A PROSPECTIVE SECONDARY MATHEMATICS TEACHER'S DEVELOPMENT OF THE MEANING OF THE CARTESIAN FORM OF COMPLEX NUMBERS

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ABSTRACT

A PROSPECTIVE SECONDARY MATHEMATICS TEACHER'S DEVELOPMENT OF THE MEANING OF THE CARTESIAN FORM OF COMPLEX NUMBERS

In this study, I articulate how a prospective secondary mathematics teacher reconstructs complex numbers upon the set of real numbers in the context of the solution sets of quadratic equations. Previous research has indicated that once asked the meaning of x and y in the Cartesian form of a complex number which is formally defined as $x + iy$ where x and y are real numbers, both students and teachers were able to state that x and y are real numbers, yet considered them separately rather than being components of a single entity. Thus, the question arises as to what x and y refer to algebraically and geometrically; why x and y have to be real numbers and what it means to be an element of the set of complex numbers. This study explicates a prospective secondary mathematics teacher's answers to these questions through the articulation of the participant's quantitative reasoning by considering Sfard's (1991) theory on the dual nature of the mathematical conceptions. With this account, I intend to contribute to mathematics education by providing evidence on how the development of the elements of complex numbers, which is through shrinking/stretching of the distance(s) between the roots and the x-coordinate of the vertex of any quadratic functions' graph, affords conceptualizing any complex number as a single entity in a well-defined set rather than only an algebraic prescription of certain operations. As the result of the instructional sequence in this study, the participant presents this well-defined set as the set consisting of the roots of quadratic equations with real coefficients.

ÖZET

BİR ORTAÖĞRETİM MATEMATİK ÖĞRETMEN ADAYININ KARMAŞIK SAYILARIN KARTEZYEN FORMUNUN ANLAMINI GELİŞTİRMESİ

Bu çalışmanın amacı bir ortaöğretim matematik öğretmeni adayının gerçek katsayılı ikinci dereceden denklemlerin köklerini çalışırken karmaşık sayıları gerçek sayılar kümesi üzerine nasıl kurduğunu incelemektir. x, y gerçek sayılar ve $i = \sqrt{-1}$ olmak üzere matematiksel olarak $x + iy$ şeklinde tanımlanan karmaşık sayıların kartezyen formundaki x ve y'nin anlamı sorulduğunda, geçmiş çalışmalarda hem öğrenciler hem öğretmenler x ve y'nin gerçek sayılar olduğunu belirtmişler ancak $x + iy$'yi tek bir çokluk olarak değil ayrı iki çokluk olarak ifade etmişlerdir. Bu nedenle kartezyen formdaki x ve y'nin cebirsel ve geometrik olarak neye karşılık geldiği, neden gerçek sayılar olması gerektiği ve $x + iy$'nin karmaşık sayılar kümesinin bir elemanı olmasının ne anlama geldiği soruları ortaya çıkmıştır. Bu çalışma, bir ortaöğretim matematik öğretmeni adayının bu sorulara cevaplarındaki nicel akıl yürütmesini Sfard'ın (1991) matematiksel kavramların ikili yapısı teorisini kullanarak incelemiştir. Bu bağlamda, bu çalışma ikinci dereceden denklemlerin grafiğindeki tepe noktasının x koordinatı ve denklemin kökleri arasındaki uzaklığın daralıp genişlemesi üzerinden karmaşık sayılar kümesinin elemanlarının nasıl ortaya çıktığına dair kanıtlar sunmuştur. Dolayısıyla karmaşık sayıların sadece cebirsel islemler sonucu oluşmadığını ve iyi tanımlanmış bir kümede tek bir çokluk olarak var olduğunu kavramsallaştırarak matematik eğitimine katkıda bulunmuştur. Çalışmadaki öğretim sonucunda katılımcı bu iyi tanımlanmış kümeyi gerçek katsayıları olan ikinci dereceden denklemin köklerinin oluşturduğu küme olarak tanımlamıştır.

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x_1	First root of a quadratic equation
x_2	Second root of a quadratic equation
z_1	First complex root of a quadratic equation
z_2	Second complex root of a quadratic equation
Δ	Discriminant

LIST OF ACRONYMS/ABBREVIATIONS

E	Prospective Secondary Mathematics Teacher 'Esra'
NCTM	National Council of Teachers of Mathematics
MEB	Ministry of National Education in Turkey
R	Researcher

1. INTRODUCTION

Understanding of a concept in the domain of mathematics is the necessary complement to its learning with reasoning in a way that one gives meaning to the constructs involved in that particular concept. From the perspective of constructivist theory, learning mathematics requires constructive work with mathematical objects (Davis *et al.*, 1990). Reasoning, on the other hand, is a necessary and powerful tool to develop understanding of any concept since it requires one to think logically and to know why and how to employ particular actions of mind which are needed to work with mathematical objects in a constructive way (Confrey, 1990; Simon *et al.*, 2010). Thus, developing an understanding of a concept depends on the individual construction of one's own knowledge about the concept through their own reasoning and mental actions in order to make use of concepts and their connections between and among each other, and to justify one's selected approach to solving a problematic situation. In this respect, constructing one's meaning of the different forms of a concept through reasoning is a need to merge various aspects of this concept into a meaningful whole.

Particularly, the concept of complex numbers in secondary school mathematics is a mathematical notion that has different forms, namely the Cartesian, exponential and polar forms as well as multiple representations of these forms both algebraic and geometric (Sfard, 1991; Karakök *et al.*, 2015). Conceptualization of complex numbers depends on blending this structure that is solidified by those different forms and representations into a consistent whole. As well a complete understanding of complex numbers dwell on one's developing these different meanings and the relationships among them, Sfard (1991) pointed that the first step in such development is one's understanding that any complex number is an element of a set superior to real numbers. Similarly, how the concept of complex numbers emerges in the curriculum offers an essential starting point to emphasize geometric meanings along with algebraic aspects of the same concept. In this regard, NCTM's Principles and Standards for School Mathematics (2000) highlighted that students' understanding of complex numbers should be based on creating a logical necessity to work with a number set superior to real numbers through working with the solutions of algebraic equations. In High School National Mathematics Curriculum of Turkey (2013b), the concept of complex numbers, in 10^{th} grade program, is also introduced under the heading of numbers and algebra as a new number set that emerged as a result of presenting particular situations where the set of real numbers are not sufficient, too. In this regard, the set of complex numbers emerges in connection with the solutions of the quadratic equations in High School National Mathematics Curriculum of Turkey (2013b) in line with NCTM standards for the grades 9-12.

Although scarce, several research studies have recently focused on conceptualization of the different forms of complex numbers (Panaoura *et al.*, 2006; Nordlander and Nordlander, 2012; Karakök *et al.*, 2015). Many of them studying conceptions of multiple forms and representations of complex numbers addressed some misconceptions (Panaoura *et al.*, 2006; Nordlander and Nordlander, 2012) and offered ways to improve understanding of complex numbers (Panaoura *et al.*, 2006; Nordlander and Nordlander, 2012; Soto-Johnson and Troup, 2014; Karakök *et al.*, 2015). However, there are not many research studies especially on how individuals reason and construct their own conception of complex numbers both algebraically and geometrically as a number set superior to real numbers through working with the solutions of algebraic equations.

In order to construct a robust conception of complex numbers as an extension of real numbers as emphasised by NCTM's Principles and Standards for School Mathematics (2000) and High School National Mathematics Curriculum of Turkey (2013b), in this study it is suggested to engage in quantitative reasoning, generated by Thompson (1994), as a means to construct mathematical objects of thought and processes involved in the development of the notion of complex number. From the perspective of the theory of quantitative reasoning, identification of quantities and quantitative operations involved in a mathematical concept such as the set of complex numbers enables one to reason through giving meaning to the mathematical objects and processes required for its better understanding. It is also recommended to introduce this new category of numbers as an extension of real numbers in the context of solutions of quadratic equations (Sfard, 1991). Thus, in this particular research study, from the perspective of quantitative reasoning I will explore a prospective mathematics teacher's developing the meaning of the Cartesian form of complex number.

In this study elaborating on the construction of the conception of complex numbers, a prospective teacher's developing the meaning of Cartesian form of complex numbers will be explored through a teaching experiment in consideration of an instructional sequence that emerged as a result of a mathematical analysis of complex numbers that focuses on quantitative reasoning.

In the next section, complex numbers in the history of mathematics, previous research on complex numbers and the theoretical framework of the study will be discussed.

2. LITERATURE REVIEW

The purpose of this section is to provide information on the history of complex numbers, previous research studies regarding the understanding of complex numbers, and the theoretical framework. First, I will briefly review the history of complex numbers. Secondly, I will present research on both students' and prospective and in-service teachers' interpretations and understandings of complex numbers. Lastly, I will describe the theoretical framework of the study building on the implications of the constructivist view in the context of mathematics education, and on Sfard's (1991) theory of the dual nature of mathematical conceptions with a particular focus on complex numbers. In the theoretical framework, I will also explain Thompson's (1990) theory of quantitative reasoning and discuss its implications within the constructivist perspective and Sfard's (1991) theory.

2.1. Complex Numbers in the History of Mathematics

Although the number of empirical studies regarding complex numbers is rare, various sources have presented historical construction of the concept of complex numbers and proposed important cognitive aspects required for improving an understanding of complex number system.

The development of complex numbers is similar to how other number systems were developed (Sfard, 1991). To illustrate, the set of negative numbers arising as an object resulting from the operation of subtracting big numbers from smaller ones, taking roots of negative numbers begot a necessity to work with complex numbers. In other words, complex numbers arose while carrying out operations following the symbols with an algorithm without regard to the meaning (Panaoura *et al.*, 2006). Particularly, Panaoura *et al.* (2006) stated that complex numbers emerged as a means to solve equations in the form of $x^3 + px = q$, in which the complex number $\sqrt{-1}$ appeared (Sfard, 1991; Usiskin *et al.*, 2003; Panaoura *et al.*, 2006). Though, it took deliberate work of some mathematicians to develop complex numbers formally. For instance, Cardano and Bombellini, first used complex numbers by calculating with a quantity whose square was -1 (Usiskin *et al.*, 2003; Panaoura *et al.*, 2006). Then, Descartes generated the 'real' and 'imaginary' terms, and later Euler introduced the letter 'i' to represent the number $\sqrt{-1}$ (Panaoura *et al.*, 2006). With the need of a formal foundation Gauss introduced a formal set of postulates from which the arithmetic properties of complex numbers could be deduced (Harkin and Harkin, 2014). It is known that all complex numbers are the numbers of the form '$x + yi$' where x and y are real numbers (Panaoura *et al.*, 2006). Thus, the expression '$x + yi$' is a way to represent complex numbers algebraically called the "binomial notation" of complex numbers (Usiskin *et al.*, 2003, p. 49). In addition, because real numbers are all the numbers that are positive, negative and zero, xy-plane can also be used to represent complex numbers geometrically as ordered pairs of real numbers (Usiskin *et al.*, 2003; Panaoura *et al.*, 2006). Particularly, real numbers are viewed as special cases of complex numbers. Algebraically speaking, they are just the numbers in the form of $x + yi$ when y equals 0; and geometrically speaking, they are the numbers on the real axis in the complex plane (Panaoura *et al.*, 2006).

Penrose (2004), after acknowledging the formal definition of complex numbers with $i = \sqrt{-1}$ stated that mathematicians treated complex numbers in the form of $x + yi$ as augmentation of two real parts, x and y. He highlighted that even though the sums can be treated as a pair of numbers, to accept complex numbers as a new category of numbers it is essential to conceptualize $x + yi$ as a single entity in a well-defined set (Sfard, 1991; Conner *et al.*, 2007). By this, Sfard (1991), Conner *et al.* (2007) refer to the fact that one has to recognize complex numbers with their binomial expressions of the form $x + y.\sqrt{-1}$ as legitimate mathematical objects in a well-defined set consisting of elements of the same kind or of a certain category. Based on these definitions, I acknowledge the definition of complex numbers as follows: 'The elements of the set of complex numbers are the roots of quadratic equations with real coefficients'. In other words, the elements of the set of complex numbers evolve from the roots of the quadratic equations with real coefficients. Since the set of complex numbers comprises all the roots of the quadratic equations, the set of real numbers is a subset of complex numbers.

Geometrically, Panaoura *et al.* (2006) stated that a complex number '$x + yi$' is represented in a plane by taking x-axis and y-axis, that is orthonormal to the x-axis, and pointing P, a point, with coordinates (x, y) where the image points of real numbers 'x' are on the x-axis and the image points of imaginary numbers 'yi' are on the y-axis. Thus, x-axis is called the real axis with a 'real unit' representing the real number 1 and y-axis is called the imaginary axis with an 'imaginary unit' i representing the complex number $\sqrt{-1}$ (Usiskin *et al.*, 2003). Even though treating a complex number as a pair of two parts, i.e. real and imaginary, facilitates its geometric representation on the coordinate plane, it is suggested that any complex number should be viewed as a single entity as an ordered pair of real numbers combining a real and an imaginary number on the complex plane (Nordlander and Nordlander, 2012; Karakök *et al.*, 2015).

According to Panaoura *et al.* (2006), John Wallis approached the concept of complex numbers as a geometric entity and tried to connect the point (x, y) with the complex quantity $x + yi$, but he could not state valid arguments about the notion of a perpendicular axis to the real axis for imaginary numbers with a common unit of $\sqrt{-1}$. Afterwards, Caspar Wessel introduced this idea of a perpendicular axis, and the definitions of operations on complex numbers with lines (Demetriadou and Gigatsis, 1996). As Gauss generated the representation of complex numbers in the plane, Wessel and Argand proposed the idea of line segments with directions (Burton, 1988). By the same token, Fauconnier and Turner (2002) argued that to conceptualize complex numbers as a new set of numbers one should have a complementary conception of complex number as both a number, which is a point in the Cartesian plane, and a vector, which is a line segment with a magnitude and direction in the Cartesian plane. That is, the notion of vectors were proposed to give a visual and physical entity to complex numbers (Panaoura *et al.*, 2006). A vector is defined as a quantity that is a directed line segment which has both magnitude and direction (Hillel, 2000).

In this regard, in this study, I acknowledge that complex numbers are the roots of quadratic equations with real coefficients such that any complex number can be represented as a point in the Cartesian plane and as a vector with some magnitude and direction. The first aspect, as mentioned earlier, implies that complex numbers belong to a well-defined set, i.e. the roots of quadratic equations, and the second aspect implies that they represent number having some measurable quantity.

2.2. Research on Complex Numbers

Some research on complex numbers has been conducted to investigate students' (Panaoura *et al.*, 2006; Nordlander and Nordlander, 2012; Soto-Johnson and Troup, 2014); prospective teachers' (Conner *et al.*, 2007; Nemirovsky *et al.*, 2012) and in-service teachers' (Karakök *et al.*, 2015) conceptions of complex numbers. In these studies, algebraic and geometric representations of different forms of complex numbers were scrutinized through participants' engagement in specific tasks related to representations of and operation on complex numbers. In particular, in these studies, the focuses were how participants' interpreted the different forms of complex numbers. Still, developing an understanding of complex numbers on the part of the learner is a rising matter of concern for the field of mathematics education because these studies point to the fact that complex numbers are understood very roughly (Panaoura *et al.*, 2006; Nemirovsky *et al.*, 2012; Soto-Johnson and Troup, 2014). In this regard, the need to develop a new set of numbers, complex numbers, that is originated in the operations on a known set of numbers, real numbers, was emphasized (Sfard, 1991; Panaoura *et al.*, 2006). In the following heading, I present these studies in detail.

2.2.1. Students' Conceptions of Complex Numbers

Students' construction of mathematical knowledge by their engagement in mathematical tasks has important implications for successful mathematics education in high schools (NCTM, 2000; Panaoura *et al.*, 2006). In this regard, Panaoura *et al.* (2006) investigated high school students' understanding of complex numbers through tasks involving algebraic and geometric representations of complex numbers and the students' preferred approaches to solve the tasks, and their fluency in translating between the two representations. Their study indicated that the students had difficulty with complex number problem solving regardless of their preferred approach (i.e. algebraic or geometric). Researchers pointed that students' difficulty resulted from their interpretation of two representations, i.e algebraic and geometric, as two distinct objects, rather than two representations of the same entity, a complex number. They contended that this might be a result of the fact that in schools the formal definition of the complex number is emphasized as '$a+bi$' which refers to a number with a real, 'a', and an unreal (imaginary) part, 'bi', including no geometric interpretation parallel to the algorithmic and symbolic interpretations. Hence, they suggested that focusing on both algebraic and geometric dimensions simultaneously would be useful for students to conceptualize complex numbers as a single entity.

Similarly, Nordlander and Norlander (2012) studied undergraduate students' conceptions of complex numbers. They classified some categories related to students concept images. Researchers also defined students' concept images as cognitive structures regarding the concept of complex numbers revealing a number of alternative conceptions (misconceptions). Their study has shown that students had difficulty with understanding that any number is an element of the complex number set. In addition, they found out that students conceived complex numbers as two separate entities consisting of a real and an imaginary part rather than having an understanding that a complex number is a single entity combining a real and an imaginary number such that it is a unique number in a well-defined set of numbers expressed binomially in the form $x + y\sqrt{-1}$ (Sfard, 1991).

As highlighted by Panaoura *et al.* (2006), another identified cognitive difficulty on the part of students was their lack of switching between different representations of complex numbers. Conner *et al.* (2007) also pointed to the same limited conception of complex numbers on the part of prospective teachers and they argued that this might have resulted from an over emphasis on the symbolic existence of i given the identity

$i^2 = -1$.

Similarly, Soto-Johnson and Troup's (2014) study with undergraduate students indicated that students focused firstly on algebraic relationships rather than geometric aspects while working with equations including complex numbers. However, they also found out that once students start thinking about the geometrical aspect of complex numbers they have been able to imagine "a complex number as a point, vector, and as an operator" (Soto-Johnson and Troup, 2014, p. 122) such that they were able to define multiplication of complex numbers as mental actions of rotation and dilation (Nemirovsky *et al.*, 2012). That is they were able to recognize complex numbers both as a point and a vector. Students also described conjugation as a reflection about the real axis.

All these studies suggested that (i) a focus on algebraic and geometric representations and (ii) the understanding of any complex number as a single entity, a single number as an ordered pair, are needed to overcome cognitive difficulties and conceptualize complex numbers. In line with these suggestions, switching flexibly between the algebraic and geometric representation of the Cartesian form was addressed as one's ability to represent a complex number as an ordered pair and the use of $x + yi$ as a binomial expression to represent any complex number to label vectors and points on the complex plane (Karakök *et al.*, 2015).

2.2.2. Prospective and In-service Teachers' Conceptions of Complex Numbers

Admitting that teachers' understanding of complex numbers and required content knowledge for teaching on the concept of complex numbers has not been studied broadly in mathematics education, there are several research studies conducted with prospective and in-service teachers to investigate their conception of complex numbers (Conner *et al.*, 2007; Nemirovsky *et al.*, 2012; Karakök *et al.*, 2015). Studying what teachers' know is important because, in order to support students' conceptualization of complex numbers, teachers should themselves also have both a significant amount of content knowledge and the knowledge about teaching complex numbers (Karakök *et al.*, 2015).

In particular, Conner *et al.* (2007) explored prospective teachers' conceptions of the arithmetic of complex numbers. Their study showed that teachers described multiplication of a real number by negative one as a reflection instead of a rotation of 180 degrees. They argued that this conception might have caused the inability to conceive multiplication by the complex number '$a + ib$' as operations of rotation and dilation since prospective teachers probably focused on the real number line rather than the complex plane. Researchers also pointed out that the conception of complex numbers is limited to the expression $i = \sqrt{-1}$. By this, similar to the high school students in the study of Panaoura *et al.* (2006) prospective teachers also considered complex numbers as pairs of real numbers rather than a single entity, a single number as an ordered pair. They concluded that although this perception might have enabled learners to interpret complex number addition geometrically by using vectors and decomposing them into the real and unreal (imaginary) parts, it does not provide meaningful geometric interpretations to conceptualize complex number multiplication.

On the other hand, Karakök *et al.* (2015) conducted a study with secondary in-service mathematics teachers to investigate their conceptualization of several forms of complex numbers, their arithmetic operations, and their translating between the different representation of these forms. Results revealed that visualization of a complex number as a point on the coordinate plane was difficult for some of the teachers in their study. One teacher even stated that the algebraic form of complex numbers as x+yi creates difficulty to think of it as a coordinate point because of the existence of an operational symbol implying addition. However, teachers need to know about multiple representations and different forms of complex numbers; understand connections between them; and navigate flexibly between these forms (Karakök *et al.*, 2015). In addition to Soto-Johnson and Troup (2014), Panaoura *et al.* (2006), and Nemirovsky *et al.* (2014), Karakök *et al.* (2015) also suggested that the simultaneous employment of algebraic and geometric aspects of complex numbers should be given more attention in order to work with all forms of complex numbers that will lead more powerful understanding of the concept.

In sum, all studies outlined above have focused on students' and teachers' understanding of various forms and different representations of complex numbers. Particularly, aforementioned studies point to the fact that both students and teachers relied primarily on algebraic aspects of complex numbers than geometric aspects to reason on equations rather. Yet, some students and teachers were fluent in working with the Cartesian form (Panaoura *et al.*, 2006; Karakök *et al.*, 2015). However, this does not guarantee their true conceptualization of complex numbers since whichever type of approach, i.e. geometric or algebraic, students preferred, they had difficulty in complex number problem solving (Panaoura *et al.*, 2006). This implies that given the Cartesian form, even though learners carry out the algorithms required for solving tasks, they might still have a lack of understanding of complex numbers thoroughly. In other words, this lack of understanding might result from an inability to employ the geometric approach effectively with multiple representations of complex numbers because of a focus on procedural aspects of complex numbers in particular (Karakök *et al.*, 2015). In these studies, alternative conceptions regarding complex numbers are also addressed with highlighting mental operations while working on complex numbers arithmetically. However, the construction of the set of complex numbers, on the part of the learner, upon real numbers has not been investigated to understand how one develops understandings and meanings of constructs involved in finding solutions of quadratic equations. It is in this respect that this study will be focusing on such an endeavor.

2.3. Theoretical Framework

In this section, from a philosophical perspective, first, I will briefly discuss the constructivist view of epistemology and its basic premises that provided insight into this study. From a psychological perspective, I will explain the nature of mathematical concepts and conceptions from Sfard's (1991) perspective. Afterward, I will state the constructs of a theory of quantitative reasoning since it corresponds with the operational aspect of mathematical conceptions within Sfard's theory. Lastly, in the context of one's construction of the complex numbers, I will explicate how these three distinct but related frameworks might integrate into shaping the basic principles of the design of the teaching of complex numbers for understanding.

2.3.1. Constructivism

Glasersfeld (1990) hypothesized the basic premises of constructivism as follows:

(i) Knowledge results from an individual's active cognizing and the learner is expected to construct his or her own meaning.
(ii) Learning is a process focusing on concepts through engaging in mental activity in minds, not on isolated objective facts.
(iii) Construction of meaning of a concept is not instantaneous, rather it takes time.

The first premise imply that learners actively construct their own knowledge and meaning from their experience. Such construction of knowledge (constructive processes) in cognition and perception are accessible through reflection that refers to a means of objectifying processes and products by creating a language to stabilize these processes, products and their positions in the network of other mathematical ideas. In this respect, individual or collective construction of one's own knowledge is carried out through their own conceptualization of mathematical ideas. Such mathematical abstractions (ideas) are operations that enable us to perceive experiential items and relational concepts (Glasersfeld, 1990). That is, if 'figurative' elements that form the experience withdraw, solely 'operative' elements remain, i.e. abstractions from operations (Piaget, 1969). While operations remain unobservable, symbols of operations are observable to manifest the abstract reaches of mathematics. However, understanding of symbolized operations cannot be demonstrated by some sequence of symbols that are assumed to be the documentation of an algorithm because constructivists contended that knowledge is not "...an experiencer-independent state of affairs" (Glasersfeld, 1990,

p. 27). Also, it is not a true representation of something that lives beyond our experience. That is, "the concepts and relations in terms of which we perceive and conceive the experiential world.." are generated by our own states of minds (Glasersfeld, 1990, p. 28). In this regard, based on the first premise, in this study, I acknowledge that one constructs his/her own knowledge through his/her operational aspects of mind. Based on the same premise, I also acknowledge that the discipline of mathematics has resulted from human activity as the generation of a language reflecting on one's mind activities and processes that construct mathematical ideas coming out of his/her experiences (Confrey, 1990; Simon *et al.*, 2010).

The second perimse imply that one's own cognitive acts (activities of mind) produce knowledge and understanding (Confrey, 1990). Thus "...conceptions can and do change." (Confrey, 1990, p. 108) with organizing one's network of constructions through reflections, i.e. learning. Since each learner has their own constructions, this suggests that one might view a mathematical idea qualitatively different than others (Piaget, 1969). In this regard, in this study, I also acknowledge that learners in a social group do not have the same mathematical concepts; rather, their mathematical concepts are compatible (Heinz *et al.*, 2000).Therefore, such compatibility supports the need to study students' learning through their own mathematical activity because, as mentioned earlier, constructivism also suggests that for promoting and developing meaning for concepts for different groups of learners, mental models of individuals' conceptions are needed (Glasersfeld, 1990). Thus, when teaching concepts, it is suggested that teachers should generate an appropriate model of learners' conception of an idea to assist their further restructuring these conceptualizations in appropriate ways that negotiate learner's and teacher's perspective (Confrey, 1990). In this respect, a teacher or researcher might develop an instructional sequence to form a hypothetical model of learners' conceptualization of a mathematical idea so that teachers or researchers can facilitate others' learning through their mathematical activity (Simon *et al.*, 2010).

2.3.2. Theory of the Dual Nature of Mathematical Conceptions

In this section, Sfard's (1991) theoretical framework of the dual nature of mathematical conceptions is provided. In order for exploring and identifying conceptualization of complex numbers, I also basically discuss the formation of complex numbers in terms of the constructs of Sfard's theory. I focus on this theory since in this study I not only take mathematical objects such as complex numbers as concepts but also take their operational aspect into consideration.

Sfard (1991) discussed the mathematical concept and mathematical conception by arguing that they are not mutually exclusive but complementary. In particular, Sfard's discussion focused on the acquisition(or formation) of mathematical concepts. Mathematical concept was defined as a mathematical idea "within the formal universe of ideal knowledge" (p. 3), whereas mathematical conception refers to "..the whole cluster of internal representations and associations evoked by the concept [or notion]- the concept's counterpart in the internal, subjective "universe of human knowing"" (p. 3). Sfard introduced operational conception and structural conception as two types of mathematical conception. Operational conception refers to "..processes, algorithms and actions.." (p. 4) a learner can go through. On the other hand, structural conception is considered to view mathematical concepts as abstract objects. However, it should be pointed out that structural conception and operational conception of a mathematical notion complement each other and imply a dual conception of a mathematical notion.

Therefore, according to these theoretical constructs, an abstract mathematical object might be considered to exist as a static (structural) entity from the perspective of a mathematical concept. From the perspective of a mathematical conception, an abstract mathematical object might be viewed as an operational entity. That is, once the mathematical object has been abstracted structurally, i.e. become a mathematical concept, the learner can and might use it to form other mathematical objects. Therefore, from the perspective of mathematical conception, the structural conception become a basis for the operational aspect of a conception. Therefore, in this study, I acknowledge that, as an abstract mathematical object, complex numbers, can be considered both structural and operational in terms of human activities. Also, that the concept of quadratic functions both algebraically and geometrically become a basis to be operationally activated to form complex numbers at a structural level. Acknowledging this view also corresponds with Confrey's (1990) statement and the constructivist epistemology such that mathematics is created by human activity such that operational conception precedes the structural conception of the process of concept formation, i.e. mathematical objects/notions (Sfard, 1991).

In this developmental process, Sfard structured a three-phase schema: interiorization, condensation, and reification to understand one's development from operational to structural conception, i.e. from process to object. Sfard (1990) suggested that these three developmental stages could be used as a means to decide the extent to which a learner can "...think structurally about a concept.." (p. 18). The stage of interiorization involves the learner's cognitive processes such as counting, matching, and subdividing. The condensation stage requires that the learner performing multiple processes can consider these processes as some part of a whole. That is, this stage provides evidence of progression such as navigating between different forms of representation of the same concept. The last stage, reification, is the point that one distinguishes an object from the process. In particular, reification is where multiple processes of a mathematical notion merge together and beget an instantaneous objectification, which generates an abstract mathematical object.

2.3.3. Quantitative Reasoning

In conjunction with the premises of the theory of constructivism and Sfard's (1991) theory of the dual nature of mathematical conceptions, in the following paragraphs, I discuss the nature of the quantitative reasoning theory. I focus on this theory since it allows for the articulation of how someone might reason while constructing mathematics concepts. That is, this theory affords to explicate what activities of mind one might go through while constructing mathematical ideas (concepts) and also how as a teacher/researcher we might create models of one's conceptions. Researchers also regarded quantitative reasoning as important to promote and support student learning in secondary and undergraduate mathematics (Confrey and Smith, 1995; Thompson, 2011; Moore, 2014).

Thompson (1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure" (p. 13) such that "..conceiving of situations and measurable quantities of a situation" (Moore *et al.*, 2009, p. 5) is necessary in order for students to reason quantitatively. Particularly, Moore *et al.* (2009) regarded quantitative reasoning as "..the mental actions of an individual conceiving a situation, constructing quantities of his or her conceived situation, and both developing and reasoning about relationships between these constructed quantities" (p. 3).

Based on quantitative reasoning, a quantity is "..a conceived attribute of something that admits a measurement process, where this 'something' could be image of a situation interpreted from a problem statement or a mathematical object" (Moore *et al.*, 2009, p. 3). This definition implies that a quantity depends on cognitive construction of an object whose attributes involve some measurement process where the mental image of this cognitive object of thought reflects a mathematical object allowing a measurement process (Smith and Thompson, 2008; Moore *et al.*, 2009). In line with this description, one of the central tenets of the theory of quantitative reasoning implies that quantities are not in the real world but in the mind of the learner (Thompson, 2011). Thus, a quantity exists as a "conceptual entity" in a way that thinking of a quantity refers to "..conceiving a quality of a cognitive object where this conception involves measurability of that quality" (Thompson, 1994, p. 184). Therefore, the notion of quantity provides insight into how one reasons quantitatively about situations because the definition of quantity involves a cognitive construction of an object or objectification of a phenomenon having measurable attributes (Moore *et al.*, 2009). In other words, "..to comprehend a quantity, an individual must have a mental image of an object and attributes of this object that can be measured" (Moore *et al.*, 2009,

p. 4) and such image "..could be an image of a situation interpreted from a problem statement or a mathematical object (e.g. a graph)" (Moore *et al.*, 2009, p. 3).

Quantification is therefore defined as a process, that involves explicit or implicit measurement of qualities of objects (Moore *et al.*, 2009), in which one assigns numerical values to attributes (Thompson, 1989). By the same token, a quantitative structure is a network of quantities and quantitative relationships (Thompson, 1989) generated as results of quantitative operations that are defined as "..mental operations by which one conceives a new quantity in relation to one or more already conceived quantities" (Thompson, 1994, p. 184). In other words, the fact that a quantitative structure is produced by mental actions of the learner is essential for the emergence of new mathematical concepts beyond existing ones (Smith and Thompson, 2008). According to this point of view, one can infer that the notion of quantity involves the result of actions of mind while reasoning quantitatively about situations since reasoning quantitatively about situations requires quantitative operations that originate in actions of one's mind.

As in detail will be explained in the next section, this suggests that quantity involves the notion of structural and operational aspects of a mathematical conception. In particular, Thompson (1994) stated that the notion of quantity supresses, goes beyond and above, the notion of concept since it consists of schemes; all images that comes with the concept.

2.3.4. Constructivism, The Dual Nature of Mathematical Conceptions, Quantitative Reasoning, and Complex Numbers

Aforementioned frameworks imply similar approaches to mathematical concepts and the constructions of them. First of all, basic premises of constructivism regard mathematical concepts as mental states of affairs occurring as a result of one's own meaningful conceptualization of a mathematical structure through mental activities. Thus, from the constructivist perspective, creation of mathematical concepts is viewed as one's mental actions in developing meaning for mathematical constructs and creating coherent mathematical structures. This view aligns with Sfard's theory because from her perspective mathematical concepts are developed both structurally and operationally (Sfard, 1991). That is, as in the constructivist perspective, concept development stages acknowledge mathematics as a creation of human activity.

