GENERAL UNITARY QUANTUM GROUPS AND GENERALIZED FERMION ALGEBRAS

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ABSTRACT

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Since the middle of the twentieth century, physicists have concentrated on finding quantum counterparts of classical systems. When a classical system is quantized its invariance group may still be a classical group. In the nineteen-eighties it was shown that when some classical systems are quantized, their classical group becomes a quantum group so that the system is invariant under a quantum group. So quantum groups play an important role in carrying physical properties to the quantum world.

In this thesis, after presenting structure of Hopf algebra as a quantum group, structure of matrix quantum groups are investigated using inhomogeneous quantum groups, fermionic inhomogeneous orthogonal quantum invariance groups FIO, and bosonic inhomogeneous symplectic quantum invariance groups $BISp$. Then using invariance quantum groups of orthofermion algebra a general structure for unitary quantum groups are constructed in chapter three. Commuting fermion algebra is defined using the Heisenberg spin algebra and its inhomogeneous quantum group is defined in chapter four. Chapter three and chapter four of the thesis is based on original research.

ÖZET

GENEL ÜNİTER KUANTUM GRUPLARI VE GENELLEŞTİRİLMİŞ FERMION CEBRİ

Yirminci yüzyılın ortalarından itibaren fizikçiler klasik sistemlerin kuantum karşılığının bulunmasına yoğunlaşmışlardır. Bir klasik sistem kuantize olduğu zaman onun değişmezlik grubu hala bir klasik grup olabilir. 1980'lerde gösterildi ki, bazı klasik sistemler kuantize olduğu zaman onların klasik grubu bir kuantum grubuna dönüşür. Hem de sistem kuantum grubu altında değişmezdir. Demek ki kuantum grupları fiziksel özellikleri kuantum dünyasına taşımakta önemli bir rol oynarlar.

Bu tezde, bir Kuantum Grup olarak Hopf Cebri yapısı sunulduktan sonra, homojen olmayan Kuantum Grupları fermionik homojen olmayan değişmezlik Kuantum Grubu, FIO ve bozonik homojen olmayan değişmezlik Kuantum Grubu, $BISp$ kullanılarak matris Kuantum Gruplarının yapısı incelenmiştir. Daha sonra Ortofermiyon cebrinin değişmezlik Kuantum Grubu kullanılarak üniter Kuantum Gruplar için genel bir yapısi üçünc ünitede inşa edilmiştir. Değişmeli fermiyon cebri Heisenberg spin cebri kullanılarak tanımlanmış ve değişmeli fermiyon cebrinin homojen olmayan değişmezlik Kuantum Grubu dördüncü ünitede tanımlanmıştır. Ünite üç ve dört tezin orjinal araştırmaya dayalı kısımlarıdır.

TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

A	Vector space, Algebra
A	Module
a_i, a_i^*	Orthofermion Creation/Annihilation Operators
B	Bialgebra
C	Vector Space
C^*	Coalgebra
$\mathbb{C}$	Complex Field
c_i, c_i^*	Boson/Fermion Creation/Annihilation Operators
G	Group
$\mathfrak{g}$	Lie Algebra
H	Hopf Algebra
$\mathbb{K}$	Field
M	Submatrix of a Quantum Matrix
S	Antipode for a Bialgebra
T	Quantum Groups Matrix
$T(\mathfrak{g})$	Tensor Algebra
$U, U^\dagger$	Unitary Matrix
u_{ij}, u_{ji}^*	Unitary Matrix Elements
$U(\mathfrak{g})$	Universal Enveloping Algebra
V	Vector Space
α_{ij}	Quantum Group Transformation Parameter
β_{ij}	Quantum Group Transformation Parameter
Γ	Sub-column of a Quantum Matrix, Generators of Clifford Algebra
γ_i	Quantum Group Transformation Paramter
Δ	Comultiplication Map for a Coalgebra
$\Delta_{(n)}$	Composition of Coproducts
δ_{ij}	Kronecker Symbol

ϵ	Counit Map for a Coalgebra
η	Unit Map for an Algebra
χ	Module Algebra
μ	Multiplication Map for an Algebra
ϕ	Left, Right action
σ_{ij}	Sign Factor
$\otimes$	Tensor Product
$\bigoplus$	Tensor Addition
$\circ$	Multiplication
$\dot{\otimes}$	Matrix Multiplication Type Tensor Product
$*$	Duality
$\triangleright$	Left Action of an Algebra on Its Modules
$\triangleleft$	Right Action of an Algebra on Its Comodules

FIO	Fermionic Inhomogeneous Orthogonal
BISp	Bosonic Inhomogeneous Symplectic
U_G	General Unitary
CoFI	Commuting Fermion Inhomogeneous

1. INTRODUCTION

Symmetry transformations based on Lie groups and Lie algebras are most known in all areas of physics. Symmetry transformations are algebraic objects and they contain Lie groups and Lie algebras as special cases. Theory of quantum integrable systems has initiated a new type of symmetry and mathematical objects called quantum groups. The quantum groups are related to usual Lie groups as quantum mechanics is related to its classical limit[1].

Quantum groups or in other words Hopf Algebras are generalization of group concept. They have rich mathematical structure and numerous roles in situations where ordinary groups are not adequate. What is a quantum group basically? To give an answer to this question first one should explain what a group is. Basically a group is a collection of invertible transformations. This is the role of symmetry. Invertible transformation means that every individual element of transformation is invertible. For the quantum groups instead of inverse there is a weaker statement which is antipode S. In the definition of antipode every individual elements does not have an inverse but the linear combinations of the elements are invertible[2].

Construction of quantum groups are based on different approaches. One is Drinfeld's approach[3], which is based on deformation of a Lie group. In Drinfeld's approach, quantum groups arise as Hopf algebras depending on an auxiliary parameter q or h, which become universal enveloping algebras of a certain Lie algebra, frequently semisimple or affine, when $q = 1$ or $h = 0$. Distinct but related objects, also called quantum groups, are deformations of the algebra of functions on a semisimple algebraic group or a compact Lie group.

Another approach is Woronowicz's[4]. His approach is completely different from Drinfeld's approach and this approach is the theory of quantum groups of matrix type. These are called matrix quantum groups and they generalize the idea of groups whose elements are matrices. Matrix quantum groups are abstract structures on which the "continuous functions" on the structure are given by elements of a C^* algebra. The geometry of a matrix quantum group is a special case of a noncommutative geometry. The general idea of noncommutative geometry arose in the early half of this century with an important theorem of I. Gel'fand and M.A. Naimark. The origin of Woronowicz's approach is Connes's general considerations of non-commutative geometry[5]. Gel'fand-Naimark theorem, which has been inspiration to quantum matrix groups, shows that every commutative C^*-algebra with a norm is necessarily the algebra of continuous functions on some topological space. The most common example of C^* algebra is algebra of $n \times n$ matrices and hermitian conjugates are defined with $-*$ operation. Given any space V we can consider the algebra of functions on it, and work directly with this algebra in place of V itself (so the dual quantum groups in this chapter are in this setting). The algebra is necessarily commutative because if f, g are two functions, then $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$. Every element has a norm and, over $\mathbb{C}$ there is a $*$ structure. A noncommutative version of the Gel'fand-Naimark theorem does not exist. However this idea has been a motivation for Hopf algebras.

Both Drinfeld's and Woronowicz's approaches are unified in Hopf algebra. Hopf algebras have constructed before quantum groups. The notion of Hopf algebra was constructed first in [6], and also in [7]. A striking feature of quantum group theory is the surprising connections with many, sometimes at first glance unrelated, branches of mathematics and physics. There are links with mathematical fields such as Lie groups, Lie algebras and their representations, special functions, operator algebras and noncommutative geometry[8]. Also quantum groups are used in various fields of physics, such as conformal field theory[9], quantum gravity[10] solvable lattice models in statistical physics[11].

2. QUANTUM GROUPS

The theory of quantum integrable systems has initiated a new type of symmetry and this mathematical object is called “quantum groups”. A ring of polynomial functions on an affine algebraic group G is a bialgebra with an antipode. This bialgebra is commutative. Dropping this condition of commutativity, we get the general notion of a Hopf algebra, which is a formalization of the intuitive notion of a quantum group[12]. For mathematical language of quantum groups Hopf algebra structure should be investigated.

2.1. Hopf Algebras

A Hopf algebra **H** is a bialgebra with an antipode S. A morphism of Hopf algebras is a morphism between the underlying bialgebras commuting with the antipodes. A bialgebra **B** carries both algebra and coalgebra structures in a compatible way. An algebra is given by a triple (A, μ, η) where A is vector space, μ is multiplication map $\mu : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$ and η is unit map $\eta : \mathbb{K} \to \mathrm{A}$. They satisfy the following relations.

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) \quad \text{associativity,} \tag{2.1}$$

$$\mu \circ (id \otimes \eta) = \mu \circ (\eta \otimes id) = id \quad \text{existence of unit .} \tag{2.2}$$

A coalgebra is a triple (C, Δ, ϵ) where C is a vector space, Δ is comultiplication (coproduct) map $\Delta : \mathrm{C} \to \mathrm{C} \otimes \mathrm{C}$ where they satisfy the following relations

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad \text{coassociativity,} \tag{2.3}$$

This property can also be iterated by defining,

$$\Delta_{(2)} = \Delta \circ (id \otimes \Delta) = \Delta \circ (\Delta \otimes id), \tag{2.4}$$

so that one has

$$\Delta_{(3)} = \Delta \circ (id \otimes \Delta_{(2)}) = \Delta \circ (\Delta_{(2)} \otimes id), \tag{2.5}$$

in general

$$\Delta_{(n)} = \Delta \circ (id \otimes \Delta_{(n-1)}) = \Delta \circ (\Delta_{(n-1)} \otimes id) \tag{2.6}$$

In Sweedler's notation, the coproduct for $h \in \mathcal{C}$ is the formal sum of the elements of algebra

$$\Delta_{(n)}(h) = \sum_h h_1 \otimes h_2 \otimes \cdots \otimes h_{n+1} \tag{2.7}$$

and ϵ is counit map $\epsilon : \mathrm{C} \longrightarrow \mathbb{K}$

$$(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id \quad \text{existence of counit,} \tag{2.8}$$

so that

$$\sum_h \epsilon(h_1)h_2 = \sum_h h_1\epsilon(h_2) = h \tag{2.9}$$

The concept of coalgebra is the dual to the concept of an algebra. Also the dual vector space of a finite dimensional algebra has coalgebra structure. The duality is represented by the symbol $*$. For example the dual of the multiplication map is the comultiplication (coproduct) map: $\mu^* = \Delta$. Using the equations (2.1, 2.2, 2.3, 2.8) a bialgebra structure is defined. Bialgebra structure $\mathbf{B}(\mu,\ \Delta,\ \eta,\ \epsilon)$ is obtained by introducing the coalgebra and algebra on the same vector space in a "compatible" way. This is called the connecting axiom. The connecting axiom is:

$$\Delta \circ \mu = \mu \otimes \mu \circ (\Delta \otimes \Delta) \tag{2.10}$$

In addition to these structures there are three more bialgebra structures. These are $\mathbf{B}^{\mathrm{op}}$, $\mathbf{B}^{\mathrm{cop}}$ and $\mathbf{B}^{\mathrm{op,\ cop}}$ which are obtained from $\mathbf{B}$ by taking the opposite of either the algebra or coalgebra structure or of both of them. That is $\mathbf{B}^{\mathrm{op}}$ has the opposite multiplication map $\mu_{\mathbf{B}^{\mathrm{op}}}$ and the comultiplication of $\mathbf{B}$, $\mathbf{B}^{\mathrm{cop}}$ has the multiplication of $\mathbf{B}$ and the opposite comultiplication $\Delta_{\mathbf{B}^{\mathrm{cop}}}$, and $\mathbf{B}^{\mathrm{op,\ cop}}$ carries the opposite multiplication $\mu_{\mathbf{B}^{\mathrm{op}}}$ and opposite comultiplication $\Delta_{\mathbf{B}^{\mathrm{cop}}}$.

