LMI FORMULATIONS FOR A CLASS OF NONCONVEX QUADRATIC PROGRAMMING PROBLEMS

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ABSTRACT

LMI FORMULATIONS FOR A CLASS OF NONCONVEX QUADRATIC PROGRAMMING PROBLEMS

Finding the optimal value of a polynomial function over a region determined by a finite number of polynomial constraints and determining the emptiness of such a region constitute major problems encountered in control theory and other branches of engineering. In general, such problems are known to be hard, and hence, it is very unlikely that efficient algorithms will developed for their solution. A general approach employed in the literature to overcome this difficulty is to develop approximations that can be computed efficiently or to identify special subproblems which can be solved easily due to their special structure. In this thesis, we follow the second approach and derive linear matrix inequality (LMI) formulations for some nonconvex quadratic optimization problems. To be more specific, we develop two related results. First, it is shown that the convex hull of a region determined by two quadratic inequality constraints is an LMI set and an algorithm producing the LMI description of the convex hull is obtained. By this way, it becomes possible to find the optimal value of a linear objective function over such a region efficiently and exactly. Second, we show that in $\mathbb{R}^2$ the convex hull of a region determined by a finite number of quadratic constraints is LMI set. However, this time the proof developed is not constructive, and hence, an algorithm for attaining the convex hull could not be derived.

ÖZET

BAZI İKİNCİ DERECE POLİNOM ENİYİLEME PROBLEMLERİ İÇİN DOĞRUSAL MATRİS EŞİTSİZLİĞİ FORMÜLASYONLARI

Bir polinom fonksiyonun polinom kısıtlarla belirlenen bir bölge üzerindeki en iyi değerinin bulunması ve bu tip bir bölgenin boş olup olmadığının belirlenmesi kontrol kuramında ve diğer mühendislik alanlarında önemli bir problem teşkil etmektedir. Genelede bu tip problemlerin çözümünün zor olduğu bilinmekte ve bu nedenle de makul bir sürede makul miktarda kaynaklar kullanan bir çözüm yönteminin geliştirilebilmesi mümkün görülmemektedir. Bu sorunun üstesinden gelebilmek için izlenebilecek iki yol vardır: ya yaklaşık bir çözümle yetinip bu çözümü üretebilecek bir algoritma geliştirmek yada tam olarak çözülebilecek özel alt problemleri belirlemek ve bunları çözmektir. Bu tezde ikinci yöntem izlenmiş ve bir takım ikinci derece polinom eniyileme problemleri için doğrusal matris eşitsizliği (DME) formülasyonları geliştirilmiştir. İlk olarak iki adet ikinci derce polinom eşitsizliği tarafından belirlenen bir bölgenin dışbükey zarfının DME ifadesi elde edilmiştir. Bu sayede doğrusal bir fonksiyonun bu tip bir bölge üzerindeki en iyi değerini kolayca hesaplamak mümkün olmuştur. İkinci olarak iki boyutlu bir uzayda sonlu sayıda polinom eşitsizliği tarafından belirlenen bir bölgenin dışbükey zarfının da DME olarak ifade edilebileceği gösterilmiş, fakat bu sefer DME ifadeyi bulan bir algoritma geliştirilememiştir.

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$\mathbb{S}^n$	Set of symmetric matrices of order n
$\mathbb{Z}$	Integers
$\mathbb{Z}_+$	Nonnegative integers
γ^*	Optimal objective value of an optimization problem
Δ	Uncertainty
ν	Number of negative eigenvalues of a matrix
π	Number of positive eigenvalues of a matrix
ρ^*	Optimal objective value of a relaxation
ρ_d^*	Optimal objective value of a dual relaxation
ρ_p^*	Optimal objective value of a primal relaxation
Σ	Set of sum of squares polynomials
GEV	Generalized eigenvalue
LMI	Linear matrix inequality
LTI	Linear time-invariant
MIMO	Multi-input multi-output
PSP	Polynomial system solving problem
QP	Quadratic program
SCC	Semi-convex cone
SDP	Semidefinite program
SISO	Single-input single-output
SOS	Sum of squares

1. INTRODUCTION

In the last five decades, there has been a shift in the paradigm employed for design and analysis of modern control systems. In the old classical methods, the preferred approach was to find elegant closed form analytical solutions to a given problem. However, the experience shows that this could be achieved for systems having certain structure such as linearity, time-invariance, etc., while it usually turns out to be an intractable task for more complicated system. Starting from the 1940s, some works appeared which show that it is possible to formulate a certain class of previously unsolved control problems as linear matrix inequalities (LMIs). Although LMIs cannot be solved analytically as well, they can be solved efficiently using numerical algorithms due to their convexity. This encouraged researchers to attack problems considered as hard previously from this perspective, which constitutes an active area of research today.

The S-procedure [1] and the SDP relaxation [2, 3] constitutes one of the fundamental tools that have been utilized for converting control problems into LMIs. Let's describe this briefly. Consider the set

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} p_i(x) \geq 0, & i = 1 : m_p \\ q_j(x) > 0, & j = 1 : m_q \\ t_k(x) = 0, & k = 1 : m_t \end{array} \right\},$$

where p_i, q_j, t_k are (multivariate) polynomials. Several questions in control theory can be cast as the following type of questions, which we refer to as polynomial system solving problems (PSP).

1. Is $\mathcal{S}$ empty? If not, how can we find a point of it?
2. How can we find an optimal solution of

$$\begin{array}{ll} \inf & p_0(x) \\ \text{s.t.} & x \in \mathcal{S}, \end{array}$$

where p_0 is a polynomial as well?

These problems are closely related with each other and from computational complexity theory, we know that they are decidable but NP-hard. This means there exists an algorithm that always returns the correct answer in finite time using a finite amount of resources. However, in general, they are so complex that the time required by any algorithm that can solve an arbitrary PSP is expected to grow exponentially with the problem size, which renders large problems intractable in practice. In this regard, what the S-procedure and the SDP relaxation provide is to obtain approximations or sufficient conditions in terms of LMIs that can be computed efficiently [4, 3]. Although an exact result is not guaranteed by these methods, at least one can obtain a partial answer in a reasonable amount of time.

As described above, the S-procedure and SDP relaxation, in general, only give sufficient conditions or approximations. However, there are some special occasions in which they turn out to be exact. An example of this is given by Yakubovich in [1], where it is proven that S-procedure is necessary and sufficient for showing emptiness of a set determined by two quadratic inequalities. Encouraged with this fact, researchers tried to generalize this result to more complicated problems for several years. An overview of such generalizations can be found in the recent survey paper [5]. As can be inferred from this reference, these efforts have not been very fruitful since the S-procedure is not powerful enough to tackle even a slightly bit more complicated problems.

Based on the preceding discussion, it can be seen that the S-procedure and the SDP relaxation are very conservative methods. The main reason behind this is the fact that they approach to problem in an over-simplistic manner. For example, when a feasibility problem is considered, S-procedure employs valid constraints that can be attained by taking linear combinations of the ones defining $\mathcal{S}$ in order to prove infeasibility. However, one of the fundamental results of real algebraic geometry, Positivstellensatz, states that a necessary and sufficient condition can be obtained by using multipliers which are polynomials and sum of squares (SOS) of polynomials instead of real numbers [6, 7]. This constitutes a very strong result and converts the problem of determining emptiness of $\mathcal{S}$ into an equivalent problem of searching for SOS multipliers leading to an infeasible valid constraint.

The original proof of Positivstellensatz given by Stengle [7] was not constructive. Therefore, this result has not been employed for a long time in practice. Recently, Parrilo and Lasserre realized almost simultaneously that by fixing their degrees, the search of SOS polynomials can be cast as an LMI problem [8, 9, 10]. Hence, using this approach, it would be possible to develop relaxations that can be computed efficiently. Moreover, the quality of these relaxations could be improved by increasing the degrees of polynomial multipliers. This allows one to choose between accuracy and computational effort, which is a very desirable property.

Our first contribution in this thesis is to show that if $\mathcal{S}$ is a feasible region of two quadratic inequality constraints, its convex hull, $\mathbf{co}(\mathcal{S})$, is an LMI set. The proof is constructive, and hence, gives an algorithm that produces the convex hull. Since optimizing a linear objective over $\mathcal{S}$ is equivalent to optimizing it over $\mathbf{co}(\mathcal{S})$, we convert such problems into a semidefinite program (SDP). Neither the S-procedure and SDP relaxation nor its extensions can solve this problem exactly. In order to obtain an exact result, one can try to employ SOS-based relaxation techniques since they lead to better results as described above. However, we show that for some instances of the problem considered (most probably ones having an unbounded feasible region) obtaining an accurate solution with these methods would be computationally expensive. Moreover, theoretically, still they are not guaranteed to give the exact solution for SOS polynomials having a finite degree. Available proofs guarantee convergence only in the limit (i.e. degrees of SOS polynomials do not have to be bounded) [11, 12]. The difficulty of obtaining finite degree bounds was also emphasized in [13]. Even worse, when $\mathcal{S}$ is unbounded, which may happen in our problem, convergence even in the limit is not guaranteed. As a corollary of our result, we also give a necessary and sufficient condition showing when the SDP relaxation gives the $\mathbf{co}(S)$ exactly. This gives insight into capabilities of this method.

Encouraged by the preceding result, we also asked if the convex hull of a region determined by more than two quadratic constraints is an LMI set. Unfortunately, in general, the answer is not affirmative as one can easily show by counterexamples. However, we showed that the LMI representation exists if we restrict ourselves to $\mathbb{R}^2$. But the proof is not constructive and an algorithm computing the LMI description of the convex hull could not be obtained.

The thesis is organized as follows. In Section 2, some examples of control problems that can be cast as PSPs are given to motivate the need for studying PSPs. The aim of Section 3 is to give some background material on convex analysis and LMIs. The S-procedure and the SDP relaxation are described in Section 4. In Section 5, SOS polynomials, Positivstellensatz and the moment relaxations are described, which constitutes powerful tools for dealing with PSPs. Our first contribution is introduced in Section 6, which provides a method to construct the convex hull of a region determined by two quadratic constraints. Our second contribution about convex hull of the region determined by more than two quadratic constraints is given in Section 7. The last section is conclusions where the outcomes of our work are discussed.

2. SOME CONTROL PROBLEMS THAT CAN BE EXPRESSED AS POLYNOMIAL OPTIMIZATION PROBLEMS

There are several important problems in systems and control literature that can be cast as PSPs. Absolute stability analysis, static output feedback control, fixed order controller design, μ-K_m synthesis and optimal control of system governed by polynomial dynamics are just to name a few. In below, we describe absolute stability analysis and static output feedback controller design briefly. Here, our goal is to just motivate studying PSPs. Therefore, although we will deal with the solution of PSPs in the following chapters, we will not attempt to solve all PSPs derived in this chapter.

2.1. Absolute Stability

The notion of absolute stability was developed to study the stability of linear systems having nonlinearities and/or uncertainty as the feedback connection [4, 14].

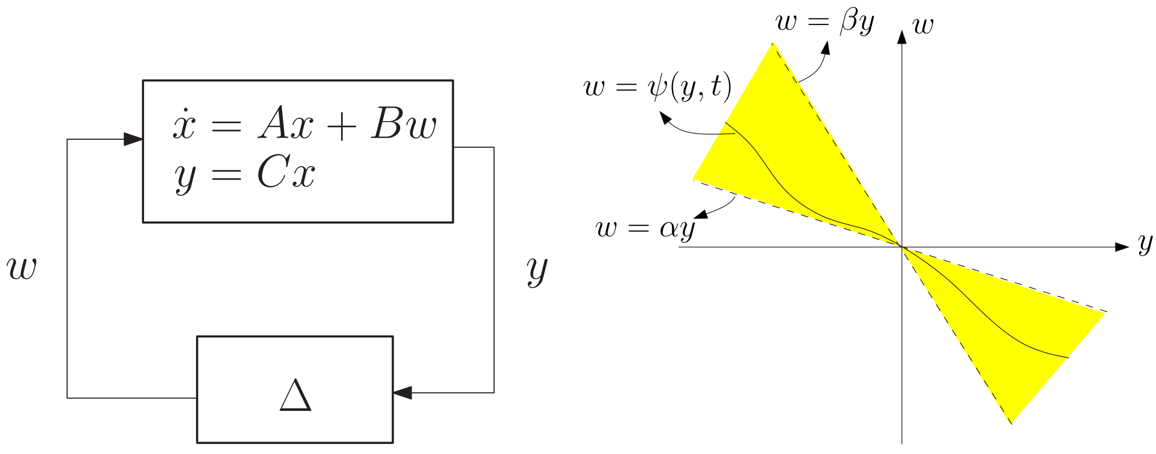
Consider the single-input single-output (SISO) linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ y &= Cx \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $w(t), y(t) \in \mathbb{R}$. This system is subject to a feedback interconnection with an uncertainty block Δ between the output y and input w as shown in Figure 2.1.a. Due to the uncertainty, for a given y, w is not determined precisely but it is known to take certain set of values. Therefore, Δ is not a function but it constitutes a relation between w and y. In standard absolute stability analysis, this relation is assumed to be determined by a quadratic form as given by

$$\Delta := \{(w, y) \in \mathbb{R}^2 |\, (\beta y - w)(w - \alpha y) \geq 0\}, \tag{2.2}$$

where α and β are real numbers. This relation is said to be a *sector* and it is a region between two lines $w = \alpha y$ and $w = \beta y$ as depicted in Figure 2.1.b. Notice that if $x = 0$, then $y = 0$, and hence, according to (2.2) $w = 0$ is the only value of w it can take. This means the origin is an equilibrium point of the closed loop system (in fact it is the only equilibrium point) and the goal of the absolute stability analysis is to determine whether this is a stable equilibrium point.



a) b)

Figure 2.1. a) A linear system having a nonlinear feedback connection; b) sector determined by the relation Δ

The most straightforward approach that can be followed to show absolute stability is to use a positive definite quadratic function

$$V(x) = x^T P x,$$

where $P \succ 0$. If we can show V is a Lyapunov function, we can ensure the stability of the system. Note that in general a system admitting a quadratic Lyapunov function is said to be quadratically stable. Clearly, showing quadratic stability is a conservative approach because even if it does not hold true, there may exist a non-quadratic Lyapunov function showing the stability.

In order to determine when V constitutes a Lyapunov function we need to find the conditions under which its time derivative is negative semidefinite along the system trajectories. The time derivative of V can be obtained as

$$\begin{aligned}
\dot{V}(x) &= x^T P\dot{x} + \dot{x}^T Px \\
&= x^T P(Ax + Bw) + (Ax + Bw)^T Px \\
&= x^T (PA + A^T P)x + x^T PBw + w^T B^T Px \\
&= \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} =: q_0,
\end{aligned} \tag{2.3}$$

which is a quadratic function of x and w. Since $(w, y) \in \Delta$, in above w and x are not free but they are related with the constraint

$$q_1 := (\beta Cx - w)(w - \alpha Cx) \geq 0$$

which is obtained by using the substitution $y = Cx$ in the inequality defining Δ. From this, we can deduce that the closed loop system is quadratically stable if and only if there exist a $P \succ 0$ such that

$$-q_0(w, x) \geq 0 \text{ for every } (w, x) \text{ satisfying } q_1(w, x) \geq 0. \tag{2.4}$$

Clearly, this condition leads to the following proposition.

Proposition 2.1.1 ([4, 14]). *The system (2.1) subject to (2.2) is quadratically stable if and only if there exists a $P \succ 0$ for which the constraints*

$$q_0(w, x) > 0, \quad q_1(w, x) \geq 0 \tag{2.5}$$

do not have a feasible solution.

This result shows that for a given Lyapunov function candidate, equivalently P, one can prove absolutely stability of the closed loop system by showing infeasibility of (2.5), which is nothing but a PSP problem.

Although the method described above is developed for the system depicted in Figure 2.1 which has a feedback uncertainty, it can also employed to show stability when the uncertain connection Δ is replaced by a possibly time-varying nonlinear feedback law $w = \psi(y,t)$ satisfying $\psi(0,t) = 0$ for every $t \in \mathbb{R}$. Here the idea is to obtain a sector as in Figure 2.1.b that contains the graph of $\psi(y,t)$ for every t. In this case, if (2.4) holds true for the sector of choice, then one can ensure the stability of the closed loop nonlinear system since absolute stability implies the stability for every kind of input out pairs (w, y) residing in the sector. Clearly, this introduce some extra conservatism since we show the stability not only for the inputs and outputs satisfying the nonlinear relation $w = \psi(y,t)$ but also for other ones lying in the sector.

Example 2.1.2. *Consider the system depicted in Figure 2.2.a. The state space realization of the feedforward transfer function is*

$$\begin{aligned} \dot{x} &= -x + w \\ y &= x \end{aligned}$$

This system has a feedback connection with saturation nonlinearity which can be expressed as

$$w = \psi(y) = \begin{cases} -y, & |y| \leq 1, \\ -\operatorname{sgn}(y), & |y| > 1. \end{cases}$$

As shown in Figure 2.2.b., this nonlinear function lies in the sector determined by the constraint

$$-(y + w)w \geq 0$$

which is obtained by choosing $\alpha = 0$ and $\beta = -1$.

Let us choose the Lyapunov function candidate as $V(x) = x^2$ (i.e. P=1). Due to Proposition 2.1.1, the closed loop system is stable if there does not exist a point

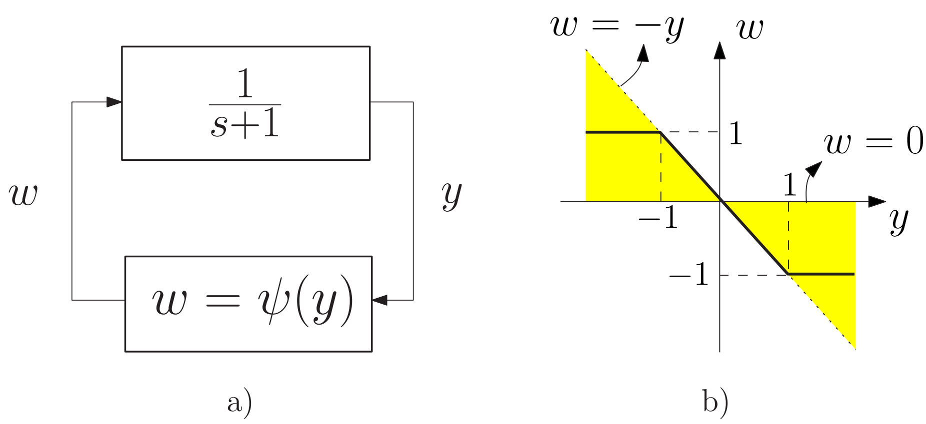


Figure 2.2. a) A linear system with nonlinear feedback connection; b) the nonlinear feedback function ψ and the sector containing it

satisfying

$$q_0(w, x) = -2x^2 + 2xw > 0,$$
$$q_1(w, x) = -(x + w)w \geq 0.$$

The first inequality implies $xw > x^2$ *while the second implies* $-w^2 \geq xw$. *This means* $-w^2 > x^2$, *which is a contradiction. Therefore the above given constraints are infeasible, and hence, the closed loop system is stable.*

Notice that in the preceding example we could check infeasibility of (2.5) by hand because the system considered was simple. In general, however, this is not an easy task. Fortunately, there are two methods that can be employed for more complicated problems. The first one is the circle criterion which is a graphical method based on the Nyquist plot of the feedforward transfer function. Since we are not interested with this method, the reader is referred to [14] for further information. The second approach is to use S-Procedure which gives a necessary and sufficient LMI condition for feasibility of a pair quadratic constraints. This method will be explained in Chapter 4 in detail.

So far, we have established absolute stability when there is a single feedback connection. This method can be generalized to the systems having multiple feedback

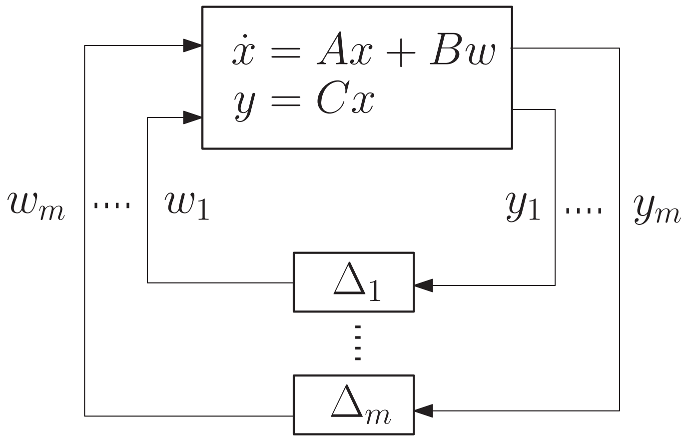


Figure 2.3. A system with multiple uncertain feedback connections

connections as well. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ y &= Cx \end{aligned} \tag{2.6}$$

where $x(t) \in \mathbb{R}^n$ and this time $w(t), y(t) \in \mathbb{R}^m$ (i.e. there are m inputs and outputs). Each output y_i is fed back to the corresponding input w_i through an uncertain channel Δ_i as shown in Figure 2.3. These connections are governed by the relations

$$\Delta_i := \{(w, y) \in \mathbb{R}^2 |\, (\beta_i y - w)(w - \alpha_i y) \geq 0\},\ i = 1 : m \tag{2.7}$$

having separate parameters $\alpha_i, \beta_i \in \mathbb{R}$.

The the conditions for absolute stability of the MIMO described above system can be obtained by following the steps we describe previously for the SISO system. Clearly, the time derivative of V will be the same, and hence, we use the same quadratic function q_0 defined in (2.3) except that the variable w appearing there is an m dimensional vector instead of a scalar. Moreover, each element of w is related with the state vector x through the constraints

$$q_i := (\beta_i Cx - w)(w - \alpha_i Cx) \geq 0,\ i = 1 : m.$$

Putting these facts together, we can infer that the closed loop system defined by (2.6) and (2.7) is quadratically stable if and only if there exists a $P \succ 0$ such that

$$-q_0(w,x) \geq 0 \text{ for every } (w,x) \text{ satisfying } q_i(w,x) \geq 0,\ i = 1:m,$$

which leads to the following result.

Proposition 2.1.3 ([4, 14])**.** *The system (2.6) subject to (2.7) is quadratically stable if and only if there exists a $P \succ 0$ for which the constraints*

$$q_0(w,x) > 0; \quad q_i(w,x) \geq 0,\ i = 1:m$$

do not have a feasible solution.

Again, for a given Lyapunov function candidate, we converted the problem of checking quadratic stability into the question of infeasibility of a system of quadratic constraints, which is a PSP. This time, however, there are more than two constraints as opposed to the SISO case. As we will see this leads to serious complications since there does not exist an efficient method that is guaranteed to solve infeasibility problem when there are more than two constraints. Indeed, as mentioned in [4], absolute stability analysis with diagonal uncertainties, which is a special case of the problem considered here, is NP-Hard. In this regard, one thing that can be done to deal with this difficulty is to use sufficient conditions provided by the S-Procedure and the relaxation methods described in the following chapters to obtain certificates. Although such an approach is not guaranteed to solve the problem, at least we will have an efficient method that may work for some problems.

2.2. Static Output Feedback Controller Design

Another important problem that can be converted into PSP is the static output feedback problem. In this problem, we are given an LTI system with the state space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx, \tag{2.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. This system is subject to a static linear output feedback law

$$u = -Ky + w \tag{2.9}$$

as shown in Figure 2.4, where $K \in \mathbb{R}^{m\times p}$ and $w \in \mathbb{R}^m$ is the external input. The state equation of the closed loop configuration composed of (2.8) and (2.9) is obtained as

$$\dot{x} = (A - BKC)x + Bw. \tag{2.10}$$

The goal of static output feedback design is to find a feedback matrix K that makes (2.10) stable if it is possible.

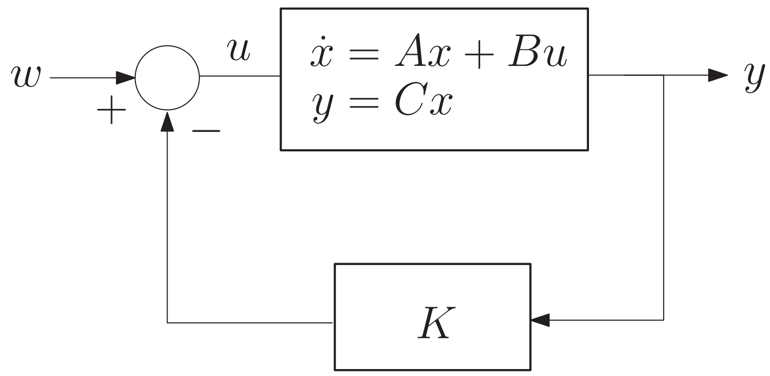


Figure 2.4. An LTI system subject to static linear output feedback

To this end, let's denote the transfer function corresponding to the state space realization (2.8) as G. That is

$$G(s) = C(sI - A)^{-1}B.$$

The following lemma states that under mild assumptions of stabilizability and detectability, an equivalent condition for the internal stability of the system depicted in Figure 2.4 can be obtained in terms of the transfer function G.

Lemma 2.2.1 ([15]). *Assume the state space model given in (2.8) is stabilizable and detectable. Then, the closed loop autonomous system (2.10) is stable if and only if all roots of the characteristic polynomial*

$$\det[\,I + KG(s)]$$

lies on the open left hand plane.

This lemma constitutes a special case of the Lemma 5.5 given in [15] and a proof can be found in this reference. As a consequence of this result, in order to determine the internal stability, we can apply the Routh-Hurwitz test to the polynomial $\det[\,I + KG(s)]$. Since the coefficients of it are polynomial functions of the elements of K, we obtain a set of polynomial constraints in these elements which is feasible if and only if the system in hand is stablizable with the static output feedback and a feasible solution gives the controller achieving the stability.

Example 2.2.2. *Assume the feedforward system depicted in Figure 2.4 has a transfer function matrix*

$$G(s) = \begin{bmatrix} \dfrac{s+1}{s^2 - 2s + 2} \\ \\ \dfrac{1}{s} \end{bmatrix}$$

which is known to arise from a stabilizable and detectable realization. Under static output feedback with the controller matrix $K = [k_1\ k_2]$, the characteristic equation corresponding to the closed loop system can be obtained as

$$1 + KG(s) = s^3 + (k_1 + k_2 - 2)s^2 + (2 + k_1 - 2k_2)s + 2k_2 = 0.$$

By applying the Routh-Hurtiwz test, we can see that K stabilizes the system if and only *if the following set of constraints has a solution*

$$\begin{aligned} &k_1 + k_2 - 2 > 0 \\ &k_2 > 0 \\ &k_1^2 - 2k_2^2 - k_1 k_2 + 4k_2 - 4 > 0. \end{aligned}$$

As can be verified easily, the point $K = [5\ 1]$ *satisfies these constraints and constitutes a stabilizing controller.*

In the preceding example obtaining the feasible solution was easy since the system considered was very simple. For more complicated system finding the stabilizing controller (if it exists) by hand turns out to be an intractable task. Therefore, one needs to develop a systematic method for dealing with PSPs.

2.3. Summary

In this section, we described two sample control problems, namely absolute stability analysis and static output feedback design, which can be cast as PSP. This shows us importance of solving PSPs in control theory. In order the explain the main idea, the examples were kept simple, and hence, could be solved by hand. However, in general, the PSPs arising from such problems cannot be solved in this manner. One needs to develop systematic methods. In the following chapters, we investigate such methods but we will not try to solve all problems investigated in this chapters completely. Here, our goal was to just show importance of studying PSPs.

3. CONVEX ANALYSIS AND LINEAR MATRIX INEQUALITIES

The aim of this chapter is to provide some background material that will be utilized through the thesis. First we will introduce some fundamental results from the convex analysis. Then, linear matrix inequalities and semidefinite programming will be investigated briefly.

3.1. Convex Analysis

In what follows, $\mathcal{S}$ is a subset of $\mathbb{R}^n$.

Definition 3.1.1. *$\mathcal{S}$ is said to be convex if for any $x, y \in \mathcal{S}$, $(1-\lambda)x + \lambda y \in \mathcal{S}$ for every $\lambda \in [0, 1]$.*

In other words, $\mathcal{S}$ is convex if and only if the line segments connecting any pair of points lying in $\mathcal{S}$ is a subset of this set. This is depicted in Figure 3.1.a and Figure 3.1.b for convex and nonconvex sets respectively.

Definition 3.1.2. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ defined over a domain $\mathcal{S}$. f is said to be convex if $\mathcal{S}$ is convex and $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$ for every $x, y \in \mathcal{S}$ and $\lambda \in [0, 1]$.*

Convexity plays a critical role in optimization theory. This is because usually the

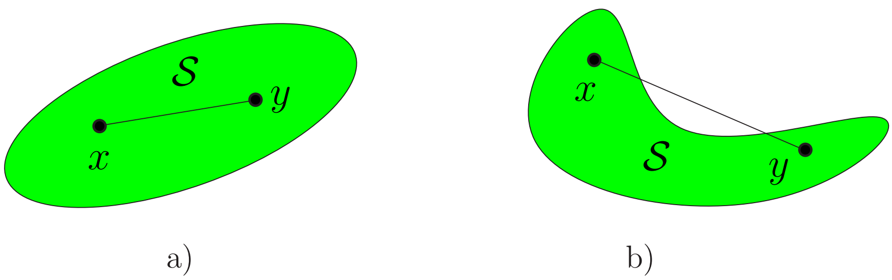


Figure 3.1. a) Convex set; b) nonconvex set

problem of optimization a convex function over a convex set can be solved efficiently using iterative algorithms.

There are some operations under which convexity is preserved. In below, we give two of them, which will be used in our developments. For a more complete list, the reader is referred to [16, 17].

Proposition 3.1.3 ([17])**.** *If $\mathcal{S}$ is convex, then its projection on to the m dimensional space*

$$\mathcal{P}_{\mathcal{S}} := \{x \in \mathbb{R}^m |\ \exists y \in \mathbb{R}^{n-m} \text{ such that } (x, y) \in \mathcal{S}\}$$

is also convex.

Proposition 3.1.4 ([17])**.** *Intersection of a collection of convex sets is convex.*

Another notion that we will make use of is convex hull. Due to the preceding proposition, the intersection of all convex sets containing a possibly nonconvex set $\mathcal{S}$ is also convex. Clearly, this intersection set is the smallest convex set containing $\mathcal{S}$.

Definition 3.1.5. *Convex hull of $\mathcal{S}$, $\mathbf{co}(\mathcal{S})$, is defined as the smallest convex set containing $\mathcal{S}$.*

Surprisingly, the closure of the convex hull can be characterized using intersection of halfspaces instead of arbitrary convex sets.

Proposition 3.1.6 ([17])**.** *The intersection of all closed halfspaces containing $\mathcal{S}$ is $\overline{\mathbf{co}(\mathcal{S})}$.*

There is also another version of this result in terms of supporting halfspaces.

Definition 3.1.7. *Consider a hyperplane $\mathcal{T} \subseteq \mathbb{R}^n$. $\mathcal{T}$ is said to be a supporting hyperplane of $\mathcal{S}$ if it has a common point with the boundary of $\mathcal{S}$ and one of the closed halfspaces determined by $\mathcal{T}$ contains $\mathcal{S}$. The closed halfspace containing $\mathcal{S}$ is defined as a supporting halfspace of $\mathcal{S}$.*

Proposition 3.1.8 ([17]). *The intersection of all supporting halfspaces of $\mathcal{S}$ is $\overline{\mathbf{co}(\mathcal{S})}$.*

Another characterization of convex hull that will be utilized in the thesis is given by Carathéodory's theorem.

Theorem 3.1.9 ([17]).

$$\mathbf{co}(\mathcal{S}) = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{n+1} \lambda_i y_i,\ \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \geq 0,\ y_i \in \mathcal{S},\ i = 1 : n+1 \right\}. \tag{3.1}$$

Apart from its theoretical importance, the notion of convex hull provides an insight that helps us to develop practical solutions to nonconvex optimization problems. Indeed, one of the main approaches in nonconvex global optimization, convex outer approximation (or relaxation) techniques [18], is based on this insight. Let us describe this briefly.

Consider the optimization problem

$$\begin{aligned} \gamma^* := \ & \inf && \ell(x) \\ & \text{s.t.} && x \in \mathcal{S}, \end{aligned} \tag{3.2}$$

where $\ell : \mathbb{R}^n \to \mathbb{R}$ is a linear function. In general, solving this kind of a problem would be hard when $\mathcal{S}$ is not convex. In order to alleviate this difficulty, one may employ a set $\mathcal{C} \subseteq \mathbb{R}^n$ which is convex and satisfies $\mathcal{S} \subseteq \mathcal{C}$ and deal with the optimization problem

$$\begin{aligned} \rho^* := \ & \inf && \ell(x) \\ & \text{s.t.} && x \in \mathcal{C}. \end{aligned} \tag{3.3}$$

Since the preceding problem is convex, it can be solved efficiently if one can obtain a suitable description of $\mathcal{C}$. However, because optimization is performed over a larger set, clearly, ρ^* constitutes a lower bound on γ^* which may not turn out to be exact. The convex problem (3.3) is said to be a convex relaxation of (3.2) and the difference between γ^* and ρ^* is called the relaxation gap. At this point one can naturally ask if

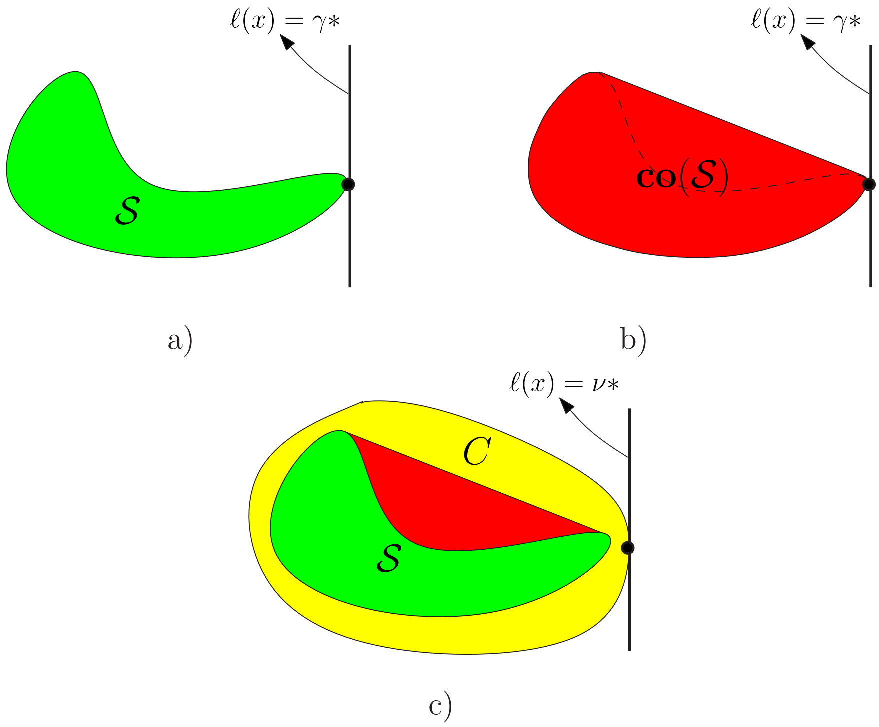


Figure 3.2. a) $\mathcal{S}$; b) convex hull of $\mathcal{S}$; c) a convex relaxation for $\mathcal{S}$

constructing relaxations constitutes a good approach for solving nonconvex problems. The following proposition shows that, in principle, the relaxation gap can be made arbitrarily small and it provides a geometric criterion for obtaining good lower bounds.

Proposition 3.1.10. *The following holds true*

$$\begin{aligned} \gamma^* = \quad & \inf \quad \ell(x) \\ & \text{s.t.} \quad x \in \mathbf{co}(\mathcal{S}). \end{aligned}$$

As can be inferred from the proposition, infimizing a linear function over $\mathcal{S}$ is equivalent to optimizing it over $\mathbf{co}(\mathcal{S})$. This is illustrated in Figure 3.2.a and b. Based on this fact, it can be inferred that γ^* can be approximated by ρ^* as much as desired by choosing $\mathcal{C}$ close enough to $\mathbf{co}(\mathcal{S})$ (see Figure 3.2.c). However, in general, although solving the relaxed problem itself is an easy task, constructing a useful description of $\mathcal{C}$ which closely approximates $\mathbf{co}(\mathcal{S})$ usually turns out to be intractable. In this regard, the most valuable advantage of relaxation method is that it offers a trade off between the computational efficiency and accuracy.

