functions ϕ and ψ , following equations can be written, after some mathematical manipulation over the Eqs. (1.37)

$$(\alpha_j^2 + \alpha_j'^2 \frac{\partial}{\partial t}) \nabla^2 \phi_j + g \frac{\partial \psi_j}{\partial x_1} = \frac{\partial^2 \phi_j}{\partial t^2} \quad (1.38)$$

$$(\beta_j^2 + \beta_j'^2 \frac{\partial}{\partial t}) \nabla^2 \psi_j - g \frac{\partial \phi_j}{\partial x_1} = \frac{\partial^2 \psi_j}{\partial t^2} \quad (1.39)$$

$$(\beta_j^2 + \beta_j'^2 \frac{\partial}{\partial t}) \nabla^2 (u_2)_j = \frac{\partial^2 (u_2)_j}{\partial t^2} \quad j = 1, 2 \quad (1.40)$$

where the suffixes 1 and 2 have been used to designate the relevant quantities for the media M_1 and M_2 respectively and

$$\alpha_j^2 = \frac{\lambda_j + 2\mu_j}{\rho_j}, \quad \alpha_j'^2 = \frac{\lambda_j' + 2\mu_j'}{\rho_j}, \quad \beta_j^2 = \frac{\mu_j}{\rho_j},$$

$$\beta_j'^2 = \frac{\mu_j'}{\rho_j} \quad (1.41)$$

The Eqs. (1.38) and (1.39) will be solved for the medium M_1 . The corresponding solutions for the medium M_2 can be obtained following a similar manner. To solve the above equations in the medium M_1 we let,

$$\phi_1 = F(x_3)\exp[i\omega(x_1 - ct)] \quad (1.42)$$

$$\psi_1 = G(x_3)\exp[i\omega(x_1 - ct)] \quad (1.43)$$

$$(u_2)_1 = H(x_3)\exp[i\omega(x_1 - ct)] \quad (1.44)$$

One gets upon substituting the Eqs. (1.42) and (1.43) into Eqs. (1.38) and (1.39)

$$\left(\frac{d^2}{dx_3^2} + h_1\right)F + \frac{ig\omega G}{\alpha_1^2 - i\omega c\alpha_1'^2} = 0 \quad (1.45)$$

$$\left(\frac{d^2}{dx_3^2} + k_1\right)G - \frac{ig\omega F}{\beta_1^2 - i\omega c\beta_1'^2} = 0 \quad (1.46)$$

where

$$h_1^2 = \frac{\omega^2 c^2}{\alpha_1^2 - i\omega c\alpha_1'^2} - \omega^2 \quad ; \quad k_1^2 = \frac{\omega^2 c^2}{\beta_1^2 - i\omega c\beta_1'^2} - \omega^2$$

From Eqs. (1.45) and (1.46) we obtain the ordinary differential equation satisfied by F or G as

$$\left[\left(\frac{d^2}{dx_3^2} + p_1^2 \omega^2 \right) \left(\frac{d^2}{dx_3^2} + q_1^2 \omega^2 \right) \right] (F, G) = 0 \quad (1.47)$$

where

$$p_1^2 + q_1^2 = \frac{h_1^2 + k_1^2}{\omega^2}$$

$$p_1^2 q_1^2 = \frac{h_1^2 k_1^2 - m_1^2}{\omega^4}$$

$$m_1^2 = \frac{\omega^2 g^2}{(\alpha_1^2 - i\omega\alpha_1'^2 c)(\beta_1^2 - i\omega\beta_1'^2 c)}$$

The solution for the above equation can be written in the form of

$$F = A_1 e^{i\omega p_1 x_3} + B_1 e^{i\omega q_1 x_3} + L_1 e^{-i\omega p_1 x_3} + N_1 e^{-i\omega q_1 x_3} \quad (1.48)$$

where A_1 , B_1 , L_1 and N_1 are unknown constants. In order to assume that we have the disturbance mostly confined to a small region close to the common surface, the function F must tend to zero as the distance from the boundary tends to infinity. This requirement is fulfilled if the real part of the argument of the exponential function is negative, which in turn demands that the imaginary parts of p_1 and q_1 should be positive. Thus, the constants L_1 and N_1 must be zero for the expressions related to region M_1 . Therefore substituting Eq. (1.48) in Eq. (1.42) we get

$$\phi_1 = [A_1 e^{i\omega p_1 x_3} + B_1 e^{i\omega q_1 x_3}] \exp[i\omega(x_1 - ct)] \quad (1.49)$$

Proceeding in a similar manner we obtain

$$\psi_1 = [C_1 e^{i\omega p_1 x_3} + D_1 e^{i\omega q_1 x_3}] \exp[i\omega(x_1 - ct)] \quad (1.50)$$

$$(u_2)_1 = E_1 \exp[i\omega(\Delta_1 x_3 + x_1 - ct)] \quad (1.51)$$

where

$$\Delta_1 = \left[\frac{p_1 c^2}{\mu_1 - i\omega c \mu_1'} - 1 \right]^{1/2}$$

Note that the imaginary part of Δ_1 must be positive, in order to have a disturbance confined to the surface, that is a surface wave.

Substitution of the functions F and G into the Eq. (1.45)

yields

$$C_1 = n_1 A_1$$

$$D_1 = \gamma_1 B_1$$

where

$$n_1 = \frac{(\omega^2 p_1^2 - h_1^2)(\alpha_1^2 - i\omega c \alpha_1'^2)}{i\omega g}$$

$$\gamma_1 = \frac{(\omega^2 q_1^2 - h_1^2)(\alpha_1^2 - i\omega c \alpha_1'^2)}{i\omega g}$$

The corresponding quantities in the medium M_2 may be determined by following the method outlined above.

The boundary conditions which must be satisfied for our problem are:

1. The components of displacement at the boundary surface between the media M_1 and M_2 must be continuous.
2. The stress components σ_{31} , σ_{33} must be continuous across the boundary surface.

Applying the boundary condition on the continuity of displacements we get

$$A_1(1 - n_1 p_1) + B_1(1 - \gamma_1 q_1) = A_2(1 + n_2 p_2) + B_2(1 + \gamma_2 q_2) \quad (1.52)$$

$$A_1(p_1 + n_1) + B_1(q_1 + \gamma_1) = A_2(-p_2 + n_2) + B_2(-q_2 + \gamma_2) \quad (1.53)$$

The stress components in a viscoelastic media of Voigt type are given as follows (See App. A):

$$(\sigma_{31})_i = (\mu_i + \mu'_i \frac{\partial}{\partial t}) (2 \frac{\partial^2 \phi_i}{\partial x_1 \partial x_3} + \frac{\partial^2 \psi_i}{\partial x_1^2} - \frac{\partial^2 \psi_i}{\partial x_3^2}) \quad (1.54)$$

$$(\sigma_{32})_i = (\mu_i + \mu'_i \frac{\partial}{\partial t}) \frac{\partial (u_2)_i}{\partial x_3} \quad (1.55)$$

$$(\sigma_{33})_i = (\lambda_i + \lambda'_i \frac{\partial}{\partial t}) \nabla^2 \phi_i + 2(\mu_i + \mu'_i \frac{\partial}{\partial t}) (\frac{\partial^2 \phi_i}{\partial x_3^2} + \frac{\partial^2 \psi_i}{\partial x_1 \partial x_3}) \quad j = 1, 2 \quad (1.56)$$

The suffixes 1 and 2 represent the relevant quantities for medium M_1 and M_2 respectively.

Imposing the condition on the continuity of the stresses across the interface yields

$$\mu_1^*[A_1(n_1 p_1^2 - 2p_1 - n_1) + B_1(\gamma_1 q_1^2 - 2q_1 - \gamma_1)] =$$

$$\mu_2^*[A_2(n_2 p_2^2 + 2p_2 - n_2) + B_2(\gamma_2 q_2^2 + 2q_2 - \gamma_2)] \quad (1.57)$$

$$A_1[\lambda_1^*(1 + p_1^2) + 2\mu_1^*p_1(p_1 + n_1)] + B_1[\lambda_1^*(1 + q_1^2) + 2\mu_1^*q_1(q_1 + \gamma_1)] =$$

$$A_2[\lambda_2^*(1 + p_2^2) + 2\mu_2^*p_2(p_2 - n_2)] + B_2[\lambda_2^*(1 + q_2^2)$$

$$+ 2\mu_2^*q_2(q_2 - \gamma_2)] \quad (1.58)$$

where

$$\theta^* = \theta - i\omega c\theta'$$

A transcendental equation is obtained by eliminating the constants A_1 , B_1 , A_2 and B_2 among the four Eqs. (1.52), (1.53), (1.57) and (1.58). Thus,

$$\begin{vmatrix} 1-n_1 p_1 & 1-\gamma_1 q_1 & -(1+n_2 p_2) & -(1+\gamma_2 q_2) \\ p_1+n_1 & q_1+\gamma_1 & p_2-n_2 & q_2-\gamma_2 \\ F_1(p_1, n_1) & F_1(q_1, \gamma_1) & F_2(p_2, n_2) & F_2(q_2, \gamma_2) \\ H_1(p_1, n_1) & H_1(q_1, \gamma_1) & H_2(p_2, n_2) & H_2(q_2, \gamma_2) \end{vmatrix} = 0 \quad (1.59)$$

where

$$F_1(x, y) = \mu_1^*(x^2 y - 2x - y)$$

$$F_2(x, y) = \mu_2^*(y - yx^2 - 2x)$$

$$H_1(x, y) = -\lambda_1^*(1 + x^2) - 2\mu_1^*x(x + y)$$

$$H_2(x, y) = \lambda_2^*(1 + x^2) - 2\mu_2^*x(y - x)$$

The roots of the Eq. (1.59) determine the wave velocity "c" of the general surface waves propagating along the common boundary of two viscoelastic solid media of Voigt type, under the influence of gravity.

In the next sections of this chapter we will study the special cases such as Rayleigh, Love and Stoneley waves.

1.3.2 Rayleigh Waves

Now we will consider the propagation of Rayleigh waves in a semi-infinite viscoelastic solid medium of Voigt type, under the influence of gravity. In this case, the medium M_2 is replaced by vacuum, so that the plane boundary now becomes the free surface. As a result, the constants A_2 and B_2 related to medium M_2 vanish which leads to the following modification of the Eqs. (1.57) and (1.58).

$$A_1(n_1 p_1^2 - 2p_1 - n_1) + B_1(\gamma_1 q_1^2 - 2q_1 - \gamma_1) = 0 \quad (1.60)$$

$$A_1[\lambda_1^*(1 + p_1^2) + 2\mu_1^* p_1(p_1 + n_1)] + B_1[\lambda_1^*(1 + q_1^2) + 2\mu_1^* q_1(q_1 + \gamma_1)] = 0 \quad (1.61)$$

The wave velocity equation is obtained by eliminating the constants A_1 and B_1 among (1.60) and (1.61)

$$(n_1 p_1^2 - 2p_1 - n_1)[(1 + q_1^2)\lambda_1^* + 2\mu_1^* q_1(q_1 + \gamma_1)] - (\gamma_1 q_1^2 - 2q_1 - \gamma_1)[(1 + p_1^2)\lambda_1^* + 2\mu_1^* p_1(p_1 + n_1)] = 0 \quad (1.62)$$

This is the wave velocity equation for Rayleigh waves in a viscoelastic solid medium of Voigt type under the influence of gravity.

1.3.3 Love Waves

We now consider the case where the half space M_1 is covered by a medium M_2 of finite thickness H (Fig. 4). Assume that the two media are welded to each other over the common surface.

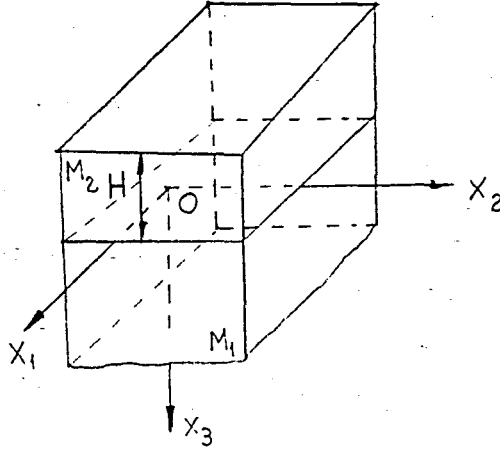


Figure 4.

Since this is the case of surface waves with displacements perpendicular to the plane of propagation, we only consider the component of displacement u_2 which is in the direction of x_2 -axis. For the medium M_1 , we proceed in the same manner as in the case of two half spaces welded to each other. Thus, with the previous notations we start by the Eq. (1.51) of the general case

$$(u_2)_1 = E_1 \exp[i\omega(\Delta_1 x_3 + x_1 - ct)] \quad (1.63)$$

Note that the imaginary part of Δ_1 must be positive, in order to have a disturbance confined to the surface.

For the medium M_2 , we have to retain the full solution as the displacement no longer diminishes with the increasing distance from the common boundary. Hence

$$(u_2)_2 = A' \exp[i\omega(\Delta_2 x_3 + x_1 - ct)] + B' \exp[i\omega(-\Delta_2 x_3 + x_1 - ct)] \quad (1.64)$$

where the imaginary part of the complex quantity Δ_2 need not to be positive as in the previous case.

The displacement component u_2 and the stress component σ_{32} must be continuous across the plane of contact.

$$\begin{aligned} (u_2)_1 &= (u_2)_2 \\ (\sigma_{32})_1 &= (\sigma_{32})_2 \end{aligned} \quad \text{at } x_3 = 0 \quad (1.65)$$

Substitution of Eqs. (1.63) and (1.64) into the Eqs. (1.65) yields

$$E_1 = A' + B' \quad (1.66)$$

$$\mu_1^* \Delta_1 E_1 = \mu_2^* \Delta_2 (A' - B') \quad (1.67)$$

Elimination of E_1 between Eqs. (1.66) and (1.67) leads to

$$A' [\Delta_2 \mu_2^* - \Delta_1 \mu_1^*] = B' [\Delta_2 \mu_2^* + \Delta_1 \mu_1^*] \quad (1.68)$$

One must also use the boundary condition that there is no stress over the free surface. Thus

$$(\sigma_{32})_2 = 0 \quad \text{at } x_3 = -H \quad (1.69)$$

applying the last boundary condition we get

$$A' e^{-i\omega \Delta_2 H} = B' e^{i\omega \Delta_2 H} \quad (1.70)$$

Eliminating A' and B' using Eqs. (1.68) and (1.70) we obtain

$$\Delta_2 \mu_2^* \tan(\omega \Delta_2 H) + i \Delta_1 \mu_1^* = 0 \quad (1.71)$$

which is the characteristic equation for the wave velocity "c".

1.3.4 Stoneley Waves

In classical theory, Stoneley waves are generalized form of Rayleigh waves propagating along the common boundary of two elastic media. Stoneley waves along the common boundary of two viscoelastic solid media of Voigt type under the influence of gravity are, therefore, determined by the roots of the Eq. (1.59).

When the media are perfectly elastic and the effect of gravity is neglected, the resulting equation is in agreement with the corresponding classical result of Stoneley [1].

II. REFLECTION AND REFRACTION OF VISCOELASTIC WAVES

2.1 INTRODUCTION

The problem of reflection of plane waves at a free surface of a semi-infinite elastic solid medium has been treated by Ewing, Jardetzky and Press [17], Bullen [11] and Jeffreys [19]. Jeffreys has studied the different cases of reflection of waves at a free surface corresponding to incident P, SV and SH waves.

Borcherdt and Buchen [4] utilized the general theory to describe the nature of plane P and SV waves in both elastic and linear anelastic media. Borcherdt [5] utilized the general theory to derive a solution for a Rayleigh-type surface wave. Their results show that the physical characteristics of both body waves and surface waves in anelastic media are significantly different from those of corresponding waves in elastic media.

Lockett (1962), Cooper and Reiss (1966), Cooper (1967), Shaw and Bugl (1969) and Schoenberg (1971) have considered the problems of homogeneous P and SV waves incident on plane boundaries between viscoelastic materials [23, 14, 31]. Their results exhibited several

characteristics of the waves not obtained in the elastic solution and showed that to study the propagation of plane waves in layered unelastic media it is necessary to consider general plane waves (that is, either homogeneous or inhomogeneous) incident on plane boundaries. In addition, it transpires that two types of inhomogeneous S waves may propagate as body waves in anelastic media. In the case of a type - I inhomogeneous S wave, the particle motion is elliptical in the plane defined by the directions of propagation and attenuation. In the case of a type - II S wave, the particle motion is linear perpendicular to this plane. With the introduction of a plane boundary perpendicular to the directions of propagation and attenuation, type-I and type-II S waves are termed SV waves and SH waves respectively, as extensions of the definitions given for elastic media.

Originally, the responses of local geological deposits to horizontal ground motions generated by earthquakes were modelled by assuming homogeneous SH waves propagating in a stack of elastic layers. Kanai [20] extended these models to Voigt type solids and derived a solution for the problem of a homogeneous SH wave normally incident on a stack of three parallel layers [20].

When the surface waves over the earth are considered, the assumptions of uniformity of material and absence of initial stress are not reasonably justified. The inner parts of the earth are under considerable stress from the weight of the matter resting upon them so that some initial stress-field will always be present in the material medium. This initial stress may be unidirectional or hydrostatic in nature.

In the present chapter, the effects of initial stress on the reflection of plane waves at a free surface of a semi-infinite viscoelastic medium of Voigt type will be investigated. The reflection and refraction of plane waves at a plane interface of two viscoelastic solid media under initial stress will be considered in the last section.