A bialgebra does not necessarily have an antipode. For a Hopf algebra, a bialgebra $\mathbf{B}$ has to have an antipode. It is proved to be an antihomomorphism. The antipode satisfies following axiom

$$(1 * S) = (S * 1) := \mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon. \tag{2.11}$$

The equation 2.11 can be written as

$$\sum S(a_{(1)})a_{(2)} = \epsilon(a)1 = \sum a_1 S(a_2). \tag{2.12}$$

The antipode of a Hopf algebra is unique and obeys

$$S(xy) = S(y)S(x), \tag{2.13}$$

$$\begin{aligned} S(x)S(y) &= \sum S(x_{(1)}\epsilon(x_{(2)}))S(y_{(1)}\epsilon(y_{(2)})) \\ &= \sum S(x_{(1)})S(y_{(1)})\epsilon(y_{(2)}x_{(2)}) \\ &= \sum S(x_{(1)})S(y_{(1)})y_{(2)}x_{(2)}S(y_{(3)}x_{(3)}) \\ &= \sum S(x_{(1)})\epsilon(y_{(1)})x_{(2)}S(y_{(2)}x_{(3)}) \\ &= \sum \epsilon(x_{(1)})\epsilon(y_{(1)})S(y_{(2)}x_{(2)}) = S(yx). \end{aligned} \tag{2.14}$$

There are four type Hopf algebras which are generated from $\mathbf{B}$, $\mathbf{B}^{\mathrm{op}}$, $\mathbf{B}^{\mathrm{cop}}$ and $\mathbf{B}^{\mathrm{op,cop}}$. They are shown as $\mathbf{H}$, $\mathbf{H}^{\mathrm{op}}$, $\mathbf{H}^{\mathrm{cop}}$ and $\mathbf{H}^{\mathrm{op,cop}}$.

2.2. Dual Pairing Of Hopf Algebras

The axioms of a Hopf algebra are self-dual by reversing arrows and interchanging $\Delta,\ \epsilon$ with $\mu,\ \eta$ gives the same set of axioms. A coalgebra Δ defines an algebra on the dual linear space and in the finite dimensional case an algebra defines a coalgebra on the dual. Thus, every finite dimensonal Hopf algebra has a dual Hopf algebra. For a finite dimensional Hopf algebra $\mathbf{H}$ the dual vector space $\mathbf{H}'$ is an algebra with respect to the multiplication

$$fg(a) := (f \otimes g)\Delta(a) \equiv \sum f(a_{(1)})g(a_2). \tag{2.15}$$

For $f \in\ \mathbf{H}'$ a functional is defined $\Delta(f) \in (\mathbf{H}\otimes\mathbf{H})'$ by $\Delta(f)(a\otimes b) := (f\circ\mu)(a\otimes b) \equiv f(ab)$ where $a,\ b\ \in \mathbf{H}$. The algebra $\mathbf{H}'$ equipped with the comultiplication Δ becomes a Hopf algebra. The antipode, the counit and unit element of this Hopf algebra H' are given by $(Sf)(a) = f(S(a))$, $\epsilon_{H'}(f) = f(1)$ and $1_{H'}(a) = \epsilon(a)$. So the Hopf algebra $\mathbf{H}'$ is obtained by "dualizing" the structure maps of the Hopf algebra $\mathbf{H}$. A dual pairing of two bialgebras $\mathbf{B}$ and $\mathbf{B}'$ is a bilinear mapping

$$\langle \Delta_{\mathbf{B}}(f), a_1 \otimes a_2\rangle = \langle f, a_1a_2\rangle, \tag{2.16}$$

$$\langle f_1f_2, a\rangle = \langle f_1 \otimes f_2, \Delta_{\mathbf{B}'}(a)\rangle, \tag{2.17}$$

$$\langle 1, a\rangle = \epsilon(a), \tag{2.18}$$

$$\epsilon(f_1) = \langle f_1, 1\rangle, \tag{2.19}$$

$$\langle Sf_1, a\rangle = \langle f_1, Sa\rangle. \tag{2.20}$$

A $*-$ vector space means that a vector space C over the field $\mathbb{K}$ endowed with mapping $a \rightarrow a^*$ called an involution and denoted by $*$. The properties of involution is: $(\alpha v + \beta\omega)^* = \overline{\alpha}v^* + \overline{\beta}\omega^*$ and $(v^*)^* = v$ where v and $\omega \in$ C and $\alpha,\ \beta \in\ \mathbb{K}$. Recall that a $*-$ algebra is an associative algebra A (with unit)together with a mapping $a \rightarrow a^*$ such that A becomes a $*$-vector space and $(ab)^* = b^*a^*$ for all $a,\ b \in$ A.

A coalgebra C^* is called a $*$- coalgebra if C^* is equipped with an involution $*$ such that C^* is a $*$-vector space. For a $*$ coalgebra the property is so obvious, $\Delta(a^*) = \Delta(a)^*$ for $a \in C^*$, the involution of $C^* \otimes C^*$ is defined by $(a \otimes b)^* = a^* \otimes b^*$.

A $*$- bialgebra is a bialgebra B with an involution for which B is both a $*$-algebra and a $*$-coalgebra. A Hopf algebra which is a $*$- bialgebra is called a Hopf $*$-algebra.

2.3. Universal Enveloping Algebra As An Example of Hopf Algebra

The universal enveloping algebra $U(\mathfrak{g})$ is defined as the quotient algebra of the tensor algebra $T(\mathfrak{g})$ of a vector space $\mathfrak{g}$ by the two sided ideal $\mathfrak{I}$, where $\mathfrak{g}$ is a Lie algebra. Universal enveloping algebra carries a cocommutative and noncommutative Hopf algebra. Tensor algebra is defined as;

$$T(\mathfrak{g}) = 1 \bigoplus_{k=1}^{\inf} \mathfrak{g}^{\otimes} k = 1 \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \tag{2.21}$$

The tensor algebra ignores the Lie algebra structure of the vector space $\mathfrak{g}$ and the universal enveloping algebra $U(\mathfrak{g})$ is generated by the relations

$$[X_i, X_j] - X_i \otimes X_j - X_j \otimes X_i = 0 \qquad \forall X_i \in \mathfrak{g}. \tag{2.22}$$

Therefore $U(\mathfrak{g})$ is the subalgebra of $T(\mathfrak{g})$ generated by the two sided ideal $\mathfrak{I}$ generated by the elements of type (2.22). This is denoted by $U(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{I}$. The universal enveloping algebra carries a natural Hopf algebra structure. The coproduct is

$$\Delta(X) = X \otimes 1 + 1 \otimes X. \tag{2.23}$$

The coproduct is an algebra morphism and

$$\Delta([X_i, X_j]) = 1 \otimes [X_i, X_j] + [X_i, X_j] \otimes 1. \tag{2.24}$$

The other Hopf algebra structure maps are $\epsilon(X_i) = 0$, and $S(X_i) = -X_i$.

2.4. FIO(2d) and BISp(2d)

The most important example for matrix quantum group is the inhomogeneous quantum invariance group of particle algebras. For fermion algebra this quantum group is called as the fermionic inhomogeneous orthogonal quantum group, $FIO(2d)$[13, 14]. Also for bosons it is called, the bosonic inhomogeneous symplectic quantum group, $BISp(2d)$. The operators of fermion(boson) algebra are considered as module algebras over a Hopf algebra. Now the module algebra should be defined. Since there are two types of modules which are left and right modules. The left **A** module algebra is an algebra χ which is as a left **A** module such that $\mu : \chi \otimes \chi \to \chi$ and $\eta : \mathbb{K} \to \chi$ are left **A** module homomorphisms. If a $a \triangleright x$ denotes the action of $a \in$ **A** on $x \in \chi$, the two latter conditions mean that

$$a \triangleright (xy) = \sum (a_i \triangleright x)(a_2 \triangleright y), \tag{2.25}$$

and

$$a \triangleright 1 = \epsilon(a)1. \tag{2.26}$$

Likewise an algebra χ is called a right **A** module algebra if it is a right **A** module with action, denoted by $\triangleleft$, such that

$$(xy) \triangleleft a = \sum (x \triangleleft a_(1))(y \triangleleft a_2), \tag{2.27}$$

and

$$1 \triangleleft a = \epsilon(a)1. \tag{2.28}$$

We should give the definiton of left and right modules. The vector pace V is called a left **A** module if there exists a linear mapping $\phi : \mathbf{A} \otimes V \to V$, written as $\phi(a \otimes v) = a \triangleright v$, such that $(ab) \triangleright v = a \triangleright (b \triangleright v)$ and $1 \triangleright v = v$ for all $a, b \in \mathbf{A}$ and $v \in V$. Then ϕ is called left action of **A** on V. A right **A** module is a vector space V with a linear mapping $\phi : V \otimes \mathbf{A} \to V$, called right action of **A** on V and written an $\phi(v \otimes a) = v \triangleleft a$, such that $v \triangleleft (ab) = (v \triangleleft a) \triangleleft b$ and $v = v \triangleleft 1$ for all $a, b \in \mathbf{A}$ and $v \in V$. By assuming some relations among the elements of the matrix Quantum Groups, one looks for the consistency conditions that yield a Hopf algebra structure. Also, the matrix Quantum Groups introduced here do not contain deformation parameter.

The boson(fermion) algebra is defined via creation and annihilation operators

$$\left[c_i, c_j^*\right]_\pm = \mathbf{1}\delta_{ij}, \tag{2.29}$$

and

$$[c_i, c_j]_\pm = 0. \tag{2.30}$$

Now, we make an inhomogeneous linear transformation on creation and annihilation operators.

$$\begin{pmatrix} c_i' \\ c_i^{*\prime} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \alpha_{ik} & \beta_{ik} & \gamma_i \\ \beta_{ik}^* & \alpha_{ik}^* & \gamma_i^* \\ 0 & 0 & \mathbf{1} \end{pmatrix} \dot{\otimes} \begin{pmatrix} c_k \\ c_k^* \\ \mathbf{1} \end{pmatrix}, \tag{2.31}$$

$$\begin{aligned} c_i' &= \alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes \mathbf{1}, \\ c_i'^* &= \alpha_{ik}^* \otimes c_k^* + \beta_{ik}^* \otimes c_k + \gamma_i^* \otimes \mathbf{1}, \qquad i,\ k = 1, 2 \cdots, d. \end{aligned} \tag{2.32}$$

The creation and annihilation operators belong to algebra A. We want that the elements of the transformation α_{ik}, β_{ik} and γ_i belong to Hopf algebra $\mathbf{H}$. The transformation matrix can be written in terms of block matrices.