3.2. Linear Matrix Inequalities and Semidefinite Programming

Let us denote the set of all symmetric matrices of order m as $\mathbb{S}^m$. Consider a linear matrix valued function $F : \mathbb{R}^n \to \mathbb{S}^m$

$$F(x) := F_0 + F_1 x_1 + \cdots + F_n x_n,$$

where $F_i \in \mathbb{S}^m$, $i = 0 : n$. A linear matrix inequality problem is to find an $x \in \mathbb{R}^n$ such that $F(x)$ is positive semidefinite, that is

$$F(x) \succeq 0. \tag{3.4}$$

On the other hand, a semidefinite program is defined as the optimization problem

$$\begin{aligned} \gamma^* := \quad & \inf \quad \ell(x) \\ & \text{s. t.} \quad F(x) \succeq 0, \end{aligned} \tag{3.5}$$

where $\ell = \ell_1 x_1 + \cdots + \ell_n x_n$ is a linear real valued function.

Using the definitions introduced in the previous section, it is easy to see that LMIs and SDPs are convex problems. Indeed, if $F(x_1) \succeq 0$ and $F(x_2) \succeq 0$ for $x_1, x_2 \in \mathbb{R}^n$, then

$$F((1-\lambda)x_1 + \lambda x_2) = (1-\lambda)F(x_1) + \lambda F(x_2) \succeq 0$$

for every $\lambda \in [0, 1]$. Therefore, the feasible region of an LMI is a convex set. The convexity of SDP follows from this in addition to the fact that a linear function is always convex. Due to the convexity property, LMIs can be solved efficiently using interior point methods [16].

Notice that in above, we used only a single matrix inequality to define LMI and SDP problems. This is because multiple matrix inequalities can be cast as a single matrix inequality using diagonalization. For instance, two constraints $F_1(x) \succeq 0$, $F_2(x) \succeq 0$ are equivalent to

$$F(x) = \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} \succeq 0.$$

A direct consequence of the diagonalization is the fact that LMIs and SDP subsumes linear programming problems as a special case since multiple linear inequalities can be converted into an LMI using this idea. Moreover, it is also known that quadratically constrained convex quadratic programming also constitutes special a case of SDP [19].

The LMI problem of finding an $x \in \mathbb{R}^n$ satisfying (3.4) has an associated dual problem of finding a $W \in \mathbb{S}^m$ such that

$$\begin{aligned} &\mathrm{tr}(WF_i) = 0,\ i = 1:m, \\ &\mathrm{tr}(WF_0) < 0, \\ &W \succeq 0, \end{aligned} \tag{3.6}$$

which can be obtained using Lagrange duality. Due to convexity, these problems are strong dual of each other. Therefore, under a mild constraint qualification assumption (see [16]) (3.4) is feasible if and only if (3.6) is infeasible. Similarly, the SDP problem (3.5) has the dual problem

$$\begin{aligned} \zeta^* := \ \sup \ & \mathrm{tr}(WF_0) \\ \mathrm{s.\,t.} \ & \mathrm{tr}(WF_i) = \ell_i,\ i = 1:m \\ & W \succeq 0. \end{aligned}$$

Under a similar constraint qualification assumption there is no relaxation gap between the primal and the dual problems, and hence, $\gamma^* = \zeta^*$.

Lastly, let us mention about a method that is employed to convert a class of nonlinear matrix inequalities into LMIs.

Proposition 3.2.1 ([16]). *The following holds true* *i)*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \tag{3.7}$$

if and only if

$$A \succ 0,\ C - B^T A^{-1} B \succeq 0; \tag{3.8}$$

ii)

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \tag{3.9}$$

if and only if

$$A \succeq 0,\ (I - AA^+)B = 0,\ C - B^T A^+ B \succeq 0, \tag{3.10}$$

where A^+ is the Moore-Penrose inverse of A.

The identities given in the preceding proposition is known as Schur formula. Suppose that A, B and C are linear matrix valued functions. Clearly, the constraints (3.8) and (3.10) are quadratic matrix inequalities, and hence, they are nonconvex. However, using Schur formula, they can be converted into LMIs (3.7) and (3.9), respectively.

3.3. Summary

The goal of this chapter was to introduce some basic concepts from convex analysis and optimization. The notions of convexity and convex hull are first defined and a number of different characterizations of the convex hull of a set are given which will be heavily used in the following chapters. Moreover, the concept of convex relaxation and its relation with the convex hull are described. Then, LMIs and SDP, which will be utilized to develop convex relaxations, are formally defined and their important properties are presented. Lastly, Schur formula is introduced which will be employed to convert some nonconvex constraints into LMIs in the subsequent chapters.

4. S-PROCEDURE AND THE SDP RELAXATION

In this section, we will describe the S-procedure and the SDP relaxation. The goal of these methods is to get some answers to following questions, which are PSPs. Consider the set

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} p_i(x) \geq 0,\ i = 1 : m_p, \\ q_j(x) > 0,\ j = 1 : m_q, \\ t_k(x) = 0,\ k = 1 : m_t \end{array} \right\}, \tag{4.1}$$

where $p_i,\ q_j,\ t_k : \mathbb{R}^n \to \mathbb{R}$ are quadratic functions that can be expressed as

$$p_i := \begin{bmatrix} x \\ 1 \end{bmatrix}^T P_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad i = 1 : m_p,$$

$$q_j := \begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad j = 1 : m_q,$$

$$t_k := \begin{bmatrix} x \\ 1 \end{bmatrix}^T T_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad k = 1 : m_t,$$

using the symmetric matrices $P_i,\ Q_j,\ T_k \in \mathbb{S}^{n+1}$.

1. Feasibility: Is $\mathcal{S}$ empty?
2. Optimization: How can we solve

$$\begin{array}{ll} \inf & p_0(x) \\ \text{s.t.} & x \in \mathcal{S}? \end{array}$$

Since, these problem are hard as we mention earlier, in general, one should not expect to obtain an exact solution efficiently. Instead, LMI-based results will be obtained which are conservative but can be computed efficiently. However, there are some special instances of these problems for which the S-procedure and the SDP relaxation turn out to be exact. We will also give such results available in the literature.

4.1. S-Procedure

The origin of S-procedure, at least in the realm of control theory, dates back to the works of Lure in 1940's. It was extensively utilized by him and the followers in absolute stability analysis and related problems [4]. Later on, it also found applications in other areas of engineering [20, 21, 22, 23]. This method basically gives certificates (sufficient conditions) that can be utilized to check emptiness of $\mathcal{S}$ defined in (4.1).

The main idea of the S-procedure is based on the classical Lagrange duality. Consider the whole collection of constraints

$$\sum_{i=1}^{m_p} \lambda_i p_i(x) + \sum_{j=1}^{m_q} \mu_j q_j(x) + \sum_{k=1}^{m_t} \xi_k t_k(x) > 0, \tag{4.2}$$

which can be obtained using multiplier vectors $\lambda \in \mathbb{R}^{m_p}$, $\mu \in \mathbb{R}^{m_q}$, $\xi \in \mathbb{R}^{m_t}$ such that $\lambda \geq 0$, $\mu \geq 0$, $\mu \neq 0$. It is straightforward to verify that all of these constraints are satisfied for every $x \in \mathcal{S}$. Therefore, if we can show that at least one of them is infeasible, this automatically implies the emptiness of $\mathcal{S}$. However, if the collection (4.2) does not contain an infeasible constraint, we cannot conclude anything. Hence, this constitutes a sufficient condition, which is a weak duality.

Now, we ask how we can find a nonegative multipliers certificating the emptiness of $\mathcal{S}$ as described above if they exist. Clearly, infeasibility of (4.2) is equivalent to

$$\sum_{i=1}^{m_p} \lambda_i p_i(x) + \sum_{j=1}^{m_q} \mu_j q_j(x) + \sum_{k=1}^{m_t} \xi_k t_k(x) \leq 0, \forall x \in \mathbb{R}^n.$$

The function on the left hand side of the above given inequality is quadratic and it can be explicitly written as

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(\sum_{i=1}^{m_p} \lambda_i P_i + \sum_{j=1}^{m_q} \mu_j Q_j + \sum_{k=1}^{m_t} \xi_k T_k \right) \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \quad \forall x \in \mathbb{R}^n.$$

It is a well known fact that a quadratic function is negative semidefinite if and only if the associated matrix is so. This leads to the inequality

$$\sum_{i=1}^{m_p} \lambda_i P_i + \sum_{j=1}^{m_q} \mu_j Q_j + \sum_{k=1}^{m_t} \xi_k T_k \preceq 0.$$

Therefore, $\mathcal{S}$ is empty if there exist $\lambda \geq 0$, $\mu \geq 0$, $\mu \neq 0$ and $\xi \in \mathbb{R}^{m_t}$ satisfying above given inequality. As a result we obtained an LMI certificate for infeasibility. This is the S-procedure and it is stated in the following proposition formally.

Proposition 4.1.1 (S-procedure [3])**.** *$\mathcal{S}$ is empty if the following LMI has a feasible solution*

$$\begin{gathered} \lambda \geq 0, \quad \mu \geq 0, \quad \mu \neq 0 \\ \sum_{i=1}^{m_p} \lambda_i P_i + \sum_{j=1}^{m_q} \mu_j Q_j + \sum_{k=1}^{m_t} \xi_k T_k \preceq 0. \end{gathered} \tag{4.3}$$

Remark 4.1.2. *Due to the inequation $\mu \neq 0$, one may object that (4.3) is an LMI. However, the two conditions $\mu \geq 0$, $\mu \neq 0$ is equivalent to the pair of inequalities $\mu \geq 0$, $\sum_{i=1}^{r} \mu_i > 0$. Hence, (4.3) is an LMI.*

Remark 4.1.3. *The preceding proposition is derived for the description given in (4.1) assuming that it has at least one nonstrict inequality, one strict inequality and one equality constraints. However, the description of $\mathcal{S}$ may not include all these types of constraints in some applications. In this case, one can add one or more of the following trivial identities into (4.1) without altering $\mathcal{S}$*

$$0 \geq 0, \quad 1 > 0, \quad 0 = 0$$

which can be obtained by choosing $p_1 = 0$, $q_1 = 1$ and $t_1 = 0$, respectively. By substi-

tuting the matrices associated with these polynomials into (4.3), one comes up with the desired relaxation.

Example 4.1.4. *Suppose we would like to check if the set of constraints*

$$\begin{aligned} p_1 &= x^2 - y^2 - 4 \geq 0 \\ p_2 &= 4 - 4x^2 - y^2 \geq 0 \\ t_1 &= 2x - 2y = 0 \end{aligned} \tag{4.4}$$

is infeasible. Let us add the trivial inequality $1 > 0$ *to this system in order to apply Proposition 4.1.1. From (4.3), it can be seen that these constraints is infeasible if there exists* $\lambda_1, \lambda_2 \geq 0$, $\mu_1 > 0$ *and* $\xi_1 \in \mathbb{R}$ *such that*

$$\lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
$$+\mu_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \xi_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \preceq 0.$$

Clearly, this is satisfied for $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 1$, $\xi_1 = 0$. *Hence, it can be inferred that (4.4) is infeasible.*

Example 4.1.5. *Now suppose* $\mathcal{S}$ *is determined by the constraints*

$$\begin{aligned} p_1(x) &= 4 - x^2 - y^2 \geq 0 \\ p_2(x) &= -x^2 + y^2 + 1 \geq 0 \\ p_3(x) &= -2y \geq 0 \\ q_1(x) &= 2x + 2y - 3 > 0. \end{aligned}$$

Using the S-procedure one can obtain

$$\lambda \geq 0, \quad \mu_1 > 0,$$
$$\lambda_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$+\lambda_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -3 \end{bmatrix} \preceq 0.$$

Unfortunately, although $\mathcal{S} = \emptyset$, the above given LMI turns out to be infeasible. Hence, we cannot prove the emptiness of $\mathcal{S}$ with this method. This shows conservatism of the S-procedure.

The second use of the S-procedure is computing lower bounds on the optimal value of the optimization problem

$$\begin{aligned} \gamma^* := \quad & \inf \quad p_0(x) \\ & \text{s.t.} \quad x \in \mathcal{S}, \end{aligned} \tag{4.5}$$

where γ^* denotes the optimal objective value. Let us describe this briefly.

Trivially, the following identity holds true

$$\begin{aligned} \gamma^* = \quad & \sup \quad t \\ & \text{s.t.} \quad p_0(x) - t \geq 0 \quad \forall x \in \mathcal{S}. \end{aligned} \tag{4.6}$$

Notice that the constraint of this new problem is the nonnegativity of the quadratic function

$$p_0 - t = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(P_0 - \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix},$$

over $\mathcal{S}$. This nonnegativity condition is satisfied if and only if the set

$$\begin{aligned}\bar{\mathcal{S}} := \;& \mathcal{S} \cap \{x \in \mathbb{R}^n | p_0(x) - t < 0\} \\ = \;& \{x \in \mathbb{R}^n |\, p_i(x) \geq 0,\; i = 1 : m_p,\; q_j(x) > 0,\; j = 1 : m_q, \\ & \quad t - p_0(x) > 0, t_k(x) = 0,\; k = 1 : m_t\}\end{aligned} \tag{4.7}$$

is empty. Hence, using S-procedure, we can replace it with a sufficient LMI condition and obtain the following relaxation.

Proposition 4.1.6. *Consider the SDP*

$$\begin{aligned}\rho_d^* := \;& \sup \quad t \\ & \text{s. t.} \quad P_0 \succeq \sum_{i=1}^{m_p} \lambda_i P_i + \sum_{j=1}^{m_q} \mu_j Q_j + \sum_{k=1}^{m_t} \xi_k T_k + \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \\ & \qquad\;\; \lambda,\, \mu \geq 0.\end{aligned} \tag{4.8}$$

ρ_d^* gives a lower bound on γ^*. That is, $\rho_d^* \leq \gamma^*$.

Proof. Due to Proposition 4.1.1, $\bar{\mathcal{S}} = \emptyset$ if there exists $\lambda \geq 0$, $(\mu, \bar{\mu}) \geq 0$, $(\mu, \bar{\mu}) \neq 0$, $t \in \mathbb{R}$ such that

$$\sum_{i=1}^{m_p} \lambda_i P_i + \sum_{j=1}^{m_q} \mu_j Q_j + \bar{\mu} \left(\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} - P_0 \right) + \sum_{k=1}^{m_t} \xi_k T_k \preceq 0.$$

By choosing $\bar{\mu} = 1$ and replacing the constraint of (4.6) with the resulting LMI, (4.8) is attained. Since the constraint of (4.6) is replaced with a sufficient condition, we have $\rho_d^* \leq \gamma^*$. □

Example 4.1.7. *Consider the optimization problem*

$$\begin{aligned}\gamma^* = \;& \inf \quad -2x_1 - 2x_2 \\ & \text{s. t.} \quad 4 - x_1^2 - x_2^2 \geq 0 \\ & \qquad\;\; -x_1^2 + x_2^2 + 1 \geq 0 \\ & \qquad\;\; -2x_2 \geq 0\end{aligned}$$

The corresponding relaxation is

$$\begin{aligned}\rho_d^* = \ & \sup t \\ \text{s. t.} & \begin{bmatrix} \lambda_1+\lambda_2 & 0 & -1 \\ 0 & -\lambda_2+\lambda_1 & \lambda_3-1 \\ -1 & \lambda_3-1 & -4\lambda_1-\lambda_2-t \end{bmatrix} \succeq 0.\end{aligned}$$

Solving this SDP the lower bound is found to be $\rho_d^* = -3.1623$ *whereas the global optimal solution is* $\gamma^* = -2$. *Therefore, the relaxation is not exact.*

Remark 4.1.8. *As can be seen from the foregoing examples the* S*-procedure and the associated relaxation developed for optimization problems lead to conservative results in general.*

Before finishing this section let us mention about a drawback of the S-procedure. It is a dual method, and thus, it does not produce a solution vector in the primal space at all. For instance, the decision variables of (4.8) are t and the multipliers λ, μ and ξ. Therefore, even if we obtain an exact result, we cannot recover the point $x^* \in \mathcal{S}$ leading to this solution since the multipliers are in the dual space. The only thing we have the optimal objective value. This may lead to difficulties for the problems in which the the optimal solution vector x^* of (4.5) is required.

4.2. SDP Relaxation

SDP relaxation provides an alternative perspective to deal with the questions invoked at the beginning of this chapter. Its main idea is to manipulate the objective and constraints in hand in order to obtain a new problem in a higher dimensional space, called the lifted problem. This lifted problem is an SDP or an LMI which produces a bound or a certificate that can be computed efficiently just as in the S-procedure.

Since it will be more convenient, we will start describing the SDP relaxation on the optimization problem

$$
\begin{aligned}
\gamma^* := \quad \inf \quad & p_0(x) \\
\text{s.t.} \quad & p_i(x) \geq 0, \;\; i = 1 : m_p, \\
& q_j(x) > 0, \;\; j = 1 : m_q, \\
& t_k(x) = 0, \;\; k = 1 : m_t.
\end{aligned}
\tag{4.9}
$$

The relaxation simply follows from the following two observations. First, trivially, the matrix inequality

$$
\begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x^T & 1 \end{bmatrix} = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \succeq 0
$$

is satisfied for all $x \in \mathbb{R}^n$. Hence, if it is used as an additional constraint in (4.9), the problem is not affected. Second, consider a quadratic polynomial p with the associated matrix P. Employing the trace operator, it can be equivalently expressed as

$$
\begin{aligned}
p \; &= \operatorname{tr}\left(\begin{bmatrix} x \\ 1 \end{bmatrix}^T P \begin{bmatrix} x \\ 1 \end{bmatrix} \right) \\
&= \operatorname{tr}\left(P \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x^T & 1 \end{bmatrix} \right) \\
&= \operatorname{tr}\left(P \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \right),
\end{aligned}
\tag{4.10}
$$

where we used the fact that, for two matrices A and B, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Based on these facts, it is easy to see that the following is equivalent to (4.9)

$$
\begin{aligned}
\gamma^* = \quad \inf \quad & \operatorname{tr}\left(P_0 \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \\
\text{s.t.} \quad & \operatorname{tr}\left(P_i \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \geq 0, \quad i = 1 : m_p, \\
& \operatorname{tr}\left(Q_j \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) > 0, \quad j = 1 : m_q, \\
& \operatorname{tr}\left(T_k \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) = 0, \quad k = 1 : m_t, \\
& \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \succeq 0, \\
& Y = xx^T,
\end{aligned}
\tag{4.11}
$$

where $Y \in \mathbb{S}^n$. As one can anticipate, the above given problem is nonconvex and not easier than the original one due to the nonlinear constraint $Y = xx^T$. However, if this constraint is removed, one attains an SDP problem whose optimal value has to be smaller than γ^* since the optimization is performed over a larger set. This is the SDP relaxation.

Proposition 4.2.1 (SDP relaxation [2]). *The optimal value of the SDP relaxation*

$$
\begin{aligned}
\rho_p^* := \quad \inf \quad & \operatorname{tr}\left(P_0 \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \\
s.t. \quad & \operatorname{tr}\left(P_i \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \geq 0, \quad i = 1 : m_p, \\
& \operatorname{tr}\left(Q_j \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) > 0, \quad j = 1 : m_q, \\
& \operatorname{tr}\left(T_k \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) = 0, \quad k = 1 : m_t, \\
& \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \succeq 0,
\end{aligned}
\tag{4.12}
$$

gives a lower bound on the optimal value of (4.9), that is, $\rho_p^* \leq \gamma^*$.

Keep in mind that, in general, the bound provided by the SDP relaxation is very conservative just like the S-procedure.

It remains to derive the SDP relaxation for feasibility problems, which is an infeasibility certificate. In fact, this certificate has already obtained in above. It is simply the constraints of (4.12).

Proposition 4.2.2. *S is empty if*

$$
\begin{aligned}
&\operatorname{tr}\left(P_i\begin{bmatrix} Y & x \\ x^T & 1\end{bmatrix}\right) \geq 0, \quad i = 1 : m_p,\\
&\operatorname{tr}\left(Q_j\begin{bmatrix} Y & x \\ x^T & 1\end{bmatrix}\right) > 0, \quad j = 1 : m_q\\
&\operatorname{tr}\left(T_k\begin{bmatrix} Y & x \\ x^T & 1\end{bmatrix}\right) = 0, \quad k = 1 : m_t,\\
&\begin{bmatrix} Y & x \\ x^T & 1\end{bmatrix} \succeq 0
\end{aligned}
\tag{4.13}
$$

is infeasible.

Proof. Suppose there exists an $\bar{x} \in \mathcal{S}$ and (4.13) is infeasible. It is immediate to check that $x = \bar{x}$ and $Y = \bar{x}\bar{x}^T$ satisfies (4.13), which contradicts with the infeasibility of the relaxation. □

4.3. Relation between the S-procedure and the SDP Relaxation

So far we have seen that both the S-procedure and the SDP relaxation provides sufficient conditions and bounds. Although, these methods look different in appearance, they, in fact, lead to the same result. In other words, the lower bound and infeasibility certificates provided by them are the same. This is due to the fact that they are related with strong duality. Therefore, one is not better than the other. Let us show this fact for the optimization problem using SDP duality.

Proposition 4.3.1. *Assume that there exists a strictly feasible solution of (4.12). The systems (4.8) and (4.12) are SDP dual of each other. Hence, $\rho_d^* = \rho_p^* =: \rho^*$.*

Proof. We can rewrite (4.12) as

$$\begin{aligned}
\inf \quad & \operatorname{tr}(P_0 Z) \\
\text{s.t.} \quad & \operatorname{tr}(P_i Z) \geq 0, \quad i = 1 : m_p, \\
& \operatorname{tr}(Q_j Z) > 0, \quad j = 1 : m_q, \\
& \operatorname{tr}(T_k Z) > 0, \quad k = 1 : m_t, \\
& \quad Z \succeq 0, \\
& \quad Z_{n+1,n+1} = 1
\end{aligned}$$

The Lagrangian corresponding to this problem is

$$\begin{aligned}
L(Z, \lambda, \mu, \xi, t, W) = \; & \operatorname{tr}(P_0 Z) - \sum_{i=1}^{m_p} \lambda_i \operatorname{tr}(P_i Z) - \sum_{j=1}^{m_q} \mu_j \operatorname{tr}(Q_j Z) \\
& - \sum_{k=1}^{m_t} \xi_k \operatorname{tr}(T_k Z) - \operatorname{tr}(WZ) + t \left(1 - \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Z \right) \right) \\
= \; & \operatorname{tr}\left(\left\{ P_0 - \sum_{i=1}^{m_p} \lambda_i P_i - \sum_{j=1}^{m_q} \mu_j Q_j - \sum_{k=1}^{m_t} \xi_k T_k \right.\right. \\
& \left.\left. - W - t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} Z \right) + t
\end{aligned}$$

Based on the Lagrangian, the dual problem $\sup\limits_{\lambda,\mu,\xi,t,W} \inf\limits_{Z} L(Z, \lambda, \mu, \xi, t, W)$ can be obtained as

$$\begin{aligned}
\sup \quad & t \\
\text{s.t.} \quad & P_0 - \sum_{i=1}^{m_p} \lambda_i P_i - \sum_{j=1}^{m_q} \mu_j Q_j - \sum_{k=1}^{m_t} \xi_k T_k - t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = W \\
& \lambda, \, \mu \geq 0, \quad \mu \neq 0, \quad W \succeq 0,
\end{aligned}$$

which clearly equivalent to the relaxation (4.8), which is based on the S-procedure. The equivalence $\rho_d^* = \rho_p^*$ follows from convexity and Lagrange duality. □

A similar argument applies to the feasibility problem. The infeasibility certificate of the SDP relaxation and the S-procedure are the same as stated in the next proposition.

Proposition 4.3.2. *The SDP relaxation (4.13) is infeasible if and only if (4.3) is feasible.*

The proof can be obtained using Lagrange duality as for the optimization problem.

4.4. Additional Properties of the SDP Relaxation

In the foregoing discussion it is shown that the S-procedure and the SDP relaxation produces the same result for a given problem, and thus, they can be considered to be equivalent. However, the latter has some advantages over the former since it works in the primal space. It gives an insight about the geometry of the problem and has a number of features which would be useful in some applications. These are explained next.

We start by describing the geometric relation between the SDP relaxation and $\mathcal{S}$. To this end, we denote the feasible set of the constraints (4.13) as $\mathcal{R}$. Clearly, $\mathcal{R}$ is composed of pairs of the form $(x, Y) \in \mathbb{R}^n \times \mathbb{S}^n$. Therefore, it is a subset of a $(n^2 + 3n)/2$ dimensional space, and thus, it lies in a higher dimensional space than $\mathbb{R}^n$ containing $\mathcal{S}$. Moreover, $\mathcal{R}$ is always a convex unlike $\mathcal{S}$. Despite these differences, there is a certain relation between these sets induced by the projection operation. Consider the orthonormal projection of $\mathcal{R}$

$$\mathcal{P}_{\mathcal{R}} := \{x \in \mathbb{R}^n | \, \exists Y \in \mathbb{S}^n \text{ such that } (x, Y) \in \mathcal{R}\}.$$

Since convexity is preserved under projection, $\mathcal{P}_{\mathcal{R}}$ is also a convex set. Moreover, $\mathcal{S}$ is a subset of $\mathcal{P}_{\mathcal{R}}$. Indeed, if $\bar{x} \in \mathcal{S}$, then $(x, Y) = (\bar{x}, \bar{x}\bar{x}^T)$ satisfies (4.13). Hence, $\bar{x} \in \mathcal{P}_{\mathcal{R}}$, which implies $\mathcal{S} \subseteq \mathcal{P}_{\mathcal{R}}$ because $\bar{x}$ is an arbitrary element of $\mathcal{S}$. Consequently, we can deduce the following important property.

Proposition 4.4.1. *Projection of $\mathcal{R}$, $\mathcal{P}_{\mathcal{R}}$, is a convex set and contains $\mathcal{S}$.*

Example 4.4.2. *Suppose that $\mathcal{S}$ is determined by the constraints*

$$\begin{aligned} p_1(x) &= 4 - x_1^2 - x_2^2 \geq 0 \\ p_2(x) &= -2x_2 \geq 0 \\ p_3(x) &= -x_1^2 + x_2^2 + 1 \geq 0. \end{aligned}$$

From (4.13), the LMI description of the relaxed set $\mathcal{R}$ is obtained as

$$\begin{gathered} 4 - y_{11} - y_{22} \geq 0, \\ -2x_2 \geq 0, \\ -y_{11} + y_{22} + 1 \geq 0, \\ \begin{bmatrix} y_{11} & y_{12} & x_1 \\ y_{12} & y_{22} & x_2 \\ x_1 & x_2 & 1 \end{bmatrix} \succeq 0. \end{gathered}$$

The set $\mathcal{S}$ is depicted in Figure 4.1.a while the projection set $\mathcal{P}_{\mathcal{R}}$ is shown in Figure 4.1.b. As can be seen from the figure, $\mathcal{P}_{\mathcal{R}}$ is convex and contains $\mathcal{S}$.

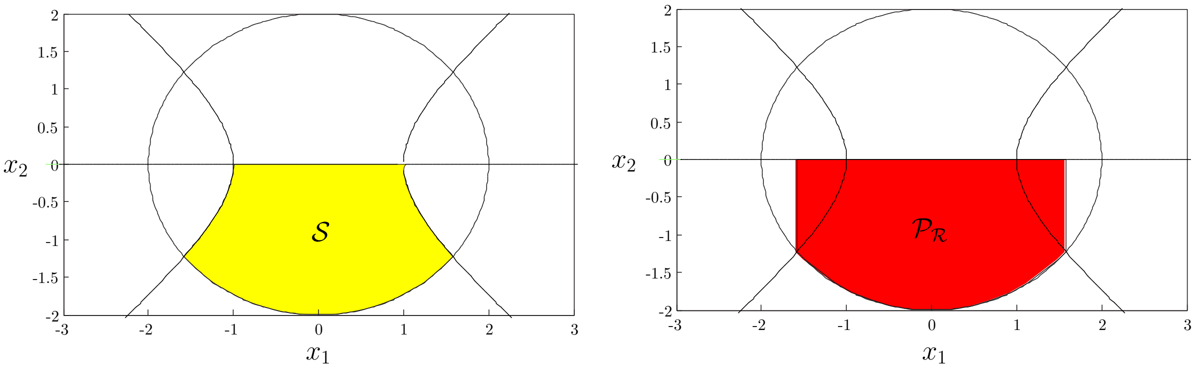


Figure 4.1. a) Feasible set $\mathcal{S}$; b) projection of the relaxation, $\mathcal{P}_{\mathcal{R}}$

Based on the foregoing result, one may naturally employ $\mathcal{P}_{\mathcal{R}}$ to obtain a convex outer approximation technique as described in Section 3.1. However, the resulting relaxation, in fact, happens to be equivalent to the SDP relaxation as shown by the following proposition. Therefore, SDP relaxation itself constitutes a convex outer approximation method.

Proposition 4.4.3. *If p_0 is a linear function, the optimal solution of the SDP relaxation (4.12), ρ_p^*, is equivalent to the optimal solution of*

$$\begin{array}{ll} \inf & p_0(x) \\ \mathrm{s.\,t.} & x \in \mathcal{P}_{\mathcal{R}}. \end{array}$$

The proof is trivial and left to the reader.

This result gives us a geometric criterion to determine the exactness of the SDP relaxation. Due to Proposition 3.1.10, when $\mathcal{P}_{\mathcal{R}} = \mathbf{co}(\mathcal{S})$, the SDP relaxation is guaranteed to produce the global optimal solution for any linear objective function. However, this constitutes a sufficient condition for the exactness. Even if $\mathbf{co}(\mathcal{S})$ is a proper subset of $\mathcal{P}_{\mathcal{R}}$, the exact solution can be obtained if the optimal solution of the relaxation occurs at a point of where the boundaries of $\mathcal{P}_{\mathcal{R}}$ and $\mathbf{co}(\mathcal{S})$ coincide.

Example 4.4.4. *Suppose we would like to minimize the objective function $p_0 = -2x_1 - 2x_2$ over $\mathcal{S}$ determined by the constraints given in Example 4.4.2. This problem was already investigated in Example 4.1.7 and the the bound was found to be $\rho_d^* = -3.1623$ whereas the global optimal solution is $\gamma^* = -2$. The convex hull of $\mathcal{S}$ and the set $\mathcal{P}_{\mathcal{R}}$ are depicted in Figure 4.2.a. As can be seen from the figure, since the boundaries of $\mathbf{co}(\mathcal{S})$ and $\mathcal{P}_{\mathcal{R}}$ does not coincide along the normal of the contours of the objective function, the SDP relaxation only gives a lower bound. This explains the reason behind the occurrence of relaxation gap in Example 4.1.7. On the other hand, if the objective was $p_0 = -x_1$ the gap would vanish and the exact solution could be recovered by the SDP relaxation even if $\mathcal{P}_{\mathcal{R}}$ does not give $\mathbf{co}(\mathcal{S})$ as depicted in Figure 4.2.b.*

There are some other aspects of the SDP relaxation which makes it attractive in practice. First, based on the geometric insight provided by the preceding proposition, if (x, Y) is a solution of (4.13), it appears to be reasonable to check if the projection of the solution, which is x, belongs $\mathcal{S}$. If this happens, we can detect the nonemptiness of $\mathcal{S}$, and hence, we obtain a feasibility certificate in addition to the infeasibility certificate provided by Proposition 4.2.2. Moreover, a point of $\mathcal{S}$ can be attained by this way,

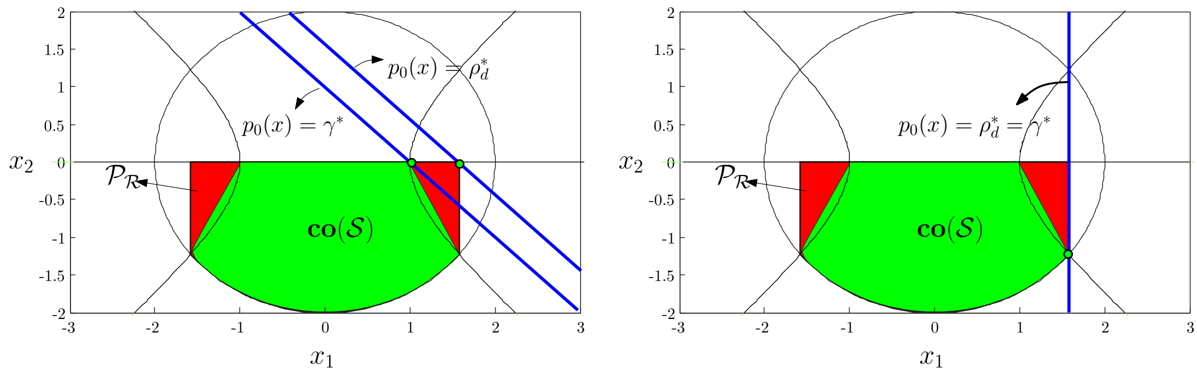


Figure 4.2. a) Problem with a relaxation gap; b) problem without a relaxation gap

which was not possible with the S-procedure as described earlier. Note that chance of finding such a point increases as $\mathcal{P}_\mathcal{R}$ gets smaller.

Second, if the objective p_0 is linear and the projection of the optimal solution of (4.12) lies in $\mathcal{S}$, we can infer that this solution gives the global optimal solution as shown by the following proposition.

Proposition 4.4.5. *Suppose p_0 is linear. Then, if (x^*, Y^*) is the optimal solution of (4.12) and $x^* \in \mathcal{S}$, then x^* is the global optimal solution of (4.9) (i.e. $\gamma^* = p_0(x^*) = \rho_p^*$).*

Proof. We already know that $\rho_p^* \leq \gamma^*$. On the other hand, since $x^* \in \mathcal{S}$

$$\gamma^* = \inf_{x \in \mathcal{S}} p_0(x) \leq p_0(x^*) = \operatorname{tr}\left(P_0 \begin{bmatrix} Y^* & x^* \\ x^{*T} & 1 \end{bmatrix}\right) = \rho_p^*, \tag{4.14}$$

where the first equality follows from the fact that P_0 can be partitioned as

$$P_0 = \begin{bmatrix} 0 & b \\ b^T & c \end{bmatrix},$$

and hence, Y^* does not contribute to the objective function of the SDP relaxation. Consequently, $\gamma^* = \rho_p^*$ and x^* is a global optimizer. $\square$

In the above given proof, the fact that the nth order leading principle submatrix of P_0 is zero was crucial for the exactness result. If this did not hold true, the first equality of (4.14) would not be satisfied due to the contribution of the term corresponding to Y^*. Consequently, this result does not extend to the case in which the objective is not linear. In this case, however, one can use the following sufficient rank condition to detect global optimality and recover the optimal solution vector.

Proposition 4.4.6. *If* (x^*, Y^*) *is the optimal solution of (4.12) and*

$$\operatorname{rank}\begin{bmatrix} Y^* & x^* \\ x^{*T} & 1 \end{bmatrix} = 1, \tag{4.15}$$

then $\rho_p^* = \gamma^*$, $x^* \in \mathcal{S}$ *and* x^* *is a global optimizer of (4.9).*

Proof. Trivially, if the rank condition is satisfied, we can infer

$$\begin{bmatrix} Y^* & x^* \\ x^{*T} & 1 \end{bmatrix} = \begin{bmatrix} x^* \\ 1 \end{bmatrix} \begin{bmatrix} x^{*T} & 1 \end{bmatrix}.$$

By substituting this decomposition in (4.12) and following (4.10) in the reverse order one can see that

$$\begin{aligned} p_i(x^*) &\geq 0,\ i = 1 : m_p, \\ q_j(x^*) &> 0,\ j = 1 : m_q, \\ t_k(x^0) &= 0,\ k = 1 : m_t. \end{aligned}$$

Hence, $x^* \in \mathcal{S}$. Applying the same reasoning to the objective of the SDP relaxation we can also infer $p_0(x^*) = \rho_p^*$. Since x^* is an element of $\mathcal{S}$, this means $\gamma^* \leq \rho_p^*$. From Proposition 4.2.1, we also know that $\rho_p^* \leq \gamma^*$. Therefore, $\rho_p^* = \gamma^*$ and x^* is a global optimizer of (4.9). □

The last property of the SDP relaxation that we would like to introduce is an alternative characterization of the projection set $\mathcal{P}_{\mathcal{R}}$ that can be obtained when $\mathcal{S}$

defined by only nonstrict inequalities. This result was developed by Kojima, et al. [24].