Similarly, the notion of quantity with respect to Thompson's theory of quantitative reasoning corresponds with Sfard's mathematical concept since Thompson (1994) highlighted that "quantities are conceptual entities" (p. 184) and dwelled on abstract mathematical objects. From Thompson's point of view, mathematical concepts also have two different but related qualities that are static, structural in Sfard's (1991) terms; and dynamic, operational in Sfard's (1991) terms. As Sfard's theory allows for the realization of mathematical concepts having two related but different aspects, Thompson's (1994) theory additionally allows for the examination of the construction of the mathematical concepts from one's point of view.

Particularly, in the context of the construction of the concept of complex numbers, in this study, embracing the idea that a mathematical concept refers to "a theoretical construct within "the formal universe of ideal knowledge"" (Sfard, 1991, p. 3), I take the formal concept definition of complex numbers in the following way: "A complex number is an expression of the form $a + bi$ or $a + ib$ where a and b are real numbers, and i is the imaginary unit" (Adams and Essex, 2009, p. A2). I also emphasize that within the formal universe of mathematics, complex numbers are thought as a ring with different axioms (Fauconnier and Turner, 2002).

However, as Sfard (1991) pointed to, the first step towards conceptualizing complex numbers is to recognize that $i = \sqrt{-1}$. In other words, I take the stance that the first step for conceptualizing complex numbers is to understand "..what complex numbers 'stand for and really are"' (Nordlander and Nordlander 2012, p. 633). In this respect, I take complex numbers as mathematical objects when the symbol x+iy, "... is interpreted as a name for a legitimate object - as an element in a certain well-defined set

- and not only (or even not at all) as a prescription for certain manipulations" (Sfard 1991, p. 20). That is, in this study, I acknowledge that the structural conception of complex numbers refers to the following: "The elements of the set of complex numbers are the roots of quadratic equations with real coefficients." Such understanding also implies that since the set of complex numbers comprises all the roots of the quadratic equations with real coefficients, the set of real numbers is a subset of complex numbers.

In this regard, in this study, I acknowledge that as a mathematical object (concept) complex numbers are elements of a new set of numbers such that it is both a number, which is a point in the Cartesian plane, and a vector, which is a line segment with a magnitude and direction in the Cartesian plane (Fauconnier and Turner, 2002). Similarly, I also acknowledge that as a mathematical conception complex numbers refer to the whole cluster of internal representations and associations evoked by it. Based on this view, in this study, I use quantitative reasoning theory in order to explicate the nature of reasoning one might undertake that results in his or her creation of complex numbers from real numbers. In this regard, operationally, I take complex numbers as conceptual entities having been emerged from a combination of metal actions one might go through.

Based on these hypotheses, therefore, in this study, while thinking of complex numbers both operationally and structurally, the development of the Cartesian form of a complex number on the part of a prospective secondary mathematics teacher will be investigated.

3. SIGNIFICANCE OF THE RESEARCH STUDY

Constructivist view of epistemology, in the field of mathematics education, emphasizes the construction of individual's own meanings and understandings of concepts through their own mental activities. It also emphasizes teachers' or researchers' constructing mental models of individuals' conceptions to promote and develop meaning for concepts on the part of others (Glasersfeld, 1990). Mathematical concepts are formally defined by mathematicians, but personal conceptualization occurs through reconstruction of definition by individuals. Thus, not to cause a distorted interpretation, subjective reconstruction of the formal definition of any mathematical concept is needed to provide students with a cognitive structure regarding the concept to understand the concept better (Confrey, 1990).

In any curriculum, the development of the number concept is considered fundamental to any of the mathematical concepts (Usiskin *et al.*, 2003). Particularly, the set of complex numbers have applications not only in mathematics but also in science and engineering (Usiskin *et al.*, 2003). In this regard, the concept of complex numbers have been given attention and students are expected to develop its meaning based on the concept of real numbers relating to the quadratic polynomials as presented by Hwang (2004), NCTM (2000) and MEB (2013b).

There is some empirical research investigating both students' and teachers' conceptions of different forms of complex numbers (Panaoura *et al.*, 2006; Soto-Johnson and Troup, 2014; Karakök *et al.*, 2015). These research studies suggested that understanding complex numbers involves algebraic and geometric representations of the rectangular (Cartesian) form along with the other forms (e.g. polar and the exponential) and transition between these forms (Karakök *et al.*, 2015). They also pointed to two important issues: First, secondary school students and teachers do not conceptualize complex numbers as a single entity. Second, they have difficulties in thinking of both algebraic and geometric standpoints which represent the same number. Hence, further research is needed to decrease such conceptual difficulties regarding complex numbers (Panaoura *et al.*, 2006; Soto-Johnson and Troup, 2014; Karakök *et al.*, 2015). Sfard (1991) suggests that the development of the concept of complex numbers could be achieved through thinking of it as reified from real numbers.

Similarly, acknowledging that Sfard's (1991) discussion on operational and structural conceptions of the Cartesian form of complex numbers result from reifying real numbers NCTM (2000) also suggests students to "... understand complex numbers as solutions to quadratic equation that do not have real roots" (Panaoura *et al.*, 2006, p .682).

Thus, in this study, I examine such development evolving from the roots of quadratic equations. The ultimate purpose of this research study is therefore to investigate and elaborate on how one constructs meanings and understandings in the field of complex numbers upon her or his conceptions of quadratic functions. This construction involves re-invention of the set of complex numbers, on the part of the learner, upon the set of real numbers. In particular, I dwell on quantitative reasoning theory (Thompson, 1994) for the conceptual development of complex numbers from real numbers based on quantities involved in the solutions of quadratic equations.In this regard, a prospective teacher's development of complex numbers is studied in this study's context, which is assumed to provide insight into mathematics education regarding making sense of and teaching of the binomial expression of the Cartesian form of complex numbers. Such articulation might also shed light on how to overcome students' and/or teachers' difficulties regarding the concept of complex numbers as well as their development of and switching between different representations of complex numbers.

4. STATEMENT OF THE PROBLEM

This study aims to generate a model of how a prospective secondary mathematics teacher develops the meaning of the Cartesian form of complex numbers based on her quantitative reasoning through working with solution sets of quadratic equations with real coefficients. Therefore, this study particularly investigates the following research questions:

(i) How does a prospective secondary mathematics teacher develop the meaning of the Cartesian form of complex numbers?

(ii) What meanings of the Cartesian form of complex numbers does a prospective secondary mathematics teacher develop during an instructional sequence based on quantitative reasoning?

5. METHOD

Since the aim is to investigate a prospective secondary teacher's construction of the meaning of complex numbers this study is based on a teaching experiment consisting of teaching sessions, pre- and post-clinical interviews and pre- and post-written assessments. In other words, the aim is to foster one's re-invention of complex numbers, beyond real numbers, with understanding based on quantitative reasoning.

The following sections consist of four main components that elaborate on the methodology implemented in this study: the design, the participants, the data gathering procedure and data analysis.

5.1. Design of the Study: Teaching Experiment Methodology

This study originates in constructivist theory of epistemology (Glasersfeld, 1995) and developing meaning with understanding. It follows the two premises, particularly: (i) knowledge is created by human activities and (ii) one's knowledge is interpretable but not fundamentally knowable to any other individual. By the implication of these premises, the purpose of this research study is to build a model of students' mathematics, referred to students' mathematical realities (separate from ours as teachers and mathematics educators) (Steffe and Thompson, 2000) because what students say and do through their engagement in mathematical activities indicate their mathematics (Zembat, 2004). These models of students are called mathematics of students referring to the models of students' mathematics that researchers hypothesize as a result of teaching experiments and the modifications students make in their ways of operating (Steffe and Thompson, 2000). In this regard, the main purpose of teaching experiments is to understand students' mathematical realities and create models explaining these realities (Steffe and Thompson, 2000).

A teaching experiment provides researchers with the tool to continually generate, test and modify hypothesized models of one's thinking through continuous interactions with students (Steffe and Thompson, 2000). These models do not correspond one-to-one representation of how one thinks, but they are the researcher's best explanations-in the framework of researcher's own understanding, perspective and operations-of the meanings of what students say and do (Moore, 2014). In the context of this research study, since the aim is to understand how a prospective secondary mathematics teacher develops the meaning of the Cartesian form of complex numbers and to generate a model of her construction of meaning, teaching experiment methodology is employed in order to access her mathematical realities regarding the concept of complex numbers.

5.2. Participants

As earlier stated, NCTM (2000) claimed that the conception of complex numbers to build upon real numbers as a result of working with the solutions sets of quadratic equations, Panaoura *et al.* (2012) and Karakök *et al.* (2014) suggested simultaneous investigation of geometric and algebraic aspects to develop better understanding of complex numbers, Soto-Johnson and Troup (2014) found out that participants of their study recognized complex numbers as points, vectors, and operator, and similarly Fauconnier and Turner (2008) mentioned that space of complex numbers is blended because in that space numbers and vectors represented the same entities, i.e. complex numbers.

Acknowledging all those suggestions and findings we proposed an instructional sequence which introduces simultaneous employment of geometric and algebraic aspects in order to build complex numbers upon real numbers. Considering the literature and the intended tasks in the instructional sequence, in order to further build on his/her knowledge I looked for a voluntary participant who had the following criteria:

(i) ability to define quadratic functions algebraically,
(ii) ability to deduce the algebraic roots of quadratic equations,

(iii) ability to represent any quadratic function geometrically,

(iv) ability to explain the meaning of $-b/2a$ algebraically, i.e. in terms of its relation to the roots of quadratic equations, and geometrically, i.e. on the graph of a quadratic function,

(v) ability to state the relationship between complex roots and quadratic equations such that when $\Delta < 0$, there are complex roots where $\Delta = b^2 - 4ac$,

(vi) ability to state the magnitude of vectors algebraically,

(vii) ability to define what a vector is,

(viii) ability of the meaning of $\sqrt{\Delta}/2a$ geometrically, i.e. in relation to the graph of a quadratic function,

(ix) lack of understanding of what x and y in the form of $z = x + y$i refer to algebraically, i.e. in relation to the roots of quadratic equations and geometrically, in relation to the graph of a quadratic function,

(x) lack of the ability to reason behind the existence of the conjugate root once a complex root exists,

(xi) lack of the ability to define complex numbers as follows: "The elements of the set of complex numbers are the roots of quadratic equations with real coefficients where $f{:}R{\to}R$, $f(x) = ax^2 + bx + c$,where a, b and $c{\in}R$ and $a{\neq}0$."

These criteria are established to assure that the participant has the prerequisite understandings and skills needed to engage in task sequence involved in teaching sessions, but lacks the concepts which are intended to be promoted in the teaching experiment sessions.

For the selection of the participant of the study, I conducted a writtten pre-assessment (see Appendix A for the pre-assessment questions) with 21 prospective mathematics teachers who were in the fourth year of her five-year undergraduate program in secondary school mathematics education in a public university in Turkey where the medium of instruction is English. The participation was voluntary. Based on the preliminary analysis of their answers in the written pre-assessment I chose seven of them who had the potential to be the best candidate for being the participant in our teaching experiment sessions. After I and my advisor (hereafter referring 'we') discussed the answers of those seven candidates to the pre-assessment questions, we decided to conduct 45 minute-long clinical interviews with them in order to further investigate their background knowledge and reasoning underlying their answers (see Appendix B for the interview protocol). I conducted pre-assessment interviews with those seven candidates and transcribed interviews. After analyzing the transcribed pre-interviews with a focus on the interviewees' eligibility for being the participant who had the closest prior knowledge corresponding to our criteria for the participant's existing knowledge needed for our study, we decided Esra (a pseudonym) to be our participant.

As stated the factor that affected the selection of the participant was the participant's current knowledge of mathematics. Therefore, rather than choosing the most capable one, a prospective secondary mathematics teacher who had a range of learned concepts and limited understandings was chosen for the study.

5.3. Procedure

5.3.1. Data Sources

Data sources included (i) transcripts of the videos of teaching sessions, (ii) digitized researcher and teacher-researcher notes taken during and between teaching sessions, (iii) digitized copies of student work during teaching experiments, (iv) transcripts of the videos of the pre-and post clinical interviews and (v) digitized copies of the pre- and post-written assessments.

Among these the main sources of data are the transcripts of the video records of teaching experiment sessions and the pre- and post-interviews with the participant. The written artifacts from the teaching sessions, the pre-post interviews, and the pre-post written assessments were considered as a secondary data sources.

5.3.2. Data Collection

Data is collected in three phases: (i) a pre-interview after a written pre-assessment, (ii) the teaching sessions, and (iii) a post-interview after a written post-assessment.

Phase I: As mentioned earlier, in order to choose the participant for the study, first I conducted a written pre-assessment (Appendix A) with 21 prospective secondary teachers who were in their third and fourth years of a five-year undergraduate program in secondary school mathematics teacher education. The written pre-assessment questions were generated by two mathematics educators and me, with regard to the participant criteria presented earlier, in order to identify current statuses of the participants' knowledge. After the analysis of their written answers to the pre-assessment questions by identifying the extent to which they satisfy the participant criteria for this study, I and my advisor, who is a mathematics educator, decided on seven of them to take the pre-assessment interview. Prior to the teaching sessions, after a written pre-assessment, I conducted 45-minutes long pre-interview (Appendix B) with seven of these prospective teachers, following the design of a clinical interview (Clement, 2000) and Goldin's (2000) principles of structured, task-based interviews. The interviews were video-recorded and later transcribed. The aim of the pre-interviews was to monitor the participant's prior understandings and meanings regarding the concept of complex numbers prior to the teaching sessions. That is, these interviews provided insight into the participants' current understandings and meanings of the nature of complex numbers as well as their prerequisite knowledge on quadratic functions and equations. Then, I selected one of them (hereafter named Esra) to participate in the study according to the participant criteria listed earlier.

Phase II: For the teaching sessions, based on our analysis of the participant's current knowledge from the pre-interview data, I and my advisor first developed a hypothetical teaching-sequence (Appendix C). Throughout the implementation of the teaching experiment consisting of three 75- to 120-minute sessions, I operated a digital video camera and an audio recorder with the purpose of recording, Esra's evolving understandings and the interactions occurring among Esra and the teacher-researcher. Two cameras recorded each session, with one camera focused on participant work and the other focused on teacher-researcher work and computer display because her reasoning was evidenced in her drawings and what she pointed out along with what she told during the teaching sessions.. Student work was collected and digitized in order to confirm written work as inferred from the video data. After each session, digitized participant work and written notes were organized chronologically and paired with appropriate video files in order to support the analysis. Later, video recordings and audio recordings were transcribed for analysis. When recording, I simultaneously observed the sessions with an attempt to capture Esra's work as much as possible for analysis. I also took notes right after the the sessions about the ideas Esra formulated and my inferences about Esra's evolving understandings. Following each teaching session, I and the teacher-researcher discussed Esra's evolving understandings. These short discussions provided instantaneous analysis that became part of the data used to hypothesize about Esra's development of mathematical notions, and how she reasoned. They also enabled us to design following teaching sessions according to the current understanding of the participant after we formed hypothesized models of her thinking. The physical configuration of the teaching sessions is diagrammed (Appendix D).

Phase III: The last step of the data collection was one-hour long structured, task-based post-interview with the participant. After the teaching sessions were done, the participant was asked to answer the written-post-assessment questions (Appendix A). Then, I interviewed Esra two week(s) later following the teaching experiment sessions. The interview was video-recorded and later transcribed. The main purpose of the post-interview was to identify the participant's current understanding of the concept of complex numbers and to monitor the development occurred along the way that Esra constructed on the concept of complex numbers during the teaching sessions. The post-interview questions were also provided (Appendix B).

5.3.3. Data Analysis

In teaching experiments, hypothesis testing and hypothesis generating related to the conceptual development of the participant were emphasized (Steffe and Thompson, 2000). Therefore, in order to conjecture about the participant's understandings, in this study, three types of data analysis are conducted: interview analysis, ongoing analysis, and retrospective analysis. Each will be explained in the following subsections in detail.

5.3.4. Analysis of Pre- and Post-interviews

In order to determine participants' initial understandings of the concepts such as quadratic functions, their geometric representations and the binomial expression of the Cartesian form of complex numbers, I marked the important instances in the interviews in which mathematical understandings revealed as the participants engaged in tasks or gave responses to certain questions. During the investigation of these responses, I focused on (i) the articulation of their thoughts while giving responses to 'Why?' questions, (ii) the way they think about their solutions, think about the mathematical ideas, focus on procedure, use an algorithm, and (iii) what resources they relied on while answering questions (e.g. representations, formulas). Once I addressed such occurrences, I looked for the patterns of certain ways of thinking or counterexamples to such patterns, if there were. I rewatched the videos to identify such occurrences. My aim was to formulate models of Esra's current mathematical knowledge. I also examined the data from the post-interview in the same way.

5.3.5. Ongoing Data Analysis

Once concluding each of the two teaching sessions, I and my advisor reflected on the sessions and interpreted Esra's evolving understandings and constructs of the targeted concepts. Regarding this, we also focused on potential understandings to be developed by Esra, and how such understandings might have been promoted. Throughout the "after teaching session meetings" we asked and focused on the following questions:

(i) The mathematical issues that appeared during the teaching sessions and the way that the participant thought about and handled such issues;

(ii) Whether there was a change in participant's thinking about those issues or in her approach at the end of the sessions and the nature of that change;

(iii) The weaknesses or difficulties and possible explanations for such weaknesses or difficulties;

(iv) The factors that afforded or limited the change or the development in her conception;

(v) The possible mathematical understandings and meanings that might have been further investigated in the following sessions.

These meetings, guided by answers to the questions stated above, enabled us to refine the goals and following teaching-session plans accordingly. The ongoing analysis included the documentation of all hypotheses made during the data collection.

5.3.6. Retrospective Analysis

During the retrospective analysis after completion of data collection, I transcribed teaching session videos along with all observable actions. I and my advisor identified interaction sequences in which Esra's actions and utterances provided information about her thinking. We employed conceptual analysis techniques in order to characterize of Esra's thinking, develop and refine hypotheses of her mental action that explain our interpretations of her observable behaviors (Steffe and Thompson, 2000; Thompson, 2008).

After formulating hypotheses about Esra's evolving understandings I tried to support compatible models that accounted for her progress and I looked for contradictory evidence in the data set in order to refine and improve my claims and models of her understandings. I iteratively engaged in discussions with my advisor about the validity of my hypotheses and claims to build more viable models of her thinking and shifts in her understanding.

6. RESULTS

To provide background information on Esra's current understanding regarding the Cartesian form of complex numbers and the way her understanding evolved, I first present pre-interview findings along with the written pre-assessment. Second, I characterize her quantification of the roots of any quadratic equation including the x-coordinate of any parabola's vertex and the roots' distances to the x-coordinate of the vertex while constructing complex numbers upon real numbers throughout the teaching sessions. I finally provide evidence of meanings she developed for the constructs such as the x-coordinate of any parabola's vertex and the roots' distances to the x-coordinate of the vertex involved in her re-invention of the Cartesian form of complex numbers. I provide such explanation based on the quantitative reasoning, her formation of the concept of any complex number as a single entity and also on her mental operations she engaged in while she invented this new set of numbers, i.e. complex numbers, beyond an existing one, i.e. real numbers.

6.1. Analysis of Pre-interviews

The aim of pre-interviews was to analyze Esra's existing knowledge regarding the definition of complex numbers and quadratic functions; quadratic equations and their sets of solutions along with their algebraic and geometric aspects, and their relations to complex numbers. Taking into consideration the previous research results on complex numbers I and my advisor (hereafter referring 'we') composed pre-assessment questions (Appendix A). The focus of analysis was to identify Esra's prior knowledge and her current meanings of the quadratic functions such as the roots, the line of symmetry and the vertex of the parabola in terms of the algebraic and geometric aspects.

In the following paragraphs I presented some evidence of Esra's knowledge at the beginning of the study to describe what she knew prior to conducting our teaching sessions. While conducting the interview, I asked Esra to articulate how she reasoned during the written pre-assessment. First, I described Esra's current knowledge on quadratic functions and the solution sets of quadratic equations, and then on complex numbers and its relation to the quadratic equations.

6.1.1. Esra's Current Knowledge on Quadratic Functions and the Solution Sets of Quadratic Equations

To build upon Esra's prior knowledge about quadratic functions in order for her to further construct the meaning for the Cartesian form of complex numbers, Esra was expected to have the ability to define quadratic functions algebraically; the ability to deduce the algebraic roots of quadratic equations; the ability to represent any quadratic function geometrically; the ability to explain algebraically the meaning of the algebraic expressions, i.e. $-b/2a$ and $\sqrt{\Delta}/2a$, in the algebraic form of the roots of quadratic equations, and to explain their meaning geometrically on the graph of a quadratic function.

During the written pre-assessment, Esra was able to define quadratic functions in the following way:

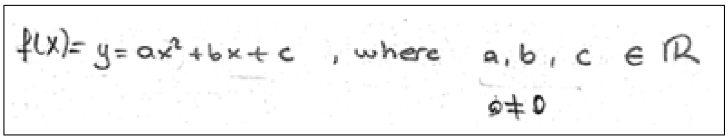

Figure 6.1. The definition of any quadratic function

During the pre-interview, once asked to explain her written statement, she was able to explain that the coefficients a,b,c are the elements of the set of real numbers and "a" should take values different than zero since otherwise the function could not be a quadratic function. She also stated that the domain and the range of any quadratic function are real numbers. This suggests that she had the ability to define quadratic functions algebraically with providing some reasoning as to why for instance the coefficient "a" has to be a non-zero real number.

When it came to represent a quadratic function geometrically in the written pre-assessment, even though she did not answer how she graphed any quadratic function, she was able to draw a few parabolic graphs:

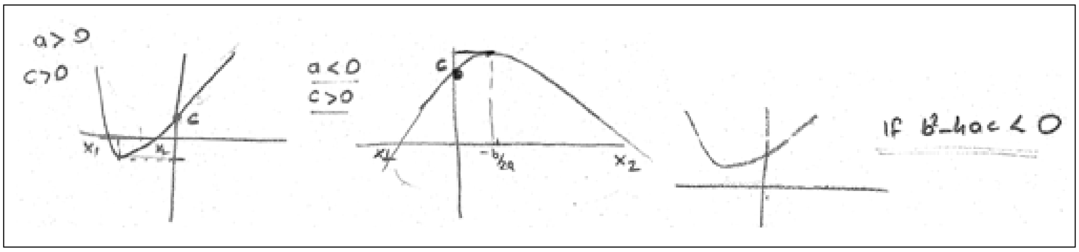

Figure 6.2. Quadratic functions' parabolas according to the changing values of a, b, c and Δ

During the interview, I asked how she drew those graphs and she explained:

> E: Now I assume $-b/2a$, as far as I have memorized, I do not know if I remember right, but did it provide me the abscissa of the vertex? I do not know x exactly. When we give 0 to x in y, $y = c$, so I indicated that as the point it touches.
>
> R: Alright.
>
> E: If a [in $ax^2 + bx + c$] is positive [graphic branches] upwards and if it is negative then I mentioned them as downwards I assume.
>
> R: OK.
>
> E: Eee, hmm. If the delta is smaller than 0 [pointing to Figure 6.3], this is so because it is not real root.

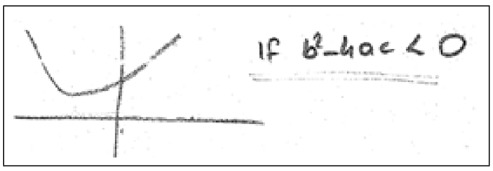

Figure 6.3. The quadratic function's parabola when $\Delta < 0$

This question was also asking for any remarkable points on the graph of any quadratic function that can be expressed in terms of a, b, c. When she was asked what x_1 and x_2 in Figure 6.2 referred to, she stated "x_1 and x_2 on the graph are the roots of the quadratic function and since it's a quadratic function there should be two roots". Her statement "if I remember right, I do not know if I remember it right, but $-b/2a$ should give me the abscissa of the vertex" suggested that she just memorized the formula without any further reasoning as to why it holds. Yet, her explanations regarding the sign of the coefficient 'a' and y-intercept of the function together with the roots suggested that she was able to graph any quadratic function.

As shown earlier, in the written pre-assessment, she pointed out that there is a point called the vertex with the algebraic form "$-b/2a$." On the other hand, in the pre-interview, she said that $-b/2a$ was the x-coordinate of the vertex. Then, I asked how she knew that $-b/2a$ was the x-coordinate of the vertex, she stated "I assume the sum of the roots, x_1 and x_2 divided by 2, it[$-b/2a$] is like right in the middle..." This, together with her written statement in Figure 6.4 suggested that she was at least able to algebraically deduce $-b/2a$ as the x-coordinate of the vertex of the quadratic function.

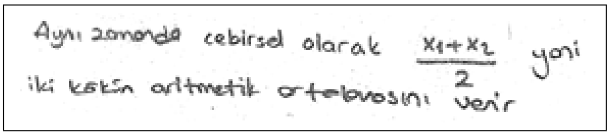

Figure 6.4. The algebraic meaning of $-b/2a$ in Esra's words as *"the arithmetical mean of the roots"*

She stated that her knowledge of the vertex was based on memorization and she also did not mention any quality of the parabola, i.e. the line of symmetry, resulting in her deduction above. When she was asked what the vertex means she defined the vertex as just the point where the quadratic function takes its minimum or maximum value. Yet, she did not define the vertex as the intersection point of the line of symmetry and the parabola. As much we acknowledge that her not stating this definition neither in the pre-assessment nor in the pre-interview does not mean that she does not have it. However, as will be further verified with the data from the first teaching session this was a limitation on her part. During the written pre-assessment, Esra was not able to deduce the roots of the quadratic equations algebraically. During the interview once asked again, she was not able to deduce it either.

E: By leaving x alone, what does x equal to? Finding it..[writing the algebraic expressions in Figure 6.5]

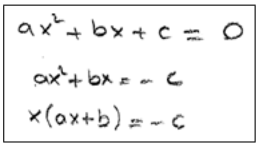

Figure 6.5. Esra's attempt to deduce the roots of any quadratic function algebraically

R: I got it.
E: I don't know..

Although Esra was not able to deduce the roots of any quadratic function with real coefficients during the written pre-assessment, she was able to mention both the algebraic and the geometric meaning of the components of it. Then during the pre-interview, I asked her to explain it again. That is, given the algebraic form of the roots, $x_1 = -b/2a - \sqrt{\Delta}/2a$ and $x_2 = -b/2a + \sqrt{\Delta}/2a$, I asked the geometric meanings of the algebraic expressions $-b/2a$ and $\sqrt{\Delta}/2a$, respectively. The reason that we asked this question was that, in our study, 'the roots' and 'the x-coordinate of the vertex' of a quadratic function's graph were considered as quantities such that they have measurable attributes such as their distances to the origin and from each other. As shown earlier, when asked what $-b/2a$ referred to algebraically, Esra stated that it equals to the average of the two roots, i.e. the sum of the two roots that is divided by two. Geometrically, she argued that $-b/2a$ is the x-coordinate of the vertex:

R: I have asked the meaning of $-b/2a$, the algebraic meaning and the geometrical one.
E: When I hear geometric, I understand that I should draw a graphic.
R: Hmm, yes, geometric means to show it in analytical plane. Yes exactly.
E: That is why I think that I have shown this [referring to what she drew and wrote in the written pre-assessment in Figure 6.6].

On the meaning of $-b/2a$, she was able to argue that, algebraically, its value is the half of the two roots' sum, and geometrically, it is the midpoint of the roots and the x-coordinate of the parabola's vertex. This suggested that $-b/2a$ was not only an algebraically deduced formula (the midpoint of the two roots) but at the same time it was a point equidistant from both of the roots such that she was able to measure such

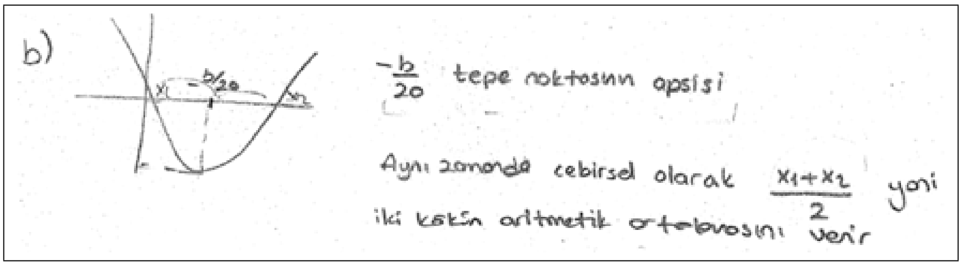

Figure 6.6. The geometric meaning of $-b/2a$ in Esra's words as *"the abscissa of the vertex"*

distance. This was evidenced also in her next explanation:

E: Yes, if we go to right and left [from $-b/2a$] this much [$\sqrt{\Delta}/2a$], then roots occur [referring to what she wrote in the written pre-assessment in Figure 6.7].

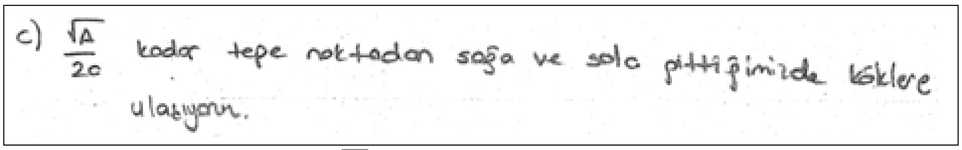

Figure 6.7. The meaning of $\sqrt{\Delta}/2a$ in Esra's words as *"if we go to right and left [from $-b/2a$] this much[$\sqrt{\Delta}/2a$] then we obtain the roots"*

Then I have reached to a conclusion from this expression [x$_1$= $-b/2a-\sqrt{\Delta}/2a$ and x$_2$= $-b/2a+\sqrt{\Delta}/2a$]. If this is x$_1$, x$_2$ [in Figure 6.8].

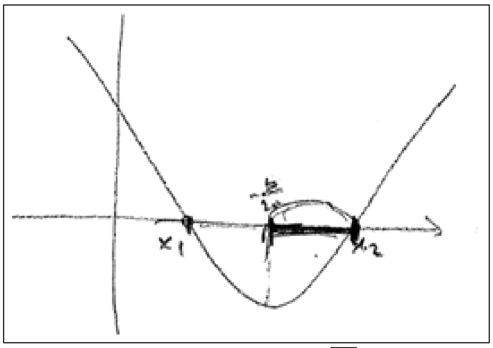

Figure 6.8. Esra's drawing of $\sqrt{\Delta}/2a$ geometrically

Eee = $-b/2a+\sqrt{\Delta}/2a$, yes when we go here [to the right] we have reached to this [x$_2$] ....What exactly are you asking geometrically about this?
R: What is $\sqrt{\Delta}/2a$?
E: It is the lentgh from the average of roots to a root or to the other root...
R: Now as I understand, a root's..
E: Eee....the distance to the midpoint of two roots...
R: ..you mean the distance to the abscissa of the vertex...
E: Exactly. Yes..

Esra knew that $\sqrt{\Delta}/2a$ referred geometrically to the distance from any of the roots to the midpoint of the roots $(-b/2a,\ 0)$. In addition, she was able to measure the distance between the roots and the x-coordinate of the vertex with $\sqrt{\Delta}/2a$. She was also able to think about the meaning of $(-b/2a{,}0)$ geometrically such that it was equidistant to both of the roots. This was evident in her statement "..if we go to right and left [from $-b/2a$] this much[$\sqrt{\Delta}/2a$], then roots occur"..

She then analyzed the roots' forms algebraically according to the changing values of discriminant (Δ), i.e. $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$. She knew algebraically that if the discriminant took values bigger than zero, there would be two real numbers as the roots of the quadratic equation; if the discriminant took values equal to zero, there would be a real number with multiplicity two as the roots of the quadratic equation; if the discriminant took values smaller than zero, there would be two unreal numbers as the roots of the quadratic equation because the algebraic expression of the roots included an expression i as in the formal definition of complex numbers.

After algebraically analyzing the values of discriminant, I asked her to reason geometrically what happens to the graph if discriminant's value is bigger than zero, i.e. $\Delta > 0$. She argued "This point [x_2 in Figure 8] is real. I get the root on the x axis, or this[x_1]."

Esra's answer implied that she could give meaning to $\sqrt{\Delta}/2a$ geometrically when $\Delta > 0$ such that there are two real roots on the real number line with a positive distance to the x-coordinate of the vertex. She, however, could not explain what happens to the geometric representation when $\Delta = 0$ and $\Delta < 0$:

R: OK. When delta is equal to 0, what do you get then?
E: Two roots are same but I did not get its shape correctly in my head...
R: OK. When Delta is smaller than 0, what kind of a geometrical meaning does it have?
E: Geometrical....I could not see it now..

This dialogue indicates that Esra regarded $\sqrt{\Delta}/2a$ as a fixed distance, not dynamic. That is, she could not argue that this distance can change, i.e. decrease or increase, for the changing values of the discriminant.