2.2 REFLECTION OF PLANE WAVES AT A FREE SURFACE OF AN INITIALLY STRESSED VISCO-ELASTIC MEDIUM

2.2.1 Unidirectional Initial Stress

We first consider the case when the initial stress is a uniaxial tension or compression. Let the free surface of a homogeneous isotropic semi-infinite viscoelastic solid medium to be the plane $x_3 = 0$ with the origin at any point on it and x_3 -axis away from the medium (Fig. 5).

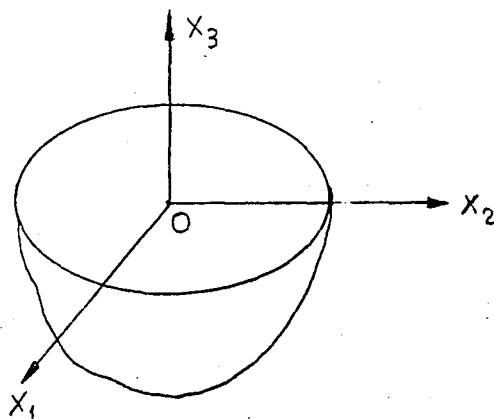


Figure 5.

Let there be a uniaxial tension (or compression) T_0 , in the direction of x_1 -axis (where $T_0 > 0$ for tension and $T_0 < 0$ for compression).

The equation of motion for a viscoelastic solid under uniaxial tension or compression are [33]

$$\begin{aligned}
 \rho \frac{\partial^2 u_1}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu')\} \frac{\partial}{\partial t} \frac{\partial \Delta}{\partial x_1} + (\mu + \mu') \frac{\partial}{\partial t} \nabla^2 u_1 \\
 &\quad + T_0 \frac{\partial^2 u_1}{\partial x_1^2} \\
 \rho \frac{\partial^2 u_2}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu')\} \frac{\partial}{\partial t} \frac{\partial \Delta}{\partial x_2} + (\mu + \mu') \frac{\partial}{\partial t} \nabla^2 u_2 \\
 &\quad + T_0 \frac{\partial^2 u_2}{\partial x_1^2} \\
 \rho \frac{\partial^2 u_3}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu')\} \frac{\partial}{\partial t} \frac{\partial \Delta}{\partial x_3} + (\mu + \mu') \frac{\partial}{\partial t} \nabla^2 u_3 \\
 &\quad + T_0 \frac{\partial^2 u_3}{\partial x_1^2}
 \end{aligned} \tag{2.1}$$

where ρ is the density of the medium and λ, μ Lamé constants.

If at any instant all the particles along a line parallel to x_2 -axis have equal displacements, then the components of displacement u_1 and u_3 at any point may be expressed in the form:

$$\begin{aligned}
 u_1 &= \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3} \\
 u_3 &= \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1}
 \end{aligned} \tag{2.2}$$

With the introduction of the potential functions ϕ and ψ , the Eqs. (2.1) reduce to the following three equations

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \phi + (\lambda' + 2\mu') \nabla^2 \frac{\partial \phi}{\partial t} + T_0 \frac{\partial^2 \phi}{\partial x_1^2} \quad (2.3)$$

$$\rho \frac{\partial^2 \psi}{\partial t^2} = \mu \nabla^2 \psi + \mu' \nabla^2 \frac{\partial \psi}{\partial t} + T_0 \frac{\partial^2 \psi}{\partial x_1^2} \quad (2.4)$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \mu \nabla^2 u_2 + \mu' \nabla^2 \frac{\partial u_2}{\partial t} + T_0 \frac{\partial^2 u_2}{\partial x_1^2} \quad (2.5)$$

The boundary conditions which must be satisfied are that all the stress-components σ_{31} , σ_{32} , σ_{33} (Eqs. (1.2), (1.13), (1.27)) should vanish at the free surface $x_3 = 0$. Thus we have at $x_3 = 0$

$$2 \frac{\partial^2 \phi}{\partial x_3 \partial x_1} - \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2} = 0 \quad (2.6)$$

$$\frac{\partial u_2}{\partial x_3} = 0 \quad (2.7)$$

$$(\lambda + \lambda' \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu + \mu' \frac{\partial}{\partial t}) (\frac{\partial^2 \phi}{\partial x_3^2} - \frac{\partial^2 \psi}{\partial x_3 \partial x_1}) = 0 \quad (2.8)$$

Incident p-wave:

Consider an incident p-wave with angle of emergence "e" which is the complement of the angle of incidence (Fig. 6)

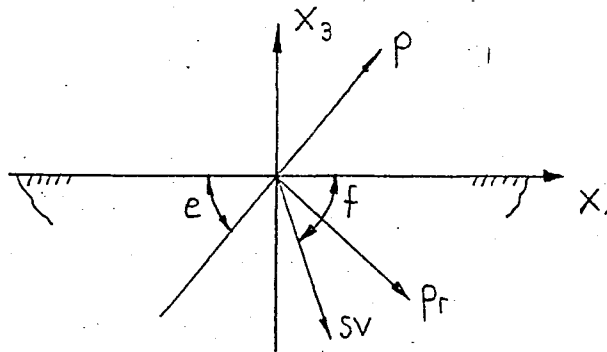


Figure 6.

On general an incident p-wave gives rise to a reflected p-wave (p_r) as well as to a reflected transverse wave with the displacement polarized in the (x_3x_2) plane (SV wave). The potential functions ϕ and ψ enables us to separate the dilatational and rotational disturbances associated with u_1 and u_3 . The component u_2 is associated with purely distortional movement. Thus ϕ and ψ correspond to p and SV waves respectively while u_2 is related to SH waves.

The solutions of the Eqs. (2.3), (2.4) and (2.5) may now be written as

$$\phi = A_1 \exp[i\xi(x_1 + x_3 \tan e - ct)] + A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \quad (2.9)$$

$$\psi = B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.10)$$

$$u_2 = C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.11)$$

The constants with the suffix 2 are related to reflected waves and the constants with the suffix 1 are related to incident waves. Substitution of the Eq. (2.11) into the Eq. (2.7) yields

$$C_2 = 0$$

which shows that there cannot be any reflected SH-wave. One gets upon substituting the Eqs. (2.9) and (2.10) into the Eqs. (2.3) and (2.4) respectively.

$$\rho c^2 = (\lambda + 2\mu) \sec^2 e - i\xi c (\lambda' + 2\mu') \sec^2 e + T_0 \quad (2.12)$$

$$\rho c^2 = (\mu - i\xi c \mu') \sec^2 f + T_0 \quad (2.13)$$

Note that the two angles e and f are related through the relations given by the Eqs. (2.12) and (2.13),

$$[(\lambda + 2\mu) - i\xi c(\lambda' + 2\mu')]\sec^2 e = (\mu - i\xi c\mu')\sec^2 f \quad (2.14)$$

We separate out the real and the imaginary parts of the complex wave velocity " c " as below,

$$c = c_1 + ic_2$$

Replacing the above equation in (2.14) and equating the real parts we get,

$$(\lambda + 2\mu)\sec^2 e + c_2\xi(\lambda' + 2\mu')\sec^2 e = c_2\xi\mu'\sec^2 f + \mu\sec^2 f$$

When the retardation time τ_2 is less than 10, c_2 can be neglected when compared by c_1 . Thus,

$$\frac{\lambda + 2\mu}{\rho} \sec^2 e = \frac{\mu}{\rho} \sec^2 f \quad (2.14')$$

Numerical results for the above equation are given in Chapter 4 by the Figure 4.1 for $\tau = 0.1$.

Applying the first boundary condition we get

$$A_1 - A_2 = \frac{1 - \tan^2 f}{2 \tan e} B_2 \quad (2.15)$$

The boundary condition (2.8), together with the Eq. (2.14) leads to

$$A_1 + A_2 = \frac{2 \tan f}{1 - \tan^2 f} B_2 \quad (2.16)$$

The amplitude ratios A_2/B_1 and B_2/A_1 can be determined through the Eqs. (2.15) and (2.16) as follows,

$$\frac{A_2}{A_1} = \frac{4 \tan e \tan f - (1 - \tan^2 f)^2}{4 \tan e \tan f + (1 - \tan^2 f)^2} \quad (2.17)$$

$$\frac{B_2}{A_1} = \frac{4 \tan e (1 - \tan^2 f)}{4 \tan e \tan f + (1 - \tan^2 f)^2} \quad (2.18)$$

Equations (2.14), (2.17) and (2.18) determine the angle f and the amplitude ratios of the incident and two reflected waves for any given angle of emergence e . The above equations agree with the classical results of elastic waves [19] and [17].

From Eq. (2.18) we observe that B_2 vanishes when $e = 0$ and $f = \pi/2$. Thus for normal incidence there can be no reflected SV-waves. In all other cases an SV-wave is reflected along with a reflected p-wave.

Note that the coefficient A_2 vanishes when

$$(1 - \tan^2 f)^2 = 4 \tan e \tan f$$

Thus, for the values of the angles e and f satisfying this relation, we do not get any reflected p-wave.

Incident SV-wave:

Consider an incident SV-wave with an angle of emergence " f ". An incident SV-wave generates a reflected p-wave as well as a reflected SV-wave thus on this case we have

$$\phi = A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \quad (2.19)$$

$$\psi = B_1 \exp[i\xi(x_1 + x_3 \tan f - ct)] + B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.20)$$

$$u_2 = C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.21)$$

From Eq. (2.21) and Eq. (2.7) we note that $C_2 = 0$, meaning that no SH-wave is reflected.

Substituting Eqs. (2.19) and (2.20) into the Eqs. (2.3) and (2.4) respectively, the angles e and f are again found to be related by Eq. (2.14).

Applying the two boundary conditions given by the Eqs. (2.6) and (2.7) we get

$$B_1 + B_2 = \frac{2 \tan e}{\tan^2 f - 1} A_2 \quad (2.22)$$

$$B_2 - B_1 = \frac{1 - \tan^2 f}{2 \tan f} A_2 \quad (2.23)$$

The amplitude ratios B_2/B_1 and A_2/B_1 can be determined through the Eqs. (2.22) and (2.23) as follows,

$$\frac{B_2}{B_1} = \frac{4 \tan e \tan f - (1 - \tan^2 f)^2}{4 \tan e \tan f + (1 - \tan^2 f)^2} \quad (2.24)$$

$$\frac{A_2}{B_1} = \frac{4 \tan f (\tan^2 f - 1)}{4 \tan e \tan f + (\tan^2 f - 1)^2} \quad (2.25)$$

Equations (2.14), (2.24) and (2.25) determine the angle e and the amplitude ratios of the incident SV-wave and the reflected wave for any given angle of emergence f . These results agree with the classical results of elastic waves [1].

From the Eqs. (2.25) and (2.7) it is observed that the reflected p-wave vanishes for $e = 0$, $e = \pi/4$ and $e = \pi/2$. For these particular values of e , the incident SV-wave is reflected as an SV-wave.

Note that the coefficient B_2 vanishes when

$$(1 - \tan^2 f)^2 = 4 \tan e \tan f$$

Thus, for the values of the angles e and f satisfying this relation, we do not get any reflected SV-wave.

Incident SH-wave:

Consider an incident SH-wave with an angle of emergence " e ". Since SH-waves are horizontally polarized transverse waves we only have to consider the displacement component in the x_2 -direction. Thus,

$$u_2 = C_1 \exp[i\xi(x_1 + x_3 \tan f - ct)] + C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.26)$$

Applying the boundary condition (2.7) we get

$$C_2 = C_1$$

which shows that the reflected wave has the same amplitude as the incident wave.

Remarks: From the preceding discussion we observe that the treatment of different cases concerning the reflection of plane waves at a free surface in the presence of an initial stress in the form of uniaxial tension or compression is the same as that if there were no initial stress. The relation between the angle of incidence and the angle of reflection and the relations among the different amplitudes remain unaffected by the initial stress. Numerical results for the relation between the angle " e " and " f " and the amplitude ratios are given in Chapter 4.

2.2.2 Hydrostatic Initial Stress

Now we consider the reflection at a free surface of a homogeneous isotropic semi-infinite viscoelastic solid medium when the initial stress is a hydrostatic tension or compression which is denoted by p_0 (tension when $p_0 > 0$ and compression when $p_0 < 0$).

The equations of motion can be written in the following form [33].

$$\begin{aligned}\rho \frac{\partial^2 u_1}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_1} + (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_1 \\ \rho \frac{\partial^2 u_2}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_2} + (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_2 \\ \rho \frac{\partial^2 u_3}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_3} + (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_3\end{aligned}\quad (2.27)$$

As before, we separate out dilatational and rotational disturbances associated with u_1 and u_3 by the use of two potential functions ϕ and ψ such that,

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3}$$

$$u_3 = \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1}$$

Then the set of Eqs. (2.27) reduce to

$$\rho \frac{\partial^2 \phi}{\partial t^2} = [(\lambda + 2\mu) - p_0 + (\lambda' + 2\mu') \frac{\partial}{\partial t}] \nabla^2 \phi \quad (2.28)$$

$$\rho \frac{\partial^2 \psi}{\partial t^2} = (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 \psi \quad (2.29)$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = (\mu - p_0 - \mu' \frac{\partial}{\partial t}) \nabla^2 u_2 \quad (2.30)$$

The boundary conditions remain the same, as in the case of uniaxial tension. That is,

$$\sigma_{31}, \sigma_{32} \text{ and } \sigma_{33} \text{ vanish at } x_3 = 0$$

We now consider the cases of incident p-wave, incident SV-wave and incident SH-wave respectively.

Incident p-wave:

The solutions of the Eqs. (2.28), (2.29) and (2.30) can be written in the following form as in the previous case

$$\phi = A_1 \exp[i\xi(x_1 + x_3 \tan e - ct)] + A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \quad (2.31)$$

$$\psi = B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.32)$$

$$u_2 = C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.33)$$

The boundary condition (2.7) gives $C_2 = 0$ so that no SH-wave is reflected.

Substitution of Eqs. (2.31) and (2.32) into Eqs. (2.28) and (2.29) respectively, yields

$$\rho c^2 = (\lambda + 2\mu - p_0) \sec^2 e - i\xi c(\lambda' + 2\mu') \sec^2 e \quad (2.34)$$

$$\rho c^2 = (\mu - p_0) \sec^2 f - i\xi c\mu' \sec^2 f \quad (2.35)$$

The two angles e and f are related through the above equations,

$$[(\lambda + 2\mu - p_0) - i\xi c(\lambda' + 2\mu')] \sec^2 e = (\mu - p_0 - i\xi c\mu') \sec^2 f \quad (2.36)$$

We separate out the real and the imaginary parts of the complex wave velocity "c" as below,

$$c = c_1 + ic_2$$

Replacing the above equation in (2.36) and equating the real parts we get,

$$\left[\frac{\lambda + 2\mu - p_0}{\rho} + c_2 \xi \frac{(\lambda' + 2\mu')}{\rho} \right] \sec^2 e = \frac{\mu - p_0}{\rho} \sec^2 f + c_2 \xi \frac{\mu'}{\rho} \sec^2 f$$

When the retardation time τ_2 is less than 10, c_2 can be neglected when compared by c_1 . Thus we have,

$$\frac{\lambda + 2\mu - p_0}{\rho} \sec^2 e = \frac{\mu - p_0}{\rho} \sec^2 f$$

Applying the boundary conditions (2.6) and (2.8) we get,

$$A_1 - A_2 = \frac{1 - \tan^2 f}{2 \tan e} B_2 \quad (2.37)$$

$$A_1 + A_2 = \frac{2\mu \tan f}{2\mu - (\lambda + 2\mu) \sec^2 e} B_2 \quad (2.38)$$

The amplitude ratios A_2/A_1 and B_2/A_1 can be determined through the Eqs. (2.37) and (2.38) as follows,

$$\frac{A_2}{A_1} = \frac{[2\mu - (\lambda + 2\mu) \sec^2 e](1 - \tan^2 f) - 4\mu \tan f \tan e}{-[2\mu - (\lambda + 2\mu) \sec^2 e](1 - \tan^2 f) - 4\mu \tan f \tan e}$$

$$\frac{B_2}{A_1} = \frac{2[2\mu - (\lambda + 2\mu) \sec^2 e] \tan e}{4\mu \tan f \tan e + (1 - \tan^2 f)[2\mu - (\lambda + 2\mu) \sec^2 e]}$$

We observe that $B_2 = 0$ when $e = 0$ or $e = \pi/2$. Thus, for these particular values of e , there will be no reflected SV-wave. On the other hand, reflected p-wave vanishes when $A_2 = 0$ which yields

$$(\tan^2 f - 1)[(\lambda + 2\mu)\sec^2 e - 2\mu] = 4\mu \tan e \tan f$$

The above equation expressed in terms of the angle "e" by means of the relation (2.36) can be written in the following form,

$$\begin{aligned} & [\lambda + (\lambda + 2\mu)\tan^2 e][\lambda + p_0 + (\lambda + 2\mu - p_0)\tan^2 e] = \\ & 4\mu \tan e (\mu - p_0)^{\frac{1}{2}} [\lambda + \mu + (\lambda + 2\mu - p_0)\tan^2 e]^{\frac{1}{2}} \end{aligned} \quad (2.39)$$

Numerical solutions for the Eq. (2.39) can be found in the numerical results section in Chapter 4.

Incident SV-wave:

In this case we write down the following set of equation as before,

$$\begin{aligned} \phi &= A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \\ \psi &= B_1 \exp[i\xi(x_1 + x_3 \tan f - ct)] + B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \\ u_2 &= C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \end{aligned}$$

Here, C_2 is again zero, applying the boundary condition (2.7) indicating that there cannot be any reflected wave of SH-type.

