COMPETITIVE INVENTORY MODELS UNDER DEMAND SUBSTITUTION

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ABSTRACT

COMPETITIVE INVENTORY MODELS UNDER DEMAND SUBSTITUTION

This work deals with the equilibrium strategies for substitutable product inventory control systems between two retailers in a finite horizon, two and single period cases. We investigate how a dynamic game framework can be used to develop various demand scenarios in a duopoly setting. We also analyze the equilibrium behavior of decentralized supply chains with competing retailers who are treated as independent agents under an effective demand uncertainty. The agents deal with a single product, they are interested in maximizing their own profits, they do not share a common inventory at retail outlets and finally the excess demand is lost. There is no pricing and the competition of the game is the ordering quantities. In this thesis, we generally make finite horizon analyses including a single period result in a Stackelberg game. First, we provide a strategy applicable in a competitive environment. We show the uniqueness of the optimality while parties gave orders at the same time with a condition on the upper bound on the total number of ordering units in a finite horizon case. Furthermore, we show the existence of the optimality in a two-period model with a more general total demand which depends on two random variables. Lastly, we investigate the uniqueness of the Nash equilibirum in a single period model with a customer satisfaction measure. In addition to this, the optimality of the Stackelberg equilibrium is shown with the same customer satisfaction setting.

ÖZET

TALEP İKAMESİ ALTINDA REKABETÇİ ENVANTER MODELLERİ

Bu tez çalışmasında, ikame edilebilen bir mal üzerinden tasarlanan envanter kontrolü modelleri oyun teoretik uygulamalarla ve sırasıyla sonlu-ufuklu, iki-periyotluk ve tek periyotluk zamanlarda incelenmiştir. Farklı talep senaryolarının iki oyunculu bir düzenlemeyle ele alındığı bu tezde, söz konusu senaryoların dinamik oyunlar çerçevesinde nasıl geliştirildiği tartışılmıştır. Merkezi olmayan tedarik zincirlerinde, kesin olmayan etkin talep altında ve rekabet eden perakendeciler eşliğinde denge davranışlarının incelenmesi tüm senaryoların ortak özelliği olarak öne çıkmaktadır. Oyundaki perakendeciler gelirlerini tek bir ürünün satışından elde etmekte ve verilen sipariş kararları, oyuncuların karlarını mümkün olan en yüksek düzeye çıkarmayı hedeflemektedir. Oyuncuların ortak bir envanteri bulunmamaktadır ve talep fazlası sistemde kaybolmaktadır. Modellerde fiyatlamaya yer verilmemiştir ve oyundaki rekabet, ürünlerin sipariş miktarları üzerinden gerçekleşmektedir. Bu tezde yer verilen oyun senaryoları genel olarak tek periyotluk bir Stackelberg oyununu da içeren sonlu-ufuklu analiz senaryoları şeklinde özetlenebilir. Teze, rekabetçi bir ortama uygulanabilen bir strateji sunularak başlanmıştır. Sipariş emirlerinin perakendeciler tarafından aynı anda verildiği ve verilen toplam sipariş miktarının bir üst sınırının olduğu bir ortamda en iyi çözümün tekliği gösterilmiştir. Ardından iki rassal değişkene bağlı daha genel bir etkin talep altında iki periyotluk model çalışılmıştır. En iyi çözümün varlığının ortaya konulduğu bu modelde ayrıca farklı periyotlardaki olası envanter seviyeleriyle, verilen sipariş miktarları arasındaki ilişkiler ortaya konmuştur. Son olarak; bir müşteri memnuniyeti ölçümüne sahip rekabetçi tek periyotluk bir modelde, hem perakendecilerin aynı anda sipariş emri verdikleri düzende hem de bir Stackelberg oyunu düzeninde, yani bir lider ve takipçinin bulunduğu bir halde, en iyi çözümlerin teklikleri gösterilmiştir.

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LIST OF SYMBOLS

b	The salvage value related to customer satisfaction in the fourth chapter,
c_i	Ordering cost per unit for retailer i, where $0 < c_i < s_i$,
D	Stationary and independent total demand which is deterministic for the third and the fifth chapters unlike the fourth chapter in which it is random in the interval $[L, H]$,
f	Continuous density function of the proportion p for retailers I and II,
h_i	Inventory holding cost per unit for retailer i,
i	Index denoting retailers, $i = 1, 2$,
I_i	Inventory level for retailer i in single period formulations,
I_n^i	Inventory level for retailer i at the beginning of the period with n periods to go where $n \neq 1$,
I_1^i	Inventory level for retailer i at the beginning of the final period,
I_0^i	End-of-horizon inventory level for retailer i,
$K(S_{1,n}, S_{2,n})$	A close and bounded strategy space,
n	Decision epochs; $\{N, \dots, 1\}$
p	A random variable denoting the proportion of total demand going to retailer I,
$1 - p$	A random variable denoting the proportion of total demand going to retailer II,
q_n^i	Order decision for retailer i with n periods to go for the replenishment where $q_n^i \geq 0$ and $S_{i,n} = q_n^i + I_n^i$,
r_i	Unit sale price for retailer i,
s_i	Unit penalty cost for retailer i,
S_i	Order-up-to level for retailer i in single period formulations where $S_i \geq I_i$,
$S_{i,n}$	Order-up-to level for retailer i with n periods to go for the first model where $S_{i,n} \geq I_n^i$ and $n \neq 1$,

u, v	Continuous joint density functions of the effective demand which depends on proportions p and the total demand D for retailers I and II respectively.
y_n^i	Order-up-to level for retailer i with n periods to go for the second and the third models where $y_n^i \geq I_n^i$,
γ	The proportion that a customer of retailer I switches to retailer II, given that the product sold by retailer I is sold out,
θ	Discount factor,
Λ_n^i	The effective demand of retailer i with n periods to go. It depends on p in the first and the third models and depends on p and D in the second model,
ν_i	Salvage value per unit for retailer i for the end-of-horizon inventory where $0 < \nu_i < s_i$.

1. INTRODUCTION

This work deals with the equilibrium strategies for substitutable product inventory control systems between two retailers in a finite horizon, two and single period cases. We discuss how a dynamic game framework can be used to develop various demand scenarios in a duopoly setting and we investigate the equilibrium behavior of decentralized supply chains with competing retailers under an effective demand uncertainty.

Our inventory systems consist of two retailers who are treated as independent agents in all three chapters. The agents deal with a single product, and are interested in maximizing their own profits, rather than that of the entire system. For simplicity, it is assumed that each retailer controls one outlet and they have their own inventories. It means none of the retailers share a common inventory (inventory left after realizing the daily demand) at retail outlets and, the excess demand is lost and not observed, there is no pricing and the competition of the game is our ordering quantities in all three chapters.

The first model, in the third chapter, addresses circumstances which meet the criteria to describe a general case of common needs. The market has a deterministic total demand for the single product and two retailers share this demand with random proportions to maximize their profits. The model consists of regular revenue and cost parameters such as sale price, ordering and holding costs. Retailers decide their ordering decisions at the beginning of each and every period and the game is played. They can also salvage their end-of-horizon inventory when the game is over. In this setting, Nagarajan and Rajagopalan [16] showed that there is an “if and only if” statement between those two conditions: the quantities ordered by two retailers are above a threshold and the total quantity ordered is larger than the total market demand. Since “if and only if” condition asserts that both statements are equally true, by using the second sentence, they showed that retailers ignore the substitution effect all together and implement monopolistic strategies (independent order-up-to-policies) as the unique equilibrium. Here, we are first interested in the reverse, that is, a lower total quantity ordered than the total demand.

Nagarajan and Rajagopalan's [16] setting, i.e. ordering decisions larger than the maximum level of market demand, may be a common behaviour for big retailers such as Carrefour or Migros which sell perishable products in the fast moving consumer goods (FMCG) sector. However, that is generally not the case for small firms which especially sell expensive products such as tablet PC's or high-end cell phones because of continuous improvement in technology and the needs of high turnover rate of inventory for that kind of business. Therefore, those companies can choose to give inadequate orders to reduce the shelf-life duration even though they consider the total demand. To analyze the above case, we will consider any level of inventory and concentrate on the ordering decisions in the first model. We characterize the equilibrium order quantities and ordering policy under certain assumptions when two firms make inventory decisions for a product that can be substitutable by customers. We formulate the single period game and we extend it to a finite horizon dynamic game by backward induction and using feedback Nash equilibria.

In game theory, Nash equilibrium [17] is a very common objective shared by players. It is a stable state of a system that involves several interacting participants, here we have two of them, in which no participant can gain by a change of strategy as long as all the other participants remain unchanged. A feedback strategy of any player i, is a decision rule, S_t^i, such that it is continuous in time t, here in periods, and uniformly Lipschitz for all S in t, here means boundedness. Those two definitions provide us the definition of the feedback Nash equilibrium as an n-tuple of feedback strategies: $\{S_t^{1*}, S_t^{2*}, ..., S_t^{n*}\}$.

In this dynamic process, we use a closed-loop setting, i.e. all past play is a common knowledge and decisions depend on historical data of effective demand and inventory left over at the beginning of any period. The reason of this choice is coming from a result by Başar and Olsder [2]; every feedback Nash equilibrium solution of an N-person discrete time infinite dynamic game is a closed-loop no-memory Nash equilibrium solution. We show the subgame perfect Nash equilibrium (SPNE) of the game by using the first order conditions and sequential relation where a subgame is that it, when seen in isolation, constitutes a game in its own right but it must reflect the strategies available to players in the larger game of which it is a subset. It must contain an initial node that is a singleton information set. When the initial node of a subgame is reached in a larger game, players can concentrate only on that subgame; they can ignore the history of the rest of the game. In addition to this, a strategy profile of the i^{th} player, $(S^i_1, \ldots, S^i_n)$, is a SPNE if it induces a Nash equilibrium in every subgame. Since every SPNE is a Nash equilibrium by definition and SPNE is equal to the set of Nash equilibrium in games of perfect information, they can be taken as the feedback Nash equilibrium in every subgame. In this chapter, the existence and uniqueness of the feedback Nash equilibrium are shown in a finite horizon for the case where substitution effect is significant and with a closed-loop setting. A state dependent order-up-to policy turns out to be a Nash equilibrium of this game, adding to a previous study where the order-up-to-policy was taken as exogenously specified [1]. In the second model, the fourth chapter, we deal with the equilibrium strategies for substitutable product inventory control systems with a stochastic demand and proportions in a two-period stationary environment between two retailers. This stationary scenario can be viewed as a dynamic game in a duopoly setting. We assume that demand and proportions are independent random variables.

The following examples motivate the strategies to be formulated in the second model.

(i) Consider a retailer placing an order for a new product in spring and summer seasons. He has some inventory and has to decide how many products will be sold before realizing the demand for the following two periods. Ordering decisions are given at the beginning of the related periods.

(ii) Consider a high-end product seller who wants to determine his inventory which consists of very expensive items at the beginning of a two-period season. Backlogging is not allowed and unsold items have very low salvage values at the end of the season. Therefore, his ordering levels should be a reasonable decision to satisfy his customers and also a good forecast to minimize the lost sales in any period.

(iii) Consider two firms selling perishable products compete with each other. Since the level of customer loyalty is not very high on perishable goods, inventory adequacy has a decisive role to maximize the profit. So, an order-up-to-level decision is an appropriate response to his competitor(s)' decisions and should satisfy all customers including the substituted ones.

These three examples give an insight of the decision types that are the foci of the fourth chapter. Two independent retailers deal with a single product again as in the third chapter, and are interested in maximizing their own profits, rather than that of the entire system. We allow the transfer of some portion of the unsatisfied customers between retailers within the same period. This portion can be defined as exogenously and depends on psychological circumstances and practical needs of customers. The game lasts two periods and retailers review their ordering strategy at the beginning of any period. We again use a closed-loop setting in this dynamic decision process and as an addition to the third chapter's setting we do not use a restriction on decisions such as upper bounds on total orderings as in [16], and special characterizations such as base-stock as in [1].

All of the papers except [16] and [18] in the literature operate under the assumption that there is either a single period or a constant order-up-to level. Only Nagarajan and Rajagopalan [16] and Olsen and Parker [18] worked in a finite horizon environment and they have first, a restriction on ordering decisions that we dealt with the contrary case and second, conditions on open loop and closed loop settings respectively. In the fourth chapter, the existence and uniqueness of the Nash equilibrium in the single period game have been taken from Parlar [21] while considering the characteristics within our model framework. Then, the existence and uniqueness of the Nash equilibrium with two periods to go are shown by using the feedback Nash equilibrium coming from the solution of the last period where substitution efiect is significant and under some density function assumptions. We show the existence of the Nash equilibrium in this two-period game by showing the existence and uniqueness of each player's best response function by finding an action that maximizes its payoff for any given action of the other player. These individual equilibria are found by using first and second order conditions. Later, we also suggest a threshold for the ordering decisions of the retailers with two periods to go in terms of parameters and inverse distribution of the effective demand. This threshold can be observed with the last period inventory level. We show that retailers order decisions with two periods to go are inversely proportional when inventory level of the last period is above this threshold in the closed-loop and dynamic setting. Then, we consider the uniqueness of the Nash equilibrium by using the contraction mapping argument and diagonal dominance as in [25] and [6] under a uniformly distributed effective demand assumption. As a result, a state dependent order-up-to policy without a decision restriction such as lower or higher than total demand turns out to be a Nash equilibrium of this game. The fourth chapter analysis continues with the comparative statics. We consider the optimal ordering levels versus the cost and revenue parameters.

In the fifth and the last chapter, we consider the single period formulations of a substitutable product inventory control with a customer satisfaction measure. Every retailer has an inventory level and a given customer satisfaction level at the beginning of the period. They try to maximize their own profit during the period and they want to finish the period with the highest customer satisfaction level as much as possible to increase the market value of their own company. This measure is related to satisfy the effective demand which consists of their own customers and/or substituted demand from the other retailer within the period. It is defined as a piecewise function due to the different realizations of the effective demand. This model refers to a brand new business structure which can be seen mostly in on-line marketing and multimedia sectors. Especially, small internet marketing firms are founded to find a virgin territory and to launch their products online rather than traditional ways. Boosting up the sales and brand awareness bring in a high working capital and corporate value. Then, within months or seasons, those kinds of short-life corporates are sold to multi-national chains or holdings with a high customer brand loyalty or attitude toward the brand due to brand awareness and customer satisfaction. Here, we try to analyze a two-person game emphasizing the eﬁect of customer satisfaction for that kind of business. Hall and Porteus [10] made an approach in their work to a finite horizon game with service competition and they analyzed the uniqueness of the Nash equilibrium. While the service sector is considered, they did not deal with the inventory level as a state variable. Therefore they analyzed the finite horizon case with a vanishing term which brings an independence among periods. Here we work on a single period game with a state variable, inventory level and we show the existence of Nash Equilibirum. We also make the comparative statics of the optimal ordering level and the customer satisfaction and we give a positive relevance between them. In the last section of this chapter, we analyze the Stackelberg game which consists of two players, a leader and a follower, with customer satisfaction measures. They compete with each other and unlike the Cournot game which contains simultaneous moves, their moves are sequential. Corporates may engage in Stackelberg competition if one has some sort of advantage enabling it to move first. The leader has commitment power. Moving first is the most common commitment: once the leader has made its move, he cannot undo it - he is committed to that action. The leader, here is retailer II, chooses the quantity first and the follower, retailer I, responds to this move by placing an order while the leader knows that the follower observes his action. We show the Stackelberg solution of the Stackelberg model by using the first order conditions and sequential relation between players. We first investigate the leader's move while taking into consideration of the follower's best response. First order conditions of the expected profit function of retailer II, the leader, give us an explicit expression in terms of retailer I's, the follower, best response. Then, finally, we state the Stackelberg solution pair in terms of parameters and inverse cumulative functions and, we show the relative advantages of the leader and the follower with a customer satisfaction measure.

To summarize, our work has the following contributions to dynamic games on stochastic inventory control:

(i) Substitution effect is important when total ordering units are less than the deterministic demand level in a dynamic finite horizon duopoly.

(ii) We introduce an ordering policy and, show that competitors have a unique feedback Nash equilibrium in a finite horizon dynamic game.

(iii) In a two-period dynamic duopoly with the effective demand which depends on total random demand and proportions, we determine an ordering policy where substitution effect is important and we show that there exists a unique feedback Nash equilibrium for all parties during periods.

(iv) We determine a threshold related to the last period inventory level and this addresses the retailers' ordering behavior with respect to each other.

(v) The equilibrium point of the two-period duopoly game is sensitive to cost and revenue parameters. It is increasing with respect to sale price and salvage value but decreasing with respect to holding cost and ordering cost.

(vi) There exists a unique Nash equilibrium in a single period closed-loop Cournot-like game with a customer satisfaction measure and this equilibrium point is increasing with customer satisfaction at the beginning of the period.

(vii) Existence of the optimality is shown in a single period Stackelberg game with a customer satisfaction measure and a state variable, inventory.

(viii) We show the leader's advantage in the Stackelberg setting.