So far the data indicated that Esra was able to define quadratic functions algebraically and represent any quadratic function geometrically as a parabola. Even tough she could not deduce the algebraic form of the roots, once she was given the algebraic expressions for the roots she was able to explain the meaning of $-b/2a$ algebraically as the half of the sum of the roots of a quadratic equation. Once asked to explain the meaning of $-b/2a$ geometrically, she was able to state that it referred to the x-coordinate of the vertex and the midpoint of the roots on the real number line. Yet, she was not able to reason as to why $-b/2a$ represented the x-coordinate of the vertex nor she was able to define the vertex in terms of the intersection of the line of symmetry and the graphical representation of any quadratic function. However, once asked to explain the meaning of $\sqrt{\Delta}/2a$, she was able to reason that it represented the distance of the roots to $-b/2a$, the midpoint between them, and she was able to relate such distance existed once Δ was bigger than zero. Still, her not being able to reason what $\Delta = 0$ and $\Delta < 0$ meant geometrically suggested that she was not able to think about the distance of the roots to $-b/2a$ as a changing quantity.

6.1.2. Esra's Current Knowledge on Complex Numbers

In this section, I show data concerning Esra's current knowledge on the definition of complex numbers and complex numbers' relation to the quadratic equations.

Particularly, during the written pre-assessment, Esra had defined complex numbers as '... the numbers in the form of $a + bi$ where a, $b \in R$ and $i = \sqrt{-1}$. During the pre-interview, when I asked her again, she repeated the same definition. Her definition was limited to the formal definition given by the mathematicians. In other words, she did not relate complex number definition to the quadratic equations in any way even though in the curriculum this form, i.e. the Cartesian form of complex numbers, was emphasized to be appeared as a result of the investigation of the solution sets of quadratic equations. Still, it is necessary to be cautious about whether Esra knew the relationship between the solution sets of quadratic equations and complex numbers. Further data is needed to clarify that.

Additionally, during the written pre-assessment, Esra had answered correctly the question “Which of the following numbers are complex numbers?” Thus, at the very beginning of the pre-interview, I asked her to explain how she reasoned while answering this question. She stated the following:

E: I think that they all are complex numbers, that is what I say.

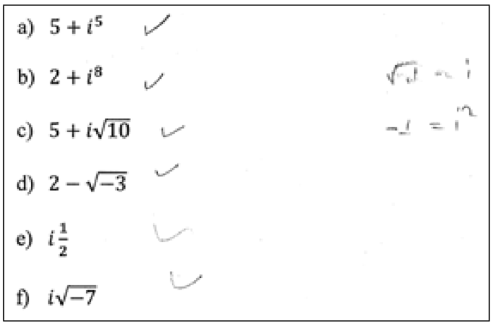
a) $5 + i^5$
b) $2 + i^8$
c) $5 + i\sqrt{10}$
d) $2 - \sqrt{-3}$
e) $i^{\frac{1}{2}}$
f) $i\sqrt{-7}$

Figure 6.9. Esra’s identification of complex numbers in a list of numbers

R: Ok, why?
E: Why..I assume I memorized until today. I am thinking...I don’t know but all the numbers are complex numbers, that is what I think.It is like that the complex numbers cover all numbers.
R: Complex numbers cover all numbers.
E: Yes.
R: What do you mean by all numbers?
E: I mean the real number. In complex numbers there are real numbers, hmm then there are rational numbers and there are natural numbers; so I think complex numbers cover them all.

Esra’s correct statement that the complex numbers included all the other numbers suggested that she knew the fact about the relationship between all the numbers and the complex numbers. Yet, her statement “I assume I memorized. . . I don’t know...” after the question ’why’ suggest that she did not have any reasoning behind it. That is, she did not know why such relationship exists. She seemed to just memorize it.

During the written pre-assessment, I also had asked Esra about the relationship between complex numbers and the roots of the quadratic equations. During the pre-interview I asked her to explain her reasoning.

R: Yes...So what was your answer? Can you explain me your answer?

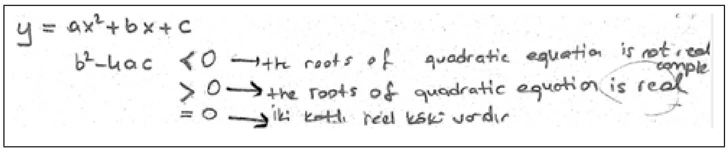

Figure 6.10. Esra's relating three cases of Δ to the roots of quadratic equations where $\Delta = 0$ corresponds to the case of having *"one real root with multiplicity two"*

E: I think this[$b^2 - 4ac$] is delta.
R: Yes, discriminant.
E: Yes, and if it is smaller than 0, it comes from complex numbers, numbers with i..If [delta] is bigger than 0 then the roots have to be real numbers and if it is equal to 0 then there is a real root with multiplicity two , that is what I said.
R: So how do you know this?
E: That is how I remembered.How do I know?
R: How do you know?
E: I got it.
R: Here [for the second case about discriminant], you mean that when delta is smaller than 0, then the roots are not real but complex, did you mean this?
E: Yes, it is complex, complex root.
R: So how do you know this? How do you know that these three cases explain those? How did you remember that? Where did you see?
E: In high school, what I memorized...
R: You know it because you memorized?
E: Exactly.

As shown in the excerpt above, her written explanation in algebraic forms and her uttering "..because this is what I memorized from high school" shows that she was able to state the three cases for the roots of the quadratic equations. However, she could not explain how she reasoned and how she came to know those statements. This again indicated her limited understanding regarding the relationship between complex numbers and the roots of the quadratic equations. As shown in the following excerpt, I then asked a follow-up question to further investigate her reasoning:

R: When delta is bigger than 0, do you get a complex number? The numbers you obtained, I mean the roots...

E: They are real, I know that complex covers real numbers too, then we can call [the roots] as complex numbers....I have only summarized what I know, but in general we call it real numbers...

R: Ok, why are real numbers the subset of complex?

E: I have no idea why real numbers are the subset of complex....That is how I learned..They [real and complex number sets] might even be disjoint now. Maybe that [relation of these sets] is on my mind because it was a visual image [in Figure 6.11].

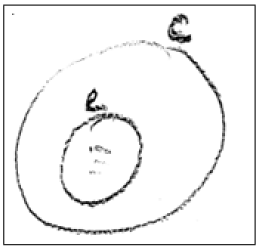

Figure 6.11. Esra's Venn scheme for the relation between the set of real numbers and the set of complex numbers

The dialogue above again suggests that she had no understanding of the relationship between the set of real numbers and the set of complex numbers. This again implied that she just heard or memorized without any reasoning that complex numbers includes real numbers.

In the written pre-assessment, Esra was asked to explain her reasoning on the following question:

For any quadratic equation $ax^2 + bx + c = 0$ where a, b and $c \in R$ and $a \neq 0$, when one root is in the form of $z = x + iy$, a complex root, the other root is $z = x - iy$ where x and y are real numbers. The complex root $z = x - iy$ is known as the conjugate of the complex root $z = x + iy$.

(i) What do x and y refer to algebraically?
(ii) Why do x and y have to be real numbers?
(iii) What do x and y refer to geometrically?
(iv) Why does the conjugate root exist? Explain your reasoning.

During the pre-interview I first asked Esra to explain what she had written for what x and y refer to algebraically. She answered:

E: So here [Figure 6.12], does this mean that it is real and imaginary?

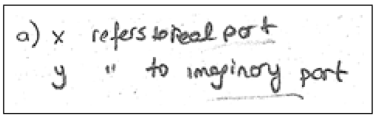

Figure 6.12. Esra's algebraic meaning for x and y in the form of $z = x + iy$

R: These x and y [x and y in $z = x + yi$] ; can you also express x and y algebraically as symbols? And for example, what is x in the question?
E: By saying as the element of real numbers?
R: Say it how you would like to answer algebraically. Say it like that. Say whatever you think.
E: x is the element of real numbers, y is the element of real numbers. Do you want a response like this?
R: Algebraically, yes, actually algebra also includes where these numbers [x and y] have come from. What is this x? What is this y algebraically?
E: Hmm.. My mind..[paused for several seconds]
R: You said x is real and y is the imaginary part. Do you think of anything else now? Does anything else come to your mind?
E: No it does not, I still think that both can be real, I am confused.

As presented in the excerpt, Esra could not reason and argue what x and y refer to algebraically in relation to any quadratic equation. Though she knew that both were real numbers. The discussion followed:

R: ..Why are x and y real? Why is this number [$z = x + iy$], is this complex number stated in a way that x and y in real [x and y as real]? It even says asks why x and y should be real?
E: Why does it have to be real?
R: Yes.
E: I don't know.

The excerpt shows that although Esra knew that x and y were real numbers she was not able to state why they had to be real numbers. I then asked her that what x and y refer to geometrically. Esra thought about their meanings geometrically as:

R: When I asked about it geometrically, you showed this [Figure 6.13.13], right? You said imaginary and real, alright. Do you have any idea what this x

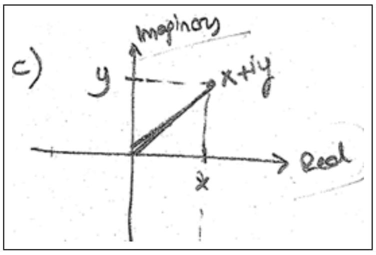

Figure 6.13. Esra's geometric meaning for x and y in the form of $z = x + iy$

and y are geometrically?

E: As I have drawn this [figure above], I thought it was this and still...

R: Not about where $x + iy$....what is x geometrically at $x + iy$? What is y geometrically?

E: I don't know.

According to the conversation above, she was not able to reason about what x and y refer to geometrically with regard to any quadratic equation, but she showed geometrically that x is on the real number line and y is on the imaginary number line. Also, data showed that x+iy referred to a point on the complex plane. The last part of the question was about the existence of conjugate roots. When she was asked about the reason for having a conjugate root if one of the roots are in the form of complex numbers, she directly stated "I have never thought about its reason, I don't know.." This illustrated that she did not know the reason underlying the existence of a conjugate roots.

As the data showed so far, Esra did not define complex numbers as follows: "The elements of the set of complex numbers are the roots of quadratic equations with real coefficients where $f{:}R{\to}R$, $f(x) = ax^2 + bx + c$,where a, b and $c{\in}R$ and $a{\neq}0$." Instead, she only provided the formal definition of complex numbers such that complex numbers are the numbers in the form of $a + bi$ where a, $b{\in}R$ and $i = \sqrt{-1}$. Although her not relating the set of complex numbers to the roots of quadratic equations should be taken cautiously, further data indicated that she really did not have any connection between the set of complex numbers and the quadratic equations. In particular, Esra stated the relationship between complex roots and quadratic equations such that for all the values of the discriminant, i.e. positive, zero, and negative, the roots of any quadratic equation, including real number roots, are complex numbers since the set of real numbers is a subset of the set of complex numbers. However, once asked she was not able to argue about why the set of complex numbers included the set of real numbers. Also, when she was asked the algebraic meanings of x and y in the form of $z = x+yi$, she just answered that they are both real numbers. Regarding the geometric meanings of x and y in the form of $z = x + yi$, she was only able to show a plane where she placed x and y in the form of $z = x + yi$ on a real and a perpendicular imaginary number line, respectively. Yet, she was not able to argue about what x and y in the form of $z = x + yi$ refer to algebraically in relation to the roots of quadratic equations, and geometrically in relation to the graph of a quadratic function. Lastly, she was not able to reason anything about the existence of the conjugate root once a complex root exists.

As the analysis presented, the pre-interview session together with the written pre-assessment was useful to identify the extent of Esra's current knowledge on quadratic functions, complex numbers, and how she reasoned regarding the concepts of quadratic functions and its constructs, and the Cartesian form of complex numbers. Based on the excerpts depicted, my advisor and I decided on the kinds of questions to be discussed and the tasks to be followed during the teaching sessions that would trigger reasoning on the part of the learner, Esra, in a way that builds upon her current understandings.

In the next section, the analysis of the three teaching experiment sessions was presented.

6.2. Analysis of Interaction

I discuss the teaching experiment sessions in three phases that are presented chronologically. The focus was on Esra's evolved quantities so that her developing understandings and meanings could be monitored.

The first teaching session focused on the deduction of the roots, the meaning of the x-coordinate of the vertex, and the meaning of the vertex. The goal was two-fold: First, I regard complex numbers as quantities obtained from the analysis of a situation involving a mathematical object of thought, such as quadratic functions and equations, into a network of quantities, i.e. the roots and the x-coordinate of the vertex, and quantitative relationships, i.e. how the roots' position and the position of the x-coordinate of the vertex relate to each other as distances and as points. Secondly, as the data from both the written pre-assessment and the pre-interview indicated , although Esra had some knowledge about quadratic functions and the roots of the quadratic equations, this knowledge was limited. That is, she had some knowledge without being able to reason about how she knew it.

The second and third teaching sessions built upon Esra's current status of knowing after the first session in a way that discussed how the roots and the x-coordinate of the vertex related to each other as distances and as points while studying the roots of quadratic equations with respect to both geometric and algebraic aspects.

6.2.1. Esra's Development of the Meaning of the Vertex

The mathematical object of thought to be used in the construction of complex numbers is hypothesized to be, as mentioned earlier, quadratic functions and equations with quantities such that the x-coordinate of the parabola's vertex and the roots of the equations whose distances to the origin and to each other can be measured.

Although Esra was able to define quadratic functions algebraically and represent any quadratic function geometrically as a parabola, she could not deduce the algebraic form of the roots. This showed that deducing the roots was beyond her reach. Therefore, we provided her with the algebraic deduction of the roots (Appendix E). We checked, given the proof, whether she could at least explain how one step followed the other. We hypothesized that if she could do so she would have been ready to be engaged in tasks that would trigger the reasoning to define vertex. This was important because as the aforementioned data indicated she lacked such reasoning. Thus, the first teaching session started with Esra's thinking about the given proof out loud Esra was able to explain the steps in the proof. Therefore, from now on, she could call on the algebraic forms of the roots when she thought she needed. That is, such knowledge was in her assimilatory scheme. At that point, acknowledging pre-interview findings, to further validate that she could call on her knowledge of the algebraic forms of the roots, she was asked to explain again what, $-b/2a$ and $\sqrt{\Delta}/2a$, refer to for any quadratic function both algebraically and geometrically. Asking such question was also important because Esra had not at all mention that $-b/2a$ was also the algebraic expression for the line of symmetry. Her knowing this was important because then she could realize that the roots were symmetric to each other about the x-coordinate of the vertex. Also, she would be able to justify why $-b/2a$ was x-coordinate of the vertex. Her knowing roots' being symmetric about the x-coordinate of the vertex was important again because in this way she would be able to start thinking that such quality was invariant for the roots of any quadratic equation. The discussion followed:

R: If you mention again, $-b/2a$ means what algebraically and geometrically?
E: Algebraically it is the arithmetical average of roots of equation..
R: Ok, fine.
E: And geometrically, it is the midpoint of the roots of equation..
R: You mentioned one more thing [during the interview].
E: This point, $[(-b/2a, 0)]$ is in equal distance to x_1 and x_2, the roots of equation and the abscissa of the vertex of function's graph is $-b/2a$.

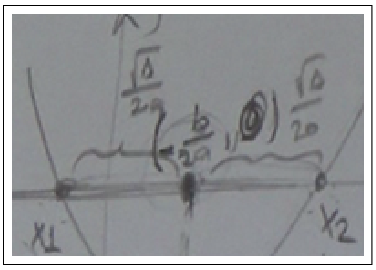

Figure 6.14. Esra's representing $-b/2a$ as a point and $\sqrt{\Delta}/2a$ as a distance on the real number line

She explained that $-b/2a$ algebraically is the average of the roots as the half of the sum of the roots' values. She also explained that $-b/2a$ geometrically meant for her the midpoint of the roots, and also the x-coordinate of the vertex of the parabola.

At that point, Esra was asked what does $\sqrt{\Delta}/2a$ referred to geometrically:

E: Here [Figure 6.14]
R: You mean?
E: This point [x_1] is here [$-b/2a-\sqrt{\Delta}/2a$], when we add this [$\sqrt{\Delta}/2a$] to this [$-b/2a-\sqrt{\Delta}/2a$] then this point $(-b/2a, 0)$ comes.. If I go as much as $\sqrt{\Delta}/2a$ to $-b/2a$, if I go here [to the right of the point $(-b/2a, 0)$ on the x-axis]and if I add [$\sqrt{\Delta}/2a$] I get the second root [$x_2= -b/2a+\sqrt{\Delta}/2a$], if I go backwards [to the left of the point $(-b/2a, 0)$ on the x-axis], if I subtract this distance [$\sqrt{\Delta}/2a$], then I will get the first root [$x_1= -b/2a-\sqrt{\Delta}/2a$] .

Esra's stating that adding $\sqrt{\Delta}/2a$ to $x_1= -b/2a-\sqrt{\Delta}/2a$ would give her the point ($-b/2a$,0) and adding $\sqrt{\Delta}/2a$ to the point ($-b/2a$,0) would give her $x_2= -b/2a+\sqrt{\Delta}/2a$, indicated that for Esra $\sqrt{\Delta}/2a$ was a number representing the distance from the x-coordinate of the vertex such that it was added to and subtracted from another number, i.e. $-b/2a$, on the real number line. Esra's relating the algebraic expressions of the roots and their representations on the parabola showed that she was simultaneously able to reason both algebraically and geometrically.

As the data showed, Esra did not mention the fact that $-b/2a$ referred to the algebraic form of the line of symmetry. Then the teacher researcher asked her the meaning of the vertex so that she might have related it to the line of symmetry from which she could have also justified why $-b/2a$ was the x-coordinate of the vertex. Esra pointed out and stated that the vertex was the lowest point on the graph of any quadratic function, parabola. At that point, her mention of vertex as only a quality of any parabola let us conjecture that Esra might not have known the definition of the vertex in terms of the line of symmetry and the parabola. That is, she had limited knowledge about it. Thus, we used the folding activity in order to trigger on her part the definition of the vertex of a parabola as the intersection point of the line of symmetry of the parabola and the parabola itself. This way, she would be able to explain why $-b/2a$ has to represent the x-coordinate of the vertex.

In the folding activity, she was given a parchment paper on which a parabola was drawn and asked to fold the paper in a way that divides the parabola into two congruent pieces and she pointed out the vertex as the point where she needed to fold:

E: I have to fold it from the the vertex.

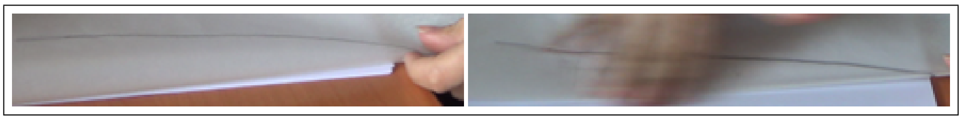

Figure 6.15. Esra folds the parchment

......

E: It seems like that it should be symmetric but when it is like this [a wider parabola], is it symmetric again? It has become symmetric from the the vertex, and we have divided it into two from there...

R: So when you consider the first parabola, if you fold it from any other point, could you divide it into two congruent parts again?

E: I don't think so, it would not be symmetric.

R: Why?

E: Because, in order to be symmetric, it should be folded only from one spot, at least for this figure from one spot...

R: What happens when you fold?

E: Same and congruent parts.

R: What are the congruent parts?

E: Congruent parts of parabola...

R: Which parts do you mean as the congruent parts of parabola?

E: This [pointing left leg of the parabola she folded]. That [pointing right leg of the parabola she folded].

R: OK. What is symmetric?

E: This [pointing left leg of the parabola] and this [pointing right leg of the parabola]

R: And, when you take a point in here [on the parabola drawn on the parchment], can you tell us this point's symmetry? Where is it?

E: The place in equal distance [dragging her pencil to the right leg of the parabola perpendicular to the line of symmetry]

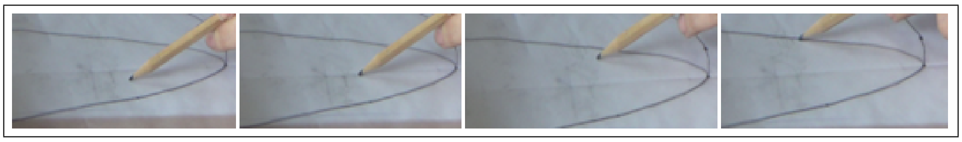

Figure 6.16. Esra finds the symmetry of any given point on the parabola

R: Can you show that equal distance?

E: These two [Figure 6.21] are equal.

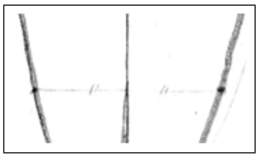

Figure 6.17. Esra's pointing to the equal distances from the points on the parabola to the line of symmetry

As the excerpt showed, Esra thought of the vertex as the mid-point of the parabola. This was evident in her folding the paper right at the vertex into two equal parts. However her statement "It seems like that it should be symmetric but when it is like this [a wider parabola], is it symmetric again? It has become symmetric from the the vertex, and we have divided it into two from there..." showed that Esra was not sure whether the symmetry would hold for other parabolas once folded at the vertex. This suggested that she had not thought of the relationship between the vertex and the line of symmetry. Yet, physically engaging in the folding activity resulted in her matching all the points on one part of the curve on the left side with all the points on the other part of the curve on the right side. Thus, folding the parabola into two congruent parts triggered the idea of symmetry on her part. That is, for her congruent parts were the same as symmetric parts which were the legs of the parabola with all the points matched with each other. In addition, once asked to show the symmetry of a point on one of the legs of the parabola on the other part, she mentioned that those points has to be equidistant from each other. Thus, being symmetric meant for her to be equidistant from each other about the line at which she folded the paper. Also, her statement that she has to fold it at the vertex, only at one point, because otherwise the symmetry does not hold, shows that she knew the uniqueness of the point at which the symmetry holds.

However, once Esra was asked to explain what the line where she folded the paper referred to algebraically, she could not express that the algebraic form of the line of symmetry was x=$-b/2a$.

To trigger on her part how to represent the line of symmetry algebraically, she was asked to place the parabola on a coordinate system. She was able to point out the roots on this coordinate system as the points where the parabola intersected the real number line. Esra, then, drew the line of symmetry on the same plane and stated that all the points on the parabola were symmetric to each other about that line. She also stated that like all the other points, the roots are also equidistant from that line where she folded the paper into two congruent parts.When asked if she knew the name of such a line, she stated that she did not know it. Then the teacher-researcher told her to call such line as the line of symmetry from now on. Then, to further validate her realization of the uniqueness of the line of symmetry and the point (the vertex) at which the parabola was folded into two congruent parts, she was asked to reason about how many line of symmetry a parabola might have:

E: In equal distance. Maybe if you fold it from here [Figure 6.18],we can find two symmetric points, but the symmetry axis should be for all points...

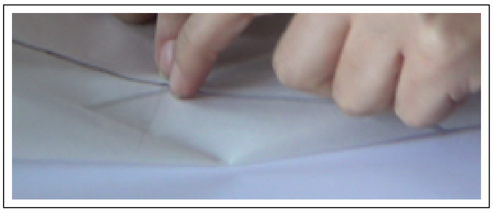

Figure 6.18. Esra folds the paper at a point different than the vertex

R: What happens to all points [on the parabola] when you fold it according to symmetry?

E: They overlap, but in here [the case of folding at a point other than the the vertex in Figure 6.18] only if we fold like this, maybe only two points can be in equal distance but the remaining points are not in equal distance to each other.

R: What is the reason? When you folded from here [the vertex] what did you say about all the points?

E: They overlap. In other one, [the case of folding at a point other than the the vertex] only two points overlap the others don't, they do not overlap...

R: Considering all of these, what did you say when you thought on the roots?

E: x_1 and x_2 [the roots] are symmetric to one another according to symmetric axis..

It is interesting that she tried to fold the parabola into two equal parts using another line through another point on the parabola, but she realized that once she did that not all the points on the two legs of the parabola would be coinciding with each other. She also reasoned that all the points should be symmetric about the line of symmetry. This implied that there should be only one line of symmetry of a parabola. It is also important to state that Esra's attempt if there is another line of symmetry suggested that she might not have related the line of symmetry and the vertex. This was also evident in her not being able to express the line of symmetry algebraically as x=$-b/2a$.

Then the teacher-researcher took Esra's attention to the roots:

> R: Considering all of these, what did you say when you thought on the roots?
> E: x_1 and x_2 [the roots] are symmetric to one another according to symmetry axis..

Esra was able to state that the roots were symmetric about this line of symmetry. This was because she already knew that for each point on the parabola she could find a point that was in equal length to the line of symmetry. That is, all points on the parabola were symmetric to each other about the line of symmetry including the roots of the quadratic equation. Then, as shown below in the excerpt, once she was asked to think about the line of symmetry in terms of the roots, she stated the following:

> E: This point ($-b/2a$,0) was in the equal distance to x_1 and x_2, the roots of equation; so the symmetry axis was the midpoint for x_1 and x_2 too. So this distance [distance of x_1 to ($-b/2a$,0)] is equal to this distance [distance to x_2 to ($-b/2a$,0)] and symmetry axis goes through the midpoints of all points including the roots, it passes from here [($-b/2a$,0)], and that is why I say x=$-b/2a$.

As the excerpt shows that she called on her knowledge of the fact that ($-b/2a$,0) was the midpoint of the two roots and therefore it was equidistant from both roots. Then, also reasoning that the line of symmetry has to be in the middle of all the points on the parabola she was able to reason that the algebraic form of the line of symmetry had to be x=$-b/2a$.

At that point, the teacher-researcher asked Esra to define the vertex the discussion followed:

> E: The symmetry axis is the line going through the vertex. And its abscissa is $-b/2a$ and the ordinate is 0. So the symmetry axis goes through these points. And the vertex is... [pausing several seconds]
> R: Could you draw the symmetry axis?
> E: The point at which the parabola intersects. The point at which the parabola intersects its symmetry axis gave us the vertex [seems puzzled]

Interestingly although Esra knew that the symmetry axis went through the point $(-b/2a,0)$ and the vertex simultaneously, she was not able to define vertex. This suggested that her focus was on the line and the vertex itself. Physically drawing the line of symmetry allowed her to mentally match the the vertex both as a point on the parabola and a point on the line of symmetry intersecting the parabola. That was why she was able to state that vertex is the point at which the parabola intersects its' symmetry axis. At that point she also stated that "..there was only one vertex since there was only one symmetry line of a parabola which goes through the vertex."

So far the data showed the following: First, she called on her knowledge that $-b/2a$ algebraically represented the midpoint of the roots and also that such point was equidistant from the roots. Then, physically folding the parabola into two congruent parts but at the same time engaging mentally in the matching all the points on the two legs of the parabola resulted in that all points were equidistant from each other, i.e. symmetric to each other. This allowed her to realize that folding the parabola physically in the middle and also mentally matching all the points on the two legs created a line from which all the points including the roots were equidistant. This then allowed her to deduce that such line of symmetry has to pass through the point $(-b/2a,0)$ and its algebraic expression had to be equal to x=$-b/2a$. Not only this but also she was able to realize that the line of symmetry goes through the vertex. Then her matching the vertex both as a point on the parabola and as a point on the line of symmetry led her to come up with the idea that "...the point at which the parabola intersects its symmetry axis gave us the vertex."

6.2.2. Esra's Development of the Definition of Complex Numbers

In the first teaching session, to discuss the changing values of the distance between the roots and the x-coordinate of the symmetry, i.e. the dynamic nature of the distance as an increasing and decreasing quantity, we asked how many parabolas having the same x-coordinate of the vertex one could draw. She stated that one could draw infinitely many parabolas having the same x-coordinate of its vertex as in Figure 6.19 she drew.

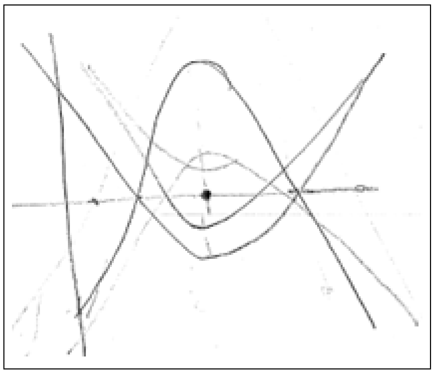

Figure 6.19. Esra draws quadratic functions having the same x-coordinate of the vertex

We then asked the question "What were changing and what were not changing in the parabolas she drew given the algebraic form, $ax^2 + bx + c$?" She was able to state that the values of a, b, c and even x and y were all changing in her examples drawn above. She also stated that since those values were changing the root's distances to the x-coordinate of the vertex was changing; but, the ratio of the value of '-b' to the value of '2a' did not change.

This was important because we conjectured that once Esra had imagined such variance and invariance simultaneously, she might have developed the idea that at some point, when ($\Delta = 0$), all of the real numbers as the roots of quadratic function would be covered. On the other hand, the quadratic function family's existence would yield to the necessity for new roots that are not real numbers with the invariant quality (i.e. the symmetry of the roots). This then might have yielded her to develop the idea that once there is a complex root of the form $x + iy$ then there has to exist another root of the form $x - iy$. Having such hypotheses and given that she reasoned on the changing and the invariant quantities such as the roots' distances to the x-coordinate of the vertex and $-b/2a$, she was asked to give some examples of quadratic functions algebraically whose parabolas had the same x-coordinate of the vertex. We asked this question to further validate whether she was able to provide concrete examples for her general drawing above. She stated "I am going to write the functions with the same $-b/2a$" and wrote the quadratic functions such as y= $2x^2 - 8x + 6$, y= $4x^2 - 16x + 7$, and y= $6x^2 - 24x + 1$.

Then we asked to explain the similarities and differences among the three function examples she gave, the dialogue followed:

> E: [She writes $y = 2x^2 - 8x + 6$, calculates Δ by writing 64-48=16, then writes $y = 4x^2 - 16x + 7$, and $y = 6x^2 - 24x + 1$]
>
> R: Could you explain what you've done while writing?
>
> E: I multiplied a by two to keep the rate $b/2a$ same and similarly I multiplied b by two [pointing to $y = 4x^2 - 16x + 7$]. Here [pointing to $y = 6x^2 - 24x + 1$] I multiplied a by 3 and b by 3 so that the rate $-b/2a$ stayed unchanged, so that it didn't change..
>
> R: You gave three examples, what are changing in these examples?
>
> E: a, b, c, x, y, they all are changing, the abscissa of the vertex isn't changing.
>
> R: When the abscissa of the vertex didn't change, if you think of other functions and the roots' states, if you think of the distance of the roots and the abscissa of the vertex, how does these distances changing or unchanging? When you think of the distance of the roots and the abscissa of the vertex are they[the distance of the roots and the abscissa of the vertex] changing or unchanging?
>
> E: They[the distance of the roots and the abscissa of the vertex] are changing because c changed; so the roots changed. The roots will be different so $\sqrt{\Delta}/2a$ will be changed.

In the excerpt above, her statement that all the values a,b, c were changing even though the x-coordinate of the vertex was the same, i.e. the value of $-b/2a$, showed that she was able to keep the x-coordinate of the vertex the same while changing the roots. Her reasoning that the changing values of 'c' would result in the changing values of the roots which in turn would bring about the changing values of $\sqrt{\Delta}/2a$ suggested that she was able to think of $\sqrt{\Delta}/2a$ as a varying quantity, i.e. distance. That is, she acknowledged that the decrease or increase in this distance was related to the changing coefficient values of quadratic functions, consequently, she was able to imagine the value of $\sqrt{\Delta}/2a$ as a changing quantity.

Since her focus was on the algebraic notation, once asked again how geometrically this was possible, she stated the following:

> E: Because if I change a, b, c; I can write infinitely many functions with only $b/2a$ as fixed...
> R: So when you consider geometrically, we said $-b/2a$ to be the same, right?
> E: Yes...
> R: You say that you can draw infinitely many parabolas whose abscissa of the vertex is the same, how can you explain the geometrical reason of this?
> E: So, for different parabolas, I can find infinitely many x_1 and x_2 in equal distance [to the point $(-b/2a, 0)$], that is why [pointing to Figure 6.19].

She knew that changing the values of a, b, c and keeping the value of $-b/2a$ the same would result in infinitely many quadratic functions. For her, infinitely many functions whose graphs on the R-plane would constitute infinitely many roots in equal distances with respect to the x-coordinate of the vertex. Her reasoning suggested that, she regarded $\sqrt{\Delta}/2a$ as a quantity that was changing and movable, i.e. dynamic and operational and simultaneously she was able to think of the two roots' having equal distances to the x-coordinate of the vertex as an invariant quantity.