Applying the boundary conditions (2.6) and (2.7) one gets

$$B_1 + B_2 = \frac{2 \tan e}{\tan^2 f - 1} A_2 \quad (2.40)$$

$$B_2 - B_1 = \frac{2(\mu - i\xi c\mu') - [(\lambda + 2\mu) - i\xi c(\lambda' + 2\mu')]\sec^2 e}{2(\mu - i\xi c\mu')\tan f} A_2 \quad (2.41)$$

The amplitude ratios A_2/B_1 and B_2/B_1 can be determined through the Eqs. (2.40) and (2.41),

$$\frac{A_2}{B_1} = \frac{4\mu \tan f (\tan^2 f - 1)}{4\mu \tan e \tan f + [(\lambda + 2\mu)\sec^2 e - 2\mu](\tan^2 f - 1)} \quad (2.42)$$

$$\frac{B_2}{B_1} = \frac{[2\mu - (\lambda + 2\mu)\sec^2 e](\tan^2 f - 1) + 4\mu \tan e \tan f}{\{[-2\mu + (\lambda + 2\mu)\sec^2 e](\tan^2 f - 1) + 4\mu \tan e \tan f\}} \quad (2.43)$$

in terms of the angle e and the angle f being determined by means of the Eq. (2.36). The numerical solutions of the Eqs. (2.42) and (2.43) can be found in Chapter 4.

There can be no reflected p-wave wave when $\tan e$ becomes zero or infinite. Also, the reflected SV-wave vanishes when $B_2 = 0$ and we again get,

$$(\tan^2 f - 1)[(\lambda + 2\mu)\sec^2 e - 2\mu] = 4\mu \tan e \tan f.$$

Incident SH-wave:

As in the previous case, we observe that the reflected wave has the same amplitude as the incident wave.

In all cases, the results agree with the corresponding classical results if we put $p_0 = 0$ i.e., when there is no initial hydrostatic tension or compression.

2.3 EFFECTS OF INITIAL STRESS ON REFLECTION AND REFRACTION OF TWO VISCOELASTIC SOLID MEDIA

In this section we will study the effects of a hydrostatic initial stress on reflection and refraction of plane interface of two viscoelastic media.

Consider two homogeneous isotropic and viscoelastic media M and M_1 welded to each other along a horizontal plane surface where the medium M_1 is above the medium M . We taken an orthogonal Cartesian coordinate system $Ox_1x_2x_3$, where the origin O is at any point on the plane boundary and Ox_3 is directed normally into the medium M_1 (Fig. 7).

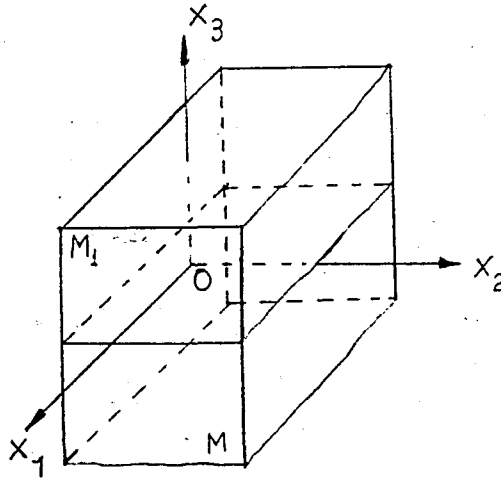


Figure 7.

The equations of motion for the medium M are given by [33].

$$\begin{aligned} \rho \frac{\partial^2 u_1}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_1} + (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_1 \\ \rho \frac{\partial^2 u_2}{\partial t^2} &= (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_2 \\ \rho \frac{\partial^2 u_3}{\partial t^2} &= \{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_3} + (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 u_3 \end{aligned} \quad (2.44)$$

With the introduction of ϕ and ψ , the set of Eqs. (2.44) reduce to the following three equations

$$\rho \frac{\partial^2 \phi}{\partial t^2} = [(\lambda + 2\mu) - p_0 + (\lambda' + 2\mu') \frac{\partial}{\partial t}] \nabla^2 \phi \quad (2.45)$$

$$\rho \frac{\partial^2 \psi}{\partial t^2} = (\mu - p_0 + \mu' \frac{\partial}{\partial t}) \nabla^2 \psi \quad (2.46)$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = (\mu - p_0 - \mu' \frac{\partial}{\partial t}) \nabla^2 u_2 \quad (2.47)$$

On using the quantities to denote those related to the medium M , the corresponding equations of motion for the medium M_1 may be written as

$$\rho_1 \frac{\partial^2 \phi_1}{\partial t^2} = [(\lambda_1 + 2\mu_1) - p_0 + (\lambda'_1 + 2\mu'_1) \frac{\partial}{\partial t}] \nabla^2 \phi_1 \quad (2.48)$$

$$\rho_1 \frac{\partial^2 \psi_1}{\partial t^2} = (\mu_1 - p_0 + \mu'_1 \frac{\partial}{\partial t}) \nabla^2 \psi_1 \quad (2.49)$$

$$\rho_1 \frac{\partial^2 (u_2)_1}{\partial t^2} = (\mu_1 - p_0 - \mu'_1 \frac{\partial}{\partial t}) \nabla^2 (u_2)_1 \quad (2.50)$$

The solutions of the Eqs. (2.44) to (2.47) may be written in the following form

$$\phi = A_1 \exp[i\xi(x_1 + x_3 \tan e - ct)] + A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \quad (2.51)$$

$$\psi = B_1 \exp[i\xi(x_1 + x_3 \tan f - ct)] + B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.52)$$

$$u_2 = C_1 \exp[i\xi(x_1 + x_3 \tan f - ct)] + C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \quad (2.53)$$

$$\phi_1 = A' \exp[i\xi(x_1 + x_3 \tan e' - ct)] \quad (2.54)$$

$$\psi_1 = B' \exp[i\xi(x_1 + x_3 \tan f' - ct)] \quad (2.55)$$

$$(u_2)_1 = C' \exp[i\xi(x_1 + x_3 \tan f' - ct)] \quad (2.56)$$

The first terms on the right hand sides of the Eqs. (2.51), (2.52) and (2.53) correspond to the incident waves and the second terms to the reflected waves. The terms on the right hand sides of the Eqs. (2.54), (2.55) and (2.56) correspond to the refracted waves.

Substitution of the Eqs. (2.51), (2.52), (2.54) and (2.55) into the Eqs. (2.45), (2.46), (2.48) and (2.49) yields,

$$\rho c^2 = (\lambda + 2\mu - p_0) \sec^2 e - i\xi c \sec^2 e (\lambda' + 2\mu') \quad (2.56)$$

$$\rho c^2 = (\mu - p_0) \sec^2 f - i\xi c \mu' \sec^2 f \quad (2.57)$$

$$\rho_1 c^2 = (\lambda_1 + 2\mu_1 - p_0) \sec^2 e' - i\xi c (\lambda'_1 + 2\mu'_1) \sec^2 e' \quad (2.58)$$

$$\rho_1 c^2 = (\mu_1 - p_0) \sec^2 f' + i\xi c \mu' \sec^2 f' \quad (2.59)$$

We separate out the real and the imaginary parts of the complex wave velocity "c" as below,

$$c = c_1 + ic_2$$

Replacing the above equation in Eqs. (2.56), (2.57), (2.58) and (2.59) we get,

$$\begin{aligned} \rho(c_1^2 + 2ic_1c_2 - c_2^2) &= (\lambda + 2\mu - p_0) \sec^2 e - i\xi c_1 (\lambda' + 2\mu') \sec^2 e \\ &\quad + \xi c_2 (\lambda' + 2\mu') \sec^2 e \end{aligned}$$

$$\rho(c_1^2 + 2ic_1c_2 - c_2^2) = (\mu - p_0)\sec^2f - i\xi c_1\mu'\sec^2f + \xi c_2\mu'\sec^2f$$

$$\begin{aligned} \rho(c_1^2 + 2ic_1c_2 - c_2^2) &= (\lambda_1 + 2\mu_1 - p_0)\sec^2e' - i\xi c_1(\lambda_1' + 2\mu_1')\sec^2e' \\ &\quad + \xi c_2(\lambda_1' + 2\mu_1')\sec^2e' \end{aligned}$$

$$\rho(c_1^2 + 2ic_1c_2 - c_2^2) = (\mu_1 - p_0)\sec^2f' + i\xi c_1 - i\xi c_2\mu'\sec^2f'$$

When the retardation time τ_2 is less than 10, c_2 can be neglected when compared by c_1 . Equating the real parts of the above equations after neglecting c_2 we obtain,

$$\rho c_1^2 = (\lambda + 2\mu - p_0)\sec^2e$$

$$\rho c_1^2 = (\mu - p_0)\sec^2f$$

$$\rho_1 c_1^2 = (\lambda_1 + 2\mu_1 - p_0)\sec^2e'$$

$$\rho_1 c_1^2 = (\mu_1 - p_0)\sec^2f'$$

The angles e , f , e' and f' are related through the above equations as follows:

$$\begin{aligned} \frac{\lambda + 2\mu - p_0}{\rho} \sec^2e &= \frac{\mu - p_0}{\rho} \sec^2f = \frac{\lambda_1 + 2\mu_1 - p_0}{\rho_1} \sec^2e' = \\ &\quad \frac{\mu_1 - p_0}{\rho_1} \sec^2f' \end{aligned} \quad (2.60)$$

The boundary conditions which must be satisfied for the present problem:

1. The displacement over the bounding surface between the media M and M_1 must be continuous at all times.
2. The stress components σ_{31} , σ_{32} and σ_{33} (See App. A) must be continuous across the bounding surface.

When expressed in terms of the potential functions and the elastic constants, the two boundary conditions above may be described by the following six equations:

$$\left[-\frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3}\right]_{x_3=0} = \left[-\frac{\partial \phi_1}{\partial x_1} + \frac{\partial \psi_1}{\partial x_3}\right]_{x_3=0} \quad (2.61)$$

$$[u_2]_{x_3=0} = [(u_2)_1]_{x_3=0} \quad (2.62)$$

$$\left[\frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1}\right]_{x_3=0} = \left[\frac{\partial \phi_1}{\partial x_3} - \frac{\partial \psi_1}{\partial x_1}\right]_{x_3=0} \quad (2.63)$$

$$\begin{aligned} (\mu + \mu' \frac{\partial}{\partial t}) \left(2 \frac{\partial^2 \phi}{\partial x_1 \partial x_3} - \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2}\right)_{x_3=0} = \\ (\mu_1 + \mu'_1 \frac{\partial}{\partial t}) \left(2 \frac{\partial^2 \phi_1}{\partial x_1 \partial x_3} - \frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{\partial^2 \psi_1}{\partial x_3^2}\right)_{x_3=0} \end{aligned} \quad (2.64)$$

$$(\mu + \mu' \frac{\partial}{\partial t}) \left(\frac{\partial u_2}{\partial x_3}\right)_{x_3=0} = (\mu_1 + \mu'_1 \frac{\partial}{\partial t}) \left(\frac{\partial (u_2)_1}{\partial x_3}\right)_{x_3=0} \quad (2.65)$$

$$\begin{aligned} (\lambda + \lambda' \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu + \mu' \frac{\partial}{\partial t}) \left(\frac{\partial^2 \phi}{\partial x_3^2} - \frac{\partial^2 \psi}{\partial x_3 \partial x_1}\right)_{x_3=0} = \\ (\lambda_1 + \lambda'_1 \frac{\partial}{\partial t}) \nabla^2 \phi_1 + 2(\mu_1 + \mu'_1 \frac{\partial}{\partial t}) \left(\frac{\partial^2 \phi_1}{\partial x_3^2} - \frac{\partial^2 \psi_1}{\partial x_3 \partial x_1}\right)_{x_3=0} \end{aligned} \quad (2.66)$$

We now consider the cases of incident p-wave, incident SV-wave and incident SH-wave and study the corresponding reflected and refracted waves.

Incident p-wave:

In general it should be anticipated that a p-wave incident on the interface of two viscoelastic solids will give rise to reflection and transmission of both p-waves and SV-waves, as shown in Figure 8.

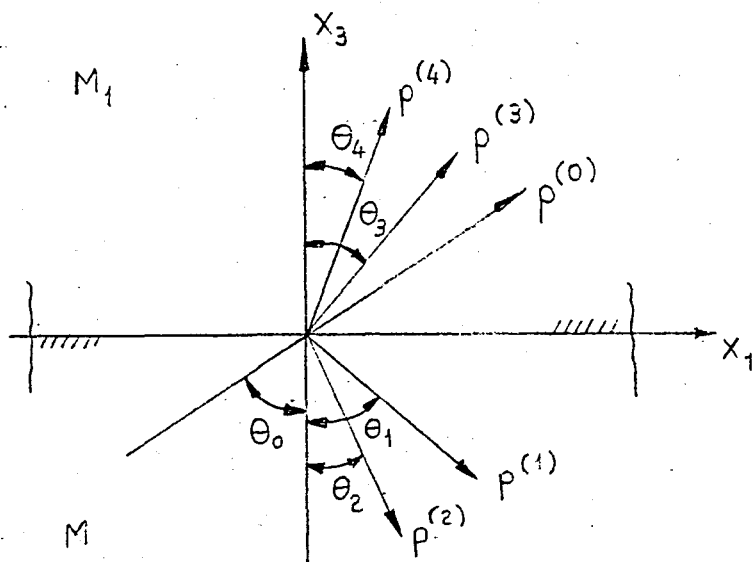


Figure 8.

The reflected longitudinal waves and the reflected transverse waves are labeled by (1) and (2) respectively. The refracted longitudinal and transverse waves are labeled by (3) and (4), respectively.

Consider an incident p-wave through the medium M with an angle of emergence "e", which is the complement of the angle of incidence " θ_0 ". Let e' and f' be the complements of the angles θ_3 and θ_4 respectively.

For an incident p-wave the Eqs. (2.52) and (2.53) can be written as

$$\psi = B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)]$$

$$u_2 = C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)]$$

Applying the boundary conditions (2.62) and (2.65) we get

$$C_2 = C' \quad (2.67)$$

$$(\mu + i\xi c_{\mu'})C_2 \tan f = -(\mu_1 + i\xi c_{\mu_1'})C' \tan f \quad (2.68)$$

From Eq. (2.67) and (2.68) we obtain $C_2 = C' = 0$ which indicates that no SH-waves are reflected or refracted for an incident p-wave.

Applying the boundary conditions (2.61), (2.63), (2.64) and (2.66) one gets the following four equations

$$A_1 + A_2 - B_2 \tan f = A' + B' \tan f' \quad (2.69)$$

$$(A_1 - A_2) \tan e - B_2 = A' \tan e' - B' \quad (2.70)$$

$$\mu[2(A_1 - A_2) - B_2(1 - \tan^2 f)] = \mu_1[2A' \tan e' - B'(1 - \tan^2 f')] \quad (2.71)$$

$$\lambda(A_1 + A_2) \sec^2 e + 2\mu[(A_1 + A_2) \tan^2 e + B_2 \tan f] = \lambda_1 A' \sec^2 e' + 2\mu_1(A' \tan^2 e' - B' \tan f) \quad (2.72)$$

The amplitude ratios A_2/A_1 , B_2/A_1 , A'/A_1 and B'/A_1 can be determined through the Eqs. (2.69), (2.70), (2.71) and (2.72) as follows,

$$\frac{A_2}{A_1} = \frac{D_1}{D} ; \quad \frac{B_2}{A_1} = \frac{D_2}{D} ; \quad \frac{A'}{A_1} = \frac{D_3}{D} ; \quad \frac{B'}{A_1} = \frac{D_4}{D} \quad (2.73)$$

where

$$D = \begin{vmatrix} 1 & -\tan f & -1 & -\tan f' \\ \tan e & 1 & \tan e' & -1 \\ 2\mu \tan e & \mu(1-\tan^2 f) & 2\mu_1 \tan e' & \mu_1(\tan^2 f' - 1) \\ \lambda \sec^2 e + 2\mu \tan^2 e & 2\mu \tan f & -(\lambda_1 \sec^2 e' + 2\mu_1 \tan^2 e') & 2\mu_1 \tan f' \end{vmatrix} \quad (2.71)$$

and D_j ($j = 1, 2, 3, 4$) is obtained from D by replacing the elements of its j th column by -1 , $\tan e$, $2\mu \tan e$ and $-(\lambda \sec^2 e + 2\mu \tan^2 e)$ respectively.

The angles of reflection and refraction can be determined in terms of the angle of incidence by means of the relations (2.60). The reflection and refraction coefficients A_2 , B_2 , A' and B' can be obtained in terms of the coefficient A_1 of the incident wave through the Eqs. (2.73) and (2.74).

There will be no reflected p-wave and SV-wave if $D_1 = 0$ and $D_2 = 0$ respectively. Also, the refracted p-wave and SV-wave vanish when $D_3 = 0$ and $D_4 = 0$ respectively.

Incident SV-wave:

Consider an incident SV-wave through the medium M with an angle of emergence " f ". In this case the Eqs. (2.52) and (2.53) can be written in the following form

$$\begin{aligned} \phi &= A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)] \\ u_2 &= C_2 \exp[i\xi(x_1 - x_3 \tan f - ct)] \end{aligned}$$

The boundary conditions (2.62) and (2.65) again lead to the relations (2.67) and (2.68) so that C_2 and C' both vanish as in the previous case. Thus, for an incident SV-wave no SH-waves are reflected or refracted.

