THE COMPOSITION FACTORS OF THE 2-BISET FUNCTOR OF KERNEL OF THE MONOMIAL LINEARIZATION MAP

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ABSTRACT

THE COMPOSITION FACTORS OF THE 2-BISET FUNCTOR OF KERNEL OF THE MONOMIAL LINEARIZATION MAP

In this thesis, we find the composition factors of the 2-biset functor of kernel of the monomial linearization map. We prove that it is equivalent to finding the composition factors of $\mathbb{C}^*$-fibered 2-biset functor $S_{C_2,1,1,1}$ as a 2-biset functor over $\mathbb{C}$. We show that there is no composition factor of the form $S_{H,V}$ unless H is a cyclic group or $C_2 \times C_{2^m}$. Moreover, we classify all composition factors of the form $S_{C_{2^m},V}$.

ÖZET

LİNEERLEŞME FONKSİYONUNUN ÇEKİRDEĞİNİN 2-İKÜME İZLECİ OLARAK BİLEŞKE ÇARPANLARI

Bu tezde, tekil linearleşme fonksiyonunun çekirdeğinin 2-iküme izleci olarak bileşke çarpanlarını bulduk. Bunun aslında $\mathbb{C}^*$-fiberli iküme izleci olan $S_{C_2,1,1,1}$'in 2-iküme olarak bileşke çarpanlarına denk olduğunu ispatladık. H'ın bir devirli grup ya da $C_2 \times C_{2^m}$'e eşit olmadığı durumda, $S_{H,V}$'nin $S_{C_2,1,1,1}$ için bileşke çarpanı olmayacağını gösterdik.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

Aut(H)	Automorphism group of H
A/X	The set of A-orbits of a A-set X
$B^A(G,H)$	Burnside ring of A-fibered (G,H)-bisets
$B^{\mathbb{C}^*}(G)$	The monomial Burnside ring of G with fiber $\mathbb{C}^*$
ch($\mathbb{C}^*, G$)	$\mathbb{C}^*$-subcharacters of G
C_{2^n}	Cyclic group of order 2^n
$\mathbb{C}$B(X,H)	$\mathbb{C}$ vector space generated by transitive (X,H)-bisets
$\mathbb{C}$I(X,H)	$\mathbb{C} \sum_{J \sqsubset H} B(X,J)B(J,X)$
$\mathbb{C}\overline{B}$(X,H)	$\mathbb{C}$B(X,H) / $\mathbb{C}$I(X,H)
$\mathcal{C}_k^A$	The category of A-fibered bisets over k
$\mathcal{C}_p$	Full subcategory of $\mathcal{C}$ containing only finite p-groups as objects
$[\mathbb{C}^* X]$	$\mathbb{C}^*$-fibred G-set
Def	Deflation fibered biset
E_G	$B_k^A(G,G)$
$\overline{E_G}$	E_G/I_G
$el(\mathbb{C}^*, G)$	$\mathbb{C}^*$-subelements of G
$e_{K,k}$	Idempotents in E_G
$e_{H,h}^G$	The primitive idempotent of $B^{\mathbb{C}^*}(G)$ associated to the $\mathbb{C}^*$-subelement (H,h)
$\mathcal{F}_k^A$	The category of A-fibered biset functors over k
${}_G Set_H^A$	The set of A-fibered (G,H)-bisets
${}_G Set^A$	The set of A-fibered G-sets
H^*	All group homomorphisms $\phi : H \to A$
H $\setminus$ G/K	The set of all double coset representatives of H and K in G
I_G	$\sum_{\lvert H\rvert < \lvert G\rvert} B_k^A(G,H)B_k^A(H,G)$
iso	Isomorphism biset
Ind	Induction fibered biset
Inf	Inflation fibered biset

$\mathrm{Irr}(kH)$	The set of isomorphism classes of irreducible kH-modules
$\mathfrak{K}$	The intersection of all kernels of deflations $\mathrm{Def}^G_{G/N}$
k[H]	Group algebra of H over k
${}_K\bigtriangledown(G)$	$\{(g_1, g_2) \in G \times G : g_1K = g_2K\}$
$\{K, \kappa\}$	G-linkage class of (K, κ)
$\mathcal{L}_{G,-}$	Left adjoint functor of ϵ_G
$\mathfrak{m}_{(G,N,g)}$	Deflation number associated to triple (G, N, g)
$\mathcal{M}_G$	All A-subcharacters of G
$\mathcal{M}_G^G$	G-fixed elements of $\mathcal{M}_G$ under the conjugation action
$\mathrm{Out}(H)$	Outer automorphism group of H
O(G)	Intersection of kernel of all linear characters of G
p_i	Projection map to the i^{th}-coordinate
Res	Restriction fibered biset
$\mathcal{R}_G$	The set of reduced pairs in $\mathcal{M}_G$
$S_{G,V}$	The Simple Biset functor associated to pair (H,V)
$\mathcal{S}_G$	A set of parametrization for the irreducible $\overline{E_G}$-modules.
$\mathbb{S}_2$	The simple $\mathbb{C}^*$-fibered 2-biset functor associated to the pair $(C_2, 1, 1, [1])$
U_ϕ	$\{(\phi(u^{-1}), u) : u \in U, \phi : U \to A\}$
$X \sqcup Y$	The categorical coproduct of X and Y
$X \otimes_{AH} Y$	Tensor product of A-fibered (G,H)-biset X and A-fibered (H,K)-biset
${}_{(G,K,\kappa}\Gamma_{(H,L,\lambda)}$	$\{[\frac{G\times H}{U,\phi}] : l(U, \phi) = (G, K, \kappa), r(U, \phi) = (H, L, \lambda)\}$
ϵ_G	Evaluation functor at G
$\mu^{\triangleleft}_{(K,\kappa),(L,\lambda)}$	Möbius coefficient corresponding to pairs (K, κ) and (L, λ)
Φ_G	Frattini subgroup of G

1. INTRODUCTION

Monomial linearization map at any finite group G is a map from monomial Burnside ring $\mathbb{C}B^{\mathbb{C}^*}(G)$, see [1] to complex character ring $\mathbb{C}\mathcal{R}_{\mathbb{C}}(G)$, see [2]. It is known that this map is surjective, see Brauer's Induction Theorem in [3]. The description of the kernel of the monomial linearization map at any finite group G has already been made topologically and can be found in [4]. We determine the structure of the kernel of the mononomial linearization map for finite 2-groups, and in future we aim to do this for finite p-groups. We make an important observation that the kernel of the monomial linearization map at a finite 2-group G is equivalent to the evaluation of G at the $\mathbb{C}^*$-fibered biset functor $S_{C_2,[1,1],[1]}$. We will use the fibered biset functor structure of $S_{C_2,[1,1],[1]}$.

Fibered bisets were introduced by Boltje and Coşkun in [5]. These objects are the combination of fibered sets which are introduced by Dress in [1], in a more general setting, and bisets which are introduced by Bouc in [6]. By constructing a category whose morphsims are fibered set or bisets, we switch our approach a more categorical point of view. This approach leads to the theory of biset functors by Bouc in [6] and it was successfully used by Bouc and Thevenaz in the determination of the structure of Dade group in [7].

One important question in mathematics is the classification of simple object of any mathematical structure. The simple A-fibered biset functors over any field have already been determined and classified by Boltje and Coşkun in [5]. On the other hand, the structure of any simple functor is not easy to determine because its evaluation at any reasonable choice of finite group contains informations which have not been fully studied. Also, any fibered biset functor can be seen as a biset functor and this approach suggests us to find the composition factors of any simple fibered biset functors as biset functor which were studied in detail before, see [8].

We study the simple $\mathbb{C}^*$-fibered 2-biset functor $S_{C_2,[1,1],[1]}$ (here we mean A-fibered 2-biset functor over $\mathbb{C}$ is the same as a A-fibered biset functor except the objects are restricted to 2-groups) . It turns out that this 2-biset functor is the same as the kernel of the monomial linearization map (which we will introduce in Chapter 3) and it is closely linked to the primitive idempotents of the Monomial Burnside Ring, see [9]. We prove that the composition factors of the kernel of the monomial linearization map as a 2-biset functor are of the form $S_{C_{2^m},V}$ or $S_{C_2\times C_{2^m},W}$ where $V \in \mathrm{Irr}(\mathbb{C}\,\mathrm{Out}(C_{2^m}))$ and $W \in \mathrm{Irr}(\mathbb{C}\,\mathrm{Out}(C_{2^m} \times C_2))$.

In Chapter 2, we give some preliminary results, introduce some notations and explain the method which we use in Chapter 3. We give the definiton of a biset functor, give some results about biset functors from [8]. Subsequently, we give a very detailed overview of A-fibered bisets from [5]. Here, we only prove some results where the proof left as an exercise in [5].

After a detailed overview of fibered biset functors, in Chapter 3 we give the definition of $\mathbb{C}^*$-fibered biset functor $\mathbb{S}_2$. At the same time, we give an overview of some results regarding monomial Burnside rings from [10] and [9]. Also, we give the definiton of monomial linearization map and state two theorems which will be essential in our composition work. We also recall primitive idempotents of monomial Burnside rings and the actions of standard bisets on primitive idempotents from [9] and [10]. At the end of Chapter 3, we briefly explain our tools to determine the composition factors of $\mathbb{S}_2$.

In Chapter 4, we determine some composition factors as a 2-biset functor of $\mathbb{S}_2$ and also we prove that there is no composition factor of the form $S_{H,V}$ in some specific cases.

In Chapter 5, we restated the main aim of the Thesis see 5.1. We also noted that the other cases will appear in the future studies which will follow from the same constructions that we did in this thesis.

2. PRELIMINARIES

2.1. Bisets

We give a brief overview of bisets and biset functors. The details in the theory of biset functor can be found in [8]. We start with the definition of a G-set. We give an overview of biset functor which is needed for the rest of the thesis via following [8].

Definition 2.1. A non-empty set X is called a G-set if there is a map $\cdot : G \times X \to X$ given by $(g, x) \to g \cdot x$ satisfies $1 \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in X$ and $g_1, g_2 \in G$.

A G-set X is called transitive G-set if there is only one G-orbit, i.e for any $x, y \in X$, we can find $g \in G$ such that $g \cdot x = y$. Moreover, there is a bijective correspondence between the isomorphism classes of transitive G-sets and the G-conjugacy classes of subgroups of G.

Definition 2.2. A left (G, H)-biset X is a left $G \times H^{op}$-set.

Note that a (G, H)-biset X is called transitive if for any $x, y \in X$, there exists $g \in G$ and $h \in H$ such that $g \cdot x \cdot h = y$. Similarly to G-sets, there is an bijective correspondence between the $G \times H^{op}$-conjugacy classes of subgroups of $G \times H^{op}$ and the isomorphism classes of transitive (G, H)-bisets.

With respect to the disjoint union of (G, H)-bisets, we define a group $B(G, H)$ which is called Grothendieck group. Note that the transitive (G, H)-bisets forms a $\mathbb{Z}$-basis for $B(G, H)$. Extending the scalar to any commutative ring R, we obtain a R-module $RB(G, H) = R \otimes_Z B(G, H)$.

We construct a multiplication which looks like a composition in a category. Let X be a (G, H)-biset and Y be a (H, K)-biset. We consider the cartesian product

$X \times Y$ and there is a natural H-action on $X \times Y$ via $h(x,y) = (xh^{-1}, hy)$. The set of H-orbits of $X \times Y$ is denoted by $X \times_H Y$ and we define a $G \times K$ action on this set as $(g,k)[x,y] = [gx, yk^{-1}]$. We obtain a (G,K)-biset. The multiplication is distributive over disjoint unions and therefore a general formula for the multiplication can be found by looking for a formula for transitive bisets. Next result which answers this problem is called Mackey's Formula and its fibered biset version will also be included in this thesis. Note that the fibered biset version of Mackey's formula is the generalization of the following result.

Theorem 2.1. *Let G, H and K be finite groups and $U \leq G \times H$, $V \leq H \times K$. Then there is an isomorphism of $G \times K$ bisets*

$$(G \times H/U) \times_H (H \times K/V) \cong \bigcup_{h \in p_2(U)/H \backslash p_1(V)} (G \times K/(U * {}^{(t,1)}V)). \tag{2.1}$$

The definitions of p_2, p_1 and $U * V$ will be given in the A-fibered biset section and it can be found on [8]. We construct an R-linear category $R\mathcal{D}$ as follows:

The objects are finite groups and the morphisms are being R-linear extensions of isomorphism classes of bisets, i.e for any groups G, H; $Mor_{\mathcal{D}}(G,H) = RB(H,G)$ and the composition of any two morphisms is an R-linear extension of the multiplication that we have defined above. Notice that it is an R-linear category. We are ready to give the definition of a biset functor over R.

Definition 2.3. An R-linear functor $F : R\mathcal{D} \to R - mod$ is called biset functor over R.

We can construct another category $R\mathcal{H}$ where the objects are biset functors over R and the morphims are natural transformations. This category is abelian and therefore we can talk about simple objects, quotient and any other constructions which are attributed to the module structure of an object.

The simple biset functors over R is any biset functor over R which is simple in the category $R\mathcal{H}$. The classification of such simple objects are done and the following is due to Bouc, see [8].

Theorem 2.2. *Given any pair* (H, V) *where* H *is a finite group and* V *is an irreducible* $R\,\mathrm{Out}(H)$*-module. There is a simple biset functor* $S_{H,V}$ *associated to the pair* (H, V)*. Two such pairs* (H, V) *and* $(H^{'}, V^{'})$ *are equivalent if* $H \cong H^{'}$ *and an isomorphism* $\phi : H \to H^{'}$ *transports* V *to* $V^{'}$*. There is a bijective correspondence* $(H, V) \longleftrightarrow S_{H,V}$ *between the equivalence classes of pairs* (H, V) *and the isomorphism classes of simple biset functors* $S_{H,V}$*. Two biset functors* $S_{H,V}$ *and* $S_{K,W}$ *are equivalent if* $S_{H,V}(K) \cong W$ *and* $S_{K,W}(H) \cong V$*.*

The construction of $S_{H,V}$ is discussed in A-fibered biset functor section. We introduce the 2-biset functor and we finish this section. Following the construction on [8], the classification theorems for 2-biset functors over any field are identical as in biset functor case.

Definition 2.4. Full subcategory of the category $R\mathcal{D}$ consists of finite p-groups as objects is denoted by $R\mathcal{D}_p$. A p-biset functor over R is a R-linear category from $R\mathcal{D}_p$ to R-modules.

2.2. A-Fibered Bisets

Throughout this chapter, G and H denote finite groups. Moreover, k denotes any commutative ring with unity and ${}^{x}H$ denotes the set $\{x^{-1}hx : h \in \ H\}$. Any $k[H]-$ module M can be seen as a $k[{}^{x}H]$ -module via extending the action ${}^{x}h.m = h.m$ k-linearly to $k[{}^{x}H]$. Moreover, any group homomorphism $\phi : H \to A$ provides a group homomorphism ${}^{x}\phi : {}^{x}H \to A$ where A is an abelian group.

2.2.1. Definition of A-fibered (G, H)-Bisets

There are two ways to let two groups act on a set X which is important in representation theory of finite groups. Firstly, two finite groups G and H act on a set X from right and left in a commuting way where we can view a set X as a left G-set and a right H-set. In this case, we call X a (G, H)-biset. This case is already studied in Bisets section and in more detail in [8].

Secondly, a group G can act on the A-orbits of a set X where A is an abelian group acting on the set X freely and X has finitely many A-orbits. In this case, we call X as a A-fibered G-set. Specifically, A-fibered G-set is a set X where G is permuting the A-orbits of X and each orbit is a copy of A as a set. A morphism between two A-fibered G-set is an $A \times G$-equivariant map. Denote ${}_G Set^A$ the category of A-fibered G-sets.

Note that any left G-set X can be seen as a right G-set X via defining the action $x.g = g^{-1}x$ for any $g \in G$ and $x \in X$. Using this result, an A-fibered (G, H)-biset is defined to be a A-fibered $(G \times H)$-set. This follows from the correspondence between the (G, H)-bisets and $(G \times H)$-sets. More specifically, an A-fibered (G, H)-biset X is a set X in which there are finitely many A-orbits with free A-action such that G acts on left on the A-orbits of X and H acts on the right on the A-orbits of X with each action commuting.

<u>2.2.1.1. The Set $\mathcal{M}_G = \mathcal{M}^A(G)$.</u> This set consists of pairs (H, ϕ) where $H \leq G$ and $\phi \in H^* = Hom(H, A)$. We can define a partial ordering on $\mathcal{M}_A$ as follows: We say that $(M, \mu) \preceq (L, \lambda)$ iff $M \leq L$ and $\lambda_{|_M} = \mu$. Also, note that G acts on $\mathcal{M}_A$ via conjugation and the conjugation respects the partial order structure on $\mathcal{M}_A$. We denote the G-orbit of the pair (M, μ) by $[M, \mu]_G$. Let H be another finite group. We define the set $\mathcal{M}_A(G \times H)) = \mathcal{M}_{G \times H}$=$\{(U, \phi) : U \leq G \times H, \phi \in U^*\}$.

Let $p_1 : G \times H \to G$ and $p_2 : G \times H \to H$ be usual projection maps. Let $(U, \phi) \in \mathcal{M}_{G\times H}$ and for such a pair, we define the following subgroups of G and H as $p_1(U) = \{g \in G :$ there exists h ∈ H such that (g,h) ∈ U$\}$, $p_2(U) = \{h \in H :$ there exists g ∈ G such that (g,h) ∈ U$\}$, $k_1(U) = \{g \in G : (g, 1) \in U\}$ and $k_2(U) = \{h \in H : (1, h) \in U\}$.

Definition 2.5. We define the subgroup $k(U) = k_1(U)\times k_2(U)$ to be largest rectangular subgroup of U.

Note that for any $\phi \in (k(U))^*$, we can write it as $\phi = \phi_1 \times \phi_2$ where $\phi_1 \in k_1(U)^*$ and $\phi_2 \in k_2(U)^*$. For the notational reasons for the future use, we will customarily write $\phi = \phi_1 \times \phi_2^{-1}$. Start with $\phi \in U^*$, restrict it to $k(U)$ and we can write its restriction as $\phi_{|k(U)} = \phi_1 \times \phi_2^{-1}$. This allows us to associate two triples $l(U, \phi) = (p_1(U), k_1(U), \phi_1)$ and $r(U, \phi) = (p_2(U), k_2(U), \phi_2)$ to the pair (U, ϕ). Call them left and right invariants of (U, ϕ). The following proposition is quite useful and gives the relation between these two triples. It is sometimes called Goursat's Theorem.

Lemma 2.1. *Let $(U, \phi) \in \mathcal{M}_{G\times H}$ and set $(P, K, \kappa) = l(U, \phi)$ and $(Q, L, \lambda) = r(U, \phi)$. Also set, $\widehat{K} = \ker \kappa$, $\tilde{K} = \widehat{K}P' \cap K$ and $\widehat{L} = \ker \lambda$, $\tilde{L} = \widehat{L}Q' \cap L$. We have $\widehat{K} \le \tilde{K} \le K \le P$ and $\widehat{L} \le \tilde{L} \le L \le Q$.*

(a) $\widehat{K}$,$\tilde{K}$ and K are normal in P and $\widehat{L}$,$\tilde{L}$ and L are normal in Q.
(b) $K/\widehat{K}$ is central in $P/\widehat{K}$ and $L/\widehat{L}$ is central in $Q/\widehat{L}$.
(c) One has group isomorphisms $P/K \cong U/(K \times L) \cong Q/L$ defined by the relation $\eta_U : Q/L \to P/K$ such that $\eta_U(qL) = pK$ if and only if $(p, q) \in U$.
(d) One has group isomorphism $\tilde{K}/\hat{K} \cong \ker(\phi_{|K\times L})/(\widehat{K} \times \widehat{L}) \cong \tilde{L}/\hat{L}$ induced by the projection maps on $\widehat{K} \times \widehat{L}$ such that $\varphi(\bar{l}\tilde{L}) = \bar{k}\tilde{K}$ if and only if $\kappa(\bar{k}) = \lambda(\bar{l})$.
(e) If $\hat{K} = 1$,$\tilde{K} = K$,$P = G$,then $|G| \le |H|$.

Proof. For part (a), We only prove the claims regarding $l(U, \phi)$, the second one follows similarly. It is trivial to check subgroup conditions so we only check normality conditions. Let $k \in K$ and $p \in P$, then $(k, 1) \in U$ and $(p, h) \in U$, then

$(p,h)^{-1}(k,1)(p,h) = (p^{-1}kp, 1) \in U$ which implies $K \trianglelefteq P$. Let $k \in \widehat{K}$ and $p \in P$, then $p^{-1}kp \in K$ and $\kappa(p^{-1}kp) = \phi(p^{-1}kp, 1) = (\phi(p,1))^{-1}\phi(k,1)\phi(p,1) = \phi(k,1) = 1$ hence $p^{-1}kp \in \hat{K}$. This proves that $\hat{K} \trianglelefteq P$. $\tilde{K} \trianglelefteq P$ follows from the facts that product of two normal subgroups is normal, commutator of a group is always normal in that group and intersection of two normal subgroups is again normal. For the part (b), notice that since the range is an abelian group A, then the result follows automatically. Part (c) is just a standard result in Group Theory. It just says that any subgroup U of $G \times H$ is characterized by the quintuple (P, K, η, Q, L). Proof of (d) can be found on [5] Part (e) follows from using part (c) and part (d), i.e we have $|G| = \frac{|K|.|Q|}{|L|} = \frac{|\tilde{L}|.|Q|}{|\widehat{L}|.|L|} \leq \frac{|Q|}{|\widehat{L}|} \leq |Q| \leq |H|$. □

2.2.1.2. Parametrization of Transitive A-Fibered Bisets. We have defined a set $\mathcal{M}_{G\times H}$ and we see that it plays an important role in the parametrization of transitive A-fibered bisets. Let X be an A-fibered (G,H)-biset. Let us denote the A-orbit of $x \in X$ by $[x]$. By the definition, $G \times H$ acts on the A-orbits of X via $(g,h)[x] = [(g,h)x] = [gxh^{-1}]$. Let S_x denote the stabilizer of $[x]$ in $G \times H$. Let us take any $(g,h) \in S_x$, then $(g,h)[x] = [x]$ that means that $(g,h)x \in [x]$, hence there exists $a \in A$ such that $(g,h)x = a.x$. Therefore, given any $(g,h) \in S_x$, there is an associated element $a_{(g,h)} \in A$ such that $(g,h)x = a_{(g,h)}x$. This allows us to define a map $\phi_x : S_x \to A$ such that $\phi_x((g,h)) = a_{(g,h)}$ if and only if $(g,h)x = a_{(g,h)}x$. Note that this is a group homomorphism which follows directly from group action axioms. Thereby, we obtain a map $X \to \mathcal{M}_{GxH}$ such that $x \to (S_x, \phi_x)$. This pair (S_x, ϕ_x) is called stabilizing pair of x. Considering $(\phi_x)_{|k(S_x)}$, we obtain two group homomorphisms $\phi_{1,x} : k_1(S_x) \to A$ and $\phi_{2,x} : k_2(S_x) \to A$ given by $\phi_{1,x}(g)x = g.x$ and $\phi_{2,x}(h)x = xh$. Note that the maps don't depend on the choice of A-orbits of x and notice that this map is constant on the orbit of $x \in X$. This allows us to define a new map $A/X \to \mathcal{M}_{G\times H}$ and observe that this map is a $G \times H$-equivariant map which follows from the fact that $(S_{(g,h)x}, \phi_{(g,h)x}) = {}^{(g,h)}(S_x, \phi_x)$.