She was then asked again to explain her reasoning geometrically on the real plane, Esra stated "Haa all real numbers [pointing to the x axis which is the real number line], the real numbers here, these real numbers on this x axis". Data suggested that once asked to think about the placement of the roots on the x-axis, she thought that all the real numbers would refer to the roots. Then, she argued the following:

> R: Fine, could you explain why they [parabolas] give [pointing to all real numbers on the x axis]?
> E: Wait a minute, I can draw [parabolas] with the same abscissa of the vertex at all points [with her two fingers pointing to any point right and left to the x-coordinate of the vertex on the x axis]; so it is true for x's, all real numbers, for real numbers on this x axis it is true; so I can draw [parabola].

Esra's thinking of the x-coordinate of the vertex as invariant and thinking of all real numbers right and left to it as potential roots implied again that Esra was able think of those roots not only as points on the real umber line but also as distances to the x-coordinate of the vertex. This again suggested that she had the understanding that any quadratic equation having real roots was intersecting the real number line or tangent to it and such existence of roots were dynamic in nature. At this point the first session ended.

In the second teaching session, she was asked to reason again on the examples she provided during the earlier session. This was done in order for Esra to call on her reasoning from previous session and to further continue to monitor her reasoning both algebraically and geometrically. Esra provided the same reasoning such that although she changed the values of 'a' and 'b' she kept $-b/2a$ the same by taking the same multiples of the values of 'a' and 'b'. She also chose the values of 'c' arbitrarily. Once asked to explain geometrically, she reasoned that the distances to the roots from the x-coordinate of the vertex was changing, i.e. it was movable on the real number line and therefore dynamic in nature.

To further build on her inference, at this point in the teaching session, we provided her with some specific examples of quadratic functions in Figure 6.20 on a mathematics software, Desmos. We chose such examples with the value of 'a' being equal to 1 so that she could focus on the changing values of the quadratic equations' roots as the parabolas were changing. She acknowledged algebraically and geometrically that for the functions $x^2 + 2x - 8$, $x^2 + 2x - 4$, $x^2 + 2x - 1$, $x^2 + 2x$, and $x^2 + 2x + 1$, the coefficients a and b did not change, so the x-coordinate of the vertex was the same for all those quadratic functions.

In addition based on the changing values of c she was able to decide which algebraic quadratic function corresponded to which parabola on the coordinate system drawn on Desmos. Similarly, she argued that the vertex for those functions were changing because y-coordinate of the vertex was changing for each quadratic function. She

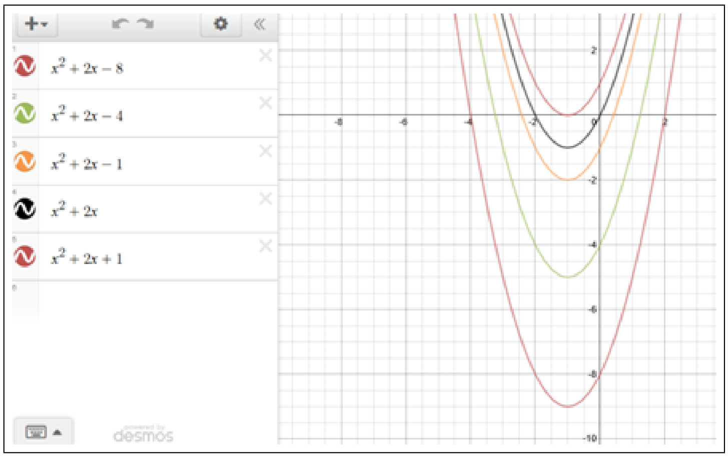

Figure 6.20. Some specific examples of quadratic functions given to Esra on a dynamic mathematics software

also added that "$\sqrt{\Delta}/2a$'s, the distances, are changing.." which implied that those distances were not fixed for her.

After presenting those specific quadratic functions on Desmos, she was presented with its printed version to point to the roots not only as quantities having distances but also as points on the real number line. Also, she would be able to focus on the changing values of the roots so that she could realize how the roots changed from one quadratic function to the other geometrically.

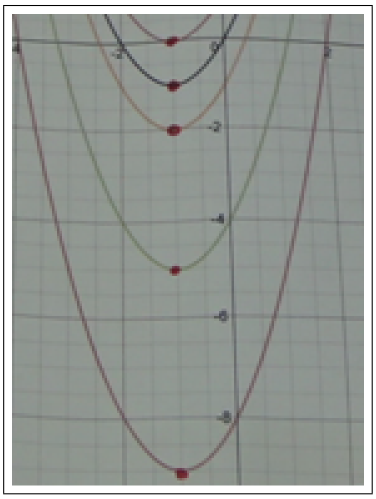

Figure 6.21. Esra marks the vertices for each given quadratic function

As in Figure 6.21, geometrically she showed the vertex for each function on the printed paper and argued that "As c changes, the roots are changing." At that point, she was asked to track the roots of those quadratic equations:

> E:$\sqrt{\Delta}/2a$ changes, the abscissa of the vertex was $-b/2a$. For this function [x^2+2x] this is x_1 root, here [on the point (-2,0) in Figure 6.23. But this function [orange parabola of the function x^2+2x-1] has this distance [the distance between the roots and the x-coordinate of the vertex].

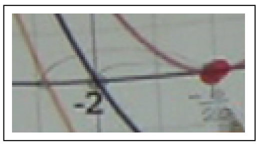

Figure 6.22. Esra points to the distance between the roots and the x-coordinate of the vertex

For Esra, since the values of c changed, the distances $\sqrt{\Delta}/2a$'s were changing. Esra also argued that the relationship between the roots for each quadratic function were not changing such that their midpoints were the same, but the roots' distances from each other were increasing. She added:

> E: According to symmetry axis, they [x_1, x_2, the roots] have the same distance, ee they are symmetric. Exactly.
> R: Symmetric, OK, fine. Then when you consider according to the abscissa of the vertex..
> E: Then [x_1, x_2, the roots] are symmetric to each other. Exactly.

Based on the excerpt, she was able to further reason that the roots have the same distances with respect to the x-coordinate of the vertex such that their being equal brought about the two roots' being reflections of each other. Also her statement "All the roots[the two roots of any quadratic equation] are symmetric to each other about the abscissa of the vertex.." suggested that she not only knew that the values of $\sqrt{\Delta}/2a$ as a quantity, representing the distance of the roots to the x-coordinate of the vertex was changing but also the symmetry of all the roots about the x-coordinate of the vertex was invariant.

She then was presented with a coordinate system without any function graph. The reason was to let her refocus on the roots not only as numbers but also as points on a number line. This was important because complex numbers are also points in a plane and as the research has suggested one has to think of them not only as numbers but also as points in a plane (Fauconnier and Turner, 2004).Regarding the quadratic function examples we gave, once asked she pointed out the real number line to put the roots of those equations. As shown in the excerpt, Esra placed the x-coordinate of the vertex for the given examples of quadratic functions and the roots of their equations on the real number line.

R: Can you show the abscissa of the vertex?
E: [writing the point (-1,0) in Figure 6.23]
R: OK, fine.
E: x_1, -4 to 0 [(-4,0) in Figure 6.23].
This is x_2, 2 to 0 [(2,0) in Figure 6.23].

For the other equations, she did not use the exact values of the roots, but she marked the roots on the real number line for each quadratic equation as (x_1,0), (x_2,0), (x_1',0), (x_2',0), (x_1",0), (x_2",0), ($x_{1,2}$"',0), respectively in Figure 6.23.

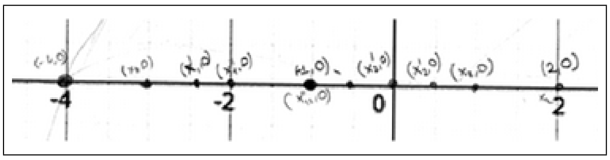

Figure 6.23. Esra marks the roots on the real number line for each given quadratic equation

She further explained that ($x_{1,2}$"',0) stood for the same two roots for the quadratic equation $x^2 + 2x + 1$, and the general algebraic form of the roots were the same for all quadratic equations, i.e. x_1= $-b/2a - \sqrt{\Delta}/2a$ and x_2= $-b/2a + \sqrt{\Delta}/2a$. She also argued that as the roots got closer to the x-coordinate of the vertex, the roots' distances to the x-coordinate of the vertex, $\sqrt{\Delta}/2a$, were decreasing. Once asked this resulted in her further investigation of the distances and the cases for the discriminant. This was important because we hypothesized that she would be able to relate three algebraic forms of Δ with the geometrical representation.

R: ..in which states of delta do these distances [$\sqrt{\Delta}/2a$] exist?

E: Delta is here [on the point (-1,0) for the quadratic function $x^2 + 2x + 1$ with its roots ($x_{1,2}$"',0)]

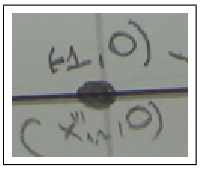

Figure 6.24. Esra points to (-1,0) corresponding the roots of the quadratic function $x^2 + 2x + 1$ she named ($x_{1,2}$"',0)

none, it[the distance $\sqrt{\Delta}/2a$] is 0. $\sqrt{\Delta}/2a$ none.

R: What does $\sqrt{\Delta}/2a$ non-existence mean, then?

E: It means the overlap of this point with the roots and with the abscissa of the vertex. Eee delta is 0.

R: What was it in the others, here [Figure 6.23] when there were two roots in here [on the x-axis]?

E: When $\sqrt{\Delta}/2a$ exists...Then the case in which it [delta] is bigger than 0.

It is important to state that earlier during the pre-interview once asked what $\Delta = 0$ meant geometrically, Esra had not been able to reason on that. She even had stated that "... but I did not get its shape correctly in my head...". However, as the data showed , at this point, she could relate geometric aspects to the algebraic aspects for the distance between the roots and the x-coordinate of the vertex. That is, for her the fact that the distances, $\sqrt{\Delta}/2a$, of the roots from the x-coordinate of the vertex existed meant that Δ was bigger than zero. Then, it decreased further and got to a point until there was no distance between the roots and the x-coordinate of the vertex This algebraically meant that $\sqrt{\Delta}/2a = 0$.

At that point the discussion became more interesting, Esra stated:

E: I can draw infinitely many parabolas having the same abscissa of the vertex, and because of that all the roots can be the numbers on the real number. I can generate parabola from all real numbers..

R: What do you mean by all real numbers? How many points do I need for a parabola?

E: Two points, and these two points have to be symmetric about the abscissa of the vertex...

R: Ok, how did you find the abscissa of that vertex if I gave you these two points [showing two random points in x axis]?

E: I can add and divide by two.

R: Yes, you find the midpoint?
E: Yes.
R: OK. Then
E: I can draw infinitely many parabolas.

Data indicated that Esra reasoned that once she had plotted the x-coordinate of vertex on the real number line, she would have thought any real number as one of the roots of any quadratic equation. This was because she could think of the roots being symmetric to each other. As the excerpt also showed Esra reversed her thinking in a way that she started from real numbers and stated that given two real numbers she could find the midpoint that would have indicated the x-coordinate of the vertex from which she would have drawn infinitely many parabolas.

Therefore, data suggested that Esra's thinking of the symmetry of the roots about the x-coordinate of the vertex allowed her to reason that, given that $\Delta = 0$ or $\Delta > 0$, not only the roots of any quadratic function could be placed as points on the real number line but also all the elements of the real number line could be potentially the roots of any quadratic equation.

Once again the relationship between the roots and the value of the discriminant was asked she answered:

E: If there is no distance between the roots and the abscissa of the vertex, if they overlap then it is $\Delta = 0$
R: Yes...
E: If there is distance, then it is $\Delta > 0$ and that is the reason of distance between them [the roots and the abscissa of the vertex]...

This suggested again that Esra was able to relate geometric aspects to the algebraic aspects for the distance between the roots and the x-coordinate of the vertex. That is, she knew that once there is no distance between the roots and the x-coordinate of the vertex, i.e. $\sqrt{\Delta}/2a = 0$, this meant that value of Δ is equal to 0. Also, once there is distance $\sqrt{\Delta}/2a$ acknowledging that changing the values of 'c' resulted in the fact that the distances $\sqrt{\Delta}/2a$ is bigger than zero which meant that $\Delta > 0$.

At this point, to further inquire the case that the value of discriminant was smaller than zero, the teacher-researcher continued the dialog in the following way:

> R: Now we have made [the distance between the roots and the abscissa of the vertex] 0. Ok, at this point, when you consider the real numbers, what is the status of real numbers?
> E: So all of them are finished now, from here [negative(-) and positive(+) infinity] if we come like this [dragging the pen to the abscissa of the vertex] now all the real numbers have finished here [on the abscissa of the the vertex in Figure 6.25].

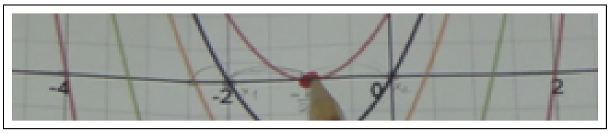

Figure 6.25. Esra points to the x-coordinate of the vertex on the real number line

Esra's moving the pencil from out in, in her own words, from positive and negative infinity to the x-coordinate of the vertex showed that she thought of the decrease of the distance from the roots to the x-coordinate of the vertex. This implied that for her once the distances got squeezed on the real number line, it got to a point that no more squeezing was possible where the value of the discriminant was zero.

At that point, she was asked whether she could write other quadratic functions having the same x-coordinate of its vertex. Esra was able to come up with examples of quadratic functions in a way that the x-coordinate of the vertex algebraically stayed invariant but the values of the discriminant continued decreasing as she calculated the values of $b^2 - 4ac$ for the functions such as $x^2 + 2x + 1$ and $x^2 + 2x + 5$. Graphically it meant that the parabolas of the quadratic functions which had their discriminant smaller than zero did not intersect the real number line in Figure 6.3.

She stated "I can write infinitely many quadratic function examples like this one[referring to $x^2 + 2x + 5$]...but we cannot place the roots of them on the real number line... Because it already finished here[on the abscissa of the vertex in Figure 6.25] and ee delta is negative, their roots cannot be real numbers anymore."

Data suggested that she knew that once there was no real distance between the roots and the x-coordinate of the vertex, she could not place the roots' distances on the real number line because the roots were not real numbers, i.e. unreal numbers. At that point, the discussion followed:

> R: From now on. OK. What happens to $\sqrt{\Delta}/2a$'s? $-b/2a$'s?
> E: ..continue to live.
> R: OK, they exist.
> E: Yes. This also [$\sqrt{\Delta}/2a$] exists but this number [$\sqrt{\Delta}$] is not a real number; so this [$\sqrt{\Delta}/2a$] is not a real number as well..
> R: OK, this [$\sqrt{\Delta}/2a$] exists. Is it on the real number line?
> E: It is not on the real number line.
> R: Why?
> E: Because inside of this [$\sqrt{\Delta}$] is negative...$\sqrt{\Delta}/2a$ distance is negative.
> R: You said these [$\sqrt{\Delta}/2a$] still exist?
> E: Yes.
> R: Do we say these [$\sqrt{\Delta}/2a$] are on the real number line?
> E: No.
> R: Why do you think they are not on the real number line?
> E: Because distance, this [$\sqrt{\Delta}/2a$] is a distance in here [on the real number line in Figure 6.25], distance is positive; so here [on the point $(-b/2a, 0)$ in Figure 6.25 corresponding the abscissa of the vertex] it [the distance] is 0. So it becomes one real root with multiplicity two, they overlap, the distance is 0.

Esra knew that that $-b/2a$ was kept the same and $\sqrt{\Delta}/2a$ seemed to represent again the distance but it was not real. Esra's thinking that $\sqrt{\Delta}/2a$ existed even when she was out of real numbers suggested that she thought of $\sqrt{\Delta}/2a$, as a quantity, referring to a distance. Yet, for her distances had to be positive and also that once the distance was zero then this would mean that all the numbers on the real number would be covered. Therefore, she argued in her own words that $\sqrt{\Delta}/2a$s "…continue to live" but such existence was not on the real number line. She even stated "This number [$\sqrt{\Delta}$] is not a real number; so $\sqrt{\Delta}/2a$ is not a real number as well..It [$\sqrt{\Delta}/2a$] is not on the real number line.." and her justification was again that the real number line had been covered, she could not put that part on the real number line anymore.

Acknowledging that Esra was not able to think geometrically anymore since for her $\sqrt{\Delta}/2a$ reached a point where its values started from being positive and ended at being zero covering all the real numbers on the real number line, we hypothesized that if Esra was able to think about $\sqrt{\Delta}/2a$ in terms of a positive factor algebraically, this could have trigger the geometrical meaning of $\sqrt{\Delta}/2a$ in a different plane, so we continued in the following way.

To build on Esra's argument about a distance being positive, the teacher-researcher then used an example to ask how to write this unreal number, $\sqrt{\Delta}/2a$, where $\Delta = b^2 - 4ac$ was smaller than zero, in a way that it had a positive factor:

R: Then if I use such an analogy; -8 for instance, how can you state it with 8?
E: It will be absolute value.

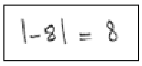

Figure 6.26. Esra mentions the absolute value of -8

R: OK, how else can you define -8 in terms of 8?
E: This is how I will

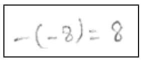

Figure 6.27. Esra mentions -8 multiplied by -1

R: OK, any other way?
E: 8 times -1.

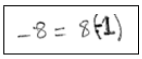

Figure 6.28. Esra mentions 8 multiplied by -1

R: If you want to state this same way [$b^2 - 4ac$] $b^2 - 4ac$.
E: I got it, if $b^2 - 4ac$ is negative; will I do with minus of this and times -1[Figure 6.29]?
R: Then could you place this expression into this [the general algebraic expression of the root $x_1 = -b/2a + \sqrt{\Delta}/2a$]?
E: Then [Figure 6.30]
I can write it separately[Figure 6.31]
R: You said you wrote it [$\sqrt{-1}.\sqrt{4ac - b^2}/2a$] separately.
E: Yes.
R: OK, can you take -1 out of the square root in real numbers?
E: Normally I cannot take it out.

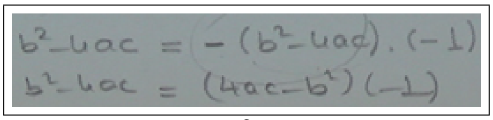

Figure 6.29. Esra writes $b^2 - 4ac$ as $(4ac - b^2).(-1)$

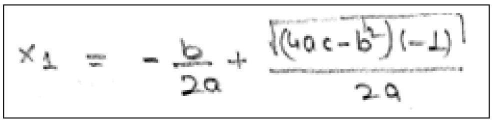

Figure 6.30. Esra rewrites the first root's algebraic expression

R: OK, are we in real numbers now?
E: We aren't.
R: OK, [you] assumed that you write it separately.
E: Yes.
R: You mean it is in this form [algebraic expression in Figure 6.31]?
E: Yes.
R: OK, where is the second root then?
E: Similarly [Figure 6.32].
R: OK, when you wrote them in this way, what did you say $4ac - b^2$ was?
E: Delta with a minus..
R: How do you express it then?
E: Negative delta, and x_2 is [writing $x_1 = -b/2a + \sqrt{-1}.\sqrt{-\Delta}/2a$ and $x_2 = -b/2a - \sqrt{-1}.\sqrt{-\Delta}/2a$].
R: OK, does this abscissa of the vertex $-b/2a$ belong to only one parabola?
E: No, it can be the abscissa of the vertex of many functions..
R: OK if you think $\sqrt{-\Delta}/2a$, what do you say?
E: It refers to the distance. It can change.

The excerpt showed that although Esra was not able to think to write Δ in terms of a positive number at first, once she was asked to think about writing -8 in terms of a positive number this allowed her to represent Δ in terms of $-\Delta$, $(\Delta = (-\Delta).(-1))$. Also she knew that she could not write $\sqrt{(-1).(4ac - b^2)}/2a$ as equal to $\sqrt{-1}.\sqrt{4ac - b^2}/2a$ in real numbers. Yet, she wanted to assume it like that. This was because she already knew that she had run out of all the real numbers and therefore the numbers were not on the real number line anymore. So far the data is important for two reasons: First, Esra's calling on her knowledge of representing -8 in terms

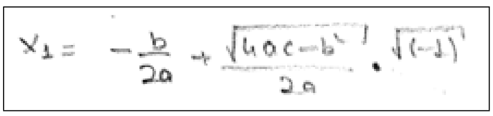

Figure 6.31. Esra rewrites the first root's algebraic expression separately

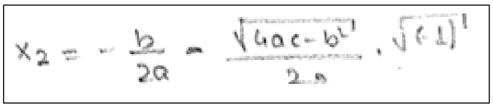

Figure 6.32. Esra writes the second root's algebraic expression separately

of 8 let her assimilate this situation into the new one (i.e. writing Δ in terms of $-\Delta$). Second, her calling on her knowledge of $\sqrt{-1}$ as being equal to 'i' let her write $\sqrt{(-1).(4ac-b^2)}/2a = \sqrt{-1}.\sqrt{4ac-b^2}/2a$. Therefore, once $\Delta < 0$, she could write as $\sqrt{\Delta}/2a = \sqrt{(-1).(4ac-b^2)}/2a = \sqrt{-1}.\sqrt{4ac-b^2}/2a$ because algebraic expression $\sqrt{4ac-b^2}/2a$ was a positive real number, and so the roots' algebraic expressions could be written as $x_{1,2} = -b/2a \pm \sqrt{-\Delta}/2a.\sqrt{-1}$. She then explained that $-b/2a$ might be the x-coordinate of the vertex of many quadratic functions, and $\sqrt{-\Delta}/2a$s stood for the distances which might be different for different quadratic functions.

At this point, it is important to state that from our point of view, rather than $\sqrt{-\Delta}/2a$, $\sqrt{-\Delta}/2a.\sqrt{-1}$ represents the distance. $\sqrt{-\Delta}/2a$ as being a positive number represents the value of the measure of such distance. This issue will be further elaborated later.

At that point, once asked if Esra could rewrite the roots' algebraic expression with new variables, she stated:

> R: If you are asked to write these [the algebraic expressions of the roots] with general expressions..
> E: OK.
> R: How can you write x_1 and x_2?
> E: For example, this[$-b/2a$]. Something else..
> R: Yes, variable.
> E: OK, let's say [$-b/2a$ is] t.
> R: OK, t.
> E: Let's say this [$\sqrt{-\Delta}/2a$] is m.[Figure 6.33]

She explained what t and m referred to "Let's say there are infinitely many [quadratic] functions, and the x-coordinate of the vertex of any quadratic function is t...and m is the distance from one of the roots to the x-coordinate of its vertex." What is interesting is that Esra was able to make sense of the values of 't' and 'm' not only

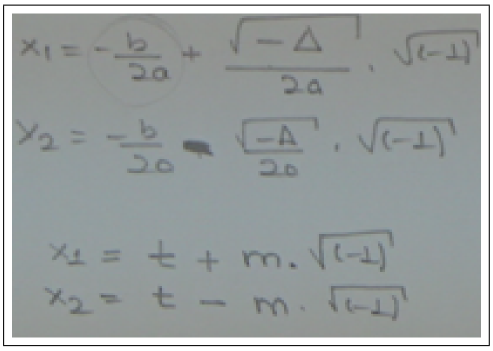

Figure 6.33. Esra rewrites the roots with the variables t and m as she choses

algebraically but also geometrically. She knew that ‘t’ stood for $-b/2a$ and ’m’ stood for $\sqrt{-\Delta}/2a$. She also was able to relate those values to the quadratic functions such that those values referred to the x-coordinate of the vertex of any quadratic function and the distances of the roots to the x-coordinate of its vertex. This made sense to her because as she had mentioned earlier, she thought that distances have to be either zero or positive. Though, it is important to state again that Esra’s geometrically making sense of ‘m’ was limited because not the value ‘m’ on its own but ‘m.$\sqrt{-1}$’ referred to the distance of the roots to the x-coordinate of the vertex. Though at this point in the teaching session, we did not attempt at investigating this idea further. In fact, her ideas on the fact that “m is the distance from one of the roots of any quadratic function with real coefficients to the abscissa of its vertex” was re-examined after her incorporation of the vector aspect of complex numbers, later on.

When she was asked what kind of numbers t and m were, she stated “..ee they are real..” and her explanation was:

> E: Because we have taken a and b as real numbers, $-b/2a$ becomes real [$\sqrt{-\Delta}$], and here this number inside [$-\Delta$] is a real number and it becomes real number outside the root. And when we divide it by 2a, which is real, it[m] is real number again.

Since Esra knew what ‘t’ and ‘m’ referred to algebraically, she was able to explain why ‘t’ and ‘m’ had to be real numbers: That is, she was able to reason that since ’t’ referred to $-b/2a$ for any quadratic equation with real coefficient ’a,b, and c’, it had to be a real number. By the same token, since the value of $-\Delta$ was positive and so 'm' had to be a real number.

At this point, Esra was asked to plot the roots, i.e. $x_1 = t + m.\sqrt{-1}$ and $x_2 = t - m.\sqrt{-1}$, on the real number line. She argued "I cannot show them[the roots] here [on the real axis] because I've already finished[all the points on the real axis]." She then was asked how to represent them geometrically. After she drew a real number line, she again argued she would not use the points on that real number line to represent the unreal roots. The dialogue got more interesting:

E: [draws any plane whose horizontal axis is the x axis on which there are real numbers] As I cannot use the x's in here [on the horizontal real axis]and the abscissa of the vertex is here [Figure 6.34], on the real x axis. I cannot show the roots here [on the real axis]... Ha, real numbers finished. Right, where are the roots? Umm.

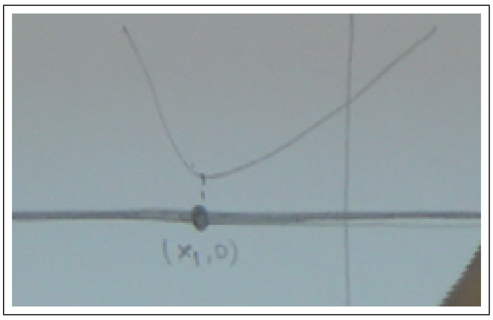

Figure 6.34. Esra draws a quadratic function whose roots are not real

R: What did those ['t' and 'm' in the algebraic expressions]e refer to?
E: t=$-b/2a$, the abscissa of the vertex.
R: OK where is that?
E: On the x axis, the real axis.
R: Where is the other part [of the root]?
E: m or m. $\sqrt{-1}$? Because if it is m, here [the distance of x_1 to $-b/2a$ on the real x-axis in figure 6.35] is it, right? Because as here [-Δ] is positive, divide it by 2a, so distance is this, right here. Then, here [the distance from the two roots to the abscissa of the the vertex in Figure 6.35] is m.

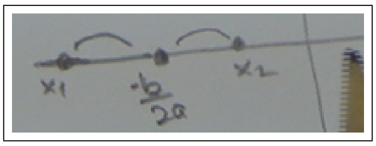

Figure 6.35. Esra first represents the distance of the complex roots to the x-coordinate of the vertex on the real number line

It is interesting that Esra first wanted to plot the unreal roots on the real axis although she had stated that she would not know where to place those roots since she was out of all the real numbers as the real roots. Though her thinking of the value of 'm' as being positive and that it represented the distance of the roots ot the x-coordinate of the vertex made sense to her to plot those unreal roots on the real axis. Only after when she reasoned about the distance on the real number line, she realized that it was the case of quadratic equations having real roots. However, as again she knew, the real roots on the real number line were covered once she was done with the real roots. She was working on the idea of where to plot those unreal roots. Then she again placed $(-b/2a,0)$ on the real number line as the x-coordinate of the vertex as in Figure 6.36 and continued reasoning.

E: Real numbers are only in here [on the real number line in Figure 6.36]. x_1 and x_2 [unreal roots] might be here[above and below the real number line].

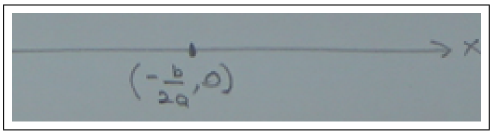

Figure 6.36. Esra draws the real number line and places the vertex

R: Where can x_1 and x_2 be?

E: So if x_1 is our bigger root, here x_2 is something like in here [Figure 6.37]. Could this distance [between the abscissa of the vertex and [x_1] be m.$\sqrt{-1}$? [The roots are] symmetric about the symmetry axis which passes through the abscissa of the vertex, they were symmetric.

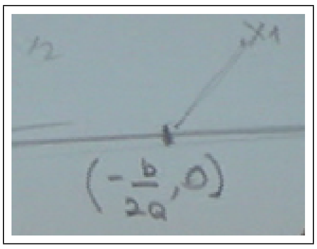

Figure 6.37. Esra draws two unreal roots above the real number line

R: And [the roots are] symmetric about the point $(-b/2a,0)$?

E: Yes. Because this [drawing a perpendicular line to the real line on the point (-b/2a,0)] is the symmetry axis and they[the roots] are symmetric about it.

R: Didn't you say they are symmetric about that [$(-b/2a,0)$] too?

E: Yes, correct, is it like this then [Figure 6.38]? Because in order for them[the roots] to be symmetric, if x_1 is here according to [$(-b/2a,0)$] then x_2 will be here [as shown in Figure 6.38].

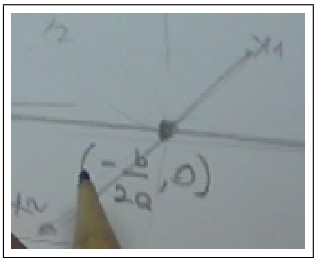

Figure 6.38. Esra draws one of the unreal roots above and the other below the real number line

As shown in the excerpt, it is interesting that only after her plotting $-b/2a$ on the real ax;s c together with her knowledge that she finished the real numbers as the roots caused her to think of using the plane above the real number line. Since she knew that the two roots still had to be in the same distance to the x-coordinate of the vertex she placed the roots as shown in the figure above. Also, it was interesting that only after she started to think about the placement of the roots geometrically she asked herself "..could this distance be m.$\sqrt{-1}$?" Still she changed her mind again. Although her reasoning had flaws in itself, it indicated that for her the two roots were dynamic in and the roots' and the vertex's existence and the roots' being symmetric to each other about the x-coordinate of the vertex stayed in variant. Then she stated:

> E: t is real, but where is the starting point, if I take it like this [draws a perpendicular line to the real axis on the starting point she chose]. t is real [drawing the distance of the point ($-b/2a$,0) to the point (0,0) a few times in Figure 6.39] x_1 is here [Figure 6.39]
>
> R: OK.
>
> E: Symmetry, this is t [drawing a big circle on the point ($-b/2a$,0) in Figure 6.39] So, therefore x_2 is here [Figure 6.39]. I did not put the first root here[on the real axis] because it [the root] was not real.

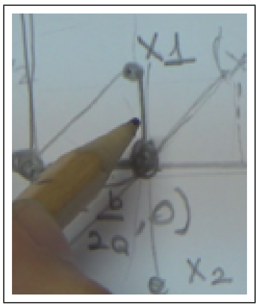

Figure 6.39. Esra places two unreal roots perpendicular to the real number line at the x-coordinate of the vertex

> R: Why?

E: Because aa the abscissa of the vertex is here [on the point $(-b/2a,0)$ in Figure 6.39]

R: Hhmm.

E: I took here [Figure 6.40] as the point 0. These..[x_1 and x_2] would be in equal distance to the abscissa; so I put x_2 here [Figure 6.39].

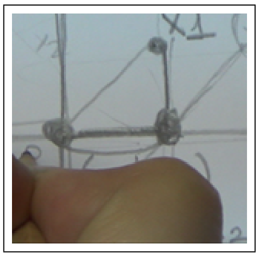

Figure 6.40. Esra places the origin on as the point (0,0)

R: OK. Why did you put x_1 and x_2 like this? Could you repeat?

E: I took the abscissa of the vertex here. I didn't put x_1 here [on the real axis in Figure 6.39] because it was not real. So I put it here [at a perpendicular distance to the abscissa of the vertex]. If I did not place the first root here [on a line perpendicular to the real axis at the point $(-b/2a,0)$ in Figure 6.39], then the abscissa of the vertex cannot be $-b/2a$ here, it[the abscissa of the vertex] would be something different. Then because the roots are symmetric about the abscissa of the vertex I put the second root x_2 like this[Figure 6.39].

R: Hhmm, OK. How can you write it [x_1]?

E: Like this [$x_1 = t + m.\sqrt{-1}$]

R: If you write point representation..We're talking about points now, right?

E: Yes, t is here [Figure 6.41], then for here[writing $m.\sqrt{-1}$]..

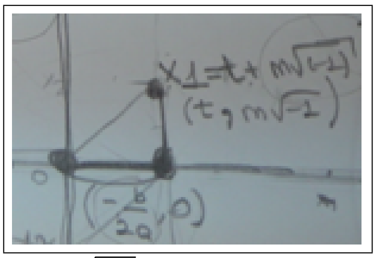

Figure 6.41. Esra places t and $m.\sqrt{-1}$ on the plane as pairs of numbers to represent the complex root

Ahh m, m [changing her representation as in Figure 6.42]

R: Why?

