MODULAR REPRESENTATIONS OF GL(2,p)

by

Turan Karakurt

B.S., Mathematics, Ankara University, 2015

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2018

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor Prof. Olcay Coşkun of the Department of Mathematics at Boğaziçi University for his calmness, patience and guidance. I would also like to thank to Prof. Burak Gürel for his sincere opinions.

I would like to thank Prof. Kazım İlhan İkeda and Assoc. Prof. Ergün Yaraneri for participating in my thesis committee.

Finally, I would thank to Fulya Taştan, R. Altar Çiçeksiz, Erol Barut, D. Ulaş Budak, Gözde Sert, Feyza Arzu and Arda Başarıcı for distracting my mind whenever I need.

This thesis is supported by Tübitak 1001-115F156.

ABSTRACT

MODULAR REPRESENTATIONS OF GL(2,p)

In this thesis, we investigate representations of $GL(2,p)$ by using indecomposable projective modules for $GL(2,p)$. Actually, this is pure module theoretic approach in parallel with [1] since we do not refer any character theoretic study. We follow [2] to obtain the structures of indecomposable projective modules. Then, we try to apply our results coming from indecomposable projectives to the blocks of $GL(2,p)$. Finally, we give the structure of Brauer tree algebras.

ÖZET

GL(2,p)'NİN MODÜLER TEMSİLLERİ

Bu savda, eleman sayısı p olan sonlu cisim üzerindeki genel lineer grubun modüler temsillerini araştıracağız ve bu çalışmada ayrışmaz projektif modüller esas aracımız olacak. Aslında bu çalışmadaki yaklaşım [1] ile paralel olarak karakter teorik bir yaklaşımın aksine tamamen modül teorik bir yaklaşım. Ayrışmaz projektif modüller için gerekli bilgiyi topladıktan sonra $GL(2,p)$ nin bloklarını belirlemeye çalışacağız. Son olarak da, Brauer ağaç cebirlerinden bahsedeceğiz.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

ker	kernel
$\|G : H\|$	index of subgroup H of the group G in the group G
$N_G(H)$	normaliser of the subgroup H in G
$C_G(H)$	centralizer of the subgroup H in G
$\mathbb{F}_{p^n}$	finite field with order p^n
$\mathrm{GL}(n, p)$	general linear group of invertible $n \times n$ $\mathbb{F}_p$-matrices

1. INTRODUCTION

1.1. Notations

Throughout this text we denote by k an algebraically closed field with characteristic p where p is a prime number. Moreover, we denote kG as a group algebra where G is a finite group and we assume that all the kG-modules are finitely generated.

1.2. Preliminaries

To understand modular representations of $GL(2,p)$ we need some knowledge on module theory (representation theory) and block theory. In this section, firstly we deal with simple algebras then we reach most important tool in our study: Indecomposable projective modules and their properties. In introduction part we follow [1].

1.2.1. Semisimple Modules

Simple modules are basic tools in module theory due to playing an important role to characterize other modules and algebras. In this section, we give the basic facts of the simple modules and simple algebras. Then we shall state the famous Artin-Wedderburn Theorem for the characterization of semisimple algebras.

Definition 1.1. *Let A be an algebra over k. An A-module U is called simple if it has no non-zero proper A-submodule. Semisimple A-modules are defined as direct sum of simple modules.*

Definition 1.2. *The radical of a ring A, denoted by* $\mathrm{rad}(A)$, is the smallest submodule of A such that quotient of A by $\mathrm{rad}A$ is semisimple. The radical of the A-module U is denoted by $\mathrm{rad}U$ and it is equal to $\mathrm{rad}(A)U$.

We can take consecutive radicals to form radical series of the A-module U,

$$U = rad^0(U) \supseteq rad^1(U) \supseteq rad^2(U) \cdots \tag{1.1}$$

where

$$rad^i(U) = rad(rad^{i-1}(U)) \tag{1.2}$$

Definition 1.3. *The largest semisimple submodule of the A-module U is called socle of U and denoted by $socU$*

Moreover, if the algebra A has zero radical then A is semisimple. Next theorems in this section are very important in terms of classification of semisimple modules and algebras. We shall skip the proofs of these theorems since they can be found in lots of representation theory books.

Theorem 1.1 (Artin-Wedderburn)**.** *Let A be a finite dimensional algebra over a field k with the property that every finite dimensional module is semisimple. Then A is a direct sum of matrix algebras over division rings.*

Now we shall pass from arbitrary algebra A to the specific algebra which is called group algebra to mention finite group representations.

Definition 1.4. *The group algebra kG is defined as the formal sums of the form*

$$\left\{ \sum_{g \in G} a_g g : a_g \in k \right\} \tag{1.3}$$

with operations

$$\left(\sum a_g g\right) + \left(\sum b_g g\right) = \left(\sum (a_g + b_g) g\right) \tag{1.4}$$

and

$$\left(\sum a_g g\right)\left(\sum b_h h\right) = \sum_{g,h} (a_g b_h) gh \tag{1.5}$$

For given group algebra, it is important to determine whether it is semisimple or not. Next theorem determines when the group algebra is semisimple.

Theorem 1.2 (Maschke)**.** *The algebra kG is semisimple if and only if $p \nmid |G|$.*

As it is expected we may not study with the semisimple group algebras due to Maschke's theorem. However, we can determine number of simple modules in modular case similar to ordinary case. For this purpose, we take into consideration p-regular elements of the given group G which are elements whose order not divisible by p.

Lemma 1.1. *The number of the isomorphism classes of simple kG-modules equals to the number of the conjugacy classes of p-regular elements of G.*

As a corollary, we can state that if G is a p-group then the trivial kG-module k is the only simple module.

If H is a subgroup of the group G and U is a kG-module then the restriction of U to H is denoted by U_H. Cllifford's Lemma states when we choose normal subgroup of the given group G, semisimplicity is preserved.

Lemma 1.2 (Clifford)**.** *If U is a semisimple kG-module and N is a normal subgroup of the group G then U_N is semisimple.*

In modular representation theory, classifying indecomposable modules is one of the biggest and hardest problems. However, there are some modules which are easy to handle in many ways. One of them is called uniserial and they have following requirements.

Let U be an A-module for a given algebra A. If U satisfies the following equivalent conditions then U is called uniserial.

(i) U has a unique composition series.

(ii) The successive quotients of the radical series of U are simple.

As an example, if G is cyclic then all the indecomposable kG-modules are uniserial.

Since we deal with characteristic p case, we may obtain modules for a given algebra which is not semisimple. To analyze non-semisimple cases we need other tools such as projective modules. We see that after this section if the algebra A is semisimple then all the A-modules are semisimple hence projective. In modular representation theory, for a given structure (module, algebra) it is important to determine how this structure differs from being semisimple and to achieve these projective modules have significant effect. We begin with the definition of the projective modules and the other equivalent conditions. Then we mention indecomposable projective modules which are keystone of this study.

1.2.2. Projective Modules

Lemma 1.3. *If U is an A-module then the following are equivalent*

(i) U is a direct summand of a free module.

(ii) If φ is a surjective homomorphism of the A-module V to the A-module W and ψ is a homomorphism of U to W then there exists an A-module homomorphism ρ of U to V such that $\varphi\rho = \psi$.

(iii) The following short exact sequence of A-modules splits

$$0 \to W \xrightarrow{\iota} V \xrightarrow{\rho} U \to 0 \tag{1.6}$$

for any A-modules W and V.

If U satisfies above statements then U is called a projective A-module.

Now we give one of the important correspondences in representation theory then we shall mention the dimensions of projective modules.

Lemma 1.4. *There is a one-to-one correspondence between the isomorphism classes of indecomposable projective A-modules and the isomorphism classes of simple A-modules, given by,*

$$P \to P/\text{rad } P \tag{1.7}$$

Lemma 1.5. *If a Sylow p-subgroup of G has order p^a then every projective kG-module has dimension divisible by p^a.*

Now we turn back to uniserial modules. Next two corollaries provide some important data which we use later.