$$T = \left(\begin{array}{cc|c} \alpha_{ik} & \beta_{ik} & \gamma_i \\ \beta_{ik}^* & \alpha_{ik}^* & \gamma_i^* \\ \hline 0 & 0 & \mathbf{1} \end{array}\right) = \left(\begin{array}{c|c} M & \Gamma \\ \hline 0 & \mathbf{1} \end{array}\right), \tag{2.33}$$

where M is homogeneous part of transformation matrix T and Γ is inhomogeneous part. Using transformed operators in the algebra one can get the relation between the elements of transformation. For example;

$$\begin{aligned} \left[c_i', c_j'\right]_\pm &= \left[\left(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes \mathbf{1}\right), \left(\alpha_{jl} \otimes c_l + \beta_{jl} \otimes c_l^* + \gamma_j \otimes \mathbf{1}\right)\right]_\pm \\ &= \alpha_{ik}\alpha_{jl} \otimes c_k c_l \pm \alpha_{jl}\alpha_{ik} \otimes c_l c_k \\ &+ \beta_{ik}\beta_{jl} \otimes c_k^* c_l^* \pm \beta_{jl}\beta_{ik} \otimes c_l^* c_k^* \\ &+ \alpha_{ik}\beta_{jl} \otimes c_k c_l^* \pm \beta_{jl}\alpha_{ik} \otimes c_l^* c_k \\ &+ \beta_{ik}\alpha_{jl} \otimes c_k c_l^* \pm \alpha_{jl}\beta_{ik} \otimes c_l^* c_k + [\gamma_i, \gamma_j^*]_\pm \otimes \mathbf{1} \\ &+ \{[\beta_{ik}, \gamma_j]_\pm + [\gamma_i, \beta_{jk}]_\pm\} \otimes c_k^* + \{[\alpha_{ik}, \gamma_j]_\pm + [\gamma_i, \alpha_{jk}]_\pm\} \otimes c_k. \end{aligned} \tag{2.34}$$

From equation (2.34) it can be seen that

$$[\alpha_{ik}, \alpha_{jl}] = 0, \tag{2.35}$$

$$[\alpha_{ik}\beta_{jl}] = 0, \quad (2.36)$$

$$[\beta_{ik}, \beta_{jl}] = 0, \quad (2.37)$$

$$[\alpha_{jk}, \gamma_i]_{\pm} = 0, \quad (2.38)$$

$$[\beta_{ik}, \gamma_j]_{\pm} = 0. \quad (2.39)$$

$$\alpha_{ik}\beta_{jk} \pm \beta_{ik}\alpha_{jk} + [\gamma_i, \gamma_j]_{\pm} = 0. \quad (2.40)$$

Using the transformed operators in equation (2.30) the relation between the elements of transformation can be found. The relations are,

$$[\alpha_{ik}, \alpha_{jl}] = 0, \quad (2.41)$$

$$\left[\alpha_{ik}\beta^*_{jl}\right] = 0, \quad (2.42)$$

$$\left[\beta_{ik}, \beta^*_{jl}\right] = 0, \quad (2.43)$$

$$[\alpha_{jk}, \gamma^*_i]_{\pm} = 0, \quad (2.44)$$

$$\left[\beta_{ik}, \gamma^*_j\right]_{\pm} = 0. \quad (2.45)$$

$$\alpha_{ik}\alpha^*_{jk} \mp \beta_{ik}\beta^*_{jk} + \left[\gamma_i, \gamma^*_j\right]_{\pm} = \delta_{ij}\mathbf{1}. \quad (2.46)$$

To summarize one can write, the set $\alpha_{ij},\ \beta_{ij},\ \alpha^*_{ij},\ \beta^*_{ij}$ commute with each other but they commute(anti-commute) with the set $\gamma_i,\ \gamma^*_i$ for bosons(fermions). The relation between inhomogeneous elements

$$[\gamma_i, \gamma_j]_{\pm} = -\alpha_{ik}\beta_{jk} \mp \beta_{jk}\alpha_{ik}, \quad (2.47)$$

$$\left[\gamma_i, \gamma^*_j\right]_{\pm} = \delta_{ij}\mathbf{1} - \alpha_{ik}\alpha^*_{jk} \mp \beta_{ik}\beta^*_{jk}. \quad (2.48)$$

In order to show the transformation is a quantum group the transformation has to satisfy Hopf algebra relations. The coproduct of the matrix elements is found by matrix multiplication rule

$$\Delta(T) = T\dot{\otimes}T. \tag{2.49}$$

The coproducts for the elements of T can be written

$$\Delta(M_{ij}) = \sum_{k=1} M_{ik} \otimes M_{kj}, \tag{2.50}$$

$$\Delta(\Gamma_i) = \sum_{k} M_{ik} \otimes \Gamma_k + \Gamma_i \otimes \mathbf{1} \tag{2.51}$$

Coproduct of the relations between the elements of transformation matrix has to remain unchanged. Indeed, it can be easily seen that the coproduct of relations are satisfied. Counit is

$$\epsilon(T) = \mathbf{I}, \tag{2.52}$$

the antipode is inverse of transformation matrix,

$$S(T) = T^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}\Gamma \\ 0 & \mathbf{1} \end{pmatrix}, \tag{2.53}$$

since all elements of homogeneous part M, commutes among themselves the inverse can be found using ordinary matrix inverse rule.

3. GENERAL UNITARY QUANTUM GROUPS

Mishra and Rajasekharan [15] have demonstrated that one can construct consistently a quantum-mechanical system of spin $\frac{1}{2}$ particles, which satisfies a new exclusion principle that is more restrictive than Pauli's principle: It allows the occupation of a single particle (either spin up or down) in any orbital state, but forbids the appearance of both spin up and down particles in the same orbital state. This means that for a system of particles, obeying this new exclusion principle, the antisymmetry under the simultaneous exchange of orbital and spin coordinates in the state vector is broken [16].

This new symmetry of the state vector leads to a quantum statistics, which is called orthofermi statistics. The particles obeying the orthofermi statistics are called orthofermions. The orthofermion statistics is particularly suitable for the strongly correlated electron systems, described by the Hubbard model in the limit $U \to \infty$ [16]. The Hubbard model is very important in condensed matter physics for it describes strongly correlated electron systems. It describes electrons that can hop between nearest neighbor sites of the chain and interact if two of them on the same site have opposite spins[17]. Orthofermi statistics is a generalization of Fermi-statistics in the sense that an orbital state cannot contain more than one particle regardless of its spin [18].

The orthofermion algebra is defined [19] as

$$a_i a_j^\dagger + \delta_{ij} \sum_{k=1}^{N} a_k^\dagger a_k = \mathbf{1}\delta_{ij}, \tag{3.1}$$

$$a_i a_j = 0, \qquad a_i^\dagger a_j^\dagger = 0. \tag{3.2}$$

Then an explicit realization of a_i can be written [18] in terms of $(N+1) \times (N+1)$ matrices as

$$[a_i]_{jk} = \delta_{j,0}\delta_{k,i}, \quad i = 1, \cdots, N, \quad j,k = 0, \cdots, N. \tag{3.3}$$

In orthofermion algebra linearly independent terms are a_i, a_i^*, $a_i^* a_j$ and 1. If one defines the matrix algebra $M_{N+1}(\mathbb{C})$,

$$e_{ij} e_{kl} = \delta_{jk} e_{il}, \tag{3.4}$$

$$e_{ij}^* = e_{ji}, \quad i,j,k,l = 0, \cdots, N \tag{3.5}$$

then a_i, a_i^* and $a_i^* a_j$ can be expressed through matrices e_{ij} in the following way:

$$\begin{aligned} e_{0i} &= a_i, & (3.6)\\ e_{i0} &= a_i^*, & (3.7)\\ e_{ij} &= a_i^* a_j, & (3.8)\\ e_{00} &= 1 - \sum_k a_k^* a_k. & (3.9) \end{aligned}$$

Using these relations one can show that relations (3.1) and (3.2) directly follow from (3.4), (3.5) and vice versa. But the matrices e_{ij} can be transformed as

$$\widetilde{e_{ij}} = u_{ik}^* u_{jl} \otimes e_{kl}, \tag{3.10}$$

where u_{ik}^* and u_{jl} are elements of unitary matrix U, $\mathrm{UU}^\dagger = \mathrm{U}^\dagger\mathrm{U} = \mathbb{I}$, namely

$$\begin{aligned} \sum_k u_{ik} u_{jk}^* &= \delta_{ij}, & (3.11)\\ \sum_k u_{ki}^* u_{kj} &= \delta_{ij}. & (3.12) \end{aligned}$$

Observe that we do not require here that elements u_{ik} and u^*_{lj} should be commutative. The elements u_{ik} and u^*_{lj} are the elements of Hopf algebra $\mathbf{H}$. Also note that the elements e_{ij} are the elements of algebra $M_{N+1}(\mathbb{C})$

Now the question arises: what are requirements on the transformations (3.10) in order that transformed matrices $\widetilde{e_{ij}}$ enjoy the same properties (3.4). Thus, to verify that $\widetilde{e_{ij}}\widetilde{e_{kl}} = \delta_{jk}\widetilde{e_{il}}$ holds, one evaluates

$$\begin{aligned} u^*_{im}u_{jn} \otimes e_{mn}(u^*_{kp}u_{lr} \otimes e_{pr}) &= \delta_{np}u^*_{im}u_{jn}u^*_{kp}u_{lr} \otimes e_{mr} && (3.13) \\ &= \delta_{jk}u^*_{im}u_{lr} \otimes e_{mr} && (3.14) \\ &= \delta_{jk}\widetilde{e_{il}}. && (3.15) \end{aligned}$$

Similarly, to check $\widetilde{e^*_{ij}} = e_{ji}$, one shows that

$$\begin{aligned} \widetilde{e^*_{ij}} &= u^*_{jl}u_{ik} \otimes e^*_{kl} && (3.16) \\ &= u^*_{jl}u_{ik} \otimes e_{lk} && (3.17) \\ &= \widetilde{e_{ji}}. && (3.18) \end{aligned}$$

Consequently, the transformation (3.10) does not change the algebra $M_{N+1}(\mathbb{C})$.

Now let us look at the property of comultiplication:

$$\Delta(u_{ik}) = \sum_m u_{im} \otimes u_{mk}, \tag{3.19}$$

$$\Delta(u^*_{jk}) = \sum_n u^*_{jn} \otimes u^*_{nk}. \tag{3.20}$$

By using these comultiplication relations together with their complex conjugates, one can verify that coproducts in (3.19) and (3.20) are consistent with equations (3.11).

Also one can easily check that all Hopf algebra axioms are satisfied with the coinverse

$$S(u_{ij}) = u^*_{ji} \tag{3.21}$$

and the counit

$$\epsilon(u_{ij}) = \delta_{ij}. \tag{3.22}$$

Since a Hopf algebra is a quantum group we get a new quantum group and we may call this quantum group $U_G(2)$[20], general unitary quantum group.

Now we should explain our results for an explicit example. For one particle case the algebra becomes;

$$\begin{array}{cccc} e_{00}e_{00} = e_{00}, & e_{00}e_{01} = e_{01}, & e_{00}e_{10} = 0, & e_{00}e_{11} = 0, \\ e_{01}e_{00} = 0, & e_{01}e_{01} = 0, & e_{01}e_{10} = e_{00}, & e_{01}e_{11} = e_{01}, \\ e_{10}e_{00} = e_{10}, & e_{10}e_{01} = e_{11}, & e_{10}e_{10} = 0, & e_{10}e_{11} = 0, \\ e_{11}e_{00} = 0, & e_{11}e_{01} = 0, & e_{11}e_{10} = e_{10}, & e_{11}e_{11} = e_{11}. \end{array} \tag{3.23}$$

$$e^*_{00} = e_{00}, \quad e^*_{01} = e_{10}, e^*_{10} = e_{01}, e^*_{11} = e_{11}. \tag{3.24}$$

Now the transformed operators are;

$$\widetilde{e_{00}} = u^*_{00}u_{00} \otimes e_{00} + u^*_{00}u_{01} \otimes e_{01} + u^*_{01}u_{00} \otimes e_{10} + u^*_{01}u_{01} \otimes e_{11}, \tag{3.25}$$

$$\widetilde{e_{01}} = u^*_{00}u_{10} \otimes e_{00} + u^*_{00}u_{11} \otimes e_{01} + u^*_{01}u_{10} \otimes e_{10} + u^*_{01}u_{11} \otimes e_{11}, \tag{3.26}$$

$$\widetilde{e_{10}} = u_{10}^* u_{00} \otimes e_{00} + u_{10}^* u_{01} \otimes e_{01} + u_{11}^* u_{00} \otimes e_{10} + u_{11}^* u_{01} \otimes e_{11}, \tag{3.27}$$

$$\widetilde{e_{11}} = u_{10}^* u_{10} \otimes e_{00} + u_{10}^* u_{11} \otimes e_{01} + u_{11}^* u_{10} \otimes e_{10} + u_{11}^* u_{11} \otimes e_{11}, \tag{3.28}$$

we should look at whether transformed operators satisfy the relations (3.23) and (3.24) and then the transformation should satisfy Hopf algebra axioms.