Let us partition P_i as

$$P_i = \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix},$$

where $A_i \in \mathbb{S}^n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$. Define the collection of multipliers

$$\Omega := \left\{ \lambda \in \mathbb{R}^{m_p} \,\middle|\, \lambda \geq 0,\ \sum_{i=1}^{m_p} \lambda_i = 1, \sum_{i=1}^{m_p} \lambda_i A_i \preceq 0 \right\}.$$

Proposition 4.4.7 ([24]). *Consider the set*

$$\mathcal{S} = \{x \in \mathbb{R}^n |\, p_i(x) \geq 0,\ i = 1 : m_p\}.$$

The following holds true

$$\mathcal{P}_{\mathcal{R}} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{m_p} \lambda_i p_i(x) \geq 0,\ \forall \lambda \in \Omega \right\}$$

This result shows that $\mathcal{P}_{\mathcal{R}}$ can be described as the set of points satisfying all concave constraints that can be obtained by taking convex combinations of the inequalities defining $\mathcal{S}$. A special case of this proposition will be employed in the later chapters to establish a relation between our findings and the SDP relaxation.

4.5. Exactness Results

The S-Procedure and the SDP Relaxation only give sufficient conditions or bounds. However, there are some special problems for which they turn out to be exact. In this case, these methods are said to be lossless. Here, we present some of the losslessness results available in the literature. Note that, however, we only give the ones related with our work, and hence, the exposition of the material is not complete. For more complete review of the subject, the reader is referred to the survey paper [5].

Losslessness results on the S-procedure has its roots on the works about the sign definiteness of the matrix pencils, a survey of which can be found in [25]. However, the fist such result which has profound effect on the control community is due to Yakubovich. In [1, 26], he showed that the S-procedure is necessary and sufficient to prove the nonnegativity of a quadratic polynomial over a domain determined by a quadratic inequality constraint. In below, we present this result in feasibility form and in a slightly more general settling which may involve both strict and nonstrict inequalities.

Suppose

$$\mathcal{S} = \{x \in \mathbb{R}^n |\, p_1(x) \rhd_1 0,\, p_2(x) \rhd_2 0\},$$

where $\rhd_i \in \{>, \geq\}$, $i = 1:2$. Assume that if there exist nonstrict inequality/inequalities in the definition of $\mathcal{S}$ at least for one of them is satisfied strictly for an $\bar{x} \in \mathbb{R}^n$.

Lemma 4.5.1 (S-Lemma, [26]). *The set $\mathcal{S}$ is empty if and only if*

i) there exists a strict inequality in the definition of $\mathcal{S}$ and there exists a $\lambda \in \mathbb{R}^2$ such that

$$\begin{gathered} \lambda \geq 0, \quad \lambda \neq 0 \\ \lambda_1 P_1 + \lambda_2 P_2 \preceq 0, \end{gathered}$$

where $\lambda_i > 0$ if $\rhd_i$ is the only strict inequality appearing in the definition of $\mathcal{S}$, or

ii) both inequalities defining $\mathcal{S}$ are nonstrict and there exists a $\lambda \in \mathbb{R}^2$ and a $\mu \in \mathbb{R}$

such that

$$\lambda \geq 0, \quad \mu > 0$$

$$\lambda_1 P_1 + \lambda_2 P_2 + \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix} \preceq 0.$$

The preceding result initiated an interest in the losslessness studies and it is followed by some extensions. A remarkable one is given by Polyak.

Lemma 4.5.2 ([20]). *Consider the set*

$$\mathcal{S} = \{x \in \mathbb{R}^n |\, p_1(x) \geq 0,\ p_2(x) \geq 0,\ q_1(x) > 0\}.$$

Assume there exits $\mu \in \mathbb{R}^2$ such that $\mu_1 P_1 + \mu_2 P_2 > 0$. Then, $\mathcal{S} = \emptyset$ if and only if there exists $\lambda \in \mathbb{R}^2$ such that

$$\lambda_1 P_1 + \lambda_2 P_2 + Q_1 \preceq 0, \quad \lambda \geq 0.$$

Keep in mind that the assumption made above about the existence of a positive definite linear combination of P_1 and P_2 is fairly restrictive.

A more recent extension is due to Sturm and Zhang.

Lemma 4.5.3 ([27]). *Consider the set*

$$\mathcal{S} = \{x \in \mathbb{R}^n |\, q_1(x) > 0,\ t_1(x) = 0\}.$$

Assume q_1 is a strictly concave function and $q_1(\bar{x}) = 0$ for an $\bar{x} \in \mathbb{R}^n$. Then, $\mathcal{S}$ is empty if and only if there exists $\xi \in \mathbb{R}$ such that

$$Q_1 + \xi_1 T_1 \preceq 0.$$

Remark 4.5.4. *The losslessness results introduced so far are given for the S-procedure.*

However, since this method is coupled with the SDP relaxation via strong duality, they also apply to the SDP relaxation. In other words, the corresponding SDP relaxations also lead to the exact solution.

Remark 4.5.5. *The losslessness of the S-procedure also implies the exactness of the associated relaxation (4.8). To be more specific, if the set $\bar{\mathcal{S}}$ defined in (4.7) satisfies conditions of one of the foregoing propositions, then the corresponding relaxation (4.8) leads to the global optimal solution γ^* since the LMI utilized in it constitutes a necessary and sufficient condition for the emptiness of $\bar{S}$ instead of just a sufficient condition.*

Another losslessness is developed for the SDP relaxation (4.12).

Lemma 4.5.6 ([28]). *Consider the optimization problem*

$$\begin{aligned} \gamma^* = \quad & \inf \quad p_0(x) \\ & \text{s.t.} \quad p_1(x) \geq 0,\ p_2(x) \geq 0 \end{aligned} \tag{4.16}$$

and the corresponding relaxation

$$\rho_p^* = \inf \quad \operatorname{tr}\left(P_0 \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \tag{4.17a}$$

$$\text{s.t.} \quad \operatorname{tr}\left(P_i \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \geq 0, \quad i = 1:2, \tag{4.17b}$$

$$\begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \succeq 0 \tag{4.17c}$$

having an optimal solution (x^*, Y^*). If at least one of the scalar constraints in (4.17b) is strictly positive for $(x, Y) = (x^*, Y^*)$ (i.e. not binding at optimality), then $\gamma^* = \rho_p^*$.

The assumption being not binding at optimality may not hold for various problems, which restricts validity of this extension seriously. Similar to the preceding lemmas, due to strong duality, this result also implies the exactness of the corresponding

S-procedure based relaxation (4.8) when there are two constraints under the stated condition. However, in order to see if this condition hold true one has to solve the SDP relaxation and check if its constraints are binding at the optimal solution. Therefore, in order to determine losslessness one has to use the SDP relaxation.

The last result we will present does not show the losslessness of the original SDP relaxation. Instead, it gives a modified form of the relaxation which yield to the exact result for another special instance of (4.9).

Theorem 4.5.7 ([27]). *Consider (4.16). Assume p_1 is a concave function and $p_2 = [x^T \; 1]a$, where $a \in \mathbb{R}^{n+1}$, $a \neq 0$, is an affine function. Then optimal solution of (4.16) is equal to that of*

$$\gamma^* = \inf \quad \operatorname{tr}\left(P_0 \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \tag{4.18a}$$

$$\text{s. t.} \quad \operatorname{tr}\left(P_1 \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix}\right) \geq 0 \tag{4.18b}$$

$$p_1\left(\begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} a\right) \geq 0 \tag{4.18c}$$

$$Y \succeq 0. \tag{4.18d}$$

Notice that the above given problem has mixture of LMIs and a concave quadratic constraint. This type of problems can be solved using available software such as SeDuMi. Alternatively, one can convert the constraint (4.18c) into an LMI as described in [19] and solved an SDP problem. However, the first approach is easier and better from computational point of view. Another aspect of this relaxation is the fact that it is not the original SDP relaxation as mentioned above. Therefore, it does not imply the losslessness of the S-procedure under the assumptions made about the polynomials p_1 and p_2.

Before finishing this section, we would like to emphasize that, in general, the S-procedure and the SDP relaxation are extremely conservative methods as evident from the exactness results presented above which are limited with at most three polynomials. This fact further supported with a counter example given in [29] which shows that achieving the losslessness for even four constraints is hopeless.

4.6. A Control Application

A classical application of the S-procedure is the absolute stability analysis, which is first established by Yakubovich in [1]. Recall that in Section 2.1 the question of quadratic stability was converted into a PSP problem but a method for the solution of this type of a PSP was not given. Yakubovich showed that this problem can be resolved using the S-procedure as described in below.

From Proposition 2.1.1, we known that the system (2.1) subject to (2.2) is quadratically stable if and only if there exists a $P \succ 0$ such that the constraints

$$p_0(x,w) = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} > 0, \tag{4.19a}$$

$$p_1(x,w) = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} -\alpha\beta C^T C & \frac{\alpha+\beta}{2} C^T \\ \frac{\alpha+\beta}{2} C & -1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \tag{4.19b}$$

do not have a feasible solution. By applying Lemma 4.5.1 to (4.19), it can be inferred that the system is quadratically stable if and only if there exists a $P \in \mathbb{S}^n$ and $\lambda \in \mathbb{R}^2$ satisfying

$$P \succ 0, \quad \lambda_1 > 0, \quad \lambda_2 \geq 0, \tag{4.20a}$$

$$\lambda_1 \begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -\alpha\beta C^T C & \frac{\alpha+\beta}{2} C^T \\ \frac{\alpha+\beta}{2} C & -1 \end{bmatrix} \prec 0. \tag{4.20b}$$

The above given system constitutes and LMI in λ but not in both P and λ. However, this difficulty can be alleviated. To this end, define $\theta := \lambda_2/\lambda_1$ and divide both sides of (4.20b) by λ_1. From this, it can be seen that quadratic stability is equivalent to the feasibility of

$$P \succ 0, \quad \theta \geq 0, \tag{4.21a}$$

$$\begin{bmatrix} PA + A^T P - \theta\alpha\beta C^T C & \theta\frac{\alpha+\beta}{2}C^T \\ \theta\frac{\alpha+\beta}{2}C & -\theta \end{bmatrix} \prec 0. \tag{4.21b}$$

Clearly, this is an LMI both in θ and P and it is feasible if and only if the the system considered is quadratically stable. Hence, the problem of determining quadratic stability can be solved efficiently by determining the feasibility of the LMI.

The approach described above can also be employed when there are multiple feedback connections as in (2.6) and (2.7). In this case, one can obtain a similar LMI using the S-procedure. However, this time there will be more than two constraints, and hence, S-procedure will not be lossless. Therefore, only a sufficient LMI condition for quadratic stability can be obtained unlike the previous case for which (4.21) gives a necessary and sufficient condition. Note that we already expect this since when there are multiple feedback connections determining quadratic stability becomes NP-hard as mentioned before.

4.7. Summary

In this chapter, we introduced the S-procedure and the SDP-relaxation. These methods are fundamentally equivalent and can be employed to find efficiently computable feasibility/infeasibility certificates for feasibility problem or lower bound for optimization problems. We described basic properties of these methods, which shows that the SDP-relaxation has certain advantages over the S-procedure since it works in the primal space. Although, S-procedure and the SDP-relaxation give sufficient conditions or approximation bounds in general, there are some results showing that they turn out to be exact for some specific problems. A list of such results are also given, which will constitute a basis for demonstrating the advantages of our findings described in the subsequent chapters. Lastly, some simple control applications of the S-procedure are presented.

5. POSITIVSTELLENSATZ AND THE SOS RELAXATIONS

In the previous chapter, we investigated the S-procedure and the SDP relaxation. Although, these methods are used extensively in the literature, they only give sufficient conditions or approximation which turn out be very conservative in general. The main reason behind this conservatism is the fact the underlying approach is over-simplistic since it employs real multipliers to create infeasibility certificates. However, a fundamental result from real algebraic geometry, Positivstellensatz, states that a necessary and sufficient condition can be obtained not only for quadratic programming problems but also for more general polynomial problems by utilizing sum of squares (SOS) of polynomials as the multipliers. In this chapter, our goal is to describe this result and explain how it can be employed to solve feasibility and optimization problems. To this end, first we will describe SOS polynomials and their characterization based on positive semidefinite matrices. Then, we will give Positivstellensatz and explain how the search for SOS multipliers can be converted into an LMI problem to attain certificates of infeasibility. Lastly, it will be shown that how some special forms of Positivstellensatz can be utilized to attain convergent lower bounds for global optimization problems involving polynomial functions.

5.1. Notation

Before proceeding further, we need to introduce some notation and definitions that will be employed not only in this chapter but also in the following ones. The set of nonnegative integers is denoted as $\mathbb{Z}_+$. If $\alpha \in \mathbb{Z}_+^n$, then $|\alpha| := \alpha_1 + \cdots + \alpha_n$. A monomial is defined as a finite product of nonnegative integer powers of a given set of variables. In particular, if the variables of interest are $x_1, \cdots, x_n$ a monomial is represented as

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

using an appropriate vector $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial x^α is defined as $|\alpha|$. A polynomial p in $\mathbb{R}^n$ is a finite linear combination of monomials and can be expressed as

$$p = \sum_{\alpha \in \mathcal{I}_p} p_\alpha x^\alpha,$$

where p_α are real numbers indexed by the elements of $\mathcal{I}_p \subseteq \mathbb{Z}_+^n$ which is a finite set of nonnegative vectors corresponding to the monomials appearing in p. The degree of a polynomial, $\deg(p)$, is defined as the largest of degrees of the monomials appearing in p which have a nonzero coefficient. Lastly, we denote the collection of all polynomials in n variables having real coefficients as $\mathbb{R}[x_1, \cdots, x_n]$. Clearly, $\mathbb{R}[x_1, \cdots, x_n]$ is a ring.

5.2. SOS Polynomials

Definition 5.2.1. *An $s \in \mathbb{R}[x_1, \cdots, x_n]$ is said to be an SOS polynomial if there exists a positive integer r and $f_i \in \mathbb{R}[x_1, \cdots, x_n]$, $i = 1 : r$ such that*

$$s = \sum_{i=1}^{r} f_i^2.$$

In what follows, the collection of all SOS polynomials will be denoted as Σ.

SOS polynomials constitute an important subject in mathematics. This is because they are closely related with the notions of positive semidefiniteness and feasibility. Indeed, trivially, an SOS polynomial always takes nonnegative values. Therefore, given a multivariate polynomial $p \in \mathbb{R}[x_1, \cdots, x_n]$, if one needs to prove that $p(x) \geq 0$ for every $x \in \mathbb{R}^n$ or equivalently that

$$\mathcal{S} = \{x \in \mathbb{R}^n | p(x) < 0\} = \emptyset,$$

it suffices to show that p is SOS. Notice that this problem looks like a special instance of the feasibility question investigated in the previous chapter except an important difference that p is an arbitrary polynomial instead of a quadratic function, which renders the situation considerably difficult. Indeed, this simple looking problem is known to be NP-hard [30].

Since every SOS polynomial is positive semidefinite as mentioned above, one can naturally ask if the inverse holds true. That is, is every positive semidefinite polynomial SOS? In other words, are SOS polynomials powerful enough to characterize all positive semidefinite polynomials? This question is investigated by Hilbert in 1933 and he showed that the answer is negative. In his proof, he used a contradictive argument and did not give a counter example. The first explicit examples are due to Motzkin, one of which is

$$m(x, y, z) = (x^2 + y^2 - 3z^2)x^2y^2 + z^6.$$

This function always takes nonnegative values but it is not SOS [31]. Moreover, today it is well known that the class of such functions is considerably large, and hence, SOS polynomials are not powerful enough to directly characterize positive semidefiniteness [32]. However, there is a remedy for this as we will see.

Based on the preceding facts, one can at least try to use SOS polynomials as certificates of positive semidefiniteness or feasibility. To this end, we should have a method to determine if a given polynomial is SOS or not. Fortunately, the necessary and sufficient condition for this exists and the SOS decomposition can be obtained efficiently, which is described next.

Before we proceed further, let us mention some facts about SOS polynomials. Consider

$$s = \sum_{i=1}^{r} f_i^2. \tag{5.1}$$

Clearly, $\deg(s) = \max_{i\in 1:r}\{2\deg(f_i)\}$. This is because squares of monomials having the largest degrees in f_i, $i = 1 : r$ will also have the largest degree in f_i^2, $i = 1 : r$ and their coefficient will be positive. Therefore, they do not cancel out in (5.1) and the ones having the maximum degree will determine the degree of s. Consequently, we can deduce that an SOS polynomial always has an even degree. That is, $\deg(s) = 2d$ for a $d \in \mathbb{Z}_+$. Moreover, we have $\deg(f_i) \leq d$, $i = 1 : r$.

Now, consider an $s \in \mathbb{R}[x_1, \cdots, x_n]$ with $\deg(s) = 2d$, $d \in \mathbb{Z}_+$. Suppose that u is a vector composed of all monomials having degree at most d. For example, if $d = 3$ and $n = 1$

$$[1,\ x,\ x^2,\ x^3]^T$$

while if $d = 2$ and $n = 2$

$$[1,\ x,\ y,\ x^2,\ xy,\ y^2]^T.$$

Let's denote the length of u as ℓ. The next proposition gives a necessary and sufficient condition for s to be SOS.

Proposition 5.2.2 ([33]). *$s \in \Sigma$ if and only if there exists a $W \in \mathbb{S}^\ell$, $W \succeq 0$ such that*

$$s = u^T W u.$$

Proof. Let us start with the if part. If $W \succeq 0$ and $\operatorname{rank}(W) = r$, there exists a matrix $V \in \mathbb{R}^{r\times\ell}$ such that $W = V^T V$. Define $f := Vu$. Clearly, elements of f, f_i, $i = 1 : r$, are polynomials. Thus, we can infer that

$$s = u^T W u = u^T V^T V u = f^T f = \sum_{i=1}^{r} f_i^2,$$

which implies s is SOS.

Conversely, if s is SOS, then

$$s = \sum_{i=1}^{r} f_i^2 = f^T f. \tag{5.2}$$

Since $\deg(f_i) \leq d$, $i = 1 : r$ and u involves all monomials having degree up to d, f can be expressed as $t = Vu$ for an appropriate constant matrix $V \in \mathbb{R}^{r\times\ell}$. By substituting this into (5.2), we can infer that $s = u^T V^T V u = u^T W u$. Clearly, since $W = V^T V$, W is positive semidefinite. $\square$

The above given proposition is introduced by Powers in [33]. However, it was Parrilo [9] who first realized that it may be used to convert the SOS problem into an LMI problem which can be solved efficiently by interior point methods. Indeed, suppose we want to check if a polynomial s of degree $2d$ is SOS. Simply construct the monomial vector u and perform the matrix multiplication $u^T W u$ to obtain the polynomial whose coefficients have to be linear functions of the elements of W (W_{ij}). By equating coefficients of this polynomial to that of s we obtain a number of linear equality constraints in W_{ij}. If these constraints are satisfied for a $W \succeq 0$ we conclude s is SOS. Otherwise, s is not SOS. Clearly, the existence of such a W can be easily verified by solving an LMI problem. Moreover, if s is SOS, the SOS decomposition can be extracted from W as follows. Using an appropriate numerical methods such as Cholesky decomposition obtain a matrix $V \in \mathbb{R}^{r\times\ell}$ such that $W = V^T V$. From the proof of Proposition 5.2.2 it follows that the polynomials $f_i = (Vu)_i$, $i = 1 : r$ satisfies the relation

$$s = \sum_{i=1}^{r} f_i^2.$$

Example 5.2.3. *Suppose we want to check if*

$$s = 5x^2y^2 - 4x^2y - 4xy^2 + 2x^2 + y^2 - 4xy + 2x + 10 \tag{5.3}$$

is SOS. Since $\deg(s) = 4$ *we need to employ a monomial vector* u *composed of all* *monomials having degree up to two. Hence, $s = u^T W u$ is given by*

$$s = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} \\ W_{12} & W_{22} & W_{23} & W_{24} & W_{25} & W_{26} \\ W_{13} & W_{23} & W_{33} & W_{34} & W_{35} & W_{36} \\ W_{14} & W_{24} & W_{34} & W_{44} & W_{45} & W_{66} \\ W_{15} & W_{25} & W_{35} & W_{45} & W_{55} & W_{56} \\ W_{16} & W_{26} & W_{36} & W_{46} & W_{56} & W_{66} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix} \tag{5.4}$$

By equating the coefficient of the monomials belonging to polynomials appearing in (5.3) and (5.4), one obtains the following set of linear constraints

$$\begin{gathered} W_{11} = 10, \quad 2W_{12} = 2, \quad 2W_{13} = 0, \quad W_{22} + 2W_{14} = 2, \\ 2W_{15} + 2W_{23} = -4, \quad 2W_{16} + W_{33} = 1, \quad 2W_{24} = 0, \\ 2W_{25} + 2W_{34} = -4, \quad 2W_{26} + 2W_{35} = -4, \quad 2W_{36} = 0, \\ W_{44} = 0, \quad 2W_{45} = 0, \quad W_{55} + 2W_{56} = 5, \quad 2W_{56} = 0, \quad W_{66} = 0. \end{gathered}$$

Therefore, s is SOS if and only if there exists a $W \succeq 0$ satisfying these constraints. This is an LMI and it can be verified that the matrix

$$W = \begin{bmatrix} 10 & 1 & 0 & 0 & -3 & 0 \\ 1 & 2 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & -2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

constitutes a feasible solution. Moreover, it can be decomposed as $W = V^T V$, where

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \end{bmatrix}.$$

From this SOS representation of s can be obtained as

$$s = (x+1)^2 + (xy-3)^2 + (x+y-2xy)^2.$$

5.3. Positivstellensatz

After showing every positive semidefinite polynomial cannot be expressed as sum of squares, Hilbert asked if such functions admit a representation as a ratio of SOS polynomials. That is, given a positive semidefinite polynomial p, do there exist $s_1, s_2 \in \Sigma$ such that $s_1 p = s_2$? This was one of his famous twenty three problems (the 17th problem) that he introduced in 1900 in Paris congress. In 1927, Emil Artin answered it affirmatively by developing the theory of real closed fields [34]. This result showed the importance of SOS polynomials for characterizing positive semidefiniteness and feasibility. Moreover, it constitutes the basis of a more general theorem that we will investigate in this section, namely the Positivstellensatz which is developed by Stengle in 1974 [7].

Positivstellensatz is one of the fundamental theorems of real algebraic geometry. Basically, it gives a necessary and sufficient condition for determining feasibility of a system of polynomial constrains. To be more specific, consider the set

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} p_i(x) \geq 0, & i = 1 : m_p \\ r_j(x) \neq 0, & j = 1 : m_r \\ t_k(x) = 0, & k = 1 : m_t \end{array} \right\},$$

where p_i, r_j, $t_k \in \mathbb{R}[x_1, \cdots, x_n]$. We would like to determine if $\mathcal{S} = \emptyset$. Positivstellensatz answers this question completely. Notice that unlike (4.1), the description of $\mathcal{S}$ given above does not include strict inequalities explicitly. However, it subsumes such constraints since a strict inequality such as $f(x) > 0$ can be easily expressed by using a non-strict inequality and an inequation as $f(x) \geq 0$, $f(x) \neq 0$.

The main idea of Positivstellensatz is similar to that of the S-procedure, and in fact, constitutes a serious generalization of this method. Recall that the underlying approach of the S-procedure is to look for the new constraints which are generated from the ones defining $\mathcal{S}$ and satisfied for every point of $\mathcal{S}$. Such constraints are called valid constraints. Trivially, if a valid constraint turns out to be infeasible, it automatically implies emptiness of $\mathcal{S}$, and hence, gives a sufficient condition for infeasibility. Because the S-procedure employs only linear combinations for this purpose, it rarely turns out to be necessary. On the other hand, Positivstellensatz shows that a necessary and sufficient condition can be obtained by employing a much more richer collection of valid constraints which can be obtained using SOS multipliers, polynomial multipliers and product of polynomials defining $\mathcal{S}$.

Let's describe this briefly. Define the index set $\mathcal{B} := \{0,1\}^{m_r}$ (i.e. set of vectors of length m_r composed of zeros and ones). Consider the collection of constraints

$$\sum_{\beta\in\mathcal{B}} s_\beta(x) p_1^{\beta_1}(x)\cdots p_{m_p}^{\beta_{m_p}}(x) + (r_1^{\zeta_1}(x)\cdots r_{m_r}^{\zeta_{m_r}}(x))^2 + \sum_{k=1}^{m_t} h_k(x)t_k(x) > 0, \tag{5.5}$$

where $s_\beta \in \Sigma$, $\beta \in \mathcal{B}$; $\zeta \in \mathbb{Z}_+^{m_r}$ and $h_k \in \mathbb{R}[x_1,\cdots,x_n]$, $k = 1 : m_t$. Realize that the elements of this collection are generated by multipliers which are SOS polynomials s_β and arbitrary real polynomials h_k. Moreover, products of the polynomials defining $\mathcal{S}$ are also employed. Clearly, this collection is much more richer than the one generated in the S-procedure, (4.2), which is composed of only nonnegative linear combination of the constraints defining $\mathcal{S}$. Now, we claim that all inequalities of the form (5.5) are satisfied over $\mathcal{S}$. Indeed, the SOS polynomials always take nonnegative values and $p_i(x) \geq 0$ for every $x \in \mathcal{S}$. Therefore,

$$\sum_{\beta\in\mathcal{B}} s_\beta(x) p_1^{\beta_1}(x)\cdots p_{m_p}^{\beta_{m_p}}(x) \geq 0 \quad \forall x \in \mathcal{S}.$$

Similarly, since r_k, $k = 1 : m_r$ are nonzero over $\mathcal{S}$, we have

$$r_1^{\zeta_1}(x)\cdots r_{m_r}^{\zeta_{m_r}}(x) \neq 0 \quad \forall x \in \mathcal{S}$$

for any $\zeta \in \mathbb{Z}_+^{m_r}$. Lastly, it can also be inferred that

$$\sum_{k=1}^{m_t} h_k(x)t_k(x) = 0 \quad \forall x \in \mathcal{S}.$$

Putting all together, validity of (5.5) for every point of $\mathcal{S}$ can be verified immediately. As as result, if we can show that a constraint of the form (5.5) is infeasible, we automatically prove the emptiness of $\mathcal{S}$. Clearly, a sufficient condition for this is the existence of $s_\beta \in \Sigma$, $\beta \in \mathcal{B}$; $\zeta \in \mathbb{Z}_+^{m_r}$ and $h_k \in \mathbb{R}[x_1, \cdots, x_n]$, $k = 1 : m_t$ such that

$$\sum_{\beta \in \mathcal{B}} s_\beta p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}} + (r_1^{\zeta_1} \cdots r_{m_r}^{\zeta_{m_r}})^2 + \sum_{k=1}^{m_t} h_k t_k = 0. \tag{5.6}$$

That is, the above given summation adds up identically to the zero polynomial . Indeed, if $\mathcal{S}$ is not empty, (5.5) must hold true for an $x \in \mathcal{S}$, which clearly cannot happen if (5.6) is satisfied. Therefore, we can conclude that $\mathcal{S} = \emptyset$ if (5.6) holds true.

Example 5.3.1 ([6]). *We ask if the constraints*

$$(x-y)^2 > 0, \; x^3 - y^3 = 0, \; x = 0$$

have a feasible solution. By replacing the strict inequality given above with an inequation, the following equivalent system is obtained

$$\begin{aligned} r_1(x,y) &= (x-y)^2 \neq 0, \\ t_1(x,y) &= x^3 - y^3 = 0, \\ t_2(x,y) &= x = 0. \end{aligned}$$

By choosing $\zeta = 2$, $h_1 = y$ and $h_2 = x^3 - 3x^2y + 6xy^2 - 4y^3$, it can be seen that

$$((x-y)^2)^2 + y(x^3 - y^3) - (x^3 - 3x^2y + 6xy^2 - 4y^3)x = 0.$$

Hence, the system investigated is infeasible.

Surprisingly, the sufficient condition described above also constitutes a necessary condition. This fact is a fundamental theorem of real algebraic geometry and its proof is nontrivial.

Theorem 5.3.2 (Positivstellensatz [7, 35]). *S is empty if and only if there exist $s_\beta \in \Sigma$, $\beta \in \mathcal{B}$; $\zeta \in \mathbb{Z}_+^{m_r}$ and $h_k \in \mathbb{R}[x_1, \cdots, x_n]$, $k = 1 : m_t$ such that*

$$\sum_{\beta \in \mathcal{B}} s_\beta p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}} + (r_1^{\zeta_1} \cdots r_{m_r}^{\zeta_{m_r}})^2 + \sum_{k=1}^{m_t} h_k t_k = 0. \tag{5.7}$$

Positivstellensatz constitutes a very strong theorem since it answers the question of emptiness completely. However, the proof given by Stengle in [7] is nonconstructive. Therefore, it does not lead to an algorithm that produces the multiplies s_β and h_k for which (5.7) holds true. It only ensures the existence of such polynomials when $\mathcal{S} = \emptyset$. Due to this very reason, Positivstellensatz did not find a considerable practical application until recently. In 2000, Parrilo realized that among the multipliers s_β and h_k having fixed degrees, one can search for the ones satisfying (5.7) by solving an LMI problem. If such an LMI turns out to be feasible, emptiness of $\mathcal{S}$ is ensured.

The approach of Parrilo is based on a very simple idea. Basically, the aim is to search for $s_\beta \in \Sigma$ and $h_k \in \mathbb{R}[x_1, \cdots, x_n]$ satisfying (5.7). However, since Σ and $\mathbb{R}[x_1, \cdots, x_n]$ are infinite dimensional spaces this constitutes an intractable task. In order to alleviate this difficulty, he proposed to perform a search over a finite dimensional subspace of multipliers composed of the polynomials having bounded degrees. Let's describe this in detail.

Let N be a nonnegative integer satisfying

$$2N \geq \max\{\deg(p_1), \cdots, \deg(p_{m_p}), \deg(t_1), \cdots, \deg(t_{m_t})\}.$$

Define the index set

$$\mathcal{I} := \{\alpha \in \mathbb{Z}_+^n \| \alpha| \leq 2N\}.$$

In (5.7), instead of arbitrary SOS multipliers, we will employ $s_\beta \in \Sigma$ whose degrees are bounded as

$$\deg(s_\beta p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}}) \leq 2N,\ \beta \in \mathcal{B}.$$

Here, if $\deg(p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}}) > 2N$, the multiplier s_β is chosen as $s_\beta = 0$. Moreover, recall that a nonzero s_β admits the parametrization

$$s_\beta = u_\beta^T W_\beta u_\beta,\ \beta \in \mathcal{B},$$

where $W_\beta \succeq 0$ is the parameter matrix and u_β are the vectors composed of all monomials having degrees up to $\deg(s_\beta)/2$. Similarly, the degrees of the multipliers t_k will be bounded such that

$$\deg(h_k t_k) \leq 2N,\ k = 1 : m_t.$$

Under this condition, they can be expressed as

$$h_k = \sum_{\alpha \in \mathcal{I}_{h_k}} (h_k)_\alpha x^\alpha, \quad k = 1 : m_t,$$

where

$$\mathcal{I}_{h_k} := \{\alpha \in \mathbb{Z}_+^n \| \alpha| \leq \deg(t_k)\}, \quad k = 1 : m_t.$$

Lastly, we employ the decomposition

$$(r_1^{\zeta_1} \ldots r_{m_r}^{\zeta_{m_r}})^2 = \sum_{\alpha \in \mathcal{I}} r_{\alpha\zeta} x^\alpha, \tag{5.8}$$

where $|\zeta| \leq N$ is an integer vector. Here, the coefficient $r_{\alpha\zeta}$ are obtained from the coefficients of the polynomials r_j through (5.8). We did not give this relation since $r_{\alpha\zeta}$ are not parameters. They are just constants that can be obtained from the polynomials r_j.

We are ready to derive equivalent LMI conditions for the existence of multipliers satisfying (5.7) which are subject to degree bounds imposed above. To this end, define $C_{\alpha\beta}$, $\alpha \in \mathcal{I}$, $\beta \in \mathcal{B}$ as the real symmetric matrices resulting from the decomposition

$$u_\beta u_\beta^T p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}} = \sum_{\alpha \in \mathcal{I}} C_{\alpha\beta} x^\alpha, \; \beta \in \mathcal{B},$$

where we could use the index set $\mathcal{I}$ since the degrees of the monomials appearing in above are not larger than $2N$ as one can easily verify. Using this decomposition, one can obtain identities

$$\begin{aligned} s_\beta p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}} &= u_\beta^T W_\beta u_\beta p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}} \\ &= \operatorname{tr}(W_\beta u_\beta u_\beta^T p_1^{\beta_1} \cdots p_{m_p}^{\beta_{m_p}}) \\ &= \sum_{\alpha \in \mathcal{I}} \operatorname{tr}(W_\beta C_{\alpha\beta}) x^\alpha, \quad \beta \in \mathcal{B}. \end{aligned} \tag{5.9}$$

Moreover, it is immediate to verify that

$$h_k t_k = \sum_{\alpha \in \mathcal{I}} \left(\sum_{\xi_1 + \xi_2 = \alpha} (h_k)_{\xi_1} (t_k)_{\xi_2} \right) x^\alpha, \quad k = 1 : m_t.$$

Substituting the last two equalities and (5.8) into (5.7), it can be seen that the polynomial appearing on the left hand side of (5.7) can be expressed as

$$\sum_{\alpha \in \mathcal{I}} \left(\sum_{\beta \in \mathcal{B}} \operatorname{tr}(W_\beta C_{\alpha\beta}) + r_{\alpha\zeta} + \sum_{\xi_1 + \xi_2 = \alpha} (h_k)_{\xi_1} (t_k)_{\xi_2} \right) x^\alpha. \tag{5.10}$$

Keeping in mind that $W_\beta \succeq 0$, we obtain the following sufficient condition for emptiness of $\mathcal{S}$.

Proposition 5.3.3. *S is empty if the LMI*

$$\begin{gathered}\sum_{\beta\in\mathcal{B}} \operatorname{tr}(W_\beta C_{\alpha\beta}) + r_{\alpha\zeta} + \sum_{\xi_1+\xi_2=\alpha} (h_k)_{\xi_1}(t_k)_{\xi_2} = 0, \quad \alpha \in \mathcal{I}, \\ W_\beta \succeq 0, \quad \beta \in \mathcal{B}\end{gathered} \tag{5.11}$$

with variables W_β and h_k has a feasible solution.

Proof. In the light of Theorem 5.3.2, the result can be obtained by equating coefficients of (5.10) zero. □

Keep in mind that the approach described above only gives sufficient conditions. Therefore, if (5.11) turns out to be infeasible, we cannot conclude nonemptiness of $\mathcal{S}$. The main reason is the fact that to be able to obtain the LMI (5.11) we fixed the relaxation order N, and hence, the exponent vector ζ and degrees of s_β and h_k. However, there would be multipliers having higher degrees which satisfies (5.7), and hence, $\mathcal{S}$ would be empty. In order to alleviate this difficulty, one should have a knowledge about bounds on the degrees of the multipliers appearing in Positivstellensatz. More specifically, consider a function that gives an N depending on the dimension n and the maximum of the degrees of the polynomials defining $\mathcal{S}$ for which Positivstellensatz is satisfied when $\mathcal{S} = \emptyset$. Obviously, if such a function exists, we can obtain a necessary and sufficient condition from (5.11) by setting N accordingly. Unfortunately, such a result does not exists to the best of our knowledge. However, a similar result exists for Nullstellensatz, which constitutes a counterpart of Positivstellensatz developed for complex solution of system of polynomial equality constraints [36]. Therefore, it is reasonable to expect finding degree bounds for Positivstellensatz.

Note that the certificate (5.11) subsumes the S-procedure as a special case and, in general, its leads to much more better results than the S-procedure as N is increased. However, we should note that the size of this LMI increases very rapidly along with N. Therefore, obtaining better certificates become computationally intensive for even problems of moderate size.

5.4. Solving Optimization Problems

As in the previous chapter, we also want to develop relaxations for optimization problems. In order to keep the technical details at the minimum level and to be in line with the works in the literature, this time we will concentrate to the set determined by only nonstrict inequalities

$$\mathcal{S} := \{x \in \mathbb{R}^n |\ p_i(x) \geq 0,\ i = 1 : m_p\}$$

where $p_i \in \mathbb{R}[x_1, \cdots, x_n]$, $i = 1 : m_p$. The goal is to find approximation bounds on the optimization problem

$$\begin{aligned} \gamma^* := \quad & \inf \quad p_0(x) \\ & \text{s.t.} \quad x \in \mathcal{S}, \end{aligned} \tag{5.12}$$

where $p_0 \in \mathbb{R}[x_1, \cdots, x_n]$. In below, we describe two related approaches that can utilized for this purpose, which are generalizations of the S-procedure based relaxation and SDP relaxation.