Corollary 1.1. *If G is cyclic of order $p^a e$, $p \nmid e$, then kG is the direct sum of e uniserial modules of dimension p^a.*

Corollary 1.2. *If R is a cyclic normal Sylow p-subgroup of G then each indecomposable projective kG-module is uniserial (hence all the indecomposable kG-modules are). Moreover, there exists one dimensional simple kG-module W which gives us the composition factors of each indecomoposable projective kG-module. For instance, if S is the simple kG-module corresponding to indecomposable projective kG-module P then*

$$P/\text{rad(P)} \cong S, radP/\text{rad}^2\text{(P)} \cong S \otimes W, rad^2(P)/\text{rad}^3\text{(P)} \cong S \otimes W \otimes W, \cdots \tag{1.8}$$

Now we have adequate background to study indecomposable projective modules for $kGL(2,p)$

2. STRUCTURE OF THE INDECOMPOSABLE PROJECTIVES

In this chapter we analyze the structure of the indecomposable projective modules for $GL(2,p)$ and this is going to be the main target of our study. Before starting the projective indecomposables, we have to determine simple $kGL(2,p)$-modules so we have to determine the conjugacy classes of p-regular elements of $GL(2,p)$. In this chapter we generally follow [2].

Throughout this chapter, G denotes the group $GL(2,p)$ and S denotes the group $SL(2,p)$. Later on, we use specific one dimensional kG-module D^j which acts as j-th power of the determinant of the given group element.

2.1. Simple Modules

Remark 2.1. *Conjugacy Classes of $GL(2,p)$ There are four types of conjugacy classes and three of them correspond to conjugacy classes of p-regular elements.*

(i) First type of conjugacy classes consist of the elements of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where $a \neq b$. Hence, diagonalizable matrices with distinct eigenvalues form these conjugacy classes. Since $a \neq b$, these elements are p-regular and there are $\frac{(p-1)(p-2)}{2}$ such conjugacy classes of this form.

(ii) Second type of conjugacy classes of G have representative of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

where $a \neq 0$*. So, these are just diagonalizable matrices with one eigenvalue and obviously they have order divisible by* p*.*

(iii) Third type of conjugacy classes of G *consists of matrices which are not diagonalizable, they have eigenvalues in* $\mathbb{F}_p$*. So these matrices have repeated eigenvalue and their representative is of the form*

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

where $a \neq 0$ *and they are* p*-regular. There are* $p-1$ *such conjugacy classes.*

(iv) Finally, consider elements of G *with eigenvalues in the quadratic extension* $\mathbb{F}_{p^2}$ *of* $\mathbb{F}_p$*. There are* $\frac{p(p-1)}{2}$ *such conjugacy classes and these elements are* p*-regular.*

Hence there are

$$\frac{(p-1)(p-2)}{2} + (p-1) + \frac{p(p-1)}{2} = p(p-1) \tag{2.1}$$

conjugacy classes of p-regular elements.

Corollary 2.1. *There are* $p(p-1)$ *isomorphism classes of simple* kG*-modules.*

Now our next aim is to determine these $p(p-1)$ simple modules.

Let V be the polynomial algebra in variables X and Y over k. Let V_m be the subspace of homogeneous polynomials of degree $m-1$. V_m has a basis consisting of the monomials $X^{m-1}, X^{m-2}Y, \cdots, Y^{m-1}$, so it has dimension m. Besides, each V_m is a kG-module by the action

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} X = aX + cY$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} Y = bX + dY$$

where

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in G$$

So each $g \in G$ induces k-algebra endomorphism of V.

Proposition 2.1. *The restriction of the kG-module V_m to S, $(V_m)_S$, is a simple kG-module for $1 \leqslant m \leqslant p$.*

Proof. Let $g = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in P$ where P is a Sylow-p subgroup of G, so g generates P. Let g_m be the matrix corresponding to the action of g on V_m (considering the given basis of V_m). Hence $gX = X + Y$ and $gY = Y$. Therefore, we obtain a matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ (m-1) & 1 & 0 & \cdots & 0 \\ \binom{m-1}{2} & (m-2) & 1 & \cdots & 0 \\ & * & \vdots & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

Then, g_m -I_m has rank $m - 1$ so its nullspace is one dimensional and this nullspace gives us the fixed points under the action of g. Since P is cyclic p-group, all the simple kP-modules are one dimensional and P acts trivially on them. This means that

$$null(g_m - Im) = soc((V_m)_P) \tag{2.2}$$

and clearly $soc((V_m)_P) =< Y^{m-1} >$. Since socle is one dimensional, $((V_m)_P)$ is indecomposable, otherwise dimension of the socle would be bigger than one. Therefore, $((V_m)_P)$ is uniserial since all the indecomposables for cyclic groups are uniserial. We have the following composition series of $((V_m)_P)$

$$(V_1)_P =< Y^{m-1} >\subseteq \cdots \subseteq< X^{m-1}, X^{m-2}Y, \cdots, Y^{m-1} > \tag{2.3}$$

and this composition series is unique since $((V_m)_P)$ is uniserial. This means that every nonzero submodule of $((V_m)_P)$ contains Y^{m-1} but no proper submodule can contain X^{m-1}.

Now, let $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and we can do the same process using h instead of g since h is also generator of the Sylow-p subgroup P. Then we obtain that soc$((V_m)_P)$=$\langle X^{m-1}\rangle$. So, if W is submodule of $((V_m)_P)$ then W must both contain X^{m-1} and Y^{m-1} so W can not be proper. Hence, $((V_m)_P)$ is simple. Consequently, $((V_m)_S)$ is also simple so this completes the proof. $\square$

Now, we shall generalize the above proposition for bigger m's.

Proposition 2.2. *For* $m \geqslant 1$ *write* m=np+r *with* $0 \leqslant r \leqslant p-1$. *Then* $((V_m)_P)$ *is the direct sum of* $((V_r)_P)$ *and* n *copies of* kP.

Proof. Let g_m be defined as in the previous proposition. Since the entries of the g_m are in $\mathbb{F}_p$, this time, we have a representative matrix consist of n p by p blocks and for each one we can imitate the previous proposition. Hence, for each p by p block we have one dimensional fixed point space. Thus g_m - I_m has rank $(np+r)-(n+1)$ if $r \geqslant 0$ and $(np+r)-n$ if r=0.

Now, consider the subspace U of $(V_m)_P$ generated by first np basis element. So, this np-dimensional submodule has fixed point space of dimension n. Thus, U is the direct sum of at most n indecomposables, each of dimension at most p. Hence, U is exactly the direct sum of n indecomposables each of dimension p since the dimension of indecomposable can not exceed p. So

$$U = \underbrace{kP \oplus \cdots \oplus kP}_{n\text{-times}} \tag{2.4}$$

Besides, if $r \geqslant 0$ then remaining r basis element also induce one dimensional fixed point space. Thus, this submodule is also indecomposable and obviously isomorphic to $(V_r)_P$. Hence

$$U = \underbrace{kP \oplus \cdots \oplus kP}_{n\text{-times}} \oplus (V_r)_P \tag{2.5}$$

□

Corollary 2.2. *If $p \mid m$ then V_m is projective and if $p \nmid m$ then V_m is a direct sum of projective module and a nonprojective indecomposable module.*

Proof. Clearly, by previous proposition, if $p \mid m$ then, $(V_m)_P$ is projective and we know that V_m is relatively P-projective, which means that $V_m \mid ((V_m)_P)^G$ and a module which is induced from a projective module is also projective. Hence, V_m is projective. On the other hand, if $p \nmid m$ then V_m is not projective otherwise its restriction would be projective. If V_m had more than one nonprojective indecomposable summand, then, so would their restrictions to P. As a result, V_m has exactly one nonprojective summand. □

Now, we shall give the details of local subgroups of G, i. e, Sylow-p subgroup P and its normalizer $N_G(P) = L$. In G, L corresponds to lower triangular matrices. Moreover, L is generated by P and elements of the form $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ where a is nonzero element of $\mathbb{F}_p$ so, obviously basis elements of V_m appear as eigenvalue for these two matrices. In particular $\langle Y^{m-1} \rangle$ is a kL-submodule of $(V_m)_L$.