Obviously $\widetilde{e_{01}}^* = \widetilde{e_{10}}$, $\widetilde{e_{00}}^* = \widetilde{e_{00}}$ and $\widetilde{e_{11}}^* = \widetilde{e_{11}}$.

Now let us look at some examples from the equation (3.23),

$$\begin{aligned}
\widetilde{e_{01}}\widetilde{e_{10}} = \; & [u_{00}^* u_{10} \otimes e_{00} + u_{00}^* u_{11} \otimes e_{01} + u_{01}^* u_{10} \otimes e_{10} + u_{01}^* u_{11} \otimes e_{11}] \\
& [u_{10}^* u_{00} \otimes e_{00} + u_{10}^* u_{01} \otimes e_{01} + u_{11}^* u_{00} \otimes e_{10} + u_{11}^* u_{01} \otimes e_{11}]
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\widetilde{e_{01}}\widetilde{e_{10}} = \; & u_{00}^* u_{10} u_{10}^* u_{00} \otimes e_{00} e_{00} + u_{00}^* u_{10} u_{10}^* u_{01} \otimes e_{00} e_{01} \\
+ \; & u_{00}^* u_{10} u_{11}^* u_{00} \otimes e_{00} e_{10} + u_{00}^* u_{10} u_{11}^* u_{01} \otimes e_{00} e_{11} \\
+ \; & u_{00}^* u_{11} u_{10}^* u_{00} \otimes e_{01} e_{00} + u_{00}^* u_{11} u_{10}^* u_{01} \otimes e_{01} e_{01} \\
+ \; & u_{00}^* u_{11} u_{11}^* u_{00} \otimes e_{01} e_{10} + u_{00}^* u_{11} u_{11}^* u_{01} \otimes e_{01} e_{11} \\
+ \; & u_{01}^* u_{10} u_{10}^* u_{00} \otimes e_{10} e_{00} + u_{01}^* u_{10} u_{10}^* u_{01} \otimes e_{10} e_{01} \\
+ \; & u_{01}^* u_{10} u_{11}^* u_{00} \otimes e_{10} e_{10} + u_{01}^* u_{10} u_{11}^* u_{01} \otimes e_{10} e_{11} \\
+ \; & u_{01}^* u_{11} u_{10}^* u_{00} \otimes e_{11} e_{00} + u_{01}^* u_{11} u_{10}^* u_{01} \otimes e_{11} e_{01} \\
+ \; & u_{01}^* u_{11} u_{11}^* u_{00} \otimes e_{11} e_{10} + u_{01}^* u_{11} u_{11}^* u_{01} \otimes e_{11} e_{11}.
\end{aligned} \tag{3.30}$$

Using equation (3.23) the equality becomes,

$$\begin{aligned}\widetilde{e_{01}}\widetilde{e_{10}} = \;& u^*_{00}(u_{10}u^*_{10} + u_{11}u^*_{11})u_{00} \otimes e_{00} + u^*_{00}(u_{10}u^*_{10} + u_{11}u^*_{11})u_{01} \otimes e_{01} \\ +\;& u^*_{01}(u_{10}u^*_{10} + u_{11}u^*_{11})u_{00} \otimes e_{10} + u^*_{01}(u_{10}u^*_{10} + u_{11}u^*_{11})u_{01} \otimes e_{11}.\end{aligned} \tag{3.31}$$

Then using (3.11), it can be easily seen that $\widetilde{e_{01}}\widetilde{e_{10}} = \widetilde{e_{00}}$. Now let us look at another example

$$\begin{aligned}\widetilde{e_{01}}\widetilde{e_{01}} = \;& [u^*_{00}u_{10} \otimes e_{00} + u^*_{00}u_{11} \otimes e_{01} + u^*_{01}u_{10} \otimes e_{10} + u^*_{01}u_{11} \otimes e_{11}] \\ & [u^*_{00}u_{10} \otimes e_{00} + u^*_{00}u_{11} \otimes e_{01} + u^*_{01}u_{10} \otimes e_{10} + u^*_{01}u_{11} \otimes e_{11}]\end{aligned} \tag{3.32}$$

$$\begin{aligned}\widetilde{e_{01}}\widetilde{e_{01}} = \;& u^*_{00}u_{10}u^*_{00}u_{10} \otimes e_{00}e_{00} + u^*_{00}u_{10}u^*_{00}u_{11} \otimes e_{00}e_{01} \\ +\;& u^*_{00}u_{10}u^*_{01}u_{10} \otimes e_{00}e_{10} + u^*_{00}u_{10}u^*_{01}u_{11} \otimes e_{00}e_{11} \\ +\;& u^*_{00}u_{11}u^*_{00}u_{10} \otimes e_{01}e_{00} + u^*_{00}u_{11}u^*_{00}u_{11} \otimes e_{01}e_{01} \\ +\;& u^*_{00}u_{11}u^*_{01}u_{10} \otimes e_{01}e_{10} + u^*_{00}u_{11}u^*_{01}u_{11} \otimes e_{01}e_{11} \\ +\;& u^*_{01}u_{10}u^*_{00}u_{10} \otimes e_{10}e_{00} + u^*_{01}u_{10}u^*_{00}u_{11} \otimes e_{10}e_{01} \\ +\;& u^*_{01}u_{10}u^*_{01}u_{10} \otimes e_{10}e_{10} + u^*_{01}u_{10}u^*_{01}u_{11} \otimes e_{10}e_{11} \\ +\;& u^*_{01}u_{11}u^*_{00}u_{10} \otimes e_{11}e_{00} + u^*_{01}u_{11}u^*_{00}u_{11} \otimes e_{11}e_{01} \\ +\;& u^*_{01}u_{11}u^*_{01}u_{10} \otimes e_{11}e_{10} + u^*_{01}u_{11}u^*_{01}u_{11} \otimes e_{11}e_{11}.\end{aligned} \tag{3.33}$$

Again using equation (3.23) one can see that

$$\begin{aligned}\widetilde{e_{01}}\widetilde{e_{01}} = \;& u^*_{00}(u_{10}u^*_{00} + u_{11}u^*_{00})u_{10} \otimes e_{00} + u^*_{00}(u_{10}u^*_{00} + u_{11}u^*_{01})u_{11} \otimes e_{01} \\ +\;& u^*_{01}(u_{10}u^*_{00} + u_{11}u^*_{01})u_{10} \otimes e_{10} + u^*_{01}(u_{10}u^*_{00} + u_{11}u^*_{01}+)u_{11} \otimes e_{11}.\end{aligned} \tag{3.34}$$

With the help of (3.11) one can see that $\widetilde{e_{01}}\widetilde{e_{01}} = 0$.

The coproduct of the elements of the transformation can be written as,

$$\Delta(u_{00}) = u_{00}\otimes u_{00} + u_{01}\otimes u_{10}, \tag{3.35}$$

$$\Delta(u_{01}) = u_{00}\otimes u_{01} + u_{01}\otimes u_{11}, \tag{3.36}$$

$$\Delta(u_{10}) = u_{10}\otimes u_{00} + u_{11}\otimes u_{10}, \tag{3.37}$$

$$\Delta(u_{11}) = u_{10}\otimes u_{01} + u_{11}\otimes u_{11} \tag{3.38}$$

also

$$\Delta(u^*_{00}) = u^*_{00}\otimes u^*_{00} + u^*_{01}\otimes u^*_{10}, \tag{3.39}$$

$$\Delta(u^*_{01}) = u^*_{00}\otimes u^*_{01} + u^*_{01}\otimes u^*_{11}, \tag{3.40}$$

$$\Delta(u^*_{10}) = u^*_{10}\otimes u^*_{00} + u^*_{11}\otimes u^*_{10}, \tag{3.41}$$

$$\Delta(u^*_{11}) = u^*_{10}\otimes u^*_{01} + u^*_{11}\otimes u^*_{11}. \tag{3.42}$$

The coproduct of equations (3.11) and (3.12) should be satisfied using the coproducts of the elements of transformation. As an example let us look at $\Delta(u_{00}u^*_{00} + u_{01}u^*_{01}) = \Delta(1)$

$$\begin{aligned}\Delta(u_{00}u^*_{00} + u_{01}u^*_{01}) = {} & \Delta(u_{00})\Delta(u^*_{00}) + \Delta(u_{01})\Delta(u^*_{01}) \\ = {} & \left[u_{00}\otimes u_{00} + u_{01}\otimes u_{10}\right]\left[u^*_{00}\otimes u^*_{00} + u^*_{01}\otimes u^*_{10}\right] \\ + {} & \left[u_{00}\otimes u_{01} + u_{01}\otimes u_{11}\right]\left[u^*_{00}\otimes u^*_{01} + u^*_{01}\otimes u^*_{11}\right].\end{aligned} \tag{3.43}$$

$$\Delta(u_{00}u^*_{00} + u_{01}u^*_{01}) = u_{00}u^*_{00}\otimes u_{00}u^*_{00} + u_{00}u^*_{01}\otimes u_{00}u^*_{10} \tag{3.44}$$

$$
\begin{aligned}
&+ \quad u_{01}u_{00}^{*} \otimes u_{10}u_{00}^{*} + u_{01}u_{01}^{*} \otimes u_{10}u_{10}^{*} \\
&+ \quad u_{00}u_{00}^{*} \otimes u_{01}u_{01}^{*} + u_{00}u_{01}^{*} \otimes u_{01}u_{11}^{*} \\
&+ \quad u_{01}u_{00}^{*} \otimes u_{11}u_{01}^{*} + u_{01}u_{01}^{*} \otimes u_{11}u_{11}^{*}.
\end{aligned}
$$