2. LITERATURE REVIEW

Substitutable product inventory problem was first studied by McGillivray and Silver [13] in the Economic Order Quantity (EOQ) context. Later, Parlar and Goyal [22] gave single period formulations for an inventory system with two substitutable products independent of each other. Also Khouja and Mehrez and Rabinowitz [11] formulated a two-item newsboy problem with substitution, but they identified the optimality with a Monte Carlo simulation without an analytical solution. In [21], Parlar proposed a Markov Decision Process model to find the optimal ordering policies for perishable and substitutable products from the point of view of a single retailer. Parlar also studied in [20] a game theoretic analysis of inventory control under substitutable demand. He modeled a two-product single period problem as a two-person nonzero-sum game and showed that there exists a unique Nash equilibrium. All the chapters in this thesis, we deal with a single product similar to that in Pasternack and Drezner's [23] work. Although they considered a stochastic model for two products with a single-period inventory structure, those products can be used as substitutes for each other should the need arise. They proved that the expected profit function is concave to find optimal stocking levels for the two products in a single period with different revenue levels. Netessine and Rudi [25] considered centralized and competitive inventory models with demand substitution. They used deterministic proportions for unsatisfied demand and showed that the total profit is decreasing in demand correlation when demand is multivariate normal. McKelvey and McLennan reviewed in [14] the methods for numerical computation of Nash equilibria for finite period n-person games. They showed that none of the methods which were described were capable of characterizing the entire set of Nash equilibria but only a method which derive from the theory of semi-algebraic sets were required for finding all equilibria. A semi algebraic set is a subset S of $\mathbb{R}^n$ for some real closed field R defined by a finite sequence of polynomial equations and inequalities, or any finite union of such sets. First and second order conditions of the models that we study in this thesis are elements of semi algebraic sets. Papadimitriou and Roughgarden [19] initiated a systematic study of algorithmic issues involved in finding Nash and correlated equilibria in games with a large number of players. They presented a polynomial-time algorithm for finding a Nash equilibrium in symmetric games without an algebraic approach. Güllü, Houtum, Sargut and Erkip [9] analyzed a decentralized supply chain consisting of a supplier and two independent retailers. Retailers can cooperate at the end of the supplier lead time and adjust their orders. They proved the existence of a unique Nash equilibrium for the retailer order-up-to levels. Avşar and Baykal-Gürsoy [1] showed that competition between retailers for a substitutable demand leads to a Nash equilibrium characterized by a pair of stationary base stock strategies which are expressed by constant order-up-to-levels for the infinite horizon problem. Nagarajan and Rajagopalan [15] examined the nature of optimal inventory policies in a system where a retailer manages substitutable products. They showed that a basestock policy is optimal in a multi-period problem for known D. After that, they [16] also showed in their later work that retailers can ignore the substitution effect all together and implement monopolistic strategies (independent order-up-to-policies) as the unique equilibrium when the total ordering units are above a threshold. They made a finite period analysis and showed the uniqueness of the Nash equilibrium with a lower bound on total ordering units. Fudenberg and Levine [8] showed a result about the comparison of static and dynamic approach in their work coming from 1988. They showed on deviations of optimal reactions in dynamic games. They investigated the conditions in which the closed-loop and open-loop equilibria are approximately the same. Finally, Olsen and Parker [18] analyze a dynamic duopoly with stockout-based substitution in a closed-loop infinite horizon setting. They give conditions for the stationary infinite horizon (open-loop) equilibrium to be a Markov perfect (closed-loop) equilibrium. In a static game setting, ordering decisions are given once at the beginning of the game and they closely depend on each other. Therefore we confront with algebraic dificulties to define the comparative relation between periodic decisions when we want to define the difierences between static and dynamic settings in our work. This comparison is left for further study.

3. DUOPOLY COMPETITION UNDER DEMAND SUBSTITUTION IN FINITE HORIZON

3.1. Definitions and the Model

A policy defines the action to be chosen in every state; it is a mapping from the set of states to the set of actions. An immediate reward, which may be positive, negative or zero, is earned in transitioning from one state to another under the influence of an action. The performance metric of a policy is usually a function (the objective function) of the immediate rewards earned when the associated policy is followed over a pre-determined time horizon. The time horizon could be finite or infinite, depending on what is intended by the designer of the system.

3.2. The Single Period Model

Total demand D is deterministic and is shared among the two retailers with proportions. The proportion, p is a random variable and p and $1-p$, are the demand shares that belong to retailer I and II respectively. We drop the time index as there is only a single period. There is an inventory holding cost h_i paid per unit by retailer i if excess inventory occurs. There is also a penalty cost s_i charged to both of the retailers and it is based on the unsatisfied customers who are not willing to substitute when they face a stockout and those who substitute but can not find the other product. We assume the beginning inventory $I_i \geq 0$. Any retailer decides his order decision as q_i for the replenishment with respect to the estimated demand and brings his inventory position to S_i. Then the demand is realized. Costs are accrued and profit or losses are collected for the single period. For each different product sold by different retailers, substitution between these products may occur. Retailers have to decide on their order quantities simultaneously in this competitive environment as they try to maximize their expected profit. We begin with the single period formulations.

The single period profit function is:

$$\begin{aligned} k_1^1(S_1, S_2, I_1, I_2) &= -c_1 q_1 + r_1 \min\big(pD + \gamma[(1-p)D - S_2]^+, S_1\big) \\ &\quad + (-h_1 + \theta v_1)\,[I_{1,0}]^+ - s_1\,[pD - S_1]^+ \end{aligned} \tag{3.1}$$

where $[x]^+$ denotes $max(0, x)$; $I_{1,0}$ denotes the end-of-horizon inventory which is $S_1 - pD - \gamma[(1-p)D - S_2]^+$, and is equal to any real number; the first term is the cost of goods replenished, the second term is the revenue from sales, the third term is the holding cost with the discounted and salvaged end-of-horizon inventory and the fourth term is the penalty cost per unit demand. Superscripts and subscripts in k are for the retailers and the periods respectively.

The single period expected profit function for retailer I is found by taking expectations. Here, we consider order-up-to-levels which make the expected end-of-horizon inventory zero, expected profit for the single period can be written as follows:

$$\begin{aligned} & E_p\left[k_1^1(S_1, S_2, I_1, I_2)\right] = \\ & \quad - c_1 q_1 + r_1 E_p\left[\min\big(pD + \gamma[(1-p)D - S_2]^+, S_1\big)\right] \\ & \quad + (-h_1 + \theta v_1) E_p\left[\left[S_1 - pD - \gamma[(1-p)D - S_2]^+\right]^+\right] \\ & \quad - s_1 E_p\left[[pD - S_1]^+\right] \end{aligned} \tag{3.2}$$

Since p is the share of market demand for retailer I, the single period expected profit function for retailer I per unit demand can be written by dividing all terms by D:

$$\begin{aligned} E_p\left[\frac{k_1^1}{D}\right] &= \Pi_1^1\left(\frac{S_1}{D}, \frac{S_2}{D}\right) \\ &= -c_1 \frac{(S_1 - I_1)}{D} + r_1 E\left[\min\left(p + \gamma\left[1 - p - \frac{S_2}{D}\right]^+, \frac{S_1}{D}\right)\right] \\ &\quad + (-h_1 + \theta v_1) E\left[\left[\frac{S_1}{D} - p - \gamma\left[1 - p - \frac{S_2}{D}\right]^+\right]^+\right] \\ &\quad - s_1 E\left[\left[p - \frac{S_1}{D}\right]^+\right] \end{aligned} \tag{3.3}$$

Here, expectations are taken over random proportion p. Players both follow general strategies in the single period game denoted as S_1 and S_2 respectively.

In order to understand the dynamics of the problem, first of all two possible cases should be considered for the given demand figure D. First case is $S_1 + S_2 \geq D$. Nagarajan and Rajagopalan [16] used newsvendor ratios which express the best responses in terms of parameters and a threshold which is the minimum value of the newsvendor ratio at which $S_1+S_2 \geq D$ in [16]. They took D as random and they showed in their study that there is an "if and only if" statement between newsvendor ratios higher than the above threshold and the order-up-to-levels among best responses under some distributions on D. As we mentioned in the introduction, they also showed that the equilibrium order quantities are such that the retailers can ignore the substitution demand and their competitor's decision under those assumptions in that case. The second case is $S_1 + S_2 < D$. In this case, substitution demand can not be fully satisfied. They [16] define a p-value at which excess inventory of retailer I is equal to the inventory level when substitution occurs from retailer II as $\frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma}$ and this p-value is coming from the equation $S_1 = pD + \gamma\left[(1-p)D - S_2\right]^+$. A similar p-value for retailer II is $1 - \frac{\hat{S}_2}{D} = \frac{1-\frac{S_2}{D}-\gamma\left(\frac{S_1}{D}\right)}{1-\gamma}$ and it is coming from the equation $S_2 = (1-p)D + \gamma\left[pD - S_1\right]^+$. Here, $\frac{\hat{S}_1}{D} \to \frac{S_1}{D}$ and $1 - \frac{\hat{S}_2}{D} \to 1 - \frac{S_2}{D}$ in case of the limiting situation: $\frac{S_1}{D} + \frac{S_2}{D} \to 1$. Since we study the case of $S_1 + S_2 < D$ then $\frac{\hat{S}_1}{D} \leq \frac{S_1}{D}$, $1 - \frac{\hat{S}_2}{D} \geq 1 - \frac{S_2}{D}$ and $\frac{\hat{S}_1}{D} \leq 1 - \frac{\hat{S}_2}{D}$. We will show that substitution and competitor's decision which do not exist in the case of [16] are important.

The nonnegative random variable p, $p \in [0,1]$, has a continuous density function, f, with a finite expectation. The cumulative and complementary cumulative functions will be denoted by F and $\overline{F} = 1 - F$ respectively where $F(0) = 0$ and $F(1) = 1$.

We, first, consider concavity of the single period expected profit function for the case $S_1 + S_2 < D$. Contrapositive of the "if and only if" statement in [16] which was mentioned in the previous paragraph, gives us the relation between best responses and the threshold $S_1 + S_2 < D$. We analyze the case under this information (i.e., when the total inventory presented to the market is less than the potential maximum deman in the market). First order conditions give us that the optimal ordering decision depends on substitution in that case. Concavity gives us the existence of unique response for retailer I(II) to maximize the expected profit function for all possible strategies of retailer II(I). We will give an explicit expression for the single period formulation of the expected payoff of retailer I from Equation 3.3 by taking expectations.

$$\begin{aligned} E_p\left[\frac{k_1^1}{D}\right] &= \Pi_1^1(\frac{S_1}{D},\frac{S_2}{D}) \\ &= \int_0^{\hat{\alpha}_1}\Bigg[r_1p_1 + r_1\gamma(1-p_1-\frac{S_2}{D}) \\ &\qquad\qquad + (-h_1+\theta v_1)\left(\frac{S_1}{D}-p_1-\gamma\left(1-p_1-\frac{S_2}{D}\right)\right)\Bigg]f(p)\,dp \\ &\quad + \int_{\hat{\alpha}_1}^{1-\beta}\left[r_1\frac{S_1}{D}-s_1\left(\gamma\left(1-p_1-\frac{S_2}{D}\right)-\left(\frac{S_1}{D}-p_1\right)\right)\right]f(p)\,dp \\ &\quad + \int_{1-\beta}^{1}\left[r_1\frac{S_1}{D}-s_1\left(p_1-\frac{S_1}{D}\right)\right]f(p)\,dp - c_1\frac{(S_1-I_1)}{D} \end{aligned} \tag{3.4}$$

where

$$\hat{\alpha}_1 = \frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma} \tag{3.5}$$

and

$$1-\beta = 1-\frac{S_2}{D} \tag{3.6}$$

The above integrals represent three cases. First; retailer II has a stockout but retailer I has inventory left over after satisfying his own demand and substitution demand coming from retailer II. Second integral is the case where both retailers are stocked out. The last one is where retailer I can not meet his own demand while retailer II satisfies his own customers' demand. $\hat{\alpha}_1$ is a specific proportion and expresses the excess inventory of retailer I at which the substitution occurs from retailer II. $1-\beta$ denotes the same proportion for retailer II. Now the first step will be the existence of best responses which are:

$$R_1^i\left(\frac{S_i}{D},\frac{S_j}{D}\right) = \arg\max_{S_i \geq I_i} \Pi_1^i(\frac{S_i}{D},\frac{S_j}{D}) \tag{3.7}$$

where superscript in R is for the retailers and $i, j = 1, 2$ and $i \neq j$.

Parlar [20] proposed the uniqueness of the Nash equilibrium in the single period case. We use this result in our case where total ordering units is less than the maximum level of market demand.

Proposition 3.2.1. *If* $S_1 + S_2 < D$ *then the best response function for any retailer is unique in the single period two-player game where* p *is random and in* $[0, 1]$ *then Retailer I's best response,* $R_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)$*, can be expressed implicitly in terms of the cumulative function of proportions as:*

$$F(\hat{\alpha}_1)\frac{1}{D}(-r_1 - s_1 - h_1 + \theta v_1) + \frac{1}{D}(r_1 + s_1 - c_1) = 0. \tag{3.8}$$

where $\hat{\alpha}_1 = \frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D} - \gamma\left(1 - \frac{S_2}{D}\right)}{1-\gamma}$.

As a special case where the proportion density function, f*, is uniform in* $[0, 1]$*, the best response of retailer I can be expressed by parameters and the order decision of retailer II,* S_2*, explicitly as*

$$R_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right) = \gamma(D - S_2) + \frac{[D(1-\gamma)(r_1 + s_1 - c_1)]}{(r_1 + s_1 + h_1 - \theta v_1)} \tag{3.9}$$

Proof. First derivative of $\Pi_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)$ is:

$$\begin{aligned}\frac{\partial \Pi_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1} &= \frac{1}{1-\gamma}\Bigg[r_1\left(\frac{\hat{S}_1}{D} + \gamma\left(1 - \frac{\hat{S}_1}{D} - \frac{S_2}{D}\right) - \frac{S_1}{D}\right) \\ &\quad + s_1\left(\gamma\left(1 - \frac{\hat{S}_1}{D} - \frac{S_2}{D}\right) - \frac{S_1}{D} + \frac{\hat{S}_1}{D}\right) \\ &\quad + (-h_1 + \theta v_1)\left(\frac{S_1}{D} - \frac{\hat{S}_1}{D} - \gamma\left(1 - \frac{\hat{S}_1}{D} - \frac{S_2}{D}\right)\right)\Bigg] f\left(\frac{\hat{S}_1}{D}\right)\end{aligned}$$

$$+\int_0^{\hat{\alpha}_1} (-h_1+\theta v_1)\frac{1}{D} f(p)\,dp + \int_{\hat{\alpha}_1}^{\beta} \left(r_1\frac{1}{D}+s_1\frac{1}{D}\right) f(p)\,dp$$
$$+\int_{\beta}^{1} \left(r_1\frac{1}{D}+s_1\frac{1}{D}\right) f(p)\,dp - c_1\frac{1}{D} \tag{3.10}$$

Since

$$\frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma} \tag{3.11}$$

then,

$$\frac{S_1}{D}-\frac{\hat{S}_1}{D}-\gamma\left(1-\frac{\hat{S}_1}{D}-\frac{S_2}{D}\right)=0 \tag{3.12}$$

so the final form of Equation 3.10 is the following:

$$\frac{\partial \Pi_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1} = \frac{-h_1+\theta v_1}{D}\left[F(\hat{\alpha}_1)-F(0)\right]+(r_1+s_1)\frac{1}{D}\left[F(1)-F(\hat{\alpha}_1)\right]-c_1\frac{1}{D} \tag{3.13}$$

The properties of cumulative functions provide: $F(1) = 1$ and $F(0) = 0$, then the above expression is

$$\frac{\partial \Pi_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1} = F(\hat{\alpha}_1)\frac{1}{D}\left(-r_1-s_1-h_1+\theta v_1\right)+\frac{1}{D}(r_1+s_1-c_1) \tag{3.14}$$

where $\hat{\alpha}_1 = \frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma}$. Solution of the equation $\frac{\partial \Pi_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1} = 0$ with respect to S_1 provides the existence of best response of retailer I. Then the best response $R_1^1\left(\frac{S_1}{D},\frac{S_2}{D}\right)$ can be expressed implicitly in terms of parameters and the cumulative function of proportions by using Equation 3.14 as:

$$F(\hat{\alpha}_1)\frac{1}{D}\left(-r_1-s_1-h_1+\theta v_1\right)+\frac{1}{D}(r_1+s_1-c_1)=0 \tag{3.15}$$

A similar analysis gives the best response of retailer II as:

$$F(\hat{\beta}_1)\frac{1}{D}\left(-r_2-s_2-h_2+\theta v_2\right)+\frac{1}{D}(r_2+s_2-c_2)=0. \tag{3.16}$$

Then the best response pairs for the single period two-player non-zero sum game are the zeros of Equation 3.15 and Equation 3.16. For our case, we determine the best response pairs of which their sum is less than or equal to demand level in the strategy space K by using $\hat{\alpha}_1 = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma}$ and $\hat{\beta}$ from Equation 3.15 and Equation 3.16 respectively.

$$R_1^1 = \gamma(D - S_2) + D(1-\gamma)F^{-1}\left(\frac{r_1 + s_1 - c_1}{r_1 + s_1 + h_1 - \theta v_1}\right) \tag{3.17}$$

$$R_1^2 = (D - \gamma S_1) - D(1-\gamma)F^{-1}\left(\frac{r_2 + s_2 - c_2}{r_2 + s_2 + h_2 - \theta v_2}\right). \tag{3.18}$$

As a special case, if the proportion density function, f, is uniform in $[0,1]$ then the best response function of retailer I can be expressed by parameters and the order decision of retailer II as in Equation 3.11

$$F(\hat{\alpha}_1) = \frac{(r_1 + s_1 - c_1)}{(r_1 + s_1 + h_1 - \theta v_1)} \tag{3.19}$$

Since f is uniformly distributed in $[0,1]$ then $F(a) = a, \forall a \in [0,1]$. Therefore $F(\hat{\alpha}_1) = \hat{\alpha}_1$ and Equation 3.15 is:

$$\frac{S_1}{D} - \gamma\left(1 - \frac{S_2}{D}\right) = \frac{(1-\gamma)(r_1 + s_1 - c_1)}{(r_1 + s_1 + h_1 - \theta v_1)} \tag{3.20}$$

Then the best response of retailer I under the uniformly distributed density function assumption is:

$$R_1^1\left(\frac{S_1}{D}, \frac{S_2}{D}\right) = \gamma(D - S_2) + \frac{[D(1-\gamma)(r_1 + s_1 - c_1)]}{(r_1 + s_1 + h_1 - \theta v_1)} \tag{3.21}$$

Here, the best response pairs of which their sum is less than or equal to demand level can be chosen by using above right hand-side ratio where $\hat{\alpha} = \frac{\hat{S}_1}{D} = \frac{\frac{S_1}{D}-\gamma\left(1-\frac{S_2}{D}\right)}{1-\gamma}$.