Applying the boundary conditions (2.61), (2.63), (2.64) and (2.66) we get

$$A_2 + (B_1 - B_2)\tan f = A' + B'\tan f' \quad (2.75)$$

$$A_2\tan e + B_1 + B_2 = B' - A'\tan e' \quad (2.76)$$

$$\mu[2A_2\tan e + (B_1+B_2)(1 - \tan^2 f)] = \mu_1[B'(1 - \tan^2 f') - 2A'\tan e'] \quad (2.77)$$

$$A_2(\lambda\sec^2 e + 2\mu\tan^2 e) - 2\mu(B_1 - B_2)\tan f = A'(\lambda_1\sec^2 e' + 2\mu_1\tan^2 e') - 2\mu_1 B'\tan f' \quad (2.78)$$

On solving the Eqs. (2.75), (2.76), (2.77) and (2.78) we get

$$\frac{A_2}{B_1} = \frac{D_5}{D} ; \frac{B_2}{B_1} = \frac{D_6}{D} ; \frac{A'}{B_1} = \frac{D_7}{D} ; \frac{B'}{B_1} = \frac{D_8}{D} \quad (2.79)$$

where

$$D = \begin{vmatrix} 1 & -\tan f & -1 & -\tan f' \\ \tan e & 1 & \tan e' & -1 \\ 2\mu\tan e & \mu(1-\tan^2 f) & 2\mu_1\tan f' & \mu_1(\tan^2 f'-1) \\ \lambda\sec^2 e + 2\mu\tan^2 e & 2\mu\tan f & -(\lambda_1\sec^2 e' + 2\mu_1\tan^2 e') & 2\mu\tan f' \end{vmatrix} \quad (2.80)$$

and D_j ($j = 5, 6, 7, 8$) is obtained from D , by replacing the elements of its k th column by $-\tan f$, -1 , $\mu_1(\tan^2 f'-1)$ and $2\mu\tan f$ respectively ($k = 1, 2, 3, 4$).

The angles of reflection and refraction can be determined in terms of the angle f by means of the relations (2.60). The reflection and refraction coefficients A_2 , B_2 , A' and B' can be obtained in terms of B_1 through the Eqs. (2.79) and (2.80).

There will be no reflected p-wave and SV-wave if $D_5 = 0$ and $D_6 = 0$. Also, the refracted p-wave and the SV-wave vanish when $D_7 = 0$ and $D_8 = 0$ respectively.

Incident SH-wave:

In this case, we consider an incident SH-wave through the medium M. In general, for an incident SH-wave, both the reflected and the transmitted waves are SH-waves (Fig. 9).

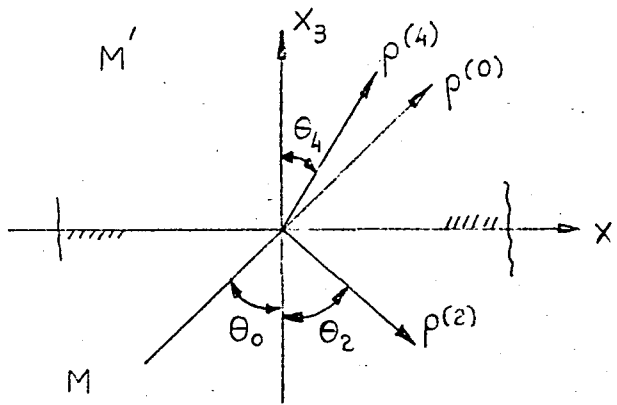


Figure 9.

The Eqs. (2.51) and (2.52) can be written as

$$\phi = A_2 \exp[i\xi(x_1 - x_3 \tan e - ct)]$$

$$\psi = B_2 \exp[i\xi(x_1 - x_3 \tan f - ct)]$$

The expressions for u_2 , ϕ_1 , ψ_1 and $(u_2)_1$ are given by the Eqs. (2.53), (2.54), (2.55) and (2.56) respectively.

The boundary conditions (2.61), (2.63), (2.64) and (2.66) involving ϕ and ψ show that the coefficients A_2 , B_2 , A' and B' have no dependence on C_1 . Hence, for an incident SH-wave, we may have reflected and refracted waves of SH-type only.

Applying the boundary conditions (2.62) and (2.65) we get

$$C_1 + C_2 = C' \quad (2.81)$$

$$\mu(C_1 - C_2)\tan f = \mu' C' \tan f' \quad (2.82)$$

Amplitude ratios C_2/C_1 and C'/C_1 are obtained through the Eqs. (2.81) and (2.82)

$$\frac{C_2}{C_1} = \frac{\mu \tan f - \mu' \tan f'}{\mu \tan f + \mu' \tan f'} \quad (2.83)$$

$$\frac{C'}{C_1} = \frac{2\mu \tan f}{\mu \tan f + \mu' \tan f'} \quad (2.84)$$

The angle f' can be determined in terms of the angle f by means of the relations (2.60). Also the coefficients of reflection and refraction can be obtained from the Eqs. (2.83) and (2.84) in terms of the amplitude C_1 of the incident wave.

There will be no reflected SH-wave if C_2 vanishes. Thus we have,

$$\mu \tan f = \mu' \tan f' \quad (2.85)$$

In this section it is observed that the presence of the hydrostatic initial stress considerably modifies the results of reflected and refracted waves in the classical case. Also the results agree with corresponding results where the media are perfectly elastic.

Numerical solutions for the Eq. (2.60) can be found in the "Numerical Results" section in Chapter 4.

III. EFFECT OF GRAVITY ON VISCOELASTIC WAVES

3.1 INTRODUCTION

In the classical theory of elastic waves and vibrations, the effects of gravity and viscosity are not considered in detail. Long before, Bromwich [6] considered the influence of gravity on elastic waves and, in particular on the vibration of an elastic globe. The effect of gravity on Rayleigh waves in an incompressible isotropic elastic solid medium was investigated by Biot [3] who treated the force of gravity as a type of initial stress of hydrostatic nature, the initial stress being produced in the body by a slow process of creep. Recently, De and Sengupta [15] have investigated the influence of gravity on plane Lamb's problem by the method of double Fourier integral transform technique. The influence of gravity on wave propagation in an elastic layer and on steady-state response to moving loads in a semi-infinite isotropic elastic solid medium are also considered by the same authors.

The propagation of monochromatic waves in an elastic plate with a stress-free surface was investigated by Rayleigh [29], Lamb [22], Prescott [28], Gogoladze [18] and Sato [30]. However, the effects of

gravity, curvature and viscosity of the material were not considered in these papers. The influence of gravity on waves in an elastic globe was first considered by Bromwich [8]. Biot studied the influence of gravity on Rayleigh waves for an incompressible medium. Following Biot's theory of initial stress, De and Sengupta [17] investigated the effect of gravity on elastic waves and vibrations.

The first section of this chapter is concerned with the investigation of plane Lamb's problem in a viscoelastic solid medium of Voigt type under the influence of gravity. In the last section, the influence of gravity on the propagation of plane waves in a viscoelastic plate of Voigt type is considered.

3.2 PLANE LAMB'S PROBLEM IN A VISCOELASTIC MEDIUM UNDER THE INFLUENCE OF GRAVITY

In this section, plane Lamb's problem in a viscoelastic solid medium of Voigt type under the influence of gravity will be studied. The wave equations will be derived starting from the equations of motion for a homogeneous isotropic and viscoelastic solid of Voigt type under initial stress.

Consider an isotropic homogeneous viscoelastic half space of Voigt type under the influence of gravity. Let there act the loading $F(x_2, t)$ normal to the plane boundary $x_1 = 0$ of the half space (Fig.10).

The components of the body force are given by $x_1 = 0$, $x_2 = g$ where g is the acceleration of gravity.

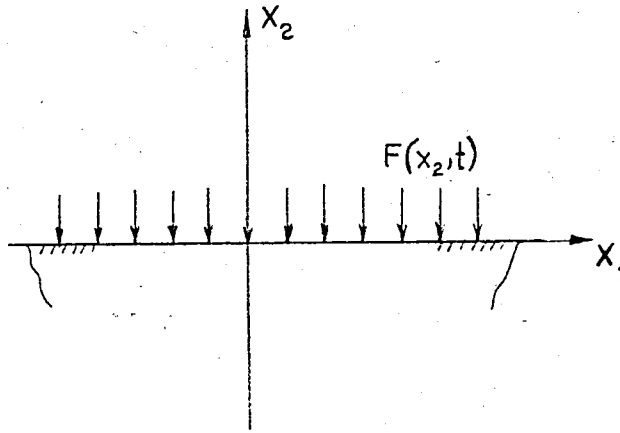


Figure 10.

Assuming the initial stress due to gravity to be hydrostatic in nature, the states of initial stress can be written as [3]

$$\sigma_{11} = \sigma_{22} = \sigma, \quad \sigma_{12} = 0 \quad (3.1)$$

where σ is a function of the depth. The equilibrium conditions of the initial stress field are [3]

$$\frac{\partial \sigma}{\partial x_1} = 0, \quad \frac{\partial \sigma}{\partial x_2} + \rho g = 0 \quad (3.2)$$

The dynamical equations of the two dimensional problem under initial stress are given by [3].

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \rho g \frac{\partial u_2}{\partial x_1} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} - \rho g \frac{\partial u_1}{\partial x_1} &= \rho \frac{\partial^2 u_2}{\partial t^2} \end{aligned} \quad (3.3)$$

with the stress components (See App. A)

$$\sigma_{ij} = 2(\mu + \mu' \frac{\partial}{\partial t})e_{ij} + \delta_{ij}(\lambda + \lambda' \frac{\partial}{\partial t})e$$

$$e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$$

$$e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$
(3.4)

Introducing the two potential functions [1]

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}$$

$$u_2 = \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}$$
(3.5)

we write down the following wave equation which we derive from (3.3) by using the Eqs. (3.4)

$$\alpha_1^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \psi}{\partial x_1} = 0$$
(3.6)

$$\alpha_2^2 \nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} - g \frac{\partial \phi}{\partial x_1} = 0$$
(3.7)

where

$$\alpha_1^2 = \frac{1}{\rho}[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t}]$$

$$\alpha_2^2 = \frac{1}{\rho}(\mu + \mu' \frac{\partial}{\partial t})$$
(3.8)

Eliminating ϕ or ψ among Eqs. (3.6) and (3.7), we obtain,

$$[(\alpha_1^2 \nabla^2 - \frac{\partial^2}{\partial t^2})(\alpha_2^2 \nabla^2 - \frac{\partial^2}{\partial t^2}) + g^2 \frac{\partial^2}{\partial x_1^2}](\phi, \psi) = 0$$
(3.9)

Thus the mathematical formulation of the two-dimensional Lamb's problem under the influence of gravity consists in finding the solutions of the Eqs. (3.6) and (3.7) under the boundary conditions

$$\sigma_{11} = -F(x_2, t) \quad , \quad \sigma_{12} = 0 \quad \text{at} \quad x_1 = 0 \quad (3.10)$$

where

$$\begin{aligned} \sigma_{11} &= (\lambda + \lambda' \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu + \mu' \frac{\partial}{\partial t}) (\frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2}) \\ \sigma_{22} &= (\lambda + \lambda' \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu + \mu' \frac{\partial}{\partial t}) (\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_2}) \\ \sigma_{12} &= (\mu + \mu' \frac{\partial}{\partial t}) (2 \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_1^2}) \end{aligned} \quad (3.11)$$

and also where the loading acting on the boundary is assumed to be bounded i.e., $\phi = 0 = \psi$ at infinity. (3.12)

Towards solving the problem, the method of double Fourier integral transform technique will be applied.

The Fourier transform of an arbitrary function $f(x)$ is defined by [2]

$$\bar{F}(f(x); x \rightarrow \xi) \equiv F(\xi) \equiv \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

where

$$F^{-1}(F(\xi); \xi \rightarrow x) \equiv f(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi$$

is the inverse transform formula.

Performing double Fourier integral transform below

$$\bar{\phi}(x_1, \xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x_1, x_2, t) e^{i(x_2 \xi + \eta t)} dx_2 dt \quad (3.13)$$

$$\phi(x_1, x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\phi}(x_1, \xi, \eta) e^{-i(x_2 \xi + \eta t)} d\xi d\eta$$

on the Eq. (3.9) we obtain

$$\left(\frac{d^2}{dx_1^2} - \gamma_1^2 \right) \left(\frac{d^2}{dx_1^2} - \gamma_2^2 \right) (\bar{\phi}, \bar{\psi}) = 0 \quad (3.14)$$

where

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 &= 2\xi^2 - k_1^2 - k_2^2 - n^2 \\ \gamma_1^2 \gamma_2^2 &= \xi^4 - (k_1^2 + k_2^2) \xi^2 + k_1^2 k_2^2 \\ c_1^2 k_1^2 &= n^2 = c_2^2 k_2^2 \end{aligned} \quad (3.15)$$

$$c_1^2 c_2^2 n^2 = g^2$$

with

$$c_1^2 = \frac{1}{\rho} [(\lambda + 2\mu) - i\eta(\lambda' + 2\mu')]$$

$$c_2^2 = \frac{1}{\rho} (\mu - i\eta\mu')$$

In view of the condition (3.12) we obtain the solution of the Eq. (3.14) in the following form.

$$\begin{aligned} \bar{\phi} &= Ae^{-\gamma_1 x_1} + Be^{-\gamma_2 x_1} \\ \bar{\psi} &= Ce^{-\gamma_1 x_1} + De^{-\gamma_2 x_1} \end{aligned} \quad (3.16)$$

where the real parts of the complex quantities γ_1 and γ_2 should be positive.

Performing the double Fourier integral transform on Eq. (3.6) or Eq. (3.7) and substituting the values of $\bar{\phi}$ and $\bar{\psi}$ as given by the Eq. (3.16), we note that the constants C, A and D, B are related as follows:

$$C = m_1 A$$

$$D = m_2 B$$

where

(3.17)

$$m_j = \frac{C_1^2}{g\gamma_j} (\gamma_j^2 - \xi^2 + k_1^2) \quad , \quad j = 1, 2$$

The unknown quantities are determined by considering the boundary conditions. Fourier transform of the boundary conditions given by the Eq. (3.10) yield,

$$[2(\mu - i\eta\mu') \left(\frac{d^2 \bar{\phi}}{dx_1^2} + i\xi \frac{d\bar{\psi}}{dx_1} \right) + (\lambda - i\eta\lambda') \left(\frac{d^2 \bar{\phi}}{dx_1^2} - \xi^2 \bar{\phi} \right)]_{x_1=0} = -\bar{F}(\xi, \eta) \quad (3.18)$$

$$[\xi^2 \bar{\psi} + \frac{d^2 \bar{\psi}}{dx_1^2} - 2i\xi \frac{d\bar{\phi}}{dx_1}]_{x_1=0} = 0$$

where

$$\bar{F}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x_2, t) e^{i(x_2 \xi + \eta t)} dx_2 dt \quad (3.19)$$

Substituting the expressions for $\bar{\phi}$ and $\bar{\psi}$ from the Eq. (3.16) into the Eqs. (3.18) we get

$$Ap_1 + Bp_2 = -\bar{F}(\xi, \eta)$$

$$Aq_1 + Bq_2 = 0$$

leading to

$$A = - \frac{q_2 \bar{F}}{p_1 q_2 - p_2 q_1} \quad (3.20)$$

$$B = \frac{q_1 \bar{F}}{p_1 q_2 - p_2 q_1}$$

$$p_j = 2(\mu - i\eta\mu')\gamma_j^2 + (\lambda - i\eta\lambda')(\gamma_j^2 - \xi^2) - i\xi\gamma_j m_j \quad (3.21)$$

$$q_j = m_j(\xi^2 + \gamma_j^2) + 2i\xi\gamma_j, \quad j = 1, 2$$

One now has to perform the inverse Fourier transform to obtain ϕ and ψ , thus,

$$\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [Ae^{-\gamma_1 x_1} + Be^{-\gamma_2 x_1}] e^{-i(\xi x_1 + \eta t)} d\xi d\eta \quad (3.22)$$

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [Am_1 e^{-\gamma_1 x_1} + Bm_2 e^{-\gamma_2 x_1}] e^{-i(\xi x_2 + \eta t)} d\xi d\eta$$

With the help of ϕ and ψ as given in Eq. (3.22) the displacement and the stress components can be determined through the Eqs. (3.4) and (3.5).

The expressions for the potential functions, displacement and stress components agree with the corresponding results for the elastic case when the effect of viscosity is ignored [15].

3.3 PROPAGATION OF WAVES IN A VISCOELASTIC PLATE UNDER THE INFLUENCE OF GRAVITY

In this section, the influence of gravity on the propagation of plane waves in a viscoelastic plate of Voigt type will be studied. Starting from the dynamical equations of motion for a homogeneous and elastic solid medium under initial stress as presented by Biot, the equations of motion will be formulated by assuming the effect of gravity to create an initial stress of hydrostatic nature. Then the wave velocity equation will be deduced.

Consider a homogeneous and isotropic plate of a viscoelastic material of Voigt type under the influence of gravity with stress-free bounding planes. As the reference system, let us take an orthogonal Cartesian system $Ox_1x_2x_3$, where the origin O is at any point on the middle plane of the infinite plate which is to be held in horizontal position and Ox_3 is directed downward. The boundary planes are given by $x_3 = \pm H$ where $2H$ is the thickness of the plate (Fig. 11).

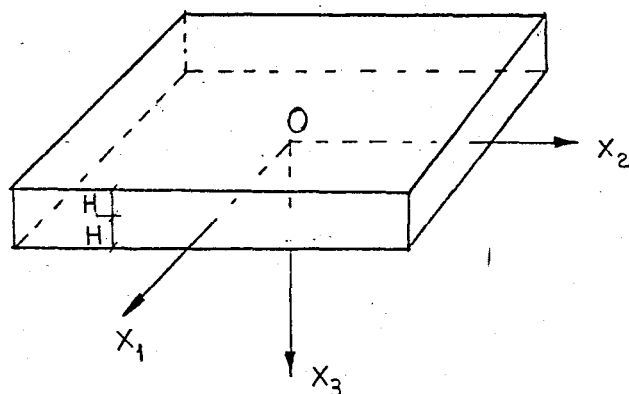


Figure 11.