Now by arranging the terms one can get

$$
\begin{aligned}
\Delta(u_{00}u_{00}^{*} + u_{01}u_{01}^{*}) = \quad & u_{00}u_{00}^{*} \otimes (u_{00}u_{00}^{*} + u_{01}u_{01}^{*}) + u_{00}u_{01}^{*} \otimes (u_{00}u_{10}^{*} + u_{01}u_{11}^{*}) \\
+ \quad & u_{01}u_{00}^{*} \otimes (u_{10}u_{00}^{*} + u_{11}u_{01}^{*}) + u_{01}u_{01}^{*} \otimes (u_{10}u_{10}^{*} + u_{11}u_{11}^{*}),
\end{aligned}
\tag{3.45}
$$

using equation (3.11)one can get

$$
\Delta(u_{00}u_{00}^{*} + u_{01}u_{01}^{*}) = \Delta(1). \tag{3.46}
$$

Now we look at another example which comes from (3.12),

$$
\begin{aligned}
\Delta(u_{01}^{*}u_{00} + u_{11}^{*}u_{10}) = \quad & \Delta(u_{01}^{*})\Delta(u_{00}) + \Delta(u_{11}^{*}\Delta(u_{10}) \qquad (3.47) \\
= \quad & [u_{00}^{*} \otimes u_{01}^{*} + u_{01}^{*} \otimes u_{11}^{*}]\,[u_{00} \otimes u_{00} + u_{01} \otimes u_{10}] \\
+ \quad & [u_{10}^{*} \otimes u_{01}^{*} + u_{11}^{*} \otimes u_{11}^{*}]\,[u_{10} \otimes u_{00} + u_{11} \otimes u_{10}]\,.
\end{aligned}
$$

$$
\begin{aligned}
\Delta(u_{01}^{*}u_{00} + u_{11}^{*}u_{10}) = \quad & u_{00}^{*}u_{00} \otimes u_{01}^{*}u_{00} + u_{00}^{*}u_{01} \otimes u_{01}^{*}u_{10} \qquad (3.48) \\
+ \quad & u_{01}^{*}u_{00} \otimes u_{11}^{*}u_{00} + u_{01}^{*}u_{01} \otimes u_{11}^{*}u_{10} \\
+ \quad & u_{10}^{*}u_{10} \otimes u_{01}^{*}u_{00} + u_{10}^{*}u_{11} \otimes u_{01}^{*}u_{10} \\
+ \quad & u_{11}^{*}u_{10} \otimes u_{11}^{*}u_{00} + u_{11}^{*}u_{11} \otimes u_{11}^{*}u_{10}
\end{aligned}
$$

By arranging the terms,

$$\begin{aligned}\Delta(u^*_{01}u_{00} + u^*_{11}u_{10}) \;&= (u^*_{00}u_{00} + u^*_{10}u_{10}) \otimes u^*_{01}u_{00} + (u^*_{00}u_{01} + u^*_{10}u_{11}) \otimes u^*_{01}u_{10} \\ + \quad & (u^*_{01}u_{00} + u^*_{11}u_{10}) \otimes u^*_{11}u_{00} + (u^*_{01}u_{01}u^*_{11}u_{11}+) \otimes u^*_{11}u_{10}\end{aligned} \tag{3.49}$$

using equation (3.12)one can get

$$\begin{aligned}\Delta(u^*_{01}u_{00} + u^*_{11}u_{10}) = &\quad 1 \otimes (u^*_{01}u_{00} + u^*_{11}u_{10}) + 1 \otimes (u^*_{01}u_{10} + u^*_{11}u_{00}) \\ = &\quad 1 \otimes 0 + 1 \otimes 0 = 0.\end{aligned} \tag{3.50}$$

All q-deformed unitary groups are quantum subgroups of $U_G(2)$. To give an example let us examine $SU_q(2)$

$$M = \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix} , \tag{3.51}$$

where $M \in SU_q(2)$ and the relations between the elements of matrix M are given as[4]

$$ab = qba\,, \qquad ab^* = qb^*a\,, \qquad bb^* = b^*b\,, \tag{3.52}$$

$$aa^* + bb^* = 1\,, \qquad a^*a + q^{-2}b^*b = 1\,. \tag{3.53}$$

The elements of matrix M can be written as $u_{11} = a$, $u_{12} = b$, $u_{21} = -q^{-1}b^*$ and $u_{22} = b^*$. By using equation (3.11) one gets equations (3.52) and (3.53). So, one concludes that $SU_q(2)$ is a quantum subgroup of $U_G(2)$. Following the same procedure one can derive that $U_G(2) \supset U_{p,q}(2) \supset U_{q,\bar{q}}(2) \supset SU_q(2)$. Here $U_{p,q}(2)$, where pq is real, is the two parameter deformed unitary quantum group [21] and $U_{q,\bar{q}}(2)$ is the two parameter deformed unitary quantum group for $p = \bar{q}$ [22].

Although all known unitary quantum groups are quantum subgroups of $U_G(n)$, we do not introduce any q parameter in $U_G(n)$. In the classical limit the matrix elements commute and $U_G(n)$ becomes the ordinary unitary group U(2).

Orthofermions form a system of particles, which are governed by a rule, more restrictive than Pauli's exclusion principle. The definition of orthofermions and explicit realization of orthofermion operators are given. Moreover, the matrix algebra of this explicit realization of orthofermions is defined. Unitary transformations, which do not change the matrix algebra, have been presented and shown to form a quantum group.

4. THE INHOMOGENEOUS QUANTUM INVARIANCE GROUP OF COMMUTING FERMION ALGEBRA

One of the fundamental systems in quantum physics is the fermion algebra. Normally the creation (or annihilation) operators for two different fermion states are taken to be anticommuting. However this is not necessary. As long as a single fermion creation (or annihilation) operator is taken to satisfy $c_i^2 = c_i^{*^2} = 0$ the Pauli exclusion principle is satisfied.

There is a simple relation between Heisenberg spin algebra and commuting fermions as;

$$\sigma_x^i = c_i + c_i^*, \quad \sigma_y^i = -i(c_i - c_i^*), \quad \sigma_z^i = c_i c_i^* - c_i^* c_i. \tag{4.1}$$

Heisenberg spin algebra for the local spin operator σ^m associated with the $m-th$ lattice is written as,

$$\left[\sigma_p^m, \sigma_q^n\right] = 2i\epsilon_{pqr}\delta_{mn}\sigma_r^m. \tag{4.2}$$

In order to define spin operators at different sites, we should define creation and annihilation operators acting at these different sites. This necessity gives us discrete degrees of freedom and in this case fermions can be taken to be commuting. We should note that commuting fermion algebra can not be defined for continuous degrees of freedom, because it prevents the construction of a proper field theory. The operators of commuting fermion algebra is defined as,

$$c_i = \underbrace{1 \otimes \cdots \otimes c}_{i} \overbrace{\otimes 1 \otimes \cdots \otimes 1}^{d-i}. \tag{4.3}$$

We will show that there is an invariance quantum group for commuting fermions. We note that integrable closed spin 1/2 XXZ model is invariant under the quantum group $U_q(sl(2))$ and open spin 1/2 XXZ model is invariant under the quantum group $U_q(su(2))$ [23, 24]. These quantum groups are used to solve these integrable models.

Fermions can be described in terms of creation and annihilation operators.

$$
\begin{aligned}
c_i c_j + \sigma_{ij} c_j c_i &= 0, && (4.4)\\
c_i c_j^* + \sigma_{ij} c_j^* c_i &= \mathbf{1}\delta_{ij} \qquad i,j = 1,2,...,d. && (4.5)
\end{aligned}
$$

$c_i = \underbrace{1 \otimes \cdots \otimes c}_{i}\overbrace{\otimes 1 \otimes \cdots \otimes 1}^{d-i}$ are called commuting fermions. For standard fermions $\sigma_{ij} = 1$, whereas for commuting fermions:

$$
\sigma_{ij} = \begin{cases} 1 & i = j, \\ -1 & i \neq j. \end{cases} \tag{4.6}
$$

When $i = j$ in (4.4) $c_i^2 = 0$, ${c_i^*}^2 = 0$.

Quantum groups are generalizations of the fundamental symmetry concepts of classical Lie groups and Lie algebras [1]. So quantum groups play an important role to carry classical properties to the quantum structure. The standard fermion algebra has an inhomogeneous invariant group FIO(2d,$\mathbf{R}$), the fermionic inhomogeneous orthogonal quantum group [13, 14]. Therefore it is natural to ask the question if there exists an inhomogeneous invariance quantum group for commuting fermions.

For the single fermion, d=1 case, a general inhomogeneous linear quantum transformation is defined by:

$$
\begin{pmatrix} c' \\ c^{*'} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & \mathbf{1} \end{pmatrix} \dot{\otimes} \begin{pmatrix} c \\ c^* \\ \mathbf{1} \end{pmatrix} \tag{4.7}
$$

Where α, β, γ and their hermitian conjugates belong to a Hopf Algebra **H** whose coproduct is given by the matrix multiplication rule [27].
For d> 1

$$c_i' = \sum_{j=1}^{d} \alpha_{ij} \otimes c_j + \sum_{j=1}^{d} \beta_{ij} \otimes c_j^* + \gamma_i \otimes \mathbf{1}, \tag{4.8}$$

$$c_i^{*'} = \sum_{j=1}^{d} \alpha_{ij}^* \otimes c_j^* + \sum_{j=1}^{d} \beta_{ij}^* \otimes c_j + \gamma_i^* \otimes \mathbf{1}. \tag{4.9}$$

here d is number of commuting fermions. When we use (4.8) and (4.9) in Equations (4.4) and (4.5) we get the relations:

$$\alpha_{ij}\alpha_{kl} = \sigma_{ik}\sigma_{jl}\alpha_{kl}\alpha_{ij}, \tag{4.10}$$

$$\alpha_{ij}\alpha_{kl}^* = \sigma_{ik}\sigma_{jl}\alpha_{kl}^*\alpha_{ij}, \tag{4.11}$$

$$\alpha_{ij}\beta_{kl} = \sigma_{ik}\sigma_{jl}\beta_{kl}\alpha_{ij}, \tag{4.12}$$

$$\alpha_{ij}\beta_{kl}^* = \sigma_{ik}\sigma_{jl}\beta_{kl}^*\alpha_{ij}, \tag{4.13}$$

$$\alpha_{ij}\gamma_k = -\sigma_{ik}\gamma_k\alpha_{ij}, \tag{4.14}$$

$$\alpha_{ij}\gamma_k^* = -\sigma_{ik}\gamma_k^*\alpha_{ij}, \tag{4.15}$$

$$\beta_{ij}\beta_{kl} = \sigma_{ik}\sigma_{jl}\beta_{kl}\beta_{ij}, \tag{4.16}$$

$$\beta_{ij}\beta_{kl}^* = \sigma_{ik}\sigma_{jl}\beta_{kl}^*\beta_{ij}, \tag{4.17}$$

$$\beta_{ij}\gamma_k = -\sigma_{ik}\gamma_k\beta_{ij}, \tag{4.18}$$

$$\beta_{ij}\gamma_k^* = -\sigma_{ik}\gamma_k^*\beta_{ij}. \tag{4.19}$$

Then the constraints:

$$\gamma_i\gamma_j + \sigma_{ij}\gamma_j\gamma_i = -\sum_k \alpha_{ik}\beta_{jk} - \sum_k \beta_{ik}\alpha_{jk}, \tag{4.20}$$

$$\gamma_i\gamma_j^* + \sigma_{ij}\gamma_j^*\gamma_i = \delta_{ij} - \sum_k \alpha_{ik}\alpha_{jk}^* - \sum_k \beta_{ik}\beta_{jk}^* \, . \tag{4.21}$$

Also, hermitian conjugates of the relations and the constraints are valid. We look for a Hopf algebra such that under this transformation commuting fermion algebra remains invariant. In order to do this one should first check the coproduct.