<u>2.2.1.3. Transitive A-Fibered Bisets.</u> It is clear that an A-fibered (G, H)-biset X is transitive as a $A \times G \times H$-set if and only if $G \times H$ acts on the A-orbits of X transitively. Therefore, we call a A-fibered (G, H)-biset X a transitive A-fibered (G, H)-biset if it is a transitive $A \times G \times H$-set, equivalently the action of $G \times H$ on the A-orbits of X is transitive.

Let X be any transitive A-fibered (G, H)-biset, then for any $x \in X$, we associate X to the pair (S_x, ϕ_x). Conversely, given any pair $(U, \phi) \in \mathcal{M}_{G\times H}$, we define a transitive A-fibered (G, H)-biset $A \times G \times H/U_\phi$ where $U_\phi = \{(\phi(u^{-1}), u) : u \in U\}$. It is easy to check that the set $A \times G \times H/U_\phi$ is a transitive A-fibered (G, H)-biset. Thereby, we can associate any pair (U, ϕ) to a transitive A-fibered (G, H)-biset $A \times G \times H/U_\phi$. It is a standard task to see that the $G \times H$-conjugacy classes of $\mathcal{M}_{G\times H}$ is in one-to-one correspondence between the isomorphism classes of transitive A-fibered (G, H)-bisets. From the definition of transitive A-fibered bisets, it is clear that any A-fibered (G, H)-biset can be written as a union of transitive A-fibered (G, H)-bisets. This gives us the following Theorem.

Theorem 2.3 (Classification of A-fibered bisets)**.** *There is a one-to-one correspondence between the $G \times H$-conjugacy classes of the set $\mathcal{M}_{G\times H}$ and the isomorphism classes of transitive A-fibered (G, H)-bisets. Furthermore, any transitive A-fibered (G, H)-biset X is isomorphic to $(A \times G \times H)/U_\phi$ for some $(U, \phi) \in \mathcal{M}_{G\times H}$. We denote this transitive fibered biset as $(\frac{G\times H}{U,\phi})$. The isomorphism class of this fibered biset is denoted by $[\frac{G\times H}{U,\phi}]$.*

<u>2.2.1.4. Basis of A-Fibered Bisets as an Abelian Group.</u> Let X and Y be two A-fibered (G, H)-bisets. We denote the disjoint union of X and Y by $X \sqcup Y$ as the categorical coproduct of X and Y. Also, let $[X]$ denote the isomorphism classes of X. In that setting, we define an addition on the isomorphism classes of A-fibered bisets by $[X] + [Y] = [X \sqcup Y]$. With respect to this operation, the isomorphism classes of A-fibered (G, H)-bisets induces a monoid structure and the Grothendieck group induced by this monoid is an abelian group and denoted by $B^A(G, H)$. It is easy to see that elements $[\frac{G\times H}{U,\phi}]$ where $[U, \phi] \in G \times H/\mathcal{M}_{G\times H}$ form a $\mathbb{Z}$-basis for $B^A(G, H)$. The group

$B^A(G,H)$ is called the Burnside group of A-fibered (G,H)-bisets.

2.2.2. Tensor Products of A-Fibered Bisets

In biset theory, we know that two biset can be composed or in other words, we can multiply two bisets in a suitable manner. However, this product doesn't work in the fibered biset case because the biset product may produce some A-orbits which are not free. To fix this problem, we define the product of two A-fibered biset as follows: Let G, H and K be finite groups and let $X \in {}_GSet_H^A$ and $Y \in {}_HSet_K^A$ then the Cartesian product $X \times Y$ becomes a $A \times H$-set and we consider the set $X \times_{AH} Y$ which is the set of $A \times H$-orbits of $X \times Y$. A acts on the set $X \times_{AH} Y$ via $a[x,y]_{AH} = [ax,y]_{AH} = [x,ay]_{AH}$ and we also have $[xh,y]_{AH} = [x,hy]_{AH}$. Taking free A-orbits of the set $X \times_{AH} Y$, we define a $G \times K$ action on the A-orbits of the set $X \times_{AH} Y$ via $(g,k)[x,y]_{AH} = [gx, yk^{-1}]_{AH}$. It is easy to show that $G \times K$ action on the set $X \times_{AH} Y$ permutes the free A-orbits and thereby the union of free A-orbits of $X \times_{AH} Y$ is a A-fibered (G,K)-biset. Denote a free orbit $[x,y]_{AH}$ via $x \otimes_{AH} y$.

Definition 2.6. Let X be A-fibered (G,H)-biset and Y be (H,K)-biset. Tensor product of X and Y, denoted by $X \otimes_{AH} Y$, is defined to be the union of all free A-orbits of the set $X \times_{AH} Y$.

We will give some properties of tensor product. It is important to note that these properties imply that our constructions are functorial so that it is reasonable to study the objects in a categorical setting.

- The tensor product is functorial in X and Y, i.e we have two functors
$$- \otimes_{AH} Y : {}_Gset_H^A \to {}_Gset_K^A, X \to X \otimes_{AH} Y \tag{2.2}$$
and we apply X from left, we obtain another functor
$$X \otimes_{AH} - : {}_Hset_K^A \to {}_Gset_K^A, Y \to X \otimes_{AH} Y. \tag{2.3}$$
- For any $X, X' \in {}_Gset_H^A$ and $Y, Y' \in {}_Hset_K^A$, we have that $(X \sqcup X') \otimes_{AH} Y \cong (X \otimes_{AH} Y) \sqcup (X' \otimes_{AH} Y)$ and $X \otimes_{AH} (Y \sqcup Y') \cong (X \otimes_{AH} Y) \sqcup (X \otimes_{AH} Y')$.

- For $X \in {}_G set_H^A$, $Y \in {}_H set_K^A$ and $Z \in {}_K set_L^A$ we have that $X \otimes_{AH} (Y \otimes_{AK} Z) \cong (X \otimes_{AH} Y) \otimes_{AK} Z$ as A-fibered (G, L)-sets.

Using the results above, it is enough to investigate the transitive A-fibered bisets to derive a formula for the tensor product of two A-fibered bisets. Yet, we have to determine which A-orbits remain free under the A-action on $X \times_{AH} Y$. For this and an exact formula for the tensor product, we need the following notations and results.

2.2.2.1. Free A-Orbits and Mackey Formula. Let G, H and K be finite groups and $U \leq G \times H$, $V \leq H \times K$, $\phi \in U^*$ and $\varphi \in V^*$. Assume that $(\phi_2)_{|k_2(U) \cap k_1(V)} = (\varphi_1)_{|k_2(U) \cap k_1(V)}$. We define the star product of U and V as

$$U * V = \{(g, k) \in G \times K : \text{There exists h} \in \text{H such that (g,h)} \in \text{U and (h,k)} \in \text{V} \}. \tag{2.4}$$

Also we define the star product of ϕ and φ as $(\phi * \varphi)(g, k) = \phi(g, h) \cdot \varphi(h, k)$ for some $h \in H$ where $(g, k) \in U * V$. Notice that due to the assumption, this map is independent of the choice of h and produces a group homomorphism. The following proposition gives us a parametrization for the set of free A-orbits of $X \times_{AH} Y$. Using the previous results, it is enough to do this for two transitive A-fibered bisets.

Proposition 2.1. *Let X be a transitive A-fibered (G, H)-biset and Y be a transitive A-fibered (H, K)-biset, let $x \in X$ and $y \in Y$. Set (S_x, ϕ_x) and (S_y, ϕ_y) be corresponding stabilizing pairs of x and y respectively. Also, set $H_x = p_2(S_x)$ and $H_y = p_1(S_y)$. Then we have:*

(a) Any element $[\alpha, \beta]_{AH} \in X \times_{AH} Y$ is in the same $A \times G \times K$-orbit as $[x, hy]_{AH}$ for some $h \in H$.

(b) Let $t, t' \in H$, then $[x, hy]_{AH}$ and $[x, h'y]_{AH}$ are in the same $A \times G \times K$-orbit of $X \times_{AH} Y$ if and only if $H_x t H_y = H_x t' H_y$.

(c) Let $t \in H$. The stabilizer of $[x, ty]_{AH}$ in A is trivial if and only if $(\phi_{x,2})_{|H_t} = {}^t\phi_{y,1|H_t}$ where $H_t = k_2(S_x) \cap {}^t k_1(S_y)$.

(d) The elements $[x, hy]_{AH}$ where h runs through the set of representative of doubles

cosets $H_x\backslash H/H_y$ satisfying (c) forms a set of representatives of $A\times G\times K$-orbit of $X\otimes_{AH}Y$. For an element $h\in H$, the stabilizing pair of $[x,hy]_{AH}$ is equal to $(S_x *^{(t,1)} S_y, \phi_x *^{(t,1)} \phi_y)$.

Proof. It can be found on [5]. □

Using the above Proposition, one can easily compute the $X\otimes_{AH}Y$ and this formula is called Mackey's formula. Also, it is easy to see that it is indeed a generalization of the product of two biset functors where we take the fiber set to be $A=\{1\}$ and $\phi=i$ where i is the trivial group homomorphism for any choice of A and $\phi\in Hom(U,A)$.

Theorem 2.4 (Mackey's formula)**.** *Let $(U,\phi)\in\mathcal{M}_{G\times H}$ and $(V,\varphi)\in\mathcal{M}_{H\times K}$, then*

$$\Big(\frac{G\times H}{U,\phi}\Big)\otimes_{AH}\Big(\frac{H\times K}{V,\varphi}\Big)\cong\bigcup_{t\in H_x/H\backslash H_y}\Big(\frac{G\times K}{U*^{(t,1)}V,\phi*^{(t,1)}\varphi}\Big) \tag{2.5}$$

such that $(\phi_2)_{H_t}={}^{t}(\varphi_1)_{H_t}$ where $H_t=k_2(U)\cap{}^{t}k_1(V)$.

2.2.2.2. Standard A-Fibered Bisets. As in case of bisets, there are special A-fibered bisets, namely restriction, inflation, deflation and induction. Similar to bisets, one may wish to write any transitive A-fibered biset as a tensor product of these standard A-fibered bisets. Let us define the standard A-fibered bisets.

Definition 2.7. Let $H_1\le H_2\le G$ and $N\trianglelefteq G$, $f:H_1\to H_2$ and consider $\Delta_f H_1=\{(h,f(h)):h\in\ H_1\}\le H_1\times H_2$, ${}_f\Delta H_1=\{(f(h),h):h\in\ H_1\}\le H_2\times H_1$. Then if f is the inclusion map, just write $\Delta(H_1)$. The standard A-fibered bisets are defined as follows:

- For $H\le G$, Induction is defined as $Ind_H^G=(\frac{G\times H}{\Delta(H),1})$.
- For $H\le G$, Restriction is defined as $Res_H^G=(\frac{H\times G}{\Delta(H),1})$.
- For $N\trianglelefteq G$, Inflation is defined as $Inf_{G/N}^G=(\frac{G\times G/N}{\Delta(G)_\pi,1})$ where π is the canonical surjection.
- For $N\trianglelefteq G$, Deflation is defined as $Def_{G/N}^G=(\frac{G/N\times G}{{}_\pi\Delta(G),1})$ where π is the canonical surjection.

Theorem 2.5. *Let* $(U,\phi) \in \mathcal{M}_{G\times H}$*, set* $(P,K,\kappa) = (p_1(U), k_1(U), \phi_1)$ *and* $(Q,L,\lambda) = (p_2(U), k_2(U), \phi_2)$*, also set* $\hat{K} = ker(\kappa)$ *and* $\hat{L} = ker(\lambda)$*. Any transitive A-fibered* (G,H)*-biset* $\left(\frac{G\times H}{U,\phi}\right)$ *can be written as a tensor product of standard A-fibered bisets as*

$$\left(\frac{G\times H}{U,\phi}\right) \cong Ind_P^G \otimes_{AP} Inf_{\frac{P}{P/\hat{K}}}^{P} \otimes_{AP/\hat{K}} X \otimes_{AQ/\hat{L}} Def_{Q/\hat{L}}^{Q} \otimes_{AQ} Res_Q^H \tag{2.6}$$

where $X \cong \left(\frac{P/\widehat{K}\times Q\ \widehat{L}}{U/\widehat{K}\times\widehat{L}}\right)$.

2.2.3. A-Fibered Biset Functors

To study the A-fibered bisets, it is enough to study the category whose morphisms are A-fibered bisets since the tensor product provides a composition of two A-fibered bisets. Let $\mathcal{C}_k^A = \mathcal{C}$ denote the category whose objects are finite groups. For two finite group G and H, let us define $Mor_C(G,H) = k \otimes_{\mathbb{Z}} B^A(H,G)$. Also, the composition is defined as

$$- \cdot_H - : B_k^A(K,H) \times B_k^A(H,G) \to B_k^A(K,G), (x,y) \to x \cdot_H y \tag{2.7}$$

where the product denotes the k-linear extension of the tensor product of A-fibered bisets. To illustrate the composition, let us take $k = \mathbb{R}$ and G, H and K be any finite groups with $(U,\phi) \in \mathcal{M}_{G\times H}$ and $(V,\varphi) \in \mathcal{M}_{H\times K}$, then for any real numbers $a, b \in \mathbb{R}$, we compute $a \cdot \left[\frac{G\times H}{U,\phi}\right] \cdot b \cdot \left[\frac{H\times K}{V,\varphi}\right] = ab \cdot (\left[\frac{G\times H}{U,\phi}\right] \otimes_{AH} \left[\frac{H\times K}{V,\varphi}\right])$. Note that the identity element for each finite group G in this composition is $Ind_G^G = Res_G^G = \left[\frac{G\times G}{\Delta G,1}\right]$.

Definition 2.8. An A-fibered biset functor over k is a k-linear functor(See [8] for Definition) from the k-linear category $\mathcal{C}$ to k-mod.

Let $\mathcal{F} = \mathcal{F}_k^A$ denote the category whose objects are A-fibered biset functors over k and the morphisms are natural transformations between two A-fibered biset functors. Since the target category in each fibered biset functor F is an abelian category, namely k-mod, the category which equips with the objects which are functors satisfying this property is an abelian category. Therefore, it allows us to consider simple functors, quotient functors and other constructions which are available on abelian categories. Our aim in the remaining section is to classify all simple objects in the category $\mathcal{F}$.

Let $f : A \to A'$ be any group homomorphism, then any A-fibered biset can be viewed as a A'-fibered biset. If $\left(\frac{G\times H}{U,\phi}\right)$ is any transitive A-fibered biset, then $\phi' = f \circ \phi$ leads us to define a transitive A-fibered biset $\left(\frac{G\times H}{U,\phi'}\right)$. Extending it linearly to k-vector space $B_k^A(G,H)$, we obtain a k-vector space map $B_k^A(G,H) \to B_k^{A'}(G,H)$. This produces a k-linear functor $\mathcal{C}_k^A \to \mathcal{C}_k^{A'}$ and this gives us a k-linear functor $\mathcal{F}_k^A \to \mathcal{F}_k^{A'}$. More precisely, let us consider $\iota : torA \to A$ be inclusion map. Since the morphisms $\phi : U \to A$ have torsion image, this implies that any element in $\mathcal{F}_k^A$ can be found on $\mathcal{F}_k^{torA}$. Therefore, we conclude that $\mathcal{F}_k^A \cong \mathcal{F}_k^{torA}$. It is enough to work with the torsion part of the abelian group A to determine the simple objects in $\mathcal{F}_k^A$.

2.2.3.1. Classification of Simple A-Fibered Biset Functors. We will use the same construction as in Chapter 4, [8]. Let G be a finite group and consider the Grothendieck group $B_k^A(G,G) = E_G$. Note that it becomes a k-algebra with multiplication induced by the tensor product. Let $F : \mathcal{C} \to k$-mod be any fibered biset functor. Then its evaluation at G is endowed with a E_G-module structure such that for any $X \in E_G$, the action of X on $F(G)$ is determined by the map $F(X) : F(G) \to F(G)$, i.e $X.m = F(X)(m)$ where $m \in F(G)$. Therefore, we have a functor $\mathcal{E}_G : \mathcal{F} \to E_G$-mod given by the evaluation of the functor F at G. Given any E_G module V, let's consider the following functor $\mathcal{L}_{G,V} : \mathcal{C} \to k$-mod such that $H \to B_k^A(H,G) \otimes_{E_G} V$ and any morphism $X \in B_k^A(H,K)$ is sent to the k-module homomorphism $\mathcal{L}_{G,V}(X) : B_k^A(H,G) \otimes_{E_G} V \to B_k^A(K,G) \otimes_{E_G} V$ given by $Y \otimes_{E_G} v$ to $(X \otimes_{AH} Y) \otimes_{E_G} v$ for any $Y \otimes_{E_G} v \in B_k^A(H,G) \otimes_{E_G} V$. Note that that construction allows us to define a functor

$$\mathcal{L}_{G,-} : E_G - mod \to \mathcal{F} \tag{2.8}$$

such that $V \to L_{G,V}$ and any k-mod homomorphism $\phi : V \to W$ corresponds to a natural transformation $L_{G,\phi} : L_{G,V} \to L_{G,W}$ such that for each object H, we have a family of morphisms $(L_{G,\phi})_H$ in k-mod such that for any $f : H \to K$ in $Mor(\mathcal{C})$, i.e $f = X \in B_k^A(K,H)$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_{G,V}(H) & \xrightarrow{(\mathcal{L}_{G,\phi})_H} & \mathcal{L}_{G,V'}(H) \\ {\scriptstyle \mathcal{L}_{G,V}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_{G,V'}(f)} \\ \mathcal{L}_{G,V}(K) & \xrightarrow{(\mathcal{L}_{G,\phi})_K} & \mathcal{L}_{G,V'}(K) \end{array} \tag{2.9}$$

Using a similar argument in [8], notice that the functor $\mathcal{L}_{G,-}$ is a left adjoint to the evaluation functor $\mathcal{E}_G$. In general, it is hard to describe the structure of the k-module $\mathcal{L}_{G,V}(H)$ for any $V \in E_G$ and for any finite group H. Specifically, we don't know the ring structure of E_G and therefore it is quite difficult to tell something about this k-module. However, when V is a simple E_G-module, then the module structure of $\mathcal{L}_{G,V}(H)$ is relatively easy. The following is from Chapter 3, [5].

Lemma 2.2. *Let V be a simple E_G-module. The A-fibered biset functor $\mathcal{L}_{G,V}$ has a unique maximal subfunctor $J_{G,V}$ which is computed at a finite group H as follows:*

$$J_{G,V}(H) = \Big\{ \sum_i x_i \otimes_{E_G} v_i \in B_k^A(H,G) \otimes_{E_G} V : \; y \in B_k^A(G,H); \sum_i (y \cdot x_i) v_i = 0 \Big\}. \tag{2.10}$$

Moreover, simple head $S_{G,V} = L_{G,V}/J_{G,V}$ is a simple A-fibered biset functor and $S_{G,V}(G) \cong V$ as a E_G-module.

If we show that any simple functor $F \in \mathcal{F}$ is of the form $S_{G,V}$, then it will be quite essential to study the ring structure of E_G. One interesting ideal of E_G which will play an important role on the classification of simple biset functors is the following:

Definition 2.9. For a finite group G, we set $I_G = \sum_{|H|<|G|} B_k^A(G,H) \cdot B_k^A(H,G)$ where the dot $\cdot$ refers to the composition of two morphism in the category $\mathcal{C}$. The sum runs over all finite groups H whose order is smaller than G.

Note that it is trivial to check that $I_G \subset E_G$ and I_G is an ideal of E_G thereby has a k-algebra structure. The factor k-algebra $\tilde{E}_G = E_G/I_G$ is called essential algebra of G. The following theorem answers the question that any simple functor $F \in \mathcal{F}$ is of the form $S_{G,V}$.

Theorem 2.6. *(a) Let $S \in \mathcal{F}$ be a simple functor and let G be a finite group such that $S(G) \neq \{0\}$. Then V is a simple E_G-module and $S_{G,V} \cong S$ in $\mathcal{F}$.*

(b) Let $S \in \mathcal{F}$ be a simple functor and let G be a group of smallest order such that $V = S(G) \neq \{0\}$. Then the simple E_G-module $S(G)$ is annihilated by I_G, i.e $S(G)$ is an $\tilde{E}_G$-module. In particular, every simple functor $S \in \mathcal{F}$ is isomorphic

to $S_{G,V}$ for some finite group G and a E_G-module V such that $V \cong S(G)$ which is annihilated by I_G.

(c) Let G be a finite group and V is an E_G-module which is annihilated by I_G. Then G is a minimal group for $S_{G,V}$, that is G is the group of smallest order such that $S_{G,V}(H) \neq \{0\}$.

(d) For finite groups G and H and simple E_G-module V and simple E_H-module W, we have that $S_{G,V} \cong S_{H,W}$ as elements in $\mathcal{F}$ if and only if $S_{G,V}(H) \cong S_{H,W}(H)$ as an E_H-module if and only if $S_{G,V}(G) \cong S_{H,W}(G)$ as an E_G-module.

Proof. The proof of part (a) follows from a similar argument as in [6]. For the part (b), let S be a simple functor and G be a finite group of smallest order satisfying $S(G) \neq \{0\}$. Let us show that $S(G)$ is annihilated by I_G, equivalently for any $x \in I_G$ and $y \in S(G)$, then $y \cdot x = 0$. Let $\left[\frac{G\times H}{U,\phi}\right]\left[\frac{H\times G}{V\varphi}\right]$ be any basis element in I_G. Then, for any $y \in S(G)$, we know that $S \cong S_{G,S(G)}$ therefore we can view y as an element in the quotient of $\mathcal{L}_{G,S(G)}$. Notice that $S(H) = 0$ and the element $\left[\frac{H\times G}{V,\varphi}\right] \cdot y$ can be seen as an element in the coset of $\mathcal{L}_{G,V}(H)$ in $J_{G,V}(H)$. Using the associativity of the tensor product, we get that $(\left[\frac{G\times H}{U,\phi}\right]\left[\frac{H\times G}{V\varphi}\right]) \cdot y = \left[\frac{G\times H}{U,\phi}\right](\left[\frac{H\times G}{V\varphi}\right] \cdot y)$.

Furthermore, we have $(\left[\frac{H\times G}{V\varphi}\right] \cdot y) \in S_{G,V}(H) = S(H) \neq \{0\}$, hence $(\left[\frac{H\times G}{V\varphi}\right] \cdot y) \in J_{G,V}(H)$, therefore by the definition, we immediately obtain that $(\left[\frac{G\times H}{U,\phi}\right]\left[\frac{H\times G}{V\varphi}\right]) \cdot y = 0$, hence I_G annihilates $S(G)$ that means $S(G)$ can be viewed as a $\tilde{E_G}$-module. Due to the simple condition, the functor S is non-zero and therefore there must be a minimal group G for S and by the above fact, such group G satisfies $S(G) \neq \{0\}$ such that $S(G)$ is annihilated by I_G. For part (c), Let V be a simple E_G-module such that V is annihilated by I_G. Assume that $|H| < |G|$. Then for any $x \otimes_{E_G} v \in B_k^A(H,G) \otimes_{E_G} V$ and for any $y \in B_k^A(G,H)$, $y \cdot x \in I_G$ and therefore $(y \cdot x) \cdot v = 0$ by assumption. This implies that $x \otimes_{E_G} v \in J_{G,V}(H)$. Hence, $S_{G,V}(H) = 0$. Thus, $S_{H,V}(G) \neq \{0\}$ implies $|H| \geq |G|$.