We consider a plane surface wave parallel to x_2 -axis moving in positive x_1 direction. Further, we assume that at any instant all the particles along any line parallel to the x_2 -axis have equal displacements. As a result of this assumption all the partial derivatives with respect to x_2 vanish. Thus the three-dimensional elastic continuum of the layer is reduced to a set of two dimensional equations.

The dynamical equations of the two-dimensional problem under the effect of gravity are:

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho g \frac{\partial u_3}{\partial x_1} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} - \rho g \frac{\partial u_1}{\partial x_1} &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (3.23)$$

Substitution of the expressions for the stress components given by the Eqs. (3.4) into the Eqs. (3.23) yields

$$\begin{aligned} [(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial x_1} + (\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_1 \\ + \rho g \frac{\partial u_3}{\partial x_1} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ [(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial x_3} + (\mu + \mu' \frac{\partial}{\partial t}) \nabla^2 u_3 \\ - \rho g \frac{\partial u_1}{\partial x_1} &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (3.24)$$

We now assume that the components of displacement u_1 and u_3 can be obtained from the displacement potentials F and G , according to the expressions,

$$u_1 = \frac{\partial F(x_1, x_3)}{\partial x_1} - \frac{\partial G(x_1, x_3)}{\partial x_3} \quad (3.25)$$

$$u_3 = \frac{\partial F(x_1, x_3)}{\partial x_3} + \frac{\partial G(x_1, x_3)}{\partial x_1}$$

In view of the above expressions, the Eqs. (3.24) may be written in terms of the displacement potentials F and G , in the following forms:

$$(\alpha^2 + \alpha'^2 \frac{\partial}{\partial t}) \nabla^2 F + g(\frac{\partial G}{\partial x_1}) = \frac{\partial^2 F}{\partial t^2} \quad (3.26)$$

$$(\beta^2 + \beta'^2 \frac{\partial}{\partial t}) \nabla^2 G - g(\frac{\partial F}{\partial x_1}) = \frac{\partial^2 G}{\partial t^2} \quad (3.27)$$

where

$$\begin{aligned} \alpha^2 &= (\lambda + 2\mu)/\rho, & \alpha'^2 &= (\lambda' + 2\mu')/\rho \\ \beta^2 &= \mu/\rho, & \beta'^2 &= \mu'/\rho \end{aligned} \quad (3.28)$$

Since the boundary planes of the plate are free of stress, we formulate the following boundary conditions which must be satisfied for the present problem:

$$\sigma_{13} = \sigma_{33} = 0 \quad \text{at} \quad x_3 = \pm H \quad (3.29)$$

where

$$\sigma_{13} = (\mu + \mu' \frac{\partial}{\partial t}) (2 \frac{\partial^2 F}{\partial x_1 \partial x_3} + \frac{\partial^2 G}{\partial x_1^2} - \frac{\partial^2 G}{\partial x_3^2}) \quad (3.30)$$

$$\sigma_{33} = (\lambda + \lambda' \frac{\partial}{\partial t}) \nabla^2 F + 2(\mu + \mu' \frac{\partial}{\partial t}) (\frac{\partial^2 F}{\partial x_3^2} + \frac{\partial^2 G}{\partial x_1 \partial x_3})$$

Thus, we will find the solutions to the Eqs. (3.26) and (3.27) subject to the boundary conditions given by the Eq. (3.29).

In order to solve the Eqs. (3.26) and (3.27) we assume,

$$F(x_1, x_3, t) = F_1(x_3) \exp[-i\omega(x_1 - ct)] \quad (3.31)$$

$$G(x_1, x_3, t) = G_1(x_3) \exp[-i\omega(x_1 - ct)] \quad (3.32)$$

Introducing the Eqs. (3.31) and (3.32) into the Eqs. (3.26) and (3.27) we obtain

$$\left(\frac{d^2}{dx_3^2} - h^2\right)F_1 - \frac{i\omega G_1}{\alpha^2 + i\omega c\alpha'^2} = 0 \quad (3.33)$$

$$\left(\frac{d^2}{dx_3^2} - k^2\right)G_1 + \frac{i\omega g F_1}{\beta^2 + i\omega c\beta'^2} = 0 \quad (3.34)$$

where

$$h^2 = \omega^2 - \frac{\omega^2 c^2}{\alpha^2 + i\omega c\alpha'^2} \quad (3.35)$$

$$k^2 = \omega^2 - \frac{\omega^2 c^2}{\beta^2 + i\omega c\beta'^2}$$

From Eqs. (3.33) and (3.34) we obtain the differential equations that F_1 and G_1 satisfy,

$$\left[\left(\frac{d^2}{dx_3^2} - h^2\right)\left(\frac{d^2}{dx_3^2} - k^2\right) - \omega^2 g^2 / (\alpha^2 + i\omega c\alpha'^2)(\beta^2 + i\omega c\beta'^2)\right](F_1, G_1) = 0$$

which can be written in the following form

$$\left[\left(\frac{d^2}{dx_3^2} - \gamma_1^2\right)\left(\frac{d^2}{dx_3^2} - \gamma_2^2\right)\right](F_1, G_1) = 0 \quad (3.36)$$

where

$$\gamma_1^2 + \gamma_2^2 = h^2 + k^2$$

$$\gamma_1^2 \gamma_2^2 = h^2 k^2 m^2 \quad (3.37)$$

$$m^2 = \omega^2 g^2 / [(\alpha^2 + i\omega c \alpha'^2)(\beta^2 + i\omega c \beta'^2)]$$

The solutions of the Eqs. (3.36) may be written as follows:

$$\begin{aligned} F_1 = & A_1 \sinh(\gamma_1 x_3) + A_2 \cosh(\gamma_1 x_3) + A_3 \sinh(\gamma_2 x_3) \\ & + A_4 \cosh(\gamma_2 x_3) \end{aligned} \quad (3.38)$$

$$\begin{aligned} G_1 = & B_1 \sinh(\gamma_1 x_3) + B_2 \cosh(\gamma_1 x_3) + B_3 \sinh(\gamma_2 x_3) \\ & + B_4 \cosh(\gamma_2 x_3) \end{aligned} \quad (3.39)$$

Upon substituting the expressions for F_1 and G_1 as given by the Eqs. (3.38) and (3.39) into the Eqs. (3.33) and (3.34), we find that the constants A_1, A_2, A_3, A_4 are related to the constants B_1, B_2, B_3, B_4 by means of the following relations.

$$B_j = in_1 A_j \quad (j = 1, 2) \quad (3.40)$$

$$B_j = in_2 A_j \quad (j = 1, 2)$$

where

$$n_j = (h^2 - \gamma_j^2)(\alpha^2 + i\omega c \alpha'^2) / \omega g \quad (j = 1, 2) \quad (3.41)$$

The displacement potentials $F(x_1, x_3, t)$ and $G(x_1, x_3, t)$ can now be determined from Eqs. (3.31) and (3.38) and from Eqs. (3.32) and (3.39) respectively.

To determine the wave velocity equation we have to apply the boundary conditions (3.29). Utilizing the newly obtained values of

the displacement potentials $F(x_1, x_3, t)$ and $G(x_1, x_3, t)$ the following four equations are obtained:

$$A_1 p_1(q_1, -p_1) + A_2 p_1(p_1, -q_1) + A_3 p_2(q_2, -p_2) + A_4 p_2(p_2, -q_2) = 0 \quad (3.42)$$

$$A_1 p_1(q_1, p_1) - A_2 p_1(p_1, q_1) + A_3 p_2(q_2, p_2) - A_4 p_2(p_2, q_2) = 0 \quad (3.43)$$

$$A_1 Q_1(p_1, q_1) + A_2 Q_1(q_1, p_1) + A_3 Q_2(p_2, q_2) + A_4 Q_2(q_2, p_2) = 0 \quad (3.44)$$

$$A_1 Q_1(-p_1, q_1) + A_2 Q_1(q_1, -p_1) + A_3 Q_2(-p_2, q_2) + A_4 Q_2(q_2, -p_2) = 0 \quad (3.45)$$

where

$$\begin{aligned} p_j &= \sinh(\gamma_j H) \\ q_j &= \cosh(\gamma_j H) \end{aligned} \quad (3.46)$$

$$P_j(x, y) = 2\gamma_j x + n_j \gamma_j^2 y + n_j y$$

$$Q_j(x, y) = 2\mu^* \gamma_j^2 x + \lambda^* (\gamma_j^2 - 1)x - 2\mu^* n_j \gamma_j y$$

$$\lambda^* = \lambda + i\omega c \lambda^t \quad ; \quad \mu^* = \mu + i\omega c \mu^t$$

The constants A_1 , A_2 , A_3 and A_4 will have a nontrivial solution of the following transcendental equation, obtained by taking the determinant of the coefficient matrix.

$$\begin{vmatrix} P_1(q_1, -p_1) & P_1(p_1, -q_1) & P_2(q_2, -p_2) & P_2(p_2, -q_2) \\ P_1(q_1, p_1) & -P_1(p_1, q_1) & P_2(q_2, p_2) & -P_2(p_2, q_2) \\ Q_1(p_1, q_1) & Q_1(q_1, p_1) & Q_2(p_2, q_2) & Q_2(q_2, p_2) \\ Q_1(-p_1, q_1) & Q_1(q_1, -p_1) & Q_2(-p_2, q_2) & Q_2(q_2, -p_2) \end{vmatrix} = 0 \quad (3.47)$$

The roots of the Eq. (3.47) determine the velocity "c" of a wave propagating in a viscoelastic plate of Voigt type under the influence of gravity. When the effect of gravity and viscosity is ignored, the Eq. (3.47) agrees with the classical result [17].

IV. NUMERICAL RESULTS AND CONCLUSIONS

The general theory of viscoelastic surface waves are now investigated and the wave velocity equations are deduced. The numerical values for the wave velocities corresponding to different values of the retardation time τ_2 will be presented in this chapter.

The effects of initial stress on the reflection of plane waves at a free-surface of a semi-infinite viscoelastic medium and the reflection and refraction of plane waves at a plane interface of two viscoelastic solid media under initial stress are investigated in Chapter 2. The numerical calculations are carried out for the relations between the angle of incidence and the angles of reflection and refraction. The results of the amplitude ratios of the incident waves and the reflected and refracted waves for incident p, SV and SH waves are given through the chapter. The solutions of the case where the reflected p-wave vanishes for an incident p-wave under hydrostatic initial stress are also included.

The imaginary part of the complex wave velocity "c" is neglected when the values of the retardation time τ_2 is less than 10 seconds.

4.1 SURFACE WAVES IN VISCOELASTIC MEDIA

The roots of the Eq. (1.16) which determines the velocity "c" of the surface waves propagating along the common boundary of two semi-infinite viscoelastic solids of Voigt type are calculated for different values of the retardation time τ_2 (Fig. 4.1) with the following material constants for the medium M_2 :

$$\rho_2 = 930 \text{ kg/m}^3$$

$$\mu_2 = 7 \times 10^5 \text{ N/m}^2$$

$$\mu'_2 = 7 \times 10^5 \text{ N/m}^2$$

$$\lambda_2 = 1 \times 10^9 \text{ N/m}^2$$

$$\lambda'_2 = 1 \times 10^9 \text{ N/m}^2$$

The above constants are kept constant for the medium M_2 while the constants for the medium M_1 are changed by multiplying the ones for the medium M_2 by τ_2 and following results are obtained:

TABLE 1 - Velocity of the surface waves propagating along the common boundary of two semi-infinite viscoelastic solids of Voigt type.

τ_2 (sec)	c (m/sec) Real Part
0.1	4454.1
1.0	1411.9

When the retardation time τ_2 is less than 10, the imaginary part of the complex wave velocity is neglected. Thus, the results obtained above are the real parts of the wave velocities for different materials.

From the above results it is observed that the wave velocity " c " increases as τ_2 decreases (medium M_2 becomes more elastic) and the value of the wave velocity agrees with the velocity of elastic waves [21].

Equation (1.21) is solved by using the same material constants of the previous case for medium M_1 and the following values are obtained for Rayleigh wave speed;

TABLE 2 - Rayleigh wave-speed in a viscoelastic medium of Voigt type.

τ_1 (sec)	c_1 (m/sec) Real Part	c_2 (m/sec) Imaginary Part
0.1	4062.1	2.00
1.0	1291.3	5.38
10.0	287.3	313.74
25.0	181.2	497.42

From the above results it is observed that the real part of the Rayleigh wave-speed decreases when the medium becomes more viscous by the increasing value of τ_1 . When τ_1 is less than 10 the imaginary part of the wave velocity can be neglected.

4.2 REFLECTION AND REFRACTION OF VISCOELASTIC WAVES

For the case of reflection of plane waves at a free surface of an initially stressed viscoelastic medium following numerical results are obtained for p and SV waves. In each case the imaginary part of the complex wave velocity is neglected.

When the initial stress is uniaxial tension or compression the relation between the angle "e" (complement of the angle of incidence) and the angle "f" (complement of the angle of reflection) is given by the Eq. (2.14') and the amplitude ratios are determined through the Eqs. (2.17), (2.18) and (2.25). Numerical results are shown by the Figures 4.1, 4.2 and 4.3 .

Incident p-wave: $\lambda = 1 \times 10^9$ $\lambda' = 1 \times 10^8$ $\mu = 7 \times 10^5$ $\mu' = 1 \times 10^4$

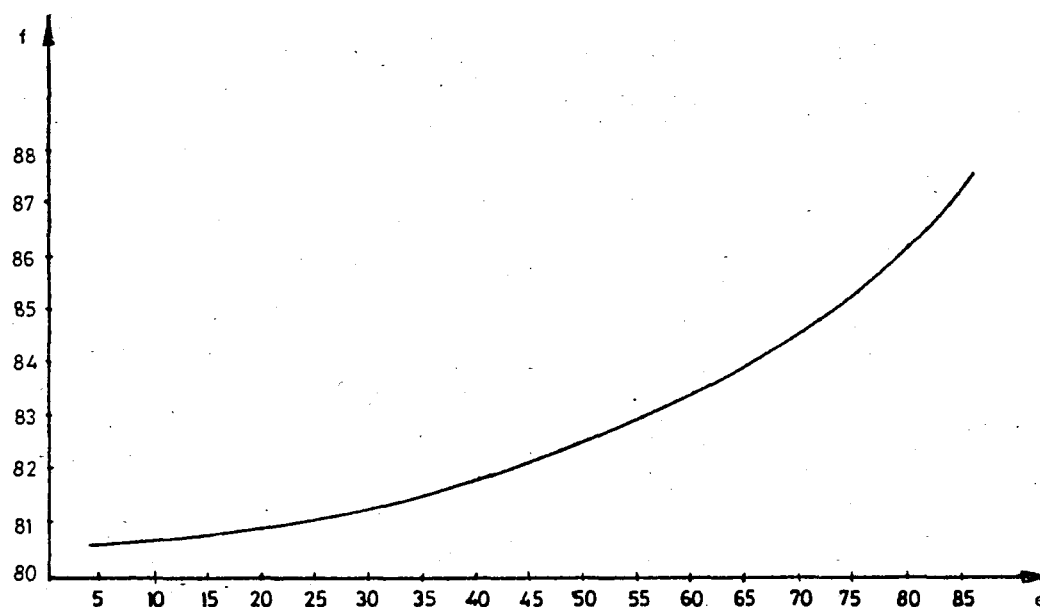
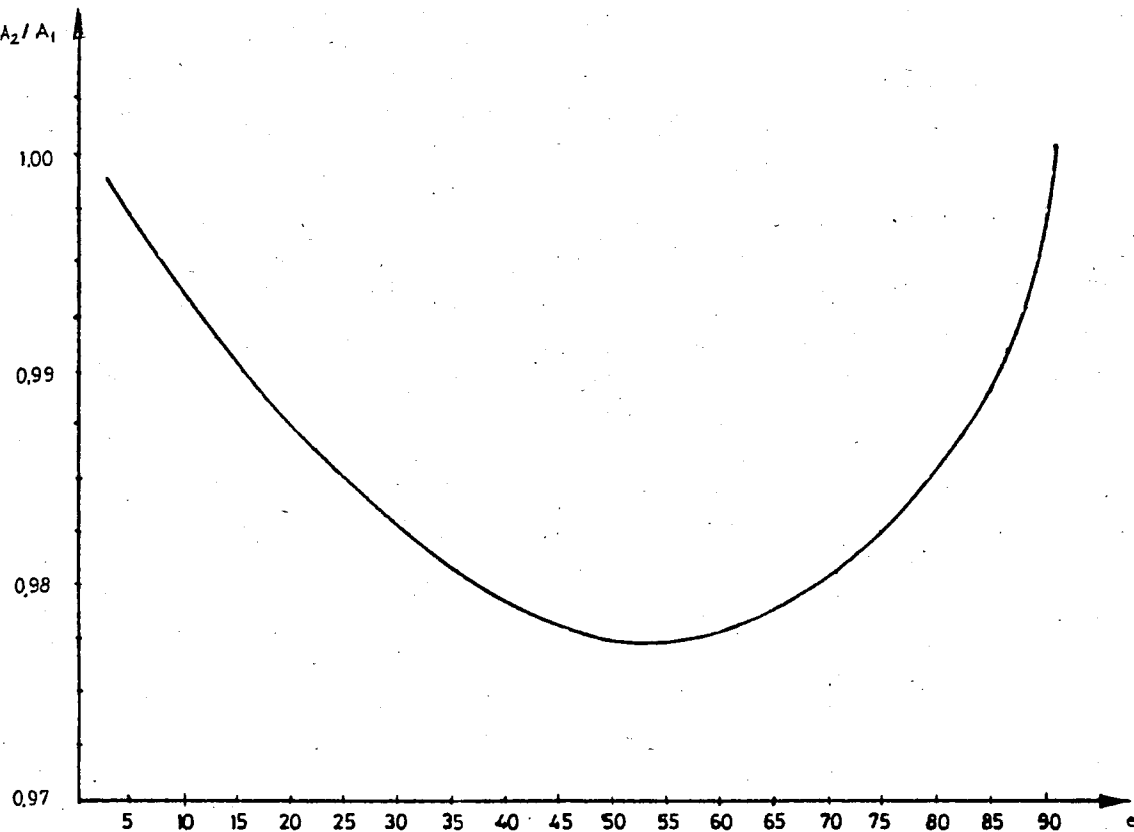


Figure 4.1 - Relation between the complement of the angle of emergence and the angle of reflection in the case of reflection of plane waves at a free surface of an initially stressed (uniaxial tension or compression) viscoelastic medium.