The first approach that will be investigated is based on Positivstellensatz. Remember that (5.12) is equivalent to

$$\begin{aligned} \gamma^* = \quad & \sup \quad t \\ & \text{s.t.} \quad p_0(x) - t \geq 0 \quad \forall x \in \mathcal{S}, \end{aligned} \tag{5.13}$$

the constraint of which is nonnegativity of the function $p_0 - t$ over $\mathcal{S}$. Because $p_i(x) \geq 0$ for every $x \in S$ and s_i are positive semidefinite polynomials, this constraint is satisfied if the representation

$$p_0 - t = \sum_{i=1}^{m_p} s_i p_i, \tag{5.14}$$

is attained for some $s_i \in \Sigma$, $i = 1 : m_p$. As can be anticipated, this constitutes a weaker form of Positivstellensatz. Therefore, it gives a sufficient condition for nonnegativity of $p_0 - t$ over $\mathcal{S}$. Hence, if we replace the constraint of (5.13) with it, we attain the relaxed problem

$$\begin{aligned} \sup \quad & t \\ \text{s.t.} \quad & p_0 - t = \sum_{i=1}^{m_p} s_i p_i, \\ & s_\beta \in \Sigma, \end{aligned} \tag{5.15}$$

the optimal solution of which gives lower bound on γ^*. However, as will be shown below, (5.15) yields to the exact solution when certain assumptions are satisfied. Note that in above we used (5.14) instead of the original Positivstellensatz since the former leads to an LMI while the latter does not for the optimization problem considered.

Clearly, (5.15) is again an infinite dimensional optimization problem, and hence, it is hard to solve in practice. In order to alleviate this difficultly, we employ the same trick demonstrated earlier by restricting the degrees of the SOS polynomials. To this end, let N be the order of the relaxation, which is a positive integer such that

$$2N \geq \max\{\deg(p_0), \cdots, \deg(p_{m_p})\}.$$

We employ the SOS polynomials $s_i \in \Sigma$, $i = 0 : m_p$ with the following restriction on their degrees. Choose

$$\deg(s_0) = 2N \tag{5.16}$$

and $\deg(s_i)$, $i = 1 : m_p$ as the largest even integers satisfying

$$\deg(s_i p_i) \leq 2N, \; i = 1 : m_p. \tag{5.17}$$

The vectors u_i, $i = 0 : m_p$ are composed of all monomials having degrees up to

$\deg(s_i)/2$. Define $C_{\alpha i}$ as the matrices in the decompositions

$$\begin{aligned} u_0 u_0^T &= \sum_{\alpha\in\mathcal{I}} C_{\alpha 0} x^\alpha \\ u_i u_i^T p_i &= \sum_{\alpha\in\mathcal{I}} C_{\alpha i} x^\alpha, \quad i = 1 : m_p. \end{aligned} \tag{5.18}$$

Proposition 5.4.1. *The optimal solution of the SDP*

$$\begin{aligned} \rho_d^*(N) := \quad \sup \quad & (p_0)_0 - \sum_{i=1}^{m_p} \operatorname{tr}(W_i C_{0i}) \\ \text{s. t.} \quad & \sum_{i=1}^{m_p} \operatorname{tr}(W_i C_{\alpha i}) = p_\alpha, \; \alpha \neq 0, \; \alpha \in \mathcal{I} \\ & W_i \succeq 0, \quad i = 1 : m_p. \end{aligned} \tag{5.19}$$

satisfies $\rho_d^*(N) \leq \gamma^*$.

Proof. Using the multipliers degrees of which are constrained as above, (5.14) can be expressed as

$$\begin{aligned} \sum_{\alpha\in\mathcal{I}} (p_0)_\alpha x^\alpha - t = \; & u_0^T W_0 u_0 + \sum_{i=1}^{m_p} u_i^T W_i u_i p_i \\ = \; & \operatorname{tr}(W_0 u_0 u_0^T) + \sum_{i=1}^{m_p} \operatorname{tr}(W_i u_i u_i^T p_i) \\ = \; & \sum_{\alpha\in\mathcal{I}} \left(\sum_{i=0}^{m_p} \operatorname{tr}(W_i C_{\alpha i}) \right) x^\alpha. \end{aligned}$$

By equating the coefficients of the polynomials on the both sides of the preceding equation, it can be seen that if there exist $W_i \succeq 0$, $i = 1 : m_p$ such that

$$\begin{aligned} t = (p_0)_0 - \sum_{i=0}^{m_p} \operatorname{tr}(W_i C_{0i}), \\ \sum_{i=0}^{m_p} \operatorname{tr}(W_i C_{\alpha i}) = p_\alpha, \; \alpha \neq 0, \; \alpha \in \mathcal{I}, \end{aligned}$$

then (5.14) is satisfied for $s_i = u_\beta W_\beta u_\beta \in \Sigma$. Replacing the constraint of (5.15) with the above given system, (5.19) is attained. Since the constraint of (5.15) is replaced with a sufficient condition, it can be inferred that $\rho_d^*(N) \leq \gamma^*$. $\square$

The second approach that will be developed is related with the theory of moments and can be considered as an extension of the SDP relaxation. It is proposed by Lasserre in [8]. Let's describe the underlying idea.

Clearly, the matrix inequalities

$$u_i(x)u_i^T(x) \succeq 0,\ i = 0 : m_p$$

hold true for every $x \in \mathbb{R}^n$. Therefore, it is easy to see that (5.12) is equivalent to

$$\begin{aligned} \gamma^* = \quad & \inf \quad p_0(x) \\ & \text{s.t.} \quad u_0(x)u_0^T(x) \succeq 0, \\ & \qquad\ \ u_i(x)u_i(x)^T p_i(x) \succeq 0,\ i = 1 : m_p. \end{aligned} \tag{5.20}$$

Using (5.18), this can be written as

$$\begin{aligned} \gamma^* = \quad & \inf \quad \sum_{\alpha \in \mathcal{I}} (p_0)_\alpha x^\alpha \\ & \text{s.t.} \quad \sum_{\alpha \in \mathcal{I}} C_{i\alpha} x^\alpha \succeq 0,\ i = 0 : m_p. \end{aligned} \tag{5.21}$$

Defining the new variables $y_\alpha := x^\alpha$, the following relaxation is obtained.

Proposition 5.4.2. *The SDP*

$$\begin{aligned} \rho_p^*(N) := \quad & \inf \quad \sum_{\alpha \in \mathcal{I}} (p_0)_\alpha y_\alpha \\ & \text{s.t.} \quad \sum_{\alpha \in \mathcal{I}} C_{i\alpha} y_\alpha \succeq 0,\ i = 0 : m_p. \end{aligned} \tag{5.22}$$

satisfies $\rho_p^*(N) \leq \gamma^*$.

Proof. If (5.21) is infeasible, there is nothing to prove. Therefore, suppose x^* is an optimal solution of (5.21). Then, trivially, $y_\alpha = (x^*)^\alpha$ is a feasible point of (5.22) with the objective value γ^*, which leads to the desired result. □

The relaxation introduced above is called the moment relaxation. Notice that if p_i are nonconstant quadratic polynomials and $N = 1$, it can be inferred that $u_0 = [1, x_1, \cdots, x_n]^T$ and $u_i = 1$, $i = 1 : m_p$ due to the degree bounds (5.16) and (5.17), respectively. From this, it is easy to see that the first order moment relaxation just gives the SDP relaxation (4.12).

Note that (5.19) and (5.22) in fact lead to the same result just like in the S-procedure and the SDP relaxation. This is justified by the next proposition

Proposition 5.4.3. *Assume there exist a strictly feasible solution of (5.19). The relaxations (5.19) and (5.22) are SDP dual of each other, and hence,* $\rho_p^*(N) = \rho_d^*(N) =: \rho^*(N)$.

Proof. This follows from standard SDP duality. □

The approaches described above produce a sequence of lower bounds $\rho^*(N)$ such that the next element in this sequence is never worse than the previous one. That is, $\rho^*(N) \leq \rho^*(N+1)$. This can be verified using any of these methods. For instance, the set of SOS multipliers s_i employed to construct the Nth order relaxation Proposition 5.4.1 constitute a subset of the ones employed in $N+1$th order relaxation. Hence, the bound provided by the latter cannot be smaller than that of the former. Based on this fact one naturally expects to get better approximations by increasing N. The following theorem shows that this expectation holds true under a moderate assumption.

Theorem 5.4.4 (Putinar, [11])**.** *Assume there exists polynomials* $\bar{s}_i \in \Sigma$, $i = 0 : m_p$ *such that the set*

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \bar{s}_0(x) + \sum_{i=1}^{m_p} \bar{s}_i(x) p_i(x) \geq 0 \right\}$$

is compact. If $p(x) > 0$*, where* $p \in \mathbb{R}[x_1, \cdots, x_n]$*, for every* $x \in \mathcal{S}$*, there exist* $s_i \in \Sigma$*,*

$i = 0 : m_p$ such that

$$p = s_0 + \sum_{i=1}^{m_p} s_i p_i.$$

Remark 5.4.5. *The assumption of the theorem does not restrict the problem seriously. For example, if we known $\mathcal{S}$ is bounded with the radius R, which would be available for many physical applications, then simply adding the inequality $p_{m_p} = R^2 - x^T x \geq 0$ will ensure that the assumption will be satisfied for $\bar{s}_i = 0$, $i = 0 : m_p - 1$ and $\bar{s}_{m_p} = 1$.*

The following is a direct consequence of the theorem.

Corollary 5.4.6. *If the assumption of the preceding theorem hold true,*

$$\gamma^* = \lim_{N\to\infty} \rho^*(N).$$

Proof. Suppose t is strictly smaller than γ^* and arbitrarily close to γ^*. Because $t < \gamma^*$, $p_0 - t$ is positive over $\mathcal{S}$, and hence, due to the above given theorem there exist multipliers $\hat{s}_i$, $i = 0 : m_p$ of certain degrees such that (5.14) is satisfied. Choose N such that $\deg(s_\beta) \geq \deg(\hat{s}_\beta)$, $i = 1 : m_p$. This ensures that $\gamma^* - \rho^*(N) \leq \gamma^* - t$ since the multipliers $\hat{s}_\beta$ are contained in the Nth order relaxation. Because t is arbitrarily close to γ^*, we are done. $\square$

The foregoing corollary shows that the global optimal solution can be approximated as much as desired by increasing the order of the relaxation. Clearly, this is a very desirable property. However, it does not ensure to obtain the exact result for a finite N. It only ensures convergence in the limit. Moreover, as in the previous section, as N increases, the complexity of the relaxation grows very rapidly. Hence, in general, obtaining an accurate result would be computationally intensive. This is supported with the fact that a degree bound does not exist as function of the degrees of the polynomials defining the problem [13].

The useful features of the SDP relaxation described in Section 4.4 also extend to the moment relaxation. With some abuse of notation, let us denote the feasible region of (5.22) as $\mathcal{R}(N)$ and its projection on the n dimensional space of the variables $(y_{10\cdots 0}, \cdots, y_{0\cdots 01})$ as $\mathcal{P}_R(N)$. From (5.20) and (5.21), it is easy to see that if $\bar{x} \in \mathcal{S}$ then $\bar{y}_\alpha = \bar{x}^\alpha \in \mathcal{R}(N)$, $\alpha \in \mathcal{I}$, which implies $(\bar{y}_{10\cdots 0}, \cdots, \bar{y}_{0\cdots 01}) = \bar{x} \in \mathcal{P}_R(N)$. Therefore, $\mathcal{P}_R(N)$ is a convex set containing $\mathcal{S}$. Moreover, as in Proposition 4.4.3, it can be shown that optimizing a linear function p_0 over $\mathcal{P}_R(N)$ is equivalent to optimizing it over $\mathcal{R}(N)$. Hence, moment relaxation constitutes a convex outer approximation technique just like the SDP relaxation. In fact, the former subsumes the later as a special case (clearly, $\mathcal{R}(1)$ gives us the feasible set of the SDP relaxation) and $\mathcal{P}_{\mathcal{R}}(N)$ converges to the convex hull as the order N is increased. Intuitively, the convergence follows from Corollary 5.4.6 because for any linear objective function the relaxation gap can be made arbitrarily small by choosing N large enough.

Example 5.4.7. *Consider the optimization problem investigate in Example 4.1.7 and Example 4.4.4. The first order relaxation corresponding to this problem just gives the SDP relaxation, which turned out to be non-exact. The second order primal relaxation (5.22) is*

$$
\begin{aligned}
\rho^*(2) = \quad \inf \quad & -2y_{10} - 2y_{01} \\
\text{s. t.} \quad & \begin{bmatrix} 4y_{20} - y_{40} - y_{22} & 4y_{11} - y_{31} - y_{13} & 4y_{10} - y_{30} - y_{12} \\ 4y_{11} - y_{31} - y13 & 4y_{02} - y_{22} - y_{04} & 4y_{01} - y_{21} - y_{03} \\ 4y_{10} - y_{30} - y_{12} & 4y_{01} - y_{21} - y_{03} & 4 - y_{20} - y_{02} \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} -y_{40} + y_{22} + y_{20} & -y_{31} + y_{13} + y_{11} & -y_{30} + y_{12} + y_{10} \\ -y_{31} + y_{13} + y11 & -y_{22} + y_{04} + y_{02} & -y_{21} + y_{03} + y_{01} \\ -y_{30} + y_{12} + y_{10} & -y_{21} + y_{03} + y_{01} & -y_{20} + y_{02} + 1 \end{bmatrix} \succeq 0 \\
& -2 \begin{bmatrix} y_{21} & y_{12} & y_{11} \\ y_{12} & y_{03} & y_{02} \\ y_{11} & y_{02} & y_{01} \end{bmatrix} \succeq 0
\end{aligned}
$$

.

The optimal solution of this relaxation is $\rho^*(2) = \gamma^* = -2$, and hence, it gives the global optimal solution of the original problem. Moreover, it can be verified that, in fact, $\mathcal{P}_{\mathcal{R}}(2) = \mathbf{co}(\mathcal{S})$. Therefore, the second order relaxation yields to the exact result for optimizing any linear objective function over the feasible set considered.

The global optimality tests described in Section 4.4 extends to moment relaxations as well. Suppose y^*_α, $\alpha \in \mathcal{I}$ is an optimal solution of (5.22). If the objective p_0 is linear and the projection $x^* := (y^*_{10\cdots0}, \cdots, y^*_{0\cdots01})$ satisfies $x^* \in \mathcal{S}$, then x^* constitutes a global optimal solution of (5.12), the proof of which is a simple generalization of that of Proposition 4.4.5. In case p_0 is not linear, one can check the rank condition

$$\operatorname{rank}\left(\sum_{\alpha\in\mathcal{I}} C_{i\alpha} y_\alpha\right) = 1, \; i = 0 : m_p.$$

If it is satisfied, $x^* = (y^*_{10\cdots0}, \cdots, y^*_{0\cdots01}) \in \mathcal{S}$ and x^* is a global optimal solution, which is a simple extension of Proposition 4.4.6.

5.5. Summary

In this chapter we introduced two fundamental concepts. One is Positivstellensatz which gives necessary and sufficient conditions for feasibility of a system of polynomial constraints. The other is a class of relaxation methods which are based on Positivstellensatz and developed by Parrilo [9] and Lasserre [8]. Although the Positivstellensatz gives necessary and sufficient conditions, the relaxations only provide certificates of feasibility or lower bounds. However, the bounds produced can be improved successively and they are guaranteed to converge to the global optimal solution in the limit under some mild assumptions. Therefore, they can be employed to approximate the global optimal solution of a polynomial optimization problem as much as desired. But as the order of relaxation increased the corresponding LMIs gets complicated rapidly. Thus, in practice, one may employ these methods to obtain crude approximations of complicated problems. Note that the relaxation methods introduced subsumes the S-procedure and SDP relaxation as special cases and share some common geometric properties. In particular, the moment relaxation produces outer approximations of the convex hull of the feasible region, which converges to the convex hull as the relaxation order is increased.

6. CONVEX HULL OF THE REGION DETERMINED BY TWO QUADRATIC CONSTRAINTS IN $\mathbb{R}^n$

In this chapter, we introduce one of our main contributions. Consider the set

$$\mathcal{S} := \{x \in \mathbb{R}^n |\, q_1(x) > 0,\ q_2(x) > 0\}, \tag{6.1}$$

where p_1, $p_2 : \mathbb{R}^n \to \mathbb{R}$ are quadratic polynomials. It is shown that, except for some cases having trivial solutions, $\mathbf{co}(\mathcal{S})$ is the intersection of two quadratic constraints which are the convex combinations of the ones defining $\mathcal{S}$ itself. The proof is constructive and leads to an algorithm for computing the quadratic description, which does not have to be composed of concave constraints. However, we propose a method to replace them with LMIs, and hence, show that $\mathbf{co}(\mathcal{S})$ admits an LMI representation.

An important consequence of the results described above is the fact that the problem of finding the optimal value of a linear function over $\mathcal{S}$ given in (6.1) can be cast as an SDP and solved exactly and efficiently. As will be shown through this chapter, in general, none of the relaxation methods described in preceding chapters is guaranteed to solve this specific problem exactly. Moreover, they also behave worse than our approach in terms of computational complexity.

6.1. Preliminaries

Recall that

$$q_i = \begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad i = 1:2,$$

where $Q_1, Q_2 \in \mathbb{S}^{n+1}$. When deemed necessary, the following expression will also be utilized

$$q_i = x^T A_i x + 2b_i^T x + c_i, \quad i = 1:2,$$

where $A_i \in \mathbb{S}^n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ arises from the partitioning

$$Q_i = \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix}, \quad i = 1:2.$$

We frequently make use of the pencil of quadratics

$$\mathbf{q}_\lambda := (1-\lambda)q_1 + \lambda q_2,$$

which is a parameterized family of quadratic functions, and the associated symmetric matrix pencil

$$\mathbf{Q}_\lambda := (1-\lambda)Q_1 + \lambda Q_2.$$

Moreover, the parameterized family of the sets

$$\mathcal{S}_\lambda := \{x \in \mathbb{R}^n |\, \mathbf{q}_\lambda(x) > 0\}, \quad \lambda \in [0,1]$$

will be employed to characterize $\mathbf{co}(\mathcal{S})$. The following is a standing assumption through this chapter.

Assumption 6.1.1. *$\mathcal{S}$ is not empty or, equivalently, there does not exists a $\lambda \in [0,1]$ such that $\mathbf{Q}_\lambda \preceq 0$. Trivially, the latter condition is a direct consequence of Lemma 4.5.1.*

It is a well known fact that for a pencil $\mathbf{Q}_\lambda$, rank$(\mathbf{Q}_\lambda)$ is a constant function of the parameter $\lambda \in \mathbb{C}$ except at a finite number of points where rank drops [37]. The points at which this happens are defined as the *generalized eigenvalues* (GEV) of the pencil. Note that although eigenvalues of a symmetric matrix are always real, GEVs of a symmetric matrix pencil may take complex values. In what follows, the ascending sequence of numbers $\alpha_i \in \mathbb{R}$, $i = 1 : n_G$ will be utilized to denote the real GEVs of $\mathbf{Q}_\lambda$ lying in $[0, 1]$ interval. Note that a GEV is assumed to appear in this sequence as many times as its algebraic multiplicity. For a formal definition of multiplicity of a GEV see [37, 38, 39]. An equivalent definition that will be utilized in the thesis is given in Remark A.2.4. For the sake of notational simplicity, we also define $\alpha_0 := 0$ and $\alpha_{n_G+1} := 1$.

The number of positive eigenvalues of a matrix A is denoted as $\pi(A)$. Consider the set

$$\Lambda := \{\lambda \in [0, 1] \mid \pi(\mathbf{Q}_\lambda) = 1\}. \tag{6.2}$$

This set constitutes an important object in our study. Main reason for this is the fact that for any $\lambda \in \Lambda$, $\mathcal{S}_\lambda$ possesses a nice geometric property. It is either a convex set or union of two disjoint convex sets, which follows from Proposition 6.3.1 that will be given later in this chapter. Due to this very property, it will be possible to characterize $\mathbf{co}(\mathcal{S})$ in terms of such sets. The following lemma reveals the structure of Λ under Assumption 6.1.1.

Lemma 6.1.2. *If $\Lambda \neq \emptyset$, then Λ is either a closed interval*

$$\Lambda = [\alpha_\ell, \alpha_{\ell+1}] \tag{6.3}$$

or the union of two closed intervals such that

$$\Lambda = [\alpha_\ell, \alpha_{\ell+1}] \cup [\alpha_{\ell+2}, \alpha_{\ell+3}]. \tag{6.4}$$

The proof is not trivial and given in Appendix A.2. In the following sections, we use the connected components of Λ,

$$\Lambda_i := [\underline{\lambda}_i, \overline{\lambda}_i], \quad i = 1 : n_c, \tag{6.5}$$

where $n_c \leq 2$. Realize that when (6.4) holds true, even if it is composed of two intervals, Λ has a single connected component if $\alpha_{\ell+1} = \alpha_{\ell+2}$. In this case we have $\underline{\lambda}_1 = \alpha_\ell$ and $\overline{\lambda}_1 = \alpha_{\ell+3}$.

Lastly, define

$$\Omega := \{\lambda \in [0, 1] | (1 - \lambda)A_1 + \lambda A_2 \preceq 0\}.$$

Clearly, this set is a closed interval. Moreover, it has two important properties given in the next proposition.

Proposition 6.1.3. *The following hold true:*

i) $\mathcal{S}_\lambda$ *is a convex set if and only if* $\lambda \in \Omega$*,*

ii) $\Omega \subseteq \Lambda_i$ *for an* $i \in 1 : n_c$*.*

Proof. i) $\mathcal{S}_\lambda$ is convex if and only if $\mathbf{q}_\lambda$ is a concave function or equivalently its Hessian, $(1 - \lambda)A_1 + \lambda A_2$, is negative semidefinite

ii) Due to Poincaré separation theorem, Corollary 4.3.16 in [40], $(1-\lambda)A_1+\lambda A_2 \preceq 0$ implies $\pi(\mathbf{Q}_\lambda) \leq 1$ for every $\lambda \in \Omega$. Moreover, Assumption 6.1.1 implies $\pi(\mathbf{Q}_\lambda) \geq 1$, $\lambda \in [0, 1]$. Hence, $\pi(\mathbf{Q}_\lambda) = 1$ for every $\lambda \in \Omega$. Consequently, we can infer that $\Omega \subseteq \Lambda$. Because Ω is an interval, it must be contained in a connected component of Λ. □

6.2. Main Results

We are ready to introduce the main outcomes of our work. The first one is the following theorem which gives a characterization of $\mathbf{co}(\mathcal{S})$.

Theorem 6.2.1.

$$\mathbf{co}(\mathcal{S}) = \begin{cases} \bigcap_{\lambda \in \Lambda_j} \mathcal{S}_\lambda = \mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j}, & \textit{if } \Omega \neq \emptyset \\ \\ \mathbb{R}^n, & \textit{otherwise,} \end{cases}$$

where j is the index satisfying $\Omega \subseteq \Lambda_j$, $j \in 1 : n_c$ provided that $\Omega \neq \emptyset$.

As can be anticipated, this theorem lends itself to an algorithm for the construction of the convex hull, which will be given at the end of the section. Therefore, if p_0 is a liner function, the optimal value of p_0 over $\mathcal{S}$ can be attained exactly by solving

$$\begin{aligned} \inf \quad & p_0(x) \\ \text{s. t.} \quad & x \in \mathbf{co}(\mathcal{S}), \end{aligned} \tag{6.6}$$

due to Proposition 3.1.10. However, there is a little issue to be addressed here. If $\mathcal{S} \neq \emptyset$ and $\mathbf{co}(\mathcal{S}) \neq \mathbb{R}^n$, $\mathbf{co}(\mathcal{S})$ admits the quadratic description

$$\begin{aligned} \mathbf{co}(\mathcal{S}) = \; & \mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j} \\ = \; & \{x \in \mathbb{R}^n | \; \mathbf{q}_{\underline{\lambda}_j}(x) > 0, \; \mathbf{q}_{\overline{\lambda}_j}(x) > 0\}. \end{aligned}$$

Due to the first part of Proposition 6.1.3, when Ω is a strictly proper subset of Λ_j, at least one of the sets $\mathcal{S}_{\underline{\lambda}_j}$ or $\mathcal{S}_{\overline{\lambda}_j}$ will not be convex, which in turn implies non-concavity of the corresponding quadratic function(s) $\mathbf{q}_{\underline{\lambda}_j}$ and/or $\mathbf{q}_{\overline{\lambda}_j}$. Therefore, in this case, although $\mathbf{co}(\mathcal{S})$ is a convex set, (6.6) will be a nonconvex optimization problem if the quadratic constraints are utilized for the description of the convex hull. Fortunately, we can alleviate this difficulty. As will be shown in Section 6.3, it is always possible to derive an LMI representation of $\mathbf{co}(\mathcal{S})$ from $\mathbf{q}_{\underline{\lambda}_j}$ and $\mathbf{q}_{\overline{\lambda}_j}$. Therefore, the optimal solution of (6.6) can be attained exactly by solving an SDP problem.

When the losslessness results presented in Section 4.5 is investigated, it can be seen that assumptions of none of them are guaranteed to be satisfied for two general quadratic functions and a linear function. Therefore, the relaxations based on the S-procedure or its extensions given there are not guaranteed to solve the optimization problem discussed above exactly. In fact, one can easily find counter examples for which S-procedure leads to very bad bounds, an instance of which is given below. Therefore, our work constitutes a considerable improvement over the S-procedure. On the other hand, as we will see the moment relaxations may also behave much more worse than our approach for some instances of the problem considered in terms of computational complexity.

Our second contribution is another exactness condition for the S-procedure and the SDP relaxation that can be added to the list of losslessness results presented Section 4.5. This condition, in fact, directly follows from the preceding theorem and the below given proposition, which is a modified version of Proposition 4.4.7 for two strict inequality constraints.

Proposition 6.2.2.

$$\mathcal{P}_{\mathcal{R}} = \bigcap_{\lambda \in \Omega} \mathcal{S}_{\lambda} = \mathcal{S}_{\underline{\omega}} \cap \mathcal{S}_{\overline{\omega}},$$

where $\underline{\omega}$ *and* $\overline{\omega}$ *are minimum and maximum elements of* Ω*.*

Proof is given in Appendix A.1.

This proposition reveals why the SDP relaxation does not always lead to the exact solution for the problem considered. The approximation of $\mathbf{co}(\mathcal{S})$ given by the SDP relaxation is equivalent to the intersection of the all convex sets in the family $\{\mathcal{S}_{\lambda}| \ \lambda \in [0,1]\}$. However, Theorem 6.2.1 suggests that in order to obtain the convex hull it is necessary to take into account some nonconvex members of this family when Ω is a proper subset of Λ_j. A direct consequence of this fact is the following losslessness result and its corollary which constitute the second main outcome of our work.

Lemma 6.2.3. *The following are equivalent:*

i) $\mathcal{P}_{\mathcal{R}} = \mathbf{co}(\mathcal{S})$.

ii) $\Omega = \Lambda_i$ *for an* $i \in 1 : n_c$.

iii) $\mathbf{co}(\mathcal{S})$ *is feasible region of two concave quadratic constraints.*

Corollary 6.2.4. *Consider the optimization problem*

$$\begin{aligned} \gamma^* = \quad & \inf \quad p_0(x) \\ & \text{s. t.} \quad p_1(x) > 0,\ p_2(x) > 0, \end{aligned} \tag{6.7}$$

where the objective function p_0 *is linear. The SDP relaxation (4.12) gives the global optimal solution* $(i.e. \gamma^* = \rho^*)$ *if* $\Omega = \Lambda_i$ *for an* $i \in 1 : n_c$.

Notice that in the corollary, the condition $\Omega = \Lambda_j$ is sufficient for the exactness of the optimization problem but not necessary. Main reason behind this is the fact that even if $\mathcal{P}_{\mathcal{R}} \neq \mathbf{co}(\mathcal{S})$ there may be some part of the boundary of $\mathcal{P}_{\mathcal{R}}$ where it coincides with the convex hull. This was described earlier and demonstrated in Example 4.4.4.

Let us demonstrate our results on a simple example.

Example 6.2.5. *Consider the pair of constraints*

$$\begin{aligned} p_1 &= -x_1^2 + x_2^2 - 2x_1 - 2 > 0 \\ p_2 &= x_1^2 - x_2^2 - 2x_1 + 2 > 0. \end{aligned}$$

Before trying to obtain the convex hull, first, one needs to ensure that the set of points satisfying this system is not empty. This can be done simply using the S-Lemma (Proposition 4.5.1). It turns out that the feasible region $\mathcal{S}$ *is not empty, which is shown in Figure 6.1.a.*

Next we need to find out if the convex hull is $\mathbb{R}^2$ *by checking wether* Ω *is empty* *or not. For the constraints in hand, Ω is the set of points satisfying*

$$(1-\lambda)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \preceq 0, \quad \lambda \in [0,1],$$

which is the singleton $\Omega = \{0.5\}$. Since, $\Omega \neq \emptyset$, the convex hull problem has a nontrivial solution. Therefore, we need to attain Λ which is determined by the inertia constraint

$$\pi\left((1-\lambda)\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\right) = 1, \quad \lambda \in [0,1].$$

Λ is found to be the union of the intervals $\Lambda_1 = [0, 1/2 - 1/(2\sqrt{2})]$ and $\Lambda_2 = [1/2, 1/2 + 1/(2\sqrt{2})]$. Since, $\Omega \subseteq \Lambda_2$, $\mathbf{co}(\mathcal{S})$ is $\mathcal{S}_{\underline{\lambda}_2} \cap \mathcal{S}_{\overline{\lambda}_2}$, or equivalently the feasible region of the constraints

$$\begin{aligned} -2x_1 &> 0, \\ x_1^2 - x_2^2 - 2\sqrt{2}x_1 + 2 &> 0. \end{aligned} \tag{6.8}$$

The set of points satisfying each constraint given above are depicted separately in Figure 6.1.b and c while the convex hull is illustrated in Figure 6.1.d. Notice that the feasible region of the first constraint is convex since $1/2 \in \Omega$. On the other hand, that of second one is nonconvex because $1/2 + 1/(2\sqrt{2}) \notin \Omega$.

Now, let us compare the convex hull with it outer approximation given by the SDP relaxation. The SDP relaxation is

$$\begin{aligned} -y_{11} + y_{22} - 2x_1 - 2 &> 0, \\ y_{11} - y_{22} - 2x_1 - 2 &> 0, \\ \begin{bmatrix} y_{11} & y_{12} & x_1 \\ y_{12} & y_{22} & x_2 \\ x_1 & x_2 & 1 \end{bmatrix} &\succeq 0. \end{aligned}$$

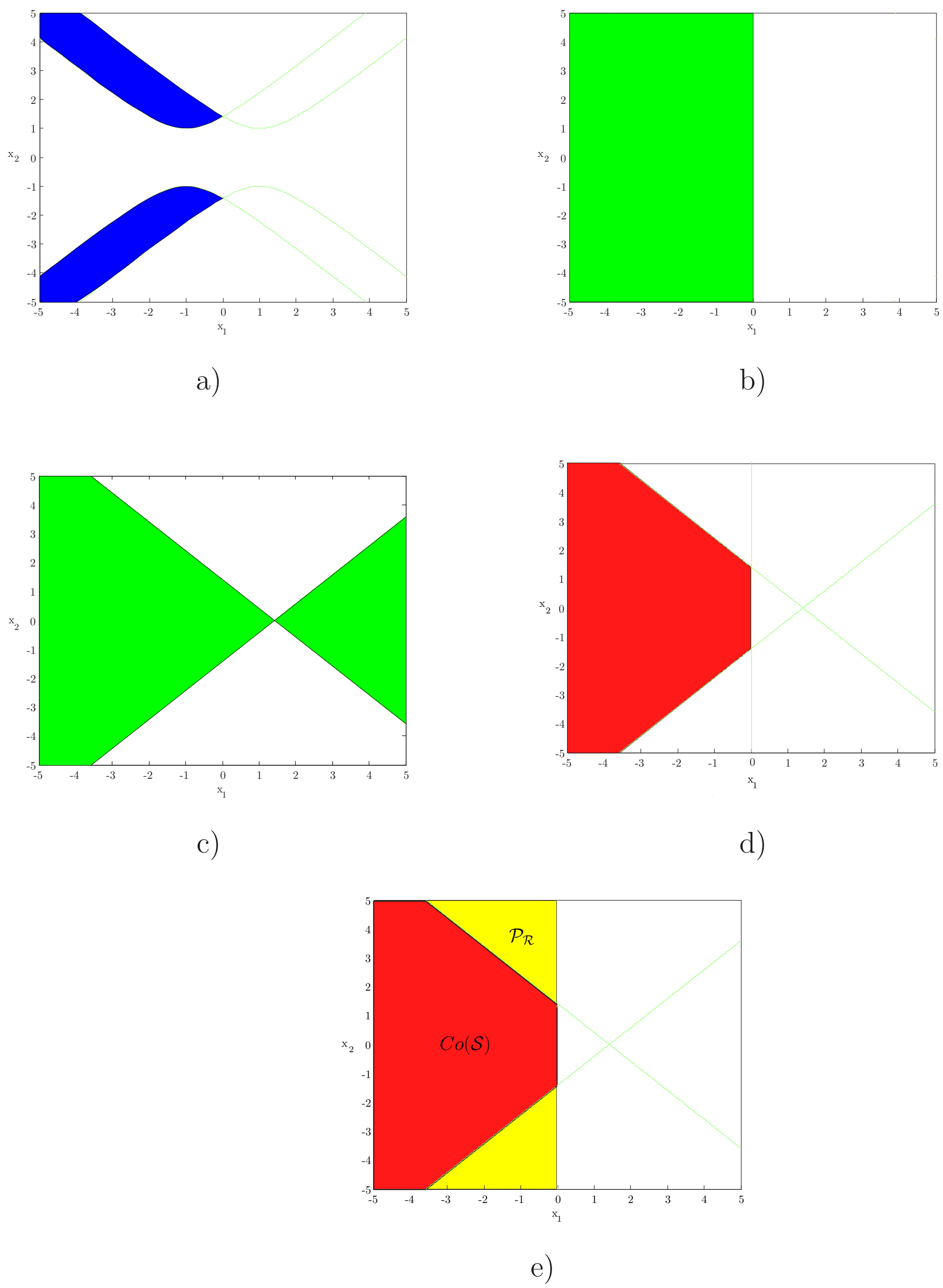


Figure 6.1. Sets from Example 6.2.5: a) $\mathcal{S}$; b) $\mathcal{S}_{\underline{\lambda}_2}$; c) $\mathcal{S}_{\overline{\lambda}_2}$; d) $\mathbf{co}(\mathcal{S})$; e) $\mathbf{co}(\mathcal{S})$ and $\mathcal{P}_{\mathcal{R}}$

Using Proposition 6.2.2, it can be seen that $\mathcal{P}_{\mathcal{R}}$ is the region $x_1 < 0$, which is depicted in Figure 6.1.e along with the convex hull. As can be anticipated, $\mathcal{P}_{\mathcal{R}}$ does not constitute good approximation of $\mathbf{co}(\mathcal{S})$ because the convex hull has a non-concave constraint in this definition. Moreover, it can be seen from the figure that the optimal value of several linear objective functions take finite values over $\mathbf{co}(\mathcal{S})$ while their optimal value over $\mathcal{P}_{\mathcal{R}}$ is $-\infty$. Therefore, for these problems the bounds provided by the SDP relaxation become practically useless.

Table 6.1. Computation of convex hull

input: matrices Q_1 and Q_2
output: $\underline{\lambda}_j$ and $\overline{\lambda}_j$ (endpoints of the interval Λ_j containing Ω)
if there does not exist a $\lambda \in [0,1]$, $(1-\lambda)Q_1 + \lambda Q_2 \preceq 0$ **then**
    return $\mathbf{co}(\mathcal{S}) = \emptyset$;
else
    if there exist $\omega \in [0,1]$, $(1-\omega)A_1 + \omega A_2 \preceq 0$ **then**
        return $\mathbf{co}(\mathcal{S}) = \mathbb{R}^n$;
    else
        if there exists a ℓ such that $\alpha_\ell \le \omega \le \alpha_{\ell+1}$, $\alpha_\ell \neq \alpha_{\ell+1}$, $\pi_\ell = 1$ **then**
            $\underline{\ell} = \min i$ such that $\max\{0, \ell-2\} \le i \le \ell$ and $\pi_j = 1$, $j = i : \ell$;
            $\overline{\ell} = \max i$ such that $\ell \le i \le \min\{\ell+2, n_G+1\}$ and $\pi_j = 1$, $j = \ell : i$;
            return $\underline{\lambda}_j = \alpha_{\underline{\ell}}$, $\overline{\lambda}_j = \alpha_{\overline{\ell}}$;
        else
            return $\underline{\lambda}_j = \overline{\lambda}_j = \omega$;
        end if
    end if
end if

Based on Theorem 6.2.1 and the preceding examples, we are ready to obtain the algorithm that computes $\mathbf{co}(\mathcal{S})$. Before this let us fix some notation. The inertia of a matrix pencil does not change between its GEVs. Therefore, if $\alpha_i < \alpha_{i+1}$, we denote the number of positive eigenvalues of $\mathbf{Q}_\lambda$ along the open interval (α_i, α_{i+1}) as π_i for

$i \in \{0 : n_G\}$. On the other hand, if $\alpha_i = \alpha_{i+1}$, we use π_i to denote $\pi(\mathbf{Q}_{\alpha_i})$.