For part (d) Assume $S_{G,V} \cong S_{H,W}$, then $S_{G,V}(H) \cong S_{H,W}(H) \cong W$ as E_H-modules. Other if part also follows similarly. Conversely, asssume that $S_{G,V}(H) \cong W \cong$

$S_{H,W}(H)$ then by part (a), notice that $S_{G,V}(H) \neq \{0\}$, thereby $S_{G,V} \cong S_{H,V} \cong S_{H,W}$ where $V = S_{G,V}(H)$ and that is isomorphic to W, which finishes the proof. □

Observe that we don't know nothing about the structure of S on large groups, yet we know exactly how we can evaluate the small groups on this functor. Later, we will develop a way to understand the structure of biset functors and this will be a reason why we want to view a fibered biset functor as a biset functor.

2.2.4. Idempotents in E_G

By theorem in the last section, some modules of the k-algebra E_G parametrizes the simple functors $S \in \mathcal{F}$ such that G appears to be a minimal set for this simple functor. It could be very useful to study the ring structure of E_G and thereby looking for primitive central idempotents is particularly important. In this section, we will provide a set of idempotents which provide us a decomposition of the k-algebra E_G. Recall that G acts on $\mathcal{M}_G$ via conjugation and let us classify the pairs (K, κ) which is fixed by the conjugation action of G. The following result is an easy consequence and classifies fixed points of $\mathcal{M}_G$.

Lemma 2.3. *The fixed points of $\mathcal{M}_G$ are the pairs (K, κ) such that ${}^gK = K$ for all $g \in G$ and the map κ is fixed on the conjugacy classes of K on the action of G. Equivalently, pairs (K, κ) such that $K \trianglelefteq G$ and κ is fixed on the conjugacy classes of K on the action G are the fixed points of $\mathcal{M}_G$. The set of fixed points in $\mathcal{M}_G$ are denoted by $\mathcal{M}_G^G$.*

Let $(U, \phi) \in \mathcal{M}_{G\times H}$ with $p_1(U) = G$, then the pair $l_0(U, \phi) = (k_1(U), \phi_1) \in \mathcal{M}_G^G$. By a symmetric argument, note that if $p_2(U) = H$, then $r_0(U, \phi) = (k_2(U), \phi_2) \in \mathcal{M}_H^H$. Elements of $\mathcal{M}_G^G$ parametrize an idempotents of E_G. We define such idempotents.

Definition 2.10. For $(K, \kappa) \in \mathcal{M}_G^G$, we define a transitive A-fibered (G, G)-biset as $E_{(K,\kappa)} = \left(\frac{G\times G}{\triangledown_K G, \phi_\kappa}\right)$ where $\triangledown_K(G) = \{(g_1, g_2) \in G\times G : g_1K = g_2K\} = (K\times\{1\})\cdot\triangledown(G)$ and for any $(g_1, g_2) \in \triangledown_K(G)$, the map $\phi_\kappa(g_1, g_2) = \kappa(g_1 \cdot g_2^{-1})$.

In this definition, the fact that $\bigtriangledown_K(G)$ is a subgroup of $G \times G$ follows from the fact that $K \lhd G$ and homomorphism property of the map ϕ_κ follows from the fact that κ is constant on the conjugacy classes of K on G. As the next proposition suggests, not only such A-fibered bisets are primitive idempotents of E_G but also tensor product of any such fibered biset is relatively easy. This next result is from [5].

Proposition 2.2. *Let* $(K,\kappa),(K',\kappa^{'}) \in \mathcal{M}_G^G$ *and* $(U,\phi) \in \mathcal{M}_{G\times H}$ *such that* $l(U,\phi) = (G, K', \kappa^{'})$.

(a) $l(\bigtriangledown_K(G),\phi_\kappa) = (G,K,\kappa) = r(\bigtriangledown_K(G),\phi_\kappa)$.
(b) One has $E_{(K,\kappa)} \otimes_{AG} \left(\frac{G\times H}{U,\phi}\right) \cong \left(\frac{G\times H}{(K\times 1)U,\kappa.\phi}\right)$ *when* $\kappa_{\big|K\cap K'} = \kappa^{'}_{\big|K\cap K'}$*, otherwise it is equal to* $\varnothing$*. More precisely,* $E_{(K,\kappa)} \otimes_{AG} E_{(K',\kappa')} \cong E_{(KK',\kappa.\kappa')}$ *in case* $\kappa_{\big|K\cap K'} = \kappa^{'}_{\big|K\cap K'}$ *otherwise it is equal to* $\varnothing$*.*
(c) Assume that $(K,\kappa) \leq (K',\kappa^{'})$*, then we have that* $E_{(K,\kappa)} \otimes_{AG} \left(\frac{G\times H}{U,\phi}\right) = \left(\frac{G\times H}{U,\phi}\right)$*, in particular we have* $E_{(K,\kappa)} \otimes_{AG} E_{(K',\kappa')} = E_{(K',\kappa')}$*.*
(d) Assume that $p_2(U) = H$*, then* $\left(\frac{G\times H}{U,\phi}\right) \otimes_{AH} \left(\frac{G\times H}{U,\phi}\right)^{op} \cong E_{(K',\kappa')}$*.*
(e) $E_{(K,\kappa)} \cong (E_{(K,\kappa)})^{op}$*.*

Proof. The proof of part (a) is just an easy verification and the other parts follow easily from the Mackey's formula. □

Let us define $e_{(K,\kappa)} = [E_{(K,\kappa)}] \in B_k^A(G,G) = E_G$ for any $(K,\kappa) \in \mathcal{M}_G^G$. Using part (c) and part (e) of previous proposition, we have that $e_{(K,\kappa)} \cdot e_{(K,\kappa)} = e_{(K,\kappa)}$ and for any two $(K,\kappa),(K',\kappa^{'}) \in \mathcal{M}_G^G$, one has $e_{(K,\kappa)}e_{(K',\kappa')} = e_{(KK',\kappa.\kappa')} = e_{(K',\kappa')}e_{(K,\kappa)} =$ and notice that $e_{1,1} = Ind_G^G = 1_{E_G}$ thus we obtain a commutative k-subalgebra $\bigoplus_{(K,\kappa)\in\mathcal{M}_G^G} k.e_{(K,\kappa)}$. However, there may exist $(K,\kappa),(L,\lambda) \in \mathcal{M}_G^G$ such that $e_{(K,\kappa)}e_{(L,\lambda)} \neq 0$ so the set of such elements cannot provide a set of orthogonal set of primitive idempotents. To look for such idempotents in the commutative k-subalgebra $\bigoplus_{(K,\kappa)\in\mathcal{M}_G^G} k.e_{(K,\kappa)}$, we define the following idempotents:

Definition 2.11. Let $\mu^{\lhd}_{(K,\kappa),(L,\lambda)}$ denote the Mobius coefficient of the two G-fixed pairs $(K,\kappa),(L,\lambda)$ with respect to the poset structure on $\mathcal{M}^G_G$. We define a set of elements in $\bigoplus\limits_{(K,\kappa)\in\mathcal{M}^G_G} k.e_{(K,\kappa)}$ as follows:

$$f_{(K,\kappa)} = \sum_{(L,\lambda)\geq(K,\kappa)} \mu^{\lhd}_{(K,\kappa),(L,\lambda)} \cdot e_{L,\lambda}. \tag{2.11}$$

Note that elements $f_{(K,\kappa)}$ are idempotent elements and by using Mobius inversion formula, $e_{K,\kappa} = \sum\limits_{(L,\lambda)\geq(K,\kappa)} f_{(L,\lambda)}$ so that the set $\{f_{(K,\kappa)} : (K,\kappa) \in \mathcal{M}^G_G\}$ forms a k-basis for the commutative subalgebra $\bigoplus\limits_{(K,\kappa)\in\mathcal{M}^G_G} k.e_{(K,\kappa)}$ with $e_{(1,1)} = \sum\limits_{(K,\kappa)\in\mathcal{M}^G_G} f_{(K,\kappa)}$. The next proposition shows us that the set of elements $f_{(K,\kappa)}$ does form a central primitive orthogonal basis for the commutative k-subalgebra $\bigoplus\limits_{(K,\kappa)\in\mathcal{M}^G_G} k.e_{(K,\kappa)}$ and tensor product of this elements is even easier to compute.

Proposition 2.3. *For $(K,\kappa),(L,\lambda) \in \mathcal{M}^G_G$, one has the followings:*

(a) $e_{(K,\kappa)} \cdot f_{(L,\lambda)} = f_{(L,\lambda)} \cdot e_{(K,\kappa)} = f_{(L,\lambda)}$ *if* $(K,\kappa) \leq (L,\lambda)$ *, otherwise it is equal to 0.*

(b) $f_{(K,\kappa)} \cdot f_{(L,\lambda)} = f_{(K,\kappa)}$ *if* $(K,\kappa) = (L,\lambda)$ *, otherwise it is 0.*

2.2.5. Linkage

Definition 2.12. Let G and H be finite groups. We call pairs $(K,\kappa) \in \mathcal{M}^G_G$ and $(L,\lambda) \in \mathcal{M}^H_H$ linked if there exists $(U,\phi) \in \mathcal{M}_{G\times H}$ such that $l(U,\phi) = (G,K,\kappa)$ and $r(U,\phi) = (H,L,\lambda)$. In particular, if $G = H$, then two linked pairs (K,κ) and $(K^{'},\kappa^{'})$ are called G-linked. We write $\{K,\kappa\}_G$ for the G-linkage classes of (K,κ).

It is easy to see that it is an equivalence relation. Also, we observe that if (K,κ) and $(K^{'},\kappa^{'})$ are G-linked, then $|K| = |K^{'}|$. Partial ordering on M^G_G can be reformulated to a partial ordering on the G-linkage classes. We say that $\{K,\kappa\} \leq \{L,\lambda\}$ if and only if there exists $(K^{'},\kappa^{'}) \in \{K,\kappa\}$ and $(L^{'},\lambda^{'}) \in \{L,\lambda\}$ such that $(K^{'},\kappa^{'}) \leq (L^{'},\lambda^{'})$.

Verification of the partial ordering can be easily shown. Let $(K,\kappa) \in \mathcal{M}_G^G$, let us define

$$e_{\{K,\kappa\}_G} = \sum_{(K',\kappa') \in \{(K,\kappa)\}} e_{(K',\kappa')} \text{and} f_{\{K,\kappa\}_G} = \sum_{(K',\kappa') \in \{(K,\kappa)\}} f_{(K',\kappa')}. \tag{2.12}$$

For such elements in $B_k^A(G,G)$, we have the following:

Lemma 2.4. *Let* $(K,k),(L,\lambda) \in \mathcal{M}_G^G$, *then*

(i) $e_{\{K,\kappa\}_G} \cdot f_{\{L,\lambda\}_G} = 0$ *unless* $\{K,\kappa\} \le \{L,\lambda\}$.
$e_{\{K,\kappa\}_G} \cdot f_{\{L,\lambda\}_G} = f_{\{L,\lambda\}_G}$ *otherwise.*

(ii) $f_{\{K,\kappa\}_G} \cdot f_{\{L,\lambda\}_G} = 0$ *if* $\{K,\kappa\} \ne \{L,\lambda\}$

(iii) $f_{\{K,\kappa\}_G} \cdot f_{\{K,\kappa\}_G} = f_{\{K,\kappa\}_G}$.

Proof. See [5]. □

2.2.6. Structure of $E^c(G)$

We have already shown that the classification of simple A-fibered biset functors is closely related to the irreducible $\overline{E_G}$-modules. Since the structure of E_G seems to be complicated and we have nothing to classify it, therefore we look for a classification for $\overline{E_G}$. The exact classification of $\overline{E_G}$ is already made and can be found on [5]. We record some important notions and some theorems regarding the structure of E_G^c which appears to be a subalgebra of E_G^c which covers $\overline{E_G}$.

Definition 2.13. We define E_G^c to be the $\mathbb{C}$-subalgebra of $E_G = B_{\mathbb{C}}^{\mathbb{C}^*}(G,G)$ generated by the transitive $\mathbb{C}^*$-fibered bisets $\left[\frac{G\times G}{U,\phi}\right]$ where U has full projections, i.e $p_1(U) = p_2(U) = G$.

Notice that when we have such two pairs (U,ϕ) and (V,φ), then the star product $(U * V, \phi * \varphi)$ has full projections. Hence, we conclude that the subgroup E_G^c is actually a $\mathbb{C}$-subalgebra. We show that the subalgebra E_G^c actually covers $\overline{E_G}$ (that means we can find a map from E_G to $\overline{E_G}$ that sends E_G^c isomorphically to $\overline{E_G}$). Moreover, the essential algebra $\overline{E_G}$ is isomorphic to direct product of matrix algebras over certain group algebras. Along with some definitions, we record some results which are essential to the classification of E_G^c.

Lemma 2.5. *(i) We call a pair* $(U, \phi) \in \mathcal{M}_{G\times H}$ *covering if* $p_1(U) = G$ *and* $p_2(U) = H$*. Note that if we have two pairs* $(U, \phi) \in \mathcal{M}_{G\times H}$ *and* $(V, \varphi) \in \mathcal{M}_{H\times K}$ *both are covering such that* $r_0(U, \phi) = l_0(V, \varphi) = (K, \kappa)$*, then we have that* $(U * V, \phi * \varphi) \in \mathcal{M}_{G\times K}$*.*

(ii) We have $e_{(K,\kappa)}, e_{\{K,\kappa\}}, f_{(K,\kappa)}, f_{\{K,\kappa\}} \in E_G^c$ *for any* $(K, \kappa) \in \mathcal{M}_G^G$*.*

(iii) Let G be a finite group and $(K, \kappa) \in \mathcal{M}_G^G$*. Then, we define* ${}_{(G,K,\kappa)}\mathit{\Gamma}_{(G,K,\kappa)}$ *to be set of all standard basis elements* $\left[\frac{G\times G}{U,\phi}\right]$ *where* $l_0(U, \phi) = r_0(U, \phi) = (K, \kappa)$*. This set is a group with respect to the multiplication of fibered bisets. The identity element in this set is* $e_{(K,k)}$ *and the inverse of any element is equal to its opposite fibered biset.*

(iv) Assume that $(K, \kappa) \in \mathcal{M}_G^G$*,*$(L, \lambda) \in \mathcal{M}_H^H$*. We set*

$$ {}_{(G,K,\kappa)}\mathit{\Gamma}_{(H,L,\lambda)} = \{\left[\frac{G\times H}{U,\phi}\right] : l(U, \phi) = (G, K, \kappa), r_0(U, \phi) = (H, L, \lambda)\}. \quad (2.13)$$

This set is non-empty if and only if $(G, K, \kappa) \sim (H, L, \lambda)$*. Also, even if this set is non-empty, the set doesn't have a natural group structure induced by the multiplication of fibered bisets.*

Assuming that $(G, K, \kappa) \sim (H, L, \lambda)$, then the set ${}_{(G,K,\kappa)}\mathit{\Gamma}_{(H,L,\lambda)}$ becomes a $(\mathit{\Gamma}_{(G,K,\kappa)}, \mathit{\Gamma}_{(H,L,\lambda)})$-biset where the actions are given by left and right multiplication of bisets. We prove that the action of both groups are free and transitive.

Lemma 2.6. *Groups* $\mathit{\Gamma}_{(G,K,\kappa)}$ *and* $\mathit{\Gamma}_{(H,L,\lambda)}$ *act freely and transitively on* ${}_{(G,K,\kappa)}\mathit{\Gamma}_{(H,L,\lambda)}$*.*

Proof. It is enough to prove these results for the action of $\mathit{\Gamma}_{(G,K,\kappa)}$ on ${}_{(G,K,\kappa)}\mathit{\Gamma}_{(H,L,\lambda)}$. We start by showing that this action is free.

Let $\left[\frac{G\times G}{V,\varphi}\right] \in \mathit{\Gamma}_{(G,K,\kappa)}$ and $\left[\frac{G\times H}{U,\phi}\right] \in {}_{(G,K,\kappa)}\mathit{\Gamma}_{(H,L,\lambda)}$ such that $\left[\frac{G\times G}{V,\varphi}\right] \cdot \left[\frac{G\times H}{U,\phi}\right] = \left[\frac{G\times H}{U,\phi}\right]$. By Mackey's formula, we have $V * U = U$ and we will prove that $\bigtriangledown_K(G) = V$.

Let $(g_1, g_2) \in V$, then there exists $(g_2, h), (g_1, h) \in U$. Hence, $(g_1.g_2^{-1}, 1) \in U$ hence $g_1.g_2^{-1} \in K$, therefore $(g_1, g_2) \in \bigtriangledown_K(G)$. Thereby, $V \leq \bigtriangledown_K(G)$. Conversely, let us take $(g_1, g_2) \in \bigtriangledown_K(G)$. By definition, we have $g_1 g_2^{-1} \in K$ so $(1, g_1 g_2^{-1}) \in V$ and $(g_2, a) \in V$ for some $a \in G$ with $(a, b) \in U$ for some $b \in H$. So these relations imply $g_2 a^{-1} \in K$ and thereby $(1, g_2 a^{-1}) \in V$ with $(g_2, a)(1, g_2 a^{-1}) = (g_2, g_2) \in V$. Moreover, we have that $(g_1.g_2^{-1}, 1)(g_2, g_2) = (g_1, g_2) \in V$. Consequently, we conclude that $\bigtriangledown_K(G) \leq V$. Hence, $\bigtriangledown_K(G) = V$.

Next we prove that $\phi_\kappa = \varphi$. Let $(g_1, g_2) \in \bigtriangledown_K(G)$. Therefore, we can find $g \in G$ such that $(g_2, g) \in V$ and $(g_1, g) \in V$. Hence, $\phi(g_1, g) = \varphi(g_1, g_2)\phi(g_2, g)$ so that $\varphi(g_1, g_2) = \phi(g_1 \cdot g_2^{-1}, 1) = \kappa(g_1 g_2^{-1})$ which concludes that $\varphi = \phi_\kappa$. Hence, the action is free.

Next, we prove that the action is transitive. Let us take any two element $\left[\frac{G\times H}{U,\phi}\right], \left[\frac{G\times H}{V,\varphi}\right] \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$. We have that $\left[\frac{G\times H}{U,\phi}\right] \cdot \left[\frac{G\times H}{V,\varphi}\right]^{op} \in \Gamma_{(G,K,\kappa)}$ and $\left[\frac{G\times H}{U,\phi}\right]^{op} \cdot \left[\frac{G\times H}{U,\phi}\right] = e_{L,\lambda}$ which give us the fact that $\left[\frac{G\times H}{U,\phi}\right] \cdot \left[\frac{G\times H}{V,\varphi}\right]^{op} \cdot \left[\frac{G\times H}{V,\varphi}\right] = \left[\frac{G\times H}{U,\phi}\right]$. Thus, the action is transitive. □

Observe that being free and transitive as left G-set and right H-set imply that we can view this set as a transitive (G, H)-biset. Also, as a $\Gamma_{(G,K,\kappa)}$-set, $\Gamma_{(G,K,\kappa)}$ and ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ are isomorphic. Also, by the same reasoning, we can see that as a $\Gamma_{(H,L,\lambda)}$-set, $\Gamma_{(H,L,\lambda)}$ and ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ are isomorphic. We obtain the following result:

Lemma 2.7. *There exists a group isomorphism γ between $\Gamma_{(H,L,\lambda)}$ and $\Gamma_{(G,K,\kappa)}$ given by $y \to \left[\frac{G\times H}{U,\phi}\right] \cdot_H y \cdot_H \cdot \left[\frac{G\times H}{U,\phi}\right]^{op}$ for some $\left[\frac{G\times H}{U,\phi}\right] \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$. Moreover, if $\left[\frac{G\times H}{U',\phi'}\right] \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$, then the resulting isomorphism γ' satisfies $\gamma' = c_z \circ \gamma$ where $z = \left[\frac{G\times H}{U',\phi'}\right] . \frac{G\times H}{U,\phi}]^{op}$ where c_z is an conjugation isomorphism of $\Gamma_{(G,K,\kappa)}$.*

Notice that since ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ is transitive as a left $\Gamma_{(G,K,\kappa)}$-set and transitive as a right $\Gamma_{(H,L,\lambda)}$-set with this isomorphism on these groups, we observe that

$$ {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)} \cong \left[\frac{\Gamma_{(G,K,\kappa)} \times \Gamma_{(H,L,\lambda)}}{\triangle_\gamma}\right] \tag{2.14}$$

where $\triangle_\gamma := \{(\gamma(y), y) : y \in \mathit{\Gamma}_{(H,L,\lambda)}\}$. This induces a canonical bijection

$$\mathrm{Irr}(k\mathit{\Gamma}_{(H,L,\lambda)}) \to \mathrm{Irr}(k\mathit{\Gamma}_{(G,K,\kappa)}) \tag{2.15}$$

and therefore we conclude that two linked pairs have an isomorphic $\mathit{\Gamma}$-groups and the corresponding group algebras have isomorphic irreducible modules. This will be used when we construct our irreducible fibered biset functor. We see that the elements $f_{\{K,k\}}$'s are the elements which parametrize the $\mathbb{C}$-subalgebra E_G^c. To prove the main theorem regarding the structure of E_G^c, we record some results there whose proofs can be found in [5]. The following two theorems are very important for the parametrization we aim to make.

Theorem 2.7. *(a) Let $(U, \phi) \in \mathcal{M}^c_{G\times G}$ and let $(K, \kappa), (L, \lambda) \in \mathcal{M}_G^G$. If*

$$f_{\{K,\kappa\}} \cdot \left[\frac{G \times G}{U, \phi}\right] \cdot f_{\{L,\lambda\}} \neq 0 \tag{2.16}$$

, then $\{K, \kappa\}_G = \{L, \lambda\}_G$.

(b) The elements $f_{\{K,\kappa\}}$ where $\{K, \kappa\}_G \in \mathcal{M}_G^G/\sim$ are mutually orthogonal central idempotents of E_G^c and their sum is equal to 1. One has a decomposition

$$E_G^c = \bigoplus_{\{K,\kappa\}\in\mathcal{M}_G^G/\sim} f_{\{K,\kappa\}_G} E_G^c \tag{2.17}$$

of E_G^c into two-sided ideals.

(c) We have another decomposition of E_G^c into $\mathbb{C}$-submodules which is given by

$$E_G^c = \bigoplus_{\{K,\kappa\}\in\mathcal{M}_G^G/\sim} E_G^{c,\{K,\kappa\}} \tag{2.18}$$

where $E_G^{c,\{K,\kappa\}}$ is generated by the transitive $\mathbb{C}^*$-fibered bisets $\left[\frac{G\times G}{U,\phi}\right] \in \mathcal{M}^c_{G\times G}$ where $l_0(U, \phi) \in \{K, \kappa\}_G$.

(d) Let $(K, \kappa) \in \mathcal{M}_G^G$, $(U, \phi) \in \mathcal{M}^c_{G\times G}$ with $r_0(U, \phi) = (K, \kappa)$ and let (L, λ) with $(K, \kappa) \not\leq (L, \lambda)$. Then $\left[\frac{G\times G}{U,\phi}\right] \cdot f_{(L,\lambda)} = 0$.

(e) One has the following decomposition

$$\bigoplus_{\{K,\kappa\}\leq\{L,\lambda\}\in\mathcal{M}_G^G/\sim} E_G^{c,\{L,\lambda\}} = \bigoplus_{\{K,\kappa\}\leq\{L,\lambda\}\in\mathcal{M}_G^G/\sim} E_G^c f_{\{L,\lambda\}_G}. \tag{2.19}$$

(f) The projection map which is from $E_G^{c,\{K,\kappa\}}$ to $E_G^c f_{\{K,\kappa\}_G}$

$$\omega : E_G^{c,\{K,\kappa\}} \to E_G^c f_{\{K,\kappa\}_G} \tag{2.20}$$

given by $b \to bf_{\{K,\kappa\}_G}$ is an isomorphism of k-modules whose inverse is the pro-

jection map with respect to the decomposition $E^c_G = \bigoplus_{\{K,\kappa\} \in \mathcal{M}^G_G/\sim} E^{c,\{K,\kappa\}}_G$.