The transformation matrix T is given in terms of sub-matrices by

$$T = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} A & \Gamma \\ 0 & \mathbf{1} \end{pmatrix} \tag{4.22}$$

$$A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{4.23}$$

$$\Gamma = \begin{pmatrix} \gamma \\ \gamma^* \end{pmatrix}. \tag{4.24}$$

Here the matrices $\alpha = (\alpha_{ij})$, $\alpha^* = (\alpha^*_{ij})$, $\beta = (\beta_{ij})$, $\beta^* = (\beta^*_{ij})$ are d×d, A is 2d×2d, $\gamma = (\gamma_i)$, $\gamma^* = (\gamma^*_i)$ are d×1 and Γ is 2d×1. We want the elements of the matrix T to belong to a Hopf algebra **H** where coproduct is given by a matrix multiplication

$$\Delta(T) = T\dot{\otimes}T. \tag{4.25}$$

$$\Delta(\alpha_{ij}) = \sum_n \alpha_{in} \otimes \alpha_{nj} + \sum_n \beta_{in} \otimes \beta^*_{nj}, \tag{4.26}$$

$$\Delta(\alpha^*_{ij}) = \sum_n \alpha^*_{in} \otimes \alpha^*_{nj} + \sum_n \beta^*_{in} \otimes \beta_{nj}, \tag{4.27}$$

$$\Delta(\beta_{ij}) = \sum_n \alpha_{in} \otimes \beta_{nj} + \sum_n \beta_{in} \otimes \alpha^*_{nj}, \tag{4.28}$$

$$\Delta(\beta^*_{ij}) = \sum_n \alpha^*_{in} \otimes \beta^*_{nj} + \sum_n \beta^*_{in} \otimes \alpha_{nj}, \tag{4.29}$$

$$\Delta(\gamma_i) = \sum_n \alpha_{in} \otimes \gamma_n + \sum_n \beta_{in} \otimes \gamma^*_n + \gamma_i \otimes \mathbf{1}, \tag{4.30}$$

$$\Delta(\gamma_i^*) = \sum_n \beta_{in}^* \otimes \gamma_n + \sum_n \alpha_{in}^* \otimes \gamma_n^* + \gamma_i^* \otimes \mathbf{1}, \tag{4.31}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}. \tag{4.32}$$

The coproduct should preserve the relations (4.10-4.19) and the constraints (4.20-4.21). Now we show coproduct of some relations as an example.

$$\begin{aligned}\Delta(\alpha_{ij}\alpha_{kl}) &= \textstyle\sum_n \left[\alpha_{in} \otimes \alpha_{nj} + \beta_{in} \otimes \beta_{nj}^*\right] \sum_m \left[\alpha_{km} \otimes \alpha_{ml} + \beta_{km} \otimes \beta_{ml}^*\right] \\ &= \textstyle\sum_{m,n} \left[\alpha_{in}\alpha_{km} \otimes \alpha_{nj}\alpha_{ml} + \alpha_{in}\beta_{km} \otimes \alpha_{nj}\beta_{ml}^*\right] \\ &+ \textstyle\sum_{m,n} \left[\beta_{in}\alpha_{km} \otimes \beta_{nj}^*\alpha_{ml} + \beta_{in}\beta_{km} \otimes \beta_{nj}^*\beta_{ml}^*\right].\end{aligned} \tag{4.33}$$

Then using equations (4.10), (4.12) and (4.16) one can easily see that

$$\begin{aligned}\Delta(\alpha_{ij}\alpha_{kl}) &= \textstyle\sum_{m,n} \left[\sigma_{ik}\sigma_{nm}\sigma_{nm}\sigma_{jl} \left(\alpha_{km}\alpha_{in} \otimes \alpha_{ml}\alpha_{nj} + \beta_{km}\alpha_{in} \otimes \beta_{ml}^*\alpha_{nj}\right)\right] \\ &+ \textstyle\sum_{m,n} \left[\sigma_{ik}\sigma_{nm}\sigma_{nm}\sigma_{jl} \left(\alpha_{km}\beta_{in} \otimes \alpha_{ml}\beta_{nj}^* + \beta_{km}\beta_{in} \otimes \beta_{ml}^*\beta_{nj}^*\right)\right] \\ &= \sigma_{ik}\sigma_{jl} \textstyle\sum_m \left[\alpha_{km} \otimes \alpha_{ml} + \beta_{km} \otimes \beta_{ml}^*\right] \sum_n \left[\alpha_{in} \otimes \alpha_{nj} + \beta_{in} \otimes \beta_{nj}^*\right],\end{aligned} \tag{4.34}$$

$$\Delta(\alpha_{ij}\alpha_{kl}) = \sigma_{ik}\sigma_{jl}\Delta(\alpha_{kl}\alpha_{ij}).$$

Another example is

$$\begin{aligned}\Delta(\beta_{ij}\gamma_k) &= \textstyle\sum_n \left[\alpha_{in} \otimes \beta_{nj} + \beta_{in} \otimes \alpha_{nj}^*\right] \sum_m \left[\alpha_{km} \otimes \gamma_m + \beta_{km} \otimes \gamma_m^* + \gamma_k \otimes \mathbf{1}\right] \\ &= \textstyle\sum_{m,n} \left[\alpha_{in}\alpha_{km} \otimes \beta_{nj}\gamma_m + \alpha_{in}\beta_{km} \otimes \beta_{nj}\gamma_m^* + \alpha_{in}\gamma_k \otimes \beta_{nj}\right] \\ &+ \textstyle\sum_{m,n} \left[\beta_{in}\alpha_{km} \otimes \alpha_{nj}^*\gamma_m + \beta_{in}\beta_{km} \otimes \alpha_{nj}^*\gamma_m^* + \beta_{in}\gamma_k \otimes \alpha_{nj}^*\right].\end{aligned} \tag{4.35}$$

Then using equations (4.10), (4.12), (4.14), (4.15), (4.16), (4.18) and (4.19) one can easily see that

$$\begin{aligned}\Delta(\beta_{ij}\gamma_k) &= \textstyle\sum_{n,m}\left[\sigma_{ik}\sigma_{nm}(-\sigma_{nm})\alpha_{km}\alpha_{in}\otimes\gamma_m\beta_{nj}+\sigma_{ik}\sigma_{nm}(-\sigma_{nm})\beta_{km}\alpha_{in}\otimes\gamma^*_m\beta_{nj}\right]\\ &+\textstyle\sum_{n,m}\left[(-\sigma_{ik})\gamma_k\alpha_{in}\otimes\beta_{nj}+\sigma_{ik}\sigma_{nm}(-\sigma_{nm})\alpha_{km}\beta_{in}\otimes\gamma_m\alpha^*_{nj}\right]\\ &+\textstyle\sum_{n,m}\left[\sigma_{ik}\sigma_{nm}(-\sigma_{nm})\beta_{km}\beta_{in}\otimes\gamma^*_m\alpha^*_{nj}+(-\sigma_{ik})\gamma_k\beta_{in}\otimes\alpha^*_{nj}\right]\\ &= -\sigma_{ik}\textstyle\sum_m\left[\alpha_{km}\otimes\gamma_m+\beta_{km}\otimes\gamma^*_m+\gamma_k\otimes\mathbf{1}\right]\sum_n\left[\alpha_{in}\otimes\beta_{nj}+\beta_{in}\otimes\alpha^*_{nj}\right],\end{aligned}\tag{4.36}$$

$$\Delta(\beta_{ij}\gamma_k)=-\sigma_{ik}\Delta(\gamma_k\beta_{ij}).$$

The most difficult example is coproduct of equation (4.20). We should check the coproduct.

$$\begin{aligned}\Delta(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i) &= \textstyle\sum_n\left[\alpha_{in}\otimes\gamma_n+\beta_{in}\otimes\gamma^*_n+\gamma_i\otimes\mathbf{1}\right]\\ &\quad\textstyle\sum_m\left[\alpha_{jm}\otimes\gamma_m+\beta_{jm}\otimes\gamma^*_m+\gamma_j\otimes\mathbf{1}\right]\\ &+\sigma_{ij}\textstyle\sum_m\left[\alpha_{jm}\otimes\gamma_m+\beta_{jm}\otimes\gamma^*_m+\gamma_j\otimes\mathbf{1}\right]\\ &\quad\textstyle\sum_n\left[\alpha_{in}\otimes\gamma_n+\beta_{in}\otimes\gamma^*_n+\gamma_i\otimes\mathbf{1}\right]\\ &= \textstyle\sum_{n,m}\left[\alpha_{in}\alpha_{jm}\otimes\gamma_n\gamma_m+\alpha_{in}\beta_{jm}\otimes\gamma_n\gamma^*_m+\alpha_{in}\gamma_j\otimes\gamma_n\right]\\ &+\textstyle\sum_{n,m}\left[\beta_{in}\alpha_{jm}\otimes\gamma^*_n\gamma_m+\beta_{in}\beta_{jm}\otimes\gamma^*_n\gamma^*_m+\beta_{in}\gamma_j\otimes\gamma^*_n\right]\\ &+\textstyle\sum_{n,m}\left[\gamma_i\alpha_{jm}\otimes\gamma_m+\gamma_i\beta_{jm}\otimes\gamma^*_m+\gamma_i\gamma_j\otimes\mathbf{1}\right]\\ &+\sigma_{ij}\textstyle\sum_{n,m}\left[\alpha_{jm}\alpha_{in}\otimes\gamma_m\gamma_n+\alpha_{jm}\beta_{in}\otimes\gamma_m\gamma^*_n+\alpha_{jm}\gamma_i\otimes\gamma_m\right]\\ &+\sigma_{ij}\textstyle\sum_{n,m}\left[\beta_{jm}\alpha_{in}\otimes\gamma^*_m\gamma_n+\beta_{jm}\beta_{in}\otimes\gamma^*_m\gamma^*_n+\beta_{jm}\gamma_i\otimes\gamma^*_m\right]\\ &+\sigma_{ij}\textstyle\sum_{n,m}\left[\gamma_j\alpha_{in}\otimes\gamma_n+\gamma_j\beta_{in}\otimes\gamma^*_n+\gamma_j\gamma_i\otimes\mathbf{1}\right].\end{aligned}\tag{4.37}$$

Then using necessary relations one can get

$$\begin{aligned}
\Delta(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i) \quad &= \textstyle\sum_{n,m}\left[\alpha_{in}\alpha_{jm}\otimes\gamma_n\gamma_m+\alpha_{in}\beta_{jm}\otimes\gamma_n\gamma_m^*+\alpha_{in}\gamma_j\otimes\gamma_n\right]\\
&+\textstyle\sum_{n,m}\left[\beta_{in}\alpha_{jm}\otimes\gamma_n^*\gamma_m+\beta_{in}\beta_{jm}\otimes\gamma_n^*\gamma_m^*+\beta_{in}\gamma_j\otimes\gamma_n^*\right]\\
&+\textstyle\sum_{n,m}\left[\gamma_i\alpha_{jm}\otimes\gamma_m+\gamma_i\beta_{jm}\otimes\gamma_m^*+\gamma_i\gamma_j\otimes\mathbf{1}\right]\\
&+\sigma_{ij}\textstyle\sum_{n,m}\left[\sigma_{ji}\sigma_{mn}\left(\alpha_{in}\alpha_{jm}\otimes\gamma_m\gamma_n+\beta_{in}\alpha_{jm}\otimes\gamma_m\gamma_n^*\right)\right]\\
&+\sigma_{ij}\textstyle\sum_{n,m}\left[(-\sigma_{ji})\gamma_i\alpha_{jm}\otimes\gamma_m+(-\sigma_{ij})\gamma_i\beta_{jm}\otimes\gamma_m^*\right]\\
&+\sigma_{ij}\textstyle\sum_{n,m}\left[\sigma_{ji}\sigma_{mn}\left(\alpha_{in}\beta_{jm}\otimes\gamma_m^*\gamma_n+\beta_{in}\beta_{jm}\otimes\gamma_m^*\gamma_n^*\right)\right]\\
&+\sigma_{ij}\textstyle\sum_{n,m}\left[(-\sigma_{ij})\alpha_{in}\gamma_j\otimes\gamma_n+(-\sigma_{ij})\beta_{in}\gamma_j\otimes\gamma_n^*+\gamma_j\gamma_i\otimes\mathbf{1}\right],
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\Delta(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i) \quad &= \textstyle\sum_{n,m}\left[\alpha_{in}\alpha_{jm}\otimes(\gamma_n\gamma_m+\sigma_{nm}\gamma_m\gamma_n)\right]\\
&+\textstyle\sum_{n,m}\left[\alpha_{in}\beta_{jm}\otimes(\gamma_n\gamma_m^*+\sigma_{nm}\gamma_m^*\gamma_n)\right]\\
&+\textstyle\sum_{n,m}\left[\beta_{in}\alpha_{jm}\otimes(\gamma_n^*\gamma_m+\sigma_{nm}\gamma_m\gamma_n^*)\right]\\
&+\textstyle\sum_{n,m}\left[\beta_{in}\beta_{jm}\otimes(\gamma_n^*\gamma_m^*+\sigma_{nm}\gamma_m^*\gamma_n^*)\right]\\
&+(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i)\otimes\mathbf{1}.
\end{aligned} \tag{4.39}$$