The algorithm is given in Table 6.1. Let us explain some details to clarify the algorithm. Notice that there always exists a ℓ such that $\alpha_\ell \leq \omega \leq \alpha_{\ell+1}$, $\pi_\ell = 1$. In the ninth line we enforce to find a ℓ such that $[\alpha_\ell, \alpha_{\ell+1}]$ constitutes an interval not a single point. If such an interval does not exists, this means Λ_j is a point which must be equal to ω. Therefore, line fourteen we return ω as $\underline{\lambda}_j$ and $\overline{\lambda}_j$. The tenth line the algorithm checks at most three intervals/points to determine the lower end of Λ_j instead of all intervals formed by consecutive GEVs. This follows from Lemma 6.1.2 as one can easily verify. The same argument also applies to the eleventh line. We should also note that this algorithm is developed to explain the main principle to be followed. We did not put too much effort to make it numerically stable. In practical implementation, one should also pay attention to these aspects.

6.3. LMI Representation of the Convex Hull

In this section, we will describe how the LMI representation of $\mathbf{co}(\mathcal{S})$ can be attained when it is not empty or not $\mathbb{R}^n$ as promised earlier. Consider the quadratic constraint

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T P \begin{bmatrix} x \\ 1 \end{bmatrix} > 0 \tag{6.9}$$

which can also be written as

$$x^T A x + 2b^T x + c > 0 \tag{6.10}$$

using the partitioning

$$P = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}.$$

Let us denote the feasible set of this constraint as $\mathcal{P}$, which is assumed to be nonempty. The following proposition shows that $\mathcal{P}$ can be expressed in terms of LMIs when the matrix P has a single positive eigenvalue. Note that in below ν represents the number of negative eigenvalues of P.

Proposition 6.3.1. *If $\pi(P) = 1$, the following holds true:*

i) If $A \preceq 0$, A can be decomposed as $A = -UU^T$, where $U \in \mathbb{R}^{n\times\nu}$, and $\mathcal{P}$ admits the LMI representation

$$\begin{bmatrix} I & U^T x \\ x^T U & 2b^T x + c \end{bmatrix} \succ 0. \tag{6.11}$$

ii) If $A \npreceq 0$, $\mathcal{P}$ is a nonconvex set and using the decomposition $P = u\, u^T - V\, V^T$, where $u \in \mathbb{R}^{n+1}$ and $V \in \mathbb{R}^{(n+1)\times\nu}$, it can be expressed as the union of the two disjoint nonempty convex sets determined by the LMIs

$$\left[\begin{array}{c|c} \pm[x^T\ 1]\, u\, I & ([x^T\ 1]V)^T \\ \hline [x^T\ 1]V & \pm[x^T\ 1]u \end{array}\right] \succ 0. \tag{6.12}$$

Proof. i) Using the spectral composition of A, one can obtain the matrix of eigenvectors U such that $A = -UU^T$. Then (6.10) can be written as

$$2b^T x + c - x^T U U^T x > 0.$$

Employing the Schur complement formula given in Proposition 3.2.1, it can be easily verified that the feasible region of the above given constraint is equal to that of the LMI (6.11).

ii) Clearly, using its spectral decomposition, P can be expressed as $P = u\, u^T -$

$V\ V^T$. Thus, (6.9) can be written as

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T u\ u^T \begin{bmatrix} x \\ 1 \end{bmatrix} - \begin{bmatrix} x \\ 1 \end{bmatrix}^T V\ V^T \begin{bmatrix} x \\ 1 \end{bmatrix} > 0.$$

From this expression, it can be easily seen that the hyperplane $[x^T\ 1]u = 0$ does not intersect $\mathcal{P}$. Hence, the sets

$$\mathcal{P}^{\pm} := \{x \in \mathcal{P} |\ \pm [x^T\ 1]u > 0\},$$

constitutes a partition on $\mathcal{P}$. More importantly, using the Schur complement formula, it is immediate to infer that they are feasible regions of the LMI constraints (6.12). In addition, we know that under the condition $A \not\preceq 0$, the feasible region of (6.10) is nonconvex. Therefore, both $\mathcal{P}^+$ and $\mathcal{P}^-$ must be nonempty. Otherwise, $\mathcal{P}$ would be a convex set. □

Remark 6.3.2. *If $A = 0$, we have a simple linear inequality constraint $2b^T x + c > 0$, which is already an LMI. Therefore, there is no need to employ (6.11) since it will lead to redundancy. Indeed, when $A = 0$, the matrix of this LMI becomes the diagonal matrix which is composed of the identity matrix and the linear function $2b^T x + c$. Therefore, we will have an LMI composed of the constraint $2b^T x + c > 0$ and the trivial inequality $I > 0$.*

Making use of the foregoing result, it is possible to show that $\mathbf{co}(\mathcal{S})$ admit an LMI representation. We know that

$$\mathbf{co}(\mathcal{S}) = \mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j}.$$

Moreover, $\pi(\mathbf{Q}_{\underline{\lambda}_\mathbf{j}}) = \pi(\mathbf{Q}_{\overline{\lambda}_\mathbf{j}}) = \mathbf{1}$ as can be inferred from the definition (6.2). Due to the preceding proposition, this means the sets $\mathcal{S}_{\underline{\lambda}_j}$ and $\mathcal{S}_{\overline{\lambda}_j}$ are either LMI sets or union of two disjoint LMI sets. If the latter hold true, denote their convex components as $\mathcal{S}^{\pm}_{\underline{\lambda}_j}$ and $\mathcal{S}^{\pm}_{\overline{\lambda}_j}$, respectively. Based on these facts, exactly one of the following is possible:

- Both $\mathcal{S}_{\underline{\lambda}_j}$ and $\mathcal{S}_{\overline{\lambda}_j}$ are LMI sets, and hence, $\mathbf{co}(\mathcal{S}) = \mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j}$,
- $\mathcal{S}_{\underline{\lambda}_j}$ is an LMI set but not $\mathcal{S}_{\overline{\lambda}_j}$. In this case,

$$\mathbf{co}(\mathcal{S}) = (\mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}^+_{\overline{\lambda}_j}) \cup (\mathcal{S}_{\underline{\lambda}_j} \cap \mathcal{S}^-_{\overline{\lambda}_j}),$$

- $\mathcal{S}_{\underline{\lambda}_j}$ is not an LMI set while $\mathcal{S}_{\overline{\lambda}_j}$ is an LMI set. Then,

$$\mathbf{co}(\mathcal{S}) = (\mathcal{S}^+_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j}) \cup (\mathcal{S}^-_{\underline{\lambda}_j} \cap \mathcal{S}_{\overline{\lambda}_j}),$$

- Both $\mathcal{S}_{\underline{\lambda}_j}$ and $\mathcal{S}_{\overline{\lambda}_j}$ are not LMI sets. Hence,

$$\mathbf{co}(\mathcal{S}) = (\mathcal{S}^+_{\underline{\lambda}_j} \cap \mathcal{S}^+_{\overline{\lambda}_j}) \cup (\mathcal{S}^+_{\underline{\lambda}_j} \cap \mathcal{S}^-_{\overline{\lambda}_j}) \cup (\mathcal{S}^-_{\underline{\lambda}_j} \cap \mathcal{S}^+_{\overline{\lambda}_j}) \cup (\mathcal{S}^-_{\underline{\lambda}_j} \cap \mathcal{S}^-_{\overline{\lambda}_j}).$$

As can be seen from above, in the first case we already have the LMI representation. For the others, $\mathbf{co}(\mathcal{S})$ is the union of at most four intersection sets each of which is an LMI. Clearly, these intersection sets are disjoint, and hence, exactly one of them must be nonempty. Otherwise, $\mathbf{co}(\mathcal{S})$ would be empty or nonconvex, which is a contradiction. Consequently, we can infer that $\mathbf{co}(\mathcal{S})$ is an LMI set, the representation of which can be obtained using Proposition 6.3.1. Note that, in above, which alternative holds true can be easily determined by checking negative semidefiniteness of $(1-\underline{\lambda}_j)A_1 + \underline{\lambda}_j A_2$ and $(1-\overline{\lambda}_j)A_1 + \overline{\lambda}_j A_2$.

Remark 6.3.3. *The formula (6.11) is used to obtain LMI representation of when the sets $\mathcal{S}_{\underline{\lambda}_j}$ and $\mathcal{S}_{\overline{\lambda}_j}$ when they are convex. This is done to show that* $\mathbf{co}(\mathcal{S})$ *is an LMI set. In optimization applications, however, if this happens, it would be better to use the original constraints instead of the LMI description. The reason is that, in this case, we have a convex quadratic constraint. It is well known that problems involving such constraints can be solved more efficiently than the ones with LMIs [41]. Moreover, we will get rid of the computations required to obtain (6.11). Note that this does not lead to any problems if only one of the set $\mathcal{S}_{\underline{\lambda}_j}$ or $\mathcal{S}_{\overline{\lambda}_j}$ is nonconvex, for which we have to use an LMI description. The readily available software like SeDuMi can deal with mixture of LMIs and convex quadratic constraints.*

Example 6.3.4. *Let us find the LMI representation of* $\mathbf{co}(\mathcal{S})$ *given earlier in (6.8). The first constraint appearing in this system, which determines* $\mathcal{S}_{\underline{\lambda}_2}$*, is already linear. Therefore, it is enough to obtain representation of the other one which can be written in the matrix form as*

$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -\sqrt{2} \\ 0 & -1 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} > 0.$$

Since the second order leading principle submatrix of the above given matrix is not negative semidefinite, it can be inferred that $\mathcal{S}_{\overline{\lambda}_2}$ *is the union of the two sets* $\mathcal{S}^{\pm}_{\overline{\lambda}_2}$ *determined by the LMIs*

$$\begin{bmatrix} \pm(x_1 - \sqrt{2}) & x_2 \\ x_2 & \pm(x_1 - \sqrt{2}) \end{bmatrix} > 0,$$

which are obtained using (6.12). As described above, only one of these LMIs can have a feasible point which also satisfies the linear inequality $-2x_1 > 0$*. It turns out this holds true for the one with the negative sign, and hence, we come up with the representation of* $\mathbf{co}(\mathcal{S})$

$$\begin{bmatrix} -(x_1 - \sqrt{2}) & x_2 & 0 \\ x_2 & -(x_1 - \sqrt{2}) & 0 \\ 0 & 0 & -2x_1 \end{bmatrix} > 0. \tag{6.13}$$

6.4. Comparison with Relaxation Methods

In the preceding section, it is shown that the SDP relaxation may lead to very bad approximations of the convex hull and the optimization problem (6.7). In order to attain betters results, one may try to employ moment relaxation introduced in Section 5.4. For this reason, here, we will compare this method with our approach in terms of accuracy. Moreover, it will also be shown that, from computational complexity point of view, our approach behaves better than the SDP relaxation and the moment relaxations asymptotically.

Example 6.4.1. *Consider the optimization problem*

$$\begin{aligned} \gamma^* = \quad & \inf \quad -x_1 - 0.9x_2 \\ & \text{s.t.} \quad -x_1^2 + x_2^2 - 2x_1 - 2 > 0 \\ & \qquad\quad x_1^2 - x_2^2 - 2x_1 + 2 > 0. \end{aligned}$$

Realize that the feasible region of this problem is the set $\mathcal{S}$ investigated in Example 6.2.5. Therefore, we can attain the optimal value of the objective by infimizing it over $\mathbf{co}(\mathcal{S})$, the LMI representation of which is already given in (6.13). Solving the corresponding SDP problem, it is found to be $\gamma^* = -1.2728$. Approximations of γ^* are calculated using moment relaxations up to the fifth order and the relative errors are enumerated in Table 6.2 along with the number of variables and the size of the matrices involved. For numerical results, we utilize the YALMIP interface and the LMILAB solver available in MATLAB environment. The relative tolerance of the solver is set to 1e-12.

Table 6.2. Comparison with moment relaxation

Relaxation order (N)	1*	2	3	4	5
Relative error ($\frac{\gamma^*-\rho^*(N)}{\gamma^*}$)	$+\infty$	22.9834	0.7958	0.0024	2.3042e-008
Number of variables	5	14	27	44	65
Maximum matrix size	3	6	10	15	21

* Same as the SDP relaxation

As can be seen from the above given example, the error of the SDP relaxation, which corresponds to the first order moment relaxation, is infinity. But it could be improved significantly by increasing the order and eventually a very accurate result is attained for the fifth order relaxation. This does not show, however, wether the exact representation of $\mathbf{co}(\mathcal{S})$ is achieved or not. The related theory presented in Section 5.4 only guaranties the convergence in the limit. Therefore, although the relaxation may get very close to the convex hull, it may not give $\mathbf{co}(\mathcal{S})$ itself in finite number of steps. Recently, there has been some efforts to determine special problems for which the convex hull can be obtained in finite steps [42]. Nevertheless, to our best knowledge, such a result covering the the problem investigated in this chapter has not appeared yet. Even worse, $\mathcal{S}$ considered in the example is unbounded and we known that moment relaxation is not guaranteed to converge even in the limit in this case (recall the assumption of Theorem 5.4.4) although this did happen for this problem.

From a computational complexity point of view, our approach also behaves better than SDP relaxation and moment relaxations when it is used for optimization purposes. Indeed, as described in [16] (see page 619) solving an SDP requires $\max\{mp^3, m^2p^2, m^3\}$ flops asymptotically, where m is the number of variables and p is matrix dimension. For the relaxation given in (4.12), $m = n(n+1)/2 + n$ and $p = n+1$. Therefore, it can be inferred that complexity of solving this SDP is $O(n^6)$. Let's find the complexity of the method we propose. First, we need to find out if the set $\mathcal{S}$ is empty using the S-Procedure which takes $O(n^3)$ flops. Similarly, one should to check if $\mathbf{co}(\mathcal{S}) = \mathbb{R}^n$ which can also be done in $O(n^3)$ steps. Then, we compute the GEVs of the matrix pencil and obtain the eigenvalues of at most two matrices to find the end points of the interval Λ_j both of which has time complexity $O(n^3)$ [39, 43]. In order to find the LMI description from the quadratic constraints, it is necessary to compute the spectral decomposition of two matrices at the cost of $O(n^3)$ and check feasibility of at most four LMI constraints in n variables having matrix dimension $n+1$ taking $O(n^4)$ flops. Finally, we need to solve an SDP problem requiring $O(n^4)$ steps to compute γ^*. From this analysis, it can be inferred that the complexity of our method is $O(n^4)$ which has a considerably better performance when compared with the SDP relaxation of complexity $O(n^6)$, which does not always produce the convex hull.

It is clear that our approach also less complex than moment relaxation since the first order relaxation of this methods is the SDP relaxation. Moreover, as can be seen from Table 6.2 the number of variables and size of the matrices involved in moment relaxation grows very rapidly with the order. Even worse, these results for only a two dimensional problem. It is known that the complexity of the relaxation grows exponentially as the dimension increases, while we can always obtain the solutio in polynomial time.

6.5. Results on Homogenous Constraints

In this section, we will try to solve a problem in the homogenous space closely related with the one stated in the main theorem. The solution obtained will constitute the basis for proving the theorem. Before describing this problem we need to introduce some definitions and notation.

6.5.1. Preliminaries

The collection of all linear hyperplanes in $\mathbb{R}^{n+1}$ is shown as $\mathbb{L}$. Given a $\mathcal{T} \in \mathbb{L}$, the open halfspaces separated by it are denoted as $\mathcal{T}^+$ and $\mathcal{T}^-$. In other words, if $\mathcal{T}$ is the set of points satisfying $n^T x = 0$, then

$$\mathcal{T}^{\pm} := \{x \in \mathbb{R}^{n+1} | \pm n^T x > 0\}.$$

Notice that as long as the normal vector is not specified explicitly, it is not clear which side of $\mathcal{T}$ is $\mathcal{T}^+$ (or $\mathcal{T}^-$). However, usually, this will not lead to any problem and we fix the halfspaces when it is necessary.

Consider a set $\mathcal{C} \subseteq \mathbb{R}^{n+1}$.

Definition 6.5.1. *$\mathcal{C}$ is said to be cone if $x \in \mathcal{C}$ then $\alpha x \in \mathcal{C}$ for any $\alpha > 0$ (i.e. $\mathcal{C}$ is closed under multiplication with a positive scalar).*

Here, we follow the convention used in [17] for the definition of a cone. According to this, a cone does not necessarily contain the origin in contrast to the usual definition the reader would be familiar with in which closedness with respect to multiplication with a nonnegative scalar is required.

Definition 6.5.2. *The symmetric reflection of $\mathcal{C}$ with respect to the origin is defined as $-\mathcal{C} := \{-x | x \in \mathcal{C}\}$.*

Definition 6.5.3. *C is said to be symmetric if $\mathcal{C} = -\mathcal{C}$.*

An illustration of the preceding definitions in $\mathbb{R}^2$ is given in Figure 6.2. The sets $\mathcal{C}$, $-\mathcal{C}$ and $\mathcal{C} \cup -\mathcal{C}$ depicted in this figure constitute examples of a cone. The set $-\mathcal{C}$ is the symmetric reflection of $\mathcal{C}$ with respect to the origin. Lastly, $\mathcal{C} \cup -\mathcal{C}$ is a symmetric set.

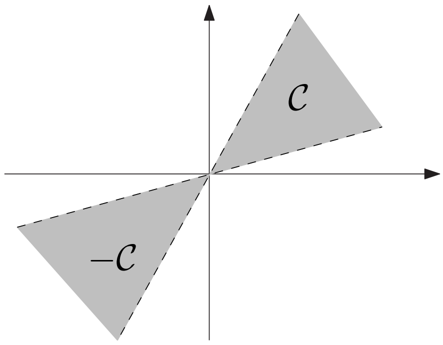


Figure 6.2. Examples of a cone, symmetric reflection and symmetric set

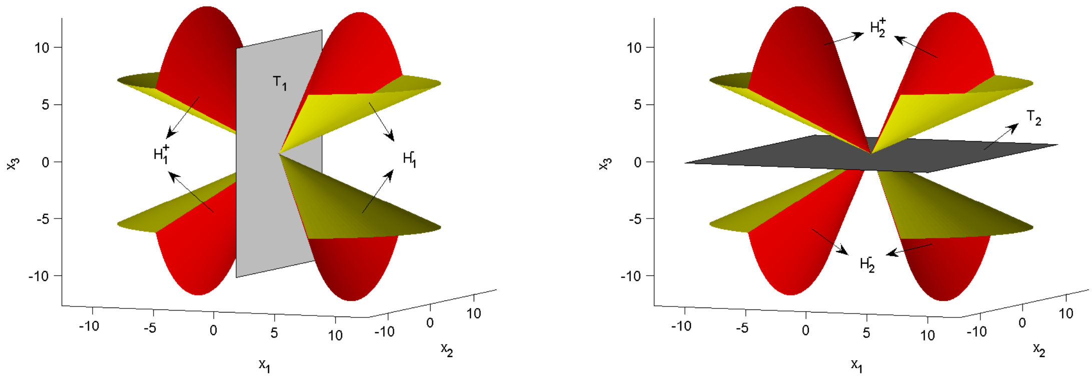
An important notion that we base our development on is separation of a symmetric cone. Since we will deal with only open cones, the definition is restricted to such sets for the sake of simplicity.

Definition 6.5.4. *Consider an open symmetric nonempty cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$. If there exists a linear hyperplane that does not intersect $\mathcal{C}$, we say $\mathcal{C}$ admits a separation. A separation of $\mathcal{C}$ is defined as the pair of nonempty disjoint cones $\mathcal{C}^{\pm} := \mathcal{C} \cap \mathcal{T}^{\pm}$, where $\mathcal{T} \in \mathbb{L}$ such that $\mathcal{C} \cap \mathcal{T} = \emptyset$. $\mathcal{T}$ is said to separate $\mathcal{C}$ and the sets $\mathcal{C}^{\pm}$ are called slices. Note that we denote a separation either enlisting its slices as $\langle \mathcal{C}^{+}, \mathcal{C}^{-} \rangle$ or making use of the linear hyperplane inducing it as $\langle \mathcal{C} \rangle_{\mathcal{T}}$.*

Example 6.5.5. *Consider $\mathcal{H} \subseteq \mathbb{R}^3$ determined by the constraints*

$$\begin{aligned} 4x_1^2 - x_2^2 - x_3^2 > 0, \\ 4x_3^2 - x_1^2 - x_2^2 > 0. \end{aligned} \tag{6.14}$$

Clearly $\mathcal{H}$ is an open symmetric cone, and hence, we can talk about its separations. As can be seen from Figure 6.3, it admits two different separations: one is $\langle \mathcal{H}_1^{+}, \mathcal{H}_1^{-} \rangle$ while *the other is* $\langle \mathcal{H}_2^+, \mathcal{H}_2^- \rangle$. *Examples the hyperplanes that induce these separations are* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *respectively.*



a) b)

Figure 6.3. Separations of $\mathcal{H}$

Based on the preceding definition and example, one can easily infer that if an open symmetric cone $\mathcal{C}$ admits a separation $\langle \mathcal{C}^+, \mathcal{C}^- \rangle$, its slices satisfy the following properties:

i) $\mathcal{C}^- = -\mathcal{C}^+$.

ii) $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

iii) $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$.

The last two equalities listed above imply that the sets $\mathcal{C}^\pm$ constitute a partition on $\mathcal{C}$.

Definition 6.5.6. *A semi-convex cone (SCC) is the union of two convex cones which are symmetric reflections of each other with respect to the origin.*

As an example of an SCC in $\mathbb{R}^3$ consider the set of points satisfying the constraint $x_1^2 - x_2^2 - x_3^2 > 0$, which is depicted in Figure 6.4. This set is the union of two second order cones which are the symmetric reflection of each other. Apparent from this example and the foregoing definition, an SCC is always symmetric. Moreover, an SCC is separated in a unique manner as stated in the following proposition.

Proposition 6.5.7. *Let* $\mathcal{C} \in \mathbb{R}^{n+1}$ *be an open SCC. Assume there exists a* $\mathcal{T} \in \mathbb{L}$ *such*

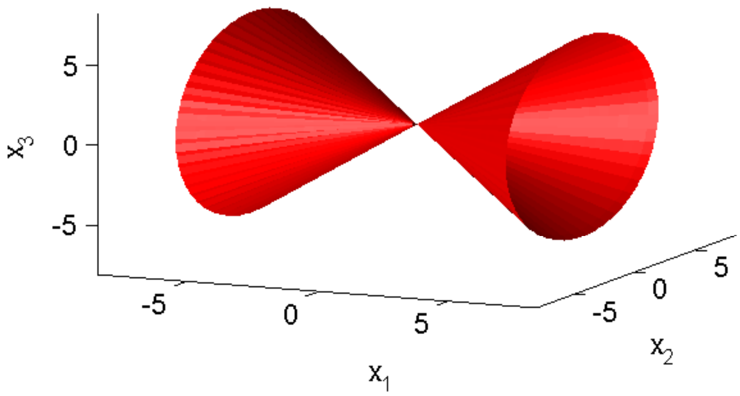


Figure 6.4. Example of an SCC

that $\mathcal{T} \cap \mathcal{C} = \emptyset$. Then $\mathcal{C}$ admits a unique separation the slices of which are the convex connected components of $\mathcal{C}$.

Proof. Due to the assumption, $\mathcal{C}$ admits a separation. Let's show it is unique. By definition, $\mathcal{C}$ is the union of two convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_2 = -\mathcal{C}_1$. Consider a $\mathcal{T} \in \mathbb{L}$ such that $\mathcal{C} \cap \mathcal{T} = \emptyset$. Clearly, $\mathcal{C}_1$ does not intersect $\mathcal{T}$, and thus, it is a subset of either $\mathcal{T}^+$ or $\mathcal{T}^-$. Without loss of generality we may assume $\mathcal{C}_1 \subseteq \mathcal{T}^+$. Then one can easily infer that $\mathcal{C}_2 \subseteq \mathcal{T}^-$ due to symmetry. Hence, it is not hard to see that $\mathcal{T}$ separates $\mathcal{C}$ as $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. Since $\mathcal{T}$ was an arbitrary hyperplane not intersecting $\mathcal{C}$, it can be seen that $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is the unique separation of $\mathcal{C}$. Note that because $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex, they are connected sets. Moreover, they are strictly separated by $\mathcal{T}$, and hence, they are different connected components of $\mathcal{C}$. □

Interestingly, for an open symmetric cone defined by a single quadratic inequality,

$$\mathcal{P} = \{x \in \mathbb{R}^{n+1} | x^T P x > 0\},$$

the notions of separation and being an SCC are closely related with each other as evident from the following key result.

Proposition 6.5.8. *Assume $\mathcal{P} \neq \emptyset$. The following statements are equivalent:*

i) *There exists a linear hyperplane that does not intersect $\mathcal{P}$ (i.e. $\mathcal{P}$ admits a separation),*

ii) $\pi(P) = 1$,

iii) *$\mathcal{P}$ is an SCC.*

See Appendix A.3 for a proof.

6.5.2. Problem Formulation and Solution in the Homogenous Space

Now, we are ready to describe the problem that will be investigated in the homogenous space. Consider the homogenized version of $\mathcal{S}$

$$\mathcal{H} := \{x \in \mathbb{R}^{n+1} |\, \tilde{q}_i(x) > 0,\ i = 1 : 2\},$$

where $\tilde{q}_i := x^T Q_i x,\ i = 1 : 2$. Clearly, $\mathcal{H}$ is an open symmetric cone. Moreover, $\mathcal{H} \neq \emptyset$ since $\mathcal{S}$ is not empty due to Assumption 6.1.1. Therefore, we can talk about separations of $\mathcal{H}$. We ask the following related questions.

i) How may different separations does $\mathcal{H}$ admit and how can we identify the linear hyperplanes inducing each separation?

ii) Given a separation $\langle \mathcal{H}^+, \mathcal{H}^- \rangle$, how can we find a useful characterization for $\mathbf{co}(\mathcal{H}^\pm)$?

It can be seen that, these questions, especially the second one, is closely related with the main theorem. Therefore, in this section our mail goal is to answer these questions.

To this end, let us define the homogenization of the quadratic pencil $\mathbf{q}_\lambda$

$$\tilde{\mathbf{q}}_\lambda := (1 - \lambda)\tilde{q}_1 + \lambda \tilde{q}_2$$

and that of $\mathcal{S}_\lambda$

$$\hat{\mathcal{H}}_\lambda := \{x \in \mathbb{R}^{n+1} |\, \tilde{\mathbf{q}}_\lambda(x) > 0\}. \tag{6.15}$$

Trivially, $\hat{\mathcal{H}}_\lambda$ satisfies the properties given in the next proposition.

Proposition 6.5.9. *The following hold true:*

i) $\mathcal{H} \subseteq \hat{\mathcal{H}}_\lambda$, $\lambda \in [0,1]$,

ii) $\hat{\mathcal{H}}_\lambda \neq \emptyset$, $\lambda \in [0,1]$.

Consider the collections of sets

$$\Gamma_i := \{\hat{\mathcal{H}}_\lambda |\, \lambda \in \Lambda_i\},\ i = 1 : n_c, \tag{6.16}$$

and the collections of linear hyperplanes

$$\Theta_i := \{\mathcal{T} \in \mathbb{L} |\ \exists \hat{\mathcal{H}}_\lambda \in \Gamma_i \text{ such that } \hat{\mathcal{H}}_\lambda \cap \mathcal{T} = \emptyset\}, \quad i = 1 : n_c, \tag{6.17}$$

where Λ_i is defined in (6.5) and $n_c \leq 2$. For $\lambda \in \Lambda$ (recall that Λ is the union of Λ_i, $i = 1 : n_c$), we have $\pi(Q_\lambda) = 1$. Therefore, from definition (6.15), Proposition 6.5.8 and Proposition 6.5.9, it can be inferred that all members of Γ_i, $i = 1 : n_c$ are nonempty SCCs containing $\mathcal{H}$. On the other hand, by definition, given a $\mathcal{T} \in \Theta_i$, there exists an $\hat{\mathcal{H}}_\lambda \in \Gamma_i$ that does not intersect $\mathcal{T}$. This means, $\mathcal{T}$ separates $\hat{\mathcal{H}}_\lambda$, and hence, separates $\mathcal{H}$ since $\mathcal{H} \subseteq \hat{\mathcal{H}}_\lambda$. Consequently, we can infer that all linear hyperplanes lying in a collection Θ_i separates $\mathcal{H}$.

The following is a standing assumption through this section.

Assumption 6.5.10. $\Lambda \neq \emptyset$.

This assumption means that Γ_1 and Θ_1 are nonempty. The former directly follows from the definition (6.16) while the latter can be inferred from the nonemptiness of Γ_1 and the first two statements of Proposition 6.5.8. Note that the assumption does not lead to any limitation. This is because the results presented in below will eventually be utilized for the proof of Theorem 6.2.1 and the assumption is automatically satisfied when $\mathbf{co}(\mathcal{S}) \neq \mathbb{R}^n$ as one can easily see from the statement of the theorem.

The collections Γ_i and Θ_i having the properties described above constitute the key objects to answer the questions invoked at the beginning of the section. Indeed, the following two lemmas give the solutions of these questions in terms of Γ_i and Θ_i.

Lemma 6.5.11. *All linear hyperplanes inducing each separation of $\mathcal{H}$ exactly make up one of the collections Θ_i, $i \in 1 : n_c$. Therefore, there exists exactly n_c different separations of $\mathcal{H}$.*

The preceding lemma shows that there is a one to one correspondence between the separations of $\mathcal{H}$ and the collections Θ_i, $i \in 1 : n_c$. In the rest of the section, we denote the separation induced by the elements of Θ_i as $\langle \mathcal{H}_i^+, \mathcal{H}_i^- \rangle$.

For $i = 1 : n_c$, define

$$\mathcal{G}_i := \bigcap_{\lambda \in \Lambda_i} \hat{\mathcal{H}}_\lambda = \hat{\mathcal{H}}_{\underline{\lambda}_i} \cap \hat{\mathcal{H}}_{\overline{\lambda}_i}, \tag{6.18}$$

which is the intersection of all elements of the collection Γ_i.

Lemma 6.5.12. *For each $i \in 1 : n_c$, $\mathcal{G}_i$ is an SCC and its convex components give $\mathbf{co}(\mathcal{H}_i^\pm)$.*

The rest of this section is devoted to the proofs of these lemmas. Before, this however, let us demonstrate them and the properties of the the collections Γ_i and Θ_i on a simple example.

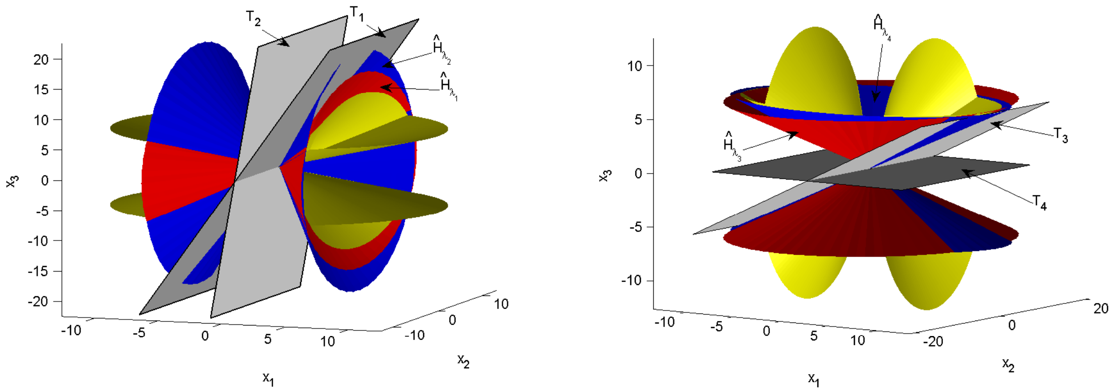
Example 6.5.13. *Consider the pair of polynomials investigated in (6.14). For these polynomials, Λ is the set of points lying in $[0, 1]$ interval satisfying*

$$\pi \left((1-\lambda) \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) = 1.$$

Therefore, it can be seen that Λ is the union of the intervals $\Lambda_1 = [0, 1/5]$ and $\Lambda_2 = [4/5, 1]$, and hence, $n_c = 2$ which means $\mathcal{H}$ admits two different separations.

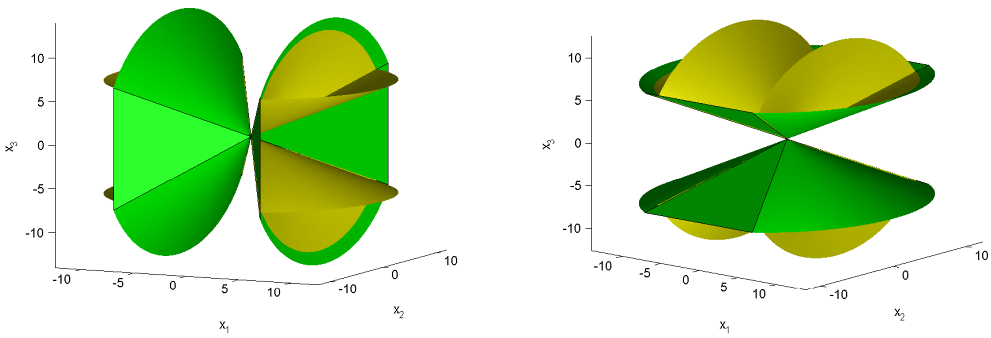
Two elements of Γ_1, $\hat{\mathcal{H}}_{\lambda_1}$ *and* $\hat{\mathcal{H}}_{\lambda_2}$, *are depicted in Figure 6.5.a. As can be seen from the figure, they are SCCs and contain* $\mathcal{H}$. *It can also be seen that the linear hyperplanes* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are members of* Θ_1 *since they do not intersect an element of* Γ_1, *namely* $\hat{\mathcal{H}}_{\lambda_1}$ *(the latter also does not intersect* $\hat{\mathcal{H}}_{\lambda_2}$*). Notice that both* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *separate* $\mathcal{H}$ *in the same manner as* $\langle \mathcal{H}_1^+, \mathcal{H}_1^- \rangle$ *while the convex components of* $\hat{\mathcal{H}}_{\lambda_1}$ *and* $\hat{\mathcal{H}}_{\lambda_2}$ *contain the slices* $\mathcal{H}_1^{\pm}$ *but not* $\mathcal{H}_2^{\pm}$. *Lastly, the intersection set* $\mathcal{G}_1$ *constitutes an SCC whose convex components gives* $\mathbf{co}(\mathcal{H}_1)^{\pm}$, *which is depicted in Figure 6.6.a.*

Arguments similar to the ones described above also hold true for $\hat{\mathcal{H}}_{\lambda_3}, \hat{\mathcal{H}}_{\lambda_4} \in \Gamma_2$ *and* $\mathcal{G}_2$*; and we have* $\mathcal{T}_3, \mathcal{T}_4 \in \Theta_2$. *However, this time they apply to the other separation,* $\langle \mathcal{H}_2^+, \mathcal{H}_2^- \rangle$, *as illustrated in Figure 6.5.b and Figure 6.6.b.*



a) b)

Figure 6.5. a) Some elements of Γ_1 and Θ_1; b) some elements of Γ_2 and Θ_2



a) b)

Figure 6.6. a) $\mathcal{G}_1$; b) $\mathcal{G}_2$

Let us start with the proof of Lemma 6.5.11. To make the presentation clear, the proof is divided into few steps and each step studied separately. Define

$$\Theta := \bigcup_{i=1}^{n_c} \Theta_i.$$

The first step is to show that Θ is made up all linear hyperplanes separating $\mathcal{H}$.