Since L consists of the matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ there naturally exist two simple kL-module A^i and B_j via actions $gx = c^i x$ and $gy = d^j y$ where $g \in L$, $x \in A^i$ and $y \in B^j$. Also, $A^i \otimes B^j \cong A^k \otimes B^l$ if and only if $i \equiv k(\mod p-1)$ and $j \equiv l(\mod p-1)$. Particularly, $\langle Y^{m-1} \rangle$ as a kL-module isomorphic to B^{m-1} thus, $< Y^{p-1} >$ is a submodule of $(V_p)_L$ isomorphic to $(V_1)_L$ since the action becomes trivial. Therefore, $(V_p)_L \otimes A^i \otimes B^j$ has a submodule isomorphic with $A^i \otimes B^j$. As $((V_p)_L \otimes A^i \otimes B^j)_P = (V_p)_P$ (since the restriction of both A^i and B^j are trival kP-modules) $(V_p)_L \otimes A^i \otimes B^j$ is indecomposable and we know that L contains cyclic normal Sylow-p subgroup P hence, $(V_p)_L \otimes A^i \otimes B^j$ is uniserial. This means that, this module has unique minimal submodule $A^i \otimes B^j$ so, $(V_p)_L \otimes A^i \otimes B^j$ is the indecomposable projective corresponding to $A^i \otimes B^j$. As the sum of dimensions of $(V_p)_L \otimes A^i \otimes B^j$'s where $i, j \in \{0, 1, 2, \cdots, p-2\}$, is equal to $p(p-1)^2$ which is the order of L. Hence,

$$kL = \bigoplus_{i,j} (V_p)_L \otimes A^i \otimes B^j \tag{2.6}$$

.

Since all the indecomposable kL-modules are uniserial, they appear in the composition series of one of the indecomposable projectives $(V_p)_L \otimes A^i \otimes B^j$. Hence, there are at most $p(p-1)^2$ isomorphism classes of indecomposable kL-modules.

Actually, there are exactly $(p-1)^2$ isomorphism classes of indecomposable kL-modules. Indeed, $(V_r)_L \otimes A^i \otimes B^j$ where $1 \leqslant r \leqslant p$, are all indecomposables since their restrictions to P are uniserial. Therefore, we obtain following result.

Corollary 2.3. *Every indecomposable kL-module is isomorphic to $(V_r)_L \otimes A^i \otimes B^j$ for $1 \leqslant r \leqslant p$ and $i, j \in$ {0, 1, 2, . . . , p-2}.*

Proposition 2.3. *For $m \geqslant 1$, let $m = np + r$ with $0 \leqslant r \leqslant p-1$. Then*

$$(V_m)_L \cong (V_r)_L \otimes B^n \bigoplus_{l=0}^{n-1} ((V_p)_L \otimes A^{n+r-1-l} \otimes B^l) \tag{2.7}$$

Proof. For each $l \in \{0, 1, ..., n-1\}$ let U_{l+1} denotes the first $p(l+1)$ elements of the basis of V_m and define U_0=0, U_{n+1}=$(V_m)_L$. So, clearly $U_1, U_2, \cdots, U_n$ are kP-modules and they are isomorphic to $(V_p)_P, (V_{2p})_P, \cdots, (V_{np})_P$ respectively. Moreover, $U_1, U_2, \cdots, U_n$ are kL-modules since the actions of $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ are well defined. Additionally, since $(U_1)_P, \cdots, (U_n)_P$ are projective, so are $U_1, U_2, \cdots, U_n$. Also, we know that we have series of kL-modules

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n \subseteq U_{n+1} = (V_m)_L \tag{2.8}$$

Hence

$$(V_m)_L \cong \bigoplus_{l=0}^{n} (U_{l+1}/\mathrm{U_l}) \tag{2.9}$$

□

In this decompositon each $U_{l+1}/\mathrm{U_l}$ has an simple submodule $\langle X^{m-1-lp}Y^{lp} + U_l \rangle$. Now, let g=$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ hence, obviously $g(X^{m-1-lp}Y^{lp}) - X^{m-1-lp}Y^{lp} \in U_l$ so this one dimensional submodule is isomorphic to $A^{m-1-lp} \otimes B^{lp}$ and since $m-1-lp \equiv n+r-1-l($ mod $p-1)$ we have

$$A^{m-1-lp} \otimes B^{lp} \cong A^{n+r-1-l} \otimes B^{l} \tag{2.10}$$

Also, $(V_m)_L$ is exactly the direct sum of n+1 modules; otherwise $(V_m)_P$ has more or less direct sum factors which is impossible by proposition 2. 2. Hence the decomposition $(V_m)_L \cong \bigoplus_{l=0}^{n} (U_{l+1}/\mathrm{U_l})$ is unrefinable. Therefore,

$$U_{l+1}/\mathrm{U_l} \cong ((V_p)_L \otimes A^{n+r-1-l} \otimes B^{l}) \tag{2.11}$$

since each indecomposable kL-module is identified by its dimension and its socle. There only remains

$$U_{l+1}/\mathrm{U}_1 \cong (V_r)_L \otimes B^n \tag{2.12}$$

so this completes the proof.

2.2. Exact Sequences

In this section we shall analyze some exact sequences to derive information about projective indecomposables and finally we will reach their structures. As in the previous section, we again adhere to Glover's article([2]).

Next theorem will be the most important tool in this section.

Theorem 2.1. *For all $m, n \geqslant 1$, there exists an exact sequence*

$$0 \to V_m \otimes V_n \xrightarrow{\theta} V_{m+1} \otimes V_{n+1} \xrightarrow{\varphi} V_{m+n+1} \to 0 \tag{2.13}$$

Proof. Let φ be defined as $\varphi(u \otimes v) = uv$ where u and v are basis elements of V_{m+1} and V_{n+1}, respectively. Since

$$\begin{aligned} g(\varphi(u \otimes v)) &= g(uv) \\ &= g(u)g(v) \\ &= \varphi(gu \otimes gv) \\ &= \varphi(g(u \otimes v)) \end{aligned}$$

where $g \in G$, so the action of G is preserved and clearly φ is surjective.

Secondly, define θ as $\theta(u \otimes v \otimes w) = (uX \otimes vY) - (uY \otimes vX)$. Now, let g= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\in G$. Then,

$$\begin{aligned} g\theta(u \otimes v \otimes w) &= g((uX \otimes vY) - (uY \otimes vX) \\ &= guX \otimes gvY - guY \otimes gvX \\ &= gu(aX + cY) \otimes gv(bX + dY) - gu(bX + dY) \otimes gv(aX + cY) \\ &= (aguX + cguY) \otimes bgvX + dgvY - (bguX + dguY) \otimes (agvX + cgvY) \\ &= (ad - bc)(guX \otimes gvY) + (bc - da)(guY \otimes gvX) \\ &= detg(g((ux \otimes vY) - (uY \otimes vX))) \\ &= detg(\theta(gu \otimes gv \otimes w)) \\ &= \theta(gu \otimes gv \otimes w) \\ &= \theta(g(u \otimes v \otimes w)) \end{aligned}$$

so, θ is a kG homomorphism. Since the algebra V has no zero divisors, if we choose an element of $V_m \otimes V_n \otimes D$ such as

$$\sum X^{m-i} Y^{i-1} \otimes v_i \otimes w \tag{2.14}$$

where $X^{m-i}Y^{i-1}$ and w are basis elements of V_m and D, respectively and v_i is uniquely determined element of V_i with respect to given basis elements of V_m and D so, if this element in $ker\theta$ then, obviously, all the v_i's must be zero. Hence, θ is injective. Therefore, the sequence is exact. □

Next lemma gives us an idea when the above sequence splits.