From Figure 4.1 we observe that the complement of the angle of reflection increases as the complement of the angle of incidence increases.



$$\lambda = 1 \times 10^9 \quad \lambda' = 1 \times 10^8 \quad \mu = 7 \times 10^5 \quad \mu' = 7 \times 10^4$$

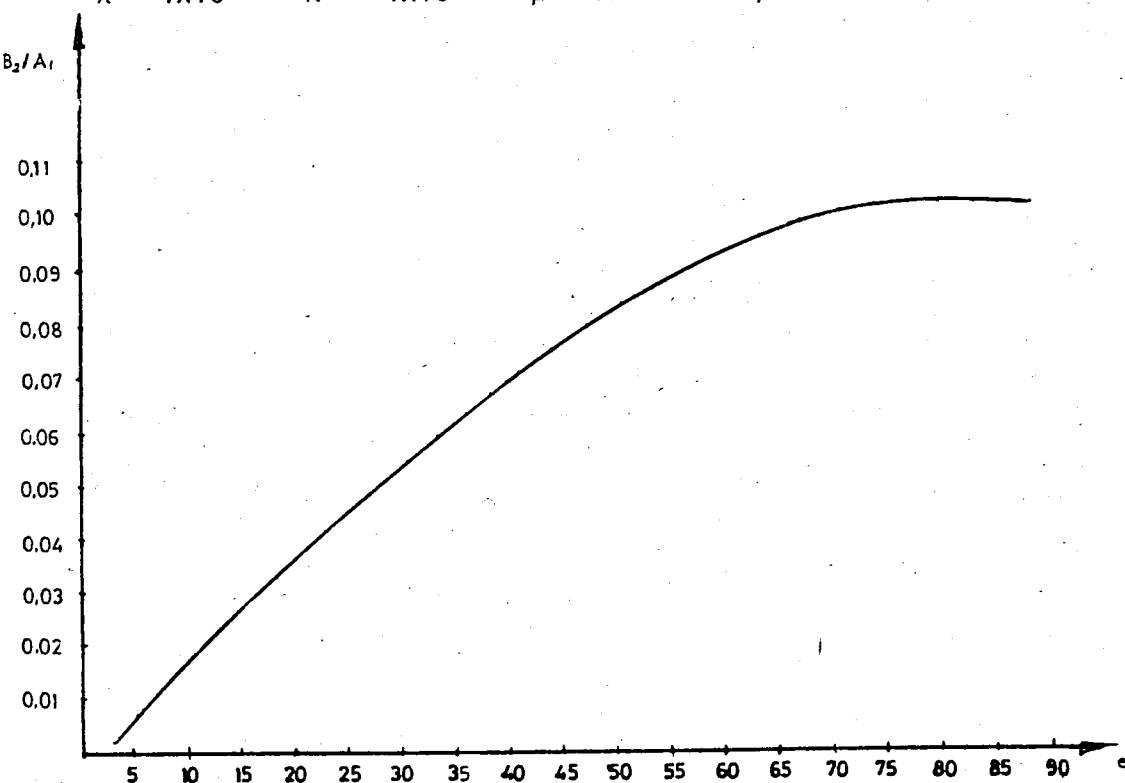


Figure 4.2 - A_2 - Amplitude of the reflected p-wave.
 B_2 - Amplitude of the reflected SV-wave.
 A_1 - Amplitude of the incident p-wave.

Note that the value of angle "f" changes between 80 and 85 degrees while the angle "e" changes between 5 to 85 degrees. The amplitude of the reflected p-wave is always less than the amplitude of the incident wave and it assumes its minimum value when $e = 50^\circ$ (Fig. 4.2). The amplitude of the reflected SV-wave is always less than the incident wave. It increases as "e" increases and the ratio stays constant when the value of e is between 70 and 85° (Fig. 4.2).

Incident SV-wave: $\lambda = 1 \times 10^9$ $\mu = 7 \times 10^5$ $\lambda' = 1 \times 10^8$ $\mu' = 7 \times 10^4$

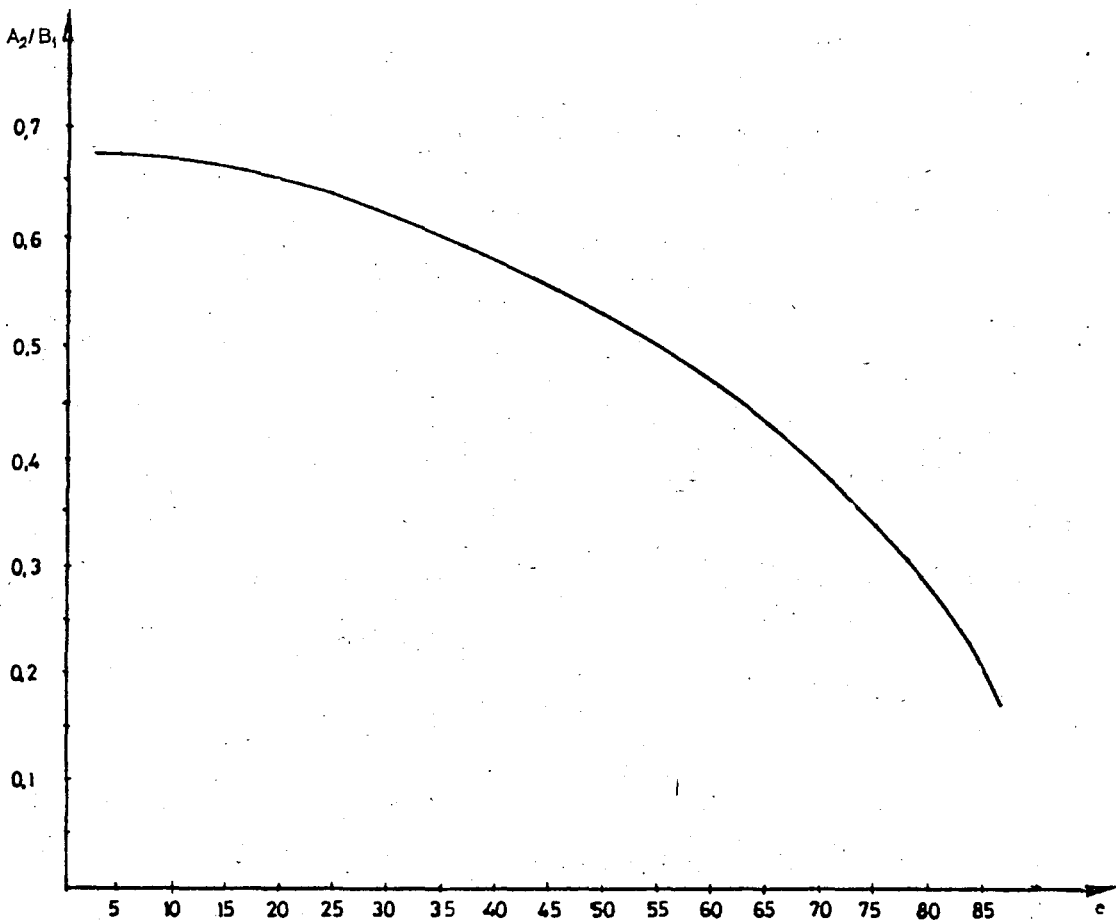


Figure 4.3 - A_2 - Amplitude of the reflected p-wave.
 B_1 - Amplitude of the incident SV-wave.

We observe that the amplitude of the reflected p-wave for an incident SV-wave is always greater than the amplitude of the incident wave. The reflected wave takes its greatest value when the angle of emergence is minimum ($e = 5^\circ$) (Fig. 4.3).

In the case of hydrostatic initial stress the reflected p-wave vanishes for an incident p-wave if Eq. (2.33) is satisfied. The solutions for the Eq. (2.39) for the cases when $\lambda = \mu = 1 \times 10^9$ [N/m²] and the initial stress $p_0 = \lambda/k$ assumes the values $k = 10, 15, 20, 25, 50, 100$ are shown in Fig. 4.4.

$p_0 = \lambda/k$	0	$\lambda/100$	$\lambda/50$	$\lambda/25$	$\lambda/20$	$\lambda/15$	$\lambda/10$
first angle	12°36'	13°0'	13°6'	13°48'	14°12'	14°54'	16°42'
second angle	29°54'	29°35'	29°0'	28°24'	28°3'	27°15'	25°12'

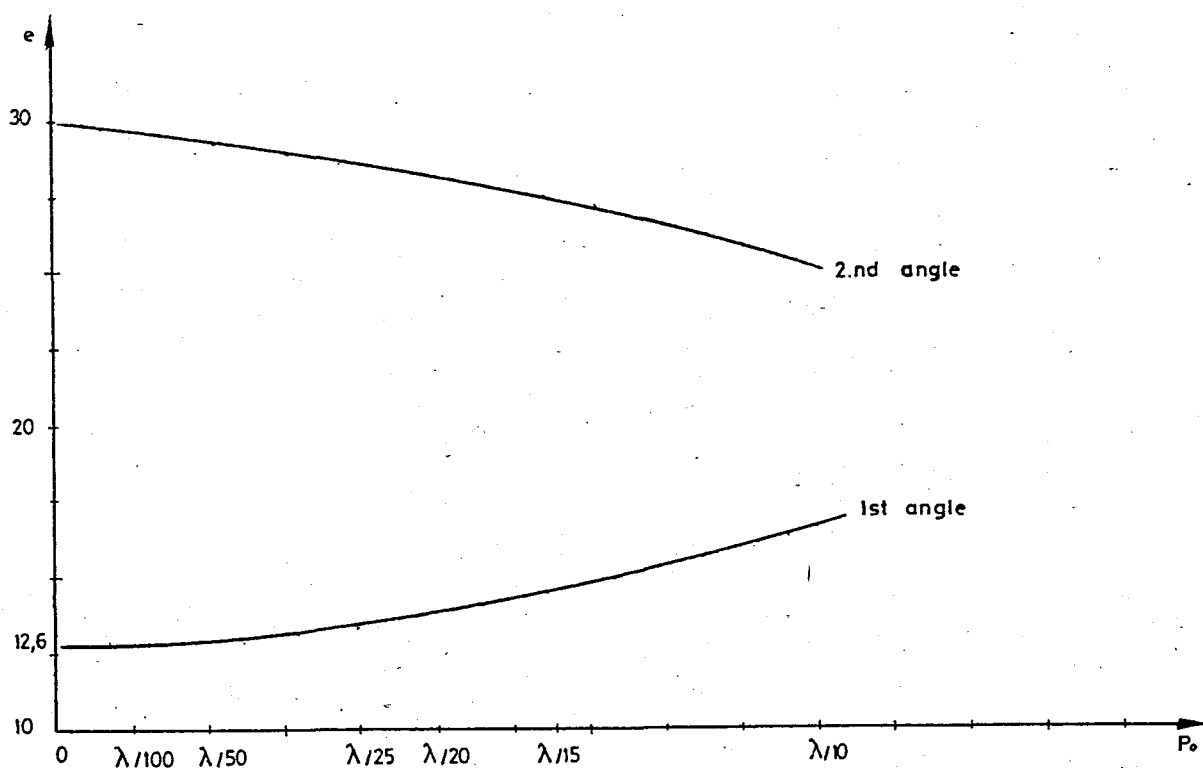


Figure 4.4 - Relation between the hydrostatic initial stress and the complement of the angle of incidence when the reflected p-wave vanishes for an incident p-wave.

From the above results we observe that the deviation of the angles from the classical values increases as the initial stress increases in magnitude. The first angle gradually increases with the increase of the initial stress while the second angle gradually decreases with the increase of the initial stress.

In the case of reflection at a free surface of a homogeneous isotropic semi-infinite viscoelastic solid medium under hydrostatic initial stress the angles "e" and "f" are related by the Eq. (2.36). The numerical solutions for Eq. (2.36) are shown in Fig. 4.5 when $\lambda = 1 \times 10^9$, $\mu = 1 \times 10^9$ and $p_0 = \lambda/50$.

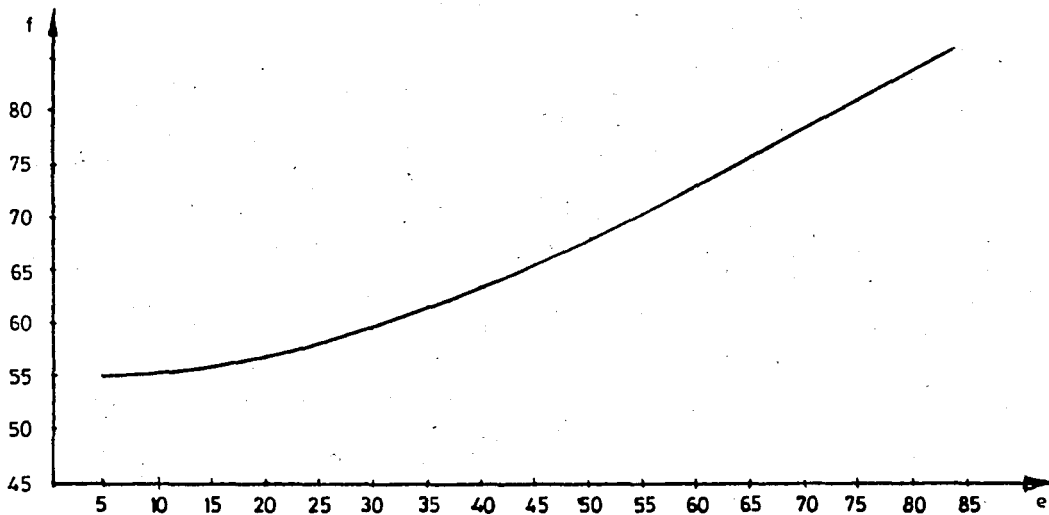


Figure 4.5 - Relation between the complement of the angle of incidence and the angle of reflection in the case of reflection of plane waves at a free surface of an initially stressed (hydrostatic) viscoelastic medium.

The angle "f" gradually increases starting from 55 degrees as the angle of emergence increases. The values of "e" and "f" come closer around 80 degrees. In this case, the numerical results for the amplitude ratios of the reflected waves to the incident SV-wave obtained from the Eqs. (2.42) and (2.43) are given by the Figures 4.6 and 4.7.

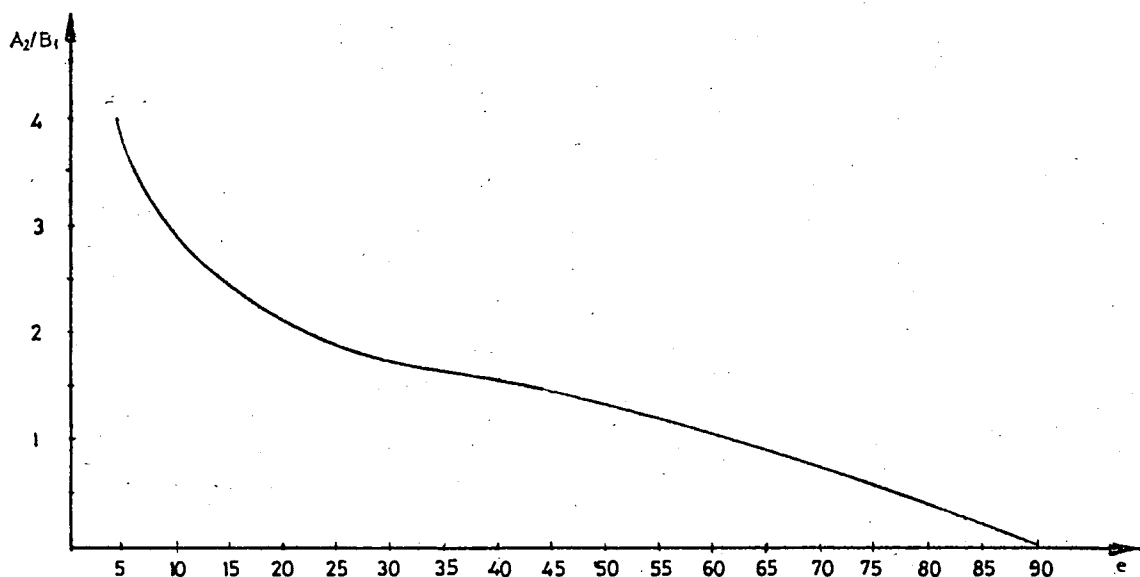


Figure 4.6 - A_2 - Amplitude of the reflected p-wave.
 B_1 - Amplitude of the incident SV-wave.

From Fig. 4.6 we observe that the amplitude of the reflected p-wave is always greater than the amplitude of the incident SV-wave. It decreases as the value of the angle "e" increases. The graph has a greater slope when $e = 5^\circ - 30^\circ$. When $e > 30$, A_2/B_1 gradually decreases and finally approaches to zero.