Lemma 6.5.14. *Consider a $\mathcal{T} \in \mathbb{L}$. $\mathcal{H} \cap \mathcal{T} = \emptyset$ if and only if $\mathcal{T} \in \Theta$.*

Proof. If part is trivial and explained above. It remains to prove only if part. We can parameterize $\mathcal{T}$ as $x = Uw$, where $w \in \mathbb{R}^n$, using an appropriate full column rank matrix $U \in \mathbb{R}^{(n+1)\times n}$. If $\mathcal{H} \cap \mathcal{T} = \emptyset$, the pair of constraints in w

$$\tilde{q}_i(Uw) = w^T U^T Q_i U w > 0, \quad i = 1:2$$

is infeasible. Hence, using to Lemma 4.5.1, it can be inferred that there exists a $\lambda \in [0, 1]$ such that

$$U^T((1-\lambda)Q_1 + \lambda Q_2)U \preceq 0.$$

This means, for this particular λ, there does not exist an $x \in \mathcal{T}$ such that $\tilde{\mathbf{q}}_\lambda(x) > 0$, and thus, $\mathcal{T} \cap \hat{\mathcal{H}}_\lambda = \emptyset$. Due to Proposition 6.5.8, the last result implies $\hat{\mathcal{H}}_\lambda \in \Gamma_i$ for an $i \in 1 : n_c$. Consequently, $\mathcal{T} \in \Theta$. □

Next, it will be shown that the linear hyperplanes belonging to the same collection, let us say Θ_i, separate $\mathcal{H}$ in the same manner. This is done in the following lemma and its corollary.

Consider an $i \in 1 : n_c$ such that $\Gamma_i \neq \emptyset$ or, equivalently, $\Theta_i \neq \emptyset$, which exists due to Assumption 6.5.10 and the discussion following it. Because of Proposition 6.5.8, for every $\hat{\mathcal{H}}_\lambda \in \Gamma_i$, there exits a $\mathcal{T} \in \Theta_i$ such that $\mathcal{T} \cap \hat{\mathcal{H}}_\lambda = \emptyset$, which may be different for different choices of the set $\hat{\mathcal{H}}_\lambda$. Surprisingly, the nontrivial and key result stated in the next lemma shows that Θ_i always contains a common hyperplane separating all members of Γ_i.

Lemma 6.5.15. *There exists a $\mathcal{K} \in \Theta_i$ such that $\hat{\mathcal{H}}_\lambda \cap \mathcal{K} = \emptyset$ for every $\hat{\mathcal{H}}_\lambda \in \Gamma_i$.*

The proof of the the lemma is given in Appendix A.2. Some sets from Γ_1 and Γ_2 for the polynomials investigated in the previous example are depicted in Figure 6.7.a and Figure 6.7.b, respectively. As can be seen from the figure, the hyperplane $x_1 = 0$ separates all elements of Γ_1 illustrated while $x_3 = 0$ separates that of Γ_2.

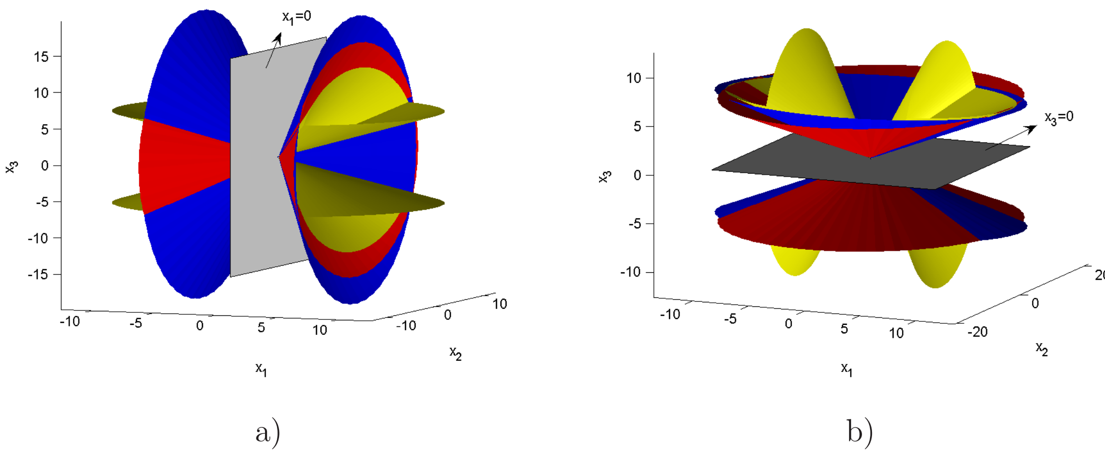


Figure 6.7. a) A common hyperplane separating elements of Γ_1; b) a common hyperplane separating elements of Γ_2

Corollary 6.5.16. *For any $i \in 1 : n_c$, if $\mathcal{T}_1, \mathcal{T}_2 \in \Theta_i$, then $\langle \mathcal{H} \rangle_{\mathcal{T}_1} = \langle \mathcal{H} \rangle_{\mathcal{T}_2}$.*

Proof. Let $\mathcal{K}$ be a linear hyperplane such that $\hat{\mathcal{H}}_\lambda \cap \mathcal{K} = \emptyset$ for every $\hat{\mathcal{H}}_\lambda \in \Gamma_i$, which exists due to Lemma 6.5.15. Since $\mathcal{T}_1 \in \Theta_i$, there exists a $\hat{\mathcal{H}}_{\lambda_1} \in \Gamma_i$ such that $\mathcal{T}_1 \cap \hat{\mathcal{H}}_{\lambda_1} = \emptyset$. Thus, we can see that both $\mathcal{K}$ and $\mathcal{T}_1$ separates $\hat{\mathcal{H}}_{\lambda_1}$. Because $\hat{\mathcal{H}}_{\lambda_1}$ is a SCC it admits a unique separation, which implies that $\langle \hat{\mathcal{H}}_{\lambda_1} \rangle_{\mathcal{T}_1} = \langle \hat{\mathcal{H}}_{\lambda_1} \rangle_{\mathcal{K}}$. In other words,

$$\hat{\mathcal{H}}_{\lambda_1} \cap \mathcal{T}_1^{\pm} = \hat{\mathcal{H}}_{\lambda_1} \cap \mathcal{K}^{\pm}$$

In above, $\mathcal{T}_1^+$ and $\mathcal{K}^+$ are determined by choosing appropriate sides of $\mathcal{T}_1$ and $\mathcal{K}$, respectively. By taking the intersection of both sides of the last equation with $\mathcal{H}$, we attain

$$\mathcal{H} \cap \hat{\mathcal{H}}_{\lambda_1} \cap \mathcal{T}_1^{\pm} = \mathcal{H} \cap \hat{\mathcal{H}}_{\lambda_1} \cap \mathcal{K}^{\pm}.$$

Because $\mathcal{H} \subseteq \hat{\mathcal{H}}_{\lambda_1}$, we have $\mathcal{H} \cap \mathcal{T}_1^{\pm} = \mathcal{H} \cap \mathcal{K}^{\pm}$. Therefore, $\mathcal{K}$ and $\mathcal{T}_1$ separates $\mathcal{H}$ in the same manner, i.e. $\langle \mathcal{H} \rangle_{\mathcal{T}_1} = \langle \mathcal{H} \rangle_{\mathcal{K}}$. Similarly, it can be shown that $\langle \mathcal{K} \rangle_{\mathcal{T}_2} = \langle \mathcal{H} \rangle_{\mathcal{K}}$. As a result, we can infer that $\langle \mathcal{H} \rangle_{\mathcal{T}_1} = \langle \mathcal{H} \rangle_{\mathcal{T}_2}$. □

In order to complete the proof of Lemma 6.5.11, it remains to show that linear hyperplanes belonging to the different collections separate $\mathcal{H}$ in different manner. Since $n_c \leq 2$, it will be enough to consider the case $n_c = 2$.

Lemma 6.5.17. *If $n_c = 2$, for any $\mathcal{T}_1 \in \Theta_1$ and $\mathcal{T}_2 \in \Theta_2$, we have $\langle \mathcal{H} \rangle_{\mathcal{T}_1} \neq \langle \mathcal{H} \rangle_{\mathcal{T}_2}$.*

The proof is given in the Appendix A.2.

Proof of Lemma 6.5.11. Note that in below we make use of the fact that all linear hyperplanes separating $\mathcal{H}$ belong to Θ, which is due to Lemma 6.5.14.

If $n_c = 1$, $\Theta = \Theta_1$. Moreover, all elements of Θ_1 separates $\mathcal{H}$ in the same manner because of Corollary 6.5.16. Therefore, $\mathcal{H}$ admits only one separation and all linear hyperplanes inducing it makes up $\Theta = \Theta_1$.

Now, suppose that $n_c = 2$. From Lemma 6.5.17, it can be inferred that elements of Θ_1 separates $\mathcal{H}$ different than elements of Θ_2. Moreover, from Corollary 6.5.16, we know that all hyperplanes in Θ_1 separates $\mathcal{H}$ in the same manner. The same applies to Θ_2. Since any hyperplane separating $\mathcal{H}$ must belong to Θ, it can be seen that $\mathcal{H}$ admits two separations and all linear hyperplanes inducing one of them makes up Θ_1 while all hyperplanes inducing the other makes up Θ_2. Since $n_c \leq 2$, we are done.

□

Remark 6.5.18. *Consider the halfspaces $\mathcal{T}^{\pm}$ corresponding to a hyperplane $\mathcal{T} \in \Theta_i$. As one can anticipate, there is a ambiguity regarding such halfspaces that we did not address till now since it was not important. It is not clear which side of $\mathcal{T}$ is $\mathcal{T}^+$ (or $\mathcal{T}^-$) as mentioned earlier. At this point, it is necessary to resolve this ambiguity among the members of a collection Θ_i to make the following discussions precise. For this purpose, the following convention will be employed. For every $\mathcal{T} \in \Theta_i$, by $\mathcal{T}^+$, we mean the halfspace containing $\mathcal{H}_i^+$. Similarly, $\mathcal{T}^-$ denotes the halfspace containing $\mathcal{H}_i^-$. By this way, we will obtain an agreement among the halfspaces associated with the elements of Θ_i. Note that, however, although this problem is solved, there is also a similar ambiguity regarding the slices of a separation. It is not clear $\mathcal{H}_i^+$ (or $\mathcal{H}_i^-$) represents which slice. But, in below, this will not be important for most of the time and we will fix it when deemed necessary.*

Next, we will prove Lemma 6.5.12.

Proof of Lemma 6.5.12. We start by showing that $\mathcal{G}_i$ are SCCs. Consider a $\mathcal{K} \in \Theta_i$ such that $\mathcal{K} \cap \hat{\mathcal{H}}_\lambda = \emptyset$ for every $\hat{\mathcal{H}}_\lambda \in \Gamma_i$, which exists due to Lemma 6.5.15. In particular, we have $\mathcal{K} \cap \hat{\mathcal{H}}_{\underline{\lambda}_i} = \mathcal{K} \cap \hat{\mathcal{H}}_{\overline{\lambda}_i} = \emptyset$, and hence, from the definition (6.18), it can be seen that $\mathcal{K} \cap \mathcal{G}_i = \emptyset$. Therefore, $\mathcal{K}$ separates $\mathcal{G}_i$, $\hat{\mathcal{H}}_{\underline{\lambda}_i}$ and $\hat{\mathcal{H}}_{\overline{\lambda}_i}$ as $\langle \mathcal{G}_i^+, \mathcal{G}_i^- \rangle$, $\langle \hat{\mathcal{H}}_{\underline{\lambda}_i}^+, \hat{\mathcal{H}}_{\underline{\lambda}_i}^- \rangle$ and $\langle \hat{\mathcal{H}}_{\overline{\lambda}_i}^+, \hat{\mathcal{H}}_{\overline{\lambda}_i}^- \rangle$, respectively. Based on these facts it can be inferred that

$$
\begin{aligned}
\mathcal{G}_i^{\pm} =& \mathcal{G}_i \cap \mathcal{K}^{\pm} = (\hat{\mathcal{H}}_{\underline{\lambda}_i} \cap \hat{\mathcal{H}}_{\overline{\lambda}_i}) \cap \mathcal{K}^{\pm} \\
=& (\hat{\mathcal{H}}_{\underline{\lambda}_i} \cap \mathcal{K}^{\pm}) \cap (\hat{\mathcal{H}}_{\overline{\lambda}_i} \cap \mathcal{K}^{\pm}) = \hat{\mathcal{H}}_{\underline{\lambda}_i}^{\pm} \cap \hat{\mathcal{H}}_{\overline{\lambda}_i}^{\pm}.
\end{aligned}
$$

Since $\hat{\mathcal{H}}_{\underline{\lambda}_i}$ and $\hat{\mathcal{H}}_{\overline{\lambda}_i}$ are SCCs, the slices $\hat{\mathcal{H}}_{\underline{\lambda}_i}^{\pm}$ and $\hat{\mathcal{H}}_{\overline{\lambda}_i}^{\pm}$ are convex cones. Consequently, from the preceding equation, $\mathcal{G}_i^{\pm}$ are also convex cones, which, in turn, implies that $\mathcal{G}_i$ are SCCs.

Now, we will show that $\mathcal{G}_i^{\pm} = \mathbf{co}(\mathcal{H}_i^{\pm})$. Let $\mathcal{T}$ be an element of Θ_i. Recall that $\mathcal{T}$ separates $\mathcal{H}$ as $\langle \mathcal{H}_i^+, \mathcal{H}_i^- \rangle$ and there exits an $\hat{\mathcal{H}}_\mu \in \Gamma_i$ such that $\mathcal{T} \cap \hat{\mathcal{H}}_\mu = \emptyset$. Moreover, we know that $\mathcal{H} \subseteq \hat{\mathcal{H}}_\mu$. Therefore,

$$\mathcal{H}_i^\pm = \mathcal{H} \cap \mathcal{T}^\pm \subseteq \hat{\mathcal{H}}_\mu \cap \mathcal{T}^\pm \subseteq \mathcal{T}^\pm.$$

On the other hand, consider a $\mathcal{K} \in \Theta_i$ such that $\mathcal{K} \cap \hat{\mathcal{H}}_\lambda = \emptyset$ for every $\hat{\mathcal{H}}_\lambda \in \Gamma_i$. Both $\mathcal{T}$ and $\mathcal{K}$ separate $\hat{\mathcal{H}}_\mu$, which admits a unique separation because it is an SCC. Thus,

$$\hat{\mathcal{H}}_\mu \cap \mathcal{T}^\pm = \hat{\mathcal{H}}_\mu \cap \mathcal{K}^\pm$$

By substituting this into the preceding equation, we come up with

$$\mathcal{H}_i^\pm \subseteq \hat{\mathcal{H}}_\mu \cap \mathcal{K}^\pm \subseteq \mathcal{T}^\pm. \tag{6.19}$$

Since $\mathcal{T}$ was an arbitrary element of Θ_i, we have show that for every $\mathcal{T} \in \Theta_i$, there exists an $\hat{\mathcal{H}}_\mu \in \Gamma_i$ (or a $\mu \in \Lambda_i$) such that (6.19) is satisfied. Employing this fact, it is easy to see that

$$\mathcal{H}_i^\pm \subseteq \bigcap_{\mu \in \Lambda_i} \hat{\mathcal{H}}_\mu \cap \mathcal{K}^\pm \subseteq \bigcap_{\mathcal{T} \in \Theta_i} \overline{\mathcal{T}^\pm},$$

or equivalently,

$$\mathcal{H}_i^\pm \subseteq \mathcal{G}_i^\pm \subseteq \bigcap_{\mathcal{T} \in \Theta_i} \overline{\mathcal{T}^\pm}. \tag{6.20}$$

The preceding equation gives us pretty much what we want. In order to finish the proof of the lemma, we will show that the intersections appearing in (6.20) are $\overline{\mathbf{co}(\mathcal{H}_i^\pm)}$. This means $\mathcal{G}_i^\pm$ are the open convex sets contained in $\overline{\mathbf{co}(\mathcal{H}_i^\pm)}$ and containing $\mathcal{H}_i^\pm$. Hence, they must be equal to $\mathbf{co}(\mathcal{H}_i^\pm)$.

The fact that

$$\overline{\mathbf{co}(\mathcal{H}_i^\pm)} = \bigcap_{\mathcal{T} \in \Theta_i} \overline{\mathcal{T}^\pm}$$

can be shown as follows. It is easy to see that a $\mathcal{T} \in \mathbb{L}$ separates $\mathcal{H}$ as $\langle \mathcal{H}_i^+, \mathcal{H}_i^- \rangle$ (i.e. $\mathcal{T} \in \Theta_i$) if and only if it is a supporting hyperplane of $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ (see Definition 3.1.7 for the definition of supporting hyperplane). Therefore, the intersections appearing in the preceding equation is, in fact, the intersections of all closed halfspaces supporting these slices. Due to Proposition 3.1.8, the intersection of all closed halfspaces supporting a cone gives closure of its convex hull, which leads to desired equality. □

6.6. Proof of the Main Theorem

Using the results introduced in the previous section, we can give the proof of the theorem. In the proof, we employ the hyperplane

$$\mathcal{E} := \{x \in \mathbb{R}^{n+1} |\, x_{n+1} = 0\}$$

and the associated halfspaces

$$\mathcal{E}^{\pm} := \{x \in \mathbb{R}^{n+1} |\, \pm x_{n+1} > 0\} \tag{6.21}$$

Moreover, we make use of the following two propositions.

Proposition 6.6.1. $\Omega = \emptyset$ *if and only if* $\mathcal{E} \cap \mathcal{H} \neq \emptyset$.

Proof. Due to Proposition 6.5.14, we know that $\mathcal{E} \cap \mathcal{H} \neq \emptyset$ if and only if $\mathcal{E} \notin \Theta$. Moreover, from the definition (6.17), it can be seen that $\mathcal{E} \notin \Theta$ if and only if there does not exist an $\hat{\mathcal{H}}_\lambda \in \bigcup_{i=1}^{n_c} \Gamma_i$ such that $\hat{\mathcal{H}}_\lambda \cap \mathcal{E} = \emptyset$. From these facts, it can be deduced that

$$\mathcal{E} \cap \mathcal{H} \neq \emptyset \Leftrightarrow \mathcal{E} \cap \hat{\mathcal{H}}_\lambda \neq \emptyset,\ \forall \lambda \in \Lambda. \tag{6.22}$$

On the other hand, we can parameterize $\mathcal{E}$ as

$$x = \begin{bmatrix} I \\ 0 \end{bmatrix} w$$

where $w \in \mathbb{R}^n$. Therefore, given an $\hat{\mathcal{H}}_\lambda$, $\mathcal{E} \cap \hat{\mathcal{H}}_\lambda \neq \emptyset$ if and only if there exists a $w \in \mathbb{R}^n$ such that

$$\begin{aligned} \tilde{\mathbf{q}}_\lambda([\ I \quad 0\ ]^T w) = & \ w^T \begin{bmatrix} I \\ 0 \end{bmatrix}^T \mathbf{Q}_\lambda \begin{bmatrix} I \\ 0 \end{bmatrix} w \\ = & \ w^T [(1-\lambda)A_1 + \lambda A_2] w > 0 \end{aligned}$$

or equivalently, $(1-\lambda)A_1 + \lambda A_2$ is not negative semidefinite. Employing this fact in (6.22), we can see that $\mathcal{E} \cap \mathcal{H} \neq \emptyset$ if and only if

$$(1-\lambda)A_1 + \lambda A_2 \npreceq 0$$

for every $\lambda \in [0, 1]$, or equivalently $\Omega = \emptyset$. □

Proposition 6.6.2. *If there does not exist a halfspace containing $\mathcal{S}$, then $\mathbf{co}(\mathcal{S}) = \mathbb{R}^n$.*

Proof. Assume $\mathbf{co}(\mathcal{S}) \neq \mathbb{R}^n$. Then, there is an $x \notin \mathbf{co}(\mathcal{S})$. Since the convex hull is a convex set, it can be strictly separated from x by a hyperplane. This means there exists a halfspace containing $\mathbf{co}(\mathcal{S})$, and thus, containing $\mathcal{S}$.

□

Proof of Theorem 6.2.1. We start by proving that $\mathbf{co}(\mathcal{S}) = \mathbb{R}^n$, if $\Omega = \emptyset$.

Consider a hyperplane

$$\mathcal{U} := \{x \in \mathbb{R}^n |\, a^T x + b = 0\}$$

and its homogenous counterpart

$$\mathcal{T} := \{[\ x^T \quad t\ ] \in \mathbb{R}^{n+1} | a^T x + bt = 0\}.$$

Let's define the associated halfspaces

$$\mathcal{U}^{\pm} := \{x \in \mathbb{R}^n | \pm (a^T x + b) > 0\}$$

and

$$\mathcal{T}^{\pm} := \{[\ x^T \quad t\ ] \in \mathbb{R}^{n+1} | \pm (a^T x + bt) > 0\}.$$

Our strategy will be to show that there does not exist a halfspace containing $\mathcal{S}$ (see Proposition 6.6.2) if $\Omega = \emptyset$. Assume the contrary, that is $\Omega = \emptyset$ but for an appropriate choice of a and b,

$$\mathcal{S} \subseteq \mathcal{U}^{+}. \tag{6.23}$$

This means $\mathcal{S} \cap \mathcal{U} = \emptyset$, which implies $\mathcal{H} \cap \mathcal{T} = \emptyset$. Therefore, $\mathcal{T} \in \Theta_i$ for an $i \in 1 : n_c$. Consequently,

$$\mathcal{H}_i^{\pm} = \mathcal{H} \cap \mathcal{T}^{\pm}. \tag{6.24}$$

On the other hand, because $\Omega = \emptyset$, $\mathcal{E} \cap \mathcal{H} \neq \emptyset$ due to Proposition 6.6.1. Thus, $\mathcal{E} \cap \mathcal{H}_i^{\pm} \neq \emptyset$ because $\mathcal{H}$ is a symmetric cone. Therefore, there exists a $[\ y^T \quad 0\ ] \in \mathcal{H}_i^{-}$ and since $\mathcal{H}_i^{-}$ is open, there exists a $[\ y^T \quad t\ ] \in \mathcal{H}_i^{-}$ such that $t > 0$. Since $[\ y^T \quad t\ ] \in \mathcal{H}$,

$$\begin{bmatrix} y \\ t \end{bmatrix}^T Q_j \begin{bmatrix} y \\ t \end{bmatrix} > 0, \quad j = 1 : 2$$

and due to the fact that $t > 0$

$$\begin{bmatrix} y/t \\ 1 \end{bmatrix}^T Q_j \begin{bmatrix} y/t \\ 1 \end{bmatrix} > 0, \quad j = 1:2,$$

or equivalently $y/t \in \mathcal{S}$. Moreover, we know that $[\ y^T \quad t\ ] \in \mathcal{T}^-$ since it is an element of $\mathcal{H}_i^-$ (see (6.24)). In other words, $-(a^T y + bt) > 0$. Since $t > 0$, we have $-(a^T y/t + b) > 0$. This means y/t is also an element of $\mathcal{U}^-$. Thus, y/t is an element of both $\mathcal{S}$ and $\mathcal{U}^-$. This contradicts with (6.23), and hence, $\mathbf{co}(\mathcal{S}) = \mathbb{R}^n$.

Now, we will prove that if $\Omega \neq \emptyset$,

$$\mathbf{co}(\mathcal{S}) = \bigcap_{\lambda \in \Lambda_i} \mathcal{S}_\lambda = \mathcal{S}_{\underline{\lambda}_i} \cap \mathcal{S}_{\overline{\lambda}_i}.$$

$\mathcal{S}$ can be expressed as

$$\mathcal{S} = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{H} \right\}.$$

Since $\Omega \neq \emptyset$, $\mathcal{E} \cap \mathcal{H} = \emptyset$, and hence, $\mathcal{E} \in \Theta_i$ for an $i \in 1 : n_c$. From this it can be inferred that $\Gamma \neq \emptyset$. Hence, Assumption 6.5.10 is satisfied and the results developed in the previous section can be employed safely in below. Notice that in (6.21), it is precisely determined which side of $\mathcal{E}$ is $\mathcal{E}^+$. This also resolve ambiguity associated with the slices $\mathcal{H}_i^\pm$ pointed out in Remark 6.5.18. They satisfy $\mathcal{H}_i^\pm \subseteq \mathcal{E}^\pm$. Based on this fact, we can see

$$\mathcal{S} = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{H}_i^+ \right\}$$

since the last components of the vectors lying in $\mathcal{H}_i^-$ are always negative. Hence, the convex hull can be expressed as

$$
\begin{aligned}
\mathbf{co}(\mathcal{S}) &= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} = \sum_{j=1}^{n+1} \theta_j \begin{bmatrix} y_j \\ 1 \end{bmatrix},\ \sum_{j=1}^{n+1} \theta_j = 1, \right. \\
&\qquad\qquad \left. \theta_j \geq 0,\ \begin{bmatrix} y_j \\ 1 \end{bmatrix} \in \mathcal{H}_i^+,\ j = 1 : n+1 \right\} \\
&= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} = \sum_{j=1}^{n+1} \theta_j \tilde{y}_j,\ \theta_j \geq 0,\ \tilde{y}_j \in \mathcal{H}_i^+,\ j = 1 : n+1 \right\} \\
&= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbf{co}(\mathcal{H}_i^+) \right\} \\
&= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{G}_i^+ \right\}
\end{aligned}
$$

using Carathéodory's theorem, Theorem 3.1.9. In addition, we can infer the following. Since $\mathcal{E} \in \Theta_i$, there exists a $\hat{\mathcal{H}}_\lambda \in \Gamma_i$ such that $\mathcal{E} \cap \hat{\mathcal{H}}_\lambda = \emptyset$. Therefore, it is easy to see that $\mathcal{G} \cap \mathcal{E} = \emptyset$, and thus, $\mathcal{G}_i^\pm \subseteq \mathcal{E}^\pm$. As a result it can be deduced that

$$
\mathbf{co}(\mathcal{S}) = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{G}_i \right\}.
$$

Based on this it is immediate to see that

$$
\mathbf{co}(\mathcal{S}) = \bigcap_{\lambda \in \Lambda_i} \mathcal{S}_\lambda = \mathcal{S}_{\underline{\lambda}_i} \cap \mathcal{S}_{\overline{\lambda}_i}.
$$

as desired. $\square$

6.7. Summary

In this chapter we presented one of our main contributions. We concentrated to the set $\mathcal{S}$ determined by two quadratic inequality constraints and showed that $\mathbf{co}(\mathcal{S})$ is an LMI set. Moreover an algorithm that efficiently computes the LMI description of $\mathbf{co}(\mathcal{S})$ is proposed. Using this description, it is possible to obtain the optimal value of linear objective function over $\mathcal{S}$ exactly in practice. As demonstrated in this chapter and Chapter 4 neither the S-procedure and the SDP relaxation nor their extensions can produce the convex hull or give the optimal value of a linear objective function over $\mathcal{S}$ exactly. Moreover, if one tries to use the moment relaxation, it can be seen that for some problems obtaining an accurate result would be computationally demanding. Therefore, our approach outperforms all these methods for the problem considered. As a by product of our findings, we also obtained a necessary and sufficient condition for determining when the SDP relaxation produces $\mathbf{co}(\mathcal{S})$ exactly. This leads to another losslessness result for the SDP relaxation which, to the best of our knowledge, did not appear in the literature previously.

7. CONVEX HULL OF THE REGION DETERMINED BY QUADRATIC CONSTRAINTS IN $\mathbb{R}^2$

Consider the set

$$\mathcal{S} := \{x \in \mathbb{R}^n |\, p_i(x) \geq 0, i = 1 : m_p\}, \tag{7.1}$$

where $p_i : \mathbb{R}^n \to \mathbb{R}$ are quadratic polynomials. In the previous chapter, constructively it was showed that $\mathbf{co}(\mathcal{S})$ admits and LMI representation if $m_p = 2$ and the inequalities employed in the description of $\mathcal{S}$ were strict. In this chapter, we ask for which other cases the LMI representation exists. As an attempt answering this question We will show that $\overline{\mathbf{co}(\mathcal{S})}$ can always be expressed as the intersection of a finite number of LMI sets (i.e. the feasible regions of LMI constraints) and some halfspaces supporting $\mathcal{S}$ along a subset of its boundary. Unfortunately, it turns out that the halfspaces involved are infinitely many in general and an LMI representation cannot be attained. However, if one restricts himself to $\mathbb{R}^2$, the intersection of the halfspaces of concern becomes a polyhedral set. Thus, in $\mathbb{R}^2$, $\overline{\mathbf{co}(\mathcal{S})}$ constitutes an LMI set. Unfortunately, since our proof is not constructive, we do not give an algorithm producing the convex hull in this case.

7.1. Notation and Preliminaries

For a real matrix A, A^+ denotes its Moore-Penrose pseudoinverse. A neighborhood of $x \in \mathbb{R}^n$ is an open ball centered at x. That is,

$$N_x := \{y \in \mathbb{R}^n | \quad \|y - x\| < r\} \quad \text{for some } r > 0.$$

For a set $\mathcal{V} \subseteq \mathbb{R}^n$, $\overline{\mathcal{V}}$ and $\mathbf{int}(\mathcal{V})$ denotes the closure and interior of $\mathcal{V}$, respectively. Moreover, the boundary of $\mathcal{V}$ is defined as $\partial\mathcal{V} := \overline{\mathcal{V}} \setminus \mathbf{int}(\mathcal{V})$.

Let $\mathcal{V} \subseteq \mathbb{R}^n$ be nonempty and assume $x \in \partial V$. The intersection of all halfspaces supporting $\mathcal{V}$ at x denoted as $\mathbf{C}(\mathcal{V}, x)$ (see Definition 3.1.7 for the definition of a supporting halfspace). If no supporting halfspace exists, then $\mathbf{C}(\mathcal{V}, x) = \mathbb{R}^n$.

To avoid technical difficulties in the rest of the chapter, we put the expression given in (7.1) into a more suitable form and make use of a number of assumptions. Note that these assumptions by no means put any restriction on our results as described below. They are used to simplify the presentation of the material.

If an inequality constraint and a negative multiple of it exist, they are replaced with a single equality constraint. By an appropriate choice of indexing, this leads to the following formulation

$$\mathcal{S} = \{x \in \mathbb{R}^n |\, p_i(x) \geq 0, i = 1 : \ell,\ p_j(x) = 0, j = \ell + 1 : m\}.$$

Assumption 7.1.1. *For $j = \ell + 1 : m$, the matrix P_j associated with the constraint $p_j(x) = 0$ satisfies $\nu(P_j) \geq \pi(P_j)$.*

Note that this assumption clearly does not lead to any limitations. If it is not satisfied, one can simply multiply the both sides of the equality constraint by a negative number.

Assumption 7.1.2. *For $i = 1 : m$, P_i is indefinite.*

In order to see that this assumption is not restrictive, first suppose $P_i \succeq 0$. In this case, p_i cannot appear in an equality constraint because the assumption $\nu(P_i) \geq \pi(P_i)$ cannot be satisfied for a nonzero polynomial. On the other hand, if it is used in an inequality constraint, the constraint is satisfied for every $x \in \mathbb{R}^n$. Therefore, it can be removed from the description. Next, suppose $P_i \preceq 0$. As can be anticipated from the spectral decomposition of P_i, for both equality and inequality constraints, the feasible set turns out to be a linear subspace of $\mathbb{R}^n$. Therefore, without altering $\mathcal{S}$, the constraint associated with p_i can be replaced by a number of equality constraints, for which the corresponding matrix has to be indefinite.

Lastly, given a quadratic polynomial p, we make use of the sets

$$\mathcal{Q}_p := \{x \in \mathbb{R}^n |\, p(x) \geq 0\} \text{ and } \mathcal{Z}_p := \{x \in \mathbb{R}^n |\, p(x) = 0\}.$$

It is clear that $\mathcal{Z}_p = \partial \mathcal{Q}_p$ as long as P (the matrix associated with p) is not positive semidefinite. We also use the compact notation

$$\hat{\mathcal{Q}}_p := \{x \in \mathbb{R}^n | p(x) \rhd 0\},$$

where $\rhd \in \{\geq, =\}$. That is, if the constraint appearing in the above given definition is an equality constraint, then $\hat{\mathcal{Q}}_p = \mathcal{Z}_p$ while for an inequality constraint $\hat{\mathcal{Q}}_p = \mathcal{Q}_p$.

7.2. Main Result - Characterization of $\overline{\mathbf{co}(\mathcal{S})}$

Define the set

$$\mathcal{I} := \left(\bigcup_{i \neq j} (\mathcal{Z}_{p_i} \cap \mathcal{Z}_{p_j}) \right) \cap \mathcal{S}. \tag{7.2}$$

The main result of this chapter is the following theorem, which shows that $\overline{\mathbf{co}(\mathcal{S})}$ can be described in terms of a combination of LMIs and supporting halfspaces

Theorem 7.2.1. *The set $\mathcal{I}$ is a subset of $\partial \mathcal{S}$. Furthermore, $\overline{\mathbf{co}(\mathcal{S})}$ can be expressed as the intersection of a finite number of LMIs and the set*

$$\mathcal{F} := \begin{cases} \bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{S}, x) & \textit{if } \mathcal{I} \neq \emptyset \\ \\ \mathbb{R}^n & \textit{if } \mathcal{I} = \emptyset. \end{cases}$$

The following is an immediate corollary of the theorem above.

Corollary 7.2.2. *If* $\mathcal{S} \in \mathbb{R}^2$*, then,* $\overline{\mathbf{co}(\mathcal{S})}$ *is an LMI set in* $\mathbb{R}^2$*.*

Proof. In $\mathbb{R}^2$, the following statements hold true. The set $\mathcal{I}$ is simply composed of a finite number of intersection points. Moreover, the intersection of all halfspaces supporting a set at a common boundary point is a polyhedral set. Therefore,

$$\bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{S}, x)$$

is an LMI set. □

7.2.1. Outline of the Proof of Theorem 7.2.1

We base the proof of Theorem 7.2.1 on the characterization of the convex hull given in Proposition 3.1.8 which states that

$$\overline{\mathbf{co}(\mathcal{S})} = \bigcap_{x \in \partial\mathcal{S}} \mathbf{C}(\mathcal{S}, x). \tag{7.3}$$

It is easy to see that the boundary points of $\mathcal{S}$ belong to the zero sets $\mathcal{Z}_{p_i}$, $i = 1 : m$. Using this fact, one can separate these points into two disjoint groups, namely, those lying at the intersections of more than one zero set, which are exactly the elements of $\mathcal{I}$ defined in (7.2), and those which are elements of only one zero set, namely $\partial\mathcal{S}\backslash\mathcal{I}$. Notice that $\partial\mathcal{S}$ is smooth along the latter while points where the boundary is not smooth lie in the former. To prove the theorem, we show that the halfspaces supporting $\mathcal{S}$ along $\partial\mathcal{S} \setminus \mathcal{I}$ can be replaced by a finite number of LMI sets which are, except for a few special cases, directly induced by p_is themselves. This leads to a mixed description of $\overline{\mathbf{co}(\mathcal{S})}$ in terms of LMIs and the intersection of possibly infinitely many halfspaces supporting $\mathcal{S}$ along $\mathcal{I}$.

7.3. An LMI Description for Supporting Halfspaces

In this section, our goal is to develop the basic results which will be employed to show that halfspaces supporting $\mathcal{S}$ along $\partial \setminus \mathcal{I}$ can be replaced by LMIs in the description of the convex hull given in (7.3). Towards this end, we concentrate on a single quadratic constraint determined by a polynomial p and the corresponding feasible region $\hat{\mathcal{Q}}_p$, where p satisfies Assumptions 7.1.1 and 7.1.2. We begin by introducing some propositions related with the LMI representation of $\mathcal{Q}_p$.

Proposition 7.3.1. *If $\pi(P) = 1$, $\mathcal{Q}_p$ is either an LMI set, or the union of two LMI sets, $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$, which have disjoint interiors and which are symmetric with respect to a point $x_c \in \mathbb{R}^n$.*

The proof of the proposition is given in Appendix B.1. For $\pi(P) = 1$, an explicit formulation of LMI regions of $\mathcal{Q}_p$ in terms of the matrix P can be found in the proof. Examples of $\mathcal{Q}_p$ under this condition are convex regions bounded by an ellipsoid, a paraboloid and a hyperboloid. The first two are LMI sets while the last one is the union of two LMI sets each of which bounded by a sheet of the hyperboloid.

Proposition 7.3.2. *If $\pi(P) = 1$ and $\mathcal{Q}_p$ is composed of two LMI components, no halfspace containing $\mathcal{Q}_p$ exists.*

As an example for Proposition 7.3.2, consider the region bounded by a hyperboloid mentioned previously. There does not exist a halfspace containing this region even though there are hyperplanes separating its two LMI components.