Using equations (4.20), (4.21) and their complex conjugates, one can get

$$\begin{aligned}
\Delta(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i) \quad &= \textstyle\sum_{n,m}\left[\alpha_{in}\alpha_{jm}\otimes\sum_l(-\alpha_{nl}\beta_{ml}-\beta_{nl}\alpha_{ml})\right]\\
&+\textstyle\sum_{n,m}\left[\alpha_{in}\beta_{jm}\otimes(\delta_{nm}-\sum_l(\alpha_{nl}\alpha_{ml}^*+\beta_{nl}\beta_{ml}^*))\right]\\
&+\textstyle\sum_{n,m}\left[\beta_{in}\alpha_{jm}\otimes(\delta_{nm}-\sum_l(\alpha_{nl}^*\alpha_{ml}+\beta_{nl}^*\beta_{ml}))\right]\\
&+\textstyle\sum_{n,m}\left[+\beta_{in}\beta_{jm}\otimes\sum_l(-\alpha_{nl}^*\beta_{ml}^*-\beta_{nl}^*\alpha_{ml}^*)\right]\\
&+(-\textstyle\sum_n\alpha_{in}\beta_{jn}-\sum_n\beta_{in}\alpha_{jn})\otimes\mathbf{1}.
\end{aligned} \tag{4.40}$$

By arranging the terms it can be written as

$$\begin{aligned}
\Delta(\gamma_i\gamma_j+\sigma_{ij}\gamma_j\gamma_i) \quad &= -\textstyle\sum_{n,\,l}\left[\alpha_{in}\otimes\alpha_{nl}+\beta_{in}\otimes\beta_{nl}^*\right]\sum_{m,\,l}\left[\alpha_{jm}\otimes\beta_{ml}+\beta_{jm}\otimes\alpha_{ml}^*\right]\\
&-\textstyle\sum_{n,\,l}\left[\alpha_{in}\otimes\beta_{nl}+\beta_{in}\otimes\alpha_{nl}^*\right]\sum_{m,\,l}\left[\alpha_{jm}\otimes\alpha_{ml}+\beta_{jm}\otimes\beta_{ml}^*\right].
\end{aligned} \tag{4.41}$$

and finally

$$\Delta(\gamma_i\gamma_j + \sigma_{ij}\gamma_j\gamma_i) = \Delta(-\sum_l \alpha_{il}\beta_{jl} - \sum_l \beta_{il}\alpha_{jl}).$$

The counit is

$$\varepsilon(T) = I \tag{4.42}$$

and antipode is

$$S = T^{-1} \tag{4.43}$$

$$T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & \mathbf{1} \end{pmatrix}. \tag{4.44}$$

However as all the elements of matrix A do not commute with each other, ordinary inverse and determinant rules can not be used. In order to find A^{-1}, we should first state the determinant rule of matrix A. The determinant rule is given by [25]

$$Det(A) = \sum \left[\prod_{\alpha<\beta,\, i_\beta<i_\alpha,\, j_\beta<j_\alpha} (-\sigma_{i_\beta i_\alpha})(-\sigma_{j_\beta j_\alpha}) \right] A_{i_1}^{j_1} A_{i_2}^{j_2} ... A_{i_{2d}}^{j_{2d}}. \tag{4.45}$$

This determinant can also be found by starting from the determinant rule which has been constructed to calculate the determinant in $GL_{p,q}(d)$ [28]. In ordinary determinant rule the sign factor is ϵ tensor. Instead of ϵ tensor we use $\prod_{\alpha<\beta,\, i_\beta<i_\alpha,\, j_\beta<j_\alpha}(-\sigma_{i_\beta i_\alpha})(-\sigma_{j_\beta j_\alpha})$ term as a sign factor in equation (4.45). Here j refers to a column and i refers to a row. The indices $i_1, i_2, ..., i_{2d}$ and $j_1, j_2, ..., j_{2d}$ are permutations of $1, 2, ..., 2d$. We take $i_1, i_2, ..., i_{2d}$ to be fixed whereas the summation runs over all permutations $j_1, j_2, ..., j_{2d}$. The indices σ, i_α and j_α should be written modulo d. The σ- symbol is the same as in (4.6).

Let us explain the term which is in the square bracket. As the square brackets comes from permutation we call it the permutation factor. For example set d=2, $i_1 = 1,\ i_2 = 2,\ i_3 = 3$ and $i_4 = 4$. For every $\alpha < \beta$ case there is not any $i_\beta < i_\alpha$. Therefore the term which is in the square bracket is 1. If we take $i_1 = 3,\ i_2 = 2,\ i_3 = 1$ and $i_4 = 4$ for d=2, for $\alpha = 1$ and $\beta = 2$, $i_2 = 2 < i_1 = 3$ the factor $(-\sigma_{21})$ appears. Then for $\alpha = 1$ and $\beta = 3$, $i_3 = 1 < i_1 = 3$ the factor $(-\sigma_{11})$ appears. Moreover $\alpha = 2$ and $\beta = 3$, $i_3 = 1 < i_2 = 2$, the factor $(-\sigma_{12})$ appears. We have to be careful about the indices of σ, because indices should be written modulo d.

The proof that equation (4.45) is indeed the correct determinant is the calculation of the inverse by using this determinant. Using the determinant rule (4.45) the inverse of A can be found.

$$A^{-1} = \frac{1}{Det(A)} \begin{bmatrix} \Delta_1^1 & \Delta_2^1 & \cdots & \Delta_{2d}^1 \\ \Delta_1^2 & \Delta_2^2 & \cdots & \Delta_{2d}^2 \\ \vdots & & \ddots & \vdots \\ \Delta_1^{2d} & & \cdots & \Delta_{2d}^{2d} \end{bmatrix}. \tag{4.46}$$

Then Δ_i^j can be defined as

$$\Delta_i^j = C \sum \left[\prod_{\alpha<\beta,\ i_\beta<i_\alpha,\ j_\beta<j_\alpha} (-\sigma_{i_\beta i_\alpha})(-\sigma_{j_\beta j_\alpha}) \right] A_{i_2}^{j_2} ... A_{i_{2d}}^{j_{2d}}. \tag{4.47}$$

The sign factor C can be written as

$$C = \prod_{i_\alpha<i,\ j_\alpha<j} (-\sigma_{i_\alpha i})(-\sigma_{j_\alpha j}). \tag{4.48}$$

Here the index $\alpha = 2, 3, .., 2d$. As with the usual determinant rule the summation in (4.47) is over the indices $j_2,\ j_3 \ldots j_{2d} \neq j$ which are all different. The indices $i_2,\ i_3 \ldots i_{2d} \neq i$ which again are all different, are all kept fixed.

As the elements of the matrix T belong to a Hopf algebra, coproduct, counit and antipode satisfy Hopf algebra axioms. Since a Hopf algebra is a quantum group we get a new quantum group. We may call this quantum group CoFI(2d,**R**)[29], the commuting fermion inhomogeneous quantum group.

In order to know whether the construction of CoFI(2d,**R**) has any meaning one should look at its representations. To find the representations of CoFI(2d), representations of Clifford Algebra which generate spinor representation of $SO(d+1)$[31] should be discussed. The generators of the Clifford Algebra satisfy anticommutation relations:

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}, \quad \text{for } i, j = 1 \text{ to } d+1. \tag{4.49}$$

Above equation means that $\Gamma_i^2 = 1$ and $\Gamma_i\Gamma_j = -\Gamma_j\Gamma_i$. Note that one can also write

$$\Gamma_i\Gamma_j = \sigma_{ij}\Gamma_j\Gamma_i, \tag{4.50}$$

which is useful for our situation. As is well known the Γ_i can be represented by $2^{[\frac{d+1}{2}]} \times 2^{[\frac{d+1}{2}]}$ matrices. Using the spinor representations one can write the representations of CoFI(2d) as:

$$\alpha_{ij} = A_{ij}\Gamma_i \otimes \Gamma_j, \tag{4.51}$$

$$\beta_{ij} = B_{ij}\Gamma_i \otimes \Gamma_j, \tag{4.52}$$

$$\gamma_i = C_i\Gamma_i \otimes \Gamma_{d+1}. \tag{4.53}$$

Here the coefficients, A_{ij}, B_{ij}, C_i, are complex numbers. They should satisfy the relations:

$$2C_iC_j = -A_{ik}B_{jk} - B_{ik}A_{jk}, \tag{4.54}$$

$$2C_i\overline{C_j} = \delta_{ij} - A_{ik}\overline{A_{jk}} - B_{ik}\overline{B_{jk}}, \tag{4.55}$$

so that (4.20,4.21)are also satisfied in addition to (4.10-4.19) which are satisfied due to (4.50). To check the representations we should give some examples. For eqution (4.10) left hand side can be written:

$$\alpha_{ij}\alpha_{kl} = A_{ij}\Gamma_i \otimes \Gamma_j \left(A_{kl}\Gamma_k \otimes \Gamma_l\right), \tag{4.56}$$

now by arranging the terms one can get

$$\alpha_{ij}\alpha_{kl} = A_{ij}A_{kl}\Gamma_i\Gamma_k \otimes \Gamma_j\Gamma_l, \tag{4.57}$$

then using equation (4.50) it can be seen that

$$\alpha_{ij}\alpha_{kl} = \sigma_{ik}\sigma_{jl}A_{kl}\Gamma_k \otimes \Gamma_l \left(A_{ij}\Gamma_i \otimes \Gamma_j\right), \tag{4.58}$$

then one can see that equation (4.10) is satisfied.