(g) *Enumerating the elements of* $\{K,\kappa\}_G$ *as* $(K_1,\kappa_1),...,(K_n,\kappa_n)$, *the isomorphism* ω *is the direct sum of k-module isomorphisms*

$$\omega_{ij} : {}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)} \to f_{(K_i,\kappa_i)} E^c_G f_{(K_j,\kappa_j)}, b_{ij} \to f_{(K_i,\kappa_i)} b_{ij} f_{(K_j,\kappa_j)}. \tag{2.21}$$

(h) *Let* $i,j,l,m \in \{1,..,n\}, b_{ij} \in \mathbb{C}\,{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}$ *and* $b_{lm} \in \mathbb{C}\,{}_{(G,K_l,\kappa_l)}\Gamma_{(G,K_m,\kappa_m)}$. *If* $j = l$,*then* $b_{ij}b_{lm} \in \mathbb{C}\,{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_m,\kappa_m)}$. *In any case, one has the following*

$$\omega_{ij}(b_{ij})\omega_{lm}(b_{lm}) = \begin{cases} \omega_{il}(b_{ij}b_{lm}), & \text{if } j = l, \\ 0, & \text{otherwise.} \end{cases} \tag{2.22}$$

(i) *The map* $k[\Gamma_{(G,K,\kappa)}] \to {}_{(G,K,\kappa)}\Gamma_{(G,K,\kappa)}$ *given by* $a \to f_{(K,\kappa)} a f_{(K,\kappa)}$ *is an k-algebra isomorphism.*

Theorem 2.8 (Structure of E^c_G)**.** *There exists a* $\mathbb{C}$*-algebra isomorphism*

$$\bigoplus_{\{K,\kappa\} \leq \{L,\lambda\} \in \mathcal{M}^G_G/\sim} \mathrm{Mat}_{|\{K,\kappa\}_G|}(\mathbb{C}\Gamma_{(G,K,\kappa)}) \xrightarrow{\sim} E^c_G \tag{2.23}$$

with the following property:

For every $(K,\kappa) \in \mathcal{M}^G_G$, *writing* $\{K,\kappa\} = \{(K_1,\kappa_1),...,(K_n,\kappa_n)\}$, *the element* $f_{(K_i,\kappa_i)}$ *is mapped to the diagonal idempotent matrices* $e_i \in \mathrm{Mat}_{|\{K,\kappa\}_G|}(\mathbb{C}\Gamma_{(G,K,\kappa)})$.

We see that using this theorem along with Morita equivalances, irreducible $\overline{E_G}$-modules are characterized by the irreducible $\mathbb{C}\Gamma_{(G,K,\kappa)}$-modules. We need to understand the structure of $\Gamma_{(G,K,\kappa)}$. We see that it is heavily dependent on the group structure of the quotient group G/K. Under some conditions, we have the following theorem.

Theorem 2.9 (Structure of $\Gamma_{(G,K,\kappa)}$)**.** *Let* $(K,\kappa) \in \mathcal{M}^G_G$ *and assume that* κ *is faithful. One has a short exact sequence*

$$1 \to (G/K)^* \xrightarrow{\iota} \Gamma_{(G,K,\kappa)} \xrightarrow{\pi} \mathrm{Out}^\circ(G/K) \to 1 \tag{2.24}$$

where $\iota : (G/K)^* \to \Gamma_{(G,K,\kappa)}$ *given by* $\theta \to \left[\frac{G\times G}{\triangle_K(G),\overline{\theta}}\right]$ *with* $\overline{\theta}(gk,g) = \kappa(k)\theta(gK)$ *and* $\pi : \Gamma_{(G,K,\kappa)} \to Out^\circ(G/K)$ *such that* $\left[\frac{G\times G}{U_{\eta,\phi}}\right] \to \overline{\eta} \to \overline{\eta}Inn(G/K)$.

2.2.7. Reduced Pairs and the Simple $\overline{E_G}$-Modules

A special subset of $\mathcal{M}_G^G$ plays a central role in the classification of E_G^c. It occurs as a parametrizing set for E_G^c when we decompose it as a $\mathbb{C}$-algebra. Let us define this set and give some results regarding this set.

Definition 2.14. We define the subset $\mathcal{R}_G = \mathcal{R}_k^A(G)$ by

$$\mathcal{R}_G = \{(K,\kappa) \in \mathcal{M}_G^G | e_{(K,\kappa)} \notin I_G\}. \tag{2.25}$$

We call $(K,\kappa) \in \mathcal{M}_G$ reduced pair if $(K,\kappa) \in \mathcal{R}_G$.

Let us record some easy but very useful results on reduced pairs. The first lemma says that partial order respects the being reduced and the second lemma will be particularly important for the description of the generators of I_G in terms of the set $\mathcal{R}_G$.

Lemma 2.8. *(a) Assume that (K,κ) and (K',κ') are G-linked. Then (K,κ) is reduced if and only if (K',κ') is reduced. Hence, the equivalence relation on $\mathcal{M}_G^G$ is inherited to $\mathcal{R}_G$.*

(b) Assume that $(K,\kappa) \leq (K',\kappa') \in \mathcal{R}_G$ implies $(K,\kappa) \in \mathcal{R}_G$.

Proof. Suppose $(K',\kappa') \notin \mathcal{R}_G$, let $(K,\kappa) \sim (K',\kappa')$ and $(K,\kappa) \in \mathcal{R}_G$. Then there exists $(U,\phi) \in \mathcal{M}_{G\times G}^c$ with $l_0(U,\phi) = (K,\kappa)$ and $r_0(U,\phi) = (K',\kappa')$. We have $e_{(K',\kappa')} \in I_G$, then $e_{(K,k)} = \left[\frac{G\times G}{U,\phi}\right] \cdot \left[\frac{G\times G}{U,\phi}\right]^{op} = \left[\frac{G\times G}{U,\phi}\right] \cdot e_{(K',\kappa')}\left[\frac{G\times G}{U,\phi}\right]^{op} \in I_G$ which leads a contradiction. For the second part, if $(K,\kappa) \notin \mathcal{R}_G$, then $e_{(K,\kappa)} \in I_G$ therefore $e_{(K,\kappa)}.e_{(K',\kappa')} = e_{(K',\kappa')} \in I_G$. This is a contradiction. □

Lemma 2.9. *The ideal I_G of E_G is generated as a $\mathbb{C}$-module by the standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with (U,ϕ) satisfying*

$$p_1(U) \neq G \tag{2.26}$$

or in case that we have $p_1(U) = G$, we should have

$$l_0(U,\phi) \notin \mathcal{R}_G. \tag{2.27}$$

Proof. See [5] for the proof. □

Since I_G is stable under the map $(-)^{op} : E_G \to E_G$, the above conditions can be replaced by $p_2(U) \neq G$ or $p_2(U) = G$ and $r_0(U, \phi) \notin \mathcal{R}_G$. Now we find a parametrizing set for $\overline{E_G}$. Detailed proof of the next theorem can be found on [5].

Theorem 2.10. *(a) One has E_G is the sum of E_G^c and I_G and we have decomposition*

$$E_G^c \cap I_G = \bigoplus_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim, (K,\kappa) \notin \mathcal{R}_G} f_{\{K,\kappa\}_G} E_G^c \tag{2.28}$$

where second summand ranges over a set of elements which is an equivalence class and their representative is not in $\mathcal{R}_G$.

(b) The canonical epimorphism $E_G \to \overline{E_G}$ maps the subalgebra

$$\bigoplus_{\{K,\kappa\}_G \in \mathcal{R}_G/\sim} f_{\{K,\kappa\}_G} E_G^c \tag{2.29}$$

of E_G^c isomorphically onto $\overline{E_G}$.

(c) For each $(K, \kappa) \in \mathcal{R}_G$, the map

$$\mathbb{C}\Gamma_{(G,K,\kappa)} \to \overline{f}_{(K,\kappa)} \overline{E_G} \overline{f}_{(K,\kappa)}, a \to \overline{f}_{(K,\kappa)} \overline{a} \overline{f}_{(K,\kappa)} \tag{2.30}$$

is a $\mathbb{C}$-algebra isomorphism.

2.2.8. Classfication of Simple Fibered Biset Functors

In this subsection, we will parametrize the simple $\mathbb{C}$-fibered biset functors. We start by introducing some notations.

Definition 2.15. Set $\mathcal{S}_G = \{((K,\kappa),[V]) | (K,\kappa) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(\mathbb{C}\Gamma_{(G,K,\kappa)})\}$. Call two elements $((K,\kappa),[V]), ((K',\kappa'),[V']) \in \mathcal{S}_G$ equivalent if (K,κ) and (K',κ') are G-linked and $[V]$ corresponds to $[V']$ via the canonical bijection

$$\mathrm{Irr}(\mathbb{C}\Gamma_{(G,K',\kappa')}) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{C}\Gamma_{(G,K,\kappa)}). \tag{2.31}$$

We note that the equivalence relation on pairs and isomorphism classes of the irreducible modules of group algebras $\mathbb{C}\Gamma_{(G,K,\kappa)}$ give rise to an equivalence relation on $\mathcal{S}_G$. The next result provides a classification of simple $\overline{E_G}$-modules. The proof of this result can also be found on [5] which follows from the previous theorems we have stated and the Morita equivalence of matrix ring $M_n(R)$ and a ring R. Also, the above theorem tells us the importance of the quotient algebra $\overline{E_G}$.

Theorem 2.11 (Classification of $\mathrm{Irr}(\overline{E_G})$)**.** *The map $\mathcal{S}_G \xrightarrow{\sim} \mathrm{Irr}(\overline{E_G})$ given by*

$$((K,\kappa),[V]) \to \tilde{V} := \overline{E_G}.\overline{f}_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V \tag{2.32}$$

induces a bijection between the set of equivalence classes of $\mathcal{S}_G$ and $\mathrm{Irr}(\overline{E_G})$.

The above theorem tells us that if we are able to find reduced pairs for any finite group G, then we classify all fibered biset functors. This is a very hard task and there is a classification of reduced pairs(see [5]) when the fiber is a nice group. The next proposition gives us a necessary condition and a sufficient condition for a pair to be reduced.

Proposition 2.4. *Let $(K,\kappa) \in \mathcal{M}_G^G$*

(i) If $(K,\kappa) \in \mathcal{R}_G$,then κ is faithful and $K \leq Z(G)$.
(ii) If $K \leq G'$ and κ is faithful, then $(K,\kappa) \in \mathcal{R}_G$.

Proof. For the first part, assume $(K,\kappa) \in \mathcal{R}_G$. Thus, $(\bigtriangledown_K(G),\phi_\kappa) \in \mathcal{M}_{G\times G}$ and we have a decomposition into standard bisets as

$$\left[\frac{G\times G}{\bigtriangledown_K(G),\phi_\kappa}\right] \cong \mathrm{Ind}_P^G \otimes_{AP} Inf_{P/\widehat{K}}^P \otimes_{AP/\widehat{K}} X \otimes_{AQ/\widehat{L}} \mathrm{Def}_{Q/\widehat{L}}^Q \otimes_{AQ} Res_Q^G. \tag{2.33}$$

Since $e_{(K,\kappa)} \notin I_G$, we must have $G = Q = P$ and also $\widehat{K} = \{id\}$. Hence, $\mathrm{Ker}(\kappa) = \{id\}$. This shows that κ is faithful. By Goursat's Lemma, we know that $K/\widehat{K}$ is central in $P/\widehat{K}$. In our situation, it is equivalent to the fact that K is central in G, i.e $K \leq Z(G)$.

For the second part, assume κ is faithful and $K \leq G'$. Assume that $e_{(K,\kappa)} \in I_G$, then $e_{(K,\kappa)}$ occurs as a summand of the multiplication of two transitive fibered bisets $\left[\frac{G\times H}{V,\varphi}\right]$ and $\left[\frac{H\times G}{W,\mu}\right]$ with $|H| < |G|$. Hence, there exists $\left[\frac{H\times G}{W',\mu'}\right]$ with $e_{(K,\kappa)} = \left[\frac{G\times H}{V,\varphi}\right] \cdot \left[\frac{H\times G}{W',\mu'}\right]$ and therefore $\bigtriangledown_K(G),\phi_\kappa$ is equal to $(V*W',\varphi*\mu')$. We have $p_1(V*W') \leq p_1(V)$

hence we conclude that $p_1(V) = G$ and $l_0(V * W', \varphi * \mu') \leq l_0(V, \varphi)$ so that $(K', \kappa') \leq (K, \kappa)$ where $(K', \kappa') = l_0(V, \varphi)$. Let's consider the product $e_{(K,\kappa)} \cdot \left[\frac{G\times H}{V,\varphi}\right] = \left[\frac{G\times H}{V',\varphi'}\right]$ where the pair (V', φ') satisfying $l_0(V', \varphi') = (G, K, \kappa)$ and using the Goursat's Lemma, we conclude that $|G| \leq |H|$, a contradiction. □

We give an equivalent condition of being reduced which seems to be impractical for the computational purposes because it requires to check so many cases for a pair to be reduced. Despite its complication on calculations, it does imply that the reduced pairs are independent of the choice of k.

Theorem 2.12. *For $(K, \kappa) \in \mathcal{M}_G^G$, the following are equivalent.*

(i) $(K, \kappa) \notin \mathcal{R}_G$.
(ii) There exists a finite group H with $|H| < |G|$ and $(L, \lambda) \in \mathcal{M}_H^H$ such that $(G, K, \kappa) \sim (H, L, \lambda)$.

2.2.9. Classification of Simple A-Fibered Biset Functors

Section 9 in [5] gives a full classification of simple A-fibered biset functors. We need few results which are important in the remaining of the thesis.

Theorem 2.13. $\omega : ((K, \kappa), [V]) \to \tilde{V} := \overline{E_G}.\overline{f}_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V$ *induces a bijection*

$$\overline{w} : \overline{\mathcal{S}} \to \mathrm{Irr}(\mathcal{F}), [G, K, \kappa, [V]] \to [S_{(G,K,\kappa,V)}]. \tag{2.34}$$

This theorem gives us the full classification of simple A-fibered biset functors. In the remaining of the thesis, we are mainly interested with the $\mathbb{C}^*$-fibered biset functor $S_{C_2,1,1,[1]}$. Since $(1, 1) \in \mathcal{R}_G$ as $1 \leq G'$ and 1 is a faithful map with [1] being irreducible trivial $\mathbb{C}\Gamma_{(G,K,\kappa)}$)-module, then such simple A-fibered biset functor exists.

Since $\mathbb{C}^*$ is a divisible group, the Hypothesis 10.1 on [1] holds for the rest of the thesis. Hence, the only reduced pair of $\mathcal{M}_{C_2}$ is $(1, 1)$. Using theorems, $S_{C_2,1,1,[1]}(G)$ is equal to simple head $\mathcal{L}_{C_2,\overline{V}}(G)/J_{(C_2,\overline{V})}(G)$. Note that $\overline{V}$ is equal to $\overline{f}_{1,1}\overline{E_{C_2}} \otimes_{\mathbb{C}\mathit{\Gamma}_{((G,1,1))}} \overline{1}$ where $\overline{1}$ denotes the irreducible trivial $\mathbb{C}\mathit{\Gamma}_{(G,1,1)}$-module. Note that in this case, we have that $\overline{E_{C_2}} \cong \mathbb{C}\mathit{\Gamma}_{(G,1,1)}$ and $\overline{f}_{(1,1)} \cong \overline{e}_{(1,1)}$ which is the identity element in $\overline{E_{C_2}}$. Hence, the irreducible trivial $\mathbb{C}\mathit{\Gamma}_{(G,1,1)}$-module $\overline{1}$ is the same as the irreducible trivial $\overline{E_{C_2}}$-module. Thus, $\overline{V} \cong \overline{E_{C_2}} \otimes_{\overline{E_{C_2}}} \overline{1} \cong \overline{1}$. Hence, we investigate the simple $\mathbb{C}^*$-fibered biset functor $S_{C_2,\bar{1}}$.

3. $\mathbb{C}^*$-FIBERED 2-BISET FUNCTOR $\mathbb{S}_2$

In the last chapter, we investigated the simple A-fibered biset functors. We note that we don't use anything specific regarding the objects in the category $\mathcal{C}_k^A$. For that reason, if we restrict the objects of this category to p-groups, all results that we have stated will hold for $\mathcal{C}_p$. Here $\mathcal{C}_p$ is the full subcategory of the $\mathcal{C}_k^A$ whose objects are finite p-groups.

A $\mathbb{C}^*$-fibered p-biset functor is a $\mathbb{C}$-linear functor from $\mathcal{C}_p$ to the category of $\mathbb{C}$-vector spaces. In this case, the morphisms in $\mathcal{C}_p$ are $\mathbb{C}$-linear extension of the $\mathbb{C}^*$-fibered bisets of the p-groups. Also, the classification theorem for simple A-fibered biset functors holds for simple A-fibered p-biset functors because the proofs and the statements are independent of the choice of the groups. We choose A to be $\mathbb{C}^*$ and note that Hypothesis 10.1 in [5] holds for this choice. The aim of this thesis is to decompose the simple $\mathbb{C}^*$-fibered 2-biset functor $S_{(C_2,1,1,[1])}$ as a 2-biset functor. We denote it by $\mathbb{S}_2$.

In this section, we give some preliminary results regarding this 2-biset functor. We start with the following from [11]. From now on, $S_{(H,V)}$ denotes a simple biset functor parametrized by the pair (H, V) where H is a finite group and V is an irreducible $\mathbb{C}\,\mathrm{Out}(H)$-module.

Lemma 3.1. *Let H be a group of even order such that $p \mid |H|$ for some odd prime p. Also let G be a finite 2-group. Then we have*

$$S_{(H,V)}(G) = \{0\}. \tag{3.1}$$

Proof. See [11] for the notation we use in proof. By Theorem 4.3 in [11], one can easily deduce that $\overline{B}(G, H) = \{0\}$ because there is no subquotient of G which is isomorphic to H. Otherwise, we must have that $|H| \mid |G|$ which implies $p \mid |G|$, a contradiction. By Proposition 4.4(b) in [11], we immediately deduce that $S_{(H,V)}(G) = \{0\}$. □

Notice that it is quite hard to describe simple fibered biset functors in general. Therefore it is difficult to view it as a biset functor. The following result is quite useful in a way that gives us a sense of how we should approach the composition problem.

Lemma 3.2. *Let S (resp. $\overline{S}$) be a simple biset functor(resp. A-fibered biset functor) such that $S(G) \neq \{0\}$ (resp. $\overline{S}(G) \neq \{0\}$) . Then S is generated by $S(G)$ (resp. $\overline{S}$ is generated by $\overline{S}(G)$), that is, $S(X) = kB(X,H)S(G)$ (resp. for $\overline{S}$) for all finite groups X. More precisely, we have that if $0 \neq u \in S(G)$, then $S(X) = B(X,G) \cdot u$.*

Proof. The proof is immediate from the definition of simple functor once we define a subfunctor $S^{'}(X) = B(X,G) \cdot u$ which is a subfunctor of S. □

Therefore, it is reasonable to take a look at the actions of standard bisets on the elements of $\mathbb{S}_2(G)$. Notice that as an A-fibered biset functor, it is simple and for any non-zero element $u \in \mathbb{S}_2(G)$, the action of standard bisets on the element u will produce the set $\mathbb{S}_2(G)$. It is a good place to give the definition of a composition factor.

Definition 3.1. Let L be a biset functor. We call a simple biset functor $S_{H,V}$ a composition factor of L if there exist subfunctors $F \subseteq F^{'} \subseteq L$ such that $F^{'}/F \cong S_{H,V}$. If $S_{H,V}$ is a composition factor, we write this relation by $S_{H,V} \mid L$.

The classification of simple biset functors (resp. simple fibered biset functors) are done via finding the minimal groups G. By using the same reasoning, we want to find the composition factors by determining the minimal group G that they appear. On the other hand , we don't know at which finite group G a composition factor $S_{H,V}$ appears, i.e for which minimal group G, the module $S_{H,V}(G)$ is a summand of $\mathbb{S}_2(G)$. Luckily, the minimal group approach is useful because if such a composition factor exists, then it must occur first at itself.

Theorem 3.1. *A simple biset functor $S_{(H,V)}$ is a composition factor of $\mathbb{S}_2$ if and only if $S_{(H,V)}(H)$ is a direct summand of $\mathbb{S}_2(H)$ as E_H-module.*

Proof. The proof makes use of a theorem from an unpublished article, [12] so we skip the proof. □

Note that the remaining of the thesis relies on this result because we will look for the composition factors $S_{(H,V)}$ in $\mathbb{S}_2(H)$. Using the above theorem, all we need is to determine the summands of $\mathbb{S}_2(H)$ as a E_H-module for each finite 2-group H. However, we don't know how to decompose it as a sum of simple biset functors as using the transitive $\mathbb{C}^*$-fibered (H,C_2)-bisets, we have no information about Out(H)-structure of elements $B(H, C_2) \cdot u$ for any $0 \neq u \in \mathbb{S}_2$.

We need to find basis elements of $\mathbb{S}_2(H)$ which has Out(H)-module structures. For this, we need to introduce some notation which are central in the remaining of the thesis. We introduce the primitive idempotents of the monomial Burnside ring.

3.1. Primitive Idempotents of Monomial Burnside Rings.

In this section, we follow the notations in [13] but mainly rely on the results in [9]. Let C be an abelian group and G be a finite group. A C-fibered G-set X is a C-free $C \times G$-set with finitely many C-orbits. Any C-fibered (G, H)-biset X can be seen as a C-fibered $G \times H^{op}$-set. Let $[CX]$ denote the isomorphism class of C-fibered G-set CX. For two C-fibered G-set CX and CY, we define their tensor product to be the C-orbits of $CX \times CY$ under the C-action given by $c(x, y) = (cx, c^{-1}y)$. Note that with respect to the C-orbits of $CX \times CY$, C-action is given by $c \cdot (x \times_C y) = (cx \times_C y)$ and G-action is given by the diagonal action. We denote the tensor product by $CX \otimes CY$ and any element with $x \otimes y$.

We consider the disjoint union of two C-fibered G-sets which has an obvious C-fibered G-set structure and denote it by $CX \sqcup CY$. Monomial Burnside ring for G with the fiber group $\mathbb{C}^*$, denoted by $B^{\mathbb{C}^*}(G)$, is defined to be the ring, generated by the isomorphism classes of the finite $\mathbb{C}^*$-fibered G-sets with the multiplication and addition given by $[\mathbb{C}^*X][\mathbb{C}^*Y] = [\mathbb{C}^*X \otimes \mathbb{C}^*Y]$ and $[\mathbb{C}^*X] + [\mathbb{C}^*Y] = [\mathbb{C}^*X \sqcup \mathbb{C}^*Y]$.

We call a $\mathbb{C}^*$-fibered G-set $\mathbb{C}^*$X transitive if and only if the action of G on the $\mathbb{C}^*$-orbits of $\mathbb{C}^*$X is transitive. For a transitive $\mathbb{C}^*$-fibered G-set $\mathbb{C}^*\ X$, let $[x]$ denote any $\mathbb{C}^*$-orbit and consider $U = Stab_G([x])$. The C-fiber structure gives rise to a group homomorphism $\mu : U \to \mathbb{C}^*$ given by $\mu(u) = c$ if and only if $ux = cx$. Conversely, any pair (U, μ) gives rise to a transitive $\mathbb{C}^*$-fibered G-set $\Big[U, \mu\Big]_G$ given by the coset representatives of the subgroup $\{(\mu(u^{-1}), u) : u \in U\}$ in $\mathbb{C}^* \times G$.