Lemma 2.1. *If $p \nmid m$ then $V_m \otimes V_2 \cong (V_{m-1} \otimes D) \oplus V_{m+1}$.*

Proof. We need a convention that V_0=0 especially for m=1. Then it is obviously valid for this case. Let $m \geqslant 1$, by the previous theorem we have an exact sequence, by taking

$m-1$ instead m and n=1,

$$0 \to V_m \otimes D \xrightarrow{\theta} V_m \otimes V_2 \xrightarrow{\varphi} V_{m+1} \to 0. \tag{2.15}$$

So, if there exists $\psi : V_{m+1} \to V_m \otimes V_2$ such that $\varphi\psi = Id_{V_{m+1}}$ then the sequence splits. For this reason define $\delta_m : V_{m+1} \to V_m \otimes V_2$ as

$$\delta_m(X^l Y^{m-l}) = l(X^{l-1} Y m - l \otimes X) + (m-l)(X^l Y^{m-l-1} \otimes Y) \tag{2.16}$$

Hence, $\varphi\delta_m = mId_{V_{m+1}}$. So, it remains to show that δ_m is a kG homomorphism for all m. Then, choose $\psi = \frac{1}{m}\delta_m$ to complete the proof. Let $g \in G$, then we need to show that for each monomial $w \in V_{m+1}$, and for all m $\delta_m(gw) = g(\delta_m(w))$. Now, let $u = X^j Y^{i-j} \in V_{i+1}$ $v = X^k Y^{m-i-k} \in v_{m-i+1}$. So, by using the exact sequence showed in theorem 2.1 and taking m=i , n= $m-i$, we obtain

$$\begin{aligned}
\delta_m(X^{k+j} Y^{m-j-k}) &= (k+j)(X^{k+j-1} Y^{m-j-k} \otimes X) + (m-j-k)(X^{k+j} Y^{m-j-k-1} \otimes Y) \\
&= (X^j Y^{i-j} \otimes (k(X^{k-1} Y^{m-i-k} \otimes X)) \\
&+ (m-i-k)(X^{k-1} Y^{m-i-k-1} \otimes Y) \\
&+ (X^k Y^{m-i-k} \otimes (j(X^{j-1} Y^{i-j} \otimes X)) + (i-j)(X^{j-1} Y^{i-j-1} \otimes Y) \\
&= (X^j Y^{i-j} \otimes k X^{k-1} Y^{m-i-k} \otimes X) \\
&+ X^j Y^{i-j} \otimes (m-i-k)(X^{k-1} Y^{m-k-1} \otimes Y) \\
&= (\varphi(u) \otimes \delta_{m-i}(v)) + (\varphi(v) \otimes \delta_i(u))
\end{aligned}$$

On the other hand, for m=1 δ_1 sends X to $1 \otimes X$ and sends Y to $1 \otimes Y$ so it is clear. For m >1, suppose that the monomial $w = \varphi(u \otimes v)$ since φ is surjective it is plausible for $u \in V_{i+1}$ and $v \in V_{m-i+1}$. By the induction hypothesis $\delta_i and \delta_{m-i}$ are kG

homomorphisms. Therefore,

$$\begin{aligned}
\delta_m(gw) &= \delta_m(g(\varphi(u \otimes v))) \\
&= \delta_m(\varphi(gu \otimes gv)) \\
&= (\varphi(gu) \otimes \delta_{m-i}(gv) + (\varphi(gv) \otimes \delta_i(gu) \\
&= (g\varphi(u) \otimes g\delta_{m-i}(v) + (g\varphi(v) \otimes g\delta_i(u) \\
&= g(\varphi(u) \otimes \delta_{m-i}(v) + \varphi(v) \otimes \delta_i(u)) \\
&= g\delta_m\varphi(u \otimes v) \\
&= g\delta_m(w)
\end{aligned}$$

□

Lemma 2.2. *$V_m \otimes V_p \cong V_{mp}$ for all m.*

Proof. Consider Frobenius homomorphism β from V_{m+1} into V_{mp+1} which sends u to u^p, hence,

$$V_{m+1} \xrightarrow{\beta} V_{mp+1} \xrightarrow{\delta_{mp}} V_{mp} \otimes V_2 \tag{2.17}$$

For m=1 it is trivial. Let m >1 and replace m by $m+1$. Then we have another sequence

$$V_{m+1} \otimes V_p \xrightarrow{\beta \otimes Id_p} V_{mp+1} \otimes V_p \xrightarrow{\varphi} V_{(m+1)p} \tag{2.18}$$

Now, take the monomials $X^l Y^{l-m} \in V_{m+1}$ and $X^i Y^{p-1-i} \in V_p$. Obviously, $\varphi(\beta \otimes Id_p)$ is surjective. By comparing dimensions we can deduce that

$$V_{m+1} \otimes V_p \cong V_{(m+1)p} \tag{2.19}$$

□

Corollary 2.4. *If $p \nmid m$ and p^k is any power of p we have*

$$V_2 \otimes V_{mp^k} \cong (V_{(m-1)p^k} \otimes D) \oplus V_{(m+1)p^k} \tag{2.20}$$

Proof. By using theorem 2.1 and lemma 2.1 we can deduce

$$\begin{aligned} V_2 \otimes V_{mp^k} &\cong V_2 \otimes V_m \otimes V_{p^k} \\ &\cong ((V_{m-1} \otimes D) \oplus V_{m+1}) \otimes V_{p^k} \\ &\cong (V_{m-1} \otimes D \otimes V_{p^k}) \oplus V_{(m+1)p^k} \\ &\cong (V_{(m-1)p^k} \otimes D) \oplus V_{(m+1)p^k} \end{aligned}$$

□

Hence we are almost ready to determine indecomposable projective modules and their structures. However, we need a couple of theorems before identify these strucutures and these theorems as stated in the Glover's article not only about the action of $GL(2,p)$ but the group M which is 2 by 2 matrices over $\mathbb{F}_p$. Giving all the details of the group M and its actions are out of scope of this thesis so we shall skip these proofs.

Theorem 2.2. *For $r \in \{1, 2, \cdots, p-2\}$ V_{p+r} is indecomposable and*

$$rad(V_{p+r}) = soc(V_{p+r}) \cong (V_{r-1} \otimes D) \oplus V_{r+1} \tag{2.21}$$

and

$$(V_{p+r})/\mathrm{rad(V_{p+r})} \cong V_{p-r} \otimes D^r \tag{2.22}$$

Following theorem provides us to construct indecomposable projective modules.

Theorem 2.3. *Let X^m be the span of the $(m-1)$st powers of the elements of V_2 and suppose that $1 \leqslant k \leqslant p$. Then*

(i) $rad(X_{(k+1)p}) \cong V_{p-2} \otimes D^k$ *and* $X_{(k+1)p}/\mathrm{rad}(X_{(k+1)p}) \otimes V_{k+1}$ *with* $X_p = V_p$

(ii) There is a submodule U_k in $V_{(k+1)p}$ containing $X_{(k+1)p}$ for which $V_{(k+1)p} \cong U_k \oplus V_{(k-1)p} \otimes D$, so with the convention that $U_0 = V_p$ we have

$$V_{(k+1)p} \cong \bigoplus_{m=0}^{[k/2]} (U_{k-2m}) \otimes D^m \tag{2.23}$$

(iii) $rad^2(U_k) = soc(U_k) \cong U_k/\mathrm{rad}(U_k) \cong V_{p-k} \otimes D^k$ *while*

$$rad(U_k)/\mathrm{soc}(U_k) \cong (V_{k-1} \oplus D) \oplus V_{k+1} \tag{2.24}$$

Actually if $k < p-1$ U is the indecomposable projective corresponding to $V_{p-k} \otimes D^k$ and if $k = p-1$ then it is the direct sum of V_p and indecomposable projective corresponding to V_1.

Now let $1 \leqslant m \leqslant p$ and $0 \leqslant n \leqslant p-2$. Let $P_{m,n}$ denote the indecomposable projective corresponding to simple module $V_m \otimes D^n$. Next theorem is the main result of this thesis.