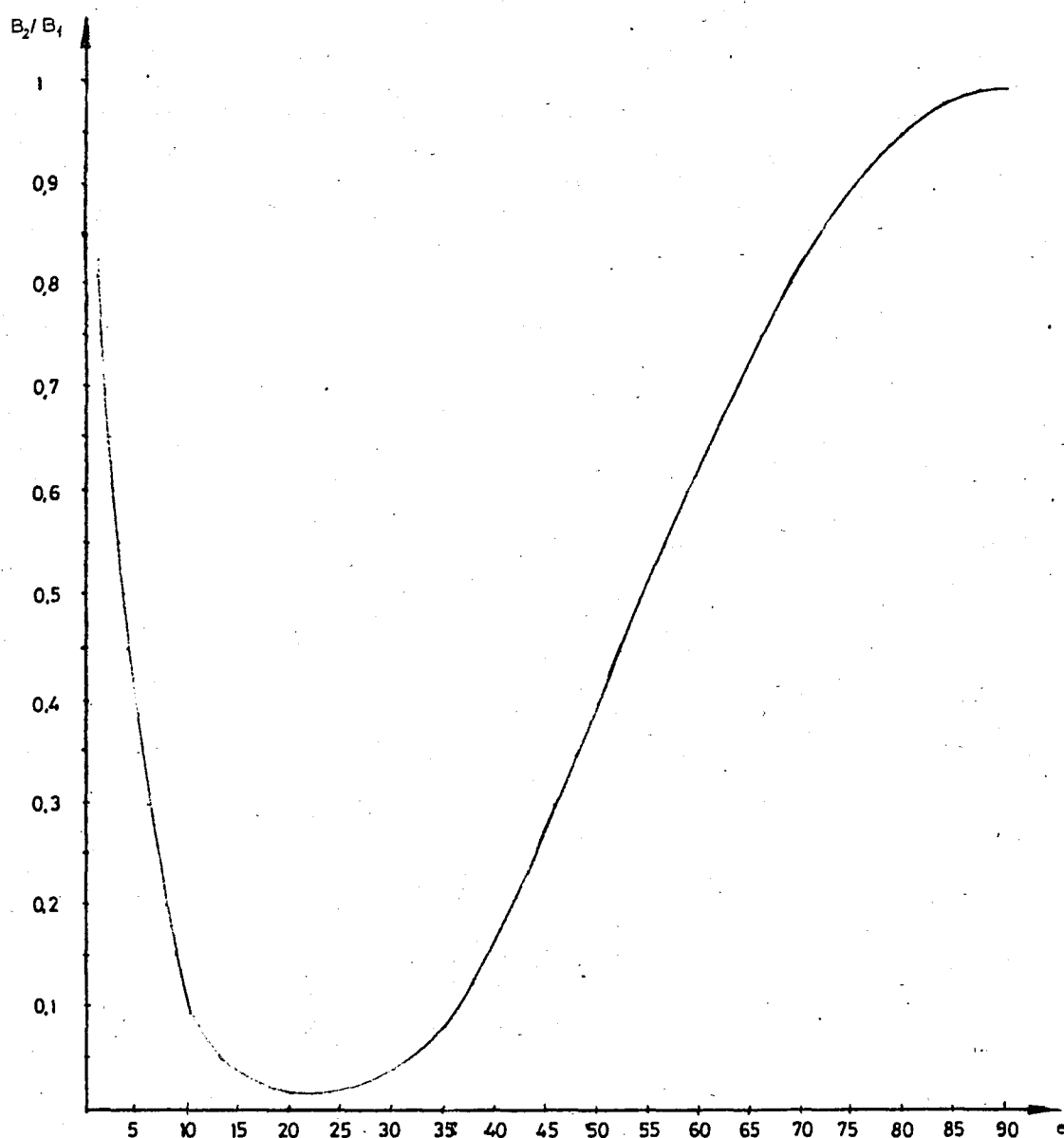


Figure 4.7 - B_2 - Amplitude of the reflected SV-wave.
 B_1 - Amplitude of the incident SV-wave.

Note that the magnitude of the reflected SV-wave for an incident SV-wave decreases for the values of e less than 25° . When $e > 25^\circ$ the magnitude of the reflected SV-wave increases and approaches to 1 as e approaches to 90° .

The effects of initial stress on reflection and refraction of two viscoelastic solid media are studied and relations between the incident waves and the reflected and the refracted waves are obtained in Chapter 2. The numerical results for these equations are given by the Figures 4.8 and 4.9 .

Numerical results for the Eq. (2.60) are given below.

$$\lambda = 1 \times 10^9 \quad \mu = 1 \times 10^9 \quad \rho = 930 \quad \lambda' = 1 \times 10^9 \quad \mu' = 1 \times 10^9 \quad \rho' = 930$$

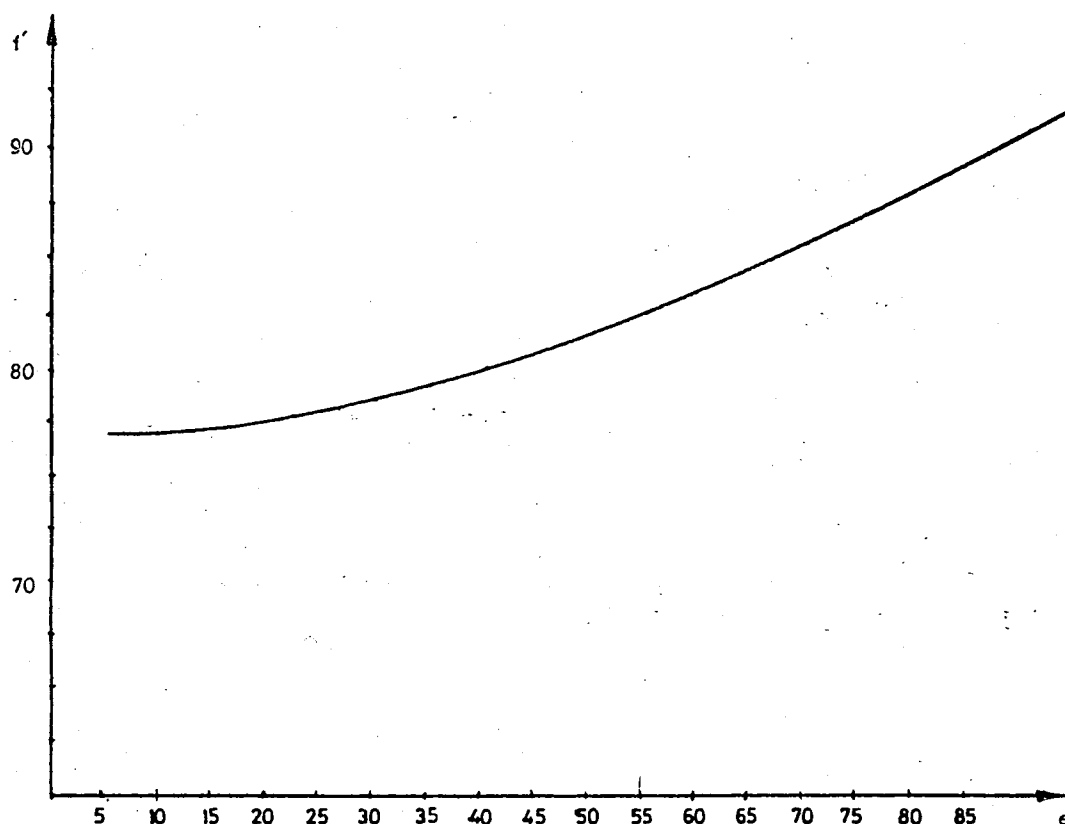


Figure 4.8 - The relation between the complement of the angle of incidence and the angle of refraction in the case of reflection and refraction of plane waves at an interface of two viscoelastic solid media under hydrostatic initial stress.

From Fig. 4.8 we observe that the angle of refraction (f') increases gradually as the angle of emergence (e) increases and approaches to 90° together with " e ".

$$\lambda = 1 \times 10^9 \quad \mu = 1 \times 10^9 \quad \rho = 930 \quad \lambda' = 1 \times 10^9 \quad \mu' = 1 \times 10^9 \quad \rho = 930$$

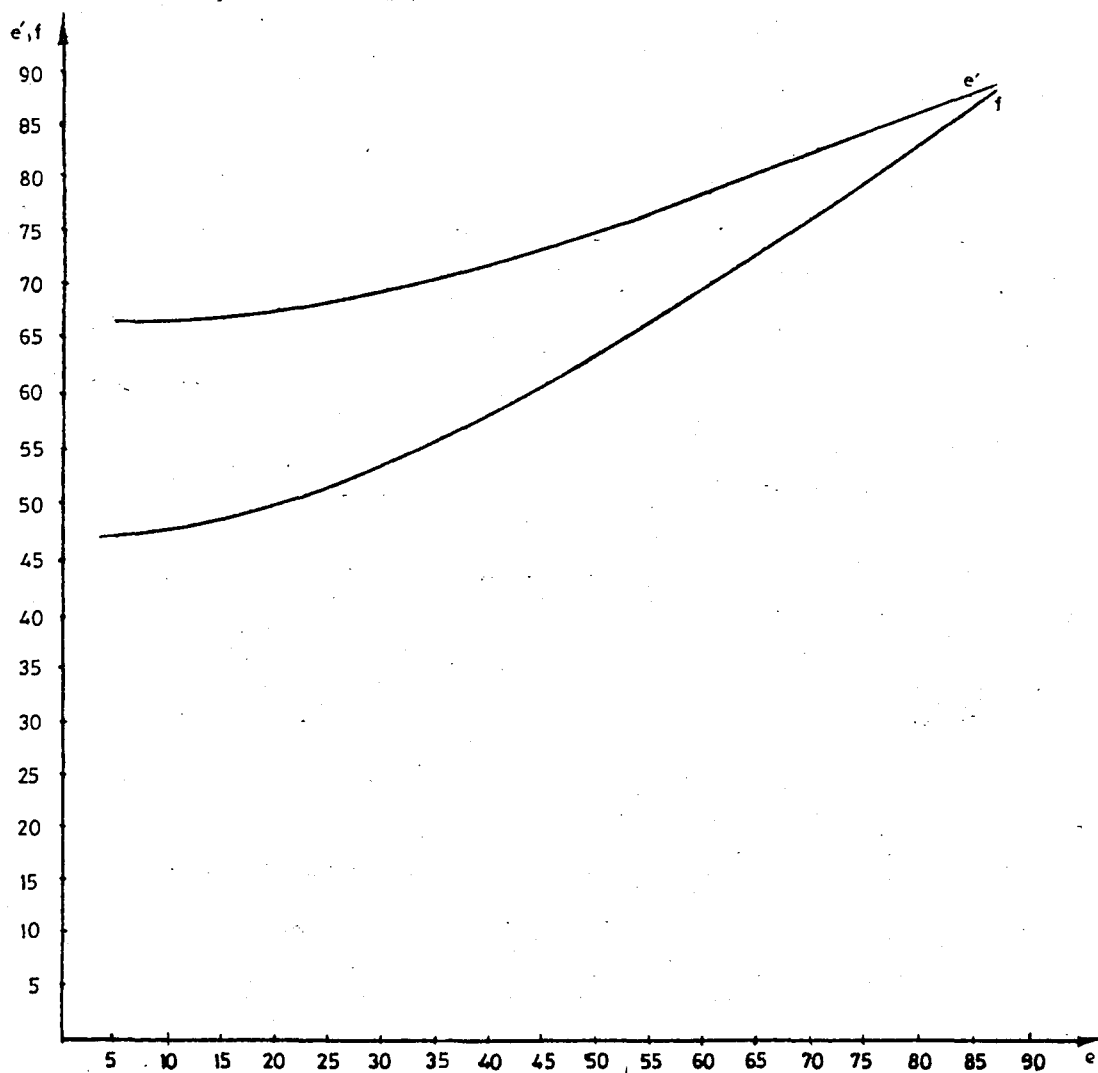


Figure 4.9 - e' - Complement of the angle of the refracted p-wave.
 f - Complement of the angle of the reflected SV-wave.

Note that the complement of the angle of the refracted p-wave (e') and the complement of the angle of the reflected SV-wave increases as the angle of emergence (e) increases. They approach to 90° together with " e ".

The amplitude ratios are obtained by the equations (2.83) and (2.84) for an incident SH-wave. Numerical solutions are given in Fig. 4.10.

$$\lambda = 1 \times 10^9 \quad \mu = 1 \times 10^9 \quad \rho = 930 \quad \lambda' = 1 \times 10^9 \quad \mu' = 1 \times 10^9 \quad \rho' = 930$$

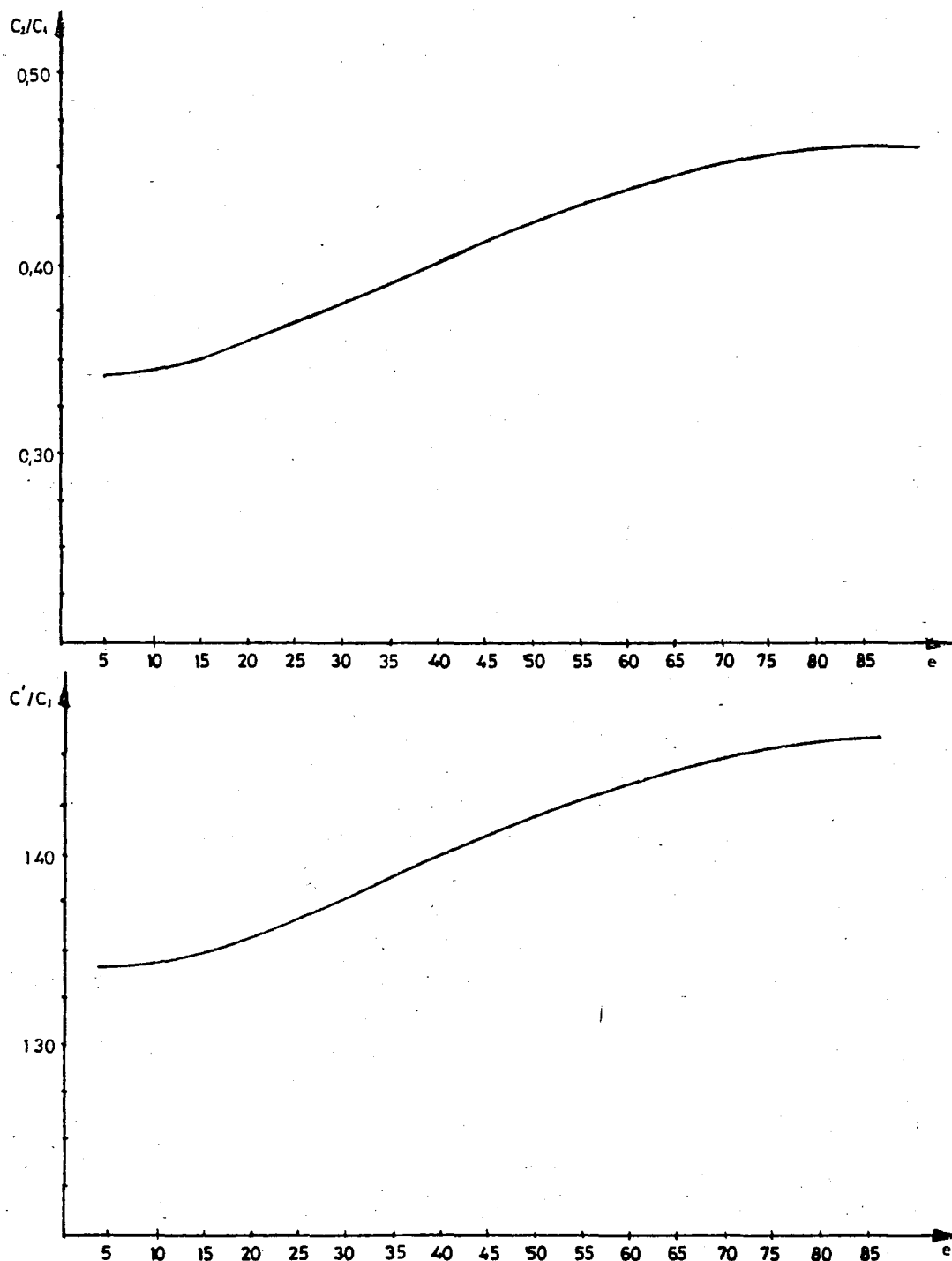


Figure 4.10 - C_2 - Amplitude of the reflected SH-wave.
 C' - Amplitude of the refracted SH-wave.
 C - Amplitude of the incident SH-wave.

From Figure 4.10 we observe that the amplitude of the reflected SH-wave is always less than the amplitude of the incident SH-wave. When " e " approaches to 90° , the amplitude of the reflected SH-wave is approximately half of the amplitude of the reflected SH-wave. The amplitude of the refracted SH-wave is always greater than the amplitude of the incident SH-wave and it increases by " e " finally approaching to a value 1.46 times C_1 .

APPENDIX A

Viscoelastic Materials [16]

All solids under the influence of external loads not only deform but also flow to a certain extent and dissipate energy. Fluent bodies also possess a certain amount of rigidity. Moreover, deformation and flow produce heat, and conversely, the addition of heat to a body causes deformation and flow. Thus physical bodies possess the characteristics of both solids and fluids.

One of the simplest rate-dependent solids is the Voigt solid. The fundamental assumption underlying this material is that stress is a function of the strain and deformation-rate tensors. For a one-dimensional theory this relation can be written as,

$$t = E\tilde{e} + E_v\dot{\tilde{e}}$$

where t is the tension, e is the strain, and E and E_v are material constants. Note that this model is a combination of an elastic solid with a viscous fluid. It is often represented as a parallel spring-dashpot model as shown in Figure A.1 with E being the spring constant and E_v the viscosity of the dashpot.