E: Eee when we express point-wise [t] is real, this [second part of the ordered pair] should be real. This[m in $m.\sqrt{-1}$] is real.

As the data indicated there was a change in her reasoning about the placement of the unreal roots. Since Esra's idea was to place the roots on the plane other than the real number line, Esra needed an origin to identify the points on the plane as on

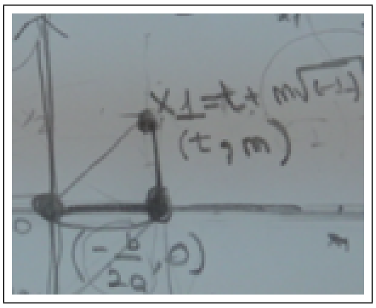

Figure 6.42. Esra places t and m on the plane as pairs of numbers to represent the complex root

a coordinate system. She then placed a perpendicular axis to the real axis creating an origin to identify the points on the plane as pairs of numbers. After thinking a reference point, the origin, she thought of the point($-b/2a$,0) as a distance from (0,0). This was evidenced in her over-crossing that distance with her pencil several times. Her thinking of ($-b/2a$,0) as a distance on the real number line with respect to the origin and of the roots' symmetry and their distances to the x-coordinate of the vertex being perpendicular distances allowed her to plot the first unreal root in the figure above. Her thinking of perpendicular distances and the symmetry was evidenced in her statement that "If I did not place the first root here[perpendicular to the real axis], then the abscissa of the vertex cannot be $-b/2a$ here, it[the abscissa of the vertex] would be something different." She then placed the second root below the real axis with a perpendicular distance to the x-coordinate of the vertex since the roots would be in equal distance to the x-coordinate of the vertex.

As shown, after she showed the first root as a point with the expression (t,m.$\sqrt{-1}$), she suddenly changed her mind and wrote (t,m) in order to represent the first root on the new coordinate plane she constructed. Once the researcher asked why she represented the roots as the ordered pair (t,m), she argued that while representing the points as ordered pairs the pairs should be real numbers as t was a real number she argued that m should be a real number too.

It is important to restate at this point that Esra was aware of the fact multiplying a positive real number by $\sqrt{-1}$ could have some meaning but she stated:

> R: The distance from the roots to the x-coordinate of the vertex?
> O: I think it's m[$\sqrt{-\Delta}/2a$]. I'm thinking it is m, now, I don't know what $\sqrt{-1}$ does to it[m] when it multiplies.

Data from the third teaching session revealed how she developed a meaning of the multiplication by $\sqrt{-1}$. At that point,the teacher-researcher asked Esta to validate one more time the meaning of (t,m) for her:

> R: OK you said (t,m) when you think point-wise, what does this point-wise expression [(t,m)] refer to? What does this (t,m) represent?
> E: x_1.
> R: What is x_1?
> E: $x_1 = t + m.\sqrt{-1}$.
> R: Hhmm. What was x_1?
> E: x_1 ee the root of the function, of the equation..
> R: What are x_1 and x_2?
> E: The roots. They are the roots of the parabola or function, quadratic function. Eee when they [the roots] are not real delta is smaller than 0. Eee when they [the roots] are real delta is bigger than 0.
> R: What do you call the algebraic expressions when delta is smaller than 0?
> E: Hhmm.. Complex numbers.
> R: Okay. Where do you get those complex numbers?
> E: From real numbers. All real numbers, on the real x axis..
> R: Okay. What were those all real numbers?
> E: The roots of eee roots of quadratic equations ee with real coefficients..
> R: OK, when you think in general how could you express complex numbers?
> E: Ee I obtain them from the real roots of quadratic equations. If they are eee...OK, correct, I obtain them from their real roots. OK, I obtain [complex numbers] from unreal ones [the unreal roots] as well.
> R: OK. Are these [the complex algebraic expressions of complex roots'] complex numbers?
> E: Yes they are complex numbers. The numbers obtained from the roots of all quadratic equations are complex numbers. Exactly. They give complex numbers.

After geometrically representing the complex algebraic expression on a new plane, in the dialogue, when Esra was asked what those algebraic expressions referred to, she called on the mathematical object of thought quadratic equations and their roots as quantities. That is, she knew that those algebraic expressions referred to both real and unreal roots of the quadratic equations She acknowledged them as quantities whose distances to the origin and to the x-coordinate of the vertex could be measured. Focusing further on the fact that the real numbers and the unreal ones were in fact the roots of quadratic equations with real coefficients allowed her to deduce the idea that all those roots constituted complex numbers. That is how she was able to define complex numbers as "...The numbers obtained from the roots of all quadratic equations..."

Once asked to also write down the definition of complex numbers again, she wrote the following definition:

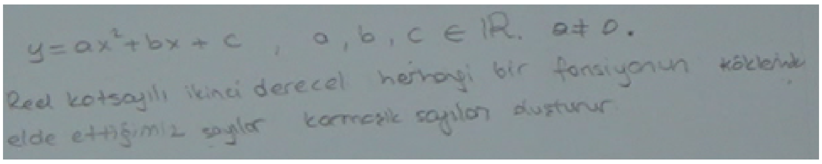

Figure 6.43. The definition of complex numbers in Esra's words as *"the numbers obtained from the roots of any quadratic equation with real coefficients"*

Then, she explained the relationship between the set of complex numbers and the set of real numbers:

> R: Alright. If I ask the relationship between complex numbers and real numbers?
> E: Complex numbers obtained from real numbers...The relationship?
> R: What kind of a relationship is there between complex and real numbers?
> E: Complex numbers includes real numbers.
> R: Why does it include?
> E: Eee because in this plane [Figure 6.44] if all of them are complex numbers, it also includes the x, the real one [the real x axis].
> R: Why?
> E: Because complex number is $x + iy$; the numbers in such a form, if y is 0, x.. y is zero, x is complex number, and x is a real number, so it [the set of complex numbers] includes [real numbers]... Eee [complex numbers are] the numbers obtained from the roots of quadratic equations; so because it consists of these [the roots], these roots can be real numbers as well or complex numbers as well... Exactly. Complex numbers consists of the roots of quadratic equations, so they include real numbers.

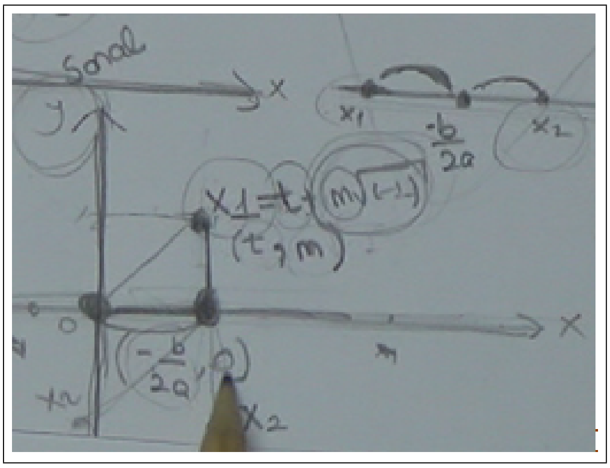

Figure 6.44. Esra constructs a new plane with a real and a perpendicular imaginary number line

Her statements indicated that her defining complex numbers through the roots of quadratic equations provided her with the reasoning about the relationship between the set of complex and real numbers in terms of the algebraic and geometric aspects. Since she thought geometrically the new plane, i.e. complex plane, included the real number line as the roots of quadratic equations and since the real numbers were algebraically embedded in the set of complex numbers depending on the values of the imaginary part of any complex number, i.e. $y = 0$ in her own words, she realized that complex number had to include all real numbers.

At that point, after Esra defined the set of complex numbers in relation to the quadratic equations she was asked to further explain the geometric and algebraic meanings of the parts of complex algebraic expressions, i.e. complex numbers. She stated again "t is the abscissa of the vertex[in t+m.$\sqrt{-1}$].. All, infinitely many [quadratic functions]with the same abscissa of the vertex...All, they[quadratic functions] are infinitely many, let me say any, t is the abscissa of the vertex of any quadratic function, exactly.." and "..m is the distance between the roots of the [quadratic] equation and the abscissa of its vertex, that[distance] was m, yes."

As the data indicated she stated that while t referred to the x-coordinate of the vertex, m referred to the roots' distances to it. We account for her meaning of the t and m in the algebraic form of $x = t + mi$, together with what she had stated earlier "..can this[the distance] be m.$\sqrt{-1}$?" in the following way. When one consider the positive real roots, the distance (as a physical length) from the roots to the x-coordinate of the vertex and the magnitude(the numerical estimation) of that distance correspond to the same numerical value which is $\sqrt{\Delta}/2a$. Having such thought Esra also tried to make sense of that distance and its magnitude in the complex plane in the same way. This is why she stated that she had to work with positive real numbers to refer to the magnitude of that distance. However, in the complex plane the distance (the physical length) from the roots to the x-coordinate of the vertex corresponds to $\sqrt{-\Delta}/2a.\sqrt{-1}$ while its magnitude corresponds to the numerical value of $\sqrt{-\Delta}/2a$ which is a positive real number. Therefore, at this point Esra was not able to distinguish the difference between the distance(the physical length) and its magnitude. One might argue that such distance does not exist in a physical sense. However, in the complex plane complex numbers are represented no only as points but also as vectors. Thus, such distances exist.

At that point, we did not ask her to think about the meaning of m and m.$\sqrt{-1}$ further. We continued with why the conjugate root exists once a complex root exists for a quadratic equation and how she reasoned geometrically on the new complex coordinate plane she constructed. She stated the following:

R: If there is one root in this form [complex root] what do we call the other[root]?

E: The other [root] exists, we call it the conjugate.

R: OK, can you explain why there exists a conjugate root? What is the reason in your opinion?

E: Because we obtained this [complex root] from a quadratic equation and because any quadratic equation has two roots, there should be two roots we obtained from there [quadratic equations].

R: OK. How do you explain this geometrically? How do you express it geometrically with regard to here [the complex plane in Figure 6.42]? Why is there a conjugate root as the second root?

E: If there is a the vertex of that parabola, and this, the abscissa of the vertex is the midpoint of the two roots, so if there is x_1 it should be x_2. They[the roots] are symmetric to each other about the abscissa of the vertex and this relation [symmetry of the roots] does not change whether it [the root] is a real number or not.

R: OK, do we know the reason that the roots are symmetric?

E: Symmetry axis overlapped with the vertex of parabola, x_2 is symmetric to x1.

As the data have shown, in the pre-interview Esra did not know why there should exist conjugate roots once there is a complex root. On the contrary, at this point in the teaching session, the data indicated that Esra knew that there should exist two roots, one of which was the conjugate of the other because algebraically she was working with the functions to the second degree. This was geometrically because the x-coordinate of the vertex was the midpoint of the roots and the roots were to be symmetric about the x-coordinate of the vertex. She even knew that the roots had to be symmetric about the x-coordinate of the vertex because the vertex was the intersection of the parabola with its line of symmetry. Also, the symmetry of the roots was invariant whether the roots were real or unreal.

She then was asked to represent the roots geometrically according to the three cases of discriminants' values, i.e. positive, negative, and zero, on the new plane she had just constructed. She drew and wrote down the following in Figure 6.45.

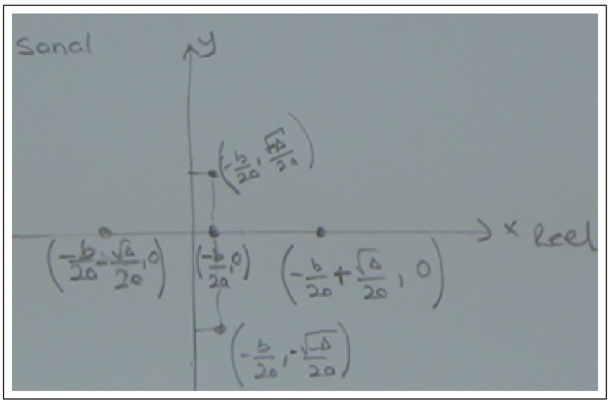

Figure 6.45. Esra places the roots of any quadratic function with real coefficients as pairs of real numbers on the complex plane

While drawing the figure above, on the newly constructed plane, she could identify all the roots of any quadratic function. She explained that $(-b/2a - \sqrt{\Delta}/2a,0)$ and $(-b/2a + \sqrt{\Delta}/2a,0)$ on the real number line corresponded to the case of discriminant's being positive, i.e. $\Delta > 0$; $(-b/2a,0)$ corresponded to the case of discriminant's being zero, i.e. $\Delta = 0$; $(-b/2a, \sqrt{-\Delta}/2a)$ and $(-b/2a, \sqrt{-\Delta}/2a)$ corresponded to the case of discriminant's being negative, i.e. $\Delta < 0$. This showed that she was able to represent the algebraic expressions of the real and unreal roots geometrically, i.e. as points on the complex plane. It is important to emphasize that she used the symbols x and y for naming the axes and she called them real and imaginary, i.e. 'sanal' in Turkish.

Once asked about the geometric meaning of the operations such as addition and subtraction, that were involved in the algebraic expressions of the roots of any quadratic equation she stated "It [addition] means that we're going this much[$\sqrt{-\Delta}/2a$] from the abscissa of the vertex [on the y axis upwards] or going this much [$\sqrt{\Delta}/2a$] [to the right on the x axis] or when we say this[subtraction] it means his much [$\sqrt{-\Delta}/2a$] [on the y axis downwards] and we're going backwards this much [$\sqrt{\Delta}/2a$] [to the left on the x axis]. These two roots [two unreal roots] are symmetric about the abscissa of the vertex, and on the x axis they [the real roots] are symmetric about the abscissa of the vertex as well."

This implied that addition and subtraction triggered the operation of reflection on her part, i.e. the symmetry of the roots about the x-coordinate of the vertex . That is, for her the symbols of addition and subtraction meant how much in perpendicular distance the roots were away from the x-coordinate of the vertex. She knew this because for her the two roots were symmetric to each other about the x-coordinate of the vertex and such symmetry attribute could be shown by adding and subtracting the same amount of distance from the value of the x-coordinate of the vertex.

6.2.3. Esra's Development of the Vectorial Aspect of Complex Numbers

In the previous teaching session, Esra had developed the definition of Complex numbers such that she knew what the Cartesian form represented in regards to quadratic equations and was able to represent the it as a point on the Complex plane. In the third teaching session, the researcher asked Esra the definition of vectors. The reason for asking her definition of vectors was to re-activate her previous knowledge on vectors. We hypothesized that as the data indicated from the previous teaching session Esra had constructed complex numbers from the roots of quadratic equations. By the same token, she was able to represent the algebraic expressions of the roots geometrically as points, i.e. as ordered pairs, and also as distances. Therefore, if she had recalled on her knowledge of vectors, she could have related vectors with complex numbers represented as points on the complex plane.

Esra was able to define vectors as "Vector is a line segment which has a starting point,a length, and is going in a direction." She argued that the vector she drew had two components, a and b, on the axes and she used the origin to identify where the vector began and placed its components on the axes in the figure below. She was also able to state the magnitude of the components as |a| and |b|, and calculate the magnitude of the vector based on the Pythagorean theorem as $\sqrt{a^2+b^2}$ since she argued the angle between the vector a and b was 90 degrees.

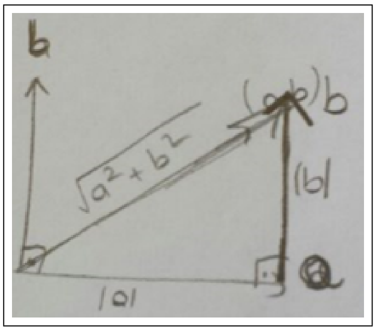

Figure 6.46. Esra's representition of a vector and its magnitude

At that point, the teacher-researcher continued the teaching session asking her to define complex numbers in her own words. First she was able to define complex numbers as "the elements in the set of complex numbers are the roots of quadratic equations with real coefficients" and then went on deducing the roots algebraically. She said that once the discriminant was smaller than zero the roots were complex numbers, and wrote the roots as $x_1= -b/2a + \sqrt{-\Delta}/2a.i$ and $x_1= -b/2a - \sqrt{-\Delta}/2a.i$ where $i = \sqrt{-1}$. The researcher then asked what $\sqrt{-\Delta}/2a.\sqrt{-1}$ was. This was because as the data from the earlier session suggested in the complex plane she thought of $\sqrt{-\Delta}/2a$ as the distance between the roots and the x-coordinate of the vertex rather than its magnitude. Thus, the researcher asked her what $\sqrt{-\Delta}/2a.\sqrt{-1}$ referred to geometrically:

R: OK. I will ask two questions. If we think of the algebraic expression [pointing to Figure 6.47] ee you said this ee $\sqrt{-\Delta}/2a$ multiplied by $\sqrt{-1}$.

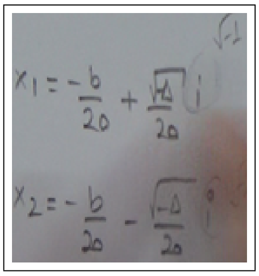

Figure 6.47. Algebraic expressions of the unreal roots

E: Hhmm.
R: What does it [$\sqrt{-\Delta}/2a.\sqrt{-1}$] mean? And what does it [$\sqrt{-\Delta}/2a$] mean?
E: This[$\sqrt{-\Delta}/2a$] and that[$\sqrt{-\Delta}/2a.\sqrt{-1}$]?
R: Hhmm, exactly.
E: We [$\sqrt{-\Delta}/2a$] have already said that this is the length, I mean the distance, distance between a root and the abscissa of the vertex. This statement [$\sqrt{-\Delta}/2a.\sqrt{-1}$] is not real. It means..This the root [x$_1$= $-b/2a + i.\sqrt{-\Delta}/2a$] there are two parts of this vector [in Figure 6.48]. One of which is this [the horizontal part], and the other is that[the vertical part] which is this part[$\sqrt{-\Delta}/2a.\sqrt{-1}$] of that number [in Figure 6.48] Yes, one part of this root, of this number [complex number] is a complex number.

Figure 6.48. Esra's vector representation of the unreal root

R: Hmm.
E: This part [in Figure 6.48]. Complex numbers, when this number [complex number] as a whole is expressed like this [referring to the arrow in the middle in Figure 6.48]. This [horizontal component she drew] is its real part [$-b/2a$ in Figure 6.48]. That [vertical component she drew] is its unreal part[$\sqrt{-\Delta}/2a.\sqrt{-1}$ in Figure 6.48].

It is interesting that although Esra still had the same confusion on the distance and its magnitude, her refocus on the meaning of m.$\sqrt{-1}$ triggered the idea that complex numbers could be represented as vectors. This was because for her $\sqrt{-\Delta}/2a$ was already a real number representing the distance and $-b/2a$ was already a real number too. But her thinking of the $[\sqrt{-\Delta}/2a.\sqrt{-1}]$ as the imaginary part of the complex number triggered the idea that the complex number had two parts. This was evidenced in her stating that "Complex numbers, when all this number[the root] is represented like this[in the figure 6.85.85] this$[-b/2a]$ is its real part[horizontal component she drew above]...and that$[\sqrt{-\Delta}/2a]$ is unreal part[vertical component she drew]." To further validate whether her representing complex numbers by arrows was only based on superficial aspects such as vectors' visual representation, the teacher researcher asked her how she could represent the complex numbers as she drew, she explained:

> E: So this $[-b/2a]$ part is on the real axis. The other one is on the imaginary, the unreal [axis]... We have divided it as real and unreal so it was like this [in Figure 6.49]...

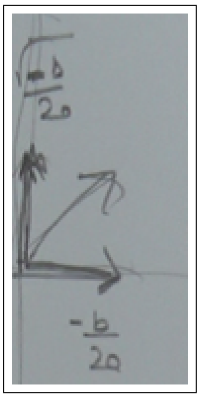

Figure 6.49. Esra represents the horizontal component as $-b/2a$ and the vertical component as $\sqrt{-\Delta}/2a$

> R: Could you explain more?
> E: Here when we state [the complex roots] as point-wise, we state and separate as $-b/2a$ and $\sqrt{-\Delta}/2a$..This[the complex root] is a length, a line segment. When we separate here, this [horizontal component she showed as $-b/2a$] is the length, magnitude of one line segment and this [vertical component she showed as $\sqrt{-\Delta}/2a$] is the other one's [length, magnitude in Figure 6.49].
> R: It seems like you thought of something?
> E: This. I thought that we can represent complex numbers as vectors with the starting point (0,0) and with a length going to a point [pointing the arrow in the middle in Figure 6.49].

As the data indicated she was able to represent the algebraic expressions of the roots geometrically as points, i.e. as ordered pairs. For her the ordered pairs represented the two parts of the root such that the roots' real and unreal parts had magnitudes, an origin, and some direction (evidenced in her drawing arrows and her stating “going somewhere”). All these aspects of the Cartesian form of complex numbers triggered on her part to recall on her knowledge of vectors. Therefore, she was able to represent complex numbers as vectors.

At that point she seemed thinking and the researcher asked:

R: Hmm, could you explain again?
E: Eee normally we would represent the number -5 on the real number line starting from 0 with the magnitude of 5.
R: How?
E: I take the starting point as 0 and when I take it directed to -x and when I take its length [from 0] to that point [-5 in Figure 6.50]. The point 5, I can represent the number -5 as a vector.

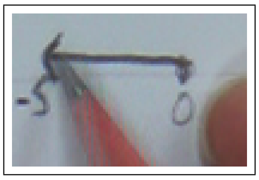

Figure 6.50. Esra’s representation of the number -5 geometrically

R: Yes.
E: In the same way, when I consider the complex number as such if I take a starting point, in the same way, I can represent this complex number as a vector. It has a specific length, a magnitude [Figure 6.51] and it has a specific starting point, a certain direction, I can represent it [a complex number] like that[pointing to the arrow in the middle she drew].

Figure 6.51. Esra shows the magnitude of a complex number

It is interesting that Esra called on her knowledge of real numbers as vectors. She thought that real numbers represented as points on the real number line could be thought as vectors because they had an origin, direction and magnitude. Similar to real numbers once she thought complex numbers as points on the complex plane locating their starting point at the origin and thinking of its magnitude allowed her to think that she could represent complex numbers as vectors too. This was important because as the data indicated Esra seemed to justify her reasoning calling on her knowledge of real numbers as vectors.

Additionally, once asked what made her think of real numbers as vectors she claimed that she thought of real numbers as roots and real numbers as roots would already act as vectors with respect to the origin. Interestingly, she further elaborated:

> E: Eee, I will draw [a parabola], this $-b/2a$ is here. $\sqrt{\Delta}/2a$ is right here. In this figure [Figure 6.52]. It makes [a vector] like this [drawing the vector highlighted by blue pen in Figure 6.52], right? We already took the point 0 as starting point...

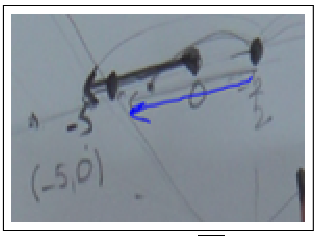

Figure 6.52. Esra's representing $-b/2a$ and $\sqrt{\Delta}/2a$ on the real number line as vectors

First, she thought of a parabola. Secondly, she pointed to the x-coordinate of the vertex on the real number line she drew. Third, she pointed to the distance between the roots and the x-coordinate of the vertex. Then she could represent such distance as a vector starting from the x-coordinate of the vertex, which was represented as ($-b/2a$,0), and ending at the points where the roots were on the real number line. This was important because as the earlier data indicated she not only was able to represent the Cartesian form of a complex number as a vector with a starting point at the origin but also was able to represent the component of it -the roots' distances to the x-coordinate of the vertex- as a vector. At that point we hypothesized that such acknowledgment together with the fact that such distance was dynamic in nature might have allowed her to think that complex numbers as vectors were dynamic in nature too. This will be further discussed later in the text.

When she was asked what further characteristics complex numbers had she argued that there was an imaginary unit in the complex numbers:

E: It has an imaginary unit. So if the factor[m] is 0, that [imaginary part in $x_1 = t + m.\sqrt{-1}$ and $x_2 = t - m.\sqrt{-1}$ does not come. It has an imaginary unit. And all the numbers consist of this part [t] and this part [$m.\sqrt{-1}$] in that way...
R: What do you mean by a unit? What does the imaginary unit refer to ?
E: Ya I mean this $m.\sqrt{-1}$. When I say unit I mean something constant..unit means..this $m.\sqrt{-1}$ constant number. I don't know anything else when I think of unit...

As data showed she claimed that $\sqrt{-1}$ was invariant, but she did not know what it meant as a unit. Therefore the discussion followed:

R: OK. Let's investigate $\sqrt{-1}$. First, where is $\sqrt{-1}$ on the plane? Where do we put it? What does it refer to point-wise?
E: Eee as we mentioned i was here [on the perpendicular imaginary axis]. Yes i was here.
R: OK, could you represent it point-wise?
E: Eee (0, i) on the imaginary axis.[Figure 6.53]

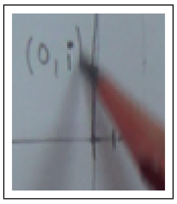

Figure 6.53. Esra's representation of i geometrically

Although her use of $(0, i)$ as the representation of $\sqrt{-1}$ might not necessarily mean that she did not really know that (0,1) was the point-wise representation of $\sqrt{-1}$ on the complex plane, her stating that she did not know what $\sqrt{-1}$ meant as a unit together with her representation of it as $(0, i)$ might be taken as that she had a limited knowledge on the imaginary unit and its representation. Having known this we hypothesized that if she could think of the algebraic representation of $\sqrt{-1}$ in terms of the roots of quadratic equation this might have allowed her to represent $\sqrt{-1}$ as the point $(0, i)$. That is, we hypothesized that she might not have been thinking $\sqrt{-1}$ as a root of a quadratic equation algebraically and geometrically at that moment.

R: Now, I'll give you two roots, +i, -i, $+\sqrt{-1}$,$-\sqrt{-1}$ [writes $x_{1,2}= \pm\sqrt{-1}$]. In these expressions [the algebraic expression of the complex root as $x_1= t+m.\sqrt{-1}$ and $x_2= t-m.\sqrt{-1}$] could you explain again what t was?

E: $-b/2a$ and the other[m. $\sqrt{-1}$], $\sqrt{-\Delta}/2a$ multiplied by $\sqrt{-1}$.

R: Considering the root [$\sqrt{-1}$] and their [the algebraic expressions in the complex root form] meanings her e[$x_1= -b/2a + \sqrt{-1}.\sqrt{-\Delta}/2a$] can you find what $\sqrt{-1}$ refers to point-wise on the plane here [pointing to the complex plane]?

E: I didn't get the question.

R: Here (0, i) what was i?

E: $\sqrt{-1}$.

R: Now given that you wrote (0, $\sqrt{-1}$) on the complex plane. Can you now explain what this [$x_1=\sqrt{-1}$] is when you think algebraically? [pointing to $x_1= -b/2a+\sqrt{-1}.\sqrt{-\Delta}/2a$] Can you represent it point-wise? I mean what does $\sqrt{-1}$ refer to as a point? Can you explain or find?

E: On the graph or here [pointing to the algebraic expression]?

R: First in the algebraic expression then on the plane.

E: OK, now $-b/2a$ is 0.

R: Why?

E: Because here [$x_1= \sqrt{-1}$] except an imaginary unit there is plus 0. So the only part missing is the real part, so it is 0.

R: Alright.

E: + ...ummm..This [$\sqrt{-\Delta}/2a$] is 1 [Figure 6.54]. Yes. 1 multiplied by this [$\sqrt{-1}$].

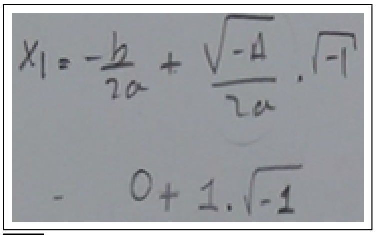

Figure 6.54. Esra writes $\sqrt{-1}$ considering the unreal roots' algebraic expressions

R: What is this[x_1=0+1.$\sqrt{-1}$]?

E: x_1.

R: Now, normally how did you represent x_1 on the plane when we asked? Accordingly can you represent that x_1 [x_1=0+1.$\sqrt{-1}$] again? Can you show it?

E:[drawing in Figure 6.55] Because $-b/2a$ is 0 [drawing in Figure 6.56]

R: And what is $\sqrt{-\Delta}/2a$? What does $\sqrt{-\Delta}/2a$ refer to numerically?

E: 1. Here [the length from the origin to the point (0, $\sqrt{-\Delta}/2a$)], this length is 1 right?

R: Can you represent it numerically [referring to the point(0,$\sqrt{-\Delta}/2a$)]?

E: Ah (0,1) [Figure 6.57]. Eee 1 unit, it is in the distance of 1 unit. Its magnitude is already 1 [writing (0,1) in Figure 6.57].

First she was asked to think of $\sqrt{-1}$ as a root of a quadratic equation and then was asked how she could write $\sqrt{-1}$ considering the algebraic expression of the quadratic equation. At that point she reasoned about the meanings of the real and unreal parts

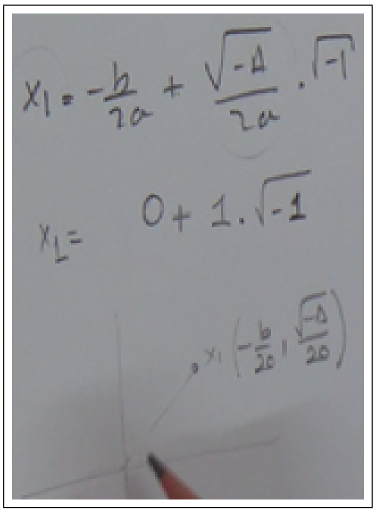

Figure 6.55. Esra's representation of the root $\sqrt{-1}$ geometrically

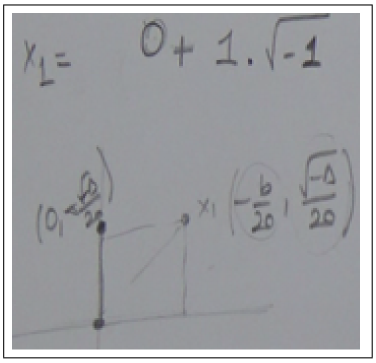

Figure 6.56. Esra's representation of the distances to the root $\sqrt{-1}$ on the imaginary number line

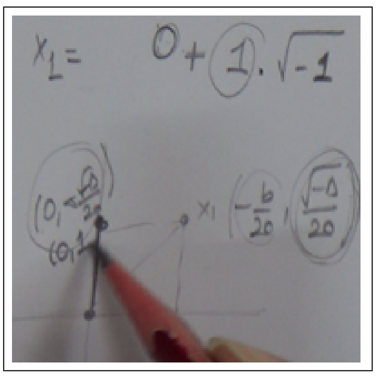

Figure 6.57. Esra's representation of i geometrically as the point (0,1) on the imaginary number line

of the root algebraically, i.e. $-b/2a$ and $\sqrt{-\Delta}/2a.\sqrt{-1}$. Then she thought of what $-b/2a$ and $\sqrt{-\Delta}/2a$ would correspond to numerically once she thought about $\sqrt{-1}$. Therefore, she came up with the values of 0 for $-b/2a$ and 1 for $\sqrt{-\Delta}/2a$. Once she was asked to represent $\sqrt{-1}$ on the plane she interestingly plotted the general expression of the roots as ordered pairs $(-b/2a,\ \sqrt{-\Delta}/2a)$. However, corresponding the value of 0 with $-b/2a$ allowed her to reason geometrically that the point should have to be placed on the imaginary axis. Thinking this she wrote down $(0,\ \sqrt{-\Delta}/2a)$ on the imaginary number line. Then focusing on the numerical value for $\sqrt{-\Delta}/2a$ in terms of $0{+}1.\sqrt{-1}$ allowed her to measure that distance as 1, which then allowed her to represent $\sqrt{-1}$ as the ordered pair (0,1). In this regard, Esra's reasoning about the algebraic and the geometric aspects of the roots of quadratic equations simultaneously to represent $\sqrt{-1}$ as a point enabled her to represent it as the ordered pair (0,1).

Since she represented the point (0,1) and measured the distance of the roots to the x-coordinate of the vertex as 1 as its magnitude, she argued that it could be represented as a vector having the following characteristics:

E: It had also a starting point and direction.
R: What can you say about the direction of $\sqrt{-1}$?
E: Its direction, like this [Figure 6.58]. Exactly like this, upwards, its starting point is 0, and it goes to the point (0,1). Its leng..its magnitude is 1.