Such pair (U, μ) is called a $\mathbb{C}^*$-subcharacter of G and $\mathbb{C}^*$-subcharacters of G admit a G-action given by the conjugation $g \cdot (U, \mu) = ({}^gU, {}^g\mu)$. It is easy to observe that there is a bijective correspondence between the G-conjugacy classes of $\mathbb{C}^*$-subcharacters of G and the isomorphism classes of transitive $\mathbb{C}^*$-fibered G-sets. We denote the set $\mathbb{C}^*$-subcharacters of G by $\mathrm{ch}(\mathbb{C}^*, G) = \{(U, \mu) : U \leq G, \mu \in Hom(U, C)\}$.

For subgroups $V, W \leq G$ and the linear characters $\nu : V \to \mathbb{C}^*, \omega : W \to \mathbb{C}^*$, we define a $\mathbb{C}^*$-linear character of $V \cap W$ via $\nu \cdot \omega : V \cap W \to \mathbb{C}^*$ given by $\nu \cdot \omega(u) = \nu(u) \cdot \omega(u)$. With respect to the above notation, we have

$$\Big[V, \nu\Big]_G \cdot \Big[W, \omega\Big]_G = \sum_{VgW \subseteq G} \Big[V \cap {}^gW, \mu \cdot {}^g\omega\Big]_G \tag{3.2}$$

where the summation runs over the set of all double coset representatives of V and W in G.

As a conseqeunce, $B^{\mathbb{C}^*}(G) = \bigoplus_{(V,\nu) \in \mathrm{ch}(\mathbb{C}^*,G)_G} \mathbb{Z}\Big[V, \nu\Big]_G$ where $\mathrm{ch}(\mathbb{C}^*, G)_G$ denotes the G-conjugacy classes of $\mathrm{ch}(\mathbb{C}^*, G)$. We may extend the scalars to any field $\mathbb{K}$, i.e $\mathbb{K}B^{\mathbb{C}^*}(G) = \mathbb{K} \otimes_{\mathbb{Z}} B^{\mathbb{C}^*}(G)$. As $\mathbb{K}$-vector spaces, we have

$$\mathbb{K}B^{\mathbb{C}^*}(G) = \bigoplus_{(V,\nu) \in ch(\mathbb{C}^*,G)_G} \mathbb{K}\Big[V, \nu\Big]_G. \tag{3.3}$$

Let $O(G)$ denote the intersection of kernels of all homomorphisms $\mu : G \to \mathbb{C}^*$. $G/O(G)$ is the largest abelian quotient whose exponent divides the order of $\mathbb{C}^*$, see [9]. Since $\mathbb{C}^*$ has infinite order, then $O(G) = G'$.

We define a set $el(\mathbb{C}^*, G) = \{(H, h) : H \leq G, hH' \in H/H'\}$ whose elements are called $\mathbb{C}^*$-subelements of G. Similar to $\mathbb{C}^*$-subcharacters of G, G acts on $el(\mathbb{C}^*, G)$

via conjugation. Note that for any $H \leq G$, we have $|H/H^{'}| = |Hom(H, \mathbb{C}^{*})|$, see [9]. This observation immediately leads us to conclude that $|el(\mathbb{C}^{*}, G)| = |ch(\mathbb{C}^{*}, G)|$. By Lemma 3.3 in [9], we immediately get that $|el(\mathbb{C}^{*}, G)/G| = |ch(\mathbb{C}^{*}, G)/G|$.

We already note that isomorphism classes of transitive $\mathbb{C}^{*}$-fibered G-sets are parametrized by G-conjugacy classes of the set $ch(\mathbb{C}^{*}, G)$. Later, we see that the G-conjugacy classes of the set $cl(\mathbb{C}^{*}, G)$ is a parametrization for the primitive idempotents in $\mathbb{C}B^{\mathbb{C}^{*}}(G)$. Now we introduce notations which are required for the construction of primitive idempotents in monomial Burnside ring. The following construction can be found in [9].

Let S be a proposition. The Kronecker value of a S is the rational integer

$$\lfloor S \rfloor = \begin{cases} 1, \text{if S holds,} \\ 0, \text{if S fails.} \end{cases} \tag{3.4}$$

Consider a finite poset P. The incidence function of P is defined to be the function $\zeta : P \times P \to \mathbb{Z}$ such that $\zeta(x, y) = \lfloor x \leq y \rfloor$ for all $x, y \in P$. For an integer $n \geq -2$, we define a function $c_n : P \times P \to \mathbb{Z}$ such that $c_{-2}(x, y) = \lfloor y = x \rfloor$, $c_{-1}(x, y) = \lfloor y < x \rfloor$ and if $n \geq 0$, then $c_n(y, x)$ is the number of chains in the form $y < z_0 < z_1 < .. < z_n < x$.

The Möbiüs function of P is defined to be the function $\mu : P \times P \to \mathbb{Z}$ such that

$$\mu(y, x) = \sum_{n=-2}^{\infty} (-1)^n c_n(y, x). \tag{3.5}$$

This sum is finite so such a function is possible. Next, we discuss the Principle of Möbius inversion.

Let A be an abelian group and θ, ϕ be functions $P \to A$. We define the totient equation and the inversion equation to be, respectively,

$$\theta(y) = \sum_{x \in P} \phi(x)\zeta(x, y) \tag{3.6}$$

and changing the role of y and x and functions,

$$\phi(y) = \sum_{y \in P} \theta(y)\mu(x, y). \tag{3.7}$$

We obtain a Möbius inversion principle which states that the totient equation holds for all $y \in P$ if and only if the inversion equation holds for all $x \in P$. Assume that P is a G-poset and we can generalize the incidence and the Möbius function to the G-orbits of P by G-invariant functions

$$\zeta_G(x,y) = \sum_{x =_G x'} \zeta(x', y), \mu_G(x,y) = \sum_{x =_G x'} \mu(x', y). \tag{3.8}$$

These functions are mutual inverses and since $\mathbb{C}^*$ contains all n^{th}-roots of unity, we define monomial incidence function and monomial Möbius function respectively as $\zeta : el(\mathbb{C}^*, G) \times ch(\mathbb{C}^*, G) \to \mathbb{C}$ where $\zeta(H, h; V, \nu) = \nu(h)\zeta(V, H)$ and we define $\mu : ch(\mathbb{C}^*, G) \times el(\mathbb{C}^*, G) \to \mathbb{C}$ where $\mu(V, \nu; H, h) = \nu^{-1}(V \cap hO(H))\mu(V, H)/|V|$ where ν^{-1} is the inverse of ν in $\widehat{V}$ and

$$\nu^{-1}(V \cap hO(H)) = \sum_{x \in V \cap hO(H)} \nu^{-1}(x). \tag{3.9}$$

Similarly, we define G-invariant functions $\zeta_G(H, h; V, \nu)$

$$\zeta_G(H, h; V, \nu) = \sum_{(H,h) =_G (H',h')} \zeta(H', h'; V, \nu) \tag{3.10}$$

and we also define G-invariant function $\mu_G(V, \nu; H, h)$ as

$$\mu_G(V, \nu; H, h) = \sum_{(V,\nu) =_G (V',\nu')} \mu(V', \nu'; H, h) \tag{3.11}$$

where $=_G$ denotes a G-conjugate pairs of $el(\mathbb{C}^*, G)$ and $ch(\mathbb{C}^*, G)$. Now we are ready to give a set of primitive idempotents. For the details, see [9].

Theorem 3.2 (Idempotent Formula)**.** *Recall that $\mathbb{C}$ satisfies the being sufficiently large condition. There is a bijective correspondence $e^G_{H,h} \longleftrightarrow [H,h]_G$ between the primitive idempotents of $\mathbb{C}B^{\mathbb{C}^*}(G)$ and the G-conjugacy classes of el($\mathbb{C}^*$,G). We have*

$$|N_G(H,h)| e^G_{H,h} = \sum_{(V,\nu) \in_G ch(\mathbb{C}^*), G} |V| \mu_G(V, \nu; H, h) \Big[V, \nu\Big]_G. \tag{3.12}$$

3.2. Kernel of Monomial Linearization Map

In this section, we introduce the monomial linearization map. We prove that $\mathbb{S}_2(G)$ is equal to kernel of monomial linearization map at finite 2-group G.

Furthermore, we prove that kernel of monomial linearization map at any finite 2-group G is generated by the primitive idempotents $e^G_{H,h}$ where $H \neq \langle h \rangle$ which is defined in the last subsection. Let us define the monomial linearization map.

Definition 3.2. The map $\text{lin}_G : \mathbb{C}B^{\mathbb{C}^*}(G) \to \mathbb{C}\mathcal{R}_{\mathbb{C}}(G)$ given by the linear extension of $\Big[V, \nu\Big]_G \to \text{Ind}^G_V \nu$ is called monomial linearization map associated to the group G.

This map is a ring homomorphism. The argument which discusses the proof of this fact can be found in [9]. The next result shows that $\mathbb{S}_2(G)$ can be seen as a subset of $B^{\mathbb{C}^*}(G)$. We use the Remark 4.2 in [13].

Theorem 3.3. *For any* 2- *group* G, $S_{C_2,1}(G)$ *can be embedded to* $B^{\mathbb{C}^*}(G)$.

Proof. Using Remark 4.2 in [13], there exists a short exact sequence

$$0 \to S_{C_2,1} \to B^{\mathbb{C}^*} \to S_{1,1} \to 0. \tag{3.13}$$

Inserting G to the exact sequence, we obtain a short exact sequence of $\mathbb{C}$-modules

$$0 \to S_{C_2,1}(G) \to B^{\mathbb{C}^*}(G) \to S_{1,1}(G) \to 0. \tag{3.14}$$

This implies that $S_{C_2,1}(G) \hookrightarrow B^{\mathbb{C}^*}(G)$. □

Our next result shows that $S_{C_2,1}$ is equal to Ker lin for any finite group G. Note that it is not true for any finite group G.

Theorem 3.4. $\mathbb{S}_2(G) = \text{Ker}\,\text{lin}(G)$ *when we work over p-groups.*

Proof. It follows from the short exact sequence given in Remark 4.2 in [13]. When we work over p-groups, $S_{1,1}$ becomes $\mathbb{C}$ character ring $\mathcal{R}_{\mathbb{C}}(G)$. □

Now, we prove that the kernel of linearization map for any group G is generated by the primitive idempotents $e^G_{H,h}$ where $H \neq \langle h \rangle$. For this purpose, we recall some results from [1]. We introduce some notation.

Definition 3.3. Let $\mathbb{K}$ be a field. $\mathbb{K}$-algebra maps $\mathbb{K}B^{\mathbb{C}^*}(G) \to \mathbb{K}$ are called species.

These maps can be used to determine the primitive idempotents of $\mathbb{K}B^{\mathbb{C}^*}(G)$. Using a similar approach, we will determine the primitive idempotents of $\mathbb{K}\mathcal{R}_{\mathbb{C}}(G)$ using such algebra maps. We construct algebra maps which are in one-to-one correspondence with the primitive idempotents of $\mathbb{C}\mathcal{R}_{\mathbb{C}}(G)$. Consider $S_g^G : \mathbb{C}\mathcal{R}_{\mathbb{C}}(G) \to \mathbb{C}$ given by $\chi \to \chi(g)$. By construction, it is a $\mathbb{C}$-algebra map. Let's determine the primitive idempotents of $\mathbb{C}\mathcal{R}_{\mathbb{C}}(G)$. Consider $e_g^G = \frac{1}{|G|} \sum_{\chi \in \text{ Irr}(G)} \chi(g^{-1})\chi$. Using the inner product property, this element is mapped to 1. It is indeed an element which can be a primitive idempotent, see [2] for the details of idempotents formulas for more general representation rings. Moreover, it is an idempotent of $\mathbb{C}\mathcal{R}_{\mathbb{C}^*}(G)$, see [2]. Next, we prove the following:

Lemma 3.3. $S_g^G(e_h^G) = \begin{cases} 0, & \textit{if } g \textit{ and } h \textit{ are not conjugate}, \\ |C_G(g)|, & \textit{otherwise}. \end{cases}$

Proof. By direct computation, we observe that this is equal to

$$\frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(g^{-1})\chi(h) \tag{3.15}$$

and we conclude the result by the second orthogonality relation. □

This result implies that idempotents of $\mathbb{C}\mathcal{R}_{\mathbb{C}}(G)$ are of the form e_g^G up to G-conjugacy classes of g. We have determined the primitive idempotents on $\mathbb{C}B^{\mathbb{C}^*}(G)$, yet we haven't described the species in detail. We follow the notation of Barker in [9].

Let $\mathbb{C}^*X$ be a fibered $\mathbb{C}^*$-set and (H, h) be a $\mathbb{C}^*$-subelement of G. Given any fiber $\mathbb{C}^*x$ stabilized by H, we have a group homomorphism $\phi_x : H \to \mathbb{C}^*$ defined by $\phi_x(h) = c$ if and only if $h \cdot x = cx$. Corresponding to $\mathbb{C}^*$-subelement (H, h), we define a species to be the linear map

$$S_{H,h}^G[CX] = \sum_{Cx} \phi_x(h) \tag{3.16}$$

where the sum runs over all fibers in CX which are stabilized by H.

The following theorem is due to Dress, see [1] but we state a special case which is due to Barker, see [9].

Lemma 3.4. *Given $\mathbb{C}^*$-subelements (H,h) and (I,i) of G, then $S^G_{H,h} = S^G_{I,i}$ if and only if $(H,h) =_G (I,i)$. Every species of $\mathbb{C}B^{\mathbb{C}^*}(G)$ is of the form $S^G_{H,h}$ and the species span the dual space of $\mathbb{C}B^{\mathbb{C}^*}(G)$.*

This result also implies that idempotents of $\mathbb{C}B^{\mathbb{C}^*}(G)$ are of the form $e^G_{H,h}$. Our aim is to compute the kernel of the linearization map. For this purpose, consider

$$\begin{array}{ccc} \mathbb{C}B^{\mathbb{C}^*}(G) & \xrightarrow{\ \mathrm{lin}_G\ } & \mathbb{C}\mathcal{R}_{\mathbb{C}}(G) \\ & \searrow^{S^G_{\langle g\rangle,g}} & \downarrow S^G_g \\ & & \mathbb{C} \end{array}$$

and we claim that this map commutes.

Lemma 3.5. *The diagram*

$$\begin{array}{ccc} \mathbb{C}B^{\mathbb{C}^*}(G) & \xrightarrow{\ \mathrm{lin}_G\ } & \mathbb{C}\mathcal{R}^{\mathbb{C}}(G) \\ & \searrow^{S^G_{\langle g\rangle,g}} & \downarrow S^G_g \\ & & \mathbb{C} \end{array}$$

commutes.

Proof. We evaluate lin_G and $S^G_k \circ \mathrm{lin}_G$ at each $\mathbb{C}$-basis element $\Big[H,\nu\Big]_G$.

$S^G_g \circ \mathrm{lin}_G(\Big[H,\nu\Big]_G) = \mathrm{Ind}^G_H\,\nu(g) = \frac{1}{|H|}\sum_{t\in G,t^{-1}gt\in H} \nu(t^{-1}gt)$. Now we compute $S^G_{\langle g\rangle,g}(\Big[H,\nu\Big]_G)$. For this, we make some observations.

Recall that any $\mathbb{C}^*$-fiber of $\Big[H,\nu\Big]_G$ is parametrized by $[1,k]H_\nu$ where k runs over all coset representatives of H in G. Moreover a $\mathbb{C}^*$-fiber $[1,k]H_\nu$ is stabilized by $\langle g\rangle$ if and only if $\mathbb{C}^*$-fiber $[1,k]H_\nu$ is stabilized by g if and only if $\langle g\rangle \le {}^kH$.

Among such coset representatives k, the corresponding group homomorphism is a restriction of $\phi_{{}^kH}$ so $g\cdot[1,k] = {}^k\nu(g) = {}^k\nu(k^{-1}gk)$. Therefore, $S^G_{\langle g\rangle,g}(\Big[H,\nu\Big]_G)$ is equal to $\sum_{k\in G\backslash H, k^{-1}gk\in H} \nu(k^{-1}gk)$. $S^G_g \circ \mathrm{lin}_G(\Big[H,\nu\Big]_G)$ is equal to $S^G_{\langle g\rangle,g}(\Big[H,\nu\Big]_G)$ because at each coset representative, there are $|H|$-many elements so we immediately get the result. □

Theorem 3.5. *For any 2-group G, we have*

$$\mathrm{Ker}\,\mathrm{lin}_G = \bigoplus_{(H,h)\in el(\mathbb{C}^*,G),\langle h\rangle\neq H} \mathbb{C}e^G_{H,h} \tag{3.17}$$

Proof. It is enough to compute the lin_G for idempotents $e^G_{H,h}$ since

$$\mathbb{C}B^{\mathbb{C}^*}(G) = \bigoplus_{H,h\in\ G/el(\mathbb{C}^*,G)} \mathbb{C}e^G_{H,h}. \tag{3.18}$$

Note that a ring homomorphism sends idempotents to idempotents. Given $g\in G$ and the idempotent $e^G_{\langle g\rangle,g}$, we have $S^G_{\langle g\rangle,g}(e^G_{\langle g\rangle,g}) = 1$ so by commuting diagram, we have $\mathrm{lin}_G\, e^G_{\langle g\rangle,g} \neq 0$.

Assume $\mathrm{lin}_G(e^G_{K,k}) = e^G_h$ where k and h are not G-conjugate and $K\neq\langle k\rangle$, then we have that $S^G_{\langle h\rangle,h}(e_{K,k}) = 0$ but $S^G_h \circ \mathrm{lin}_G(e^G_{K,k}) = 1$, a contradiction. Assume $\mathrm{lin}_G(e^G_{K,k}) = e^G_k$, then $S^G_{\langle k\rangle,k}(e_{K,k}) = 0$ but $S^G_k \circ \mathrm{lin}_G(e^G_{K,k}) = 1$, a contradiction. Therefore we must have $\mathrm{lin}_G(e^G_{K,k}) = 0$ in case $K\neq\langle k\rangle$. □

3.3. Actions of Standard Bisets on Primitive Idempotents

The actions of standard bisets $\mathrm{Res}, \mathrm{Ind}, \mathrm{Def}, \mathrm{Inf}$ are already computed in [10] and [13]. In the remaining of the thesis, we will be mainly dealing with the deflation formula because the $\mathbb{C}\,\mathrm{Out}(H)$-modules appears in the primitive idempotents $e_{K,k}$ has already been in $\mathbb{S}_2(K)$ and such idempotents will naturally provide a $\mathbb{C}\,\mathrm{Out}(K)$-module structure once they have already had a $\mathbb{C}\,\mathrm{Out}(H)$-module structure. The proof of the next results can be found in [10] and in [13].

Proposition 3.1 (**Restriction formula for primitive idempotents**). *Given $F \leq G$ and a C-subelement (H,h) of G, then*

$$\mathrm{Res}^G_F(e^G_{H,h}) = \sum_{(J,j)} e^F_{J,j} \tag{3.19}$$

where (J,j) runs over representatives of the F-classes of C-subelements of F such that (J,j) is G-conjugate to (H,h).

As a result of this proposition, for any subgroup $H < G$, we have $\mathrm{Res}^G_H\, e_{G,g} = 0$. Hence, we will neglect the computation of $\mathrm{Res}^G_H\, e_{G,g}$.

Proposition 3.2 (**Induction formula for primitive idempotents**). *Given $F \leq G$ and a C-subelement (J,j) of F,then*

$$\mathrm{Ind}^G_F\, e^F_{J,j} = |N_G(J,j) : N_F(J,j)| e^G_{J,j}. \tag{3.20}$$

Proposition 3.3 (Isogation and Inflation formulas). *(i) Let $\phi : G \to G^{'}$ be a group isomorphism. We define ther isogation formula for primitive idempotent as*

$$\mathrm{iso}_{G',G}\, e^G_{F,g} = e^{G^{'}}_{F',g'} \tag{3.21}$$

where $\phi(G) = G^{'}$ and $\phi(g) = g^{'}$.

(ii) Let $N \trianglelefteq G$ and $(\overline{F},\overline{f}) \in el(C,G/N)$,then

$$\mathrm{Inf}^G_{G/N}\, e^{G/N}_{\overline{F},\overline{f}} = \sum_{(J,j)\in_G el(C,G):(\overline{JN},\overline{j}=_{G/N}((\overline{F},\overline{f})} e^G_{J,j}. \tag{3.22}$$

We give the deflation formula of a primitive idempotent which is central in our work. For this, we need to introduce deflation number $\mathfrak{m}_{G,N,g}$.

Definition 3.4. A number γ is called a deflation number if there exists a finite group G and a normal subgroup N of G and $g \in G$ such that $\mathrm{Def}^G_{G/N}\, e^G_{G,g} = \gamma \cdot e^{G/N}_{G/N,gN}$. In this case, we denote γ by $\mathfrak{m}_{G,N,g}$.

Theorem 3.6 (**Deflation formula for the primitive idempotents**). *Let* $N \trianglelefteq G$ *then, we define the deflation formula for primitive idempotent*

$$\mathrm{Def}^G_{G/N}\, e^G_{G,g} = \mathfrak{m}_{G,N,g} e^{G/N}_{G/N,gN} \tag{3.23}$$

and the number $\mathfrak{m}_{G,N,g}$ *is equal to*

$$\mathfrak{m}_{G,N,g} = \frac{1}{|N \cdot G'|} \sum_{V \leq G : VN = G} |V \cap gG'| \mu(V, G). \tag{3.24}$$

For specific cases, deflation numbers can be calculated in a more efficient way. We give such results when it is necessary. We also omit the proofs of the special cases.

3.4. Method of Finding Composition Factors of $\mathbb{S}_p$

In this section, we develop a method which is our main tool to determine the composition factors. Firstly, recall that $\mathbb{S}_p$ is generated by $e^H_{H,h}$ where H is a finite p-group and $\langle h \rangle \neq H$. In order to find the composition factors of the form $S_{(G,V)}$ we consider the elements $\alpha \in \mathbb{S}_p(G)$ such that $\mathrm{Res}^G_H\, \alpha = \mathrm{Def}^G_{G/N}\, \alpha = 0$ for any $H < G$ and $N \triangleleft G$. The reason for such choice is that if it has a non-zero restriction or deflation, it already appears on $\mathbb{S}_2(H)$ where H is a group whose order is smaller than G which will already be in $\mathbb{S}_2(G)$ with a composition factor of the form $S_{H,V}$ if it has such composition factor.

Once, for all subgroups $H < G$ and normal subgroups $N \triangleleft G$ we determine α which belongs to the kernel of all restrictions and deflations, then among such α, we will try to construct $\mathbb{C}\,\mathrm{Out}(G)$-module structures using this elements. More precisely, we already know that $\mathrm{Res}^G_H\, e_{G,g} = 0$ and if we define a $\mathbb{C}$-vector space $K = \mathrm{span}_{\mathbb{C}} \bigcap_{N \triangleleft G} KerDef^G_{G/N}$, then we will try to construct a $\mathbb{C}\,\mathrm{Out}(G)$-module structure on K. The irreducible $\mathbb{C}\,\mathrm{Out}(G)$-modules that appears in K will corresponds to the simple biset functors $S_{H,V}(G)$.

4. COMPOSITION FACTORS OF $\mathbb{S}_2$

Note that A-fibered 2-biset functor $\mathbb{S}_2$ can be viewed as a 2-biset functor and a simple A-fibered 2-biset functor may not remain as a simple 2-biset functor. In this chapter, we find the composition factors of $\mathbb{S}_2$.