Now, we will show that the best response is unique by the concavity of the single period expected profit function. Second derivative of Equation 3.3 with respect to S_1

is equal to:

$$\frac{\partial^2 \Pi_1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1^2} = \frac{1}{1-\gamma}\frac{1}{D^2} f(\hat{\alpha})(-r_1 - s_1 - h_1 + \theta v_1) \tag{3.22}$$

Since $\hat{\alpha} > 0$, $h_1 > v_1$ and $\theta < 1$ then

$$\frac{\partial^2 \Pi_1\left(\frac{S_1}{D},\frac{S_2}{D}\right)}{\partial S_1^2} < 0 \tag{3.23}$$

So that the single period expected profit function is concave. □

Since retailers are assumed to be rational decision makers then our strategy space, K, consists of finitely many points and is said to be a compact space. The game is symmetric and payoff functions are continuous and quasiconcave by Proposition 3.2.1 (concavity implies quasiconcavity) with respect to each player's own strategy then there exists at least one symmetric pure strategy Nash equilibirium in the game by [6].

Proposition 3.2.2. *There exists a pair* (S_1^*, S_2^*) *as the Nash equilibrium of the game.*

Proof. Proof is completed by the compactness of the strategy space K and concavity of the expected profit function proven by Proposition 3.2.1. □

Now, we show uniqueness of the Nash equilibrium by using a contraction mapping argument.

Proposition 3.2.3. *There exists a unique Nash equilibrium for the single period two-player non-zero sum game.*

Proof. The proof is completed by the Implicit Function Theorem in [5]. The following inequality which is called "diagonal dominance" should be satisfied on response functions for the uniqueness of the Nash equilibrium where π_k is an arbitrary response function which belongs to k^{th} player:

$$\sum_{i=1, i\neq k}^{n} \left| \frac{\partial^2 \pi_k}{\partial x_k \partial x_i} \right| \leq \left| \frac{\partial^2 \pi_k}{\partial x_k^2} \right|, \quad \forall k. \tag{3.24}$$

Equation 3.14 is used for the cross partial derivative of the single period formulations. Cross partial derivative is:

$$\frac{\partial^2 \Pi_1}{\partial S_1 \partial S_2} = \frac{\gamma}{1-\gamma} \frac{1}{D^2} f(\hat{\alpha})(-r_1 - s_1 - h_1 + \theta v_1). \tag{3.25}$$

By taking the absolute value, the left hand side of Equation 3.2.3 is:

$$\left| \frac{\partial^2 \Pi_1}{\partial S_1 \partial S_2} \right| = \frac{\gamma}{1-\gamma} \frac{1}{D^2} f(\hat{\alpha})(r_1 + s_1 + h_1 - \theta v_1). \tag{3.26}$$

We know that $\gamma < 1$ and by taking absolute value of Equation 3.21, we satisfy the diagonal dominance condition:

$$\left| \frac{\partial^2 \Pi_1}{\partial S_1 \partial S_2} \right| < \left| \frac{\partial^2 \Pi_1}{\partial S_1^2} \right|. \tag{3.27}$$

Since the game is symmetric, the same is true for the second retailer. Then there exists a unique Nash equilibrium in the single period. □

3.3. n-Period Case

In the dynamic equations which are depicted below players both follow general strategies denoted as $S_{1,n}$ and $S_{2,n}$ respectively with n periods to-go, as $n = N, \ldots, 1$. Here the state variable $I_{i,n}$, inventory left over at retailer i when there are n periods to go, is equal to $\left[S_{i,n+1} - D^e_{i,n+1}\right]^+$, where $D^e_{i,n+1}$ is the effective demand of retailer i with $(n+1)$ periods to-go and is equal to $pD + \gamma[(1-p)D - S_{j,n+1}]^+$ where $[a]^+ = \max\{0, a\}$ and $i \neq j$, $i = 1, 2$. Excess demand is lost and $I_{i,n}$ is the inventory of the retailer i at the beginning of the period when n periods to-go. Then she decides her order decision which is $q_{i,n}$ for the replenishment with respect to the estimated demand and brings her inventory position to $S_{i,n}$. Then the demand in period n is realized. Costs are accrued and profit or losses are collected for period n. If we consider the finite horizon case, i.e. $n = N \ldots, 1$, maximization of the expected discounted profit on p of retailer I with n periods to go until the end of the planning horizon, $G'_{1,n}$, is given by the following equations:

$$G'_{1,0} = \upsilon_1 I_{1,0} \tag{3.28}$$

$$K'_{1,n}(S_{1,n}, S_{2,n}) = \Pi_1 \left(\frac{S_{1,n}}{D}, \frac{S_{2,n}}{D} \right) + \theta E_p \left[G'_{1,n-1} \left(I_{1,n-1}, S_{2,n-1} \right) \right] \tag{3.29}$$

where $n = N, \ldots, 1$ and $I_{1,n-1} = [S_{1,n} - D^e_{1,n}]^+$.

$$G'_{1,n}(I_{1,n}, S_{2,n}) = \max_{S_{1,n} \geq I_{1,n}} K'_{1,n}(S_{1,n}, S_{2,n}) \tag{3.30}$$

where $E_p[G'_{1,n}(I_{1,n}, S_{2,n})]$ is the maximum expected discounted profit of retailer I with n periods to go given that the beginning inventory of the first retailer is $I_{1,n}$, the strategy of retailer II with n periods to go is $S_{2,n}$ and $I_{1,0}$ is the end-of-horizon inventory of retailer I. Here, the end-of-horizon inventory which is equal to any real number, can be equalized to zero by the following transformation.

$$G_{1,n}(I_{1,n}, S_{2,n}) = G'_{1,n}(I_{1,n}, S_{2,n}) - E_p[\theta^n \nu_1 I_{1,0}]$$

$$G_{1,0} = \nu_1 I_{1,0} - \nu_1 I_{1,0} = 0$$

$$K_{1,1}(S_{1,1}, S_{2,1}) = \Pi_1 \left(\tfrac{S_{1,1}}{D}, \tfrac{S_{2,1}}{D} \right) + \theta E_p[\nu_1 I_{1,0}]$$

$$K_{1,n}(S_{1,n}, S_{2,n}) = \Pi_1 \left(\tfrac{S_{1,n}}{D}, \tfrac{S_{2,n}}{D} \right) + \theta E_p \left[G_{1,n-1} \left(I_{1,n-1}, S_{2,n-1} \right) + \theta^{n-1} \nu_1 I_{1,0} \right]$$

Without loss of generality discount factor θ can be taken as 1. So the explicit form of the maximum expected profit function for the case where the newsvendor ratios are less than the threshold at which $S_{1,i} + S_{2,i} < D$ for all i is:

$$\begin{aligned}
G_{1,n}(I_{1,n}, S_{2,n}) \;=\; \max_{S_{1,n} \geq I_{1,n}} \Bigg\{ & \Pi_1\left(\frac{S_{1,n}}{D}, \frac{S_{2,n}}{D}\right) \\
& + \int_0^{\max\left(\frac{\hat{S}_{1,n}}{D},0\right)} G_{1,n-1}\left(S_{1,n} - D^e_{1,n}, S_{2,n-1}\right) f(p)\,dp \\
& + \int_{\max\left(\frac{\hat{S}_{1,n}}{D},0\right)}^{1-\frac{S_{2,n}}{D}} G_{1,n-1}\left(0, S_{2,n-1}\right) f(p)\,dp \\
& + \int_{1-\frac{S_{2,n}}{D}}^{1} G_{1,n-1}\left(0, S_{2,n-1}\right) f(p)\,dp \\
& + \int_0^{\max\left(\frac{\hat{S}_{1,1}}{D},0\right)} G_{1,0}\left(S_{1,1} - D^e_{1,1}, S_{2,0}\right) f(p)\,dp \Bigg\}
\end{aligned} \tag{3.31}$$

where $\frac{\hat{S}_{1,n}}{D} = \frac{\frac{S_{1,n}}{D} - \gamma\left(1-\frac{S_{2,n}}{D}\right)}{1-\gamma}$. As in Equation 3.4, the first three integrals in the above equation represent; retailer II has a stockout, both retailers have a stockout, and retailer I has a stockout respectively. The last integral represents the case retailer I has a positive end-of-horizon inventory. A similar notation will be used for the retailer II as $G_{2,n}(S_{1,n}, I_{2,n})$.

Proposition 3.3.1. *There is a unique best response of retailer I(II) for a given ordering level of retailer II(I) in any period of a finite horizon dynamic game.*

Proof. The proof is completed by induction on periods with the feedback Nash equilibria of dynamic games. Without loss of generality discount factor θ can be taken as 1. Then Equation 3.3 can be written as:

$$G_{1,n}(I_{1,n}, S_{2,n}) = \max_{S_{1,n} \geq I_{1,n}} \Big\{ K_{1,n}\left(S_{1,n}, S_{2,n}\right) \Big\}, \tag{3.32}$$

where

$$K_{1,n}(S_{1,n}, S_{2,n}) = \Pi_1\left(\frac{S_{1,n}}{D}, \frac{S_{2,n}}{D}\right) + E_p\left[G_{1,n-1}\left(I_{1,n-1}, S_{2,n-1}\right)\right] \tag{3.33}$$

The results from the single period calculations are used in case of one period to go ($n = 1$). In terms of the above notation;

$$K_{1,1}(S_{1,1}, S_{2,1}) = \Pi_1\left(\frac{S_{1,1}}{D}, \frac{S_{2,1}}{D}\right) \tag{3.34}$$

is concave with respect to $S_{1,1}$ by Equation 3.23. This implies that $G_{1,1}(I_{1,1}, S_{2,1})$ is also concave with respect to its first variable as in [16]. Now let's assume the hypothesis is true for the period when n period to-go, that is $K_{1,n}(S_{1,n}, S_{2,n})$ is concave in $S_{1,n}$ and thus $G_{1,n}(I_{1,n}, S_{2,n})$ is concave in $I_{1,n}$. Now we consider the period when $(n+1)$ period to-go and

$$K_{1,n+1}(S_{1,n+1}, S_{2,n+1}) = \Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right) + E_p\left[G_{1,n}\left(I_{1,n}, S_{2,n}\right)\right]. \tag{3.35}$$

Our aim is to show that $K_{1,n+1}$ is concave in $S_{1,n+1}$. The derivative of Equation 3.35 is calculated with respect to three different inventory levels of retailers as in Equation 3.31:

$$\begin{aligned}
\frac{\partial K_{1,n+1}}{\partial S_{1,n+1}} \quad = \quad & \frac{\partial}{\partial S_{n+1}}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right) \\
& + \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}(S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}], S_{2,n}) f(p)\, dp \\
& + \frac{\partial}{\partial S_{1,n+1}}\left[\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)\right] \\
& \qquad \times G_{1,n}\left(S_{1,n+1} - \frac{\hat{S}_{1,n+1}}{D}D - \gamma[(1 - \frac{\hat{S}_{1,n+1}}{D})D - S_{2,n+1}], S_{2,n}\right) \\
& - \frac{\partial}{\partial S_{1,n+1}}\left[\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)\right] G_{1,n}\left(0, S_{2,n}\right) \\
& + \int_{\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)}^{1-\frac{S_{2,n+1}}{D}} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(0, S_{2,n}\right) f(p)\, dp
\end{aligned}$$

$$+ \int_{1-\frac{S_{2,n+1}}{D}}^{1} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(0, S_{2,n}\right) f(p)\, dp \tag{3.36}$$

where

$$\hat{S}_{1,i} = \frac{S_{1,i} - \gamma(D - S_{2,i})}{1-\gamma} \tag{3.37}$$

In the above derivation, the last integral which does not depend on $S_{1,n+1}$ is vanished. If $\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right) = 0$ then the 3^{rd} and the 4^{th} terms are zero. If $\frac{\hat{S}_{1,n+1}}{D} > 0$ then they are equal to each other, so they cancel out. Then the first derivative of the expected profit function is:

$$\begin{aligned}\frac{\partial K_{1,n+1}}{\partial S_{1,n+1}} &= \frac{\partial}{\partial S_{n+1}} \Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\\ &+ \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}], S_{2,n}\right) f(p)\, dp\\ &+ \int_{\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)}^{1-\frac{S_{2,n+1}}{D}} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(0, S_{2,n}\right) f(p)\, dp\\ &+ \int_{1-\frac{S_{2,n+1}}{D}}^{1} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(0, S_{2,n}\right) f(p)\, dp \end{aligned} \tag{3.38}$$

Now the first variable of $G_{1,n}$ which is $I_{1,n}$ is equal to $S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}]$ or 0 in the above equation. Let $I^1 = S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}]$. Then

$$\frac{\partial G_{1,n}}{\partial S_{1,n+1}} = \frac{\partial G_{1,n}}{\partial I^1} \frac{\partial I^1}{\partial S_{1,n+1}} \tag{3.39}$$

So the final form of the first derivative of the expected profit function is:

$$\begin{aligned}\frac{\partial K_{1,n+1}}{\partial S_{1,n+1}} &= \frac{\partial}{\partial S_{n+1}} \Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\\ &+ \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial}{\partial I^1} G_{1,n}\left(I^1, S_{2,n}\right) \frac{\partial I^1}{\partial S_{1,n+1}} f(p)\, dp\\ &+ \int_{\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)}^{1-\frac{S_{2,n+1}}{D}} \frac{\partial}{\partial S_{1,n+1}} G_{1,n}\left(0, S_{2,n}\right) f(p)\, dp\end{aligned}$$

$$+\int_{1-\frac{S_{2,n+1}}{D}}^{1}\frac{\partial}{\partial S_{1,n+1}}G_{1,n}\left(0,S_{2,n}\right)f(p)\,dp \tag{3.40}$$

where $\frac{\partial I^1}{\partial S_{1,n+1}}=1$. The second derivative of the expected profit with respect to $S_{1,n+1}$ is:

$$\begin{aligned}
\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}^2} &= \frac{\partial^2}{\partial S_{n+1}^2}\Pi_1\left(\frac{S_{1,n+1}}{D},\frac{S_{2,n+1}}{D}\right) \\
&+\int_0^{\frac{\hat{S}_{1,n+1}}{D}}\frac{\partial^2}{\partial (I^1)^2}G_{1,n}\left(I_{1,n}^1,S_{2,n}\right)\frac{\partial I^1}{\partial S_{1,n+1}}f(p)\,dp \\
&+\frac{\partial}{\partial S_{1,n+1}}\left[\max\left(\frac{\hat{S}_{1,n+1}}{D},0\right)\right] \\
&\times\frac{\partial}{\partial S_{1,n+1}}G_{1,n}\left(S_{1,n+1}-\frac{\hat{S}_{1,n+1}}{D}D-\gamma[(1-\frac{\hat{S}_{1,n+1}}{D})D-S_{2,n+1}],S_{2,n}\right) \\
&-\frac{\partial}{\partial S_{1,n+1}}\left[\max\left(\frac{\hat{S}_{1,n+1}}{D},0\right)\right]\frac{\partial}{\partial S_{1,n+1}}G_{1,n}\left(0,S_{2,n}\right) \\
&+\int_{\max\left(\frac{\hat{S}_{1,n+1}}{D},0\right)}^{1-\frac{S_{2,n+1}}{D}}\frac{\partial^2}{\partial S_{1,n+1}^2}G_{1,n}\left(0,S_{2,n}\right)f(p)\,dp \\
&+\int_{1-\frac{S_{2,n+1}}{D}}^{1}\frac{\partial^2}{\partial S_{1,n+1}^2}G_{1,n}\left(0,S_{2,n}\right)f(p)\,dp
\end{aligned} \tag{3.41}$$

As above, if $\max\left(\frac{\hat{S}_{1,n+1}}{D},0\right)=0$ then the 3^{rd} and the 4^{th} terms are zero. If $\frac{\hat{S}_{1,n+1}}{D}>0$ then they are equal to each other. So that they are cancelled out. Since $\frac{\partial I^1}{\partial S_{1,n+1}}=1$ then the second derivative of the expected profit function is:

$$\begin{aligned}
\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}^2} &= \frac{\partial^2}{\partial S_{n+1}^2}\Pi_1\left(\frac{S_{1,n+1}}{D},\frac{S_{2,n+1}}{D}\right) \\
&+\int_0^{\frac{\hat{S}_{1,n+1}}{D}}\frac{\partial^2}{\partial (I^1)^2}G_{1,n}\left(I_{1,n}^1,S_{2,n}\right)f(p)\,dp \\
&+\int_{\max\left(\frac{\hat{S}_{1,n+1}}{D},0\right)}^{1-\frac{S_{2,n+1}}{D}}\frac{\partial^2}{\partial S_{1,n+1}^2}G_{1,n}\left(0,S_{2,n}\right)f(p)\,dp \\
&+\int_{1-\frac{S_{2,n+1}}{D}}^{1}\frac{\partial^2}{\partial S_{1,n+1}^2}G_{1,n}\left(0,S_{2,n}\right)f(p)\,dp
\end{aligned} \tag{3.42}$$

Due to the concavity of $G_{1,n}$ with respect to its first variable and Π_1,

$$\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}^2} < 0 \tag{3.43}$$

Then the induction is completed. □

Theorem 3.3.2. *Ordering policy of retailer $i(j)$, i,j=1,2, is defined by the comparison with the unique best response of retailer $i(j)$, $S^*_{i,n}(S^*_{j,n})$ for a given ordering level of retailer $j(i)$, as in Equation 3.8, and the inventory level at the beginning of the period when there are n periods to-go and it is:*

$$S^*_{i,n} = \begin{cases} R^i_n & \text{if } I_{i,n} < R^i_n, \\ I_{i,n} & \text{if } I_{i,n} > R^i_n. \end{cases}$$

where $S^*_{i,n} + S^*_{j,n} < H$, $\forall n$ and the demand substitution is important.