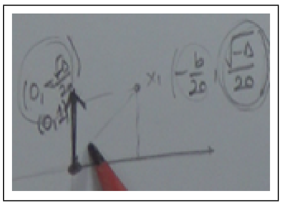

Figure 6.58. Esra draws an arrow on the imaginary axis to represent the root i

R: OK. What can you say if you think of its direction in terms of angles?
E: 90.
R: How did you find it?
E: Here, if we start from here [the real x axis]. Here is 90 degrees [Figure 6.59].

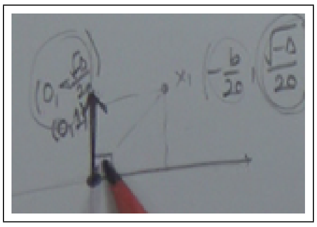

Figure 6.59. Esra represents 90 degrees between the root i and the real number line

Thinking of the ordered pair (0,1) to represent $\sqrt{-1}$ triggered the idea that (0,1) had the magnitude of 1, a starting point, and a direction, i.e. the vectorial aspect. This was consistent with her earlier reasoning once she had first thought that the ordered pairs represented the two parts of the roots. Once she was asked to explain what such direction meant in terms of an angle, she argued that it was 90 degrees counter clockwise from the real axis. The researcher did not further probe her reasoning. The data in the upcoming paragraphs will elaborate on her reasoning.

At that point building upon current knowledge that $\sqrt{-1}$ could be represented as an ordered pair and as a vector, the researcher probed her to rethink about the meaning of $\sqrt{-\Delta}/2a$ and $\sqrt{-\Delta}/2a.\sqrt{-1}$. This was also because earlier data from the second teaching session had shown that Esra could not distinguish the difference between the roots' distances to the x-coordinate of the vertex and its magnitude.

R: OK. Let me ask this again, $\sqrt{-\Delta}/2a$ and $\sqrt{-\Delta}/2a.\sqrt{-1}$. What does this whole [$\sqrt{-\Delta}/2a.\sqrt{-1}$] mean to you, this $\sqrt{-\Delta}/2a.\sqrt{-1}$?
E: It integrates direction to this, right?
R: It integrates direction to what?
E: Now we could separate this [x_1 considering it as a vector] like this [Figure 6.60]. Eee $-b/2a$ is here [on the horizontal axis] and $\sqrt{-\Delta}/2a$, this[$\sqrt{-\Delta}/2a$] is a magnitude for me a length, when we multiply this [$\sqrt{-\Delta}/2a$] by this [$\sqrt{-1}$] does it integrate direction so that I can draw this like this here [vertical component in Figure 6.60]? Eee is it because of $\sqrt{-1}$ I could draw this [the vertical part] here [on the plane]?
R: How does it [multiplying by $\sqrt{-1}$] integrate that direction?
E: What should I say..We know that $\sqrt{-1}$ was here [on the perpendicular imaginary axis in Figure 6.60]. What should I say, $\sqrt{-1}$ this, or i, this is 1 unit [writing the point (0,1) in Figure 6.60]. Does this multiplying [by $\sqrt{-1}$] make this [a number on the real number line] on the y-axis [imaginary axis]?

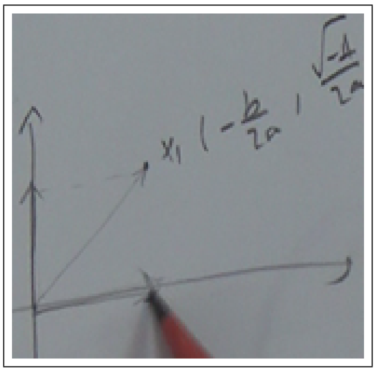

Figure 6.60. Esra's vector representation of a complex root

Her first figure suggested that she thought of both the real part and the unreal parts as vectors with a starting point at origin. Since she already knew that $\sqrt{-\Delta}/2a.\sqrt{-1}$ was represented on the imaginary number line perdicular to the real number line and that $\sqrt{-\Delta}/2a$ was positive real number, she started questioning whether multiplying the real number $\sqrt{-\Delta}/2a$ with $\sqrt{-1}$ meant a change in the direction of $\sqrt{-\Delta}/2a$ and in her own words "..put it on the imaginary axis..", i.e. 90 degrees rotation counter clockwise.

When she was asked to further explain where $\sqrt{-\Delta}/2a$ was, she stated:

R: Where is $\sqrt{-\Delta}/2a$?
E: This [$\sqrt{-\Delta}/2a$] is real and it should be on this [the real x axis], but when we multiply it by $\sqrt{-1}$, it relocates in here [on the imaginary axis in Figure 6.61].

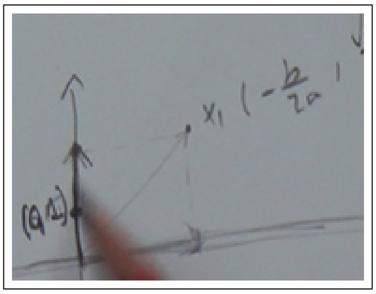

Figure 6.61. Esra rotates her pencil onto the imaginary axis

She argued that $\sqrt{-\Delta}/2a$ was a real number and existed as a vector on the real number line, which was evidenced in her over-crossing the arrow in the figure above. Knowing that $\sqrt{-1}$ existed on the imaginary number line, multiplying $\sqrt{-\Delta}/2a$ with

$\sqrt{-1}$ "..relocated $\sqrt{-\Delta}/2a$ on the imaginary number line" in her own words. When asked what relocation meant for her, she argued:

> R: Hhmm, what do you mean by relocation?
> E: Direction, it changes its direction. For instance its direction was here [on the real number line in Figure 6.62]. Because I took a starting point, and because it changed direction, exactly, its magnitude is same but its direction has changed.

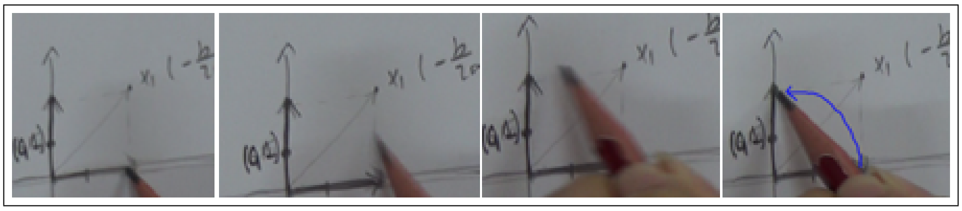

Figure 6.62. Esra's representation of the multiplication of a real number by i as rotation

> R: How does it change its direction?
> E: 90 degrees. It relocates 90 degrees counter clockwise.

Data showed that relocation meant a change in direction that was 90 degrees rotation around the origin counter clockwise. Moreover, she claimed that multiplying by $\sqrt{-1}$ made no change in the magnitude of $\sqrt{-\Delta}/2a.\sqrt{-1}$. Then she was asked one more time to explain how she reasoned:

> E: Ee $\sqrt{-\Delta}/2a$ is a real number. Because this real number [$\sqrt{-\Delta}/2a$] ee is a real number it should be on this axis [the real x axis] because all real numbers are on this axis [the real x axis]. But when I multiply this[$\sqrt{-\Delta}/2a$] eee by this[$\sqrt{-1}$] the imaginary unit, this [the vector whose magnitude is $\sqrt{-\Delta}/2a$] eee rotates 90 degrees counter clockwise and relocates on the y-axis [imaginary axis], the y axis which we put. In my words it changes its direction.

It is important to reemphasize that in the second teaching session she had stated that she did not know what multiplying $\sqrt{-\Delta}/2a$ by $\sqrt{-1}$ meant. She thought of $\sqrt{-\Delta}/2a$ as a real number existed on the real number line. She also knew that $\sqrt{-1}$ could be represented as a vector with direction of 90 degrees rotation counter clockwise from the x-axis. Therefore, multiplying $\sqrt{-\Delta}/2a$ by $\sqrt{-1}$ meant a 90 degrees rotation counter clockwise and relocated it on the imaginary number line.

In addition to what relocation meant for her, the researcher asked why she thought that the magnitude did not change when multiplied $\sqrt{-\Delta}/2a$] with $\sqrt{-1}$ she stated that "Because this[$\sqrt{-1}$] is 1 unit." Since she knew that the magnitude of $\sqrt{-1}$ was 1, multiplying $\sqrt{-\Delta}/2a$ by $\sqrt{-1}$ did not change its magnitude for her.

Knowing that in the second teaching session Esra could not distinguish the difference between $\sqrt{-\Delta}/2a$ and $\sqrt{-\Delta}/2a.\sqrt{-1}$ and considering Esra's current stage of knowing, the researcher asked her to re-explain the meaning of $\sqrt{-\Delta}/2a.\sqrt{-1}$:

R: Then what does this [$\sqrt{-\Delta}/2a.\sqrt{-1}$] mean to you?
E: For a parabola what does it mean?
R: Hhmm, considering the roots of a parabola..
E: It means the distance of root to the abscissa of the vertex, exactly. I am not so sure, this is root, I took this [point x_1] as root. $\sqrt{-\Delta}/2a.\sqrt{-1}$ here it is [Figure 6.63].

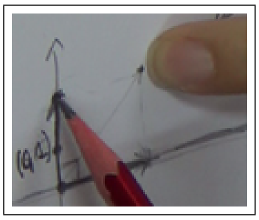

Figure 6.63. Esra's representation of the distance between the roots and the x-coordinate of the vertex on the imaginary number line as an imaginary distance

The data indicated that she knew that the distance between the roots and the x-coordinate of the vertex was $\sqrt{-\Delta}/2a.\sqrt{-1}$. Still, she was not sure about it.

At that point, going back to the second teaching session the researcher confronted Esra with her statement that $\sqrt{-\Delta}/2a$ was the distance between the roots and the x-coordinate of the vertex. She then stated:

E: Because this is [$\sqrt{-\Delta}/2a$] is a magnitude, it has to be a positive number ee any magnitude is expressed with a positive number. OK, this [$\sqrt{-1}$] may have the length of 1 unit. Let's assume that this [$\sqrt{-1}$] is not this, let's take it as -5, ee I take its magnitude as 5, and take its absolute value. So if this [$\sqrt{-1}$] has already its length as 1 unit I represent it as $\sqrt{-\Delta}/2a$, this [$\sqrt{-\Delta}/2a.\sqrt{-1}$] does not represent a magnitude.

Her argument showed that she was able to distinguish the difference between $\sqrt{-\Delta}/2a$ and $\sqrt{-\Delta}/2a.\sqrt{-1}$ such that $\sqrt{-\Delta}/2a.\sqrt{-1}$ corresponded to the distance between the roots and the x-coordinate of the vertex and $\sqrt{-\Delta}/2a$ corresponded the magnitude of that distance. This was because she knew that the magnitude of $\sqrt{-1}$ was 1. She even recalled on her knowledge of negative integers whose magnitudes differed from their numerical representation. As she had stated when the negative integer -5 was measured, its magnitude, i.e. the absolute value of -5, would be a positive real number, 5. Similarly, when $\sqrt{-\Delta}/2a.\sqrt{-1}$ was measured its magnitude would be the positive real number $\sqrt{-\Delta}/2a$.

6.3. Analysis of Post-interview

The purpose of post-interview was to examine Esra's developed knowledge regarding the definition of complex numbers and quadratic functions; quadratic equations and their sets of solutions along with their algebraic and geometric aspects, and their relations to complex numbers. The interview was conducted one week after upon the completion of the three teaching experiment sessions. While conducting the interview, I also asked Esra to articulate how she reasoned during the written post-assessment session that she completed right after the third teaching session. The focus of analysis was to characterize Esra's reasoning on and current meanings of the quantities of quadratic functions such as the roots, the line of symmetry and the vertex of the parabola in terms of the algebraic and geometric aspects, and the definition of complex numbers.

In the next paragraphs I presented some evidence of Esra's knowledge at the end of the study to describe what meanings and understandings she developed throughout the teaching sessions. To further explicate the extent of Esra's reasoning, I also compared and contrasted how she reasoned during the pre-interview and the post-interview sessions.

While conducting the post-interview, I asked Esra to articulate how she reasoned during the written post-assessment. First, I described Esra's current meanings of the vertex, and the real and imaginary parts in the Cartesian form of complex numbers, her making sense of the geometric representation of the Cartesian form as vectors, the imaginary unit 'i' in the Cartesian form of complex numbers, her knowledge on the definition of complex numbers, and her making sense of conjugate roots of any quadratic equation.

6.3.1. Esra's Current Meanings of the Vertex of Quadratic Functions' Graphs

As the data have shown earlier, compared to the beginning of the study where Esra had a limited knowledge on the definition of the vertex of a parabola, during the post-interview, she stated "The point where the quadratic function takes it lowest or highest value and the point where the parabola intersects its symmetry axis."

This was important because by this definition Esra was able to explain why $-b/2a$ was the x-coordinate of the vertex. As the data had indicated, at the beginning of the study Esra had not known why $-b/2a$ was the x-coordinate of the vertex. Esra stated:

> E: Because the symmetry axis, I know that the roots are in equal distance to this[the abscissa of the vertex], [the roots] are symmetric about this point[the abscissa of the vertex], and when I add these[roots] it gives this point, $-b/2a$.

As the data indicated, since she knew that the roots have to be in equal length to line of symmetry, once she calculated the midpoint of the roots, it was equal to $-b/2a$. That is, the line of symmetry also went through the midpoint of the roots since their perpendicular distance to the line of symmetry was equal in length. Together with her knowledge that the line of symmetry went through the vertex also, therefore, $-b/2a$ was to be the x-coordinate of the vertex.

6.3.2. Esra's Current Meanings of the Cartesian Form of Complex Numbers

At the beginning of the study, she was not able to argue about what x and y in the form of $z = x + yi$ refer to algebraically in relation to the roots of quadratic equations, and geometrically in relation to the graph of a quadratic function. She did not know why those numbers had to be real numbers either. However, in contrast to the beginning, she was able answer these questions:

> E: When I wrote it like $z_{1,2}= x \pm yi$ I found that x was $-b/2a$ here, the abscissa of the vertices, $\sqrt{-\Delta}/2a$ was the distance of one of the roots to this abscissa[the abscissa of the vertex, so x and y means this here...Geometrically it means the distance, the distance from one root of the equation to the abscissa of the vertex of the parabola, it means that distance... Now, as we stated earlier x_1 and x_2[the roots] and their midpoint, if we find it again it gave $-b/2a$ as we just did earlier. Ee the midpoint [of the roots] means that if I go this much $\sqrt{\Delta}/2a$ to the left of $-b/2a$ it gives a root, to the right it gives the other root in the same distance [to the $-b/2a$], so the midpoint of them[the roots] geometrically is $-b/2a$.

Data showed that she knew that the real part, x, referred algebraically to $-b/2a$ and geometrically to the x-coordinate of the vertex which was the midpoint of the two roots. Also, for her y algebraically referred to $\sqrt{-\Delta}/2a$ and geometrically referred to the roots' distances to the x-coordinate of the vertex. Also, as the data indicated, for Esra, $\sqrt{\Delta}/2a$ was a quantity. That is, she knew that $-b/2a$, the x-coordinate of the vertex, referred to the midpoint of the two roots such that once the distance measured by $\sqrt{\Delta}/2a$ was taken from such point in both directions then the two roots would be reached. In this regard, for her, the distance between the roots and the x-coordinate of the vertex and the roots was a quantity whose measure could be evaluated by the value of $\sqrt{\Delta}/2a$.

Once asked how she knew that x and y were real number she stated:

> E: So here $\sqrt{-\Delta}/2a$, our delta is smaller than 0 so here -Δ is positive, so this number [$\sqrt{-\Delta}/2a$] is also positive. This [$\sqrt{-\Delta}/2a$] is a positive number, so we call it real number. Positive real number, this [$\sqrt{-\Delta}$]is a real number this [2a] is real number when I divide it by real, it is a real number. For example,$-b/2a$ in the same manner this [b] is real and this [2a] is real, so [$-b/2a$] real number.

She also was able to explain why x and y had to be real numbers based on her algebraic meanings for the quantities. She knew that since discriminant is smaller than 0 then $\sqrt{-\Delta}$ had to be positive and divided by '2a' which is also a real number than would make $\sqrt{-\Delta}/2a$ a real number too. By the same token, she was able to reason that both 'b' and '2a' were real so their ratio would also be a real number.

Esra also was able to think of $z_{1,2}= x \pm yi$ and its components as vectors and as points.

> E: x [in $z_{1,2}= x \pm yi$]means this point and this vector [Figure 6.64]. I mean this length [the length from the point (0,0) to $(-b/2a,0)$]. It also means a point. This is y. y is this length [from $(-b/2a,0)$ to z_1] or that length[from (0,0) to $(0,\sqrt{-\Delta}/2a)$].

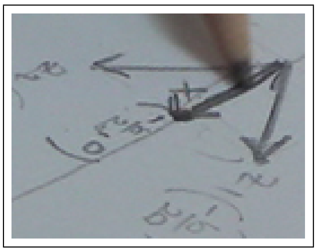

Figure 6.64. Esra's representation of x in $z = x \pm iy$ both as a vector and a point

As the excerpt indicated for Esra the complex number z represented both a point $(-b/2a,\sqrt{-\Delta}/2a)$ and a vector whose components were x and y. In particular, Esra represented x both as a point $(-b/2a,0)$ and as a vector on the real number line because x referred to a length, in her own words, whose measure could be evaluated by the value of $-b/2a$. She represented y as a point $(0,\sqrt{-\Delta}/2a)$ on the imaginary number line and as a vector in the complex plane because y referred to again the magnitude of a length, $\sqrt{-\Delta}/2a$. As she had earlier stated, $\sqrt{-\Delta}/2a.sqrt{-1}$ was the distance from the roots to the x-coordinate of the vertex, $-b/2a$. She added that "y is a perpendicular distance to x."

To provide more evidence on why she repeatedly constructed the roots as both vectors and points she was asked how she could represent the root $-b/2a+\sqrt{-\Delta}/2a.\sqrt{-1}$ as a vector on the complex plane:

E: So... This number because when we say here $\sqrt{\Delta}/2a$ we can represent it as a vector [Figure 6.65]. Two pieces of vector. So $(-b/2a, \sqrt{-\Delta}/2a)$. With a starting point and direction. Now $-b/2a$ is both a point and a part of this [complex number][pointing to the point $(-b/2a, \sqrt{-\Delta}/2a)$ and the middle vector in Figure 6.65]. It is a part indicating the length..and that $[\sqrt{\Delta}/2a]$ is also a length and a part of it [complex number], these two $[-b/2a$ and $\sqrt{\Delta}/2a]$ combine and this vector [the middle vector in Figure 6.65] is created, this vector at the same time represents this point.

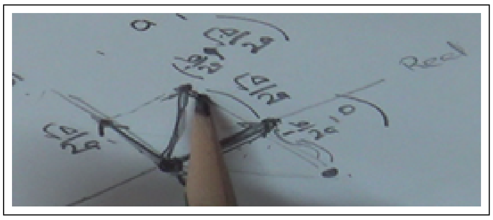

Figure 6.65. Esra's vectorial representation of complex numbers

At the beginning of the study, Esra was only able to represent a complex number as a point (x,y) without knowing what x and y in the form of $z = x + i.y$ referred to both algebraically and geometrically. However, as she related quadratic equations to complex numbers she thought $-b/2a$ and $\sqrt{\Delta}/2a$ could be represented as vectors on the complex plane because they both had a starting point, some magnitude with a direction. Thus, the combination of those two vectors gave another vector that identified a point on the complex plane to represent a complex number.

At that point, how she could regard the complex algebraic expressions as numbers was discussed. Once asked how one could suggest that the algebraic expressions in the Cartesian form were numbers, based on her knowledge of real numbers she explained:

E: When I have a starting point if I can represent normal numbers [real numbers], making inferences from what I know, like this[as vectors] then I can represent this number [complex number] in the same way, as a vector with a starting point, it has a direction, it has a length [magnitude].

Esra assumed that since real numbers could be represented by vectors on the real number line to measure some quantities, so could the complex numbers. Since the algebraic expressions of complex roots involved quantities that could be represented by vectors such as the x-coordinate of the vertex or the midpoint of the roots, and its distance to the roots, she suggested to represent complex numbers also as vectors on the complex plane.

In addition to her arguments about $-b/2a$ as a point, vector on the complex plane and a number in the set of real numbers, once asked how she reasoned $\sqrt{\Delta}/2a$ as a vector representing the roots' distances to the x-coordinate of the vertex on the complex plane:

> R: Why is it another axis? Why imaginary?
>
> E: Because real here[on the real x axis], all the numbers done in real so another axis we call him an imaginary axis, we call the unreal numbers because we call all over the real axis...All real numbers are finished here [on the real axis]. Ee there isn't another real number to represent. I assume it is here [above the real number line], or here[below the real number line]. This[$\sqrt{-\Delta}/2a$] also represents length, a magnitude; so it is in fact positive. For example, the number we call i is here [Figure 6.66].

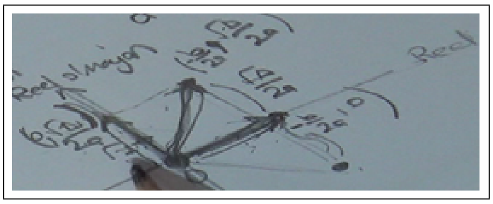

Figure 6.66. Esra's vectorial representation of complex number 'i'

> $\sqrt{-1}$ is here, these[the numbers which are negative inside a square root] are not the elements of real numbers, inside the square root is not positive. I found that this number [$\sqrt{-1}$] was here because when we multiplied 1 [1 on the real axis] by i, it [1] rotated 90 degrees. When multiplied [by i] it rotated 90 degrees, it only changed its direction, its starting point didn't change, its magnitude didn't change, because the length of this [$\sqrt{-1}$] is 1. It only changed direction and came here [on the perpendicular y axis, imaginary axis]... It [imaginary axis] has a unit, a starting point.
>
> R: Why is it imaginary? Why is that distance imaginary?
>
> E: Its unit is $\sqrt{-1}$, exactly. This is not real.

The dialogue indicated that since she was working beyond the quadratic equations which had real roots, when $\Delta < 0$, she no longer would be able to represent any unreal root on the real number line. Additionally, because the distance had an unreal factor $\sqrt{-1}$ when $\Delta < 0$ and it was an imaginary unit, she claimed that the distance $\sqrt{\Delta}/2a$ could be represented on an imaginary number line having the unit $\sqrt{-1}$. That was, with the need to create an imaginary number line to represent the distance of the unreal roots to the x-coordinate of the vertex, she created a new unreal number line to identify the unreal roots represented as points on the new complex plane.

To clarify her explanations she explained her answer to the question in the written post-assessment related to how she could think of a complex numbers as a single entity:

> E: For example, again, I can represent a real number as a vector like this because when I take a starting point, a magnitude and a direction, for example I can represent the number 5 like this [Figure 6.67]. In the same manner I divided

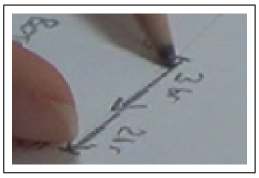

Figure 6.67. Esra's representation of a real number as a vector

> this length [the length of the complex number $2 + 6i$] into these two parts [in Figure 6.68]. I combined them [two components] and this number [the complex number $2 + 6i$] was created.

Figure 6.68. Esra's representation of complex number '$2 + 6i$' on the complex plane as a vector

After all those explanations of separating parts and representing them on the complex plane as both points and vectors, she argued that "...[$2+6i$] is a single number I think because... these[horizontal and vertical components on the complex plane] are the components of this vector and they come together and create only one point only one number. I know that I can represent a number as only one point, when I combine these[components] it gives the number $2 + 6i$, that point." She emphasized that even though it consisted of some parts the combination gave a number which she claimed it to be a single number as a single entity:

E: This point [the point (2,6) representing 2+6i] has a starting point size and direction, so it is a number.

She argued that she again was able to represent that combination of two parts as a single vector which had a starting point, a direction and a magnitude, which acknowledged her idea of representing a number.

During the pre-interview Esra was only able to state that i equals to $\sqrt{-1}$ without any further explanation. During the pre-interview, she even stated that she knew this by memorization in the formal definition of complex numbers. Also, during the teaching sessions, as the data earlier indicated, Esra did not know what $\sqrt{-1}$ represented geometrically because she tried to represent $\sqrt{-1}$ or i on the complex plane as the ordered pair of (0, $\sqrt{-1}$) or (0,i), respectively. Only after she reasoned on the algebraic form of 'i' as an unreal root she argued that 'i' is algebraically $z = 0 + 1.i$ thinking of the x-coordinate of the vertex and the distances of the roots to it, and so she was able to locate 'i' geometrically. In other words, only after she focused on the fact that in the algebraic form $z = 0 + 1.i$, the '0' meant that the x-coordinate of the vertex was zero and that the distance to it was in length of one unit, then Esra was able to state that she could represent 'i' with the point (0,1). This the allowed her to think that the magnitude of 'i' was equal to 1 unit.

Similarly, during the post-interview in contrast to the pre-interview, she was able to call on her knowledge of the representation of the imaginary unit 'i' both geometrically as a point (0,1), in Figure 6.142.148, on the complex plane and algebraically in the form of $z = 0 + 1.i$.

Not only that but also Esra provided her reasoning on the multiplication by i:

R: I got it, as a positive real number it could have been (0,2), but why was that (0,1)?

E: Because here we multiplied 1 [(1,0) on the real number line] by i and the length didn't change, it [the length] stayed unchanged. It only changed direction, nothing else... Ee when I multiply 1 [(1,0) on the real number line] by i, we obtain this number [$\sqrt{-1}$ represented as the point (0,1)]; so I decided it didn't change its magnitude. I multiplied 1 by i, it changed direction, rotated 90 degrees and came here [on the y imaginary axis in Figure 6.69]

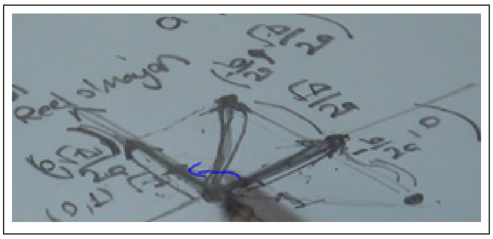

Figure 6.69. Esra's rotation of her pencil 90 degrees counter clockwise from the point 1 on the positive real number line

Esra thought of (0,1) as the representation of 'i' and in the form of $z = 0 + 1.i$, she thought 1.i as multiplication such that it was an operation of rotation without any change in its magnitude. She could state that because as she already knew 1 was a real number and she could also locate it on the real number line as (1,0). Multiplying this real number with 'i' made it representable on the unreal number line with the notation (0,1). Therefore, she was able to state that there was no change in the length but only the direction changed.

That is, she reasoned simultaneously in geometric and algebraic aspects of complex numbers and found out that multiplying a real number, say 1, by i referred to the rotation of 90 degrees counter clockwise. However, this did not provide enough evidence whether for any number she thought the same way. Once she was asked of multiplying any number, say y, by i at is, she thought of any number y as a complex number, said $y = a + bi$, and multiplied it by 'i' algebraically and represented it geometrically. Then, she drew and stated the following:

E: For example, if I say this [the number y] to be $a + bi$, when I multiply it [$a + bi$] by i, $ai + bi^2$, $ai - b$ it is $-b + ai$ [Figure 6.70].

Because it [a] is the coefficient of i, it [a] should be on the imaginary number line. -b is a real number. a and b are said to be positive here[in $a + bi$], so -b is here [on the negative direction of the real number line]. Eee in fact even though my figure doesn't show it clearly its magnitude didn't change, it rotated, it even rotated 90 degrees counter clockwise.

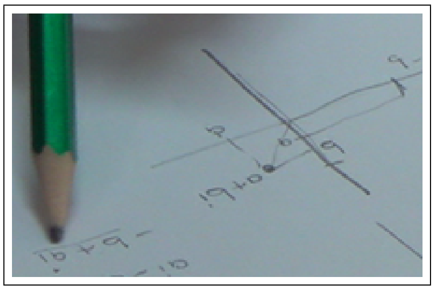

Figure 6.70. Esra's placing '$-b+ai$' on the complex plane

For her again, multiplication by i of any number caused 90 degrees counter clockwise rotation without any change in the length. She placed the resulting number -b+a.i on the complex plane accordingly. Thus, in terms of geometric reasoning, for Esra a complex number, i, was a point, a vector and an operator since she also came up with that the multiplication by i meant the rotation of 90 degrees counter clockwise. Yet, since she still did not think about the dilation, to investigate one step further how she could reason, I asked what would happen if any number $a+bi$ was multiplied by $a+bi$. She multiplied algebraically in Figure 6.71.

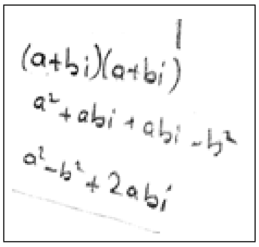

Figure 6.71. Esra's multiplication of '$a+bi$' by '$a+bi$'

She then reasoned geometrically "..then both its direction changes and its magnitude stretches."

> E: It [2ab] is the coefficient of i, it must be on the imaginary axis. It rotated and its magnitude changed, stretched [Figure 6.72].

Once she plotted this any complex number and its multiplied form by itself on the complex plane she argued that the multiplication operation on complex numbers meant rotation and dilation by showing on the complex plane as presented in the dialogue above. She further generated her meaning of the multiplication of complex numbers:

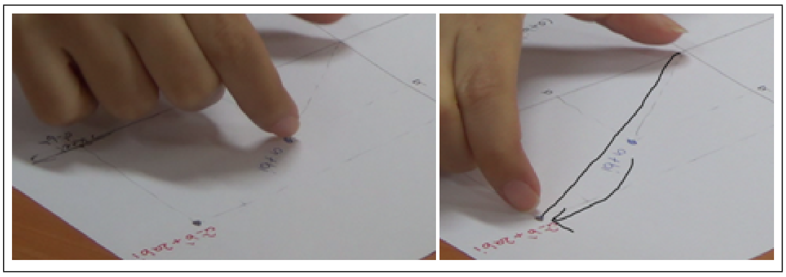

Figure 6.72. Esra's strecthing the magnitude of a complex number

E: [It is] the operation which changes the magnitude, length and direction of a number...Without changing its starting point, at any angle, it rotates its direction... It [the magnitude] can make it increase, decrease like this [Figure 6.73]. It can change its direction as well... It [the multiplication in complex numbers] is rotation, and also increasing its magnitude or decreasing, changing its magnitude.

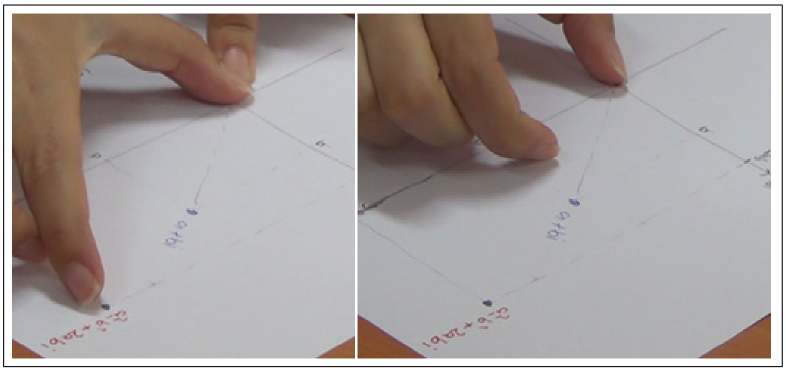

Figure 6.73. Esra's increasing or decreasing any complex number when multiplied by any complex number

As the excerpt indicated, she suggested that multiplication of complex numbers corresponded to a rotation dilation. To further validate that she thought of dilation I put the numbers $x.i$ and $y.i$, where x and y were real numbers, on the complex plane and asked the relationship between them, and she answered:

E: This [yi] is that [xi] multiplied by y/x. I multiplied xi by y/x. Its magnitude stretched. It [xi] increased, it stretched y/x much.

R: OK. How do you find the magnitude of any complex number?

E: If I draw it like this...This is real axis..This is imaginary [axis]..b is on the imaginary axis and a is on the real. This number [complex number $a + bi$] is written as (a,b) because I can represent it as a vector. Its [complex number $a + bi$] two components are a and b. And their magnitude [can be represented] with absolute value...This [the magnitude of the complex number $a+bi$] is.. from Pythagorean [theorem][Figure 6.74].

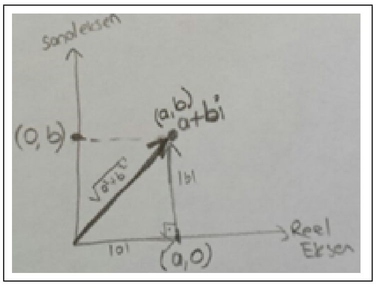

Figure 6.74. Esra's finding the magnitude of any complex number

She was able to calculate the magnitude of any complex number, $a + b.i$ on the complex plane as $\sqrt{a^2 + b^2}$ since she claimed any complex number could be represented as a vector. Therefore, the evidence in the excerpt above indicated that Esra was thinking of dilation while multiplying complex numbers along with rotation. Her statements provided evidence on that she thought of i, the imaginary unit, as a point, vector, number, and operator.