Firstly, we show that there exists no composition factor of $\mathbb{S}_2$ of the form $S_{H,V}$ where H is a non-abelian group. Before proving this, we must determine the 2-groups G for which $\mathbb{S}_2(G)$ is non-zero. Interestingly, for any 2-group G , we have $\mathbb{S}_2(G) \neq \{0\}$. Then, the next result guarantees the non-existence of a composition factor of the form $S_{G,V}$ of $\mathbb{S}_2$ where G is a non-abelian 2-group. Hence, we will restrict our attention to abelian groups.

Theorem 4.1. *Let H be a non-trivial finite 2-group. Then we have*

$$\mathbb{S}_2(H) \neq \{0\} \tag{4.1}$$

Proof. Since $\mathbb{S}_2(H) = \operatorname{Ker} \operatorname{lin}_H$ and for any non-trivial group H, we have $e^H_{H,1} \in \operatorname{Ker} \operatorname{lin}_H$. Therefore, we conclude that $\mathbb{S}_2(H) \neq \{0\}$. □

Theorem 4.2. *Let G be a non-abelian 2-group. Then $\mathbb{S}_2$ contains no composition factor of the form $S_{G,V}$ where V is any $\mathbb{C}\operatorname{Out}(G)$-module.*

Proof. If there exists a composition factor of the form $S_{G,V}$, then such composition factor is generated by the primitive idempotents $e_{G,g}$ where G, g denotes a G-conjugacy class of $\mathbb{C}^*$-subelement of G. We show that $\operatorname{Def}^G_{G/G'} e_{G,g} \neq \operatorname{Def}^G_{G/G'} e_{G,g'}$ for any distinct $\{G, g\}, \{G, g^{'}\} \in\ G/el(\mathbb{C}^*, G)$.

For p-groups, $G^{'}$ is always contained in $\Phi(G)$ and this implies that $\mathfrak{m}_{G,G',g} = \frac{|gG^{'}|}{|G^{'}.G^{'}|} = \frac{|G^{'}|}{|G^{'}|} = 1$. Hence, $\operatorname{Def}^G_{G/G'} e_{G,g} = \operatorname{Def}^G_{G/G'} e_{G,g'}$ if and only if $e_{G/G',gG'} = e_{G/G',g'G'}$ if and only if ${}^k g \cdot (g^{'})^{-1} \in G^{'}$.

This shows us that $\{G,g\} =_G \{G,g^{'}\}$. We can write any $\alpha \in KerDef^G_{G/G'}$ in the form $\alpha = \sum_{\{G,g\}} \alpha_g e_{G,g}$, so the above fact implies that $\alpha_g = 0$. We conclude that $KerDef^G_{G/G'} = \{0\}$. There is no composition factor of the form $S_{G,V}$. □

We reduce our problem to composition factors $S_{G,V}$ where G is abelian. It is a good time to determine the G-conjugacy classes of $el(\mathbb{C}^*, G)$ when G is abelian. Note that $G^{'} = 1$ for any abelian group G and two pairs (H,h) and $(H^{'},h^{'})$ belong to same G-conjugacy class if and only if there exists $g \in G$ such that ${}^g(H,h) = (H^{'},h^{'})$ if and only if ${}^gH = H^{'}$ and ${}^gh = h^{'}$ if and only $H = H^{'}$ and $h = h^{'}$. Therefore, each pair (H,h) where $H \leq G$ and $h \in H$ gives rise to a distinct idempotents $e^G_{H,h}$ of $\mathbb{C}B^{\mathbb{C}^*}(G)$. We start with the elementary abelian case.

Proposition 4.1. *The evaluation of* $\mathbb{S}_2$ *at* $(C_2)^n$ *where* $(C_2)^n = C_2 \times C_2 \times ... \times C_2$ *is equal to* $S_{C_2,1}(C_2^n) \oplus S_{C_2 \times C_2,1}(C_2^n)$ *as* $E_{(C_2)^n}$*-module.*

Proof. To ease the notation, write $G = (C_2)^n$. Then $e^G_{C_2,1} \in \mathbb{S}_2(G)$. The set $S = \mathrm{span}_{\mathbb{C}}\{e^G_{C_2,1}\}$ is isomorphic to $S_{C_2,1}(G)$ as an $\mathbb{C}\,\mathrm{Out}(G)$-module. Therefore, $\mathbb{S}_2$ has a composition factor of the form $S_{C_2,1}$.

Now, we show the following result $\mathbb{S}_2(C_2 \times C_2) = S_{C_2,1}(C_2 \times C_2) \oplus S_{C_2 \times C_2,1}(C_2 \times C_2)$ when we view it as a $E_{C_2 \times C_2}$-module. Since $\mathbb{S}_2$ has a composition factor of the form $S_{C_2,1}$, then $S_{C_2,1}(C_2 \times C_2) \subseteq \mathbb{S}_2(C_2 \times C_2)$. Our aim is to find the 2-biset structure of $S_2(C_2 \times C_2)/S_{C_2,1}(C_2 \times C_2)$. We evaluate $\mathbb{C}$-vector space dimensions of $S_2(C_2 \times C_2)$ and $S_{C_2,1}(C_2 \times C_2)$. We have

$$\mathbb{S}_2(C_2 \times C_2) = \mathrm{span}_{\mathbb{C}}\{e_{A,1}, e_{B,1}, e_{C,1}, e_{C_2 \times C_2,1}, e_{C_2 \times C_2,a}, e_{C_2 \times C_2,b}, e_{C_2 \times C_2,ab}\} \quad (4.2)$$

as a $\mathbb{C}$-vector space. Hence $\dim_{\mathbb{C}} \mathbb{S}_2(C_2 \times C_2) = 7$. Using Theorem 1 in [14], we have $\dim_{\mathbb{C}} S_{C_2,1}(C_2 \times C_2) = 6$. We claim that $S_{C_2 \times C_2,1}(C_2 \times C_2)$ has an isomorphic copy in the vector space decomposition of $\mathbb{S}_2(C_2 \times C_2)$. If we prove this, then we prove our claim. If such a summand occurs, the intersection of kernels of restriction maps and deflations maps is non-zero. Moreover, such a summand appears as a simple $\mathbb{C}\,\mathrm{Out}(G)$-submodule of $(\bigcap_{H<G} \operatorname{Ker}\operatorname{Res}^G_H)\bigcap(\bigcap_{N \lhd G} \operatorname{Ker}\operatorname{Def}^G_N)$. We evaluate the kernels of restriction maps. For $e^G_{G,g}$, $\operatorname{Res}^G_H e^G_{G,g} = 0$ for any $H < G$. Therefore, $\operatorname{span}_{\mathbb{C}}\{e^G_{G,g} : \langle g\rangle \neq G\} \subseteq \bigcap_{H<G} \operatorname{Ker}\operatorname{Res}^G_H$. Let us take $e^G_{H,h}$ where $H < G$, then $Res^G_H e^G_{H,h} = e^G_{H,h}$. If $\alpha \in \bigcap_{H<G} \operatorname{Ker}\operatorname{Res}^G_H$ and we write $\alpha = \sum_{(H,h)} \alpha_{(H,h)} e^G_{H,h}$, then $\operatorname{Res}^G_H \alpha \neq 0$ if α contains a non-zero summand of the form $e^G_{H,h}$ where $H < G$. This proves that $\bigcap_{H<G} \operatorname{Ker}\operatorname{Res}^G_H = \operatorname{span}_{\mathbb{C}}\{e^G_{G,g} : \langle g\rangle \neq G\}$.

Next we find the intersection of the kernels of the deflation maps as a subset of $\operatorname{span}_{\mathbb{C}}\{e^G_{G,g} : \langle g\rangle \neq G\}$. G has 3 non-trivial proper subgroups which are cyclic group of order 2. Notice that there is no need to check the deflation number of 1 because it will always be equal to $\frac{-1}{2}$. For the other elements other than 1, we can represent the deflation numbers which will be very important for calculations as the table below.

Table 4.1. Deflation table.

$\mathfrak{m}_{G,N,g}$	**a**	**b**	**ab**
G / $\langle a\rangle$	$\frac{1}{2}$	0	0
G / $\langle b\rangle$	0	$\frac{1}{2}$	0
G / $\langle a \cdot b\rangle$	0	0	$\frac{1}{2}$

Let $x = a_1 \cdot e_{G,1} + a_2 \cdot e_{G,a} + a_3 \cdot e_{G,b} + a_4 \cdot e_{G,ab} \in \bigcap_{1\neq g\in G} \operatorname{Ker}\operatorname{Def}^G_{G/<g>}$, then we have that $a_1 - a_2 = 0$, $a_1 - a_3 = 0$ and $a_1 - a_4 = 0$. Therefore any element in this intersection must be of the form $x = k \cdot e_{G,1} + k \cdot e_{G,a} + k \cdot e_{G,b} + k \cdot e_{G,ab}$. So the dimension of this vector space is 1 and also observe that any outer automorphism of $C_2 \times C_2$ fixes this subspace so the subspace generated by this element must be isomorphic to the trivial $\mathbb{C}\operatorname{Out}(C_2 \times C_2)$-module. Hence, the summand $S_{C_2\times C_2,1}(C_2 \times C_2)$ must occur in the decomposition of $\mathbb{S}_2(C_2 \times C_2)$. The rest of the proof lies on a counting argument using Theorem 1 in [14]. We evaluate $\dim_{\mathbb{C}}(S_{C_2,1})((C_2)^n)$ and $\dim_{\mathbb{C}}(S_{C_2\times C_2,1})((C_2)^n)$.

By using Theorem 1 in [14], the number $\dim_{\mathbb{C}}(S_{C_2,1})((C_2)^n)$ is equal to the number of sections T/S in $(C_2)^n$ where T is 2-elementary and $T/S \cong C_2$. Since any subgroup of $(C_2)^n$ is of the form $C_2 \times ... \times C_2$, there is no harm to remove the 2-elementary condition. We only count the sections T/S where $|T : S| = 2$. Moreover, any subgroup of $(C_2)^n$ is a vector space over $\mathbb{F}_2$ and to count the number of sections $|T : S|$, we may count the number of i-dimensional subspaces of $(C_2)^n$(which give us the number of subgroups of $(C_2)^n$ of order 2^i) and number of $(i-1)$-dimensional subspaces of i-dimensional vector space over $\mathbb{F}_2$. Observe that the number of i-dimensional subspaces of $(C_2)^n$ is equal to the number of i-dimensional subspaces of n dimensional vector space over $\mathbb{F}_2$.

For the sake of easing the notation, we denote the number of i dimensional subspaces of a n-dimensional vector space by $C(i,n)$. Then notice that the number of sections T/S such that $|T : S| = 2$ and $|T| = 2^i$ is equal to $C(i,n)\cdot C(i-1,i)$ where first we find the i dimensional subspaces and then for each i dimensional subspaces, we look for the sections T/S in which there are $C(i-1,i)$ many choices for each T. Therefore, doing this for each $0 < i \leq n$, we obtain that $\sum_{i=1}^{n} C(i,n)C(i-1,i) = \dim_{\mathbb{C}}(S_{C_2,1})((C_2)^n)$.

Next, we find the $\dim_{\mathbb{C}}(S_{C_2\times C_2,1})((C_2)^n)$. Note that this number is equal to the number of conjugacy classes of non-cyclic subgroups of $(C_2)^n$(See Theorem 1 in [14]). Equivalently, we may count the number of subspaces of $(C_2)^n$ having a dimension greater or equal to 2. As already noted, it is equal to $\sum_{i=2}^{n} C(i,n)$. Hence, $\dim_{\mathbb{C}}(S_{C_2\times C_2,1})((C_2)^n) = \sum_{i=2}^{n} C(i,n)$.

Let us determine $\dim_{\mathbb{C}} \mathbb{S}_2((C_2)^n)$. Note that as a $\mathbb{C}$-vector space, it is generated by the idempotents $e^G_{H,h}$ where $H \leq G$ and $h \in H$ such that $\langle h\rangle \neq H$. When a subgroup is a 2 or larger dimensional $\mathbb{F}_2$-vector space, then it is non-cyclic and each element contributes to the dimension. When, a subspace is 1-dimensional, then it is cyclic of order 2 and there is only 1 element which will contribute to the dimension. Hence, the dimension of $\mathbb{S}_2((C_2)^n)$ is equal to $\sum_{i=2}^{n} 2^i C(i,n) + C(1,n)$. Notice we have following equation

$$\dim_{\mathbb{C}}(S_{C_2,1})((C_2)^n)+\dim_{\mathbb{C}}(S_{C_2\times C_2,1})((C_2)^n) = \sum_{i=1}^{n} C(i,n)C(i-1,i)+\sum_{i=2}^{n} C(i,n) \quad (4.3)$$

.

Moreover, we have the following equalities

$$\sum_{i=1}^{n} C(i,n)C(i-1,i) + \sum_{i=2}^{n} C(i,n) = \sum_{i=1}^{n} C(i,n)(2^i - 1) + \sum_{i=2}^{n} C(i,n) \quad (4.4)$$

$$= \sum_{i=2}^{n} C(i,n) \cdot 2^i + C(1,n)C(0,1) \quad (4.5)$$

$$= \dim_{\mathbb{C}} \mathbb{S}_2((C_2)^n). \quad (4.6)$$

where the equalities follows from the results that we have proved in this proof. In particular, both composition factor have only zero as a common element, this concludes us that $\mathbb{S}_2((C_2)^n) = S_{C_2,1}((C_2)^n) \oplus S_{C_2 \times C_2,1}((C_2)^n)$. □

Also, note that the above proposition tells us that when the group is of elementary abelian of rank greater than 2 (i.e as a vector space of dimension greater than 2), then we don't have any new composition factors. Our next aim is to find all composition factors when we deal with a cyclic 2-group. Luckily, like in the elementary abelian case, we find all the composition factors.

Let $G = C_{2^n}$ be a cyclic group of order 2^n. Note that the value $\dim_{\mathbb{C}} S_{H,V}(G)$ is known when V is the trivial $\mathbb{C}\,\mathrm{Out}(H)$-module (see [6]). In general, it is quite hard to find the number $\dim_{\mathbb{C}} S_{H,V}(G)$ for arbitrary groups H, G and arbitrary irreducible $\mathbb{C}\,\mathrm{Out}(H)$-module V. The next result answers this case when both H and G are cyclic 2-groups and V is any irreducible $\mathbb{C}\,\mathrm{Out}(H)$-module.

Proposition 4.2. *For any irreducible $\mathbb{C}\,\mathrm{Out}(C_{2^n})$-module V, we have that*

$$\dim_{\mathbb{C}} S_{C_{2^n},V}(C_{2^m}) = \dim_{\mathbb{C}} S_{C_{2^n},1}(C_{2^m}) = m - n + 1. \quad (4.7)$$

Proof. We prove the result by using the techniques in [11]. Define $\mathbb{C}$-vector space $\mathbb{C}B(X,H)$ which consists of $\mathbb{C}$-linear extensions of (X,H)-bisets. Define $\mathbb{C}I(X,H) = \sum_{J \sqsubset H} \mathbb{C}B(X,J)B(J,H)$ and consider $\mathbb{C}\bar{B}(X,H) = \mathbb{C}B(X,H)/\mathbb{C}I(X,H)$. For any finite group H, we have $\mathbb{C}B(H,H)/\mathbb{C}I(H,H) \cong \mathbb{C}\,\mathrm{Out}(H)$ and we have a natural surjection map $\gamma : \mathbb{C}B(H,H) \to \mathbb{C}\,\mathrm{Out}(H)$. For any simple $\mathbb{C}\,\mathrm{Out}(H)$-module V, we also have map $\mathit{\Pi} : \mathbb{C}\operatorname{Out}(H) \to End(V)$ where each element is sent to the its action on V. Also, given any $\phi \in End(V)$, considering its matrix representative, it is natural to consider its trace τ. Using these maps, we obtain a bilinear form on $\mathbb{C}\bar{B}(X,H)$ via

$$\langle -,- \rangle_X : \mathbb{C}\bar{B}(X,H) \times \mathbb{C}\bar{B}(X,H) \to \mathbb{C}, \langle \bar{X}, \bar{Y} \rangle = (\tau \circ \mathit{\Pi} \circ \gamma)(X^{op}.Y) \tag{4.8}$$

for any $X, Y \in \mathbb{C}\bar{B}(X,H)$.

It has been proven that, when we take V as a simple $\mathbb{C}\operatorname{Out}(H)$-module, H and G be finite groups, we have $\dim_{\mathbb{C}} S_{H,V}(G) = \frac{rank\langle,\rangle_G}{\dim(V)}$(See [11]). Since we have that $\operatorname{Out}(C_{2^m}) = C_2 \times C_{2^{m-2}}$, so that any simple $\mathbb{C}\operatorname{Out}(C_{2^m})$-module is 1-dimensional. Thus, we only need to find the rank of this bilinear form.

To do that, we start by finding a basis for $\bar{B}(G,H)$. Recall that $\mathbb{C}B(G,H)$ is generated by the transitive (G,H)-bisets U and write $U = \operatorname{Ind}_S^G \operatorname{inf}_T^S \operatorname{iso}_\sigma \operatorname{Def}\operatorname{Res}_{J/K}^H$ where S/T is a section of G and J/K is a section of H such that $\sigma : J/K \to S/T$ is a group isomorphism. If $S < H$, then $\bar{U} = \bar{0}$ because restriction will not be zero and the section S/T will be a subqoutient of H. Hence we must have $S = H$ for a non-zero element $\bar{U}$ in $\mathbb{C}B(G,H)$. Similarly if $S \neq J$, then S/T will be a subqoutient of H, that means again $\bar{U} = \bar{0}$. Hence, these two facts imply that $H \cong S/T$ and Def Res is nothing but an isomorphism between H and S/T. Therefore, any non-zero transitive element U in $\mathbb{C}\bar{B}(G,H)$ must be of the form $\operatorname{Indinf}_{S/T}^G \operatorname{iso}_\sigma$ where $\sigma : S/T \to H$ is a group isomorphism.

In our case, we look for the all sections S/T such that $S/T \cong C_{2^m}$ where the sections are in C_{2^n}. Notice that there are $n-m+1$ many such sections. Let us denote the basis by $\{\operatorname{Indinf}_{S/T}^G \operatorname{iso}_\sigma$: (S,T) is a section of G and $\sigma : H \to S/T\}$. The size of this set is $(m-n+1).|\operatorname{Out}(H)| = 2^{n-1}(m-n+1)$.

Next, we evaluate the bilinear form on basis elements. For this purpose , we need to create an index on basis elements so that the matrix representation of the bilinear form will become something whose rank is quite easy to compute. Put the basis elements $\operatorname{Indinf}_{S/T}^G \operatorname{iso}_\sigma$ earlier if $|S|$ is smaller and among the basis elements

$\text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma}$ having same section, define $\text{Out}(H) = \{\sigma_1, ..\sigma_m\}$ and the basis element $\text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma_i}$ will appear before the basis element $\text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma_j}$ if and only if $i \leq j$.

Let $\alpha = \text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma}$ and $\beta = \text{Indinf}_{J/K}^{G}\,\text{iso}_{\rho}$,then using the result in [11], the bilinear form $\langle \alpha, \beta \rangle_G$ is non-zero if and only if the sections S/T and J/K are linked. (See the definition of linked in [15]) . Thus we must have $(S \cap J)T = S$, $(S \cap J)K = J$ and $S \cap K = T \cap J$. Since S is a cyclic group, then $(S \cap J)T = S$ holds if and only if $S \cap J = S$ or $T = S$. Since $S > T$, then latter cannot hold. Therefore, $S \cap J = S$, then $S \subset J$, hence $|S| \leq |J|$. So we obtain $|S| \leq |J|$. Using a similar argument, $(S \cap J)K = J$ implies $|J| \leq |S|$. Therefore, $|S| = |J|$, since there is unique subgroup of order 2^m, that implies $S = J$. Hence, the bilinear form $\langle \alpha, \beta \rangle_G$ is non-zero if and only if $S = J$ and $T = K$.

To find the matrix representative of this bilinear form, all we need to do is to compute $\langle \text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma}, \text{Indinf}_{S/T}^{G}\,.\,\text{iso}_{\rho} \rangle$. We evaluate the (H, H)-biset

$$\text{iso}_{\sigma^{-1}}\,\text{Def}\,\text{Res}_{S/T}^{G}\,\text{Indinf}_{S/T}^{G}\,\text{iso}_{\rho}\,. \tag{4.9}$$

Using the relations on standard bisets, we notice that this product is equal to the $[G : S]\,\text{iso}_{\sigma^{-1}.\rho}$ and its image with respect to the canonical surjection $B(H, H) \rightarrow \bar{B}(H, H)$ because the elements of $\bar{B}(H, H)$ is generated by the transitive bisets $\left[\frac{H \times H}{\nabla_{\sigma}(H)}\right]$ where $\sigma : H \rightarrow H$ is any automorphism of H. Thus, $\bar{B}(H, H) = \text{span}_{\mathbb{C}}\{\left[\frac{H \times H}{\nabla_{\sigma}(H)}\right] : \sigma \in \text{Out}(H)\}$. Thus the composition map defined by $\mathbb{C}B(H, H) \rightarrow \mathbb{C}\bar{B}(H, H) \rightarrow \mathbb{C}\,\text{Out}(H) \rightarrow End_{\mathbb{C}}(V)$ corresponds an element $\text{iso}_{\sigma^{-1}}\,\text{Def}\,\text{Res}_{S/T}^{G}\,\text{Indinf}_{S/T}^{G}\,\text{iso}_{\rho}$ to the character value of the $[G : S]\sigma^{-1}\rho$, $[G : S]\chi_V(\sigma^{-1}\rho)$.

The values

$$\langle \text{Indinf}_{S/T}^{G}\,\text{iso}_{\sigma}, \text{Indinf}_{S/T}^{G}\,.\,\text{iso}_{\rho} \rangle, \langle \text{Indinf}_{J/K}^{G}\,\text{iso}_{\sigma}, \text{Indinf}_{J/K}^{G}\,.\,\text{iso}_{\rho} \rangle \tag{4.10}$$

changes up to a constant multiple of $\frac{[G:S]}{[G:J]}$. Let us assume that there are l many sections S/T of G such that $S/T \cong H$ where G and H are cyclic 2-groups(in our case, $l = m - n + 1$). The matrix representation of the bilinear form with respect to the sorted basis(see basis choice) is characterized as follows:

Let $H_0 = \langle c^{2^m} \rangle$ where $\langle c \rangle = C_{2^m}$ and define $H_k = \langle c^{2^{m-k}} \rangle$. Then the sections appear on the basis element are of the form H_{n+i}/H_i where $0 \le i \le m-n$.

(i) The $|\operatorname{Out}(H)| \times |\operatorname{Out}(H)|$-submatrices appears as a block matrices in the representations. The first i,j-entries where $1 \le i,j \le |\operatorname{Out}(H)|$ are provided by

$$\langle \operatorname{Indinf}^G_{H_n/H_0} \operatorname{iso}_{\sigma_i}, \operatorname{Indinf}^G_{H_n/H_0} . \operatorname{iso}_{\sigma_j} \rangle \tag{4.11}$$

where it will correspond to the $(i,j)^{th}$-entry of the matrix.

(ii) The next block matrix appear on the entries a_{ij} where $|\operatorname{Out}(H)| < i,j \le 2|\operatorname{Out}(H)|$ and entries are provided by the values

$$\langle \operatorname{Indinf}^G_{H_{n+1}/H_1} \operatorname{iso}_{\sigma_i}, \operatorname{Indinf}^G_{H_{n+1}/H_1} . \operatorname{iso}_{\sigma_j} \rangle \tag{4.12}$$

which will corresponds to the $(|\operatorname{Out}(H)| + i, |\operatorname{Out}(H)| + j)^{th}$-entry.