Proof. Uniqueness of the response functions for any period is shown by the Proposition 3.3.1. Then the decision policy of a rational decision maker is a comparison between inventory level at the beginning of the period and the unique response. □

Proposition 3.3.3. *There is a unique feedback Nash equilibrium for every period in the finite horizon dynamic game.*

Proof. As in Proposition 3.3.1, the proof is completed by induction on periods with the feedback Nash equilibria of dynamic games. By using single period calculations, Proposition 3.2.3 and Equation 3.34, the following inequality holds.

$$\left|\frac{\partial^2 K_{1,1}}{\partial S_{1,1}\partial S_{2,1}}\right| < \left|\frac{\partial^2 K_{1,1}}{\partial S_{1,1}^2}\right|. \tag{3.44}$$

Now, by using Equation 3.44 and relation between $K_{1,1}$ and $G_{1,1}$ defined in [16] the following holds:

$$\left|\frac{\partial^2 G_{1,1}}{\partial I_{1,1}\partial I_{2,1}}\right| < \left|\frac{\partial^2 G_{1,1}}{\partial I_{1,1}^2}\right| \tag{3.45}$$

where $K_{1,1} = K_{1,1}(S_{1,1}, S_{2,1})$, $G_{1,1} = G_{1,1}(I_{1,1}, S_{2,1})$ and $S_{2,1} = I_{2,1} + q_{2,1}$. The proof is completed for $n = 1$.

The induction assumption is: the above inequalities are true with n periods to go, i.e.:

$$\left|\frac{\partial^2 K_{1,n}}{\partial S_{1,n}\partial S_{2,n}}\right| < \left|\frac{\partial^2 K_{1,n}}{\partial S_{1,n}^2}\right| \tag{3.46}$$

implies

$$\left|\frac{\partial^2 G_{1,n}}{\partial I_{1,n}\partial S_{2,n}}\right| < \left|\frac{\partial^2 G_{1,n}}{\partial I_{1,n}^2}\right| \tag{3.47}$$

Here, $I_{1,n}$ and $I_{2,n}$, inventory levels of retailer I and II respectively with n periods to go, are 0, $S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}]$ or $S_{2,n+1} - (1-p)D - (1-\gamma)(pD - S_{1,n+1})$. Let $I_{1,n} = I^1$ and $I_{2,n} = I^2$. Since $S_{2,n} = I_{2,n} + q_{2,n}$ then the following equalities hold

$$\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial I^1 \partial S_{2,n}} = \frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial I^1 \partial I^2}\frac{\partial I^2}{\partial S_{2,n}} = \frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial I^1 \partial I^2} \tag{3.48}$$

where $\frac{\partial I^2}{\partial S_{2,n}} = 1$. Then Equation 3.47 implies

$$\left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial I^1 \partial I^2}\right| < \left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial (I^1)^2}\right| \tag{3.49}$$

Now, we will show that it is also true for period $n+1$. By using Equation 3.38, derivative of $\frac{\partial K_{1,n+1}}{\partial S_{1,n+1}}$ with respect to $S_{2,n+1}$ is:

$$\begin{aligned}\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}\partial S_{2,n+1}} \quad = \quad & \frac{\partial^2}{\partial S_{n+1}\partial S_{2,n+1}}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right) \\ & + \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial^2 G_{1,n}\left(I_{1,n}^1, S_{2,n}\right)}{\partial (I^1)\partial S_{2,n+1}} f(p)\,dp \\ & + \frac{\partial}{\partial S_{2,n+1}}\left[\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)\right]\end{aligned}$$

$$\begin{aligned}
&\times \frac{\partial}{\partial S_{1,n+1}} G_{1,n} \Bigg(S_{1,n+1} - \frac{\hat{S}_{1,n+1}}{D} D \\
&\quad - \gamma[(1 - \frac{\hat{S}_{1,n+1}}{D})D - S_{2,n+1}], S_{2,n} \Bigg) \\
&+ \frac{\partial}{\partial S_{2,n+1}} \left(1 - \frac{S_{2,n+1}}{D} \right) \frac{\partial}{\partial S_{1,n+1}} G_{1,n} \left(0, S_{2,n} \right) \\
&- \frac{\partial}{\partial S_{2,n+1}} \left[\max \left(\frac{\hat{S}_{1,n+1}}{D}, 0 \right) \right] \frac{\partial}{\partial S_{1,n+1}} G_{1,n} \left(0, S_{2,n} \right) \\
&+ \int_{\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right)}^{1 - \frac{S_{2,n+1}}{D}} \frac{\partial^2}{\partial S_{1,n+1} \partial S_{2,n+1}} G_{1,n} \left(0, S_{2,n} \right) f(p)\, dp \\
&- \frac{\partial}{\partial S_{2,n+1}} \left(1 - \frac{S_{2,n+1}}{D} \right) \frac{\partial}{\partial S_{1,n+1}} G_{1,n} \left(0, S_{2,n} \right) \\
&+ \int_{1 - \frac{S_{2,n+1}}{D}}^{1} \frac{\partial^2}{\partial S_{1,n+1} \partial S_{2,n+1}} G_{1,n} \left(0, S_{2,n} \right) f(p)\, dp
\end{aligned} \tag{3.50}$$

As in Equation 3.41, if $\max\left(\frac{\hat{S}_{1,n+1}}{D}, 0\right) = 0$ then the 3^{rd} and the 5^{th} terms are zero and if $\frac{\hat{S}_{1,n+1}}{D} > 0$ then they are equal to each other. So that they are cancelled out. By using Equation 3.39 and $\frac{\partial G_{1,n}(0,S_{2,n})}{\partial I^1} = 0$, $\frac{\partial G_{1,n}(0,S_{2,n})}{\partial S_{1,n+1}} = 0$. Then the final form of the above equality is:

$$\begin{aligned}
\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1} \partial S_{2,n+1}} &= \frac{\partial^2}{\partial S_{n+1} \partial S_{2,n+1}} \Pi_1 \left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D} \right) \\
&+ \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial^2 G_{1,n} \left(I^1_{1,n}, S_{2,n} \right)}{\partial (I^1) \partial S_{2,n+1}} f(p)\, dp
\end{aligned} \tag{3.51}$$

Since $I^1 = S_{1,n+1} - pD - \gamma[(1-p)D - S_{2,n+1}]$, $S_{2,n} = I_{2,n} + q_{2,n}$ and $I^{(2)} = I_{2,n} = S_{2,n+1} - (1-p)D - (1-\gamma)(pD - S_{1,n+1})$ then

$$\frac{\partial^2 G_{1,n} \left(I^1, S_{2,n} \right)}{\partial I^1 \partial S_{2,n+1}} = \frac{\partial^2 G_{1,n} \left(I^1, S_{2,n} \right)}{\partial I^1 \partial I^{(2)}} \frac{\partial I^{(2)}}{\partial S_{2,n+1}} = \frac{\partial^2 G_{1,n} \left(I^1, S_{2,n} \right)}{\partial I^1 \partial I^{(2)}} \tag{3.52}$$

where $\frac{\partial I^{(2)}}{\partial S_{2,n+1}} = 1$. So Equation 3.51 can be written as

$$\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1} \partial S_{2,n+1}} = \frac{\partial^2}{\partial S_{n+1} \partial S_{2,n+1}} \Pi_1 \left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D} \right)$$

$$+\int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial^2 G_{1,n}\left(I^1, S_{2,[n]}\right)}{\partial(I^1)\partial I^{(2)}} f(p)\,dp \tag{3.53}$$

By taking the absolute value

$$\begin{aligned}
&\left|\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}\partial S_{2,n+1}}\right| \\
&= \left|\frac{\partial^2}{\partial S_{n+1}\partial S_{2,n+1}}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right) + \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial(I^1)\partial I^{(2)}} f(p)\,dp\right| \\
&< \left|\frac{\partial^2}{\partial S_{n+1}\partial S_{2,n+1}}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\right| \\
&\quad + \int_0^{\frac{\hat{S}_{1,n+1}}{D}} \left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial(I^1)\partial I^{(2)}}\right| f(p)\,dp
\end{aligned} \tag{3.54}$$

The following inequalities are known for the integrands of the left hand side of Equation 3.54 from the single period and the induction assumption:

$$\left|\frac{\partial^2}{\partial S_{1,n+1}\partial S_{2,n+1}}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\right| < \left|\frac{\partial^2}{\partial S_{1,n+1}^2}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\right| \tag{3.55}$$

$$\left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial(I^1)\partial I^{(2)}}\right| < \left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,n}\right)}{\partial(I^1)^2}\right| \tag{3.56}$$

Since inner parts of the right-hand side expressions are known to be concave from Proposition 3.2.1 and 3.3.1 then the below equality is satisfied.

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial S_{1,n+1}^2}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right)\right| + \left|\frac{\partial^2 G_{1,n}\left(I^1, S_{2,[n]}\right)}{\partial(I^1)^2}\right| \\
&= \left|\frac{\partial^2}{\partial S_{1,n+1}^2}\Pi_1\left(\frac{S_{1,n+1}}{D}, \frac{S_{2,n+1}}{D}\right) + \frac{\partial^2 G_{1,n}\left(I^1, S_{2,[n]}\right)}{\partial(I^1)^2}\right| \\
&= \left|\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}^2}\right|
\end{aligned} \tag{3.57}$$

Since signs are preserved under integral calculations, right-hand side of Equation 3.54 is strictly less than right-hand side of Equation 3.57 with integral calculations. Then the result follows as

$$\left|\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}\partial S_{2,n+1}}\right| < \left|\frac{\partial^2 K_{1,n+1}}{\partial S_{1,n+1}^2}\right|. \tag{3.58}$$

□

4. COMPETITION IN A TWO-PERIOD MODEL

4.1. Definitions and the Model

This work deals with the equilibrium strategies for substitutable product inventory control systems with a stochastic demand in a two-period stationary environment between two retailers. This stationary scenario can be viewed as a dynamic game in a duopoly setting. Different than the first model, we use a random demand in this chapter. The retailers share this demand with proportions and we assume that demand and proportions are independent random variables. Therefore, stochasticity of the effective demand is coming from both the market demand and proportions. Unlike the third chapter, there is no upper bound on total ordering units and we do not consider the relation between orders and possible demand levels. We develop results for any level of inventory and ordering decisions.

We formulate the single period game and extend it to the two-period dynamic game. Ordering decisions are given by using the inventory information coming from the previous period if it exists. We allow the transfer of some portion of the unsatisfied customers between retailers within the same period.

As we mentioned in the literature survey, except [16], [8] and [18], all investigations were made under a constant order-up-to level assumption or in a single period case when competitive games are considered. As an exception, Nagarajan and Rajagopalan worked with general strategies in [16] but they use a bound on total ordering decisions as being less than the maximum demand level. We characterize the equilibrium order quantities and ordering policy under certain assumptions when two firms make inventory decisions for products that are substitutable. Fudenberg and Levine [8] showed a similar result with [16] in their work coming from 1988 on deviations of optimal reactions in dynamic games. They investigated the conditions in which the closed-loop and open-loop equilibria are approximately the same. Finally, Olsen and Parker [18] analyze a dynamic duopoly with stockout-based substitution in a closed-loop infinite horizon setting. They give conditions for the stationary inifinite horizon (open-loop) equilibrium to be a Markov perfect (closed-loop) equilibrium.

Existence and uniqueness of the Nash equilibrium in the single period game have been shown by Parlar in [20]. In this work, the existence and uniqueness of the Nash equilibrium with two periods to go are shown by starting from the last period and using Parlar's result in [20] and the feedback Nash equilibrium where substitution effect is significant and under some density function assumptions.

We show the existence of the Nash equilibrium in this two-period game by showing the existence and uniqueness of each player's best response function by finding an action that maximizes its payoff for any given action of the other player. First and second order conditions are used to find those individual equilibria. We also suggest a threshold for the ordering decisions of the retailers with two periods to go in terms of parameters and inverse distribution functions of the effective demand. This threshold can be observed with the last period inventory level. We prove that retailers' order decisions with two periods to go are inversely proportional when inventory level of the last period is above this threshold. Then, at last, we consider the uniqueness of the Nash equilibrium by using the contraction mapping argument and diagonal dominance as in [5] with a uniformly distributed effective demand assumption. As a result, a state dependent order-up-to policy turns out to be a Nash equilibrium of this game, adding to a previous study [1] where the order-up-to-policy was taken as exogenously specified. Comparative statics of optimal ordering level with respect to parameters are also done at the end of this chapter.

4.2. The Single Period Model

In this chapter, there are two retailers who compete with each other in the market over the two-period time horizon. Demand is random in the interval $[L, H]$ and stationary in periods due to get rid of the different characterization of distributions for the case of non-stationary demand compared with stationary one. For any demand in $[L, H]$, p and $1 - p$ are the proportions of demand that belong to retailer I and II respectively. We assume that the random variables D and p are independent of each other. The model is based on standard cost and revenue parameters such as ordering, c_i, holding costs, h_i, and sale price, r_i for retailer i where $i = 1, 2$. There is no penalty cost charged to the retailers. We assume the beginning inventory $I_1^i \geq 0$, $\forall i$. As in the first model in the third chapter, retailers decide their order decisions simultaneously to maximize their expected profit. Replenishment is made with respect to the estimated demand and inventory positions are brought to $y_1^{i(j)}$, where $y_1^{i(j)} \geq I_1^{i(j)}$. The game is played after demand realization.

We begin with the single period expected profit function for retailer I. It can be written without a time index as follows:

$$\begin{aligned} \Pi_1(y_1^1, I_1^1, y_1^2) &= r_1 E\left[\min\left(y_1^1, \Lambda_1^1\right)\right] - h_1 E\left[(y_1^1 - \Lambda_1^1)^+\right] \\ &\quad - c_1(y_1^1 - I_1^1) + \nu_1 E\left[(y_1^1 - \Lambda_1^1)^+\right] \end{aligned} \tag{4.1}$$

where Λ_1^1 is the effective demand of retailer I and equals $pD - \gamma[(1-p)D - y^2]^+$ and $[x]^+$ denotes $max(0, x)$. In the above equation the first term is the revenue from expected sales, the second term is the expected holding cost, the third term is the cost of goods replenished and the fourth term is the expected salvage value at the end of the period. We also assume that unsatisfied (effective) demand is lost.

Expectations should be taken over the effective demand Λ_1^1 (i.e., on both p and D). Players I and II both follow general strategies in the single period denoted as y_1^1 and y_1^2 respectively. The nonnegative random variables Λ_1^1 and Λ_1^2 have continuous joint density functions u and v respectively with finite expectations. The corresponding cumulative functions will be denoted by U, V and let $U(L) = V(L) = 0$ and $U(H) = V(H) = 1$ or either both or one of them go to 0 or 1 asymptotically as Λ_1^i, $i = 1, 2$ go to L or H respectively. We, first, consider concavity of the single period expected profit function.

First order conditions give us the optimal ordering decision. Concavity also yields the existence of unique response for retailer I(II) to maximize the profit function for all possible strategies of retailer II(I). General strategies will become base stock strategies. We give an explicit expression for the single period formulation of the expected payoff of retailer I with Equation 4.1.

$$\begin{aligned}\Pi_1^1\left(y_1^1, x_1^1, y_1^2\right) &= \int_L^{y_1^1}\left[r_1 y_1^1 + (-r_1 - h_1 + \nu_1)(y_1^1 - \Lambda_1^1) - c_1(y_1^1 - x_1^1)\right] u(\Lambda_1^1) d\Lambda_1^1 \\ &+ \int_{y_1^1}^{H}\left[r_1 y_1^1 - c_1(y_1^1 - x_1^1)\right] u(\Lambda_1^1) d\Lambda_1^1 \end{aligned} \tag{4.2}$$

The above integrals represent two cases. First; retailer II may or may not have a stockout but retailer I has inventory left over after satisfying his own demand and substitution demand coming from retailer II if it exists. Second integral is the case where retailer I can not meet his own effective demand. Now the first step will be the existence of best responses which are:

$$R_1^i\left(y_1^j, I_1^i\right) = \arg\max_{y_1^i \geq I_1^i} \Pi_1^i\left(y_1^i, I_1^i, y_1^j\right) \tag{4.3}$$

where superscript in R is for the retailers and $i, j = 1, 2$ and $i \neq j$. Parlar [20] showed the existence and uniqueness of the best response using an algebraic argument and Netessine and Cachon [5] use the contraction mapping argument to show the uniqueness as we do. Therefore, we do not give proofs for the following two proposition.

Proposition 4.2.1. *The best response of retailer I(II) is unique for a given ordering level of retailer II(I) for the single period two-player game and it is given by:*

$$R_1^i\left(y_1^j, I_1^i\right) = \begin{cases} y_1^i & \textit{when } y_1^i \geq I_1^i, \\ I_1^i & \textit{when } y_1^i < I_1^i. \end{cases}$$

where y_1^i is the order up to level of retailer i.