As the data earlier showed, during the pre-interview, Esra had only provided the formal definition of complex numbers. That is, she was able to define complex numbers as the numbers in the form of $a + bi$ where a, $b \in R$ and $i = \sqrt{-1}$. During the post-interview, in addition to the formal definition of complex numbers, based on quadratic equations Esra stated "... the numbers obtained from the roots of all quadratic equations with real coefficients constitute the set of complex numbers." Along with this definition, she was also able to provide her reasoning as to why complex numbers include real numbers as follows:

> E: Because we already obtain complex numbers from the roots of quadratic equations. So [real numbers are] inside it [the set of complex numbers]. Because real numbers are included by the complex numbers.

Esra also had defined complex numbers in the written post-assessment in Figure 6.75.

This data indicated that Esra was able define complex numbers as the elements of a well-defined set. That is, she was able to state that the set of complex numbers are obtained from the roots of all quadratic equations with real coefficients. By the

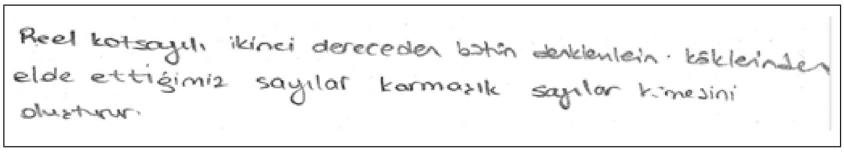

Figure 6.75. Complex numbers in Esra's words *"The numbers obtained from the roots of all quadratic equations with real coefficients constitute the set of complex numbers"*

same token, Esra was able to reason that since all the roots including the real number roots consitue the elements of the set of complex numbers, real numbers have to be embedded in the set of complex numbers. As the data from the pre-interview indicated Esra had known such relationship but was not able to provide any reasoning for that.

The following data further revealed on how Esra related the set of complex numbers to the roots of quadratic equations:

> E: If delta is bigger than 0, it was on real axis because my roots were real. Then I have taken all those real numbers in infinite number because there are a lot of quadratic equations which have delta bigger than 0, so I finished all of them in real axis. I finished all my roots on the real axis, then when it [delta] was 0, the roots with multiplicity two was here [on the point ($-b/2a$,0) on the real number line]. Then when [delta] is negative, [the roots are] on this plane as I have just told. It passed to the whole plane here. If delta is smaller than zero, it is a complex number, if delta is positive, negative or zero; all of them are complex numbers. But if it [delta] is positive, they are on real axis....and if it [delta] is zero, it is the same [they are on the real axis]. If [delta is] negative, here [on the real axis] no [numbers] are left [Figure 6.76].

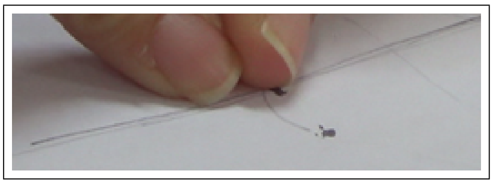

Figure 6.76. Esra's covering all the real numbers on the real number line toward the point ($-b/2a$,0)

> Ee well, I know that the roots [unreal roots] are symmetric about this point[$-b/2a$] because [the roots are] $\sqrt{\Delta}/2a$ far in the distance to $-b/2a$, I decided they [the roots] were here [below and above at the same perpendicular distance to the point ($-b/2a$,0)].

As the data indicated, Esra reasoned on the the values of the discriminant both geometrically and algebraically. Her arguments validated that she regarded the x-coordinate of the vertex, $-b/2a$, and its distance to the roots, $\sqrt{\Delta}/2a$, as quantities involved in the construction of the complex numbers geometrically. That is, she reasoned that she could find infinitely many real numbers representing the roots of quadratic equations because she was able to think of so many quadratic equations with having delta bigger than zero. The value of delta being bigger than zero in fact meant for her that there was distance, $\sqrt{\Delta}/2a$, between the roots and the x-coordinate of the vertex. And such roots would compass the real number line until it gets to a point there is no distance between the roots and the x-coordinate of the vertex. This also meant for her that $\Delta = 0$. Then, she knew that once the delta gets smaller than zero the roots have to be on a different plane because she knew that she finished all the real numbers when delta is equal to or bigger than zero. Her knowing this allowed her to not only deduce the relationship between the roots of any quadratic equation and the complex numbers but also the reasoning behind the relationship between real numbers and the complex numbers. That is, the set of complex numbers are obtained from the roots of any quadratic function with real coefficients and so such set includes all real numbers.

While Esra was representing the complex roots on the complex plane, she plotted the second root which is called the conjugate root considering the distance between the roots and the x-coordinate of the vertex:

> E: I said $\sqrt{\Delta}/2a$, if we do this [the first root] like this [below the real axis at a perpendicular distance from the abscissa of the vertex], it [the first root] is here and its conjugate is here [below the real axis at a perpendicular distance from the abscissa of the vertex in Figure 6.77]

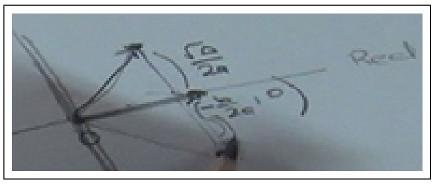

Figure 6.77. Esra's pointing to the conjugate root on the complex plane

The way she reasoned also suggested that the conjugate root could also be represented by a vector, which implied that she thought that it was also a complex number on the complex plane. However, at the beginning, she was not able to reason about the existence of the conjugate root once a complex root exists. At the end of the study, she explained that there are two roots of any quadratic equation and they are conjugates:

> E: There were two roots of two quadratic equations, I mean, if there is one[of the roots] the other[root] exists as well because it is a quadratic equation and we obtain these complex numbers from these two roots, and this root, so for example, when you give one[root], because the parabola has a vertex and this[the root] must be symmetric about the abscissa of the vertex, when x_1 is given, x_2 must exist as well because [the roots are] symmetric about this point[the abscissa of the vertex]. If I give a root, the other must exist because this parabola has a symmetry axis, has a vertex, so a second root must exist because they[the roots] are symmetric.

Her reasoning explicates how she came up with a second root for all quadratic equations. Based on both the definition of complex numbers in relation to quantities involved in the quadratic equations, the invariant qualities of quadratic functions' graphs, she reasoned algebraically and geometrically that there should exist conjugate roots for all quadratic equations.

7. CONCLUSION

The purpose of this study was to investigate how a prospective teacher, Esra, developed the meaning of the Cartesian form of complex numbers while reconstructing complex numbers from real numbers based on the solution sets of quadratic equations with real coefficients. This investigation resulted in the researcher's generation of a model of how she developed the meaning of the Cartesian form of complex numbers during the instructional sequence. In this account of the model of Esra's knowledge development, I articulated aspects of her conception of complex number. In this regard, the purpose of this section was to discuss the major findings in Esra's development of the Cartesian form of complex numbers as elements of a well-defined set along with (i) Esra's development of the meaning of the vertex and (ii) Esra's coordinating the different aspects of complex numbers such as vectors, points and operators. The major findings of this study follows in the next sections.

7.1. Esra's Development of the Cartesian Form of Complex Numbers as the Elements of a Well-defined Set

Smith and Thompson (2008) claimed "...because the central goal is to focus on quantities and how they relate in situations, and because this represents a major obstacle for many students, it is important for teachers to open discussions with questions that lead to discussions of quantities, not numbers..." (p. 36). Similarly, previous research suggests that both students and teachers think of complex numbers as "...typically ... instances of mathematical structures that seem to depend merely on formal arrangements in a system of meaningless signs, not referring to anything informal or quasi-empirical" (Glas, 1998, p. 367). That is, both students and teachers do not think of complex numbers as quantities but as formal algebraic expressions (Nordlander and Nordlander, 2012). In this regard, in this study as suggested by Smith and Thompson (2008) once a prospective teacher examined some attributes of quadratic functions not as numbers but as quantities she was able to deduce complex numbers as elements of a well-defined set. In this study I take such attributes of quadratic equations that make them unique on their own as the 'roots' (zeros) and 'the x-coordinate of the vertex' (NCTM, 2000) such that their measures could be determined both from the origin and from each other. In particular, once a prospective teacher thought that "...$-b/2a$ is the vertex and the $\sqrt{\Delta}/2a$ is the distance to the zeros (left and right)" (Hedden and Langbauer, 2003, p. 158) she was able to deduce complex numbers originated in all solutions to the quadratic equations (Panaoura *et al.*, 2006). In this regard, in this study I reported on how reasoning about $-b/2a$ and $\sqrt{\Delta}/2a$ as quantities and relationships between these quantities may support a prospective teacher in developing meanings regarding the Cartesian form of complex numbers.

7.1.1. Esra's Development of the Meaning of the Vertex

At the beginning of the study given the general algebraic expression of any quadratic function, Esra knew that $-b/2a$ geometrically represented the x-coordinate of the vertex and $\sqrt{\Delta}/2a$ represented the distance to the roots. However, she knew that it by memorization: She could not explain why $-b/2a$ represented the x-coordinate of the vertex. Similarly, she knew that it also represented geometrically the midpoint of the roots because she averaged the roots algebraically. By the same token, her knowledge on the meaning of the vertex was limited such that she was able to define it only as the point on the parabola on which the quadratic function takes its lowest or highest value, i.e. a minimum or maximum point. Although the vertex of any parabola provides information on the parabolas' maximum or minimum points, such knowing is limited since she did not relate it to the line of symmetry. In addition, the data showed that Esra did not know the algebraic representation of the line of symmetry as x= $-b/2a$ either. These data altogether indicated that not only her knowledge on $-b/2a$ was limited but also she might not have known the definition of vertex as the point at which the parabola intersects its line of symmetry (Gibson, 1998).

One might argue that Esra's not stating the formal definition of vertex of a parabola might not guarantee that she did not have it. However, her not expressing the line of symmetry as x=$-b/2a$ and not knowing why $-b/2a$ represented the x-coordinate of the vertex suggested that her knowledge was compartmented. By the same token, although Esra was able to algebraically deduce that $\sqrt{\Delta}/2a$ was the distance from the x-coordinate of the vertex to the roots and was able to geometrically represent it on a parabola only when the discriminant was bigger than zero. She was not able to imagine such distance neither when the discriminant was zero nor was smaller than zero. Her coordinating both the algebraic and the geometric meanings of $-b/2a$ with the line of symmetry was important: Such knowledge would be the basis of thinking the dynamic of the roots of infinitely many quadratic functions with the same x-coordinate of the vertex as she would regard the distance $\sqrt{\Delta}/2a$ as a varying quantity (Smith and Thompson, 2008).

In order for her to relate $-b/2a$ and vertex to the line of symmetry, Esra first engaged in physically folding the shape of the parabola into two congruent parts. This allowed her to mentally match the points on two parts of the folded shapes. The activity of matching of all the points on the two congruent parts of the parabola, allowed her to conclude that there would be only one vertical line, i.e. the line of symmetry, right up through the middle which would split the parabola into two mirrored halves. The symmetry for her meant that all the points on the two congruent halves of the parabola would be reflections of each other. This in turn enabled her to deduce that among any point on the parabola, the roots were also symmetric about the line of symmetry (Cooney, Beckmann and Llooyd, 2010). Acknowledging that the line of symmetry has to be in the middle of all the points on the parabola (i.e., equidistant from all the points symmetric to each other), Esra was able to reason that the algebraic form of the line of symmetry had to be x=$-b/2a$. She also reasoned that since she could draw only one line of symmetry for a parabola it went through the vertex of the parabola. However, she was still not able to define vertex. Then, physically drawing the line of symmetry on the shape of the parabola she had folded allowed her to mentally match the vertex both as a point on the parabola and a point on the line of symmetry intersecting the parabola. Therefore, she was not only able to define vertex as the point where the line of symmetry intersected the parabola (Gibson, 1998), but also was able to reason that $-b/2a$ had to be the x-coordinate of the vertex: The line of symmetry went through the midpoint of all the points including the roots and $-b/2a$ was equidistant from the roots, which was an invariant quality for any parabola. She was also able to deduce that the roots of the quadratic equations are symmetric about the x-coordinate of the vertex, $-b/2a$ (Cooney *et al.*, 2010). This suggested that Esra was able to coordinate her existing knowledge on $-b/2a$ with her newly constructed knowledge on the definition of vertex in relation to the line of symmetry.

7.1.2. Esra's Coordination of the Different Aspects of Complex Numbers as Vectors, Points and Operators

Acknowledging the reasoning behind why $-b/2a$ was the x-coordinate of the vertex, once prompted she was able to imagine infinitely many quadratic functions with the same x-coordinate of the vertex. Particularly, she first reasoned from the algebraic point of view such that the changing values of the real coefficients a, b, c of any quadratic function constituted infinitely many quadratic functions. By the same token, such changing values allowed her to think that although the ratio, $-b/2a$, was kept same the values of $\sqrt{\Delta}/2a$ varied. It is important to state that Esra was still thinking algebraically. Only after prompted to think about particular graphs of quadratic functions, the x-coordinate of the vertex of which was equal to -1, she was able to reason geometrically. Her thinking of the symmetry of the roots about the x-coordinate of the vertex allowed her to imagine infinitely many parabolas with the same x-coordinate of the vertex. Thinking of the existence of infinitely many parabolas with the same x-coordinate of the vertex enabled Esra to imagine the 'movability' of the distances of the roots to the x-coordinate of the vertex. A consequence of this was that Esra thought of $\sqrt{\Delta}/2a$ as a varying quantity dynamic in nature such that it shrinks and/or stretches (dilates).

Once prompted to think about the placements of the roots and the vertex on the real number line Esra was able to think of the x-coordinate of the vertex as represented by ($-b/2a$,0) and the roots as represented by ($-b/2a + \sqrt{\Delta}/2a$,0) and ($-b/2a - \sqrt{\Delta}/2a$,0). Simultaneously, Esra was able to think that although the roots' distances to the x-coordinate of the vertex were changing, both the placement of the x-coordinate of the vertex and roots' being reflections of each other about it were invariant. This was important because then Esra could imagine and reason that once there was no distance between the roots and the x-coordinate of the vertex; that is, the value of $\sqrt{\Delta}/2a$ was zero; then the roots were represented by ($-b/2a$,0). Therefore, Esra's thinking of the roots not only as distances to the x-coordinate of the vertex but also as points allowed her to place the roots on the real number line. Particularly, Esra was able to think that all the real numbers as points could represent the roots of infinitely many quadratic equations with the same x-coordinate of the vertex. Once she thought about the geometric meaning of discriminant's being smaller than zero although she knew that infinitely many quadratic functions existed with the same x-coordinate of the vertex and discriminant's being smaller than zero, she could not place the unreal roots on the real number line. She reasoned that since all the real numbers as potential roots were capsulized there was no place left on the real number line to represent the unreal roots. Therefore, she was not able to locate the unreal roots. Though she had difficulty in placing the unreal roots on the real number line such image afforded on her part the necessity to think of a plane whose horizontal axis was the real number line.

Importantly, Esra still thought of $\sqrt{\Delta}/2a$ as a distance to the x-coordinate of the vertex even when the discriminant was smaller than zero. She surmounted the difficulty in the following way: Esra first algebraically thought about the discriminant as being equal to $-1.4ac - b^2$ and the roots as being equal to $-b/2a \pm \sqrt{(-1).4ac - b^2}/2a$. Contrary to the pre-interview results Esra was able to symbolize $-b/2a$ and $\sqrt{4ac - b^2}/2a$ as variables, t and m respectively, i.e. $x_{1,2} = t \pm m.\sqrt{-1}$. Esra knew that she could write $\sqrt{(-1).4ac - b^2}/2a$ as $\sqrt{-1}.\sqrt{4ac - b^2}/2a$ because the former expression was not real anymore, yet $\sqrt{4ac - b^2}/2a$ was a positive real number. Only after Esra placed

$-b/2a$ on the real number line and called on her knowledge that all the real roots were capsulized she started to think about placing the unreal roots as points on a plane including the real number line. That is, for her "the mapping from points on a line to numbers has been extended to a mapping from points in a plane to numbers.." (Fauconnier and Turner, 2002, p. 272) such that those numbers represented the roots of quadratic equations. Although she attempted to place the unreal roots on a plane her placement did not align with complex plane. Only after she thought of the origin and the roots' distances being perpendicular to the x-coordinate of the vertex she was able to think of an axis perpendicular to the real number line. She then called it the unreal number line. Esra also thought of (t,m) as the point representation of unreal roots, reasoning that as ordered pairs she needed real numbers to represent any point on this new plane. Her thinking aligns with the fact that xy-plane can also be used to represent complex numbers geometrically as ordered pairs of real numbers (Usiskin *et al.*, 2003; Panaoura *et al.*, 2006).

Thinking of both the expression x=$-b/2a \pm \sqrt{\Delta}/2a$ and x=$-b/2a \pm \sqrt{\Delta}/2a.\sqrt{-1}$ as roots of any quadratic equations Esra was able to reason that all the roots, i.e. real and unreal, constituted the complex numbers. Contrary to the earlier research results such that students could not imagine "..what complex numbers 'stand for and really are"' (Nordlander and Nordlander 2012, p. 633), Esra was able to define complex numbers as elements of a well-defined set (Sfard, 1991). To accept complex numbers as a new category of numbers it is essential to conceptualize $x + yi$ as a single entity in a well-defined set (Conner *et al.*, 2007; Sfard, 1991). Particularly, Esra defined complex numbers as the elements of the set of numbers obtained from the roots of any quadratic equation with real coefficients. She knew that once the numbers were real they included two real parts, the x-coordinate of the vertex and the roots' distances to the vertex. Similarly, once they were complex they still included the same two parts; one is real and the other is unreal. Such reasoning allowed Esra to conceptualize that complex numbers could both be represented as $(-b/2a \pm \sqrt{b^2 - 4ac}/2a$,0) on the real number line and also $(-b/2a, \pm\sqrt{4ac - b^2}/2a)$ on the complex plane as ordered pairs of real numbers. This is important because she knew that any complex number as a single entity, whether totally real or both real and unreal, belongs to the set of the roots of quadratic equations and is always represented with two components. This aligns with that mathematicians also treated complex numbers in the form of $x + yi$ as augmentation of two real parts, x and y (Penrose, 2004). Contrary to the high school students in the study of Panaoura *et al.* (2006) and undergraduate students in the study of Norlander and Norlander (2012), Esra considered any complex number as a single entity, a single number represented geometrically as an ordered pair of real numbers and algebraically as augmentation of real and unreal parts. The whole process solidified something familiar, i.e. the real numbers representing roots of quadratic equations, in to a new object, i.e. the complex numbers representing the roots of quadratic equations. Conner *et al.* (2007) concluded that separating complex numbers into the real and unreal (imaginary) parts might enable learners to interpret complex number addition geometrically, but it might not provide meaningful geometric interpretation of multiplication in complex numbers. However, in this study, I reported that one could interpret complex number multiplication geometrically while representing complex numbers as a single entity with two components at the same time.

Such realization also afforded Esra's understanding why complex numbers involve real numbers. This was important because research has shown that students have difficulty in recognizing that any number is a complex number (Nordlander and Nordlander, 2012). Although at the beginning of the study Esra was aware that any number was a complex numbers she was not able to reason about it. At this point, she knew that since all the roots of any quadratic equation already included real numbers complex numbers had to include the real numbers.

Though from the researchers' point of view, there was a flaw in Esra's reasoning about the unreal part. From the point of view of quantitative reasoning, she confused the quantity with its numerical value. She could not distinguish the distance and its magnitude. Thompson (2011) stated "Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations childrean (and teachers) can confound them unproblematically..” (p. 42). Particularly, she did not know that $\sqrt{-\Delta}/2a$ referred to the numerical value, i.e. the absolute value, of the distance as a quantity, $\sqrt{\Delta}/2a$, to the x-coordinate of the vertex. Still, her not realizing such dichotomy in her reasoning was meaningful because she reasoned that distances had to be represented by positive real numbers. Knowing that $\sqrt{-\Delta}/2a$ was a positive real number she concluded that $\sqrt{-\Delta}/2a$ represented that distance. Once one thinks about positive real numbers on the real number line both the distance of that number to the origin and its magnitude is represented with the same expression. However, considering the case of unreal roots, $\sqrt{(-1).4ac - b^2}/2a$ as a distance and its magnitude $\sqrt{4ac - b^2}/2a$ are represented with different expressions.

Still, Esra's thinking of real $-b/2a$ and unreal parts $\sqrt{-\Delta}/2a.\sqrt{-1}$ as two components allowed her to relate complex numbers to vectors. This was because besides thinking of the point-wise representation $(-b/2a,\ \sqrt{-\Delta}/2a)$, she thought of the real part as a distance on the real number line and the unreal part as a distance on the unreal number line. She also reasoned that real numbers could be represented as vectors because they had an origin, direction and magnitude. Her reasoning was evidenced in the mathematician Wallis' observation that “...if negative numbers could be mapped onto a directed line complex numbers could be mapped onto points in a two dimensional plane..” (Fauconnier and Turner, 2002, p. 271). Similar to real numbers once she thought complex numbers as points on the complex plane locating their starting point at the origin and thinking of its real and unreal parts' magnitudes allowed her to think that she could represent complex numbers as vectors. Based on Esra's knowledge development I hypothesize that, prior to students' construction of complex numbers, they might need to know that real numbers could be represented as points and vectors on a number line.

Esra was able to overcome her difficulty about what $\sqrt{-\Delta}/2a.\sqrt{-1}$ represented only after she reasoned about what $\sqrt{-1}$ geometrically meant both as a point and a vector. Reasoning that $\pm\sqrt{-1}$ as roots of a quadratic equation she first thought about its numerical correspondence within the algebraic form of the roots $x_{1,2} = t \pm m.\sqrt{-1}$.

She concluded that the x-coordinate of the vertex had to be zero since there was no real part and m had to be equal to 1. Calling on her knowledge that the point representation of $x_{1,2}=t \pm m.\sqrt{-1}$ was (t,m) she was able to represent $\sqrt{-1}$ as the point (0,1). She also reasoned that its magnitude would be 1 unit such that it referred to the imaginary unit (Sfard, 1991). I propose that Esra's thinking of $\sqrt{-\Delta}/2a$ as a real number and $\sqrt{-1}$'s being represented on the unreal axis allowed her to reason that multiplying $\sqrt{-\Delta}/2a$ with $\sqrt{-1}$ resulted in placing it on the unreal axis. This was because she also knew that the geometric representation of the roots was already shown by $(-b/2a, \sqrt{-\Delta}/2a)$. Therefore, she was able to deduce that the geometric representation of $\sqrt{-\Delta}/2a.\sqrt{-1}$ would be shown by $(0, \sqrt{-\Delta}/2a)$. This allowed her to think of the multiplication, i.e. a numerical operation, of a real number by $\sqrt{-1}$ meant transforming the real number $\sqrt{-\Delta}/2a$ onto the unreal axis without changing its magnitude, i.e. quantitative operation. She knew that the magnitude did not change because $\sqrt{-1}$ was represented geometrically by (0,1). She defined this transformation as a 90 degrees rotation counterclockwise. That is, $\sqrt{-\Delta}/2a.\sqrt{-1}$ as a quantity was represented by $(0, \sqrt{-\Delta}/2a)$ such that it had both a magnitude and a direction. Such reasoning implied that "In the blended space of complex numbers, numbers and vectors are the same thing.." (Fauconnier and Turner, 2002, p. 273). this was evidenced in the post-interview too. That is, for Esra complex numbers are elements of the set of the roots of the quadratic equations such that it is both a number, which is a point in the Cartesian plane, and a vector, which is a line segment with a magnitude and direction in the Cartesian plane (Fauconnier and Turner, 2002). In this way, she was also able to distinguish $\sqrt{-\Delta}/2a.\sqrt{-1}$ from $\sqrt{-\Delta}/2a$.

In this study, multiplication with the imaginary unit also begets the idea that the imaginary unit as an operator such as rotation as it is evidenced in "..multiplication of numbers is just rotation and stretching of vectors" (Fauconnier and Turner, 2002, p. 273). It is in this respect that in this study, I acknowledge that for the formation of the complex numbers as a mathematical concept structurally, one needs to engage mentally in some particular activities such as constructing an imaginary unit and acquiring proficiency in using square roots, which corresponds to the interiorization stage (Sfard, 1991); and be able to relate the geometrical and algebraic forms of complex numbers simultaneously, which corresponds to the condensation stage (Sfard, 1991). I argue that the ability to think of the family of quadratic functions keeping their attributes such as the vertex, the x-coordinate of the vertex, and the existence of its distances to the roots invariant while imagining the movability of the roots' distances to the x-coordinate of the vertex corresponds to the condensation stage (Sfard, 1991) since it refers to thinking of the process as a whole. Upon moving through these stages, one can and might be able to reify complex numbers as elements of a well-defined set. That is, she could define complex numbers as the elements of the solution set of any quadratic function with real coefficients.

Thinking of the roots and the x-coordinate of the vertex as distances and coordinate points allowed Esra to conceptualize that " just as |x| is the distance from x to 0 on the real number line, |z| is the distance from the origin in the complex plane. So, |z|=$\sqrt{x^2+y^2}$ is the absolute value or modulus of $z = x + iy$" (Usiskin *et al.*, 2003). Thus, that algebraic expression represents a quantity which is measurable also confirms that these complex algebraic expressions exist as numbers because the distances between the roots and the x-coordinate of the vertex are based on "..conceiving a quality of a cognitive object where this conception involves measurability of that quality" (Thompson, 1994, p. 184). Since Karakök *et al.* (2015) argued that switching flexibly between algebraic and geometric representation of the Cartesian form was addressed also in the use of $x + yi$ as an expression to represent any complex number to label vectors and points on the complex plane.

Examining the algebraic expressions involved in the roots of quadratic equations with their geometric meanings allowed her to reason that there existed some imaginary distance to be represented on an unreal axis. I acknowledge that this also brought about Esra's realization of the necessary existence of the conjugate root because she could understand that the distances from the roots to the x-coordinate of the vertex stayed invariant yet started existing on the unreal number line. That is why the conjugates are the reflections about the x-coordinate of the vertex or the real axis (Usiskin *et al.*

2003; Soto-Johnson and Troup, 2014).

Previous research on complex numbers has shown that students consider "…the geometric and algebraic representation as two different autonomous mathematical objects and not as two means of representing the same concept" (Panaoura *et al.* 2006). Thus, it was important to employ both algebraic and geometric representation simultaneously. In addition, since "..to comprehend a quantity, an individual must have a mental image of an object and attributes of this object that can be measured.." (Moore *et al.*, 2009, p. 4), I argue that focusing on both algebraic and geometric aspects of the x-coordinate of the vertex and the roots of a quadratic equation as a distance, a vector, and a point on the number line simultaneously might be both necessary and essential in conceptualizing complex numbers' algebraic and geometric representations.

This study provided insight into how a prospective teacher reasoned while constructing the Cartesian form of complex numbers algebraically and geometrically upon real numbers considering the roots of quadratic equations with real coefficients. The articulation of a prospective secondary mathematics teacher's development of complex numbers conception presented which quantities and the quantitative relationships this conception involved in its process of construction. Quantities and quantitative relationships were presented through an instructional sequence with a focus on the algebraic and geometric aspects of quadratic equations triggering the mental actions and operations on the part of the students to develop their meanings of the Cartesian form of complex numbers while introducing the notion of complex number.

8. IMPLICATIONS

From the teaching perspective, I also argue that, thinking of the quadratic functions' attributes such as the vertex and its x-coordinate, and the roots' distances to the x-coordinate of the vertex might provide an opportunity for teachers and teacher educators with bringing their students' attention to the concrete examples. Particularly, as shared in this paper, teachers might choose to start with examples of quadratic functions such that the x-coordinate of the vertex is on the x-axis and then continue with the ones having the ones on the y-axis. This might assist students to think of quadratic functions such as $y = x^2 + 1$ and $y = x^2 + 2x + 2$ coming from the same family of functions, i.e. quadratic functions with real coefficients, and help them imagine the processes these functions go through as a whole. Also, another important point is that students could be asked to think of what stays invariant and what varies once they produce quadratic functions with the same x-coordinate of the vertex. The interesting point is that, although the line of symmetry and the x-coordinate of the vertex stay invariant, the values of 'a' and' b' in any algebraic quadratic equation might change. Although students might come up with different concrete examples of quadratic functions, while teaching I propose to keep the values of 'a' and 'b' the same whereas changing the values of 'c'. This way, it might be easier to bring students' focus on the movability of the roots' distances to the line of symmetry as presented in this study. That is, students might imagine what varies more easily. Mental actions as reflection and dilation one might engage in while constructing complex numbers from real numbers have been discussed in this study. However, in similar studies it might be further discussed why one needs an imaginary number line that is perpendicular to the real number line for a complex plane.

In addition, I suggest that the imaginary unit 'i' might be interpreted geometrically when studying the roots of quadratic equations and their distances to the x-coordinate of the vertex along with the algebraic representation of the roots. I also propose that using similar tasks, teacher educators/researchers might investigate students' and/or prospective/in-service teachers' conceptualization of the Cartesian form of complex numbers both algebraically and geometrically. As presented in this study, doing research in real context might provide further insight into and the details of how someone might reason to come to such construction of the Cartesian form of complex numbers.

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APPENDIX A: PRE- AND POST-ASSESSMENT REASONING TEST

(i) Could you define complex numbers?

(ii) What is the relationship between complex numbers and the roots of the quadratic equations?

(iii) Could you define quadratic functions algebraically?

(iv) Given any quadratic function, $f{:}R{\to}R$, $f(x) = ax^2 + bx + c$, where a, b and $c{\in}R$ and $a{\neq}0$.

  (a) Could you explain how you would draw the graph of the function?

  (b) Please state the critical points of the function in terms of "a,b,c".

  (c) Please show your work on a graph of a quadratic function.

(v) Given any quadratic function, $f{:}R{\to}R$, $f(x) = ax^2 + bx + c$, where a, b and $c{\in}R$ and $a{\neq}0$.

  (a) Deduce that the roots for the equation $ax^2 + bx + c = 0$ could be written as $\mathrm{x}_{1,2}{=}{-b}/2a \pm \sqrt{\Delta}/2a$, where $\Delta = b^2 - 4ac$ is called as discriminant.

  (b) What does -$b/2a$ mean geometrically and algebraically? Why? Explain your reasoning.

  (c) What does $\sqrt{\Delta}/2a$ mean geometrically? Why? Explain your reasoning.

(vi) For any quadratic equation $ax^2 + bx + c$, where a, b and $c{\in}R$ and $a{\neq}0$, when one root is in the form of $z = x + iy$, a complex root, the other root is $z = x - iy$ where x and y are real numbers. The complex root $z = x - iy$ is known as the conjugate of the complex root $z = x + iy$.

  (a) What do x and y refer to algebraically?

  (b) Why do x and y have to be real numbers?

  (c) What do x and y refer to geometrically?

  (d) Why does the conjugate root exist? Explain your reasoning.

(vii) Jason says that he thinks of the number $2 + 6i$ in terms of two different parts; the '2' and '$6i$'. Sharilyn, however, says that she thinks of $2 + 6i$ as a single number, '$2 + 6i$' rather than in terms of two different parts. Do you think about $2 + 6i$ like Jason does, like Sharilyn does, or a different way? Please explain your reasoning

(Conner *et al.*, 2007).

APPENDIX B: PRE- AND POST-INTERVIEW QUESTIONS

*Which of the followings are in the set of complex numbers? i.e. Which are the following numbers complex numbers? Please circle of your choice.

(i) $5 + i^5$
(ii) $2 + i^8$
(iii) $5 + i\sqrt{10}$
(iv) $2 - \sqrt{-3}$
(v) $i.(1/2)$
(vi) $i.\sqrt{-7}$

(i) Could you define complex numbers? If there is, what are complex numbers?
(ii) Can you explain your answer to the sixth question in the pre-assessment?
(iii) Can you explain your answer to the second question in the pre-assessment? Are the roots of quadratic equations complex numbers when $\Delta > 0$?
(iv) Can you explain your answer to the third question in the pre-assessment? What are the domain and the range of the quadratic function you expressed?
(v) Can you explain your answer to the fourth question in the pre-assessment? How do you know that $-b/2a$ is the x-coordinate of the vertex? In the graph you drew, is there just the x-coordinate of the vertex? What is its y-coordinate? What does the vertex mean?
(vi) Can you explain your answer to the fifth question in the pre-assessment?
   (a) Given the form of the roots of quadratic equations, what does $\Delta > 0$ and $\Delta < 0$ means algebraically?
   (b) Given the form of the roots of quadratic equations, what does $\Delta > 0$ and $\Delta < 0$ means geometrically?
(vii) How do you explain what Jason says? How do you explain what Sharilyn says? How do you show this number analytically?