(iii) The block matrices appears on the $(i,j)^{th}$-entries where $s|\operatorname{Out}(H)| \le i,j \le (s+1)|\operatorname{Out}(H)|$ and $s \in \{0, 1, ..., m-n\}$ and the $(s|\operatorname{Out}(H)|+i, s|\operatorname{Out}(H)|+j)^{th}$-entry where $1 \le i,j \le |\operatorname{Out}(H)|$ is given by

$$\langle \operatorname{Indinf}^G_{H_{n+s}/H_s} iso_{\sigma_i}, \operatorname{Indinf}^G_{H_{n+s}/H_s} .iso_{\sigma_j} \rangle. \tag{4.13}$$

(iv) The other entries of the matrix is 0.

Hence M is a block matrix with blocks $M_1, .., M_{m-n+1}$. The matrix looks like

$$M = \begin{pmatrix} M_1 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ & & & M_{m-n} & \\ & & & & M_{m-n+1} \end{pmatrix} \tag{4.14}$$

and the $\operatorname{rank}(M) = \operatorname{rank}(M)_1 + .. + \operatorname{rank}(M)_{m-n+1}$. We compute the rank of each M_i. We will show that the rank of M_i is 1. The $(jk)^{th}$-entry of M_i is given by

$$\langle \operatorname{Indinf}^G_{H_{n+i}/H_i} \operatorname{iso}_{\sigma_j}, \operatorname{Indinf}^G_{H_{n+i}/H_i} . \operatorname{iso}_{\sigma_k} \rangle = \chi_V(\sigma_j^{-1}.\sigma_k) \tag{4.15}$$

and the $(mk)^{th}$-entry of M_i is given by

$$\langle \mathrm{Indinf}^G_{H_{n+i}/H_i} \mathrm{iso}_{\sigma_m}, \mathrm{Indinf}^G_{H_{n+i}/H_i} . \mathrm{iso}_{\sigma_k} \rangle = \chi_V(\sigma_m^{-1}.\sigma_k) \tag{4.16}$$

and since χ_V is a linear character so that $(mk)^{th}$-entry and $(jk)^{th}$-entries change only by a scalar multiple $\frac{\chi_V(\sigma_m)^{-1}}{\chi_V(\sigma_j)^{-1}}$. Using this observation, we conclude that m^{th}-row of M_i is only a $\frac{\chi_V(\sigma_m)^{-1}}{\chi_V(\sigma_j)^{-1}}$-multiple of j-th row of M_i. Hence, the rank of each M_i is 1 and this leads us to conclude that $\mathrm{rank}(M) = l = m - n + 1 = \dim_{\mathbb{C}}(S_{C_{2^n},1})(C_{2^m}) = \dim_{\mathbb{C}}(S_{C_{2^n},V})(C_{2^m})$. □

Next, we try to find composition factors of $\mathbb{S}_2(C_{2^m})$ for any $m \in \mathbb{N}$. Equivalently, we would like to find the composition factors $S_{C_{2^n},V}(C_{2^n})$ which appears as a direct summand in $\mathbb{S}_2(C_{2^n})$. Note that if $S_{C_{2^n},V}$ appears as a summand in $\mathbb{S}_2(C_{2^n})$, then for each $m \geq n$, the evaluation of C_{2^m} at $\mathbb{S}_2(C_{2^n})$ has $S_{C_{2^n},V}$ as a summand which is evaluated at C_{2^m}.

We first show that the kernel of the deflation maps is non-zero and there exists an element in the kernel of the deflation maps which becomes an irreducible $\mathbb{C}\,\mathrm{Out}(C_{2^n})$-module. Before proving these results, we introduce some notations and prove some preliminary results.

Let $G = H_n = C_{2^n} = \langle c \rangle$ and we put $H_l = \langle h_l \rangle = \langle c^{2^{n-l}} \rangle$ for $0 \leq l < n$ and define $\widehat{\mathrm{Aut}(H_l)} = \{\phi : \mathrm{Aut}(H_l) \to \mathbb{C}^*\}$. For any $\phi \in \widehat{\mathrm{Aut}(H_l)}$, we define the elements:

$$e_{G,l,\phi} = \sum_{\alpha \in \mathrm{Aut}(H_l)} \phi(\alpha^{-1}) e^G_{G,\alpha(h_l)}. \tag{4.17}$$

Regarding such elements, we have a series of results. These results will be used in consecutive lemmas and they are important to reach our final result.

Lemma 4.1. *For any $\alpha \in \mathrm{Aut}(H_l)$, we have $\alpha(h_l) \in H_l$ with $\alpha(h_l) \notin H_{l-1}$.*

Proof. An automorphism sends a generator to another generator, therefore $|\alpha(h_l)| = |h_l|$ and we have $|h_l| > |H_{l-1}|$. These two observation directly prove the result. □

Lemma 4.2. *Given any element $e^G_{G,x}$ where $|x| = 2^l$ appears exactly $\frac{|H_l|}{2}$ many times as a summand in the elements $e_{G,l,\phi}$. Moreover, $e_{G,x}$ occurs as a non-zero summand of $e_{G,k,\phi}$ if and only if $k = l$.*

Proof. Given any $\phi \in \widehat{\mathrm{Aut}(H_l)}$, we define $e_{G,l,\phi} = \sum\limits_{\alpha \in \mathrm{Aut}(H_l)} \phi(\alpha^{-1}) e^G_{G,\alpha(h_l)}$. Since $|x| = 2^l$, there exists a unique $\alpha \in \mathrm{Aut}(H_l)$ such that $\alpha(h_l) = x$ provided that $\phi(\alpha^{-1}) \neq 0$ which follows from the fact that the irreducible characters of abelian groups take non-zero values on each element. Thus, x appears in each summand $e_{G,l,\phi}$. Since H_l is a cyclic group of order 2^l, the automorphism group is isomorphic to $C_2 \times C_{2^{l-2}}$ which has exactly 2^{l-1} elements. If $j \neq l$, then any automorphism in H_j sends an element of order 2^j to an element of order 2^j which means that an element of the form $e_{G,x}$ doesn't occur as a summand in $e_{G,j,\psi}$. This proves the result. □

Let $\{c_1, ..c_{2^{l-1}}\}$ be the set of elements in C_{2^l} having the order 2^l. Then by the above result, for any $\varphi \in \widehat{\mathrm{Aut}(H_l)}$, $e_{G,l,\varphi}$ is the linear combination of e_{G,c_i}'s where each e_{G,c_i} appears on each summand. The following result is quite important.

Lemma 4.3. $\mathrm{span}_{\mathbb{C}}\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\} = \mathrm{span}_{\mathbb{C}}\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\}$ *where the elements c_i are as above.*

Proof. Notice that the set $\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\}$ is linearly indepedent, therefore $\dim_{\mathbb{C}} \mathrm{span}_{\mathbb{C}}\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\} = 2^{l-1}$. Also, $|\mathrm{Aut}(H_l)| = 2^{l-1}$ and there are 2^{l-1} many irreducible characters of $\mathrm{Aut}(H_l)$. If we prove that $\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\}$ is linearly independent, then we will prove that $\mathrm{span}_{\mathbb{C}}\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\} = \mathrm{span}_{\mathbb{C}}\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\}$ because we already have $\mathrm{span}_{\mathbb{C}}\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\} \subset \mathrm{span}_{\mathbb{C}}\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\}$. With respect to the basis $\{e_{G,c_i} : 1 \leq i\ 2^{l-1}\}$, the set $\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\}$ has a matrix representation which corresponds to the character table of $\widehat{\mathrm{Aut}(H_l)}$ and since any character table is invertible, we conclude that the set $\{e_{G,l,\varphi} : \varphi \in \widehat{\mathrm{Aut}(H_l)}\}$ is linearly independent. Thus, we finish the proof. □

Lemma 4.3 says that intersection of kernel of the restriction maps to the group C_{2^l} is generated by the set $\{e_{G,c_i} : 1 \leq i \leq 2^{l-1}\}$. In other words, this result paves the way of the next theorem which will give us a basis for $\tilde{e}^G_G \mathbb{S}_2$.

Lemma 4.4. *The set* $\mathcal{D}_G = \{e_{G,l,\varphi} : 0 \leq l \leq n-1, \varphi \in \widehat{\mathrm{Aut}(H_l)}\}$ *is a basis for* $\tilde{e}^G_G \mathbb{S}_2$.

Proof. $\tilde{e}^G_G \mathbb{S}_2(G)$ consists of elements which belong to the intersection of kernel of restriction maps. The result becomes obvious with this observation. □

Now, we calculate the deflation of the basis elements in $\mathcal{D}_G$. Let A be the unique subgroup of $G = C_{2^n}$ of order 2. Since it is the minimal subgroup and Def is a transitive map, it is enough to determine the kernel of the map $\mathrm{Def}^G_{G/A}$.

Firstly, for any $x \in G$, we have that $\mathrm{Def}^G_{G/A}\, e_{G,x} = \frac{1}{2} e_{G/A,xA}$. We determine $\alpha, \alpha' \in \mathrm{Aut}(H_l)$ such that $\alpha(h_l)A = \alpha'(h_l)A$. Notice that $\alpha(h_l)c^{2^{n-1}} = \alpha'(h_l)$ since α is completely determined by h_l and α, α' being distinct implies that $\alpha(h_l) \neq \alpha'(h_l)$. Hence, this observation leads to the following result:

Lemma 4.5. $\mathrm{Def}^G_{G/A}\, e_{G,\alpha(h_l)} = \mathrm{Def}^G_{G/A}\, e_{G,\alpha'(h_l)}$ *if and only if* $\alpha = c^{2^{n-1}}\alpha'$ *if and only if* $\alpha\big|_{H_{l-1}} = \alpha'\big|_{H_{l-1}}$.

Proof. Let $x \in H_{l-1}$, then $x = (h_l)^{2m}$ where m is any integer. Then, $\alpha(x) = \alpha((h_l)^{2m}) = \alpha(h_l)^{2m} = (\alpha'(h_l)c^{2^{n-1}})^{2m} = \alpha'(h_l^{2m}) = \alpha'(x)$ □

We have that $\mathrm{Aut}(H_l) = C_2 \times C_{2^{l-2}} = \langle a \rangle \times \langle b \rangle$ and notice that above result holds for $\alpha, \alpha' \in \mathrm{Aut}(H_l)$ if and only if $\alpha = (1, g)$ implies $\alpha' = (a, g)$ or $\alpha = (a, g)$ implies $\alpha' = (1, g)$ for some $g \in C_{2^{l-2}}$. Using this result, we determine the kernel of the deflation map $\mathrm{Def}^G_{G/A}$ where A is the minimal subgroup of G. Before, we introduce an automorphism $\xi_l \in \mathrm{Aut}(H_l)$.

Definition 4.1. Let $\mathrm{Aut}(H_l) = \langle a \rangle \times \langle b \rangle$ where $\langle a \rangle \cong C_2$ and $\langle b \rangle \cong C_{2^{l-2}}$. Then we define an automorphism ξ_l of H_l given by

$$\xi_l : H_l \mapsto H_l \tag{4.18}$$

such that $x \mapsto x \cdot a$ where $\langle a \rangle = A$ is the minimal subgroup of G.

Lemma 4.6. $\mathrm{Def}^G_{G/A}\, e_{G,l,\varphi} = \frac{1+\varphi(\xi_l)^{-1}}{2} e_{G/A,l-1,\varphi\big|_{H_{l-1}}}$. *Hence,* $e_{G,l,\varphi} \in \mathrm{Ker}\,\mathrm{Def}^G_{G/A}$ *if and only if the map* $\xi_l : H_l \to H_l$ *satisfies* $\varphi(\xi_l) = -1$. *If* $\varphi(\xi_l) = 1$, *then we have that* $\mathrm{Def}^G_{G/A}\, e_{G,l,\varphi} \cong e_{G,l-1,\phi}$ *where* $\phi = \varphi\big|_{H_{l-1}}$.

Proof. Using the above result, the coefficient of $e_{G/A,\alpha(h_l)A}$ in $2 \cdot \mathrm{Def}^G_{G/A}\, e_{G,l,\varphi}$ is equal to $\varphi(\alpha^{-1}) + \varphi((\alpha^{'})^{-1}) = \varphi(\alpha^{-1}) + \varphi((\alpha\xi_l))^{-1} = \varphi(\alpha^{-1}) + \varphi(\alpha^{-1})\varphi(\xi_l^{-1})$. Hence, from this relation, the if and only if part can be easily shown. Hence, $e_{G,l,\varphi} \in \mathrm{Ker}\,\mathrm{Def}^G_{G/A}$ if and only if the map $\xi_l : H_l \to H_l$ satisfies $\varphi(\xi_l) = -1$ and the deflation formula also automatically follows from checking the coefficients of primitive idempotents $e_{G,\alpha(h_l)}$. Using the deflation formula, when $\varphi(\xi_l) = 1$, we have that

$$\mathrm{Def}^G_{G/A}\, e_{G,l,\varphi} = \sum_{\alpha \in \mathrm{Aut}(H_l)} \varphi(\alpha^{-1}) e_{G/A,\alpha(h_l)A} \tag{4.19}$$

and notice that $e_{G/A,\alpha(h_l)A}$ is the same as $e_{G,\alpha(h_{l-1})}$ and any automorphism remains fixed in smaller subgroups, so we conclude that $\mathrm{Def}^G_{G/A}\, e_{G,l,\varphi} = e_{G,l-1,\varphi\big|_{H_{l-1}}}$. ☐

Hence, kernel of the deflation is non-zero. The next fact we are going to prove is that the kernel of deflation is generated by $e_{G,l,\varphi}$ where $\varphi(\xi_l) = -1$.

Proposition 4.3. *We have* $\mathrm{Ker}\,\mathrm{Def}^G_{G/A} = \mathrm{span}_{\mathbb{C}}\{e_{G,l,\varphi} : \varphi(\xi_l) = -1\}$.

Proof. We are going to show that if $e^G_{G,x}$ is in the kernel of the deflation, then it is a linear combination of elements in $\mathrm{span}_{\mathbb{C}}\{e_{G,l,\varphi} : \varphi(\xi_l) = -1\}$. Note that we can write $e^G_{G,x} = \sum_{\varphi \in \widehat{\mathrm{Aut}(H_l)}} a_\varphi e_{G,l,\varphi}$ where $|x| = 2^l$. Then its deflation is zero when each element in its summand is in the kernel of the deflation. Notice that if there exists an summand $e_{G,l,\varphi}$ with non-zero coefficient such that $\mathrm{Def}^G_{G/A}\, e_{G,l,\varphi}$ is non-zero, then the deflation contains an element of the form $e_{G,l-1,\varphi'}$ and then due to linear independence of elements $e_{G,l-1,\phi}$, we will obtain a contradiction because the remaining elements will be different from $e_{G,l-1,\phi'}$. Thus, only summands having zero deflation will occur and this proves the result. □

We know that the dimension of $\operatorname{Ker}\operatorname{Def}^G_{G/A}$ is equal to $\left(\sum\limits_{l=2}^{n-1} \frac{|H_l|}{4}\right)+1$ using a straightforward argument. We determine which $e_{G,l,\varphi} \in \operatorname{Ker}\operatorname{Def}^G_{G/A}$ can be viewed as a $\mathbb{C}\operatorname{Out}(C_{2^n})$-module. Indeed, each element in the span of the kernel of the deflation generates an irreducible $\operatorname{Out}(C_{2^n})$-module.

Lemma 4.7. *Let* $\operatorname{Aut}(G)$ *act on the lines* $\mathbb{C}e_{G,l,\varphi}$ *via*

$$\beta \cdot e_{G,l,\varphi} = \sum_{\alpha\in\operatorname{Aut}(H_l)} \varphi(\alpha^{-1}) e_{G,(\beta\circ\alpha)(h_l)} \tag{4.20}$$

for any $\beta \in \operatorname{Aut}(G)$. *Then each line* $\mathbb{C}e_{G,l,\varphi}$ *becomes* $\mathbb{C}\operatorname{Aut}(G) = \mathbb{C}\operatorname{Out}(G)$*-module.*

Proof. Define $\gamma = \beta\circ\alpha$, so when α runs through $Aut(H_l)$,then γ runs through $Aut(H_l)$ as well. Let's calculate the action.

$$\begin{aligned}
\beta \cdot e_{G,l,\varphi} &= \sum_{\alpha\in\operatorname{Aut}(H_l)} \varphi(\alpha^{-1}) e_{G,(\beta\circ\alpha)(h_l)} &(4.21)\\
&= \sum_{\gamma\circ\beta^{-1}\in\operatorname{Aut}(H_l)} \varphi(\alpha^{-1}) e_{G,\gamma(h_l)} &(4.22)\\
&= \sum_{\gamma\circ\beta^{-1}\in\operatorname{Aut}(H_l)} \varphi(\beta\gamma^{-1}) e_{G,\gamma(h_l)} &(4.23)\\
&= \sum_{\gamma\circ\beta^{-1}\in\operatorname{Aut}(H_l)} \varphi(\beta)\varphi(\gamma^{-1}) e_{G,\gamma(h_l)} &(4.24)\\
&= \varphi(\beta) e_{G,l,\varphi}. &(4.25)
\end{aligned}$$

Hence each line $\mathbb{C}e_{G,l,\varphi}$ is an $\mathbb{C}\operatorname{Out}(G)$-module. □

Also, distinct $e_{G,l,\varphi}$'s produces distinct irreducible $\mathbb{C}\operatorname{Out}(G)$-modules because notice that the $\operatorname{Out}(G)$-action is characterized by the character

$$\chi : \operatorname{Out}(G) \to \mathbb{C}^* \tag{4.26}$$

such that $\beta \to \varphi(\beta)$ so it is completely determined by φ. As a result of consecutive theorems that we have shown above, we have proved the following theorem:

Theorem 4.3. *Let G be a cyclic group of order 2^n. Then:*

(i) $\dim_{\mathbb{C}} \operatorname{Ker} \operatorname{Def}^G_{G/A} = \frac{|G|}{4}$.

(ii) $\operatorname{Ker} \operatorname{Def}^G_{G/A} = \operatorname{span}_{\mathbb{C}}\{e_{G,l,\varphi} | \textit{where } \varphi \textit{ sends } \xi_l : \operatorname{Aut}(H_l) \to \operatorname{Aut}(H_l) \textit{ to -1}\}$.

(iii) For any $n \in \mathbb{N}$ and any $m \geq n$, the $\mathbb{C}$-vector space $\mathbb{S}_2(C_{2^m})$ contains a summand $S_{C_{2^n},V}$ in the $\mathbb{C}$-vector space decomposition where V is an irreducible $\mathbb{C}\operatorname{Out}(C_{2^n})$-module isomorphic to $\mathbb{C}e_{G,l,\varphi}$ for some (l,φ) where $0 \leq l < n$ and $\varphi \in \widehat{\operatorname{Aut}(H_l)}$. In particular, for each $n \in \mathbb{N}$, $\operatorname{Ker} \operatorname{Def}^{C_{2^n}}_{C_{2^n}/A}$ is non-zero.

(iv) $\mathbb{S}_2(C_{2^m}) = \bigoplus_{(C_{2^i},V_i)} S_{C_{2^i},V_i}(C_{2^m})$. *where the direct sum ranges over all pairs (C_{2^i}, V_i) where $i \leq m$ and V_i is an irreducible $\mathbb{C}\operatorname{Out}(C_{2^i})$-module which is isomorphic to* $\mathbb{C}e_{G,l,\varphi} \subseteq \operatorname{Ker} \operatorname{Def}^{C_{2^i}}_{C_{2^i}/A}$.

(v) There are 2^{m-1} many $\mathbb{C}\operatorname{Out}(C_{2^m})$-submodule in $\mathbb{S}_2(C_{2^m})$.

Proof. Each of these facts follow from the previous theorems and verification of the dimensions can be made from the fact that $\dim_{\mathbb{C}}(S_{C_{2^n},V})(C_{2^m}) = m - n + 1$. □

Now we will show that in $\mathbb{S}_2$, there is no composition factor of the form $S_{G,V}$ where $G \cong C_{2^{n_1}} \times C_{2^{n_2}} \times \ldots \times C_{2^{n_k}}$ with $n_i \geq 2$,$k \geq 2$ and V is an irreducible $\mathbb{C}\operatorname{Out}(G)$-module. Equivalently, we will show that the intersection of kernel of deflation maps $Def^G_{G/A}$ is zero where $A \lhd G$. Let $G \cong C_{2^{n_1}} \times C_{2^{n_2}} \times \ldots \times C_{2^{n_k}}$ with $n_i \geq 2$ for each $1 \leq i \leq k$ and $k \geq 2$. We start by computing the deflation number $\mathfrak{m}_{G,g,A}$ for any $g \in G$ and $A \lhd G$ where A is minimal.

Any minimal subgroup of G is of order 2. Moreover, the Frattini subgroup $\Phi(G)$ is isomorphic to $C_{2^{n_1-1}} \times C_{2^{n_2-1}} \times \ldots \times C_{2^{n_k-1}}$ and therefore Frattini subgroup contains every minimal subgroup of G, so every maximal subgroup of G contains every minimal subgroup of G. As a result, for any minimal subgroup A of G, $N \cdot A = G$ for some $N \leq G$ if and only if $N = G$. Using 3.6, the deflation number $\mathfrak{m}_{G,g,A}$ is always equal to $\frac{1}{2}$. Let E denote the largest elementary abelian subgroup of G. Then we have $E \cong (C_2)^k$, i.e E consists of all elements of order 2 in G with identity element. We immediately observe the following results.

Lemma 4.8. *Let $A = \langle a \rangle$ be any minimal group of order 2 with $g, h \in G$ and $g \neq h$. Then $\mathrm{Def}^G_{G/\langle a \rangle} e^G_{G,g} = \mathrm{Def}^G_{G/\langle a \rangle} e^G_{G,h}$ if and only if $g = ha$.*

Proof. It follows from the computation of deflation number and the fact that $gA = hA$ if and only if $g = ha$. □

Lemma 4.9. *Let $g, h \in G$. Then $gE = hE$ if and only if $gA = hA$ for some minimal group A in G.*

Proof. Let us assume $gE = hE$, then $ge = h$ for some $e \in E$ and E is elementary abelian so $|e| = 2$. Hence $A = \langle e \rangle$ is a minimal group with $gA = hA$. Conversely, $gA = hA$ implies $(gA)E = (hA)E$ implies $gE = hE$. □

Let $\mathcal{S}$ denote the set of all minimal subgroups of G whose largest elementary abelian subgroup has rank k. Then $\mathcal{S}$ has exactly $2^k - 1$ many elements which are generated by elements of order 2. Assume that $\alpha \in \bigcap_{A \in \mathcal{S}} \mathrm{Ker}\,\mathrm{Def}^G_{G/A}$. Let $e_{G,g}$ be a non-zero summand of α. Then, since $\alpha \in \bigcap_{A \in \mathfrak{S}} \mathrm{Ker}\,\mathrm{Def}^G_{G/A}$, so the elements $e_{G,ag}$ where $|\langle a \rangle| = |A| = 2$ must be in the summand of α. Thus, α must contain the summand $\sum_{a \in E} e_{G,ag}$. By this observation, if we prove that there is no element in $\bigcap_{A \in \mathfrak{S}} \mathrm{Ker}\,\mathrm{Def}^G_{G/A}$ of the form $\sum_{a \in E} \alpha_a e_{G,ag}$, then no such α exists.