Now, we are ready to focus on the main result. Consider a set $\mathcal{V} \subseteq \mathbb{R}^n$, which has a nonempty interior. Let's define

$$\mathcal{U} := \mathcal{V} \cap \hat{\mathcal{Q}}_p$$

and assume

$$\mathcal{B} := \mathbf{int}(\mathcal{V}) \cap \mathcal{Z}_p \neq \emptyset. \tag{7.4}$$

It is clear that $\mathcal{B} \subseteq \partial\mathcal{U}$. A conceptual picture illustrating these definitions in $\mathbb{R}^2$ is given in Figure 7.1 for an inequality constraint (i.e., $\hat{\mathcal{Q}}_p = \mathcal{Q}_p$).

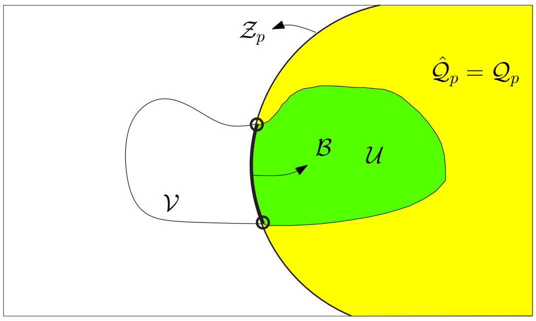


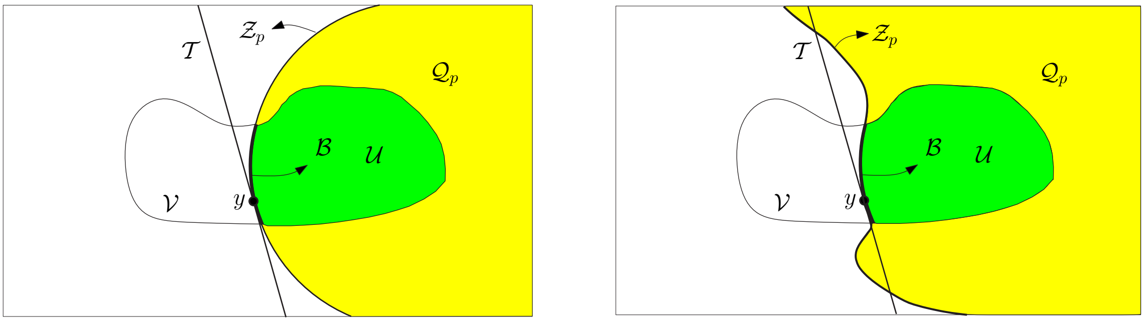
Figure 7.1. Illustration of $\mathcal{V}$, $\mathcal{Q}_p$, $\mathcal{U}$ and $\mathcal{B}$

In the rest of the section, we show that halfspaces supporting $\mathcal{U}$ along $\mathcal{B}$ can be replaced by an LMI set.

Lemma 7.3.3. *If there exists a hyperplane, $\mathcal{T}$, supporting $\mathcal{U}$ at a point $y \in \mathcal{B}$, then $\pi(P) = 1$ and $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$.*

The main idea behind Lemma 7.3.3 for an inequality constraint can be interpreted as follows. (A similar reasoning applies to an equality constraint.) If a hyperplane $\mathcal{T}$ supports $\mathcal{U}$ at a point $y \in \mathcal{B}$, clearly, $\mathcal{T} \cap \mathbf{int}(\mathcal{U}) = \emptyset$. This also implies that $\mathcal{U}$ must possess a certain convexity property around a neighborhood of y. Roughly speaking, Lemma 7.3.3 (in conjunction with Proposition 7.3.1) states that these properties also hold true globally for $\mathcal{T}$ and $\mathcal{Q}_p$.

As an example, consider Figure 7.2.a. As can be seen from the figure, there exists a supporting halfspace at a point $y \in \mathcal{B}$, and hence, due to Lemma 7.3.3 and Proposition 7.3.1, $\mathcal{Q}_p$ is composed of LMI sets and $\mathcal{T}$ does not intersect its interior (for this example, $\mathcal{Q}_p$ has only one LMI component). The lemma guarantees that a



a) b)

Figure 7.2. Examples for Lemma 7.3.3

situation like in Figure 7.2.b (*i.e.*, $\mathcal{T}$ supports $\mathcal{U}$ but intersects the interior of $\mathcal{Q}_p$) cannot occur even in $\mathbb{R}^n$.

Lemma 7.3.4. *If there exists a hyperplane supporting $\mathcal{U}$ at a point $y \in \mathcal{B}$, there exists an LMI set $\mathcal{L}_p$ such that $\mathcal{U} \subseteq \mathcal{L}_p$ and any hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$ also supports $\mathcal{L}_p$.*

Except for some technical details, Lemma 7.3.4 is a simple consequence of previous results. We describe this briefly for an inequality constraint. If there exists a supporting halfspace along $\mathcal{B}$, $\mathcal{Q}_p$ is either an LMI set or a union of two LMI sets. If the former holds true, $\mathcal{Q}_p$ itself can be chosen as $\mathcal{L}_p$ and we are done (see Figure B.1.a given in appendix). A similar reasoning applies when $\mathcal{Q}_p$ is composed of two components and $\mathcal{U}$ is a subset of one of them as can be seen from Figure B.2. It remains to consider the case in which $\mathcal{Q}_p$ is composed of two components but $\mathcal{U}$ is not a subset of one of them as in Figure B.3.a. However, for this situation, there does not exist a hyperplane supporting $\mathcal{U}$ along $\mathcal{B}$ if we disregard some technical details. In order to see this, assume there exists such an hyperplane, $\mathcal{T}$. Then, $\mathcal{T}$ cannot intersect $\mathbf{int}(\mathcal{Q}_p)$. Moreover, due to Proposition 7.3.2, a halfspace cannot contain $\mathcal{Q}_p$. Hence, $\mathcal{T}$ has to separate $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$, which are convex sets having nonempty disjoint interiors. This leads to a contradiction.

As can be seen from the sketch given above, $\mathcal{L}_p$ is usually determined by the polynomial p itself. However, there are some exceptions as described in the proof. For these and a rigorous treatment including equality constraint, refer to Appendix B.2.

The following corollary of Lemma 7.3.4 is the main result of this section which will be utilized to obtain the characterization of $\overline{\mathbf{co}(\mathcal{S})}$.

Corollary 7.3.5. *If the exists a hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$, then there exists an LMI set $\mathcal{L}_p$ such that*

$$\mathcal{U} \subseteq \mathcal{L}_p \subseteq \bigcap_{x \in \mathcal{B}} \mathbf{C}(\mathcal{S}, x). \tag{7.5}$$

Remark 7.3.6. As one can see from the definition of $\mathcal{B}$ (7.4), the formulation given in (7.5) does not include the halfspaces supporting $\mathcal{U}$ at points of $\mathcal{Z}_p \cap \partial S$. These points are marked with small circlers in Figure 7.1.

7.4. Proof of Theorem 7.2.1

We are ready to prove the main theorem employing the results developed in the previous section. Let's define

$$\mathcal{S}_{p_k} := \begin{cases} \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} p_i(x) \geq 0, i = 1 : \ell, i \neq k; \\ p_j(x) = 0, j = \ell + 1 : m, j \neq k \end{array} \right\} & \text{if } m > 1 \\ \mathbb{R}^n & \text{if } m = 1, \end{cases} \tag{7.6}$$

which is obtained by removing the constraint corresponding to polynomial p_k from the description of $\mathcal{S}$. In order to prove the theorem, for each $k \in \{1 : m\}$, we apply Corollary 7.3.5 to p_k and $\mathcal{S}_{p_k}$ regarding them p and $\mathcal{U}$ respectively. By this way, we can replace all supporting halfspaces used in description of $\overline{\mathbf{co}(S)}$ successively by LMI sets except the ones supporting $\mathcal{S}$ along the intersection set $\mathcal{I}$, and we are done. Note that we can not get rid of all supporting halfspaces because points lying in $\mathcal{Z}_{p_k} \cap \partial \mathcal{S}_{p_k}$ are not handled by Corollary 7.3.5 as stated in Remark 7.3.6. In the sequel, we give the formal proof of the theorem.

Proof of Theorem 7.2.1. We first construct a characterization of $\partial\mathcal{S}$. Because $\mathcal{S}$ is closed, $\partial\mathcal{S} \subseteq \mathcal{S}$. Moreover, as one can easily verify, an element of $\mathcal{S}$ is its boundary point if and only if it is also an element of $\mathcal{Z}_{p_i}$ for some $i = 1 : m$. Using these facts, we can express $\partial\mathcal{S}$ as the union of the two disjoint set

$$\mathcal{I} = \{x \in \mathcal{S} | x \in \mathcal{Z}_{p_i} \text{ for at least two different values of } i\}$$

and

$$\mathcal{J} := \{x \in \mathcal{S} | x \in \mathcal{Z}_{p_i} \text{ for exactly one } i\}.$$

It is clear that the former is equivalent to the definition given in (7.2).

Define

$$\mathcal{B}_{p_i} := \mathcal{Z}_{p_i} \cap \mathbf{int}(\mathcal{S}_{p_i}), \quad i = 1 : m.$$

It can be inferred that

$$\mathcal{J} = \bigcup_{i=1}^{m} \mathcal{B}_{p_i}.$$

Without loss of generality, if there exists some nonempty elements in the collection $\{\mathcal{B}_{p_i}\}$, we can assume they are the first elements in the collection by an appropriate change of indexing. Hence, one can write

$$\mathcal{J} = \bigcup_{i=1}^{r} \mathcal{B}_{p_i},$$

where r is the number of nonempty sets in $\{\mathcal{B}_{p_i}\}$. Putting all together, the boundary of $\mathcal{S}$ can be expressed as

$$\partial\mathcal{S} = \mathcal{I} \cup \left(\bigcup_{i=1}^{r} \mathcal{B}_{p_i} \right). \tag{7.7}$$

We proceed by the characterization of the closure of the convex hull of $\mathcal{S}$. By substituting (7.7) into (7.3), one obtains

$$\overline{\mathbf{co}(\mathcal{S})} = \left(\bigcap_{x\in\mathcal{I}} \mathbf{C}(\mathcal{S},x)\right) \bigcap \left(\bigcap_{i=1}^{r}\left(\bigcap_{x\in\mathcal{B}_{p_i}} \mathbf{C}(\mathcal{S},x)\right)\right). \tag{7.8}$$

If all the sets in the collection $\{\mathcal{B}_{p_i}\}$ are empty, we are done. Alternatively, assume there are nonempty elements in $\{\mathcal{B}_{p_i}\}$. By definition,

$$\mathcal{S} = \mathcal{S}_{p_i} \cap \mathcal{Q}_{p_i}, \quad i = 1:r.$$

Therefore, for each $\mathcal{S}_{p_i}$ and $\mathcal{Q}_{p_i}$, $i \in 1:r$, we can apply Corollary 7.3.5 and deduce that either there exists an LMI set $\mathcal{L}_{p_i}$ such that

$$\mathcal{S} \subseteq \mathcal{L}_{p_i} \subseteq \bigcap_{x\in\mathcal{B}_{p_i}} \mathbf{C}(\mathcal{S},x),$$

or there does not exist any hyperplane supporting $\mathcal{S}$ along $\mathcal{B}_{p_i}$. Using this result with (7.8), we come up with

$$\mathcal{S} \subseteq \left(\bigcap_{x\in\mathcal{I}} \mathbf{C}(\mathcal{S},x)\right) \bigcap \left(\bigcap_{i=1}^{s} \mathcal{L}_{p_i}\right) \subseteq \overline{\mathbf{co}(\mathcal{S})},$$

which leads to the desired result since the intersection set in the middle is a closed convex set that contains $\mathcal{S}$ and contained in $\overline{\mathbf{co}(\mathcal{S})}$ (recall that $\overline{\mathbf{co}(\mathcal{S})}$ is the smallest closed convex set containing $\mathcal{S}$). Note that, in above, we applied a change of indexing to ensure that the indices for which there exists a hyperplane supporting $\mathcal{S}$ along $\mathcal{B}_i$ are the first s ones. □

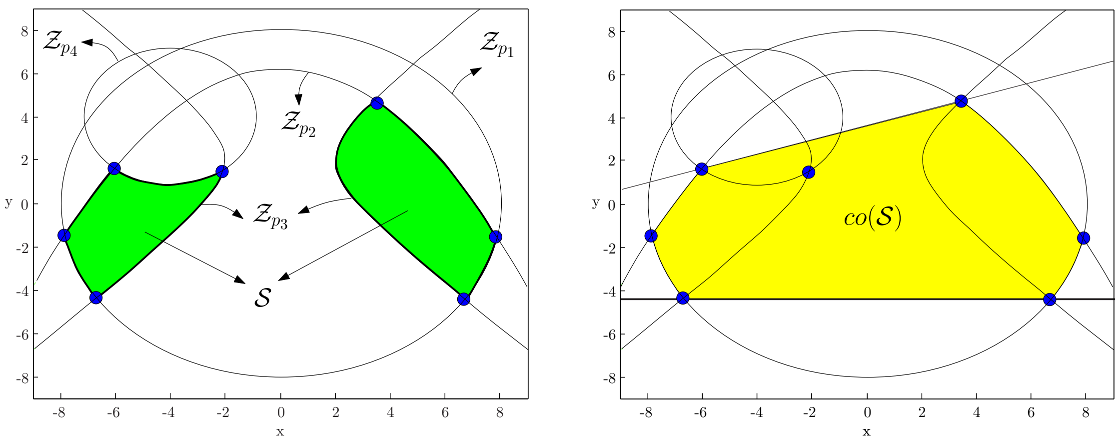
7.5. Example

In what follows, we illustrate our results on an example in $\mathbb{R}^2$.

Consider the set of constraints

$$
\begin{aligned}
p_1(x) &= 64 - x_1^2 - x_2^2 \geq 0, \\
p_2(x) &= 50 - x_1^2 - 8x_2 \geq 0, \\
p_3(x) &= x_1^2 - (x_2 - 2)^2 - 4 \geq 0, \\
p_4(x) &= (x_1 + 4)^2 + (x_2 - 4)^2 - 10 \geq 0.
\end{aligned}
$$

The corresponding feasible region, $\mathcal{S}$, is depicted in Figure 7.3.a. We know that $\overline{\mathbf{co}(\mathcal{S})}$ is the intersection of its supporting halfspaces. It is possible to separate these halfspaces into two groups – those supporting $\mathcal{S}$ at the points in $\partial\mathcal{S} \setminus \mathcal{I}$ and those supporting it at the points in $\mathcal{I}$ (points of $\mathcal{I}$ are indicated by thick dots in the figure). Corollary 7.3.5 states that the former can always be replaced by LMIs. These LMIs can be identified as follows.



a) b)

Figure 7.3. a) The region determined by quadratic inequalities; b) its convex hull

The halfspaces supporting $\mathcal{S}$ along $\mathcal{Z}_{p_1}$ and $\mathcal{Z}_{p_2}$ can be replaced by $\mathcal{Q}_{p_1}$ and $\mathcal{Q}_{p_2}$. Because these sets are determined by convex quadratic constraints, they admit the LMI representations

$$
\mathcal{L}_{p_1} : \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_1 & x_2 & 64 \end{bmatrix} \succeq 0; \quad \mathcal{L}_{p_2} : \begin{bmatrix} 1 & x_1 \\ x_1 & 50 - 8x_2 \end{bmatrix} \succeq 0
$$

which can be derived from (B.3) and (B.6) respectively. On the other hand, there does not exist any halfspace supporting $\mathcal{S}$ along the points of $\partial\mathcal{S} \setminus \mathcal{I}$ lying on $\mathcal{Z}_{p_3}$ or $\mathcal{Z}_{p_4}$.

Now it remains to consider the halfspaces supporting $\mathcal{S}$ at points of $\mathcal{I}$. As shown in the proof of Corollary 7.2.2, using only finitely many of them is enough. In this example, we have only two

$$\mathcal{L}_{I_1} : 0.3322x_1 - x_2 + 3.6438 \geq 0; \quad \mathcal{L}_{I_2} : x_2 + 4.385 \geq 0$$

which are depicted in Figure 7.3.b. As a result, the following LMI representation is obtained

$$\overline{\mathbf{co}(\mathcal{S})} = \mathcal{L}_{p_1} \cap \mathcal{L}_{p_2} \cap \mathcal{L}_{I_1} \cap \mathcal{L}_{I_2}.$$

Notice that, Corollary 7.2.2 only shows that the convex hull is an LMI set and it is not constructive. The main difficulty in obtaining an LMI representation arises in finding the halfspaces supporting $\mathcal{S}$ at the points of $\mathcal{I}$. In the example given above, we could compute the convex hull by inspection.

7.6. Summary

In this chapter, we asked when the convex hull of $\mathcal{S}$ defined in (7.1) admits an LMI representation. Towards answering this question, we showed that $\overline{\mathbf{co}(\mathcal{S})}$ can be expressed as the intersection of a finite number of LMI sets and possibly infinitely many halfspaces supporting $\mathcal{S}$ along $\mathcal{I}$. In $\mathbb{R}^2$, the intersection of the supporting halfspaces turns out to be a polyhedral set. Therefore, an LMI representation always exists for this case. Unfortunately, since the proof is not constructive, we could not give an algorithm producing the LMI description of $\overline{\mathbf{co}(\mathcal{S})}$ unlike the results developed in the previous chapter.

8. CONCLUSIONS

Several practical and theoretical questions in systems and control literature can be formulated as PSPs. Therefore, studying their solutions constitutes an important subject in control theory, which also motivated us to concentrate to this topic. A good feature of PSPs is the fact that they are decidable, and hence, there exists algorithms for their solution. On the other hand, they are also known to be NP-hard. This means it is very unlikely to find an algorithm that can solve any PSP exactly and efficiently. Therefore, there are two possible approaches that can be followed to deal with such problems in practice. One is to develop algorithms that give cheaply computable approximations; the other is to determine subproblems that can be solved efficiently due to their structure and obtain the algorithm achieving this. In this thesis, we followed the second approach. Our main goal was to convert some special nonconvex polynomial optimization problems into LMIs in order to solve them efficiently. In this regard, we made two contributions.

In the first one, we showed that the convex hull of a region determined by two quadratic inequality constraints is an LMI set and developed the algorithm producing the LMI description. By this way, it would be possible to compute the optimal value of a linear function over such a region exactly and efficiently. When the related methods in the literature in investigated, it can be seen that none of them can achieve this. They lead to inexact results and/or they are computationally inefficient. This is demonstrated in Chapter 6 in detail. Moreover, we also developed a new losslessness result for the SDP relaxation and the S-procedure which was not available in the literature.

In Chapter 7, we concentrated to a more general problem and investigated the sets determined by a finite number of quadratic inequality constraints. We tried to understand if the convex hull of such sets admit an LMI representation. Unfortunately, this turned out to be far from the truth. However, during this effort, it is shown that if we restrict ourselves to $\mathbb{R}^2$, the LMI description always exists. But the proof is not constructive and an algorithm computing the LMI representation of the convex hull could not be developed.

As a future work, one may try to develop an algorithm for constructing the convex hull of the region determined by quadratic polynomials in $\mathbb{R}^2$ for which a nonconstructive proof is attained as just described above. It seems for this purpose it is necessary to employ SOS multipliers instead of just real numbers. Moreover, using SOS multipliers, it would also be possible to identify some other problems in a higher dimensional space for which convex hull can be constructed.

APPENDIX A: PROOFS OF CHAPTER 6

A.1. Proof of Lemma 6.2.2

Consider the LMI system which is obtained by replacing all inequalities appearing in (4.12) with nonstrict ones. We denote the corresponding feasible set as $\mathcal{N}$ and its projection as $\mathcal{P}_{\mathcal{N}}$. Employing Proposition 4.4.7, $\mathcal{P}_{\mathcal{N}}$ can be characterized as

$$\mathcal{P}_{\mathcal{N}} = \{x \in \mathbb{R}^n |\ \mathbf{q}_\lambda(x) \geq 0,\ \forall \lambda \in \Omega\}.$$

It is easy to see that, for $\lambda \in \Omega$, a polynomial lying in the pencil $\mathbf{q}_\lambda$ can be expressed as a positive linear combination of the polynomials $\mathbf{q}_{\underline{\omega}}$ and $\mathbf{q}_{\overline{\omega}}$. Therefore, one can see that

$$\mathcal{P}_{\mathcal{N}} = \{x \in \mathbb{R}^n |\ \mathbf{q}_\lambda(x) \geq 0,\ \text{for } \lambda = \underline{\omega},\ \text{and } \lambda = \overline{\omega}\}.$$

On the other hand, using elementary arguments it can be shown that $\mathbf{int}(\mathcal{P}_{\mathcal{N}})$ can be described by replacing the nonstrict inequalities appearing in the preceding description with strict ones (in general, this does not always hold true for arbitrary polynomials). Thus,

$$\mathbf{int}(\mathcal{P}_{\mathcal{N}}) = \mathcal{S}_{\underline{\omega}} \cap \mathcal{S}_{\overline{\omega}} \ = \bigcap_{\lambda \in \omega} \mathcal{S}_\lambda.$$

Based on this fact, it is enough to show that $\mathcal{P}_{\mathcal{R}} = \mathbf{int}(\mathcal{P}_{\mathcal{N}})$ in order to prove our claim.

To this end, consider the set $\mathcal{V}$ which we define as the feasible region of the LMI constraint which is attained by replacing the nonstrict matrix inequality given in (4.12) by a strict one. Let's denote projection of this set as $\mathcal{P}_{\mathcal{V}}$. First, we show that $\mathcal{P}_{\mathcal{V}} = \mathcal{P}_{\mathcal{R}}$. Consider a point $x \in \mathcal{P}_{\mathcal{R}}$, which exists because $\mathcal{V} \neq \emptyset$. Since, $\mathcal{P}_{\mathcal{R}}$ is the projection set, there exists a matrix Y such that (4.12) is satisfied for the pair (x, Y). Notice that for a small enough positive number α, if we replace Y with $\tilde{Y} = Y + \alpha I$, (4.12) still holds true. However, unlike the pair (x, Y), for $(x, \tilde{Y})$

$$\begin{bmatrix} \tilde{Y} & x \\ x^T & 1 \end{bmatrix} \succ 0.$$

Therefore, every point of $\mathcal{P}_{\mathcal{R}}$ is also a point of $\mathcal{P}_{\mathcal{V}}$. The inclusion in the other direction is trivial. Consequently, we obtain the desired result.

In order to complete the proof, we show that $\mathcal{P}_{\mathcal{V}}$ is interior of $\mathcal{P}_{\mathcal{N}}$. Because $\mathcal{S} \neq \emptyset$, $\mathcal{V}$ is not empty, which implies $\mathcal{V} = \mathbf{int}(\mathcal{N})$. Since projection is a linear mapping and the sets of interest are convex, it can be deduced that $\mathcal{P}_{\mathcal{V}} = \mathbf{int}(\mathcal{P}_{\mathcal{N}})$ ([17], Theorem 6.6).

A.2. Proofs of Lemma 6.1.2, Lemma 6.5.15 and Lemma 6.5.17

Before getting into the proofs, we need to give some definitions and derive some basic results.

Suppose that Q is a real symmetric matrix, that is, $Q \in \mathbb{S}^n$.

Definition A.2.1. *Q is said to be real congruent to a $T \in \mathbb{S}^n$ if there exists a nonsingular square matrix $U \in \mathbb{R}^{n\times n}$ such that $T = U^T Q U$.*

It is a well known fact that the inertia of Q (i.e. its numbers of positive, negative and zero eigenvalues) is preserved under real congruence. Moreover, Q can always be put into a diagonal form such that each diagonal element is either $+1$, 0 or -1 and their numbers give the inertia of Q.

Consider a real symmetric matrix pencil $\mathbf{Q}_\lambda$.

Definition A.2.2. *$\mathbf{Q}_\lambda$ is said to be real congruent to a real symmetric matrix pencil $\mathbf{T}_\lambda$ if there exists a nonsingular square matrix $U \in \mathbb{R}^{n\times n}$ such that $\mathbf{T}_\lambda = U^T \mathbf{Q}_\lambda U$ for every $\lambda \in \mathbb{C}$.*

There are two important properties of a matrix pencil that are preserved under congruence. Suppose that $\mathbf{Q}_\lambda$ and $\mathbf{T}_\lambda$ are real congruent. First, due to the facts mentioned above, the inertia of $\mathbf{Q}_\lambda$ is the same as the inertia of $\mathbf{T}_\lambda$ for every $\lambda \in \mathbb{R}$. Second, since the multiplication with a nonsingular matrix does not change the rank, the GEVs of the pencil $\mathbf{Q}_\lambda$ and $\mathbf{T}_\lambda$ are the same (see definition of GEV given in Section 6.1). Apart from these facts, the symmetric pencil $\mathbf{Q}_\lambda$ can always be brought into a diagonal from by real congruence. However, unlike symmetric matrices, this form, which is given in the next theorem, is more complicated and composed of matrix blocks instead of scalars.

Define the matrices

$$E_m := \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad G_m := \begin{bmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & & 0 & 0 \\ \vdots & & \iddots & & \vdots \\ 1 & 0 & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$H_{2m} := \begin{bmatrix} 0 & \cdots & \cdots & 0 & Z \\ \vdots & & & Z & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & Z & & & \vdots \\ Z & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

where the order of each matrix is indicated in their subscripts, $m \geq 1$ and $Z := \operatorname{diag}(1, -1)$ (i.e. diagonal matrix of order two having diagonal entries $+1$ and -1). Keep in mind that $E_1 = 1$, $G_1 = 0$.

In addition, define the blocks

$$\mathbf{F}_{i\lambda} := \delta_i(E_{f_i} + \lambda G_{f_i}),\ i = 1 : n_F$$

$$\mathbf{R}_{i\lambda} := \eta_i((\lambda - \beta_i)E_{r_i} + G_{r_i}),\ i = 1 : n_R$$

$$\mathbf{S}_{i\lambda} := \left(\lambda \begin{bmatrix} 0 & 0 & E_{s_i} \\ 0 & 0_1 & 0 \\ E_{s_i} & 0 & 0 \end{bmatrix} + G_{2s_i+1} \right),\ i = 1 : n_S$$

$$\mathbf{C}_{i\lambda} := (\lambda + \sigma_i)E_{2c_i} + \omega_i H_{2c_i} + \begin{bmatrix} E_{2c_i-2} & 0 \\ 0 & 0_2 \end{bmatrix},\ i = 1 : n_C.$$

Theorem A.2.3 ([38], Theorem 9.2)**.** *Every real symmetric matrix pencil $\mathbf{Q}_\lambda$ is real congruent to the following block diagonal form*

$$\mathbf{D}_\lambda = 0_z \oplus \left(\bigoplus_{i=1}^{n_F} \mathbf{F}_{i\lambda} \right) \oplus \left(\bigoplus_{i=1}^{n_R} \mathbf{R}_{i\lambda} \right) \oplus \left(\bigoplus_{i=1}^{n_S} \mathbf{S}_{i\lambda} \right) \oplus \left(\bigoplus_{i=1}^{n_C} \mathbf{C}_{i\lambda} \right), \tag{A.1}$$

where $s_1, \cdots, s_{n_S}$, $f_1, \cdots, f_{n_F}$, $r_1, \cdots, r_{n_R}$, $c_1, \cdots c_{n_C}$ are positive integers; z, n_S, n_F, n_R and n_C are nonnegative integers; $\beta_1 \leq \cdots \leq \beta_{n_R}$ are real numbers; $\sigma_i + j\omega_i$, $i = 1 : n_C$ are non-real complex numbers with $\omega_i > 0$; $\delta_i \in \{+1, -1\}$, $i = 1 : n_F$ and $\eta_i \in \{+1, -1\}$, $i = 1 : n_R$ are signs. Note that if any of z, n_S, n_F, n_R or n_C is zero, then the corresponding block does not exist and should be omitted in decomposition of $\mathbf{D}_\lambda$.

Remark A.2.4. *In above, β_i, $i = 1 : n_R$ and $\sigma_i + j\omega_i$, $i = 1 : n_C$ represent the real and complex GEVs of the pencil $\mathbf{D}_\lambda$, equivalently that of $\mathbf{Q}_\lambda$, respectively. Consider a real GEV β_ℓ. Suppose that k is the minimum index such that $\beta_k = \beta_\ell$ while $k + m$ is the maximum index such that $\beta_{k+m} = \beta_\ell$. This means there are exactly $m + 1$ real blocks having GEV equal to β_ℓ. The number $m + 1$ gives the geometric multiplicity of this GEV. Hence, a GEV appears in the sequence β_i, $i = 1 : n_R$ as many times as its geometric multiplicity. On the other hand, the algebraic multiplicity of β_ℓ is $\sum_{i=k}^{k+m} r_i$. Recall that a real GEV in $[0, 1]$ interval appears in the sequence α_i, $i = 1 : n_G$ described in Section 6.1 as many times as its algebraic multiplicity. Therefore, this sequence and* *the part of β_i, $i = 1 : n_R$ lying in $[0,1]$ are the same except their multiplicities.*

Note that $\mathbf{S}_{i\lambda}$, $\mathbf{F}_{i\lambda}$, $\mathbf{R}_{i\lambda}$, and $\mathbf{C}_{i\lambda}$ are called the blocks corresponding to singular part of the pencil, GEVs at infinity, real GEVs and complex GEVs, respectively. In order to prove the lemmas, we need to obtain $\pi(\mathbf{D}_\lambda)$, and hence, the number of positive eigenvalues of each type of block as a function of the parameter λ. The following proposition will be useful in this regard.

Proposition A.2.5. *Consider the matrix*

$$M := \begin{bmatrix} 0 & A & B \\ A^T & C & 0 \\ B & 0 & 0 \end{bmatrix}, \tag{A.2}$$

where $A \in \mathbb{R}^{m\times p}$, $B \in \mathbb{S}^m$, $C \in \mathbb{S}^p$ and B is nonsingular. M is real congruent to

$$2B \oplus (-2B) \oplus C \tag{A.3}$$

Therefore, $\pi(M) = m + \pi(C)$.

Proof. For

$$U = \begin{bmatrix} 0 & I & I \\ -I & 0 & 0 \\ B^{-1} & I & -I \end{bmatrix},$$

we have $U^T M U = 2B \oplus (-2B) \oplus C$, which shows M is congruent to (A.3). Due to congruence,

$$\begin{aligned} \pi(M) &= \pi\left(2B \oplus (-2B) \oplus C\right) \\ &= \pi(B) + \pi(-B) + \pi(C) \\ &= \pi(B) + \nu(B) + \pi(C) \\ &= m + \pi(C). \end{aligned}$$

□

Using this proposition, we can obtain the number of positive eigenvalues of each type of block.

Lemma A.2.6. *The following hold true:*

i) $\pi(\mathbf{S}_{i\lambda}) = s_i$,

ii) $\pi\left(\mathbf{C}_{i\lambda}\right) = c_i$

iii) $\pi(\mathbf{F}_{i\lambda}) = \begin{cases} \frac{f_i}{2} & f_i \text{ is even}, \\ \frac{f_i+\delta_i}{2} & f_i \text{ is odd} \end{cases}$

iv) $\pi(\mathbf{R}_{i\lambda}) = \begin{cases} \frac{r_i-1}{2} + \pi(\eta_i(\lambda - \beta_i)) & r_i \text{ is odd}, \\ \frac{r_i}{2} & r_i \text{ is even and } \eta_i = +1, \\ \frac{r_i-2}{2} + \pi(\lambda - \beta_i) + \pi(-(\lambda - \beta_i)) & r_i \text{ is even and } \eta_i = -1, \end{cases}$

Proof. All blocks of the diagonal form (A.1), can be partitioned as in (A.2). To show this, for each case, we give the matrices A, B and C. Then, we use Proposition A.2.5 to prove the lemma.

i) Suppose that $\lambda \neq 0$. We have

$$A = [0 \cdots 0\ 1]^T, \quad B = \lambda E_{s_i} + G_{s_i}, \quad C = 0_1$$

and clearly B is not singular for any $\lambda \neq 0$. Thus, $\pi(\mathbf{S}_{\lambda_i}) = s_i + \pi(C) = s_i$. If $\lambda = 0$, then the singular block becomes G_{2s_i+1}. Using Proposition A.2.5, it can be easily shown that $\pi(G_{2s_i+1}) = s_i$, hence, we are done.

ii) Define the matrices

$$X := \begin{bmatrix} \omega_i & \lambda+\sigma_i \\ \lambda+\sigma_i & -\omega_i \end{bmatrix}, \quad Y := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In below, we use

$$B = \begin{bmatrix} 0 & \cdots & 0 & Y & X \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \iddots & \vdots \\ Y & \iddots & \iddots & & \vdots \\ X & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

of appropriate dimension which is nonsingular for any λ since $\omega_i \neq 0$. If c_i is odd, the complex block can be partitioned as

$$A = [0 \cdots 0\ Y]^T, \quad C = X$$

with B being a symmetric matrix of order $c_i - 1$, which has the form defined above. Moreover, it can be easily shown that C is always indefinite since $\omega_i \neq 0$, and hence, $\pi(C) = 1$ for every $\lambda \in \mathbb{R}$. Thus, $\pi(\mathbf{C}_{i\lambda}) = m$. If c_i is even, we again have $\pi(\mathbf{C}_{i\lambda}) = c_i$ as a result of the partitioning

$$A = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & Y \end{bmatrix}^T, \quad C = \begin{bmatrix} Y & X \\ X & 0 \end{bmatrix}$$

with the matrix B of order $c_i - 2$ and the fact that $\pi(C) = 2$.

iii) If f_i is odd,

$$A = \delta_i[0 \cdots 0\ 1]^T, \quad B = \delta_i(E_{(f_i-1)/2} + \lambda G_{(f_i-1)/2}), \quad C = \delta_i.$$

Clearly, B is not singular for any $\lambda \in \mathbb{R}$. Therefore, $\pi(\mathbf{F}_{i\lambda}) = (f_i - 1)/2 + \pi(\delta) =$

$(f_i + \delta)/2$.

On the other hand, if f_i is even, we have

$$A = \delta_i \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}^T, \quad B = \delta_i(E_{(f_i-2)/2} + \lambda G_{(f_i-2)/2}), \quad C = \delta_i \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}.$$

It is immediate to verify that C is always indefinite and hence $\pi(C) = 1$ for every $\lambda \in \mathbb{R}$. Therefore, $\pi(\mathbf{F}_{i\lambda}) = (f_i - 2)/2 + \pi(C) = f_i/2$.

iv) Suppose that r_i is odd. If $\lambda \neq \beta_i$, this block can be partitioned as

$$A = \eta_i[0 \cdots 0 \; \lambda - \beta_i \; 1]^T, \quad C = \eta_i(\lambda - \beta_i)$$
$$B = \eta_i((\lambda - \beta_i)E_{(r_i-1)/2} + G_{(r_i-1)/2}).$$

Because $\lambda \neq \beta_i$, B is not singular. Thus, $\pi(\mathbf{R}_{i\lambda}) = \frac{r_i-1}{2} + \pi(\eta_i(\lambda - \beta_i))$. If $\lambda = \beta_i$, the block becomes G_{r_i} and $\pi(G_{r_i}) = \frac{r_i-1}{2}$. Consequently, we can see that the number of positive eigenvalues of this block is $\frac{r_i-1}{2} + \pi(\eta_i(\lambda - \beta_i))$ for any $\lambda \in \mathbb{R}$.

Now, suppose r_i is even. If $\lambda \neq \beta_i$, we have the partitioning

$$A = \eta_i \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}^T, \quad C = \eta_i \begin{bmatrix} 1 & \lambda - \beta_i \\ \lambda - \beta_i & 0 \end{bmatrix},$$

$$B = \eta_i((\lambda - \beta_i)E_{(r_i-2)/2} + G_{(r_i-2)/2}).$$

Therefore, $\pi(\mathbf{R}_{i\lambda}) = \frac{r_i-2}{2} + 1 = \frac{r_i}{2}$ for $\lambda \neq \beta_i$. If $\lambda = \beta_i$, then the block becomes $\eta_i G_{r_i}$. Hence, the number of positive eigenvalues is $\frac{r_i-2}{2} + \pi(\eta_i)$ for $\lambda = \beta_i$. Combining these two results, it can be seen that the number of positive eigenvalues can be expressed as $\frac{r_i}{2}$ if $\eta_i = +1$ and $\frac{r_i-2}{2} + \pi(\lambda - \beta_i) + \pi(-(\lambda - \beta_i))$ if $\eta_i = -1$. $\square$

Based on the preceding lemma and Theorem A.2.3, it can be seen that $\pi(\mathbf{D}_\lambda)$

can be expressed as

$$\pi(\mathbf{D}_\lambda) = c + \sum_{i=1}^{n_{\bar{R}}} \pi(\bar{\eta}_i(\lambda - \bar{\beta}_i)), \tag{A.4}$$

where c is a nonnegative integer constant, $\bar{\eta}_i$, $i = 1 : n_{\bar{R}}$ are signs and $\bar{\beta}_i$, $i = 1 : n_{\bar{R}}$ are GEVs. GEVs and signs are indexed such that $i < j$ if $\bar{\beta}_i < \bar{\beta}_j$ or $\bar{\beta}_i = \bar{\beta}_j$ and $\bar{\eta}_i = -1$ and $\bar{\eta}_j = +1$.