Theorem 2.4. *Let $P_{m,n}$ is as defined above. Then*

$$rad(P_{1,n})/\mathrm{soc}(P_{1,n}) \cong V_{p-2} \otimes D^{n+1} \tag{2.25}$$

$$rad(P_{m,n})/\mathrm{soc}(P_{m,n}) \cong (V_{p-1-m} \otimes D^{m+n}) \oplus (V_{p+1-m} \otimes D^{m+n-1}) \tag{2.26}$$

when $1 < m < p$

$$P_{p,n} = V_p \otimes D^n \tag{2.27}$$

Proof. Since $V_p \otimes D^n$ is simple and projective it is the indecomposable projective $P_{p,n}$. For $1 \leqslant m < p$ by theorem 2.3 $P_{m,p-m}$ is as claimed. Moreover, $P_{m,p-m} \otimes D^{n+m-1}$ is clerly projective and $V_m \otimes D^n$ is a homomorphic image of this module. Indeed, there is a surjective map from $P_{m,p-m} \otimes D^{n+m-1}$ to $V_m \otimes D^{p-m} \otimes D^{m+n-1} \cong V_m \otimes D^n$. Also there is a bijection between the submodules of $P_{m,p-m} \otimes D^{n+m-1}$ and those of $P_{m,p-m}$ since

$$(P_{m,p-m} \otimes D^{n+m-1}) \otimes D^{2p-m-n-1} \cong P_{m,n}. \tag{2.28}$$

By this correspondence $P_{m,p-m} \otimes D^{n+m-1}$ is indecomposable so it is isomorphic to $P_{m,n}$. □

We know that $V_m \otimes D^n$ are simple. Actually, next corollary says that these are all the simple modules.

Corollary 2.5. *Any simple kG-module is isomorphic to $V_m \otimes D^n$ for some $m \in \{1, 2, \cdots, p\}$ and $n \in \{0, 1, \cdots, p-2\}$.*

In the light of previous information we shall give an example of indecomposable projective modules for $kGL(2,p)$. We take p=7 for this example since it contains enough data in what ways the composition factors appear.

2.3. Structure of the indecomposable projective kG-modules for p=7

Since there are $p(p-1)$ indecomposable projective modules (up to isomorphism) in case p=7 we have 42 modules. Now, we shall describe these modules.