Since payoff functions are continuous and retailer II has a similar density function for effective demand; quasiconcavity holds for all players by Proposition 4.3.1 (concavity implies quasiconcavity) with respect to each player's own strategy. If a set E in $\mathbb{R}^k$

is closed and bounded and every infinite subset of E has a limit point in E then the set E is compact by [24]. In our model, upper boundary of the random demand, H, is less than infinity. If a player orders infinitely many products then his ordering cost goes to infinity despite a finite expected demand and revenue. Therefore, the players as rational decision makers give ordering decisions for finitely many units and we can set an upper limit of responses, $\bar{y}_n^i$, as again a response. Then our strategy space is bounded. On the other hand, since every convergent sequence of response functions has a limit of response and it can be less than or equal to $\bar{y}_n^i$ then the strategy space contains all limit points. Therefore it is also closed.

The above two conditions imply compactness of the strategy space. We conclude that for any infinite subset of the strategy space best responses are included in the strategy space. Then there exists at least one symmetric pure strategy Nash equilibrium in the game by [6].

Now, we show uniqueness of the Nash equilibrium by using a contraction mapping argument.

Proposition 4.2.2. *There exists a unique Nash equilibrium* (R_1^{1*}, R_1^{2*}) *for the single period two-player game and the components are given by*

$$R_1^{1*}\left(y_1^{2*}, I_1^1\right) = \begin{cases} y_1^{1*} = U^{-1}\left(\frac{r_1-c_1}{r_1+h_1-\nu_1}\right) & \textit{when } y_1^{1*} \geq I_1^1, \\ I_1^1 & \textit{when } y_1^{1*} < I_1^1. \end{cases}$$

and

$$R_1^{2*}\left(y_1^{1*}, I_1^2\right) = \begin{cases} y_1^{2*} = V^{-1}\left(\frac{r_2-c_2}{r_2+h_2-\nu_2}\right) & \textit{when } y_1^{2*} \geq I_1^2, \\ I_1^2 & \textit{when } y_1^{2*} < I_1^2. \end{cases}$$

4.3. Two-Period Case

In this section we deal with a two-period dynamic game. Despite a static game, in which decisions are assumed to be made simultaneously, in dynamic games, there is an explicit time-schedule that describes when players make their decisions. The dynamic decision process is endogenous, that is, the players learn from observing and base their actions on their state of knowledge where the actions are taken. Their decision rules are reviewed and revised in response at each stage of the game. In this dynamic two-period problem, a player has information about the strategies chosen on the other player at the beginning of the game. Thus they base their play on past moves. In the dynamic equations depicted below players both follow strategies denoted as $R_2^1 = R_2^1\left(y_2^2, x_2^1\right)$ and $R_2^2 = R_2^2\left(y_2^1, x_2^2\right)$ respectively with two periods to go. Here the state variable x_n^i, inventory left over at retailer i when there are n periods to go ($n = 2, 1$), is equal to $\left[R_{n+1}^i - \Lambda_{n+1}^i\right]^+$, where Λ_{n+1}^1 and Λ_{n+1}^2 are the effective demands of retailer I and II respectively in period $(n+1)$ and are equal to $pD + \gamma[(1-p)D - y_{n+1}^j]^+$ and $(1-p)D + \alpha[pD - y_{n+1}^i]^+$. Here $[a]^+ = \max\{0, a\}$ and $i, j = 1, 2$, $i \neq j$. The density functions of effective demand of retailer I and II are u and v respectively. At the beginning of the period with two periods to go, we shall call it period two, the retailer has the inventory level at I_2^1. Then she decides her order decision for the replenishment with respect to the estimated demand and brings her inventory position to R_2^1, here $R_2^1 \geq I_2^1$. Then the demand with two periods to go is realized. The expected discounted profit of retailer I with two periods to go until the end of the planning horizon is:

$$J_2^1\left(I_2^1, y_2^1, y_2^2\right) = \Pi_2^1\left(y_2^1, I_2^1, y_2^2\right) + \theta E\left[G_1^1\left(y_2^1 - \Lambda_2^1, y_1^2\right)\right] \tag{4.4}$$

$$G_1^1\left(I_1^1, y_1^2\right) = \max_{y_1^1 \geq I_1^1} J_1^1\left(I_1^1, y_1^1, y_1^2\right) \tag{4.5}$$

where $\Pi_t^1\left(y_t^1, I_t^1, y_t^2\right)$ is the expected profit of player I with t periods to go and only in that period (single period payoff), $J_t^1\left(I_t^1, y_t^1, y_t^2\right)$ is the expected profit of player I with t periods to go, and $G_t^1(I_t^1, y_t^2)$ is the optimal value function of retailer I with two periods to go given that the beginning inventory of the first retailer is I_t^1 and the decision variable of retailer II with t periods to go is y_t^2. For the single period, expected profit function of the retailer I is

$$J_1^1\left(I_1^1, y_1^1, y_1^2\right) = \Pi_1^1\left(y_1^1, I_1^1, y_1^2\right) \tag{4.6}$$

Then by using Equation 4.4 and Equation 4.5, *the feedback Nash formulations* for the expected profit function of the retailer I with two periods to go are:

$$J_2^1\left(I_2^1, y_2^1, y_2^2\right) = \Pi_2^1\left(y_2^1, I_2^1, y_2^2\right) + \theta E\left[G_1^1\left(y_2^1 - \Lambda_2^1, R_1^{2*}\right)\right] \tag{4.7}$$

where $I_1^2 = \left(y_2^2 - \Lambda_2^2\right)^+$

$$G_1^1\left(I_1^1, R_1^{2*}\right) = \max_{y_1^1 \geq I_1^1} J_1^1\left(I_1^1, y_1^1, R_1^{2*}\right) \triangleq R_1^{1*} \tag{4.8}$$

So, the explicit form of the expected profit function of the retailer I with two periods to go can be written with respect to two possible cases of feedback Nash equilibrium of respective periods:

$$\begin{aligned} J_2^1\left(I_2^1, y_2^1, y_2^2\right) &= \Pi_2^1\left(y_2^1, I_2^1, y_2^2\right) \\ &\quad + \theta\left\{\int_L^{\left(y_2^1 - y_1^{1^*}\right)} J_1^1\left(y_2^1 - \Lambda_2^1, I_1^1, R_1^{2*}\right) u\left(\Lambda_2^1\right)\, d\Lambda_2^1\right. \\ &\qquad + \int_{\left(y_2^1 - y_1^{1^*}\right)}^{y_2^1} J_1^1\left(y_2^1 - \Lambda_2^1, y_1^{1*}, R_1^{2*}\right) u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\ &\qquad \left. + \int_{y_2^1}^{H} J_1^1\left(0, y_1^{1*}, R_1^{2*}\right) u\left(\Lambda_2^1\right)\, d\Lambda_2^1\right\} \end{aligned} \tag{4.9}$$

if $y_2^1 \geq y_1^{1^*}$, or

$$\begin{aligned} J_2^1\left(I_2^1, y_2^1, y_2^2\right) &= \Pi_2^1\left(y_2^1, I_2^1, y_2^2\right) \\ &\quad + \theta\left\{\int_L^{y_2^1} J_1^1\left(y_2^1 - \Lambda_2^1, y_1^{1*}, R_1^{2*}\right) u\left(\Lambda_2^1\right)\, d\Lambda_2^1\right. \end{aligned}$$

$$+\int_{y_2^1}^{H} J_1^1\left(0, y_1^{1*}, R_1^{2*}\right) u\left(\Lambda_2^1\right) d\Lambda_2^1 \Bigg\} \tag{4.10}$$

if $y_2^1 < y_1^{1^*}$ and where $\Lambda_2^1 = pD + \gamma[(1-p)D - y_2^2]^+$ is the effective demand of retailer I in period two and $R_1^{1*} = \max(y_1^{1^*}, I_1^1)$ is the Nash order point of the single period solution for retailer I.

As in Equation 4.2, the first and the second integrals in the first case of the above Equation 4.9 represent; retailer I has a positive inventory at the beginning of the last period. The third integral represents retailer I has a stockout. In the second case in Equation 4.10; unlike the first integral which expresses a positive inventory for the retailer, the second one represents a stockout. A similar notation will be used for the retailer II as $J_2^2(I_2^2, y_2^1, y_2^2)$.

Equation 4.9 and Equation 4.10 can be written more explicitly with respect to above cases as follows:

$$\begin{aligned}
J_2^1\left(I_2^1, y_2^1, y_2^2\right) = & \int_L^{y_2^1} \left[r_1 y_2^1 + (-r_1 - h_1)(y_2^1 - \Lambda_2^1) - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
& + \int_{y_2^1}^{H} \left[r_1 y_2^1 - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
& + \theta \Bigg\{ \int_{\left(y_2^1 - y_1^{1^*}\right)}^{H} \Bigg[\int_L^{y_1^{1^*}} \Big(r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\big(y_1^{1^*} - \Lambda_1^1\big) \\
& \qquad - c_1\left(y_1^{1^*} - I_1^1\right) \Big) u\left(\Lambda_1^1\right) d\Lambda_1^1 \\
& \qquad + \int_{y_1^{1^*}}^{H} \left[r_1 y_1^{1^*} - c_1(y_1^{1^*} - I_1^1)\right] u\left(\Lambda_1^1\right) d\Lambda_1^1 \Bigg] u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
& + \int_L^{\left(y_2^1 - y_1^{1^*}\right)} \Bigg[\int_L^{I_1^1} \left(r_1 I_1^1 + (-r_1 - h_1 + \nu_1)\left(I_1^1 - \Lambda_1^1\right)\right) u\left(\Lambda_1^1\right) d\Lambda_1^1 \\
& \qquad + \int_{I_1^1}^{H} \left(r_1 I_1^1\right) f\left(\Lambda_1^1\right) d\Lambda_1^1 \Bigg] u\left(\Lambda_2^1\right) d\Lambda_2^1 \Bigg\}
\end{aligned} \tag{4.11}$$

if $y_2^1 \geq y_1^{1^*}$, or

$$
\begin{aligned}
J_2^1\left(I_2^1, y_2^1, y_2^2\right) &= \int_L^{y_2^1} \left[r_1 y_2^1 + (-r_1 - h_1)(y_2^1 - \Lambda_2^1) - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \int_{y_2^1}^{H} \left[r_1 y_2^1 - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \theta \left\{ \int_L^H \left[\int_L^{y_1^{1^*}} \left(r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\left(y_1^{1^*} - \Lambda_1^1\right) - c_1\left(y_1^{1^*} - I_1^1\right)\right) u\left(\Lambda_1^1\right) d\Lambda_1^1 \right.\right. \\
&\qquad \left.\left. + \int_{y_1^{1^*}}^{H} \left[r_1 y_1^{1^*} - c_1(y_1^{1^*} - I_1^1)\right] u\left(\Lambda_1^1\right) d\Lambda_1^1 \right] u\left(\Lambda_2^1\right) d\Lambda_2^1 \right\} \qquad (4.12)
\end{aligned}
$$

if $y_2^1 < y_1^{1^*}$, where $y_1^{1^*}$ is the feedback Nash point of the last period and $I_1^1 = (y_2^1 - \Lambda_2^1)^+$.

Proposition 4.3.1. *If the joint probability density of the effective demand of retailer I with respect to random proportions and demand is an increasing and convex function then*

(i) J_2^1 *is concave in* y_2^1*,*

(ii) the second period best response in a two-period dynamic game can be found from

$$
\begin{aligned}
\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} &= (-r_1 - h_1)U(y_2^1) + r_1 - c_1 + \gamma c_1 + \gamma(r_1 - c_1)U\left(y_2^1 - y_1^{1*}\right) \\
&\quad - \gamma \int_L^{y_2^1 - y_1^{1*}} (+r_1 + h_1 - \nu_1)U(I_1^1)u(\Lambda_2^1)d\Lambda_2^1 = 0 \qquad (4.13)
\end{aligned}
$$

and

(iii) $R_2^{1*} \geq y_1^{1^*}$.

Proof. For the first case of ordering decisions, $y_2^1 \geq y_1^{1^*}$, the expected profit function of retailer I with two periods to go is:

$$
\begin{aligned}
&J_2^1\left(I_2^1, y_2^1, y_2^2\right) = \Pi_2^1\left(y_2^1, I_2^1, y_2^2\right) + \theta E\big[G_1^1\left(y_2^1 - \Lambda_2^1, y_1^{2*} = \max\{y_1^{2N}, I_1^2\}\right)\big] \\
&= \int_L^{y_2^1} \left[r_1 y_2^1 + (-r_1 - h_1)(y_2^1 - \Lambda_2^1) - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \int_{y_2^1}^{H} \left[r_1 y_2^1 - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \theta \Bigg\{ \int_{\left(y_2^1 - y_1^{1^*}\right)}^{H} \Bigg[\int_L^{y_1^{1^*}} \Big[r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\left(y_1^{1^*} - \Lambda_1^1\right) \\
&\qquad\qquad\qquad - c_1\left(y_1^{1^*} - I_1^1\right) \Big] u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad\quad + \int_{y_1^{1^*}}^{H} \left[r_1 y_1^{1^*} - c_1(y_1^{1^*} - I_1^1)\right] u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg] u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\
&\quad + \int_L^{\left(y_2^1 - y_1^{1^*}\right)} \Bigg[\int_L^{I_1^1} \left(r_1 I_1^1 + (-r_1 - h_1 + \nu_1)\left(I_1^1 - \Lambda_1^1\right)\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad + \int_{I_1^1}^{H} \left(r_1 x_1^1\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg] u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \Bigg\}
\end{aligned}
\tag{4.14}
$$

where $y_1^{1^*}$ is the feedback Nash point of the last period.

Derivative of the above expression with respect to retailer I's decision variable, y_2^1, is:

$$
\begin{aligned}
&\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} = \int_L^{y_2^1} (-h_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 + \int_{y_2^1}^{H} (r_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
&+ \theta \Bigg\{ - \Bigg[\int_L^{y_1^{1^*}} \left(r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\left(y_1^{1^*} - \Lambda_1^1\right)\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad + \int_{y_1^{1^*}}^{H} r_1 y_1^{1^*} u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg] u\left(y_2^1 - y_1^{1^*}\right) + \int_{y_2^1 - y_1^{1^*}}^{H} c_1 u(\Lambda_2^1) d\Lambda_2^1 \\
&\qquad + \int_L^{y_2^1 - y_1^{1^*}} \Bigg(\int_L^{I_1^1} \left[r_1 + (-r_1 - h_1 + \nu_1)\right] u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad\qquad + \int_{I_1^1}^{H} r_1 u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg) u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\
&\qquad + \Bigg(\int_L^{y_1^{1^*}} \left(r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\left(y_1^{1^*} - \Lambda_1^1\right)\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1
\end{aligned}
$$

$$+\int_{y_1^{1^*}}^{H} \left(r_1 y_1^{1^*}\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg) u\left(y_2^1 - y_1^{1^*}\right) \Bigg\} \tag{4.15}$$

When we simplify the above expression, in case of $y_2^1 \geq y_1^{1^*}$, best response of retailer I with two periods to go can be found from:

$$\begin{aligned}
\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} &= \int_L^{y_2^1} (-h_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 + \int_{y_2^1}^{H} (r_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
&\quad + \theta \Bigg\{ \int_L^{y_2^1 - y_1^{1^*}} \Bigg(\int_L^{I_1^1} (r_1 + (-r_1 - h_1 + \nu_1)) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad + \int_{I_1^1}^{H} r_1 u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg) u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\
&\quad + \int_{y_2^1 - y_1^{1^*}}^{H} c_1 u(\Lambda_2^1) d\Lambda_2^1 \Bigg\} = 0.
\end{aligned} \tag{4.16}$$

Then, by taking the integrals, we get Equation 4.13 as:

$$\begin{aligned}
\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} &= (-r_1 - h_1) U(y_2^1) + r_1 - c_1 + \theta c_1 + \theta (r_1 - c_1) U\left(y_2^1 - y_1^{1*}\right) \\
&\quad - \theta \int_L^{y_2^1 - y_1^{1*}} (+r_1 + h_1 - \nu_1) U(I_1^1) u(\Lambda_2^1) d\Lambda_2^1 = 0.
\end{aligned} \tag{4.17}$$

On the other hand, if $y_2^1 < y_1^{1^*}$ then explicit form of the expected profit function, as in Equation 4.12, is:

$$\begin{aligned}
J_2^1\left(I_2^1, y_2^1, y_2^2\right) &= \int_L^{y_2^1} \left[r_1 y_2^1 + (-r_1 - h_1)(y_2^1 - \Lambda_2^1) - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \int_{y_2^1}^{H} \left[r_1 y_2^1 - c_1(y_2^1 - I_2^1)\right] u(\Lambda_2^1) d\Lambda_2^1 \\
&+ \theta \Bigg\{ \int_L^{H} \Bigg[\int_L^{y_1^{1^*}} \left(r_1 y_1^{1^*} + (-r_1 - h_1 + \nu_1)\left(y_1^{1^*} - \Lambda_1^1\right) - c_1\left(y_1^{1^*} - I_1^1\right)\right) u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \\
&\qquad + \int_{y_1^{1^*}}^{H} \left[r_1 y_1^{1^*} - c_1(y_1^{1^*} - I_1^1)\right] u\left(\Lambda_1^1\right)\, d\Lambda_1^1 \Bigg] u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \Bigg\}.
\end{aligned} \tag{4.18}$$

First order conditions and rearrangement of the above expression give the following to find the best response of retailer I in case of $y_2^1 < y_1^{1^*}$.

$$\begin{aligned}
\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} &= \int_L^{y_2^1} (-h_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
&\quad + \int_{y_2^1}^{H} (r_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 + \theta c_1 = 0
\end{aligned} \tag{4.19}$$

Then the best response of retailer I is $R_2^{1*} = U^{-1}\left(\frac{r_1 - c_1 + \theta c_1}{r_1 + h_1}\right)$ under the assumption $y_2^1 < y_1^{1^*}$. But since y_1^{1*} is defined in Proposition 4.2.2, the result contradicts the fact that $R_2^{1*} > y_1^{1*}$. Then the solution set of best responses is empty under the case of $y_2^1 < y_1^{1^*}$ and we only have the first case to consider the concavity.