APPENDIX C: THE PLAN OF INSTRUCTION

The plan of instruction in this study is presented in the following pages.

Instruction Plan for Developing the Meaning of Complex Numbers: The Cartesian Form	
First Teaching Session (75 minutes)	
Researcher's Questions	**Participants' Activity**
At the beginning, researcher provides the student with her answer to the fifth question in the pre-assessment. **Question 5:** Given any quadratic function, $R \rightarrow R, f(x) = ax^2 + bx + c$, where a, b and $c \in R$ and $a \neq 0$. a. Deduce that the roots for the equation $ax^2 + bx + c = 0$ could be written as $x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$ is called as discriminant.	
Researcher asks student the following:	

Figure C.1. Instruction Plan-Page 1

Afterwards, the researcher asks for her own explanation for how the roots are deduced from the quadratic equation algebraically such that the roots are algebraically found in an attempt to create a perfect square.	$ax^2 + bx + c = 0 \qquad a, b, c \in R$ $x^2 + \frac{bx}{a} + \frac{c}{a} = 0 \qquad \boldsymbol{a \neq 0}$ $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ $(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ $(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ $(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$ $\left\lvert x + \frac{b}{2a} \right\rvert = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ $\left\lvert x + \frac{b}{2a} \right\rvert = \frac{\sqrt{b^2 - 4ac}}{2a}$ $x_1 + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x_2 + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$ $x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$
Researcher asks for the meaning of $\frac{-b}{2a}$ both **algebraically and geometrically**, and provides the student's own answer to this question in the pre-assessment. 1. What does $\frac{-b}{2a}$ mean geometrically and	

Figure C.2. Instruction Plan-Page 2

algebraically? Why? Explain your reasoning.	
If she had stated that $\frac{-b}{2a}$ refers to the x-coordinate of the vertex of any parabola then the researcher asks the following questions. 1. What does the vertex mean? 2. How many vertices might a parabola have? How do you think that? 3. How do we know geometrically that a parabola has only one vertex? Why is it unique?	1. The expected answer for **the definition of the vertex is** the point where the parabola intersects its axis of symmetry, which is the line of **symmetry** of a **parabola** and divides a **parabola** into two equal halves that are **reflections** of each other about the line of **symmetry**. It intersects a **parabola** at its vertex. 2. The expected answer to the question that how many vertices a parabola might have is that there is only one. The expected answer for how do we know that there is only one vertex of a parabola is that she can think of folding the parabola and bringing the legs of the parabola on top of each other making an equal shape. Then there is only one way to do it because otherwise the legs do not match on top of each other. That is, the legs have to match with each other because the points on the legs are reflections about the parabola's line of symmetry. Then that line that separates the shape into two equal pieces is the line of symmetry.
If she could answer the questions above, the researcher will ask the following questions marked with *.	In the folding activity, the student is given three parchment papers with a parabola drawn on each. Student's expected answers to the questions follow below.

Figure C.3. Instruction Plan-Page 3

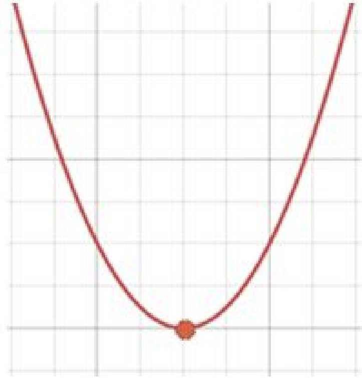
If she can answer the If the student cannot state how we know that there is only one vertex for any parabola, then the folding activity will be given with parchment paper. Three papers with a parabola drawn on each will be given and the following questions will be asked.	
1. Could you think about a point on the parabola for which the parabola is reflected and divided into two congruent parts?	1. Yes, it is the vertex. She might draw the following. 
2. Why did you choose that point? Why not some other points on the parabola?	2. Because folding the parabola at that point brings the legs of the parabola on top of each other making an equal shape. Then there is only one way to do it because at any other point the legs of the parabola do not match on top of each other. That is, the legs have to match with each other because the points on the legs are reflections about the parabola's line of symmetry. Then that line that separates the shape into two equal pieces is the line of symmetry.

Figure C.4. Instruction Plan-Page 4

3. *When you put that parabola on the coordinate system, what characteristics does that point have?	3. It is the minimum or maximum point of the parabola. At that point, the graph or function changes its behavior from increasing to decreasing or from decreasing to increasing.
4. What could you say about the line passing through that point obtained by folding?	4. It is the line of symmetry of the parabola dividing it into two congruent pieces or reflecting all the points on the left leg of the parabola to the points on the right leg of the parabola.
5. What is the relationship between the vertex and the line of symmetry?	5. The vertex of the parabola is on the parabola's symmetry line.
6. Could you define the vertex in your own words?	6. It is the intersection of the parabola and the line of symmetry.
The folding activity ends. The following questions are asked.	
7. *Could you write the algebraic expression of the line of symmetry?	7. The x-coordinate of the vertex also gives the point where the line of symmetry that is parallel to y-axis and perpendicular to the x-axis intersects the x-axis. Therefore, The line of symmetry is the line such that $x = \frac{-b}{2a}$.
8. *What is the relationship between the x-coordinate of the vertex and the line of symmetry?	8. The x-coordinate of the vertex gives the point on the x-axis where the line of symmetry passing through.

Figure C.5. Instruction Plan-Page 5

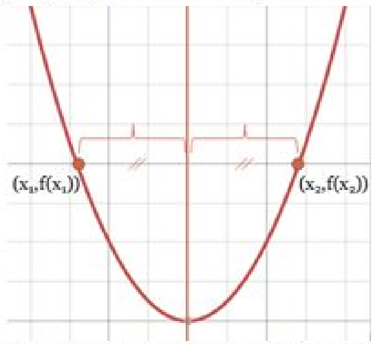
9. *How do you know all the points on the parabola are reflected with respect to the line passing through the vertex? 10. *Think about any point on the parabola on the coordinate system such that $(x_1, f(x_1))$, could you show where the reflection of that point is on the parabola. How do you know?	9. For any point on the parabola, we have a matching point on the parabola when we fold the parabola through the line of symmetry of the parabola which is the graph of the quadratic function f. 10. Let $(x_1, f(x_1))$ is given, then she is expected to draw the figure below.  She might explain that for any point on the parabola, say $(x_1, f(x_1))$, we have another point $(x_2, f(x_2))$ where $f(x_1) = f(x_2)$, so x_1 and x_2 are reflections of each other with respect to the line of symmetry of the parabola which is the graph of the quadratic function f.
Researcher asks the questions:	

Figure C.6. Instruction Plan-Page 6

1. What could you say about the points of intersection between the parabola and the x-axis?	1. They are the roots.
2. What could you say about the line of symmetry when you think of roots' placement on the x-axis?	2. It is passing through the midpoint of the roots which is the x-coordinate of the parabola's vertex. The x-coordinate of the parabola's vertex is $\frac{-b}{2a}$.
3. Could you write the roots algebraically on the parabola?	3. The roots x_1 and x_2 are in the following form: $x_1 = \frac{-b}{2a} + \frac{\sqrt{\Delta}}{2a}$ $\quad$ $x_2 = \frac{-b}{2a} - \frac{\sqrt{\Delta}}{2a}$ She might also draw the figure below:

Figure C.7. Instruction Plan-Page 7

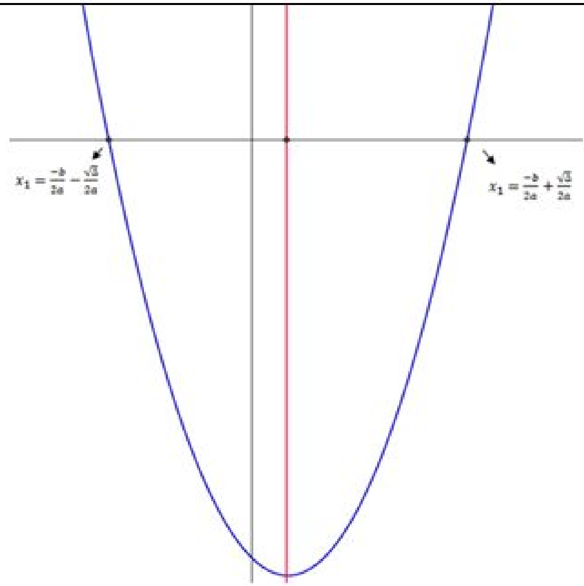
	
Researcher continues with the following questions: 1. Using the algebraic form of the roots how can we verify algebraically that $\frac{-b}{2a}$ is the x-coordinate of the vertex? Researcher lets student find out the answer	1. The expected answer is that the student uses the algebraic form of the roots x_1 and x_2, add them and divides by two to find the algebraic form of the midpoint of the roots since the vertex is on the line of symmetry and the line of symmetry is equidistant to the points having the same function value.

Figure C.8. Instruction Plan-Page 8

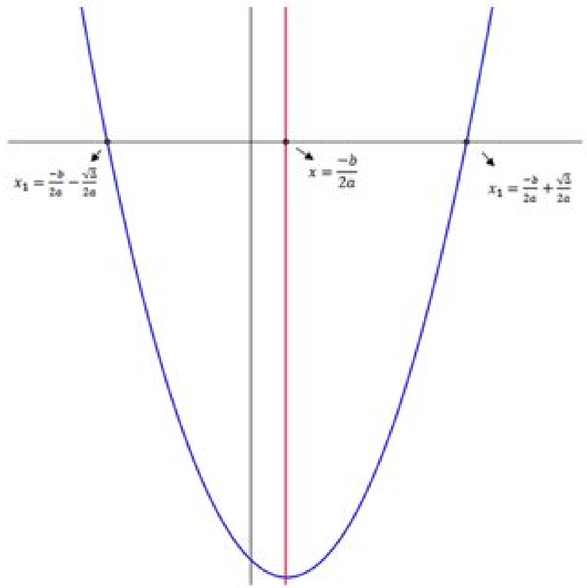
algebraically, then asks the questions:	
2. Could you explain what you have found algebraically? What does $\frac{-b}{2a}$ represent?	2. $\frac{-b}{2a}$ is the midpoint of the roots of the quadratic equation, it is $\frac{x_1+x_2}{2}$ She might draw the following graph. 
3. Using roots, can you show geometrically	3. The expected answer is that she shows it on the parabola as the following.

Figure C.9. Instruction Plan-Page 9

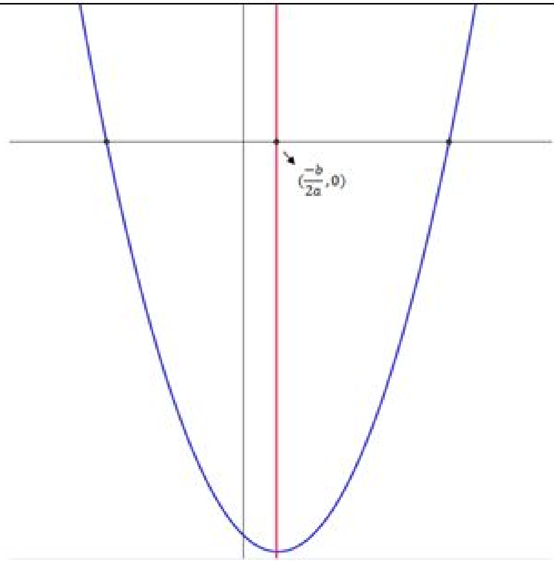
that $\frac{-b}{2a}$ is the x-coordinate of the vertex?	
4. Could you also show the roots analytically on the coordinate system?	4. The expected answer is that she shows it on the parabola as the following.

Figure C.10. Instruction Plan-Page 10

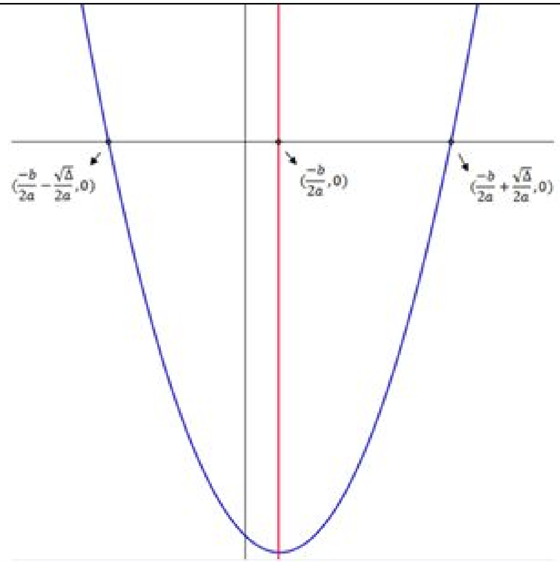
	
After discussing the meaning of $\frac{-b}{2a}$, researcher asks what $\frac{\sqrt{\Delta}}{2a}$ refers to, and asks how one knows that.	She is expected to state that $\frac{\sqrt{\Delta}}{2a}$ is the distance between one of the two roots (left and right) and the x-coordinate of the vertex as shown geometrically in the figure below in order to justify thoughts.

Figure C.11. Instruction Plan-Page 11

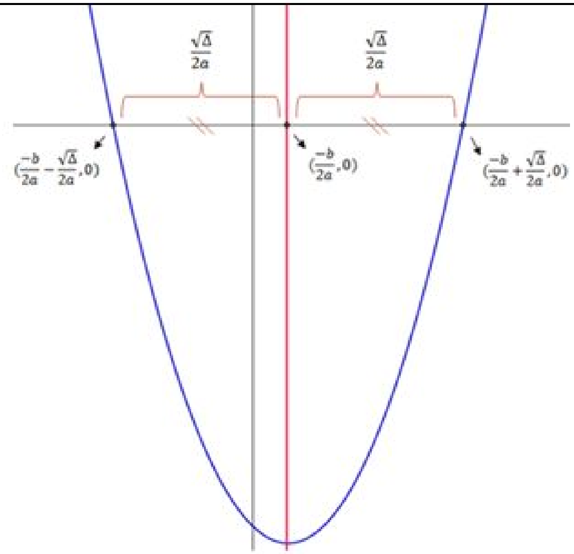
	
Researcher asks that: 1. How many parabola(s) might we draw passing through the same x-coordinate of the vertex? How could you show that?	1. The expected answer is that infinitely many parabolas we can draw which have the same x-coordinate of its vertex. She might draw the figure below.

Figure C.12. Instruction Plan-Page 12

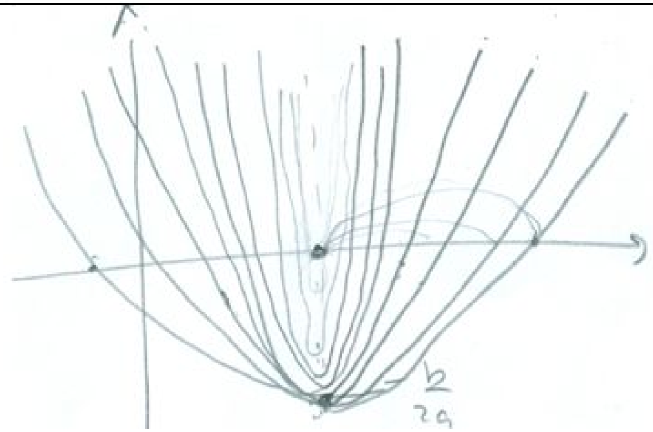
	
2. Can you give some examples of quadratic functions such that they have the same x-coordinate of the vertex? Can you also draw them analytically (on the coordinate system)? Based on the student's own examples, changes in a, b and c is examined graphically through drawing her examples on Desmos, and the following questions are asked;	2. She is expected to give some quadratic functions f such that $f(x) = ax^2 + bx + c$, where a, b and $c \in R$ and $a \neq 0$ that have the same $\frac{-b}{2a}$ even though the values of a, b, and c change.
3. When you think of the graphs of your function examples, what is not changing?	3. The expected answer is that the x-coordinate of the vertex and explicitly the line of symetry do not change.

Figure C.13. Instruction Plan-Page 13

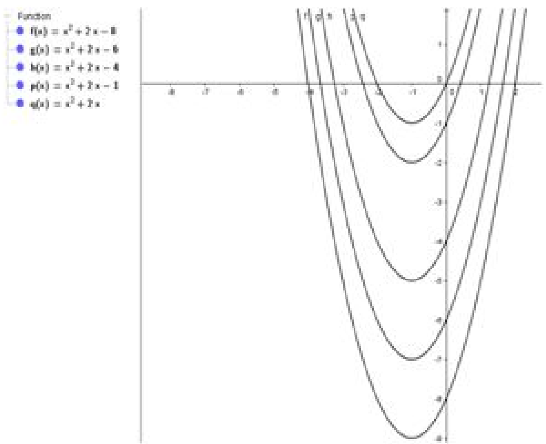
4. What is changing?	4. The expected answer is that the values for $a, b, c,$ in the quadratic equation of the form $ax^2 + bx + c$ are changing along with the roots' distances to the x-coordinate of the vertex for this particular parabola. $\frac{-b}{2a}$ is invariant although the values of a and b can change. $\frac{\sqrt{\Delta}}{2a}$ also changes.
Second Teaching Session (100 minutes)	
Researcher's Questions	**Participants' Activity**
The researcher presents some quadratic functions having the same x-coordinate of the vertex. For these two questions, several examples of quadratic functions are the following; $y = x^2 + 2x - 8$ $y = x^2 + 2x - 6$ $y = x^2 + 2x - 4$ $y = x^2 + 2x - 1$	

Figure C.14. Instruction Plan-Page 14

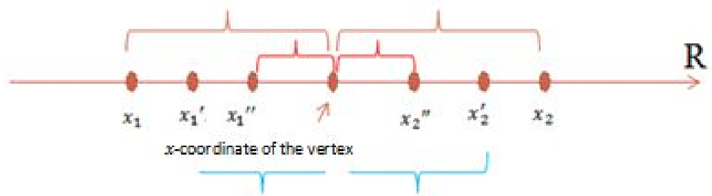
$y = x^2 + 2x$ They are drawn on GeoGebra as in the figure on the right. Then, questions follow:	
5. When you think of these graphs of quadratic function examples, what is not changing? What is changing?	5. Since the values of a and b are the same $\frac{-b}{2a}$ is the same, so the x-coordinate of vertex is invariant. Also, the distances to the roots, which is represented by $\frac{\sqrt{\Delta}}{2a}$, decreases continuously because the value of $\sqrt{\Delta}$ is changing where the value of a stays the same. This means that the value of 'c' is changing. We can also observe that in the quadratic functions: the value of the 'c' are changing from '-8' to '0'.
6. What happens when c changes?	6. The distances of the roots to the x-coordinate of the vertex is changing.
7. Only on the real number line, could you track the changes in the distances of the roots to the x-coordinate of the vertex as roots change?	7. She is expected to draw the following figure. 
8. Based on the positions of the roots on the real number line you drew, how does the distance $\frac{\sqrt{\Delta}}{2a}$ change as c changes ?	8. As we continue with changing the values of 'c' such that the distances $\frac{\sqrt{\Delta}}{2a}$ of the roots from the x-coordinate of vertex *decreases*.

Figure C.15. Instruction Plan-Page 15

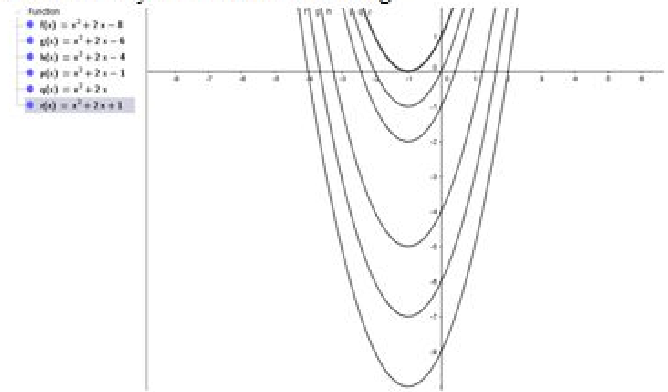
9. If we think that the roots are getting closer to the x-xoordinate of the vertex on the real number linem, how does Δ change ?	9. Discriminant should decrease. The reason is that the value of a does not change and the distances $\frac{\sqrt{\Delta}}{2a}$ of the roots from t the x-coordinate of vertex *decrease*. As the distances decrease further and gets to a point until there is *no distance* between the roots and the x-coordinate of vertex. This algebraically means that $\frac{\sqrt{\Delta}}{2a} = 0$ and it implies Δ=0. Geometrically we draw the following: 
10. How does the existence of two roots relate to the discriminant (Δ)?	10. She is expected to state the following cases for discriminant and its relation to the roots: a. If $\Delta > 0$, there are two real roots. b. If $\Delta = 0$, there is one real root and the root is repeated. c. If $\Delta < 0$, there are two complex roots.

Figure C.16. Instruction Plan-Page 16

11. What does $\Delta = 0$ mean? 12. What does $\Delta = 0$ mean if you think of the roots on the real number line?	11. This algebraically means that $\frac{\sqrt{\Delta}}{2a} = 0$ and implies that there is only one real root which is repeated. 12. There is no distance between the roots and the x-coordinate of the vertex.
Researcher reminds the student that she told there are infinitely many quadratic functions that have the same x-coordinate of the vertex. 1. If we think all those functions' roots cover the real number line, could you write any other quadratic functions that that have the same x-coordinate of the vertex and does not have rel roots? Do the quadratic functions still live? 2. If you write the algebraic form of the roots again as x_1 and x_2, what happens to discriminant after there is no distance between the roots and the x-coordinate of the vertex? 3. What does Δ mean algebraically? 4. What does $\Delta = b^2 - 4ac$ mean to be negative?	The student is expected to state the following. 1. One can continue constructing more quadratic functions with real number coefficients such as $y = x^2 + 2x + 2$, $y = x^2 + 2x + 3$ etc. 2. This algebraically means that the x-coordinate of vertex stays invariant but the value of the delta $\Delta = b^2 - 4ac$ continues decreasing then becomes 0. Discriminant still exists but it starts to take negative values. The values of $\frac{\sqrt{\Delta}}{2a}$ are not defined in the set of real numbers anymore. 3. It is in the form of $\Delta = b^2 - 4ac$. 4. This means the fact that the distances $\frac{\sqrt{\Delta}}{2a}$ of the roots from the line of

Figure C.17. Instruction Plan-Page 17

	symmetry still exist but they do not exist in the real number line any more.
5. How could we re-write $\Delta = b^2 - 4ac$ in terms of a positive number, say k? Let's say $k > 0$, we can write $\Delta = b^2 - 4ac = (-1).k$	
6. What is k in the equation above? What kind of a number is k?	6. k is a positive real number in the form of $k = 4ac - b^2$. k is a positive real number since a, b, c are real numbers.
7. What do you get if you write such discriminant in the algebraic form of the roots of quadratic equations? Then, researcher will tell the student to call $\sqrt{-1}$ as "i" to name them as imaginary since they are different from real numbers (i.e., unreal numbers).	7. So we might get $\frac{\sqrt{\Delta}}{2a} = \frac{\sqrt{(-1)(4ac-b^2)}}{2a} = \sqrt{-1}\frac{\sqrt{(4ac-b^2)}}{2a}$, so we have the roots in the following form $x_1 = -\frac{b}{2a} + \sqrt{-1}\frac{\sqrt{(4ac-b^2)}}{2a}$ and $x_2 = -\frac{b}{2a} - \sqrt{-1}\frac{\sqrt{(4ac-b^2)}}{2a}$.
8. If we think of real number line, discriminant takes negative values after the emergence of only one repeating real roots. Could we show the case of $\Delta < 0$ on the real number line?	8. Once the distances get squeezed on the real number line axis, it gets to a point that no more squeezing is possible. On the other hand, once can continue constructing more quadratic functions with real number coefficients such as $y = x^2 + 2x + 2, y = x^2 + 2x + 3$. That is, the part $-\frac{b}{2a}$ is the real number part and $\sqrt{-1}\frac{\sqrt{(4ac-b^2)}}{2a}$ is the unreal number part, so we cannot write the case of $\Delta < 0$ on the real number line.

Figure C.18. Instruction Plan-Page 18

9. What is $\frac{\sqrt{4ac-b^2}}{2a}$?	9. $\frac{\sqrt{(4ac-b^2)}}{2a} = \frac{\sqrt{(-\Delta)}}{2a}$
10. In this case of $\Delta < 0$, if we try to write the algebraic form of the root in terms of real numbers, what kind of an algebraic expression do we get?	10. Since $\frac{\sqrt{(4ac-b^2)}}{2a} = \frac{\sqrt{(-\Delta)}}{2a}$ is a real number, $x_{1,2} = -\frac{b}{2a} \pm \sqrt{-1}\frac{\sqrt{-\Delta}}{2a}$.
11. When we go back to the real number line, can we show these roots on the real number line?	11. After coming to the point that no more squeezing is possible on the real number line, there exists an unreal part in the newly formed number as a result of discriminant's being negative. That is, they are squeezed enough that as if 'some explosion' occurs and these distance(s) starts leaving in a different world out of real numbers. Hence, we need more than real number line to show those numbers.
12. Since we cover all the real number line, there is no real number to cover. How can we express the unreal part?	12. Those distances do not go anywhere; they just become alive in a different world (an imaginary world) as part of the numbers, the roots. That number as a total is not real since one part of it does not live in the real number world. That is, both the x-coordinate of vertex and the distance of the roots from it together make another number. Since the x-coordinate of vertex is real we can still represent it on the real number line, but we cannot place the roots distances on the real number line anymore.

Figure C.19. Instruction Plan-Page 19

The researcher presents a new imaginary number line which is perpendicular to the real number line. The questions follow: 1. How could you represent these points which are the roots of a quadratic polynomial analytically on this new plane? 2. How did you use + and − in the algebraic form of the roots, that have real and unreal parts and are in the form of $x_{1,2} = -\frac{b}{2a} \pm \sqrt{-1}\frac{\sqrt{-\Delta}}{2a}$, while you place these roots on this new plane? 3. What do addition and subtraction mean for you?	1. I represent them on somewhere out of the real number line. 2. I add and subtract the unreal number to position the roots because the unreal part, $\frac{\sqrt{-\Delta}}{2a}$, gives the distance of the roots to the x-coordinate of the vertex. These signs represent the reflection of the roots with respect to the x-coordinate of the vertex.
The researcher tells that let us say that the numbers in such a form are new numbers. They are called complex numbers and shown as $x_1 = x + \sqrt{-1}.y$ $x_2 = x - \sqrt{-1}.y$ 1. What do x and y refer to?	1. The real part 'x' refers to the x-coordinate of the vertex of a quadratic function that is fixed and 'y' refers to $\frac{\sqrt{(-\Delta)}}{2a}$ where $i\frac{\sqrt{(-\Delta)}}{2a}$ is *the absolute value of which is the distance* of the roots to the x-coordinate of the vertex. Since 'y' refers to $\frac{\sqrt{(-\Delta)}}{2a}$ the absolute value of which is the distance of the roots to the x-coordinate of the vertex, and we add and subtract such distance from the the x-coordinate of the vertex, then there *has to have another root* x_2 of the form of $z = x - iy$. Therefore there exists some numbers that comes out of the roots of quadratic functions. Also, one of the roots x_1 can be written as in the form of

Figure C.20. Instruction Plan-Page 20

	$z = x + iy$ (binomial form) (Usiskin et al. , 2003) such that $z = -\frac{b}{2a} + i\frac{\sqrt{(4ac-b^2)}}{2a}$ and x_2 can be written as $z = x - iy$ such that $z = -\frac{b}{2a} - i\frac{\sqrt{(4ac-b^2)}}{2a}$.
2. Why do x and y have to be real numbers in the form $z = x + iy$?	2. Because $x = \frac{-b}{2a}, y = \frac{\sqrt{(-\Delta)}}{2a}$ and a, b, c are real numbers.
3. x_1 and x_2 are the roots of a quadratic equation, if one of the roots is in the form of $x_1 = x + \sqrt{-1}.y$, then why does the other root have to be in the form of $x_2 = x - \sqrt{-1}.y$?	3. Because of the reflection, or symmetry, of the roots on a parabola with respect to the x-coordinate of the vertex.
4. What do you think about the roots' positions on the complex plane according to each other?	4. They are the reflections of each other with respect to the x-coordinate of the vertex.
5. When you think of the roots which you have written analytically, how could you represent the roots on this complex plane according to discriminant's cases of being positive, zero and negative?	5. She is expected to draw the following graph:

Figure C.21. Instruction Plan-Page 21

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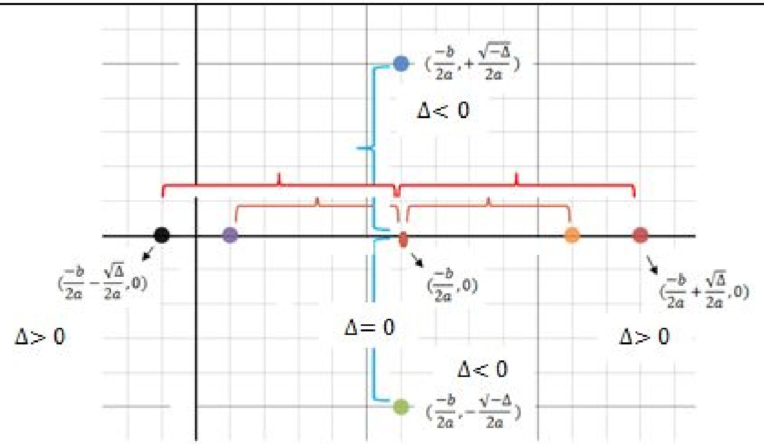
<td>
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- **If we call these new numbers complex numbers, how do you define them?**
- **How do you define these new numbers when you think of the roots of quadratic equations?**

</td>
<td>The elements of the set of Complex numbers are the roots of quadratic functions with real coefficients. That, the set of Complex numbers consists of both numbers of the form $x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$ and others of the form $x_{1,2} = -\frac{b}{2a} \pm \sqrt{-1}\frac{\sqrt{-\Delta}}{2a}$. Since these numbers could be totally real or could have both real and unreal parts. We can show complex numbers with either as a point on the real number line, also on this new plane, $(-\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}, 0)$ or as a point on the new complex plane $(-\frac{b}{2a}, \pm \frac{\sqrt{-\Delta}}{2a})$</td>
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Figure C.22. Instruction Plan-Page 22

APPENDIX D: PHYSICAL CONFIGURATION OF TEACHING SESSIONS

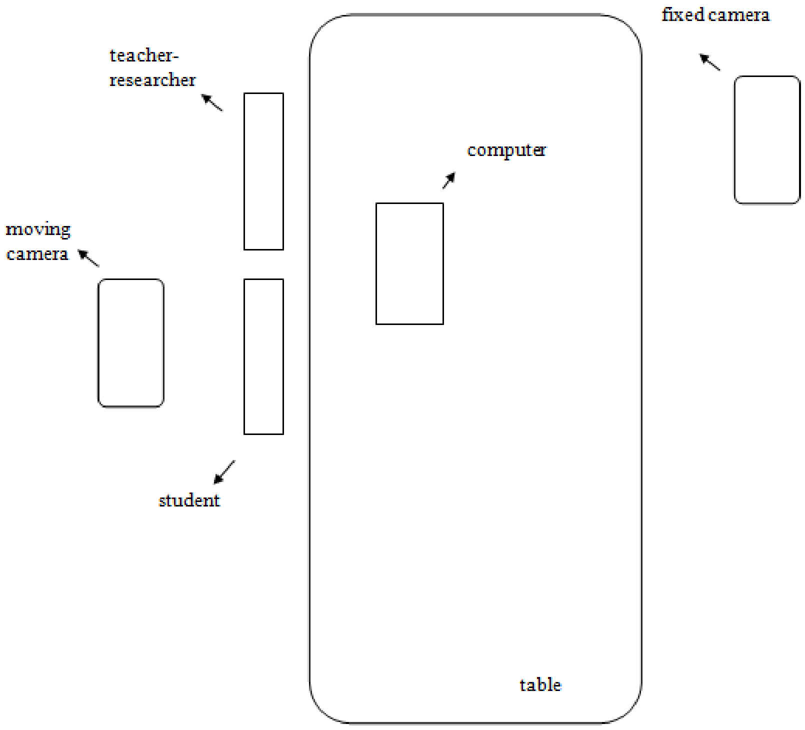

Figure D.1. Physical Configuration

APPENDIX E: DEDUCTION OF QUADRATIC EQUATIONS' ROOTS

$$ax^2 + bx + c = 0 \qquad a, b, c \in R$$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0 \qquad \boldsymbol{a \neq 0}$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\left|x + \frac{b}{2a}\right| = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x_1 + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x_2 + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Figure E.1. Algebraic Deduction of the Roots of Quadratic Equations