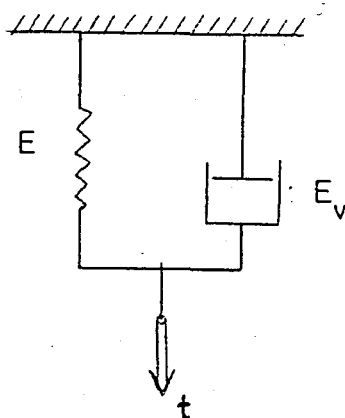


Figure A.1.

Generalization of this idea to any nonlinear, three-dimensional solid begins with a constitutive equation of the form

$$t_{kl} = f_{kl}(x_{,K}, \dot{x}_{,K}, \theta, X)$$

and similar expressions for the heat vector q , internal energy ϵ and entropy η . They have the explicit forms

$$t_{kl} = F_{KL}(C, \dot{C}, \theta, X) x_{,K,k} x_{,L,l} \quad (A.1)$$

$$q_k = Q_K(C, \dot{C}, \theta, X) x_{,K,k} \quad (A.2)$$

$$\epsilon = e(C, \dot{C}, \theta, X) \quad (A.3)$$

$$\eta = n(C, \dot{C}, \theta, X) \quad (A.4)$$

Here F and Q are tensor and vector functions, respectively, and e and n are scalar functions of the tensor variables C , $\dot{C}$ are the Green deformation tensor and its rate is given by

$$C_{KL} = x_{,K,k} x_{,L,l} \quad (A.5)$$

$$\dot{C}_{KL} = \frac{DC_{KL}}{Dt} = 2d_{kl}x_{k,K}x_{l,L}$$

where d is the deformation-rate tensor

$$d_{kl} = \frac{1}{2}(v_{k,l} + v_{l,k}) \quad (A.6)$$

More convenient forms of (A.1) and (A.4) are obtained if we write

$$\begin{aligned} F_{KL} &= j^{-1} T_{MN} C_{MK} C_{NL} \\ Q_K &= j^{-1} G_L C_{LK}, \\ j^{-1} &= [\det(C_{KL})]^{-\frac{1}{2}} = \rho/\rho_0 > 0 \end{aligned} \quad (A.7)$$

Substitution of Eqs. (A.5) and (A.7) in Eqs. (A.1) and (A.4) yields

$$t_{kl} = j^{-1} T_{KL}(C, \dot{C}, \theta, X) x_{k,K} x_{l,L} \quad (A.8)$$

$$q_k = j^{-1} G_K(C, \dot{C}, \theta, X) x_{k,K} \quad (A.9)$$

$$\epsilon = e(C, \dot{C}, \theta, X) \quad (A.10)$$

$$\eta = n(C, \dot{C}, \theta, X) \quad (A.11)$$

where T_{KL} is a tensor function and G_K is a vector function of the tensor variables C , $\dot{C}$, and scalars θ and X . We may now use these equations to define a Voigt solid:

A Voigt solid is a thermomechanical medium having constitutive equations of the form (A.8) to (A.11) subject to the axioms of admissibility, objectivity, and material invariance.

According to the axiom of entropy, t , q , ϵ and η must satisfy the Clausius-Duhem inequality

$$-\frac{\rho}{\theta}(\dot{\psi} + \eta\dot{\theta}) + \frac{1}{\theta} t_{kl} v_{l,k} + \frac{1}{\theta^2} q_k \theta_{,k} \geq 0$$

Here ψ is the Helmholtz free-energy function defined by

$$\psi \equiv \epsilon - \theta\eta$$

In order a Voigt solid be thermodynamically admissible it must be characterized by a set of constitutive equations of the following form as a result of Clausius-Duhem inequality

$$t_{kl} = 2 \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial C_{KL}} x_{k,K} x_{l,L} + \frac{\rho}{\rho_0} D_{KL}^{TKL}(C, \dot{C}, \theta, X) x_{k,K} x_{l,L} \quad (A.12)$$

$$q_k = 0 \quad (A.13)$$

$$\epsilon = \frac{1}{\rho_0} [\Sigma(C, \theta, X) - \theta \frac{\partial \Sigma}{\partial \theta}] \quad (A.14)$$

$$\eta = - \frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta} \quad (A.15)$$

where Σ is a coefficient function.

Since nonlinear theory is too complicated for most of the practical applications the simpler forms of these equations are obtained by using the coefficient function in Eq. (A.12).

For the general anisotropic solid it is appropriate to use the Lagrangian strain measure E defined by

$$2E_{KL} \equiv C_{KL} - \delta_{KL} \quad (A.16)$$

where δ_{KL} is the Kronecker symbol. Whenever the two indices are equal,

it has the value 1; otherwise it is equal to zero. For the linear theory it is appropriate to employ the infinitesimal strain tensor $\tilde{E}$ in place of E so that

$$2E_{KL} \approx 2\tilde{E}_{KL} = U_{K,L} + U_{L,K} \quad (A.17)$$

where U_K is the displacement vector.

The deformation gradient, $x_{k,K}$ can be written in the following form

$$\begin{aligned} x_{k,K} &= (\delta_{MK} + U_{M,K})\delta_{Mk} \\ &= (\delta_{MK} + \tilde{E}_{MK} + \tilde{R}_{MK})\delta_{Mk} \end{aligned} \quad (A.18)$$

where $\tilde{R}_{MK}$ is the infinitesimal rotation tensor defined by

$$\tilde{R}_{MK} = \frac{1}{2}(U_{M,K} - U_{K,M}) \quad U_{[M,K]}$$

A linear theory for the stress requires a second-order expansion for Σ in E . Hence, writing

$$\Sigma(E, \theta) \equiv \rho_0 \psi(E, \theta) = \Sigma_0(\theta) + \Sigma_{KL}(\theta)E_{KL} + \frac{1}{2} \Sigma_{KLMN}(\theta)E_{KL}E_{MN} \quad (A.19)$$

we obtain

$${}_D T_{KL}(\dot{E}, \theta) = \Gamma_{KLMN}(\theta) \dot{E}_{MN} \quad (A.20)$$

The linear theory for ${}_D T$ requires an expansion of the form

$${}_D T_{KL}(\dot{E}, \theta) = \Gamma_{KLMN}(\theta) \dot{E}_{MN} \quad (A.21)$$

where Γ_{KLMN} is a coefficient function. Due to the symmetry of the stress and strain tensors, we also have

$$\Sigma_{KL} = \Sigma_{LK}$$

$$\Sigma_{KLMN} = \Sigma_{MNKL} = \Sigma_{LKMN} = \Sigma_{KLN M} \quad (A.22)$$

$$\Gamma_{KLMN} = \Gamma_{LKMN} = \Gamma_{KLN M}$$

Thus, the number of independent coefficients for Σ_{KL} is 6, for Σ_{KLMN} is 21 and for Γ_{KLMN} is 36.

If the reference state is unstressed, then we also have

$$\Sigma_{KL} = 0 \quad (A.23)$$

We find upon writing $\tilde{E}$ for E and using Eq. (A.18) that the stress constitutive equations (A.12) take the form

$$\begin{aligned} t_{k1} = & \frac{\rho}{\rho_0} [\Sigma_{KL} + \Sigma_{ML}(\tilde{E}_{KM} + \tilde{R}_{KM}) + \Sigma_{KM}(\tilde{E}_{LM} + \tilde{R}_{LM}) \\ & + \Sigma_{KLMN}\tilde{E}_{MN} + \Gamma_{KLMN}\tilde{E}_{MN}] \delta_{KK} \delta_{L1} \end{aligned} \quad (A.24)$$

In the linear theory we have the approximations

$$\tilde{E}_{KL} \approx \tilde{e}_{k1} \delta_{KK} \delta_{L1}$$

$$\tilde{R}_{KL} \approx \tilde{r}_{k1} \delta_{KK} \delta_{L1}$$

$$\frac{\rho}{\rho_0} = 1 - \tilde{E}_{KK} = 1 - \tilde{e}_{kk} \quad (A.25)$$

so that (A.24) may be written in the Eulerian representation as

$$\begin{aligned} t_{k1} = & (1 - \tilde{e}_{mm}) \sigma_{k1} + \sigma_{m1}(\tilde{e}_{km} + \tilde{r}_{km}) + \sigma_{km}(\tilde{e}_{1m} + \tilde{r}_{1m}) \\ & + \sigma_{klmn} \tilde{e}_{mn} + \gamma_{klmn} \dot{\tilde{e}}_{mn} \end{aligned} \quad (A.26)$$

where

$$\begin{aligned}\sigma_{k1} &\equiv \Sigma_{KL} \delta_{Kk} \delta_{L1} \\ \sigma_{klmn} &\equiv \Sigma_{KLMN} \delta_{Kk} \delta_{Ll} \delta_{Mm} \delta_{Nn} \\ \gamma_{klmn} &\equiv \Gamma_{KLMN} \delta_{Kk} \delta_{Ll} \delta_{Mm} \delta_{Nn}\end{aligned}\tag{A.27}$$

are functions of θ and X and satisfy the symmetry conditions

$$\begin{aligned}\sigma_{k1} &= \sigma_{1k} \quad \sigma_{klmn} = \sigma_{mnkl} = \sigma_{1kmn} = \sigma_{klnm} \\ \gamma_{klmn} &= \gamma_{1kmn} = \gamma_{klnm}\end{aligned}\tag{A.28}$$

and

$$\begin{aligned}\tilde{e}_{k1} &= \frac{1}{2}(u_{k,1} + u_{1,k}) \equiv u_{(k,1)} \\ \tilde{r}_{ij} &= \frac{1}{2}(u_{k,1} - u_{1,k}) \equiv u_{[k,1]}\end{aligned}$$

Material symmetry places further restrictions on the constitutive coefficients σ_{k1} , σ_{klmn} and γ_{klmn} of the anisotropic solid. The constitutive coefficients are isotropic tensors, so that we may write

$$\begin{aligned}\Sigma_{KL} &= \alpha_E \delta_{KL} \\ \Sigma_{KLMN} &= \lambda_E \delta_{KL} \delta_{MN} + \mu_E (\delta_{KM} \delta_{LN} + \delta_{KN} \delta_{LM}) \\ \Gamma_{KLMN} &= \lambda_V \delta_{KL} \delta_{MN} + \mu_V (\delta_{KM} \delta_{LN} + \delta_{KN} \delta_{LM})\end{aligned}\tag{A.29}$$

where α_E , λ_E , μ_E , λ_V and μ_V in general are functions of θ and X . With this, the stress potential (A.19) and the stress constitutive equation (A.26) take respectively the forms

$$\Sigma = \rho_o \psi = \Sigma_o(\theta) + \alpha_E I_{\tilde{e}} + \frac{1}{2}(\lambda_E + 2\mu_E) I_{\tilde{e}}^2 - 2\mu_E I I_{\tilde{e}} \quad (A.30)$$

$$t_{kl} = \alpha_e \delta_{kl} + (\lambda_e - \alpha_e) I_e \delta_{kl} + 2(\mu_e - \alpha_e) \tilde{e}_{kl} \\ + \lambda_v \dot{I}_{\tilde{e}} \delta_{kl} + 2\mu_v \dot{\tilde{e}}_{kl} \quad (A.31)$$

where we also wrote

$$\alpha_e \equiv \alpha_E, \quad \lambda_e \equiv \lambda_E, \quad \mu_e \equiv \mu_E + 2\alpha_E$$

When the natural state is stress free we have $\alpha_e = 0$ thus

$$t_{kl} = \lambda_e I_{\tilde{e}} \delta_{kl} + \lambda_v \dot{I}_{\tilde{e}} \delta_{kl} + 2\mu_e \tilde{e}_{kl} + 2\mu_v \dot{\tilde{e}}_{kl} \quad (A.32)$$

where

$$I_{\tilde{e}} = \text{tr} \tilde{e} = \tilde{e}_{11} + \tilde{e}_{22} + \tilde{e}_{33}$$

Equation (A.32) can be written in the following form

$$t_{kl} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \tilde{e}_{rr} \delta_{kl} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \tilde{e}_{kl} \quad (A.33)$$

After substituting $\tilde{e}_{rr} = u_{r,r}$ in Eq. (A.33), we may write down $t_{lk,l}$ in the form of

$$t_{lk,l} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \frac{\partial}{\partial x_l} (\frac{\partial u_r}{\partial x_r}) + (\mu_e + \mu_v \frac{\partial}{\partial t}) \frac{\partial^2 u_k}{\partial x_l^2} \\ + (\mu_e + \mu_v \frac{\partial}{\partial t}) \frac{\partial}{\partial x_l} (\frac{\partial u_l}{\partial x_k}) \quad (A.34)$$

Performing the following simplifications

$$\frac{\partial \Delta}{\partial x_j} = \frac{\partial}{\partial x_l} (\frac{\partial u_l}{\partial x_k}) = \frac{\partial}{\partial x_l} (\frac{\partial u_r}{\partial x_r}) ; \quad \nabla^2 u_j = \frac{\partial^2 u_k}{\partial x_l^2}$$

we get

$$t_{1k,1} = \{(\lambda_e + \mu_e) + (\lambda_v + \mu_v) \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_j} + (\mu_e + \mu_v \frac{\partial}{\partial t}) \nabla^2 u_k \quad (A.35)$$

One gets upon substituting Eq. (A.35) into the equation of balance of momenta defined by

$$t_{1k,1} + (f_k - \ddot{u}_k) = 0$$

and setting the body force $f_k = 0$

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \{(\lambda_e + \mu_e) + (\lambda_v + \mu_v) \frac{\partial}{\partial t}\} \frac{\partial \Delta}{\partial x_j} + (\mu_e + \mu_v \frac{\partial}{\partial t}) \nabla^2 u_k \quad (A.36)$$

Equation (A.36) is the equation of motion for a Voigt type solid.

The components of stress field for a viscoelastic body of Voigt type in a orthogonal Cartesian coordinate system can be written through the Eq. (A.33) as follows:

$$\sigma_{11} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_1}{\partial x_1})$$

$$\sigma_{12} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2})$$

$$\sigma_{13} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1})$$

$$\sigma_{21} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2})$$

$$\sigma_{22} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_2}{\partial x_2})$$

$$\sigma_{23} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2})$$

$$\sigma_{31} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1})$$

$$\sigma_{32} = (\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2})$$

$$\sigma_{33} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) (\frac{\partial u_3}{\partial x_3})$$

APPENDIX B

Retardation Time

One needs to give numerical values to the elastic and viscous constants in order to get numerical results by using the equation of motion of a Voigt type solid (App.A). In this section we will study the relations between the elastic and the viscous constants starting from Voigt's definition [21]

$$\sigma = Ee + E_v \dot{e} \quad (\text{B.1})$$

In order to find the strain e we will perform the Laplace transform method.

The Laplace transform of an arbitrary function $f(x)$ is defined by

$$L(f(x); x \rightarrow s) \equiv \bar{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx$$

and the inverse formula is given by

$$f(x) = L^{-1}(\bar{f}(s); s \rightarrow x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{f}(s) e^{sx} ds$$

Performing Laplace transform on (B.1) we get

$$\bar{\sigma} = E\bar{e} + sE_v\bar{e} \quad (\text{B.2})$$

Equation (B.2) can be written in the form of

$$\bar{e} = \frac{1}{E + sE_v} \bar{\sigma} \quad (\text{B.3})$$

Let us consider that a constant stress σ_0 is applied on a Voigt solid (Fig. B.1).

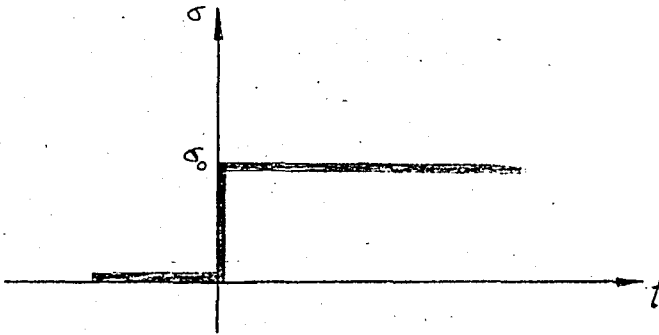


Figure B.1.

So that

$$\bar{e} = \frac{1}{E + sE_v} \bar{\sigma}_0 \quad (\text{B.4})$$

Evaluating the inverse Laplace transform we get

$$e = \frac{1}{E_v} (1 - e^{-(E/E_v)t}) \sigma_0 \quad (\text{B.5})$$

where

$$\frac{E_v}{E} = \tau_r \quad (\text{B.6})$$

is called the "retardation time" and the relation between the elastic and viscous Lamé constants are given by [21].

$$\tau_R = \frac{\lambda_v}{\lambda} = \frac{\mu_v}{\mu} \quad (\text{B.7})$$

When $\tau = \tau_R$, experimental results have shown that (Fig. B.2)

$$e = 0.632e_\infty$$

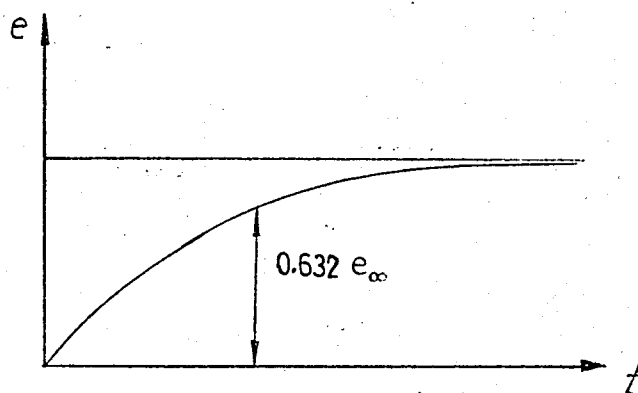


Figure B.2.

It is also experimentally determined that for an almost fully elastic material $\tau_r = 0.1$ sec and for a highly viscous material $\tau_R = 30$ min.

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