Now, we know that second retailer optimal order up to level is $y_1^{2^*} = \max\{y_1^{2N}, I_1^2\}$ where y_1^{2N} is the Nash point for the last period. Now, if $y_1^{2N} > I_1^2$ then $y_1^{2^*} = y_1^{2N}$ and $\Lambda_1^1 = pD + \gamma\left[(1-p)D - y_1^{2^*}\right]^+$. On the other hand, if $y_1^{2N} < I_1^2 = y_2^{2^*} - \Lambda_2^2$, $\left(\text{or } \Lambda_2^2 < y_2^{2^*} - y_1^{2N}\right)$ then $y_1^{2^*} = x_1^2$ and $\Lambda_1^1 = pD + \gamma\left[(1-p)D - I_1^2\right]^+$. Let Λ_1^{1-} and $\Lambda_1^{1^-}$ denote the random demand of the previous two cases respectively. Since $I_1^2 = y_2^{2^*} - \Lambda_2^2$ and $\Lambda_2^2 = \Lambda_2^2(y_2^1)$ then $I_1^2 = I_1^2(y_2^1)$ and $\Lambda_1^{1^-} = \Lambda_1^{1^-}(y_1^2)$ in case of $y_1^{2N} < I_1^2$. Therefore Equation 4.15 can be written as follows by using Equation 4.16:

$$\begin{aligned}
\frac{\partial J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial y_2^1} &= \int_L^{y_2^1} (-h_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 + \int_{y_2^1}^{H} (r_1 - c_1) u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
+\,\theta \Bigg\{ \int_L^{y_2^1 - y_1^{1^*}} & \left[\int_L^{y_2^{2^*} - y_1^{2^*}} \left(\int_L^{I_1^1} (-h_1 + b_1) u\left(\Lambda_1^{1^-}\right) d\Lambda_1^1 \right.\right. \\
&\qquad \left. + \int_{I_1^1}^{H} r_1 u\left(\Lambda_1^{1^-}\right) d\Lambda_1^1 \right) v\left(\Lambda_2^2\right) d\Lambda_2^2 \\
&\quad + \int_{y_2^{2^*} - y_1^{2^*}}^{H} \left(\int_L^{I_1^1} (-h_1 + b_1) u\left(\Lambda_1^{1-}\right) d\Lambda_1^1 \right. \\
&\qquad \left.\left. + \int_{I_1^1}^{H} r_1 u\left(\Lambda_1^{1-}\right) d\Lambda_1^1 \right) v\left(\Lambda_2^2\right) d\Lambda_2^2 \right] u\left(\Lambda_2^1\right) d\Lambda_2^1 \\
+ \int_{y_2^1 - y_1^{1^*}}^{H} c_1 u(\Lambda_2^1) d\Lambda_2^1 \Bigg\} &
\end{aligned} \tag{4.20}$$

Now the second derivative of the expected payoff with respect to y_2^1 is:

$$
\begin{aligned}
&\frac{\partial^2 J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial\left[y_2^1\right]^2} \\
&= (-h_1 - c_1)u\left(y_2^1\right) - (r_1 - c_1)u\left(y_2^1\right) \\
&\quad + \theta\Bigg\{ - c_1 u\left(y_2^1 - y_1^{1*}\right) + \int_L^{y_2^{2^*}-y_1^{2^*}} \Bigg[\int_L^{y_1^{1*}} (-h_1+\nu_1)u\left(\Lambda_1^1\right) d\Lambda_1^1 \\
&\qquad\qquad\qquad + \int_{y_1^{1*}}^H r_1 u\left(\Lambda_1^1\right) d\Lambda_1^1\Bigg] v\left(\Lambda_2^2\right)\, d\Lambda_2^2\, u\left(y_2^1 - y_1^{1*}\right) \\
&\qquad\quad + \int_{y_2^{2^*}-y_1^{2^*}}^H \Bigg[\int_L^{y_1^{1*}} (-h_1+\nu_1)u\left(\Lambda_1^1\right) d\Lambda_1^1 \\
&\qquad\qquad + \int_{y_1^{1*}}^H r_1 u\left(\Lambda_1^1\right) d\Lambda_1^1\Bigg] v\left(\Lambda_2^2\right)\, d\Lambda_2^2\, u\left(y_2^1 - y_1^{1*}\right) \\
&\quad + \int_L^{y_2^1-y_1^{1^*}} \int_L^{y_2^{2^*}-y_1^{2^*}} \Bigg[(-r_1 - h_1 + \nu_1)u\left(I_1^1\right) \\
&\qquad + \int_L^{I_1^1} (-\gamma)(-h_1+\nu_1)P\Big(pD > y_2^{1^*}\Big)P\Big((1-p)D > I_1^2\Big)u'\left(\Lambda_1^1\right) d\Lambda_1^1 \\
&\qquad + \int_{I_1^1}^H (-\gamma)(r_1)P\Big(pD > y_2^{1^*}\Big)P\Big((1-p)D > I_1^2\Big)u'\left(\Lambda_1^1\right) d\Lambda_1^1\Bigg] v\left(\Lambda_2^2\right)\, d\Lambda_2^2 \\
&\qquad \times u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\
&\quad + \int_L^{y_2^1-y_1^{1^*}} \int_{y_2^{2^*}-y_1^{2^*}}^H (-r_1 - h_1 + \nu_1)u\left(I_1^1\right) v\left(\Lambda_2^2\right)\, d\Lambda_2^2\, u\left(\Lambda_2^1\right)\, d\Lambda_2^1\Bigg\}
\end{aligned}
\tag{4.21}
$$

By taking the integrals and after some algebra, second derivative can be written as:

$$
\begin{aligned}
\frac{\partial^2 J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial\left[y_2^1\right]^2} \quad = \quad & (-r_1 - h_1)u\left(y_2^1\right) \\
& + \theta\Bigg\{ - c_1 u\left(y_2^1 - y_1^{1*}\right) \\
& \quad + \Big[(-h_1+\nu_1)U\left(y_1^{1*}\right) + r_1\left(1 - U\left(y_1^{1*}\right)\right)\Big] u\left(y_2^1 - y_1^{1*}\right) \\
& \quad + \int_L^{y_2^1-y_1^{1^*}} (-r_1 - h_1 + \nu_1)u\left(I_1^1\right) u\left(\Lambda_2^1\right)\, d\Lambda_2^1 \\
& \quad + \int_L^{y_2^1-y_1^{1^*}} \int_L^{y_2^{2^*}-y_1^{2^*}} (-\gamma)P\Big(pD > y_2^{1^*}\Big)P\Big((1-p)D > I_1^2\Big) \\
& \qquad \times \Bigg[\int_L^{I_1^1} (-h_1+\nu_1)u'\left(\Lambda_1^1\right) d\Lambda_1^1
\end{aligned}
$$

$$+\int_{I_1^1}^{H} r_1 u'\left(\Lambda_1^1\right) d\Lambda_1^1\Bigg] v\left(\Lambda_2^2\right) d\Lambda_2^2 u\left(\Lambda_2^1\right) d\Lambda_2^1\Bigg\} \quad (4.22)$$

A rearrangement on the last integral in the Equation 4.22 satisfies the following:

$$\begin{aligned}&\int_{L}^{I_1^1} (-h_1+\nu_1) u'\left(\Lambda_1^1\right) d\Lambda_1^1\\ &\qquad+\int_{L}^{H} r_1 u'\left(\Lambda_1^1\right) d\Lambda_1^1 - \int_{L}^{I_1^1} r_1 u'\left(\Lambda_1^1\right) d\Lambda_1^1\\ &=\int_{L}^{I_1^1} (r_1-h_1+\nu_1) u'\left(\Lambda_1^1\right) d\Lambda_1^1\\ &\qquad+\int_{I_1^1}^{H} r_1 u'\left(\Lambda_1^1\right) d\Lambda_1^1 - \int_{L}^{I_1^1} r_1 u'\left(\Lambda_1^1\right) d\Lambda_1^1\end{aligned} \quad (4.23)$$

The first term is positive on the right hand side of the above equation. The terms $r_1[u(H) - u(I_1^1) - u(I_1^1) + u(L)]$ are from the second and the third integrals on the right hand side. It is positive by using the convexity assumption of u. Therefore the fifth term is negative in Equation 4.22. Since u is also an increasing function from assumption then $\frac{\partial^2 J_2^1}{\partial {y_1^1}^2} < 0$. ☐

Theorem 4.3.2. *Let $y_n^{i^*}$ denote the equilibrium order-upto-level of retailer i, when there are n periods ($n = 2, 1$) until the end of the horizon of the two-period single product problem with demand substitution. The Nash equilibrium order-up-to-levels of the two retailers in periods are given by the order-up-to levels in a single-product two-period stochastic inventory model under the conditions of Proposition 4.3.1. That is,*

$$R_n^{i*}(y_n^j, I_n^i) = \begin{cases} y_n^{i^*} & \textit{when } I_n^i \le y_n^{i^*}, \\ I_n^i & \textit{when } I_n^i > y_n^{i^*}. \end{cases}$$

where I_n^i is the inventory level of the retailer i at the beginning of the period n and $i = 1, 2$.

Thus, the equilibrium point of each retailer in the two-period game is the order-up-to levels such that they can be equal to inventory level if it is high enough.

Lemma 4.3.3. *If I_1^1, the inventory level of retailer I at the beginning of the last period* *is above a threshold which is* $U^{-1}\left(\frac{r_1}{r_1+h_1-\nu_1}\right)$*, then* $\frac{dy_2^1}{dy_2^2} < 0$*. Otherwise the relation is undetermined.*

Proof. By partial derivatives of implicit functions:

$$\frac{\partial y_2^1}{\partial y_2^2} = -\frac{\frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2}}{\frac{\partial^2 J_2^1}{\partial [y_2^1]^2}} \tag{4.24}$$

where $J_2^1 = J_2^1\left(I_2^1, y_2^1, y_2^2\right)$. By the previous proposition the denominator is negative under the increasing probability density function assumption. The nominator is:

$$\begin{aligned}
&\frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2}\\
&= \theta\Bigg\{ \int_L^{{y_2^1-y_1^1}^*} \Bigg[\int_{{y_2^2-y_1^2}^*}^H \gamma\left[(-h_1+\nu_1)u(I_1^1) - r_1 u(I_1^1)\right] \\
&\qquad\qquad \times P\big((1-p)D > y_2^2\big) v\left(\Lambda_2^2\right) d\Lambda_2^2 u\left(\Lambda_2^1\right) \\
&\qquad + \int_{{y_2^2-y_1^2}^*}^H \left(\int_L^{I_1^1} (-h_1+\nu_1) u\left(\Lambda_1^1\right) \, d\Lambda_1^1 + \int_{I_1^1}^H r_1 u\left(\Lambda_1^1\right) \, d\Lambda_1^1 \right) \\
&\qquad\qquad \times v\left(\Lambda_2^2\right) d\Lambda_2^2 (-\gamma) P\big((1-p)D > y_2^2\big) u'\left(\Lambda_2^1\right) \Bigg] \, d\Lambda_2^1 \\
&\quad + \int_{{y_2^1-y_1^1}^*}^H c_1(-\gamma) P\big((1-p)D > y_2^2\big) u'(\Lambda_2^1) d\Lambda_2^1 \Bigg\}
\end{aligned} \tag{4.25}$$

The above partial derivative is negative under the assumption: $I_1^1 \geq U^{-1}\left(\frac{r_1}{r_1+h_1-\nu_1}\right)$. Then

$$\frac{\partial y_2^1}{\partial y_2^2} < 0 \tag{4.26}$$

□

Single period best response, $y_1^{1*} = U^{-1}\left(\frac{r_1-c_1}{r_1+h_1-\nu_1}\right)$ is less than the above threshold, $U^{-1}\left(\frac{r_1}{r_1+h_1-\nu_1}\right)$, since $c_1 > 0$. If inventory level of the last period is equal or higher than this threshold, then at the beginning of the game, when two periods to go, retailer

I has large amount of inventory. If retailer II increases his order point, substitution effect will decrease and retailer I will deal with a small amount of substituted demand with his own customers. But since he has a large amount of inventory, decreasing the ordering level or doing nothing will be a reasonable consequence.

On the other hand, if the last period inventory level, I_1^1, is less than the above threshold, than the above result can not be apply. In that case, comparison between y_1^{1*} and I_1^1 has to be considered. If $y_1^{1*} > I_1^1$ then it means retailer I will face enough amount of demand in the beginning period to go under y_1^{1*} because of the third statement of Proposition 4.3.1 and at the beginning of the last period there will be a replenishment. Therefore at the very beginning of the game, he may increase the first replenishment level. As a second case, if $y_1^{1*} < I_1^1$ then whatever the demand in the beginning period is retailer I may decrease the starting order.

Proposition 4.3.4. *As a special case; if the density of the effective demand of retailer I is uniformly distributed over* $[L, H]$ *then there is a unique Nash equilibrium for every period in a two-period dynamic game.*

Proof. We show the uniqueness of Nash equilibrium by using contraction mapping argument and the proof is completed by the Implicit Function Theorem in [5]. The following which is called "diagonal dominance" should be satisfied on response functions for the uniqueness of the Nash equilibrium:

$$\sum_{i=1, i\neq k}^{n} \left| \frac{\partial^2 J^k}{\partial y^k \partial y^i} \right| \leq \left| \frac{\partial^2 J^k}{\partial [y^k]^2} \right|, \quad \forall k. \tag{4.27}$$

Now the second derivative of the expected payoff function by implicit differentiation with respect to y_2^1 and y_2^2 respectively is:

$$\begin{aligned} \frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2} &= (-r_1 - h_1) u\left(y_2^1\right) \frac{\partial y_2^1}{\partial y_2^2} \\ &\quad + \theta \Bigg\{ -c_1 f\left(y_2^1 - y_1^{1*}\right) \frac{\partial y_2^1}{\partial y_2^2} \end{aligned}$$

$$\begin{aligned}
&+\left[(-h_1+\nu_1)U\left(y_1^{1*}\right)+r_1\left(1-U\left(y_1^{1*}\right)\right)\right]u\left(y_2^1-y_1^{1*}\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&+\int_L^{y_2^1-{y_1^1}^*}(-r_1-h_1+\nu_1)u\left(I_1^1\right)\left(\frac{\partial y_2^1}{\partial y_2^2}+\gamma\right)u\left(\Lambda_2^1\right)\,d\Lambda_2^1 \\
&+\int_L^{y_2^1-{y_1^1}^*}\left[r_1+(-r_1-h_1+\nu_1)U\left(I_1^1\right)\right](-\gamma)\,u'\left(\Lambda_2^1\right)d\Lambda_2^1\Bigg\}
\end{aligned} \tag{4.28}$$

The second integral can be evaluated by using integration by parts method;

$$\begin{aligned}
\frac{\partial^2 J_2^1}{\partial y_2^1\partial y_2^2} &= (-r_1-h_1)u\left(y_2^1\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&\quad+\theta\Bigg\{-c_1u\left(y_2^1-y_1^{1*}\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&\qquad+\left[r_1+(-r_1-h_1+\nu_1)U\left(y_1^{1*}\right)\right]u\left(y_2^1-y_1^{1*}\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&\qquad+\int_L^{y_2^1-{y_1^1}^*}(-r_1-h_1+\nu_1)u\left(I_1^1\right)\left(\frac{\partial y_2^1}{\partial y_2^2}+\gamma\right)u\left(\Lambda_2^1\right)\,d\Lambda_2^1 \\
&\qquad-\gamma\left[r_1+(-r_1-h_1+\nu_1)U\left(y_1^{1*}\right)\right]u\left(y_2^1-y_1^{1*}\right) \\
&\qquad+\gamma\left[r_1+(-r_1-h_1+\nu_1)U\left(y_2^1-L\right)\right]u\left(L\right) \\
&\qquad-\gamma\int_L^{y_2^1-{y_1^1}^*}(-r_1-h_1+\nu_1)u\left(I_1^1\right)u\left(\Lambda_2^1\right)\,d\Lambda_2^1\Bigg\}
\end{aligned} \tag{4.29}$$

after cancellation and grouping:

$$\begin{aligned}
\frac{\partial^2 J_2^1}{\partial y_2^1\partial y_2^2} &= (-r_1-h_1)u\left(y_2^1\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&\quad+\theta\Bigg\{-c_1u\left(y_2^1-y_1^{1*}\right)\frac{\partial y_2^1}{\partial y_2^2} \\
&\qquad+\left[r_1+(-r_1-h_1+\nu_1)U\left(y_1^{1*}\right)\right]u\left(y_2^1-y_1^{1*}\right)\left[\frac{\partial y_2^1}{\partial y_2^2}-\gamma\right] \\
&\qquad+\int_L^{y_2^1-{y_1^1}^*}(-r_1-h_1+\nu_1)u\left(I_1^1\right)\frac{\partial y_2^1}{\partial y_2^2}u\left(\Lambda_2^1\right)\,d\Lambda_2^1 \\
&\qquad+\gamma\left[r_1+(-r_1-h_1+\nu_1)U\left(y_2^1-L\right)\right]u\left(L\right)\Bigg\}
\end{aligned} \tag{4.30}$$