Theorem 4.4. *If G is a group satisfying the conditions above, then we have*

$$\mathcal{K} = \bigcap_{A \in \mathcal{S}} \mathrm{Ker}\,\mathrm{Def}^G_{G/A} = \{0\}. \tag{4.27}$$

Proof. Using the above observation, it is enough to prove that an element of the form $\sum_{a \in E} \alpha_a e_{G,ag}$ is not in the set $\mathcal{K}$ for any $g \in G$. Assume that $\sum_{a \in E} \alpha_a e_{G,ag} \in \mathcal{K}$, then for any $e \in E \setminus \{id\}$, this gives rise to the set of equations

$$\alpha_{e_1} + \alpha_{e_1 e} = 0, \tag{4.28}$$

$$\alpha_{e_2} + \alpha_{e_2 e} = 0, \tag{4.29}$$

$$\alpha_{e_3} + \alpha_{e_3 e} = 0 \tag{4.30}$$

$$...., \tag{4.31}$$

$$...., \tag{4.32}$$

$$\alpha_{e_{\frac{|E|}{2}}} + \alpha_{e_{\frac{|E|}{2}} e} = 0 \tag{4.33}$$

which produces $\frac{|E|}{2}$ many equations with $|E|$ many variables α_{e_i} where $e_i \in E$. Since any minimal subgroup is in one-to-one correspondence with the elements in E, so an element $\sum_{a \in E} \alpha_a e_{G,ag} \in \mathcal{K}$ gives rise to $(|E| - 1) \cdot \frac{|E|}{2}$ many equations with $|E|$ where the zero vector satisfies the set of equations. If we show that the matrix M whose rows associated to the equations produced by $\sum_{a \in E} \alpha_a e_{G,ag} \in \mathcal{K}$ has rank $|E|$, then we conclude that such element must be 0. Label the rows as follows: Let us write $E = \{e_1, .., e_{|E|}\}$. Given any $g \in G$, we write the element α as $\sum_{i=1}^{|E|} \alpha_{e_i} e_{G,ge_i}$. Then the first row is associated with the equation

$$\alpha_{e_1} + \alpha_{e_2} = 0 \tag{4.34}$$

and let us associate the ith row where $1 \leq i \leq |E| - 1$ with the equation

$$\alpha_{e_1} + \alpha_{e_{i+1}} = 0 \tag{4.35}$$

and let us associate the $|E|^{th}$ row with the equation

$$\alpha_{e_2} + \alpha_{e_3} = 0 \tag{4.36}$$

and first $|E| \times |E|$-entries of the associated matrix is of the form

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 0 \end{pmatrix} \tag{4.37}$$

where deleting the first column and the last row, we obtain an identity submatrix.

Firstly, subtract the first row from the each row except the last one. Then add the last row to the second row. We will obtain 2 in the $(2,3)^{th}$-entry of the submatrix A and divide this row by 2. Subtracting the second row from the last row and then adding the last row to $(|E|-1)^{th}$-row we obtain a permutation matrix of rank $|E|$. Hence, the rank of the actual matrix is at least $|E|$ and it should be $|E|$. Therefore, the system must have a unique solution which is the zero vector. Therefore, we conclude that $\mathcal{K} = \bigcap_{A \in \mathcal{S}} \operatorname{Ker} \operatorname{Def}^G_{G/A} = \{0\}$. □

The last result indeed suggests that there are few composition factors of the form $S_{G,V}$. Notice that when G has at least 2 factors with no C_2-factor, then there is no composition factor of the form $S_{G,V}$. Indeed, the next result shows that even though G has a C_2-factor, if G has rank at least 3, then there is no composition factor of the form $S_{G,V}$. Idea of the proof is similar to the previous result. Note that the proof of the previous result and this result also hold in case of any odd prime p.

Theorem 4.5. *There is no composition factor of $\mathbb{S}_2$ of the form $S_{G,V}$ where G has finite rank of at least* 3.

Proof. We will show that $\mathcal{K} = \bigcap_{A \in \mathcal{S}} \operatorname{Ker} \operatorname{Def}^G_{G/A} = \{0\}$. Let $\alpha \in \mathcal{K}$, then we have shown that if $e_{G,g}$ appears to be a non-zero summand of α, then we have a system of equations

$$\mathfrak{m}_{G,G/T,g} \cdot \alpha_g + \mathfrak{m}_{G,G/T,eg} \cdot \alpha_{eg} = 0 \tag{4.38}$$

for any $e \in E$ with $\langle e \rangle = T$ where E is the largest elementary subgroup of G, i.e the set consisting of all elements of order 2. We can write G as $G = H \times K$ where H contains no C_2 factor and K only contains C_2 factors. We set $H = H_1 \times ... \times H_l$ and let K be elementary abelian group of rank k. Let $E = \operatorname{Soc}(G)$ be largest elementary abelian subgroup of G. Then G is a finite abelian group of rank $k + l$ with Frattini group of rank l. To solve the above equations; we need the deflation number $\mathfrak{m}_{G,G/T,g}$ for any $g \in G$ and any minimal group T. For this, we have the following results given in [10] and [13].

Lemma 4.10. *Let P be a p-group and N be a normal subgroup of P. Then*

(i) $\mathfrak{m}_{P/N,P,g} = \mathfrak{m}_{\overline{P}/\overline{N.\Phi(P)},\overline{P},\bar{g}}$ *where $\overline{N}$ or $\overline{g}$ denotes the image of a element or a subgroup of P under the canonical group homomorphism*

$$\Pi : P \to P/\Phi(P) \tag{4.39}$$

(ii) Let $P \cong C_p^d$ and $N \cong C_p$ then we have

$$\mathfrak{m}_{\bar{P},P,g} = \begin{cases} \frac{1}{p} & 1 \neq g \in N \\ \frac{1-p^{d-1}}{p} & g = 1 \\ \frac{1-p^{d-2}}{p} & otherwise \end{cases} \tag{4.40}$$

Proof. The proof can be found in [10]. Note that the proofs are still valid in our case since we take our fiber to be $\mathbb{C}^*$ which contains all n^{th}-roots of unity. □

However, we need one more lemma which is useful in deflation number calculation. Define the monomial β-numbers by

$$\beta_p(c,d) = \frac{1}{p^{d-c}}.\prod_{i=c}^{d-1}(1-p^i) \tag{4.41}$$

and $\beta_p(d,d) = 1$ for any integers $0 \leq c < d$.

Lemma 4.11. *Let $P \cong C_p^d$ and $N \cong C_p^{d-c}$ then we have*

$$\mathfrak{m}_{\overline{P},P,g} = \begin{cases} \beta_p(c,d) & g = 1 \\ \beta_p(c-1,d-1) & g \notin N \\ \frac{\beta_p(c,d-1)}{p} & 1 \neq g \in N \end{cases} \tag{4.42}$$

Proof. See [10]. □

Using these results, we can now compute the deflation numbers. Let G be as above and let $e \in E$ and $\langle e \rangle = T$.

Lemma 4.12. *Assume that $T \leq \Phi(G)$, then*

$$\mathfrak{m}_{G,G/T,g} = \frac{1}{2} \tag{4.43}$$

Proof. Notice that for any abelian group G, the quotient $G/\Phi(G)$ is an elementary abelian group of rank 1 and using Lemma 4.5, we have that $\mathfrak{m}_{G,G/T,g} = \mathfrak{m}_{\overline{G},\overline{T\Phi(G)},\overline{g}} = \mathfrak{m}_{\overline{G},\overline{\Phi(G)},\overline{g}} = \frac{1}{|T \cap \Phi(G)|} \cdot 1 = \frac{1}{2}$ □

Lemma 4.13. *Assume that $T \not\leq \Phi(G)$, then*

$$\mathfrak{m}_{G,G/T,g} = \begin{cases} \frac{1}{2}(1 - 2^{k+l-1}) & g \in \Phi(G) \\ \frac{1}{2}(1 - 2^{k+l-2}) & g\Phi(G) \notin T\Phi(G) \\ \frac{1}{2} & g\Phi(G) \in T\Phi(G)/\Phi(G) \text{ and } g \notin \Phi(G) \end{cases} \tag{4.44}$$

Proof. Notice that G is of rank $k + l$ and the quotient group $G/\Phi(G)$ is elementary abelian group of rank $k + l$ and the quotient $T\Phi(G)/\Phi(G)$ is elementary abelian group of rank 1. Therefore, the formula in Lemma 4.10 is used with $d = k+l$ and $c = k+l-1$. Notice that $P = G/\Phi(G)$ so $g = 1$ condition is equivalent to $g \in \Phi(G)$. The condition $g \notin N$ is equivalent to $g\Phi(G) \notin T\Phi(G)$ and the last observation is also obvious. □

Let G be a finite abelian 2-group of rank at least 3. Then G contains a $C_2 \times C_2$ copy. Hence, we can choose elements $t_1, t_2 \in G$ which are distinct such that $|t_1| = |t_2| = 2$. Define $t_3 = t_1 t_2$. Then $t_1, t_2, t_3 \in E$. Let $\alpha \in \mathcal{K}$, and let us take a non-zero summand $\alpha_g e_{G,g}$ in α. We must have three equations, the first equation is given by

$$\mathfrak{m}_{(G,G/\langle t_1 \rangle,g)} \cdot \alpha_g + \mathfrak{m}_{(G,G/\langle t_1 \rangle,t_1 g)} \cdot \alpha_{t_1 g} = 0, \tag{4.45}$$

the second equation is given by

$$\mathfrak{m}_{(G,G/\langle t_2 \rangle,t_1 g)} \cdot \alpha_{t_1 g} + \mathfrak{m}_{(G,G/\langle t_2 \rangle,t_3 g)} \cdot \alpha_{t_3 g} = 0 \tag{4.46}$$

and the third equation is given by

$$\mathfrak{m}_{(G,G/\langle t_3 \rangle,g)} \cdot \alpha_g + \mathfrak{m}_{(G,G/\langle t_3 \rangle,t_3 g)} \cdot \alpha_{t_3 g} = 0. \tag{4.47}$$

Note that the above equations has a trivial solution, i.e $\alpha_g = \alpha_{t_1 g} = \alpha_{t_3 g} = 0$. If the matrix M defined by

$$\begin{bmatrix} \mathfrak{m}_{(G,G/\langle t_1\rangle,g)} & \mathfrak{m}_{(G,G/\langle t_1\rangle,t_1g)} & 0 \\ 0 & \mathfrak{m}_{(G,G/\langle t_2\rangle,t_1g)} & \mathfrak{m}_{(G,G/\langle t_2\rangle,t_3g)} \\ \mathfrak{m}_{(G,G/\langle t_3\rangle,g)} & 0 & \mathfrak{m}_{(G,G/\langle t_3\rangle,t_3g)} \end{bmatrix}$$

has a non-zero determinant, then we can only have a trivial solution to the system of equations. We compute the determinant and we find out that $\det(M)$ is equal to

$$\mathfrak{m}_{(G,G/\langle t_1\rangle,g)} \cdot \mathfrak{m}_{(G,G/\langle t_2\rangle,t_1g)} \cdot \mathfrak{m}_{(G,G/\langle t_3\rangle,t_3g)} + \mathfrak{m}_{(G,G/\langle t_1\rangle,t_1g)} \cdot \mathfrak{m}_{(G,G/\langle t_3\rangle,g)} \cdot \mathfrak{m}_{(G,G/\langle t_2\rangle,t_3g)}. \quad (4.48)$$

To ease the notation, let $\mathfrak{m}_{(G,G/\langle t_1\rangle,g)} = A$, $\mathfrak{m}_{(G,G/\langle t_2\rangle,t_1g)} = B$, $\mathfrak{m}_{(G,G/\langle t_3\rangle,t_3g)} = C$. Moreover, we let $\mathfrak{m}_{(G,G/\langle t_1\rangle,t_1g)} = D$, $\mathfrak{m}_{(G,G/\langle t_3\rangle,g)} = E$ and $\mathfrak{m}_{(G,G/\langle t_2\rangle,t_3}$.Also, we let $\langle t_i\rangle = T_i$. The rest of the proof investigates each possible cases and we conclude that in each case, determinant is non-zero which leads us to conclude that such a non-zero α is not possible. The cases are done because the deflation numbers vary by each cases. Note that, in each case we use the fact that $k + l > 2$.

(i) Assume that $t_1, t_2, t_3 \in \Phi(G)$. Given any $g \in G$, we have $\mathfrak{m}_{G,G/T_i,g} = \frac{1}{2}$ since $T_i \leq \Phi(G)$. Then $\det M = \frac{1}{4} \neq 0$ and that implies $\alpha_g = 0$, a contradiction.

(ii) Assume that $t_1, t_2, t_3 \notin \Phi(G)$

   (a) Assume $g\Phi(G) = t_1\Phi(G)$ and $g\Phi(G) = t_2\Phi(G)$, then $g = t_1g^{'}$ and $g = t_2g^{''}$ where $t^{'}, t^{''} \in \Phi(G)$. Thus, $g' = t_3g''$, i.e we have $t_3 \in \Phi(G)$ and we obtain a contradiction.

   In general, if $t_i\Phi(G) = t_j\Phi(G)$, then we obtain a contradiction.

   (b) Assume that $g\Phi(G) = t_1\Phi(G)$, $g\Phi(G) \neq t_2\Phi(G)$ and $g\Phi(G) \neq t_3\Phi(G)$. In this case, the deflation numbers become

   $$A = \tfrac{1}{2}, B = \tfrac{1-2^{k+l-2}}{2},\ C = \tfrac{1-2^{k+l-2}}{2}, D = \tfrac{1-2^{k+l-1}}{2}, E = \tfrac{1-2^{k+l-2}}{2},\ F = \tfrac{1}{2}$$

   and $\det(M)$ is a positive number.

   (c) Assume that $g\Phi(G) = t_2\Phi(G)$, $g\Phi(G) \neq t_1\Phi(G)$ and $g\Phi(G) \neq t_3\Phi(G)$. In this case, the deflation numbers become

   $$A = \tfrac{1-2^{k+l-2}}{2}, B = \tfrac{1-2^{k+l-2}}{2},\ C = \tfrac{1-2^{k+l-2}}{2}, D = \tfrac{1-2^{k+l-2}}{2}, E = \tfrac{1-2^{k+l-2}}{2},$$
   $$F = \tfrac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a negative number.

(d) Assume that $g\Phi(G) = t_3\Phi(G)$, $g\Phi(G) \neq t_1\Phi(G)$ and $g\Phi(G) \neq t_2\Phi(G)$. In this case, the deflation numbers become

$$A = \frac{1-2^{k+l-2}}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-1}}{2}, D = \frac{1-2^{k+l-2}}{2}, E = \frac{1}{2},\ F = \frac{1-2^{k+l-1}}{2}$$

and $\det(M)$ is a positive number.

(e) Assume that $t_i\Phi(G) \neq g\Phi(G)$. If $g \in \Phi(G)$, then deflation numbers become

$$A = \frac{1-2^{k+l-1}}{2}, B = \frac{1-2^{k+l-2}}{2},\ C = \frac{1}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-1}}{2},\ F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a positive number. If $g \notin \Phi(G)$,

then deflation numbers become

$$A = \frac{1-2^{k+l-2}}{2}, B = \frac{1-2^{k+l-2}}{2},\ C = \frac{1-2^{k+l-2}}{2}, D = \frac{1-2^{k+l-2}}{2}, E = \frac{1-2^{k+l-2}}{2},$$
$$F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a negative number.

(iii) Assume that $t_1 \in \Phi(G)$ and $t_2, t_3 \notin \Phi(G)$.

(a) The cases $t_i\Phi(G) = g\Phi(G)$ for all t_i and $t_1\Phi(G) = t_i\Phi(G)$ for $i \in \{2, 3\}$ are not possible.

(b) Assume $t_1\Phi(G) = g\Phi(G)$, i.e $g \in \Phi(G)$. Hence $g\Phi(G) \neq t_i\Phi(G)$ for $i \in \{2, 3\}$. The deflation numbers become

$$A = \frac{1}{2}, B = \frac{1-2^{k+l-1}}{2},\ C = \frac{1}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-1}}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a negative number.

(c) Assume $t_i\Phi(G) \neq g\Phi(G)$. The deflation numbers become

$$A = \frac{1}{2}, B = \frac{1-2^{k+l-2}}{2},\ C = \frac{1-2^{k+l-2}}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-2}}{2},\ F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a positive number.

(d) Assume $t_1\Phi(G) \neq g\Phi(G)$, $t_2\Phi(G) = g\Phi(G)$ and $t_3\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-1}}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-2}}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a negative number.

(e) Assume $t_1\Phi(G) \neq g\Phi(G)$, $t_2\Phi(G) \neq g\Phi(G)$ and $t_3\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-1}}{2}, D = \frac{1}{2}, E = \frac{1}{2},\ F = \frac{1-2^{k+l-1}}{2}$$

and $\det(M)$ is a negative number.

(f) Assume $t_1\Phi(G) \neq g\Phi(G)$, $t_2\Phi(G) = g\Phi(G)$ and $t_3\Phi(G) \neq g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1-2^{k+l-2}}{2},\ C = \frac{1-2^{k+l-2}}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-2}}{2},\ F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a positive number.

(iv) Assume that $t_2 \in \Phi(G)$ and $t_1, t_3 \notin \Phi(G)$

(a) The cases $t_i\Phi(G) = g\Phi(G)$ for all t_i and $t_1\Phi(G) = t_i\Phi(G)$ for $i \in \{1, 3\}$ are not possible.

(b) Assume $t_2\Phi(G) = g\Phi(G)$, i.e $g \in \Phi(G)$. Hence $g\Phi(G) \neq t_i\Phi(G)$ for $i \in \{1, 3\}$. The deflation numbers become

$$A = \frac{1-2^{k+l-1}}{2}, B = \frac{1}{2},\ C = \frac{1}{2}, D = \frac{1}{2}, E = \frac{1-2^{k+l-1}}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a negative number.

(c) Assume $t_i\Phi(G) \neq g\Phi(G)$. The deflation numbers become

$$A = \frac{1}{2}, B = \frac{1}{2},\ C = \frac{1}{2}, D = \frac{1}{2}, E = \frac{1}{2},\ F = \frac{1}{2}$$

and $\det(M) = 1/4$ is a positive number.

(d) Assume $t_2\Phi(G) \neq g\Phi(G)$, $t_1\Phi(G) = g\Phi(G)$ and $t_3\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-1}}{2}, D = \frac{1-2^{k+l-1}}{2}, E = \frac{1}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a negative number.

(e) Assume $t_2\Phi(G) \neq g\Phi(G)$, $t_1\Phi(G) \neq g\Phi(G)$ and $t_3\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1-2^{k+l-2}}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-1}}{2},\ D = \frac{1-2^{k+l-2}}{2}, E = \frac{1}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a positive number.

(f) Assume $t_2\Phi(G) \neq g\Phi(G)$, $t_1\Phi(G) = g\Phi(G)$ and $t_3\Phi(G) \neq g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1}{2},\ C = \frac{1-2^{k+l-2}}{2},\ D = \frac{1-2^{k+l-1}}{2}, E = \frac{1-2^{k+l-2}}{2},\ F = \frac{1}{2}$$

and $\det(M)$ which is a non-zero number.

(v) Assume that $t_3 \in \Phi(G)$ and $t_1, t_2 \notin \Phi(G)$

(a) The cases $t_i\Phi(G) = g\Phi(G)$ for all t_i and $t_1\Phi(G) = t_i\Phi(G)$ for $i \in \{1, 3\}$ are not possible.

(b) Assume $t_3\Phi(G) = g\Phi(G)$, i.e $g \in \Phi(G)$. Hence $g\Phi(G) \neq t_i\Phi(G)$ for $i \in \{1, 2\}$. The deflation numbers become

$$A = \frac{1-2^{k+l-1}}{2}, B = \frac{1}{2},\ C = \frac{1}{2}, D = \frac{1}{2}, E = \frac{1}{2},\ F = \frac{1}{2}$$

and $\det(M)$ is a negative number.

(c) Assume $t_i\Phi(G) \neq g\Phi(G)$. The deflation numbers become

$$A = \frac{1-2^{k+l-2}}{2}, B = \frac{1-2^{k+l-2}}{2}, \; C = \frac{1}{2}, D = \frac{1-2^{k+l-2}}{2}, E = \frac{1}{2}, \; F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a positive number.

(d) Assume $t_3\Phi(G) \neq g\Phi(G)$, $t_1\Phi(G) = g\Phi(G)$ and $t_2\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1-2^{k+l-1}}{2}, \; C = \frac{1}{2}, D = \frac{1-2^{k+l-1}}{2}, E = \frac{1-2^{k+l-2}}{2}, \; F = \frac{1}{2}$$

and $\det(M)$ is a non-zero number.

(e) Assume $t_3\Phi(G) \neq g\Phi(G)$, $t_1\Phi(G) \neq g\Phi(G)$ and $t_2\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1-2^{k+l-2}}{2}, B = \frac{1-2^{k+l-2}}{2}, \; C = \frac{1}{2}, \; D = \frac{1-2^{k+l-2}}{2}, E = \frac{1}{2}, \; F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a positive number.

(f) Assume $t_3\Phi(G) \neq g\Phi(G)$, $t_2\Phi(G) \neq g\Phi(G)$ and $t_1\Phi(G) = g\Phi(G)$, then the deflation numbers become

$$A = \frac{1}{2}, B = \frac{1-2^{k+l-1}}{2}, \; C = \frac{1}{2}, \; D = \frac{1-2^{k+l-1}}{2}, E = \frac{1}{2}, \; F = \frac{1-2^{k+l-2}}{2}$$

and $\det(M)$ is a non-zero number.

In each case, the determinant is non-zero , thus $\alpha_g = 0$ for all $g \in G$. Hence, an element in the intersection of kernel of deflation map is zero. □

5. CONCLUSION

Hence, there is no composition factor of the form $S_{G,V}$ of $\mathbb{S}_2$ when G is a finite abelian group of rank at least 3. Using previous results, we conclude the following. This theorem summarizes what have been done in this thesis.

Theorem 5.1 (**Composition factors of** $\mathbb{S}_2$)**.** *The composition factors $S_{G,V}$ of the $\mathbb{C}^*$-fibered 2-biset functor $\mathbb{S}_2$ are given as follows:*

(i) For each cyclic 2-group $C_{2^n} = G$, simple biset functors $S_{C_{2^n},V}$ appears as a composition factor of $\mathbb{S}_2$ if V is an irreducible $\mathbb{C}\operatorname{Out}(G)$-module that is isomorphic to $\mathbb{C}e_{G,l,\varphi} \in \operatorname{Ker}\operatorname{Def}^G_{G/A}$ where A is the minimal group of G. For each $\mathbb{C}e_{G,l,\varphi} \in \operatorname{Ker}\operatorname{Def}^G_{G/A} \cong V_i$ viewed as a $\mathbb{C}\operatorname{Out}(G)$-module, there is a composition factor in $\mathbb{S}_2$ of the form S_{G,V_i}

(ii) Simple biset functors $S_{C_2,1}$ and $S_{C_2\times C_2,1}$ appears as a composition factor of $\mathbb{S}_2$.

(iii) There is no composition factor of the form $S_{G,V}$ where G is finite abelian 2-group having at least 3 factors.

(iv) When G is a non-abelian finite 2-group, the simple biset functors $S_{G,V}$ doesn't appear as a composition factor in $\mathbb{S}_2$

(v) When $G \cong C_2 \times C_{2^m}$, the simple biset functor $S_{G,V}$ can appear as a composition factor of $\mathbb{S}_2$ when we view it as a 2-biset functor.

The composition factors of the form $S_{C_{2^n}\times C_2,V}$ were studied and the complete composition factors will be described in the future studies. In a similar manner, following similar ideas appears on the thesis, one tries to find the composition factors of the kernel of the monomial linearization map at any finite p-group. This question is the central one which will appear in the future studies.

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