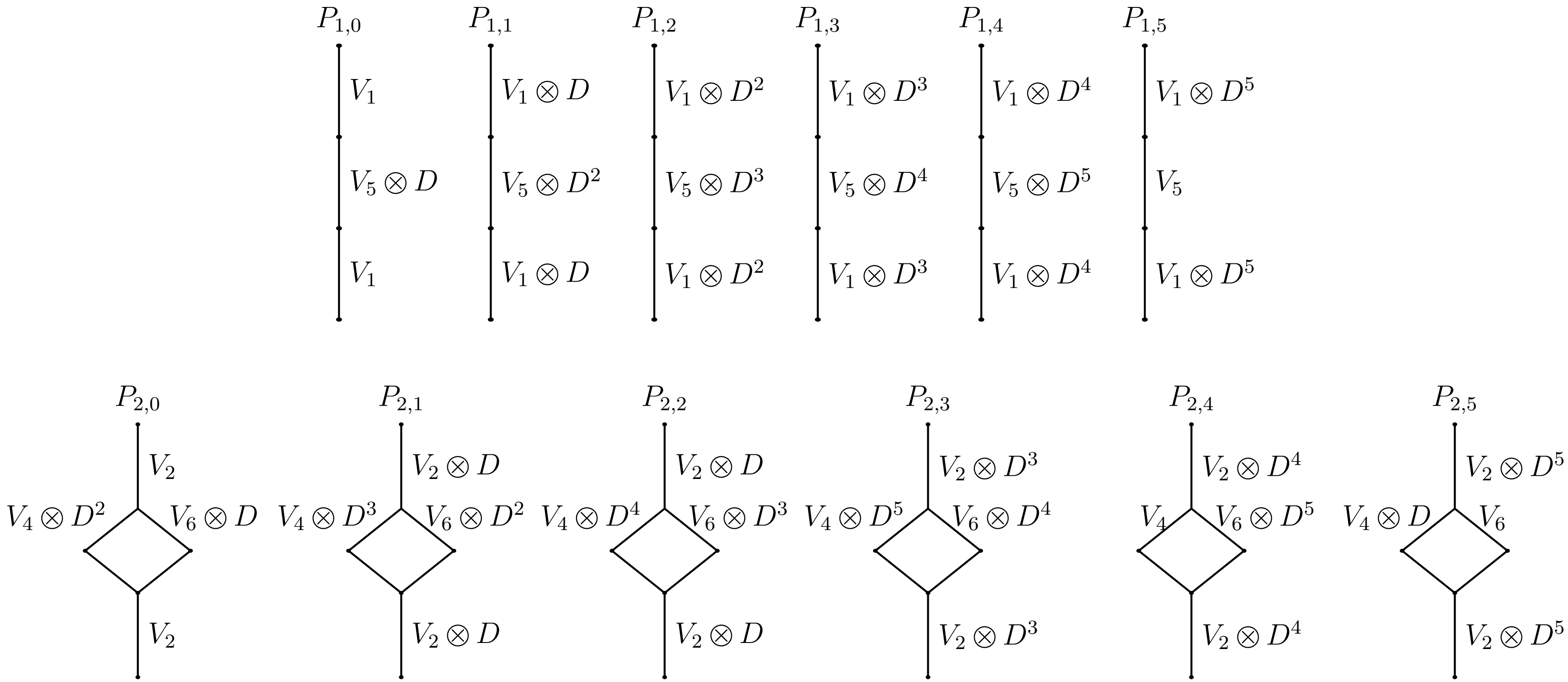


Figure 2.1. Projective Indecomposable Modules for $m = 1$ and $m = 2$

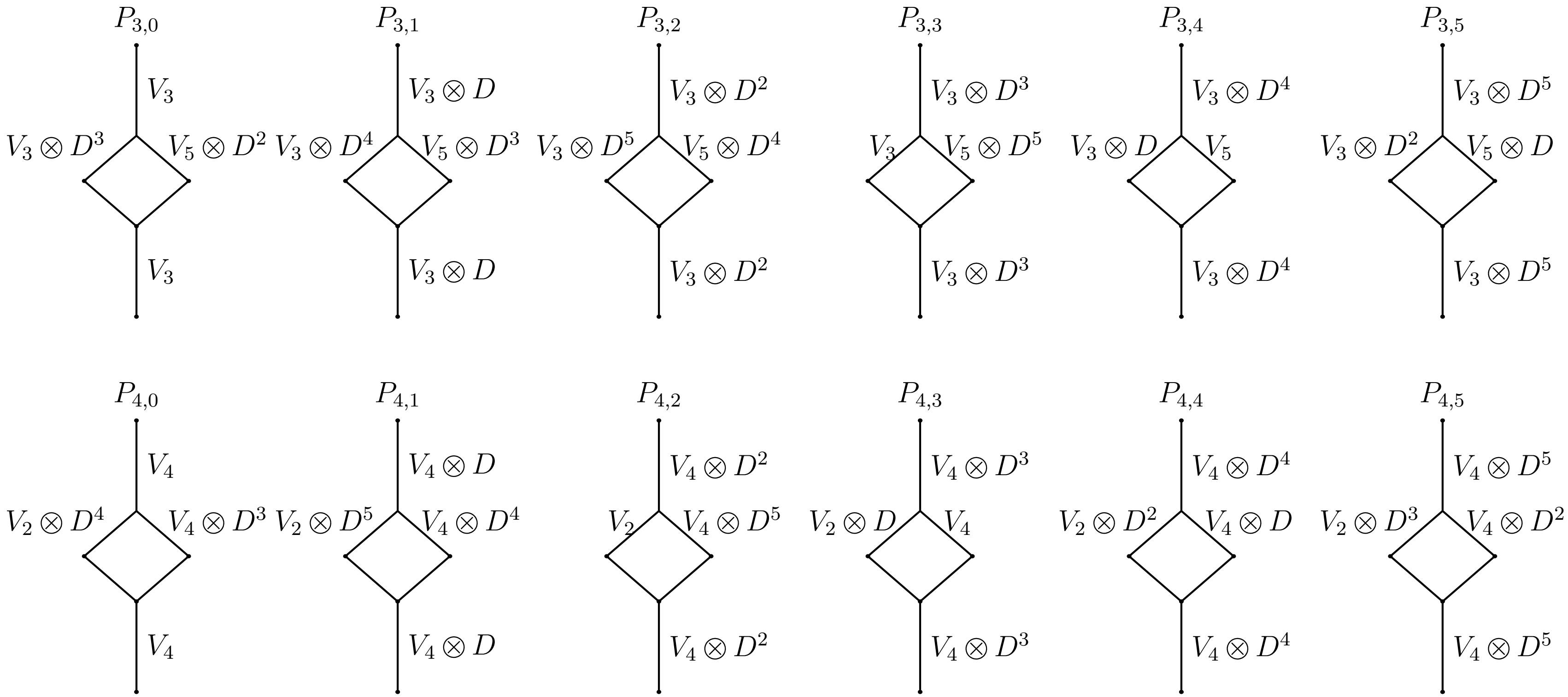


Figure 2.2. Projective Indecomposable Modules for $m = 3$ and $m = 4$

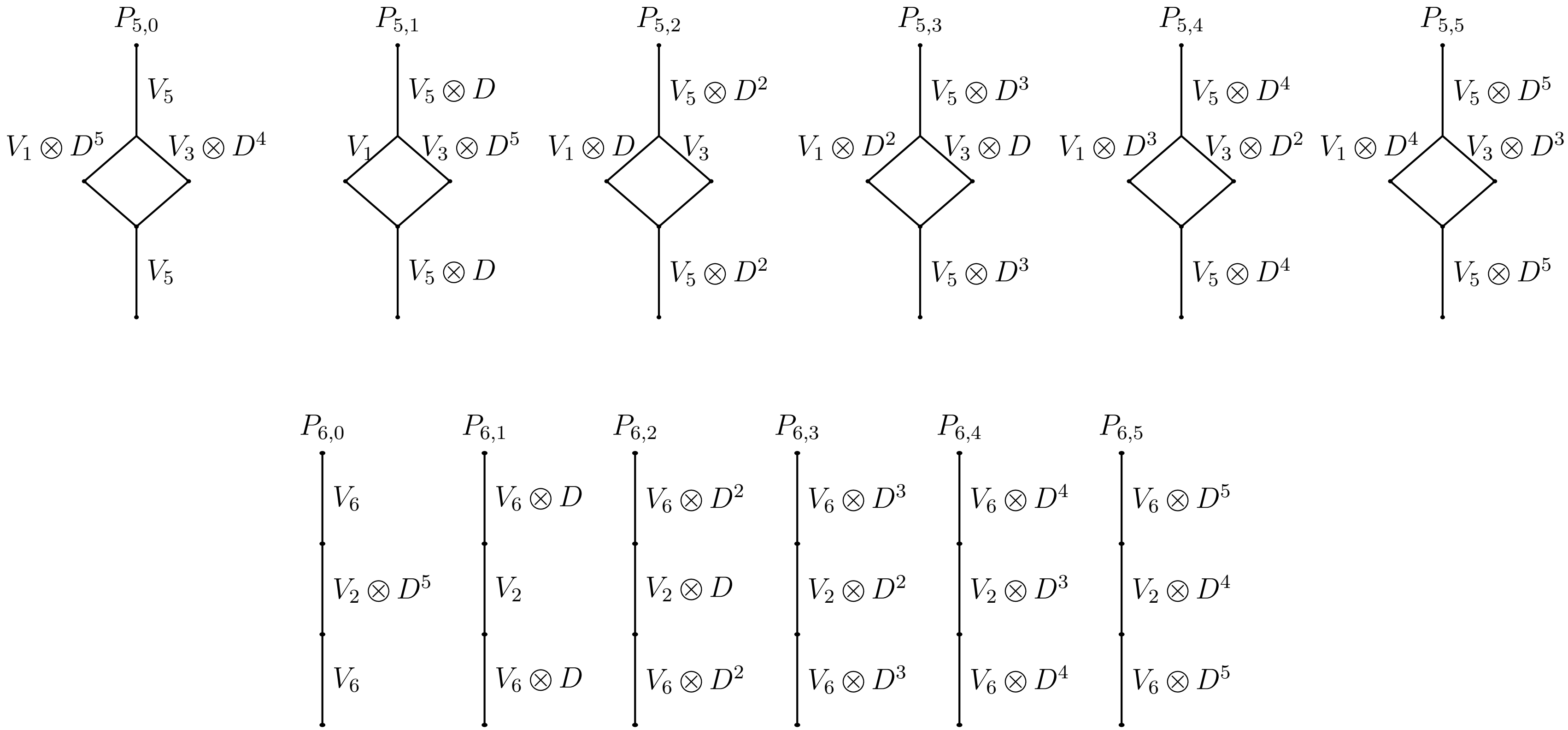


Figure 2.3. Projective Indecomposable Modules for $m = 5$ and $m = 6$

3. BLOCK THEORY

Let A be again a k-algebra. We shall state that algebra A can be decomposed into its subalgebras (ideals) in a unique way. Then, we shall apply this idea to the group algebra kG and then we are going to investigate how the simple kG-modules characterizes these summands, i.e, blocks of kG.

Before defining *block* we shall introduce idempotent elements of A.

3.1. Idempotents

We use [4] to get information about idempotents.

Definition 3.1. *An element $e \in A$ is called an idempotent if $e^2 = e$ where $e \neq 0$.*

Two idempotents e_1 and e_2 of A are called orthogonal if e_1e_2=$e_2e_1 = 0$. In this case, if e_1, $e_2 \cdots$ e_n are orthogonal then $e_1 + \cdots + e_n$ is also an idempotent of A. An idempotent $e \in A$ is called primitive if it can not be written as a sum of two orthogonal idempotents. Lastly, if the idempotent e is in the center of A then it is called central idempotent of A.

Remark 3.1. *By an idempotent decomposition of the idempotent e we mean the decomposition*

$$e = e_1 + e_2 + \cdots + e_n \tag{3.1}$$

into sum of orthogonal idempotents e_i.

Theorem 3.1. *Let e be an idempotent of A. If e=$e_1 + \cdots + e_n$ is an idempotent decomposition of e , then*

$$Ae = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_n \tag{3.2}$$

Conversely, if Ae is a direct sum of left ideals,

$$Ae = I_1 \oplus I_2 \oplus \cdots \oplus I_n \tag{3.3}$$

then there is an idempotent decomposition $e{=}e_1{+}\cdots+e_n$ *such that* $I_i{=}e_iA$.

Proof. Let e=$e_1{+}\cdots+e_n$ be an idempotent decompositon of the idempotent e. Thus e_ie=e_i since all the idempotents in the decomposition are orthogonal. Hence, $Ae_i \in Ae$, since the left ideal generated by e_i is Ae_i. But this leads us to have, $Ae = \sum_{i=1}^{n} Ae_i$ since $ae = ae_1 + ae_2 + \cdots + ae_n \in \sum_{i=1}^{n} Ae_i$ for all $a \in A$. Moreover, this sum is direct: indeed, if ae=$\sum_{i=1}^{n} a_ie_i$ then $ae_i{=}aee_i{=}a_ie_i$ hence the expression is unique. Now assume Ae is a direct sum of left ideals as stated above, and e=$e_1{+}\cdots+e_n$ where $e_i \in I_i$. Then for any $a_i \in I_i \in Ae$, so $a_i = a_ie$. Indeed, if $a_i = ae$ for some $a \in A$ then $a_ie = ae$, hence $a_i = a_ie$. □

Theorem 3.2. *Let* $I = Ae = eA$ *be a two-sided ideal of* A *generated by a central idempotent* e*, then we have a correspondence between central idempotent decomposition of* e *and the direct sum decomposition of* Ae *into two-sided ideals of* A*.*

Proof. Since e_i is central we have $Ae_i{=}e_iA$ is two-sided. Then, by the previous theorem

$$Ae = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_n \tag{3.4}$$

If $Ae = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ where each I_i is an two-sided ideal of A, then e=$e_1{+}\cdots+e_n$ and $I_i{=}Ae_i$ by the previous theorem. Since ae=ea for all $a \in A$, it follows that $ae_1{+}ae_2{+}\cdots{+}ae_n{=}\ e_1a{+}e_2a{+}\cdots{+}e_na$. This means that $ae_i{=}e_ia$ since the expression is unique. Therefore, e_i is central. □

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a direct sum decomposition of ideals of A. Then each A_i is called *block* of A. Moreover, this decomposition is unique.