As a special case, if u is uniformly distributed over $[L,H]$ then the above expression will be:

$$\begin{aligned}\frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2} &= \frac{(-r_1 - h_1)}{H-L}\frac{\partial y_2^1}{\partial y_2^2} \\ &\quad + \theta\left\{\frac{-c_1}{H-L}\frac{\partial y_2^1}{\partial y_2^2}\right. \\ &\qquad + \left[r_1 + (-r_1 - h_1 + \nu_1)\frac{y_1^{1*}}{H-L}\right]\frac{1}{H-L}\left(\frac{\partial y_2^1}{\partial y_2^2} - \gamma\right) \\ &\qquad + \int_L^{y_2^1 - y_1^{1^*}} (-r_1 - h_1 + \nu_1)\frac{1}{(H-L)^2}\frac{\partial y_2^1}{\partial y_2^2}\, d\Lambda_2^1 \\ &\qquad \left. +\gamma\left[r_1 + (-r_1 - h_1 + \nu_1)\frac{(y_2^1 - L)}{H-L}\right]\frac{1}{H-L}\right\}\end{aligned} \tag{4.31}$$

Taking the above integral and grouping the terms give us the following:

$$\begin{aligned}\frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2} &= \frac{(-r_1 - h_1)}{H-L}\frac{\partial y_2^1}{\partial y_2^2} \\ &\quad + \theta\left\{\frac{-c_1}{H-L}\frac{\partial y_2^1}{\partial y_2^2}\right. \\ &\qquad + \left[r_1 + (-r_1 - h_1 + \nu_1)\frac{y_1^{1*}}{H-L}\right]\frac{1}{H-L}\left(\frac{\partial y_2^1}{\partial y_2^2}\right) \\ &\qquad \left. +(-r_1 - h_1 + \nu_1)\left(y_2^1 - y_1^{1*} - L\right)\frac{1}{(H-L)^2}\left(\frac{\partial y_2^1}{\partial y_2^2} + \gamma\right)\right\}\end{aligned} \tag{4.32}$$

Now, Equation 4.29 is compared to $\frac{\partial^2 J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial\left[y_2^1\right]^2}$ under the uniformly distributed density function assumption. By using Equation 4.22, second derivative of the expected profit is:

$$\begin{aligned}\frac{\partial^2 J_2^1\left(I_2^1, y_2^1, y_2^2\right)}{\partial\left[y_2^1\right]^2} &= \frac{(-r_1 - h_1)}{H-L} \\ &\quad + \theta\left\{\frac{-c_1}{H-L}\right. \\ &\qquad + \left[r_1 + (-r_1 - h_1 + \nu_1)\frac{y_1^{1*}}{H-L}\right]\frac{1}{H-L} \\ &\qquad \left. +(-r_1 - h_1 + \nu_1)\left(y_2^1 - y_1^{1*} - L\right)\frac{1}{(H-L)^2}\right\}\end{aligned} \tag{4.33}$$

Now, since $\frac{\partial y_2^1}{\partial y_2^2}$ is negative and between -1 and 0 after a threshold where orders go beyond the equilibrium then $\left|\frac{\partial y_2^1}{\partial y_2^2} + \beta\right| < 1$. Therefore the triangle inequality gives the following:

$$\begin{aligned}
\left|\frac{\partial^2 J_2^1}{\partial y_2^1 \partial y_2^2}\right| \quad &\leq \quad \left|\frac{(-r_1 - h_1)}{H - L}\frac{\partial y_2^1}{\partial y_2^2}\right| + \left|\theta\frac{-c_1}{H - L}\frac{\partial y_2^1}{\partial y_2^2}\right| \\
&+ \left|\theta\left(r_1 + (-r_1 - h_1 + \nu_1)\frac{y_1^{1*}}{H - L}\right)\frac{1}{H - L}\frac{\partial y_2^1}{\partial y_2^2}\right| \\
&+ \left|\theta(-r_1 - h_1 + \nu_1)(y_2^1 - y_1^{1*} - L)\frac{1}{(H - L)^2}\left(\frac{\partial y_2^1}{\partial y_2^2} + \gamma\right)\right| \\
&< \left|\frac{(-r_1 - h_1)}{H - L}\right| + \left|\theta\frac{-c_1}{H - L}\right| \\
&+ \left|\theta\left(r_1 + (-r_1 - h_1 + \nu_1)\frac{y_1^{1*}}{H - L}\right)\frac{1}{H - L}\right| \\
&+ \left|\theta(-r_1 - h_1 + \nu_1)\left(y_2^1 - y_1^{1*} - L\right)\frac{1}{(H - L)^2}\right| \\
&= \left|\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}\right|. \qquad (4.34)
\end{aligned}$$

Since all the terms in $\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}$ are negative then taking the complete absolute value is equal to taking it termwise. So the last equality holds. □

Proposition 4.3.5. *The rate of change of the equilibrium point of retailer I in two-period case is increasing with respect to sale price, r_1, and salvage value, ν_1, but decreasing with respect to holding cost, h_1, and ordering cost, c_1.*

Proof.

$$\frac{\partial y_2^1}{\partial r_1} = -\frac{\frac{\partial^2 J_2^1}{\partial y_2^1 \partial r_1}}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}}$$

$$= \quad \frac{\left[-U(y_2^1) + 1 + \theta U(y_2^1 - y_1^{1*}) - \theta\int_L^{y_2^1 - {y_1^1}^*} U(I_1^1) f(\Lambda_2^1) d\Lambda_2^1\right]}{-\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}}$$

$$= \quad \frac{\left[-U(y_2^1) + 1 + \theta U(y_2^1 - y_1^{1*}) - \theta U(y_1^{1*}) U(y_2^1 - y_1^{1*}) + \theta\int_L^{y_2^1 - {y_1^1}^*} u(I_1^1) U(\Lambda_2^1) d\Lambda_2^1\right]}{-\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}}$$

$$= \frac{\left[-U(y_2^1) + 1 + \theta\left[1 - U(y_1^{1*})\right] U(y_2^1 - y_1^{1*}) + \int_L^{y_2^1 - y_1^{1^*}} u(I_1^1) U(\Lambda_2^1) d\Lambda_2^1\right]}{-\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} \tag{4.35}$$

Since $U(y_2^1) \leq 1$ and $U(y_1^{1*}) \leq 1$ then numerator of the first equation has a positive sign. The denominator has a negative sign from the concavity of the expected profit by Proposition 4.3.1. Then the sign of the above expression is positive.

$$\begin{aligned} \frac{\partial y_2^1}{\partial b_1} &= -\frac{\frac{\partial^2 J_2^1}{\partial y_2^1 \partial b_1}}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} \\ &= -\frac{\left[\theta \int_L^{y_2^1 - y_1^{1^*}} U(I_1^1) u(\Lambda_2^1) d\Lambda_2^1\right]}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} > 0 \end{aligned} \tag{4.36}$$

All of the below have negative signs.

$$\begin{aligned} \frac{\partial y_2^1}{\partial h_1} &= -\frac{\frac{\partial^2 J_2^1}{\partial y_2^1 \partial h_1}}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} \\ &= -\frac{\left[-U(y_2^1) - \theta \int_L^{y_2^1 - y_1^{1^*}} U(I_1^1) u(\Lambda_2^1) d\Lambda_2^1\right]}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} < 0 \end{aligned} \tag{4.37}$$

$$\begin{aligned} \frac{\partial y_2^1}{\partial c_1} &= -\frac{\frac{\partial^2 J_2^1}{\partial y_2^1 \partial c_1}}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} \\ &= -\frac{\left[-1 + \theta - \theta U(y_2^1 - y_1^{1^*})\right]}{\frac{\partial^2 J_2^1}{\partial\left[y_2^1\right]^2}} < 0 \end{aligned} \tag{4.38}$$

While the nominator of the first and the second equations have positive signs, the others have negative signs. Since the denominator of all right hand sides are negative then rate of change of the optimal order-up-to-level of retailer I with two periods to go is increasing with respect to sale price and salvage value but decreasing with respect to holding and ordering costs. □

In the static game described by Avşar and Baykal-Gürsoy in [1], Nash equilibria characterized by stationary order-up-to-levels are within stationary base stock strategies. On the contrary, our strategy space consists of decisions without restrictions such as base stock characteristics. In our model, parties review their decisions based on the historical data and choose their orders dynamically from this general set of decisions at the beginning of any period. The existence and uniqueness of the Nash equilibrium are shown with an arbitrary probability density and a uniformly distributed density functions respectively by Proposition 4.3.1 and 4.3.4 and those points are within order-up-to-levels.

5. COMPETITION UNDER SUBSTITUTION WITH CUSTOMER SATISFACTION

5.1. Definitions and the Model

In this chapter, we consider the single period formulations of a substitutable product inventory control with a customer satisfaction measure. Every retailer has an inventory level and a given customer satisfaction level at the beginning of the period. They try to maximize their own profit during the period and they want to finish the period with the highest customer satisfaction level as much as possible to increase the market value of their own company. This measure is related to satisfy the effective demand which consists of their own customers and/or substituted demand from the other retailer within the period. It is defined as a piecewise function due to the different realizations of effective demand. In the first case, we consider a higher ordering level than the effective demand which includes substituted demand. In that case, retailers' customer satisfaction measure is increased by a function of difference between the effective demand and their own customers' demand at the end of the period. In the second case, the ordering level is between the effective demand and its own customers' demand. The respective satisfaction level is affected by increase of its own customers demand considered the difference between ordering level and its own customers' demand level. Finally, the third case; the replenishment level is less than the retailer's own customers demand level, has a negative effect on the customer satisfaction measure due to the lack of goods and unsatisfied customers. This model refers to a brand new business structure which can be seen mostly in on-line marketing and multimedia sectors. Small internet marketing firms are founded to find a virgin territory to boost up their sales and brand awareness. Then, within months or seasons, those kinds of short-life corporates are sold to multi-national chains or holdings with a high customer brand loyalty or attitude toward the brand due to brand awareness and customer satisfaction. Here, we try to analyze a two-person game emphasizing the effect of customer satisfaction for that kind of business.

Hall and Porteus [10] made an approach in their work to a finite horizon game with service competition and they analyzed the uniqueness of the Nash equilibrium. All variables that they had are related to the current period. Therefore a finite period analysis has been done. Here, we work on a single period game because our state variable, inventory level, can not be fully described one period after due to possible cases of customer satisfaction measure. We first investigate the existence and uniqueness of best response functions while considering three possible circumstances of effective demand and respective customer satisfaction measures. We determine each player's best response function by finding the action that maximizes its payoff for any given action of the other player. They are found by using first order conditions. Uniqueness of best response is demonstrated by using the second order conditions. In this model, best responses depend on the other retailer's decisions, parameters and the customer satisfaction measure. Finally we show the uniqueness of the Nash equilibrium by using the contraction mapping argument and diagonal dominance as in [6]. We also give the comparative statics of the optimal ordering level and the customer satisfaction at the beginning of the period.

5.2. Customer Satisfaction in the Single Period

Total demand, D, is deterministic in this model. Since the model lasts only one period, we drop the time index in the notation. Therefore the effective demand of retailer I is $\Lambda^1 = pD + \gamma\left[(1-p)D - y^2\right]^+$ and in the interval $[L, H]$ where the proportion p is random and between [0,1]. We take pD and $(1-p)D$ as D_1 and D_2 for retailers respectively. Therefore D_1 and D_2 are negatively correlated. Single period expected profit function of retailer I with a customer satisfaction measure, Θ_j, with j period(s) to go, $j = 0, 1$, is given as:

$$\begin{aligned}\Pi^1(y^1, I^1, y^2) &= r_1 E\left[\min\left(y^1, \Lambda^1\right)\right] - h_1 E\left[(y^1 - \Lambda^1)^+\right] \\ &\quad - c_1(y^1 - x^1) + \nu_1 E\left[(y^1 - \Lambda^1)^+\right] + b\Theta_0 \qquad (5.1)\end{aligned}$$

where b is the salvage value related to satisfaction measure. As in Chapter 3, the expected profit function of retailer I per unit demand is:

$$\begin{aligned}
\pi^1 &= -c_1 \frac{y^1 - I^1}{D} \\
&\quad + \int_0^{\hat{\alpha}} \left[r_1 p + r_1 \gamma \left(1 - p - \frac{y^2}{D} \right) \right. \\
&\qquad \left. + (-h_1 + \nu_1) \left(\frac{y^1}{D} - p - \gamma \left(1 - p - \frac{y^2}{D} \right) \right) + b\lambda_0 \right] f(p)dp \\
&\quad + \int_{\hat{\alpha}}^{y^1/D} \left[r_1 \frac{y^1}{D} + b\lambda_0 \right] f(p)dp + \int_{y^1/D}^{1} \left[r_1 \frac{y^1}{D} + b\lambda_0 \right] f(p)dp \qquad (5.2)
\end{aligned}$$

where $\Theta_j / D = \lambda_j$ and

$$\lambda_0 = \begin{cases} \lambda_1 \left(1 + \delta_1 \frac{E\left[(\Lambda_1 - D_1)^+\right]}{\epsilon_\lambda} \right) & \text{when } y^1 \geq \Lambda_1, \\ \lambda_1 \left(1 + \delta_1 \frac{E\left[(y^1 - D_1)\right]}{\epsilon_\lambda} \right) & \text{when } \Lambda_1 > y^1 \geq D_1, \\ \lambda_1 \left(1 - \delta_1 \frac{E\left[D_1 - y^1\right]}{\epsilon_\lambda} \right) & \text{when } y^1 < D_1. \end{cases}$$

Here, $\hat{\alpha} = \frac{\frac{y^1}{D} - \gamma\left(1 - \frac{y^2}{D}\right)}{1-\gamma}$ is the p-value where the excess inventory of retailer I and coming from the equation $y^1 = pD + \gamma\left[(1-p)D - y^2\right]^+$ and ϵ_λ which is used to normalize the above expressions, is $H - L$.

First term in the right hand-side of the above Equation 5.2 expresses the total ordering cost. In the second term, retailer II has a stockout but retailer I has inventory left over after satisfying his own demand and substitution demand coming from retailer II if it exists. The third term is the case where retailer I can meet his own demand and at the same time retailer II can. The last integral represents that retailer I has a stockout.

Customer satisfaction measure, λ_0, has three cases. In the first case, we consider a higher ordering level than the effective demand which includes substituted demand. In that case, retailers' customer satisfaction measure is increased by the difference between the effective demand and their own customers' demand at the end of the period. In the second case, the ordering level is between the effective demand and its own customers' demand. The respective satisfaction level is affected by increase of its own customers demand while considering the difference between ordering level and its own customers' demand level. Finally, the third case; the replenishment level is less than the retailer's own customers demand level, has a negative effect on the customer satisfaction measure due to the lack of goods and unsatisfied customers.

Proposition 5.2.1. *Best response of retailer I(II) is unique for a given ordering level of retailer II(I) for the single period two-player game [20] and given by:*

$$R_1^i\left(y^j, I^i\right) = \begin{cases} y^i & \text{when } y^i \geq I_1^i, \\ I^i & \text{when } y^i < I_1^i. \end{cases}$$

where y^i*, the order up to level of retailer* i*,* $i = 1, 2$*, is* $D(1-\gamma)F^{-1}\left(\frac{r_1 - c_1 + b\lambda_1 \frac{\delta}{\varepsilon_\lambda}}{r_1 + h_1 - \nu_1 + b\lambda_1 \frac{\delta}{\varepsilon_\lambda}}\right) + \gamma(D - y^2)$ *and* λ_1^i *is a customer satisfaction measure for retailer* i *when we have one period to go.*

Proof. To show the existence of the best response function of retailer I, we use first order conditions of expected profit function of retailer I.

$$\begin{aligned} \frac{\partial \pi^1}{\partial y^1} &= -\frac{c_1}{D} - \frac{1}{D(1-\gamma)} b\lambda_1^1 \left(1 + \frac{\delta}{\epsilon_\lambda}\left(\frac{y^1}{D} - \hat{\alpha}\right)\right) u(\hat{\alpha}) \\ &\quad + \frac{1}{D(1-\gamma)} b\lambda_1^1 \left(1 + \frac{\delta}{\epsilon_\lambda}\gamma\left(1 - \hat{\alpha} - \frac{y^2}{D}\right)\right) u(\hat{\alpha}) \\ &\quad + \frac{1}{D}\int_0^{\hat{\alpha}} (-h_1 + \nu_1) u(p)dp + \int_{\hat{\alpha}}^1 \left[\frac{r_1}{D} + b\lambda_1^1 \frac{\delta}{\epsilon_\lambda}\frac{1}{D}\right] u(p)dp \end{aligned} \tag{5.3}$$

Second and third terms of the above equation can be arranged as follows:

$$\begin{aligned} \frac{\partial \pi^1}{\partial y^1} &= -\frac{c_1}{D} - \frac{1}{D(1-\beta)} b\lambda_1^1 \left(1 + \frac{\delta}{\epsilon_\lambda}\left[\frac{y^1}{D} - \hat{\alpha} - \gamma\left(1 - \hat{\alpha} - \frac{y^2}{D}\right)\right]\right) u(\hat{\alpha}) \\ &\quad + \frac{1}{D}\int_0^{\hat{\alpha}} (-h_1 + \nu_1) u(p)dp + \int_{\hat{\alpha}}^1 \left[\frac{r_1}{D} + b\lambda_1^1 \frac{\delta}{\epsilon_\lambda}\frac{1}{D}\right] u(p)dp \end{aligned} \tag{5.4}$$