Another example is equation (4.12):

$$\alpha_{ij}\beta_{kl} = A_{ij}\Gamma_i \otimes \Gamma_j \left(B_{kl}\Gamma_k \otimes \Gamma_l\right), \tag{4.59}$$

by following the previous steps it can be seen that

$$\alpha_{ij}\beta_{kl} = \sigma_{ik}\sigma_{jl}B_{kl}\Gamma_k \otimes \Gamma_l \left(A_{ij}\Gamma_i \otimes \Gamma_j\right) \tag{4.60}$$

$$\alpha_{ij}\beta_{l} = \sigma_{ik}\sigma_{jl}\beta_{kl}\alpha_{ij}. \tag{4.61}$$

In the next example we check the equation (4.14).

$$\alpha_{ij}\gamma_k = A_{ij}\Gamma_i \otimes \Gamma_k \left(C_k\Gamma_k \otimes \Gamma_{d+1}\right), \tag{4.62}$$

using $\Gamma_i \Gamma_{d+1} = -\Gamma_{d+1}\Gamma_i$ and our usual procedure one can get

$$\alpha_{ij}\gamma_k = -\sigma_{ik} C_k \Gamma_k \otimes \Gamma_{d+1} \left(A_{ij} \Gamma_i \otimes \Gamma_j \right) \quad (4.63)$$

$$\alpha_{ij}\gamma_k = -\sigma_{ik}\gamma_k \alpha_{ik}. \quad (4.64)$$

Now we should explain our results for an explicit example. Taking $d = 1$ we look at the commutation relations, coproduct, determinant, inverse and representations. For $d = 1$ case the transformed creation and annihilation operators are;

$$c' = \alpha \otimes c + \beta \otimes c^* + \gamma \otimes \mathbf{1}, \quad (4.65)$$

$$c^{*'} = \alpha^* \otimes c^* + \beta^* \otimes c + \gamma^* \otimes \mathbf{1}. \quad (4.66)$$

The transformation matrix is

$$T = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & \mathbf{1} \end{bmatrix}. \quad (4.67)$$

The commutation relations are:

$$\alpha\alpha^* = \alpha^*\alpha, \quad (4.68)$$

$$\alpha\beta = \beta\alpha, \quad (4.69)$$

$$\alpha\beta^* = \beta^*\alpha, \quad (4.70)$$

$$\alpha\gamma = -\gamma\alpha, \quad (4.71)$$

$$\alpha\gamma^* = -\gamma^*\alpha, \quad (4.72)$$

$$\beta\beta^* = \beta^*\beta \quad (4.73)$$

$$\beta\gamma = -\gamma\beta, \quad (4.74)$$

$$\beta\gamma^* = -\gamma^*\beta \quad (4.75)$$

and their hermitean conjugates. The constraints can be written as:

$$\gamma\gamma^* + \gamma^*\gamma = \mathbf{1} - \alpha\alpha^* + \beta\beta^*, \tag{4.76}$$

$$\gamma^2 = -\alpha\beta. \tag{4.77}$$

Now we should look at the coproduct of the relations and the constraints. The coproduct of α, β and γ should be found. For this example the coproducts are;

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \beta^*, \tag{4.78}$$

$$\Delta(\alpha^*) = \alpha^* \otimes \alpha^* + \beta^* \otimes \beta, \tag{4.79}$$

$$\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*, \tag{4.80}$$

$$\Delta(\beta^*) = \alpha^* \otimes \beta^* + \beta^* \otimes \alpha, \tag{4.81}$$

$$\Delta(\gamma) = \alpha \otimes \gamma + \beta \otimes \gamma^* + \gamma \otimes \mathbf{1}, \tag{4.82}$$

$$\Delta(\gamma^*) = \alpha^* \otimes \gamma^* + \beta^* \otimes \gamma + \gamma^* \otimes \mathbf{1}, \tag{4.83}$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}. \tag{4.84}$$

The coproduct must satisfy the relations and the constraints. To show this we look at the coproduct of some relations as an example.

$$\Delta(\alpha\alpha^*) = [\alpha \otimes \alpha + \beta \otimes \beta^*]\,[\alpha^* \otimes \alpha^* + \beta^* \otimes \beta] \tag{4.85}$$

$$= \alpha\alpha^* \otimes \alpha\alpha^* + \alpha\beta^* \otimes \alpha\beta + \beta\alpha^* \otimes \beta^*\alpha^* + \beta\beta^* \otimes \beta^*\beta \tag{4.86}$$

using the commutation relations one can get the following equality;

$$\Delta(\alpha\alpha^*) = \alpha^*\alpha \otimes \alpha^*\alpha + \beta^*\alpha \otimes \beta\alpha + \alpha^*\beta \otimes \alpha^*\beta^* + \beta^*\beta \otimes \beta\beta^* \tag{4.87}$$

$$= [\alpha^* \otimes \alpha^* + \beta^* \otimes \beta]\,[\alpha \otimes \alpha + \beta \otimes \beta^*], \tag{4.88}$$

and finally it can be easily seen that

$$\Delta(\alpha\alpha^*) = \Delta(\alpha^*\alpha).$$

Another example:

$$\begin{aligned} \Delta(\alpha\beta^*) &= [\alpha \otimes \alpha + \beta \otimes \beta^*] [\alpha^* \otimes \beta^* + \beta^* \otimes \alpha] && (4.89) \\ &= \alpha\alpha^* \otimes \alpha\beta^* + \alpha\beta^* \otimes \alpha\alpha + \beta\alpha^* \otimes \beta^*\beta^* + \beta\beta^* \otimes \beta^*\alpha && (4.90) \end{aligned}$$

$$\begin{aligned} \Delta(\alpha\beta^*) &= \alpha^*\alpha \otimes \beta^*\alpha + \beta^*\alpha \otimes \alpha\alpha + \alpha^*\beta \otimes \beta^*\beta^* + \beta^*\beta \otimes \alpha\beta^* && (4.91) \\ &= [\alpha^* \otimes \beta^* + \beta^* \otimes \alpha] [\alpha \otimes \alpha + \beta \otimes \beta^*], && (4.92) \end{aligned}$$

then

$$\Delta(\alpha\beta^*) = \Delta(\beta^*\alpha).$$

The last example is about the coproduct of the constraints.

$$\begin{aligned} \Delta(\gamma^2) &= [\alpha \otimes \gamma + \beta \otimes \gamma^* + \gamma \otimes \mathbf{1}] [\alpha \otimes \gamma + \beta \otimes \gamma^* + \gamma \otimes \mathbf{1}] && (4.93) \\ &= \alpha\alpha \otimes \gamma\gamma + \alpha\beta \otimes \gamma\gamma^* + \alpha\gamma \otimes \gamma + \beta\alpha \otimes \gamma^*\gamma + \beta\beta \otimes \gamma^*\gamma^* && (4.94) \\ &+ \beta\gamma \otimes \gamma^* + \gamma\alpha \otimes \gamma + \gamma\beta \otimes \gamma^* + \gamma\gamma \otimes \mathbf{1}, && (4.95) \\ & && (4.96) \end{aligned}$$

then using the commutation relations and the constraints one can write

$$\begin{aligned} \Delta(\gamma^2) &= -\alpha\alpha \otimes \alpha\beta - \alpha\beta \otimes \alpha\alpha^* - \alpha\beta \otimes \beta\beta^* - \beta^2 \otimes \alpha^*\beta^* && (4.97) \\ &= -[\alpha \otimes \alpha + \beta \otimes \beta^*] [\alpha \otimes \beta + \beta \otimes \alpha^*] && (4.98) \end{aligned}$$

and finally

$$\Delta(\gamma^2) = -\Delta(\alpha\beta).$$

Now we should look at the antipode. From equation (4.45) we can find the determinant.

$$Det(A) = A^1_1 A^2_1 + (-\sigma_{11}) A^2_1 A^1_1 = \alpha\alpha^* - \beta\beta^*.$$

Now cofactor terms

$$\Delta_1^{\ 1} = CA^2_2, \tag{4.99}$$

and the sign factor can be found from equation (4.48) as $C = 1$. Finally $\Delta_1^{\ 1} = \alpha^*$.

$$\Delta_1^{\ 2} = CA^1_2 = (-\sigma_{11})\beta^*, \tag{4.100}$$

then $\Delta_1^{\ 2} = -\beta^*$.

$$\Delta_2^{\ 1} = CA^2_1 = (-\sigma_{11})\beta, \tag{4.101}$$

so $\Delta_2^{\ 1} = -\beta$. Then the final cofactor is:

$$\Delta_2^{\ 2} = CA^1_1 = \alpha. \tag{4.102}$$

The inverse can be written as:

$$A^{-1} = \frac{1}{Det(A)} \begin{bmatrix} \Delta_1^{\ 1} & \Delta_2^{\ 1} \\ \Delta_1^{\ 2} & \Delta_2^{\ 2} \end{bmatrix}. \tag{4.103}$$

Then

$$A^{-1} = \frac{1}{\alpha\alpha^* - \beta\beta^*} \begin{bmatrix} \alpha^* & -\beta \\ -\beta^* & \alpha \end{bmatrix}. \tag{4.104}$$

Since all elements of the matrix A is commute among themselves the matrix inverse can also be found via ordinary inverse rule.

The last point is representations. In our example one can write the representations like

$$\alpha = A\Gamma_1 \otimes \Gamma_1, \tag{4.105}$$

$$\beta = B\Gamma_1 \otimes \Gamma_1, \tag{4.106}$$

$$\gamma = C\Gamma_1 \otimes \Gamma_2. \tag{4.107}$$

Where A, B and C are complex numbers satisfying

$$2C\overline{C} = 1 - A\overline{A} - B\overline{B}, \tag{4.108}$$

$$C^2 = -AB. \tag{4.109}$$

Also one can compare our transformation matrix T with the color supergroup structure[25, 26]. It can be seen that the structure of homogeneous part A of the transformation matrix T is the same as the color supergroup structure. Thus this generalizes that of [25, 26] by including an inhomogeneous part for the transformation matrix T. However although the homogeneous part of T has color supergroup structure our transformation matrix T is a quantum group.

The color supergroup is a case of generalization of Lie group. In that case the parameters of group are commuting and anticommuting parameters. They satisfy,

$$\theta_{\alpha,\, m}\theta_{\beta,\, n} - (-1)^{(\beta,\, \alpha)}\theta_{\beta,\, n}\theta_{\alpha,\, m} = 0, \tag{4.110}$$

where $\theta_{\alpha,\,m}$ is group parameter and $\alpha,\ \beta$ are grading vectors. The scalar product $(\alpha,\ \beta)$ can be symmetric or anti-symmetric. For example symmetric Z_2 case,

$$(\alpha,\ \beta) = \alpha_1\beta_1 + \alpha_2\beta_2. \tag{4.111}$$

For antisymmetric Z_2 case,

$$(\alpha,\ \beta) = \alpha_1\beta_2 - \alpha_2\beta_1. \tag{4.112}$$

Generally color supergroups are built by product of two Z_n graded matrices. It is easy to see that the relations between the elements of homogeneous part of transformation matrix has color group structure because their elements has commuting and anticommuting structure.

5. CONCLUSIONS

While the physicists and mathematicians was looking for a new type symmetries which has more general structure than group structure, an algebraic structure was introduced. This structure was called quantum groups. This name comes from Drinfeld's approach. In Drinfeld's approach a quantum group can be constructed from a usual Lie group by defining a deformation parameter. The procedure is very similar to quantization of a classical system.

Since quantum groups are generalization of group concept they have algebraic structure. The algebraic structure is Hopf algebra. Although quantum groups were discovered in the middle of eighties Hopf algebra structure was constructed in the begining of forties. Mathematically a Hopf algebra is a bialgebra with an antipode. Bialgebra is combination of algebra and coalgebra structure. The term antipode is the inverse of certain linear combinations of elements of bialgebra.

Since a quantum group is a Hopf algebra basically, the algebraic structures which satisfy Hopf algebra axioms can be a quantum group. From this fact Drinfeld's approach is not the only way to get a quantum group. Another technique is Woronowicz's approach. His technique is basically a generalization of matrix groups and it is called matrix quantum groups. A symmetry transformation can have group structure, starting from this point, one can write a transformation matrix which the elements satisfy Hopf algebra axioms and it has an antipode. So, this transformation is a matrix quantum group.

The first physical example of matrix quantum groups are the quantum groups FIO and $BISp$. These are the fermionic inhomogeneous orthogonal quantum group and the bosonic inhomogeneous symplectic quantum group respectively. The detailed consideration has been done in chapter two.

In chapter three, orthofermion algebra was defined as a new interpretation of fermion algebra and starting from the representation of orthofermion algebra a unitary quantum group has been defined. This quantum group was called general unitary quantum group, U_G, since all unitary quantum groups can be derived from this quantum group.

In chapter four, using the Heisenberg spin algebra, the fermion algebra can be written so that creation and annihilation operators for fermions in different states commute. This fermion algebra is called the commuting fermion algebra. Making a transformation on commuting fermion algebra a quantum group has been introduced and it is called the commuting fermionic inhomogeneous quantum group, $CoFI$.

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