Definition 3.2. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be the decomposition of A into blocks. If M is an A-module with A_iM=M and A_jM=0 for all $j \neq i$, then we say that M lies in the block A_i.*

So if M lies in the block A_i and 1=$e_1 + e_2 + \cdots + e_n$ where e_i's are orthogonal idempotents (in particular they are unit elements of the blocks of A) , then e_iM=M and e_jM=0 for $j \neq i$.

Conversely, if e_iM=M and e_jM=0 for $j \neq i$ then M lies in A_i since $AM = Ae_i$M= A_iM and A_jM=A_je_jM.

If M_i and M_j lie in the blocks A_i and A_j respectively and $\varphi \in \mathrm{Hom}_A(M_i, M_j)$ then $\varphi(e_im) = e_i\varphi(m) = 0$ for $m \in M_i$ so $\varphi = 0$. Consequently, $\mathrm{Hom}_A(M_i, M_j) = 0$.

From now on, A has block decomposition $A_1 + A_2 + \cdots + A_n$.

Proposition 3.1. *Let M be an A-module lying in the block A_i. Then each composition factor of M also lies in the block A_i. Conversely, if all the composition factors of M lies in A_i then M also lies in A_i.*

Proof. If S is a composition factor of M then

$$M_l/\mathrm{M}_{\mathrm{l+1}} \cong S \tag{3.5}$$

for some submodules M_l and M_{l+1} of M. Clearly, submodules of M also lie in the same block. Hence

$$e_i(m + M_{l+1}) = m + M_{l+1} \tag{3.6}$$

where $m \in M_l$. Moreover, e_j annihilates $m + M_{l+1}$ so, S lies in A_i.

On the other hand, suppose that M has a composition series

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_k = 0 \tag{3.7}$$

and each composition factor $M_l/\mathrm{M_{l+1}}$ lies in A_i. So $M_{k-1}/\mathrm{M_k} \cong M_k$ lies in A_i, then since $M_{k-2}/\mathrm{M_{k-1}}$ lies in in A_i

$$e_i(m + M_{k-1}) = e_i m + M_{k-1}$$

then

$$e_i m - m = m^{'}$$

for some $m^{'} \in M_{k-1}$

$$e_i m - e_i m = e_i m^{'} = m^{'} = 0$$

so

$$e_i m = m$$

Consequently M_{k-2} lies in A_i. So the same process apllies the other factors inductively. Finally, we obtain that M lies in A_i. □

As a result, we can say that the simple modules characterizes the blocks and the following theorem gives a way of determining blocks. Besides, this theorem is going to be very important tool in our study in order to determine the blocks of $GL(2, p)$.

Theorem 3.3. *Let S and T be simple A-mdoules. S and T lie in the same block if and only if there are simple A-modules $S = S_1, \cdots, S_n = T$ such that S_i, S_{i+1} , $1 \leqslant i \leqslant n$, are composition factors of an indecomposable projective A-module. Equivalently, S and T lie in the same block if and only if there are simple A-modules $S = S_1, \cdots, S_n = T$ such that S_i, S_{i+1} , $1 \leqslant i \leqslant n$ are equal or there is a non-split extension obtained by these two modules.*

We shall skip the proof of this theorem and it can be found in [1].

Corollary 3.1. *If P is a projective simple A-module lying in block B then P is the only simple lying in B. As a consequence, blocks which contains projective simple module are the matrix algebra summands of the algebra A.*

(i) (SL(2, p)) Since the simple $kSl(2,p)$-modules of odd index can be linked via the non-split exact sequences so, they lie in the same block. However, this is also true for the simples of even index; hence we have two blocks. Apart from these simples we have only one projective simple module V_p and this module also determines a block. Thus,

$$kSl(2,p) = B_1 \oplus\ B_2 \oplus B_3 \tag{3.8}$$

where B_1 denotes the block of the simple modules of odd index , B_2 denotes the block of the simple modules of even index and lastly B_3 denotes the block containing projective simple V_p. In this case B_3 has defect zero and the other two blocks are of full defect since $Sl(2,p)$ has cyclic Sylow-p subgroup of order p.

(ii) (Gl(2, p)) $Gl(2,p)$ has similar block structure with $Sl(2,p)$ since it also has cyclic Sylow-p subgroup of order p. However, it is slightly more complicated as expected since it has more simples and determining its structure using local techniques such as Green correspondence is more challenging. However, after drawing the pictures of the indecomposable projectives as in section? we can deduce that $GL(2,p)$ has block decompositon

$$kGL(2,p) = A_1 \oplus A_2 \oplus \cdots \oplus A_{\frac{p-1}{2}} \oplus B_1 \oplus B_2 \oplus \cdots B_{\frac{p-1}{2}} \oplus C_1 \oplus C_2 \oplus \cdots C_{p-1} \tag{3.9}$$

where A_i's containing simple modules of odd indices and B_i's containing of simple modules of even indices and lastly C_i's are the blocks corresponding to projective simples. Now we shall see these block structures of $GL(2,p)$ in detail, and in the next chapter we shall give the Brauer tree algebra structure of these subalgebras.

Remark 3.2. *Remember that $GL(2,p)$ has $p(p-1)$ simple modules (isomorphism classes) and they are of the form $V_i \otimes D^j$ where $1 \leqslant i \leqslant p, 0 \leqslant j \leqslant p-2$. Moreover, $P_{i,j}$ denotes the indecomposable projective corresponding to the simple module $V_i \otimes D^j$. Then*

$$radP_{i,j}/\mathrm{socP_{i,j}} \cong (V_{p-1-i} \otimes D^{i+j}) \oplus (V_{p+1-i} \otimes D^{i+j-1}) \tag{3.10}$$

and for i=1

$$radP_{i,j}/\mathrm{socP_{i,j}} \cong V_{p-2} \otimes D^{j+1} \tag{3.11}$$

Hence, after careful tracking of the compositon factors of the projective indecomposables we can obtain sequences of simple modules as seen in the following

$$V_1 - V_{p-2} \otimes D^1 - V_3 \otimes D^{p-2} - V_{p-4} \otimes D^2 - V_5 \otimes D^{p-3} - V_{p-6} \otimes D^3 - V_7 \otimes D^{p-4} - V_8 \otimes D^4 - \cdots \tag{3.12}$$

$$V_1 \otimes D^1 (\cong D^p) - V_{p-2} \otimes D^2 - V_3 \otimes D^0 (\cong D^{p-1}) - V_{p-4} \otimes D^3 - V_5 \otimes D^{p-2} - \cdots \tag{3.13}$$

Now we shall pass from the arbitrary algebra A to the group algebra kG and its blocks. To achieve this, we shall see the group algebra kG as a $k[G \times G]$-module via the action given by

$$(g_1, g_2)x = g_1 x g_2^{-1} \tag{3.14}$$

where g_1 and $g_2 \in G$ and $x \in kG$.

Hence, the submodules of kG as a $k[G \times G]$-module are the ideals of kG. Thus, we have the block decompositon of kG.

From now on B is assumed to be the block of kG unless otherwise stated.

Now we shall introduce the defect group of the given block of kG.

Definition 3.3. *Let B be a block of the group algebra kG. Then a p-subgroup of G is called a defect group of B, if δD is a vertex of B as a $k[G \times G]$-module where δ:G $\rightarrow G \times G$ is a group homomorphism which sends group element g to (g,g). Also defect* *groups of B are conjugate since the vertices are conjugate. If the order of the defect group D is p^l then it is said that B has defect l.*

It can be stated that D is one of the largest p-subgroups of G in the sense that it contains every normal p-subgroups of G and any kG-module lying in B has a vertex contained in D.

Corollary 3.2. *If the trivial kG-module lying in the block B then defect groups of B are Sylow-p subgroups of G since the vertex of the trivial kG-module is Sylow-p subgroup of G.*

Corollary 3.3. *Block B is semisimple algebra if and only if it has defect zero.*

Proof. Assume that block B is semisimple then all the B-modules are semisimple so B-module B has trivial group as a defect group.