Since $\frac{y^1}{D} - \hat{\alpha} - \gamma\left(1 - \hat{\alpha} - \frac{y^2}{D}\right) = 0$ then

$$\frac{\partial \pi^1}{\partial y^1} = -\frac{c_1}{D} + \frac{1}{D}\int_0^{\hat{\alpha}} (-h_1 + \nu_1)u(p)dp + \int_{\hat{\alpha}}^1 \left[\frac{r_1}{D} + b\lambda_1^1 \frac{\delta}{\epsilon_\lambda}\frac{1}{D}\right] u(p)dp \qquad (5.5)$$

The best response function of retailer I, $R_1^1(y^2, I^1)$, for any given action of retailer II, y^2, can be found explicitly in terms of $\hat{\alpha}$ and the satisfaction measure λ_1^1 by using the first order conditions, i.e. $\frac{\partial \pi^1}{\partial y^1} = 0$ in Equation 5.5.

$$R_1^1\left(y^2, I^1\right) = D(1-\gamma)U^{-1}\left(\frac{r_1 - c_1 + b\lambda_1\frac{\delta}{\varepsilon_\lambda}}{r_1 + h_1 - \nu_1 + b\lambda_1\frac{\delta}{\varepsilon_\lambda}}\right) + \gamma(D - y^2). \qquad (5.6)$$

We use concavity of the single period expected profit function of retailer I to show the uniqueness of the response function. Second derivative of the expected profit function of retailer I with respect to y^1 is:

$$\frac{\partial^2 \pi^1}{\partial {y^1}^2} = \frac{1}{D^2(1-\gamma)}(-h_1 + \nu_1)u(\hat{\alpha}) - \frac{1}{D^2(1-\gamma)}\left(\frac{r_1}{D} + b\lambda_1\frac{\delta}{\varepsilon_\lambda}\frac{1}{D}\right)u(\hat{\alpha}) \qquad (5.7)$$

Since $h_1 > \nu_1$ in the first term then all the terms of the above expression have negative signs. Therefore $\frac{\partial^2 \pi^1}{\partial {y^1}^2} < 0$ and concavity follows. $\square$

Now, to show the uniqueness of the Nash equilibrium, contraction mapping argument and Implicit Function Theorem are used [5]. As the proof of Proposition 3.2.3 in the Chapter 3, "diagonal dominance" should be satisfied on response functions for the uniqueness of the Nash equilibrium. Cross partial derivative of the expected profit of retailer I with respect to actions chosen by retailers is:

$$\frac{\partial^2 \pi^1}{\partial y^1 \partial y^2} = \frac{\gamma}{D^2(1-\gamma)}(-h_1 + \nu_1)u(\hat{\alpha}) - \frac{\gamma}{D^2(1-\gamma)}\left(\frac{r_1}{D} + b\lambda_1^1\frac{\delta}{\epsilon_\lambda}\frac{1}{D}\right)u(\hat{\alpha}) \qquad (5.8)$$

Since $\gamma < 1$ then $\left|\frac{\partial^2 \pi^1}{\partial y^1 \partial y^2}\right| < \left|\frac{\partial^2 \pi^1}{\partial {y^1}^2}\right|$.

5.3. Comparative Statics

Here, we describe how equilibrium behaviour and outcomes change as a function of the exogenous parameter; the satisfaction measure. We begin with solving for the game's unique Nash equilbrium and calculating the element of interest by using Equation 5.5 and Equation 5.8. Taking the derivative of the Nash equilibrium with respect to the satisfaction measure, λ_1^1, helps us to see how changing the parameter affects the element of interest.

Proposition 5.3.1. *The Nash equilibrium of the single period two-player game with a customer satisfaction measure increases as a function of customer satisfaction measure of retailer I,* λ_1^1.

Proof. By taking the cross partial derivative of the expected profit function of retailer I with respect to retailer I's order decision and the customer satisfaction measure respectively, the change in the Nash equilibrium is:

$$\frac{\partial^2 \pi^1}{\partial y^1 \partial \lambda_1^1} = \int_{\hat{\alpha}}^{1} b \frac{\delta}{\epsilon_\lambda} \frac{1}{D} u(p) dp \geq 0 \tag{5.9}$$

Since it is always positive, the proof is completed. □

5.4. Stackelberg Equilibria

In our Stackelberg type game, there are two players, a leader and a follower and they compete on quantity. In the Stackelberg model, despite of moving simultaneously (as in the Cournot model) the firms move sequentially. The leader chooses the quantity first and the other firm, the follower, observes the leader's quantity and then chooses his quantity. The leader knows that the follower observes his action. In our model, retailer I is the follower and retailer II is the leader. Retailer I observes the move of retailer II first and then he decides his order level. For any given ordering level of retailer II, y^2, retailer I's best response function is depicted by Equation 5.3 in terms of retailer II's action. Now we formulate the expected profit of retailer II with the customer satisfaction measure, λ_1^2, when one period to go to find the retailer II's, the leader, action. Expected profit of retailer II is:

$$\begin{aligned}\pi^2 &= -c_2\frac{y^2-I^2}{D}\\ &\quad+\int_0^{\hat{\gamma}}\left[r_2p+r_2\gamma\left(p-\frac{y^1}{D}\right)\right.\\ &\qquad\left.+(-h_2+\nu_2)\left(\frac{y^2}{D}-(1-p)-\gamma\left(p-\frac{y^1}{D}\right)\right)+b\lambda_0^2\right]u(p)dp\\ &\quad+\int_{\hat{\gamma}_1}^{y^2/D}\left[r_2\frac{y^2}{D}+b\lambda_0^2\right]u(p)dp+\int_{y^2/D}^{1}\left[r_2\frac{y^2}{D}+b\lambda_0^2\right]u(p)dp \qquad (5.10)\end{aligned}$$

where $y^1 = y^1(y^2)$, the customer satisfaction measure of retailer II is

$$\lambda_0^2 = \begin{cases}\lambda_1^2\left(1+\delta_2\frac{E\left[(\Lambda_2-D_2)^+\right]}{\epsilon_\lambda}\right) & \text{when } y^2 \geq \Lambda_2,\\ \lambda_1^2\left(1+\delta_2\frac{E\left[(y^1-D_1)\right]}{\epsilon_\lambda}\right) & \text{when } \Lambda_2 > y^2 \geq D_2,\\ \lambda_1^2\left(1-\delta_2\frac{E\left[D_1-y^1\right]}{\epsilon_\lambda}\right) & \text{when } y^2 < D_2,\end{cases}$$

and $\hat{\gamma} = \frac{1-\frac{y^2}{D}-\gamma\frac{y^1}{D}}{1-\gamma}$ is the p value at which excess inventory of retailer II is equal to the inventory level when substitution occurs from retailer I and is coming from the equation $y^2 = (1-p)D + \gamma pD - \gamma y^1$.

Proposition 5.4.1. *Stackelberg equilibrium of the model is*

$$(y^{1*}, y^{2*}) = \left(\frac{D}{1+\gamma}\left[U^{-1}(A)+\beta U^{-1}(B)\right], D-\frac{D}{1+\gamma}\left[\beta U^{-1}(A)+U^{-1}(B)\right]\right) \quad (5.11)$$

where $A = \frac{r_1-c_1+b\lambda_1^1\frac{\delta}{\varepsilon_\lambda}}{r_1+h_1-\nu_1+b\lambda_1^1\frac{\delta}{\varepsilon_\lambda}}$ *and* $B = \frac{r_2-c_2+b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}{(r_2+h_2-\nu_2)(1-\gamma^2)+b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}$.

Proof. To calculate the Stackelberg equilibrium, the best response of the leader is found by substitution for the best response of the follower. The first order condition of the expected profit function of retailer II gives us the Stackelberg solution of the system where the best response function of retailer I, Equation 5.3, the follower, depends on the move of retailer II.

Now, consider the leader's problem, Equation 5.10, substituting for $R_1^1(y^2, I^1)$ from the follower's problem and taking the first derivative with respect to order decision, y^2, is:

$$\begin{aligned}\frac{\partial \pi^2}{\partial (y^2)} &= -\frac{c_2}{D} + \frac{1}{D}\int_0^{\hat{\gamma}} \left[(-h_2+\nu_2)(1-\gamma^2) + r_2\gamma^2\right] u(p)dp \\ &\quad + \frac{1}{D}\int_{\hat{\gamma}}^1 \left[r_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}\right] u(p)dp \end{aligned} \tag{5.12}$$

First order conditions are satisfied by setting it equal to 0; $\frac{\partial \pi^2}{\partial (y^2)} = 0$.

$$-\frac{c_2}{D} + \frac{1}{D}U(\hat{\gamma})\left[(-h_2+\nu_2)(1-\gamma^2) + r_2\gamma^2\right] + \frac{1}{D}\left[r_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}\right](1-U(\hat{\gamma})) = 0 \tag{5.13}$$

$$\frac{r_2 - c_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}{D} + \frac{U(\hat{\gamma})}{D}\left[(-r_2 - h_2 + \nu_2)(1-\gamma^2) - b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}\right] = 0 \tag{5.14}$$

$$\hat{\gamma} = U^{-1}\left(\frac{r_2 - c_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}\right) \tag{5.15}$$

where $\hat{\gamma} = \frac{1-\frac{y^2}{D}-\gamma\frac{y^1}{D}}{1-\gamma}$. Then the leader's optimal action, y^{2*}, is:

$$y^{2*} = D - \gamma y^1 - D(1-\gamma)F^{-1}\left(\frac{r_2 - c_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}\right) \tag{5.16}$$

By arranging the above equation and using again the best response of the follower, $y^1 = y^1(y^2)$, Equation 5.17 and Equation 5.18 are derived respectively as:

$$y^{2*} = \gamma(D - y^1) + (1-\gamma)D\left[1 - U^{-1}\left(\frac{r_2 - c_2 + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2\frac{\delta_2}{\epsilon_\lambda}}\right)\right] \tag{5.17}$$

$$\begin{aligned} y^{2*} &= \gamma \left(D - D(1-\gamma)U^{-1} \left(\frac{r_1 - c_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}}{r_1 + h_1 - \nu_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}} \right) - \gamma(D - y^{2*}) \right) \\ &\quad + (1-\gamma)D \left[1 - U^{-1} \left(\frac{r_2 - c_2 + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}} \right) \right] \end{aligned} \tag{5.18}$$

By making a rearrangement of Equation 5.18, the best response pair of retailer II and retailer I, the leader and the follower respectively, is:

$$\begin{aligned} y^{2*} &= D - \frac{D}{1+\gamma} \left[\beta U^{-1} \left(\frac{r_1 - c_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}}{r_1 + h_1 - \nu_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}} \right) \right. \\ &\quad \left. + U^{-1} \left(\frac{r_2 - c_2 + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}} \right) \right] \end{aligned} \tag{5.19}$$

$$\begin{aligned} y^{1*} &= \frac{D}{1+\gamma} \left[U^{-1} \left(\frac{r_1 - c_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}}{r_1 + h_1 - \nu_1 + b\lambda_1^1 \frac{\delta}{\varepsilon_\lambda}} \right) \right. \\ &\quad \left. + \beta U^{-1} \left(\frac{r_2 - c_2 + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}}{(r_2 + h_2 - \nu_2)(1-\gamma^2) + b\lambda_1^2 \frac{\delta_2}{\epsilon_\lambda}} \right) \right] \end{aligned} \tag{5.20}$$

□

If all cost parameters are the same and unsatisfied customers have similar discontentment then the leader optimal ordering is higher than the follower's decision;

$$y^{2*} - y^{1*} = D - D\left[U^{-1}(A) + U^{-1}(B)\right] \tag{5.21}$$

Since U is a cumulative distribution of proportions then $U^{-1}(A)$ and $U^{-1}(B)$ are the shared proportions of retailers I and II respectively. Since the total demand is deterministic, the summation $U^{-1}(A) + U^{-1}(B)$ is less than 1. It concludes that $y^{2*} > y^{1*}$. Then the leader, retailer II, has a significant advantage to serve more customers than the follower, retailer I. On the other hand, the commitment that the leader has would be a benefit for the follower. If the proportion expectancy of the follower decreases momentarily after the leader's commitment then the follower can avoid a loss by giving orders less than his optimal.

6. CONCLUSION AND FUTURE WORK

This thesis dealt with a dynamic doupoly game of a substitutable product inventory control problem. We investigated three cases in three chapters which depend on different demand assumptions, i.e. deterministic and random, and ordering priorities. We generally made a finite horizon analysis including a single period result in a Stackelberg game. With this work, we wanted to give an insight to corporates in a competitive environment especially in the retail industry, on customers' brand loyalty which often outweighs their store loyalty. While considering service and product quality which can be seen brand image in most cases, they drive customer loyalty as measured by a customer's willingness to make shopping and/or recommend the retailer's store to other people. Service and operations management managers can improve these drivers of customer loyalty by better training, reward programs, day-to-day store operations which include job, product and store designs and most importantly adequate diversification of products within inventory limits. Therefore, we first focused on a generic demand model in a finite horizon case. Stochasticity came from the market share and substitution potential due to lack of goods gave a game theoretic feature to the problem. We showed the uniqueness of the optimality while parties gave orders at the same time with a condition on upper bound of total ordering units. Contrapositive of this condition was investigated in [16]. We proved the importance of substitution effect as a contribution in the game with our condition. This situation can be seen widely in competitive and high profile markets such as new generation communication devices or designer products. We also proposed an ordering policy to parties and showed that competitors have a unique feedback Nash point to order.

We implemented a two-period model in the fourth chapter. This chapter was an extension of the first one and had no boundaries on the ordering units. The model had again two competitive retailers and consisted of two random variables: total demand and proportion of total demand among retailers. Excess demand for any retailer can be substituted and this possibility gave this model a game theoretic profile again as in the first chapter. Olsen and Parker [18] gave the conditions for the stationary infinite horizon (open-loop) equilibrium to be a Markov perfect (closed-loop) equilibrium in their work. As an addition, one of our result is the ordering policy in a two-period inventory case. We proposed the policy which was related to the given inventory level and opponent's decision strategy at the beginning of the game with two periods to go. Here we used an increasing probability distribution on demand as an assumption. This is a very relevant point for that kind of analysis because this points out a trendy and competitive market in which players want to grab a share. So that we extended the single period formulations which have given by Parlar [20] to a two-period game. We also showed a relation between evolving inventory levels through periods and ordering decisions under the same demand distribution assumption as one of our contributions. A threshold which was in terms of parameters and distributions helped us to notice the trend on quantity which has to be ordered. We also demonstrated the uniqueness of the Nash equilibrium with a certain demand distribution. This equilibrium was shown to be as a sensitive point to cost and revenue parameters. It was increasing with sale price and salvage value but decreasing with respect to holding and ordering cost.

In the fifth chapter we dealt with a single period model with a customer satisfaction measure in the deterministic demand case. Nowadays, customer satisfaction without conflicting with the duty to maximize shareholder value or profit maximization has been the key point to merge companies. Today's mergers and acquisitions are about creating shareholder value, competitive advantage and synergy. Besides, if the company has a well-known brand then those kind of mergers are closely followed by millions of investors, analysts, media, and customers worldwide. Therefore many small or mid-cap firms, especially in the on-line business, are established to be sold or to merge with other companies. During a merger and acquisition process, pre-merger due diligence is to focus on identifying the factors that might jeopardize the success of the merger. Customer satisfaction due to corporate brand strategy are one of those factors that can significantly influence the success of a merger. It also contributes to superior post-merger performance. Hall and Porteus [10] worked on customer satisfaction in service sector without an inventory. Since all their variables were related to current period they could have made a finite period analysis and showed the uniqueness of the Nash equilibrium. Since we had a state variable, inventory level, which can not be fully described one period later due to different cases of customer satisfaction measure, we made a single period analysis. As a contribution, existence and uniqueness of the Nash equilibrium were shown with a customer satisfaction measure and inventory level. We have also shown that there was a positive correlation between the optimal point and customer satisfaction when the game begins. In the second part of this chapter, a Stackelberg game was designed and we showed the existence of the best responses in terms of parameters and inverse distributions. We also showed that leader had an advantage in the game to satisfy more customers than the follower since the leader's optimal quantity is larger than the leader's point.

As extensions of these games, open-loop and closed-loop equilibria can be investigated to coincide with each other. When players need not consider their opponents' moves or how they deviate from the equilibrium path, equilibria are more tractable and easier to solve analytically. On the other hand, when the demand is considered, a discredited market can be defined as having a decreasing tendency in effective demand. Existence and uniqueness of the Nash equilibria and policies on ordering decisions for all players can be also investigated for a discredited market in the two-period case. As an addition, generalization of the substitution effect and construction of a pattern on the market share of retailers will be studied also. In that kind of a game; expected number of unsatisfied customers will be depicted by a proxy function as in the paper [10] by Hall and Porteus. Further work will focus the Stackelberg leadership model which is a strategic game in economics in which the leader firm moves first and then the follower firm or firms move sequentially in a finite horizon case. Existence and uniqueness of the Subgame perfect Nash equilibrium or equilibria will be investigated in this model and can be shown a pattern on optimal ordering decisions during periods.

One can also interested in a three-period game to have more general results than the two-period under the same effective demand assumptions and parameters. Feedback Nash equilibria will be used and they will come from this two-period analysis. Best responses and optimal ordering policies will be probably related to the last and the middle period inventory levels. The threshold which was related to the trend of decrease/increase in optimal order decisions in this work, can be defined with a piecewise function generated by mostly best responses of related periods and also parameters. If the existence and uniqueness of the Nash equilibria can be defined in this setting, those points can be set as solutions or almost representatives for a finite period game. One can also interested in a more than two-player, i.e. multi-player setting. In this setting, deterministic demand case seems as the simplest case to analyze. Nevertheless, apart from the deterministic demand case, two-period and single period analysis with customer satisfaction will not be very smooth due to different effective demand definitions that are coming from the existence of other players.

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