On the other hand, if B has defect 0 then all the B-modules have trivial group as a vertex. Hence all the B-modules are projective. Thus all the short exact sequences of B-modules split which means that all the B-modules are semisimple so is B. □

3.2. Block induction, Brauer correspondence

Definition 3.4. *Let H be a subgroup of G. If b is the block of H and as a $k[H \times H]$-module is a summand of the $B_{H\times H}$ then it is said that B corresponds to b and it is denoted by $B = b^G$. Moreover it is required that B is unique with this property. So , b^G is not always defined. However, if the centralizer of a defect group of b, namely D, is in H then b^G is always defined. Consequently, if we choose N_GD then for any block of N_GD, b^G is defined since $C_GD \subseteq N_GD$.*

By the above definition we have a map from the blocks of H to the blocks of G. To obtain the opposite direction of this map now, we shall state the Brauer correspondence.

Even though we state that this correspondence for only the normalizer of the given defect group it can be also generalized to any subgroup which contains the normalizer of the given defect group and it is called the first main theorem of Brauer.

Theorem 3.4. *If D is a p-subgroup of G then there is a one-to-one correspondence between the blocks of $N_G D$ with defect group D and the blocks of G with defect group D via*

$$b \longmapsto b^G \tag{3.15}$$

3.3. Brauer Tree Algebras

As a conclusion, we shall give the some special structure of the blocks with cyclic defect group. Before this section we determined indecomposable projective modules of $kGL(2,p)$ due to some graphs of them(radical series). This time we try to describe blocks of $kGl(2,p)$ by drawing some graphs which are called Brauer trees. Brauer tree defined as a finite, connected, undirected graph without loops, so this makes it a tree. Brauer tree has a circular ordering of the edges with an assignment of a positive integer to one of the vertices which is called exceptional vertex. We shall give the related examples later. Before this, in the manner of representation theory, we give the definition of Brauer tree algebra, which is an algebra and its indecomposable projective modules can be described by using Brauer tree. Moreover, there is a bijection between indecomposable projective modules of this algebra or its simple modules and the edges of the Brauer tree. If S is a simple module for this algebra and P_S is the indecomposable projective corresponding to S then this structure requires that $rad(P_S)/\mathrm{soc}(\mathrm{P_S})$ is the direct sum of two uniserial modules.

Now we shall give some examples of Brauer tree algebras. First example is the most basic example of the Brauer tree algebras.

Suppose that A is a Brauer tree algebra with the Brauer tree

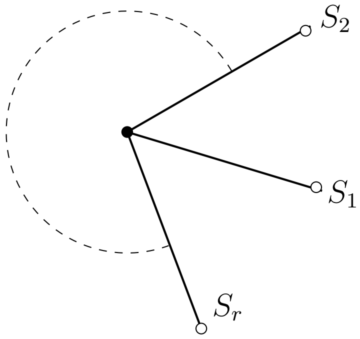
where the multiplicity m. The blackened circle indicates that it is exceptional vertex. Since there is one edge A has one simple module, namely S. So considering indecomposable projective P_S ,corresponding to S, it has the following structure

Here S appears in the composition series of $rad(P_S)/\mathrm{soc(P_S)}$ as $m-1$ times and P_S has radical length $m+1$. If G is a cyclic p-group of order p^n then kG is a Brauer tree algebra with above Brauer tree.

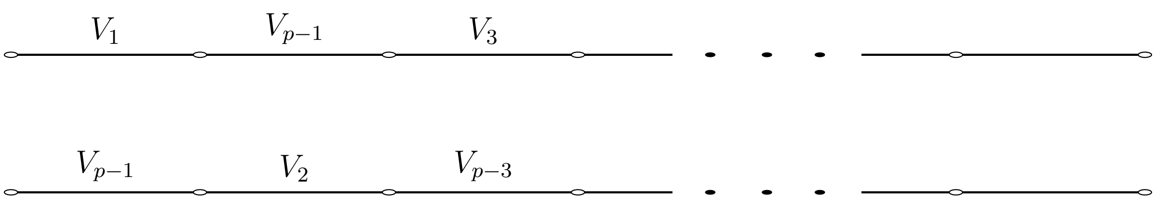
Next example is about the group G with cyclic normal Sylow p-subgroup P. So as we mentioned in section 1.2.2 corollary 1.3 there is a one dimensional simple module which leads us to determine all the other composition factors of given any indecomposable projective kG-module. To deduce Brauer tree algebra structure for this group we need a following remark.

Remark 3.3. *Suppose that a finite group G has a cyclic normal Sylow-p subgroup P of order p^n. Then simple kG-modules S and T lie in the same block if and only if $T \cong S \otimes W \otimes W \cdots \otimes W$ for some number of factors W. Indeed, when we consider indecomposable projective kG-module corresponding to trivial kG-module k, we have $k \cong W \otimes W \otimes \cdots \otimes W$ for some numbers of W. So if $T \cong S \otimes W \cdots \otimes W$ for enough number of W's, we deduce that we can link S and T via some indecomposable projectives. So if $S_1, \cdots, S_r$ simples that lie in the block B then we can list them as $S_i \otimes W \cong S_{i+1}$, $1 \leqslant i < r$ and $S_r \otimes W \cong S$.*

Consequently, B is a Brauer tree algebra for a star with r edges and exceptional vertex in the center with multiplicity $\frac{p^n-1}{r}$ and the picture is as the following



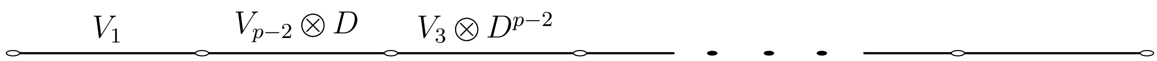
Our next example is $SL(2,p)$ and we deduced the block decomposition of this algebra before. Then the Brauer tree algebra structure of the blocks of nonzero defect as following

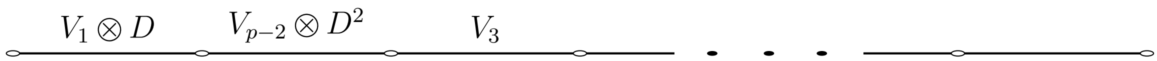


So far, all the given examples containing cyclic p-groups and then we mention their Brauer tree structure. As stated in [1] this is not a coincidence due to the following theorem and this theorem can be regarded as the main theorem of the cyclic blocks(blocks with (cyclic defect groups). To state this theorem we need a little preparation and this is our final statement. Let B be a block of the group G with defect group D cyclic of order p^n. Let b be the Brauer correspondant of the block B so it is block of $N_G(D)$. If N is a normal subgroup of G then the conjugation action of G over the blocks of N gives an algebra automorphism of kN and more precisely this action permutes the blocks of N. In our case, obviously the subgroup $DC_G(D)$ is normal in $N_G(D)$ and if we take a block β of $DC_G(D)$ covered by b then we can consider its stabilizer in $N_G(D)$. Let $e = |stab(\beta) : DC_G(D)|$. Then

Theorem 3.5. *The block B is a Brauer tree algebra for a tree with e edges and multiplicity $\frac{p^n-1}{r}$.*

Proof of this theorem is covered throughout the last chapter of [1] and we do not need techniques which used in that proof since the blocks of $GL(2,p)$ have either cyclic defect or of defect zero. So if the block of $kGL(2,p)$ is not of defect zero then it is Brauer tree algebra with $p-1$ edges and multiplicity $\frac{p-1}{p-1}= 1$. Consequently blocks of $GL(2,p)$ has similar Brauer tree structure with those of $SL(2,p)$ and have the following pictures (for the two blocks).





4. CONCLUSION

In this thesis, we try to determine indecomposable projective $kGL(2,p)$-modules and with respect to the distribution of these modules we determine blocks of $kGL(2,p)$. As a corollary, we mention Brauer tree algebras of related blocks with cyclic defect group.

REFERENCES

1. Alperin, J.L., 1986, *Local Representation Theory*, Cambridge University Press, Cambridge-New York.

2. Glover, D.J., 1978, "A Study of Certain Modular Representations", *J. Alg*, 51, pp. 425-475.

3. Schneider, P., 2013, *Modular Representation Theory of Finite Groups*, Springer, Dordrecht.

4. Nagao, H., Tsushima, Y., 1989, *Representations of Finite Groups*, Academic Press, Boston.