Remark A.2.7. *For the sake of notational simplicity, we insert $\bar{\beta}_i = 0$, $\bar{\eta}_i = -1$ and $\bar{\beta}_j = 1$, $\bar{\eta}_j = +1$ into the sequence of numbers and signs for appropriate i and j that does not violate the indexing described above. Notice that this does not change our problem since it does not affect $\pi(\mathbf{D}_\lambda)$ in $[0, 1]$ interval (indeed $\pi(-(\lambda - 0)) = \pi(\lambda - 1) = 0$ for every $\lambda \in [0, 1]$).*

Remark A.2.8. *The new signs $\bar{\eta}_i$, $i = 1 : n_{\bar{R}}$ and the sequence $\bar{\beta}_i$, $i = 1 : n_{\bar{R}}$ are the same as η_i, $i = 1 : n_R$ and β_i, $i = 1 : n_R$ except two differences (in addition to inserting $\bar{\beta}_i = 0$ and $\bar{\beta}_j = 1$ as mentioned above) which occur for real blocks of even sizes (see (iv) of Lemma A.2.6). Suppose that β_i is GEV corresponding to this type of block. First, if $\eta_i = -1$, then β_i has two copies, $\bar{\beta}_j = \bar{\beta}_{j+1} = \beta_i$, in the new sequence and we have $\bar{\eta}_j = -1$, $\bar{\eta}_{j+1} = +1$. Second, if $\eta_i = +1$, then β_i does not appear in the new sequence.*

Define the function

$$g(\lambda) := \sum_{i=1}^{n_{\bar{R}}} \pi(\bar{\eta}_i(\lambda - \bar{\beta}_i)).$$

The following propositions are the last tools we need to prove lemmas.

Proposition A.2.9. *Consider the set*

$$\Lambda^0 := \{\lambda \in \mathbb{R} \mid g(\lambda) = 0\}.$$

If Λ^0 is not empty,

i) $\bar{\eta}_i = -1$ *for* $i = 1 : k$ *while* $\bar{\eta}_i = +1$ *for* $i = k+1 : n_{\bar{R}}$,

ii) $\Lambda^0 = [\bar{\beta}_k, \bar{\beta}_{k+1}] \subseteq [0, 1]$,

where $k \in 1 : n_{\bar{R}}$.

Proof. i) Assume the contrary. Then, there exist $i, j \in 1 : n_{\bar{R}}$ such that $i < j$, $\bar{\eta}_i = +1$, $\bar{\eta}_j = -1$. Due to the indexing described just after (A.4), $\beta_i < \beta_j$. Moreover,

$$g(\lambda) \geq \pi(\lambda - \bar{\beta}_i) + \pi(-(\lambda - \bar{\beta}_j)).$$

Therefore, $g(\lambda) \geq 1$ for every λ. This implies that $\Lambda^0 = \emptyset$, which is a contradiction. Hence, we are done.

ii) Due to the first part,

$$g(\lambda) = \sum_{i=1}^{k} \pi(-(\lambda - \bar{\beta}_i)) + \sum_{i=k+1}^{n_{\bar{R}}} \pi(\lambda - \bar{\beta}_i).$$

Therefore, the set of points satisfying $g(\lambda) = 0$ is the interval $[\bar{\beta}_k, \bar{\beta}_{k+1}]$.

$\square$

Proposition A.2.10. *Consider the set*

$$\Lambda^1 := \{\lambda \in [0, 1] | g(\lambda) = 1\}. \tag{A.5}$$

Assume that $g(\lambda) \geq 1$ *for every* $\lambda \in [0, 1]$ *and* Λ^1 *is not empty. Then, either*

- $\bar{\eta}_i = -1$ *for* $i = 1 : k$, $i \neq \ell$ *while* $\bar{\eta}_i = +1$ *for* $i = k+1 : n_{\bar{R}}$, $i \neq \ell$,
- $\bar{\eta}_\ell = +1, \bar{\beta}_\ell < \bar{\beta}_k$ *or* $\bar{\eta}_\ell = -1, \bar{\beta}_\ell > \bar{\beta}_{k+1}$,
- $\Lambda^1 = [\bar{\beta}_k, \bar{\beta}_{k+1}]$

or

- *$\bar{\eta}_i = -1$ for $i = 1 : k$, $\bar{\eta}_{k+1} = +1$, $\bar{\eta}_{k+2} = -1$ and $\bar{\eta}_i = +1$ for $i = k+3 : n_{\bar{R}}$,*
- *$\Lambda^1 = [\bar{\beta}_k, \bar{\beta}_{k+1}] \cup [\bar{\beta}_{k+2}, \bar{\beta}_{k+3}]$ and $\bar{\beta}_{k+1} < \bar{\beta}_{k+2}$*

where $k, \ell \in 1 : n_{\bar{R}}$.

Proof. Define the sets

$$\Lambda^1_j := \{\lambda \in [0,1] \,|\, \pi(\bar{\eta}_j(\lambda - \bar{\beta}_j)) = 1,\ \pi(\bar{\eta}_i(\lambda - \bar{\beta}_i)) = 0,\ i \neq j\},\ j = 1 : n_{\bar{R}}. \tag{A.6}$$

Clearly, these sets are pairwise disjoint and their union is Λ^1.

Consider a Λ^1_ℓ. From (A.6), it can be seen that this set is the collection of points that lie in $[0,1]$ interval and satisfy linear inequalities

$$\bar{\eta}_\ell(\lambda - \bar{\beta}_\ell) > 0, \tag{A.7a}$$

$$\bar{\eta}_i(\lambda - \bar{\beta}_i) \leq 0,\ i \neq \ell,\ i = 1 : n_{\bar{R}}. \tag{A.7b}$$

Suppose $\Lambda^1_\ell \neq \emptyset$ and $\mu \in \Lambda^1_\ell$. The above given inequalities are satisfied for $\lambda = \mu$. Hence, from (A.7b), it can be inferred that $\bar{\eta}_i = -1$ if $\bar{\beta}_i < \mu$ and $i \neq \ell$ while $\bar{\eta}_i = +1$ if $\bar{\beta}_i > \mu$ and $i \neq \ell$. Taking into account the indexing described after (A.4), this implies that

$$\begin{aligned} \bar{\eta}_i = -1,\ i = 1 : k,\ i \neq \ell, \\ \bar{\eta}_i = +1,\ i = k+1 : n_{\bar{R}},\ i \neq \ell. \end{aligned} \tag{A.8}$$

for a $k \in 1 : n_{\bar{R}}$. Moreover, from (A.7b), it can be seen that either $\bar{\beta}_\ell < \mu$ and $\bar{\eta}_\ell = +1$ or $\bar{\beta}_\ell > \mu$ and $\bar{\eta}_\ell = -1$. Therefore, $\bar{\eta}_\ell = +1$ if $\ell \in 1 : k$ while $\bar{\eta}_\ell = -1$ if $\ell \in k+1 : n_{\bar{R}}$. Suppose that $\bar{\beta}_\ell = \bar{\beta}_k$. Due to (A.8), $\bar{\eta}_i(\bar{\beta}_\ell - \bar{\beta}_i) \leq 0$ for every $i \in 1 : n_{\bar{R}}$ This implies $g(\bar{\beta}_\ell) = 0$. On the other hand, since $\bar{\eta}_\ell = +1$, from Remark A.2.7, it can be inferred that $\bar{\beta}_\ell \geq 0$. Moreover, because $\bar{\beta}_\ell < \mu$, $\bar{\beta}_\ell < 1$. Hence, $\bar{\beta}_\ell \in [0,1]$ and $g(\bar{\beta}_\ell) = 0$, which contradicts with the fact that $g(\lambda) \geq 1$ for every $\lambda \in [0,1]$. As a result, it can be inferred that $\bar{\beta}_\ell \neq \bar{\beta}_k$. Similarly, it can be shown that $\bar{\beta}_\ell \neq \bar{\beta}_{k+1}$. Therefore, either

$\bar{\beta}_\ell < \bar{\beta}_k$ or $\bar{\beta}_\ell > \bar{\beta}_{k+1}$. Using this fact together with (A.7) and (A.8), it can be seen that $\Lambda^1_\ell = [\bar{\beta}_k, \bar{\beta}_{k+1}]$. Moreover, Remark A.2.7 implies $[\bar{\beta}_k, \bar{\beta}_{k+1}] \subseteq [0,1]$. To sum, if a Λ^1_ℓ is not empty, we showed that

- (A.8) holds true for a $k \in 1 : n_{\bar{R}}$,
- $\bar{\eta}_\ell = +1, \bar{\beta}_\ell < \bar{\beta}_k$ or $\bar{\eta}_\ell = -1, \bar{\beta}_\ell > \bar{\beta}_{k+1}$,
- $\Lambda^1_\ell = [\bar{\beta}_k, \bar{\beta}_{k+1}]$.

Since Λ^1 is not empty, there exists a nonempty set among Λ^1_j, $j = 1 : n_{\bar{R}}$. Without loss of generality, assume that $\ell \in 1 : n_{\bar{R}}$ is the largest number such that $\Lambda^1_\ell \neq \emptyset$. If Λ^1_ℓ is the only nonempty set, the results derived in above prove the proposition.

Now suppose that there exists a nonempty set Λ^1_m other than Λ^1_ℓ. Clearly, $m < \ell$ and from the preceding discussion, we know that

$$\bar{\eta}_\ell = +1,\ \bar{\beta}_\ell < \bar{\beta}_k \quad \text{or} \quad \bar{\eta}_\ell = -1,\ \bar{\beta}_\ell > \bar{\beta}_{k+1}, \tag{A.9a}$$

$$\bar{\eta}_m = +1,\ \bar{\beta}_m < \bar{\beta}_t \quad \text{or} \quad \bar{\eta}_m = -1,\ \bar{\beta}_m > \bar{\beta}_{t+1}, \tag{A.9b}$$

$$\bar{\eta}_i = -1,\ i = 1 : k,\ i \neq \ell, \quad \bar{\eta}_i = +1,\ i = k+1 : n_{\bar{R}},\ i \neq \ell, \tag{A.9c}$$

$$\bar{\eta}_j = -1,\ j = 1 : t,\ j \neq m, \quad \bar{\eta}_j = +1,\ j = t+1 : n_{\bar{R}},\ j \neq m. \tag{A.9d}$$

First, we will show that $\bar{\eta}_m = +1$. In order to see this, assume the contrary (i.e. $\bar{\eta}_m = -1$). Then, from (A.9b), $m > t + 1$. Moreover, we know that $\ell > m$. Therefore, from (A.9d), it can be inferred that $\bar{\eta}_\ell = +1$. Due to (A.9a), this implies $\ell < k$. Consequently, $t + 1 < m < \ell < k$. However, if this is true (A.9c), implies $\bar{\eta}_k = -1$ while (A.9d) implies $\bar{\eta}_k = +1$, which is a contradiction. Therefore, $\bar{\eta}_m = +1$.

Since $\bar{\eta}_m = +1$ and $\ell > m$, (A.9c) implies $\ell > m \geq k + 1$, which also implies $\bar{\eta}_\ell = -1$ due to (A.9a). Using $\bar{\eta}_\ell = -1$ in (A.9d), it can be seen that $\ell \leq t$. Consequently, $k + 1 \leq m < \ell \leq t$. Employing this fact, we can infer the following.

Since $k + 1 \neq \ell$, (A.9c) implies $\bar{\eta}_{k+1} = +1$. If $k + 1 \neq m$, (A.9d) together with the fact that $k+1 < t$ implies that $\bar{\eta}_{k+1} = -1$, which is a contradiction. Hence, $k+1 = m$. Similarly, because $t \neq m$, from (A.9d) it can be inferred that $\bar{\eta}_t = -1$. If $\ell \neq t$, we must have $\bar{\eta}_t = +1$ since $k+1 < t$ and due to (A.9c), which is a contradiction. Therefore, $\ell = t$. To sum up, we showed that $k+1 = m < \ell = t$ and $\bar{\eta}_{k+1} = +1$ while $\bar{\eta}_t = -1$. Now suppose that there exists an index u such that $k+1 = m < u < \ell = t$. Then, from (A.9c) we must have $\bar{\eta}_u = +1$ while due to (A.9d) we have $\bar{\eta}_u = -1$, which is a contradiction. Thus, $\ell = m+1$. This means if there exists a nonempty set Λ^1_m other than Λ^1_ℓ, it must be $\Lambda^1_m = \Lambda^1_{\ell-1}$, and hence, no other nonempty set exists in Λ^1_j, $j = 1 : n_{\bar{R}}$. Moreover, we must have $m = k+1$, $t = \ell = k+2$, and hence, $\Lambda^1_\ell = [\bar{\beta}_k, \bar{\beta}_{k+1}]$, while $\Lambda^1_m = [\bar{\beta}_{k+1}, \bar{\beta}_{k+2}]$.

$\square$

Proof of Lemma 6.1.2. Since $\mathbf{Q}_\lambda$ is real congruent to $\mathbf{D}_\lambda$, Λ is the set of points satisfying $\pi(\mathbf{D}_\lambda) = 1$ in $[0,1]$ interval. Therefore, in below, we find the solution set of this constraint under the condition that $\Lambda \neq \emptyset$ and Assumption 7.1.2 (i.e. $\mathbf{D}_\lambda$ is not negative semidefinite for any $\lambda \in [0,1]$).

In (A.4), either $c = 0$ or $c = 1$. Otherwise, we would have $\pi(\mathbf{D}_\lambda) \geq 1$ for every $\lambda \in [0,1]$, and hence, $\Lambda = \emptyset$. Suppose that $c = 0$. In this case, from (A.4) and the definition (A.5) it can be inferred that $\Lambda = \Lambda^1$. Therefore, due to Proposition A.2.10, either $\Lambda = [\bar{\beta}_k, \bar{\beta}_{k+1}]$ or $\Lambda = [\bar{\beta}_k, \bar{\beta}_{k+1}] \cup [\bar{\beta}_{k+2}, \bar{\beta}_{k+3}]$ with $\bar{\beta}_{k+1} < \bar{\beta}_{k+2}$ for a $k \in 1 : n_{\bar{R}}$. Moreover, since $c = 0$ and due to Lemma A.2.6, either $s_i = 1$ or $s_i = 2$ and $\eta_i = -1$ for any $\mathbf{R}_{\lambda_i}$ appearing in $\mathbf{D}_\lambda$. For the former, β_i has one copy in the sequence $\bar{\beta}_i$, $i = 1 : n_{\bar{R}}$ while for the latter it has two copies and the multiplicity of β_i is also two since $s_i = 2$. Therefore, we can infer that the sequence α_i, $i = 1 : n_G$ is the same as the part of the sequence $\bar{\beta}_i$, $i = 1 : n_{\bar{R}}$ lying in $[0,1]$ interval. Hence, either $\Lambda = [\alpha_\ell, \alpha_{\ell+1}]$ or $\Lambda = [\alpha_\ell, \alpha_{\ell+1}] \cup [\alpha_{\ell+2}, \alpha_{\ell+3}]$ with $\alpha_{\ell+1} < \alpha_{\ell+2}$ for an appropriate index ℓ, where $\alpha_{\ell+i} = \bar{\beta}_{k+i}$, $i = 0 : 3$.

Alternatively, suppose that $c = 1$. In this case, (A.4) implies that $\Lambda = \Lambda^0$, and hence, due to Proposition A.2.9, $\Lambda = [\bar{\beta}_k, \bar{\beta}_{k+1}]$. As long as there is no GEV in $(\bar{\beta}_k, \bar{\beta}_{k+1})$, we have $\Lambda = [\alpha_\ell, \alpha_{\ell+1}]$ for an appropriate index ℓ. There may exist a GEV in $(\bar{\beta}_k, \bar{\beta}_{k+1})$ if there is a real block $\mathbf{R}_{i\lambda}$ with even s_i and $\bar{\eta}_i = -1$. If this is the case, $s_i = 2$ because $c = 1$, and hence, the GEV β_i has double multiplicity. Therefore, if this happens, we have $\Lambda = [\alpha_\ell, \alpha_{\ell+1}] \cup [\alpha_{\ell+2}, \alpha_{\ell+3}]$, where $\alpha_\ell = \bar{\beta}_k$, $\alpha_{\ell+3} = \bar{\beta}_{k+1}$ and $\alpha_{\ell+1} = \alpha_{\ell+2} = \beta_i$ is the GEV having double multiplicity. $\square$

Proof of Lemma 6.5.15. From Lemma A.2.6, it can be seen that the number of positive eigenvalues only changes for real blocks and the change takes place at points of the sequence $\bar{\beta}_j$, $j \in 1 : n_{\bar{R}}$. On the other hand, as described in proof of Lemma 6.1.2, the end points of an interval Λ_i are always two consecutive numbers from this sequence. Therefore, we can seen see that for all $\lambda \in \Lambda_i$ only one of the blocks making up $\mathbf{D}_\lambda$ has a single positive eigenvalue while the others have to be negative semidefinite. Let us show this block as $\mathbf{L}_\lambda$. Using Lemma A.2.6, it can be inferred that $\mathbf{L}_\lambda$ would be only one of following:

- $\mathbf{S}_{i\lambda}$ with $s_i = 1$,
- $\mathbf{F}_{i\lambda}$ with either $f_i = 2$ or $f_i = 1$, $\delta_i = +1$ or $f_i = 3$, $\delta_i = -1$,
- $\mathbf{C}_{i\lambda}$ with $c_i = 1$,
- $\mathbf{R}_{i\lambda}$ with either $r_i = 1$ or $r_i = 2$ or $r_i = 3$ or $r_i = 4$, $\eta_i = -1$.

Otherwise, we would have $c \geq 2$ for every $\lambda \in [0, 1]$, which means $\Lambda = \emptyset$. Denote the size of $\mathbf{L}_\lambda$ as d and define $U := [\,0\ I_{d-1}]$, where $0 \in \mathbb{R}^{(d-1)\times 1}$. For each possible form of $\mathbf{L}_\lambda$ enlisted above, one can easily show that $U^T \mathbf{L}_\lambda U \preceq 0$ for every $\lambda \in \Lambda_i$. For example, if $\mathbf{L}_\lambda$ is $\mathbf{S}_{i\lambda}$, then

$$\mathbf{L}_\lambda = \begin{bmatrix} 0 & 1 & \lambda \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or if it is $\mathbf{F}_{i\lambda}$ with $f_i = 2$, then

$$\mathbf{L}_\lambda = \delta_i \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The other cases follows similarly.

Now consider the matrix $V := I_{n_1} \oplus U \oplus I_{n_2}$, where I_{n_1} and I_{n_2} are identity matrices of appropriate dimensions such that U^T and U multiply $\mathbf{L}_\lambda$ from left and right in $V^T \mathbf{D}_\lambda V$. Suppose W is the matrix satisfying $\mathbf{D}_\lambda = W^T \mathbf{Q}_\lambda W$ for every $\lambda \in \mathbb{C}$. Then, it can be seen that $V^T W^T \mathbf{Q}_\lambda W V = V^T \mathbf{D}_\lambda V \preceq 0$ for every $\lambda \in \Lambda_i$. This means the hyperplane

$$\mathcal{K} := \left\{ x \in \mathbb{R}^{n+1} | x = WVz, \ z \in \mathbb{R}^n \right\}$$

does not intersect $\hat{\mathcal{H}}_\lambda$ for every $\lambda \in \Lambda_i$. □

Proposition A.2.11. *Consider two linear hyperplanes $\mathcal{K}_1$ and $\mathcal{K}_2$ separating $\mathcal{H}$ and points $x \in \mathcal{K}_1$ and $y \in \mathcal{K}_2$ lying on them. If both the line segment between x and y, and the line segment between x and $-y$ intersects $\mathcal{H}$, then $\langle \mathcal{H} \rangle_{\mathcal{K}_1} \neq \langle \mathcal{H} \rangle_{\mathcal{K}_2}$.*

Proof. Assume $\langle \mathcal{H} \rangle_{\mathcal{K}_1} = \langle \mathcal{H} \rangle_{\mathcal{K}_2}$. Let us define $\mathcal{H}^\pm := \mathcal{H} \cap \mathcal{K}_1^\pm$. Then, by choosing the appropriate side of $\mathcal{K}$ as $\mathcal{K}_2^+$, we also have $\mathcal{H}^\pm = \mathcal{H} \cap \mathcal{K}_2^\pm$. Hence, the closed polyhedral cones

$$\overline{\mathcal{K}_1^+} \cap \overline{\mathcal{K}_2^-} \text{ and } \overline{\mathcal{K}_1^-} \cap \overline{\mathcal{K}_2^+} \tag{A.10}$$

do not have a common point with $\mathcal{H}$. On the other hand, it is easy to see that either both x and y or both x and $-y$ elements of one of the sets given in (A.10). Since these sets are convex and does not intersect $\mathcal{H}$, either line segment between x and y or between x and $-y$ does not intersect $\mathcal{H}$. □

Proof of Lemma 6.5.17. From Corollary 6.5.16, we know that all hyperplanes lying in Θ_1 separates $\mathcal{H}$ in the same manner. The same holds true for Θ_2. This implies if $\langle\mathcal{H}\rangle_{\mathcal{K}_1} = \langle\mathcal{H}\rangle_{\mathcal{K}_2}$ for a $\mathcal{K}_1 \in \Theta_1$ and $\mathcal{K}_2 \in \Theta_2$, then $\langle\mathcal{H}\rangle_{\mathcal{T}_1} = \langle\mathcal{H}\rangle_{\mathcal{T}_2}$ for every $\mathcal{T}_1 \in \Theta_1$ and $\mathcal{T}_2 \in \Theta_2$. Therefore, in order to prove the claim of the lemma, it is enough show that there exist a $\mathcal{K}_1 \in \Theta_1$ and $\mathcal{K}_2 \in \Theta_2$ such that $\langle\mathcal{H}\rangle_{\mathcal{K}_1} \neq \langle\mathcal{H}\rangle_{\mathcal{K}_2}$.

As discussed in proof of Lemma 6.1.2, since Λ is composed of two intervals, $\Lambda = \Lambda^1$ and $c = 0$. Therefore, from Lemma A.2.6 and (A.4), it can be seen that for any real block $\mathbf{R}_{j\lambda}$ appearing in $\mathbf{D}_\lambda$ either $r_j = 1$ or $r_j = 2$ and $\eta_j = -1$. Recall that for the latter, the corresponding GEV have two copies $\bar{\beta}_i = \bar{\beta}_{i+1} = \beta_j$ in the new sequence and there are two signs $\bar{\eta}_i = -1$, $\bar{\eta}_{i+1} = +1$. On the other hand, Proposition A.2.10 shows that we must have

$$\bar{\eta}_i = -1, i = 1 : k; \bar{\eta}_{k+1} = +1; \bar{\eta}_{k+2} = -1; \bar{\eta}_i = +1, i = k+3 : n_{\bar{R}}.$$

Consequently, it can be seen that $\mathbf{D}_\lambda$ can be expressed as

$$\mathbf{D}_\lambda = \bigoplus_{i=1}^{k-1} -(\lambda - \bar{\beta}_i) \oplus \mathbf{M}_{1\lambda} \oplus \mathbf{M}_{2\lambda} \oplus \bigoplus_{i=k+4}^{n_{\bar{R}}} (\lambda - \bar{\beta}_i) \oplus 0 \oplus (-I), \tag{A.11a}$$

where $\mathbf{M}_{1\lambda}$ is either

$$\begin{bmatrix} -(\lambda - \bar{\beta}_k) & 0 \\ 0 & \lambda - \bar{\beta}_{k+1} \end{bmatrix} \quad \text{or} \quad -\begin{bmatrix} 1 & \lambda - \bar{\beta}_k \\ \lambda - \bar{\beta}_k & 0 \end{bmatrix} \tag{A.11b}$$

and $\mathbf{M}_{2\lambda}$ is either

$$\begin{bmatrix} -(\lambda - \bar{\beta}_{k+2}) & 0 \\ 0 & \lambda - \bar{\beta}_{k+3} \end{bmatrix} \quad \text{or} \quad -\begin{bmatrix} 1 & \lambda - \bar{\beta}_{k+2} \\ \lambda - \bar{\beta}_{k+2} & 0 \end{bmatrix}. \tag{A.11c}$$

Therefore, $\mathbf{D}_\lambda$ take four different forms. Note that in (A.11a), $-I$ comes from the blocks $\mathbf{F}_{i\lambda}$ each of which has to be -1 since $c = 0$ which implies $f_i = 1$. Moreover, in this equation we changed ordering of blocks appearing in (A.1) to simplify the notation.

Clearly, this does not effect our results.

Let U_1 and U_2 be the matrices obtained by removing the $k+2$nd and $k+1$st rows of the identity matrix I_{n+1}, respectively. Define $\mathcal{K}_1$ and $\mathcal{K}_2$ as the range spaces of the matrices WU_1 and WU_2, respectively. Employing (A.11), one can easily show that, $U_1^T W^T \mathbf{Q}_\lambda W U_1 = U_1^T \mathbf{D}_\lambda U_1 \preceq 0$ for any $\lambda \in \Lambda_1 = [\bar{\beta}_k, \bar{\beta}_{k+1}]$ while $U_2^T W^T \mathbf{Q}_\lambda W U_2 = U_2^T \mathbf{D}_\lambda U_2 \preceq 0$ for any $\lambda \in \Lambda_2 = [\bar{\beta}_{k+2}, \bar{\beta}_{k+3}]$. Therefore, it can be seen that $\mathcal{K}_1 \in \Theta_1$ while $\mathcal{K}_2 \in \Theta_2$ (in fact they are the common hyperplanes separating all element of Γ_1 and Γ_2). In order to prove the lemma we will do the following. For each possible form of $\mathbf{D}_\lambda$ mentioned above, we give points $x \in \mathcal{K}_1$, $y \in \mathcal{K}_2$. Moreover, we pick points u, v that lie on the line segments connecting x, y and x, $-y$, respectively, and show that $\tilde{\mathbf{q}}_\lambda(u) > 0$, $\tilde{\mathbf{q}}_\lambda(v) > 0$ for every $\lambda \in \mathbb{R}$. In particular, this implies

$$\tilde{\mathbf{q}}_0(u) = \tilde{q}_1(u) > 0, \quad \tilde{\mathbf{q}}_1(u) = \tilde{q}_2(u) > 0,$$
$$\tilde{\mathbf{q}}_0(v) = \tilde{q}_1(v) > 0, \quad \tilde{\mathbf{q}}_1(v) = \tilde{q}_2(v) > 0.$$

Therefore, $u, v \in \mathcal{H}$, and hence, using Proposition A.2.11 it can be inferred that $\mathcal{K}_1$ and $\mathcal{K}_2$ separate $\mathcal{H}$ in different manner. Note that in below, e_i represents the ith standard basis vector (i.e. the ith element of e_i is one while the others are zero).

If $\mathbf{M}_{1\lambda}$ and $\mathbf{M}_{2\lambda}$ are both diagonal matrices appearing in (A.11), choose $x = We_{k+1}$, $y = We_{k+2}$, $u := (x+y)/2$ and $v := (x-y)/2$. It is easy to show that these points satisfy the properties mentioned above. Moreover,

$$\begin{aligned}\tilde{\mathbf{q}}_\lambda(u) =& u^T \mathbf{Q}_\lambda u = (\bar{\beta}_2 - \bar{\beta}_1)/4 > 0,\ \forall \lambda \in \mathbb{R}\\ \tilde{\mathbf{q}}_\lambda(v) =& v^T \mathbf{Q}_\lambda v = (\bar{\beta}_2 - \bar{\beta}_1)/4 > 0, \forall \lambda \in \mathbb{R}.\end{aligned}$$

Hence, we are done.

Suppose $\mathbf{M}_{1\lambda}$ is diagonal while $\mathbf{M}_{2\lambda}$ is not. Choose $x = We_{k+1}$, $y = W(\epsilon e_{k+2} + e_{k+3})$, $u := (1-\theta)x + \theta y$ and $v := (1-\theta)x - \theta y$ with $\theta \in (0,1)$. These points satisfy the required properties. Moreover, let $\theta \in (0,1)$ and ϵ satisfy

$$\begin{aligned} (1-\theta)^2 - 2\epsilon\theta^2 &= 0, \\ 2\epsilon(\bar{\beta}_{k+2} - \bar{\beta}_{k+1}) - \epsilon^2 &> 0. \end{aligned} \tag{A.12}$$

Such θ and ϵ always exists. Indeed, for $\epsilon = (1-\theta)^2/(2\theta^2)$, $\theta \in (0,1)$ the equality is satisfied and ϵ can be made arbitrarily small by choosing θ close enough to 1. On the other hand, for a small enough ϵ, the inequality is also satisfied since $\bar{\beta}_{k+2} - \bar{\beta}_{k+1} > 0$ and the linear term dominates the quadratic term. Now, it is immediate to verify that

$$\tilde{\mathbf{q}}_\lambda(u) = \tilde{\mathbf{q}}_\lambda(v) = [(1-\theta)^2 - 2\epsilon\theta^2]\lambda - (1-\theta)^2\bar{\beta}_{k+1} + 2\epsilon\theta^2\bar{\beta}_{k+2} - \epsilon^2\theta^2.$$

Therefore, using (A.12), it can be inferred that

$$\begin{aligned} \tilde{\mathbf{q}}_\lambda(u) = \tilde{\mathbf{q}}_\lambda(v) &= -(1-\theta)^2\bar{\beta}_{k+1} + 2\epsilon\theta^2\bar{\beta}_{k+2} - \epsilon^2\theta^2 \\ &= \theta^2(2\epsilon(\bar{\beta}_{k+2} - \bar{\beta}_{k+1}) - \epsilon^2) > 0, \ \lambda \in \mathbb{R}. \end{aligned}$$

Thus, we are done. The case for which $\mathbf{M}_{2\lambda}$ is diagonal but $\mathbf{M}_{1\lambda}$ is not can be proven similarly and left to the reader.

Lastly, suppose that both $\mathbf{M}_{1\lambda}$ and $\mathbf{M}_{2\lambda}$ are not diagonal. Choose, $x = W(\epsilon e_k + e_{k+1})$, $y = W(\epsilon e_{k+2} - e_{k+3})$, $u := (x+y)/2$ and $v := (x-y)/2$. Then,

$$\tilde{\mathbf{q}}_\lambda(u) = \tilde{\mathbf{q}}_\lambda(v) = -2\epsilon^2 + 2\epsilon(\bar{\beta}_{k+1} - \bar{\beta}_{k+2}).$$

Since $\bar{\beta}_{k+1} - \bar{\beta}_{k+2} < 0$, there exists an ϵ that make the preceding equality strictly positive, and hence, for this ϵ we obtain the desired result. $\square$

A.3. Proof of Proposition 6.5.8

Proof. [i)⇒ii)] Consider a hyperplane $\mathcal{T}$ that does not intersect $\mathcal{P}$. We can parameterize $\mathcal{T}$ as $x = Uw$, where $w \in \mathbb{R}^n$ and $U \in \mathbb{R}^{(n+1)\times n}$ is a full column rank matrix. $\mathcal{T} \cap \mathcal{P} = \emptyset$ implies $w^T U^T P U w \leq 0$ for every $w \in \mathbb{R}^n$, which means $U^T P U \in \mathbb{R}^{n\times n}$ is negative semidefinite. Due to Poincare Separation theorem, [40],

$$\pi(P) - 1 \leq \pi(U^T P U).$$

This implies $\pi(P) \leq 1$. On the other hand, since $\mathcal{P}$ is not empty P cannot be negative semidefinite. Hence, $\pi(P) = 1$ as desired.

[ii)⇒iii)] Because P is symmetric and $\pi(P) = 1$, there exists a nonsingular linear transformation $z = Dx$ for which the matrix D satisfies

$$P = D^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix} D$$

where ν is the number of negative eigenvalues of P. If $\nu \geq 1$, then $\mathcal{P}$ can be expressed as

$$\mathcal{P} = \{z \in \mathbb{R}^{n+1} | z_1^2 - \bar{z}^T \bar{z} > 0\}$$

in the transformed space, where $\bar{z} = \begin{bmatrix} z_2 & \cdots & z_\nu \end{bmatrix}$. This set is clearly union of two disjoint convex cones

$$\mathcal{P}^{\pm} = \{z \in \mathbb{R}^{n+1} | z_1^2 - \bar{z}^T \bar{z} > 0, \pm z_1 > 0\}$$

which are symmetric reflection of each other with respect to the origin. This means $\mathcal{P}$

is a SCC. On the other hand, if $\nu = 0$, then $\mathcal{P}$ is union of the sets

$$\mathcal{P}^{\pm} = \{z \in \mathbb{R}^{n+1} | \pm z_1 > 0\}.$$

Hence, $\mathcal{P}$ is, again, a SCC. Since, the result is not effected with a nonsingular linear transformation, we are done.

[iii)⇒i)] If $\mathcal{P}$ is a SCC, then it is the union of two disjoint convex cones, $\mathcal{P}^{\pm}$. Moreover, since $\mathcal{P}$ is defined by strict inequalities, these cones are also open. Therefore, there exists a linear hyperplane $\mathcal{T}$ which separates $\mathcal{P}^{+}$ and $\mathcal{P}^{-}$ strictly. Hence, $\mathcal{P} \cap \mathcal{T} = \emptyset$. $\square$

APPENDIX B: PROOFS OF CHAPTER 7

B.1. Proofs of Propositions 7.3.1 and 7.3.2

Proof of Proposition 7.3.1. By partitioning the matrix P, we can express the inequality constraint as

$$p(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0$$

Let's apply the linear transformation $x = \bar{x} - A^+ b$. The constraint in the new space can now be written as

$$\bar{p}(\bar{x}) = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^T \bar{P} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0, \quad \bar{P} := \begin{bmatrix} A & \bar{b} \\ \bar{b}^T & \bar{c} \end{bmatrix}, \tag{B.1}$$

where $\bar{b} := (I - AA^+)b$, $\bar{c} := c - b^T A^+ b$.

We proceed by investigating a number of scenarios. First, assume $b \in \mathrm{Im}(A)$. In that case, $\bar{b} = 0$ and (B.1) simplifies to

$$\bar{c} + \bar{x}^T A \bar{x} \geq 0. \tag{B.2}$$

If $\bar{c} > 0$, A has to be negative semidefinite because $\pi(P) = 1$. Therefore, A can be decomposed as $A = -VV^T$, where V is the matrix of eigenvectors corresponding to the negative eigenvalues of A. Consequently, using the Schur complement formula given in Proposition 3.2.1, it can be seen that (B.2) holds trues if and only if

$$\begin{bmatrix} I & V^T \bar{x} \\ \bar{x}^T V & \bar{c} \end{bmatrix} = \begin{bmatrix} I & V^T (x + A^+ b) \\ (x + A^+ b)^T V & c - b^T A^+ b \end{bmatrix} \succeq 0. \tag{B.3}$$

This shows $\mathcal{Q}_p$ is an LMI set, and we are done.

Now, assume $\bar{c} \leq 0$. This implies $\pi(A) = 1$. Therefore, A can be expressed as $A = uu^T - VV^T$, where u is the eigenvector corresponding to the positive eigenvalue and V is the matrix of eigenvectors corresponding to the negative eigenvalues of A. Using this decomposition in (B.1), we obtain

$$\bar{p}(\bar{x}) = \bar{x}^T uu^T \bar{x} - \bar{x}^T VV^T \bar{x} - d^2 \geq 0 \tag{B.4}$$

where $-d^2 := \bar{c}$. Employing Schur complement formula, it can be seen that the set of points satisfying (B.4) are the solutions of one of the two constraints

$$\begin{bmatrix} \pm u^T \bar{x} I & \begin{pmatrix} V^T \bar{x} \\ d \end{pmatrix} \\ \begin{pmatrix} \bar{x}^T V & d \end{pmatrix} & \pm u^T \bar{x} \end{bmatrix} = \begin{bmatrix} \pm u^T (x + A^+ b) I & \begin{pmatrix} V^T (x + A^+ b) \\ d \end{pmatrix} \\ \begin{pmatrix} (x + A^+ b)^T V & d \end{pmatrix} & \pm u^T (x + A^+ b) \end{bmatrix} \succeq 0 \tag{B.5}$$

and we denote their feasible sets as $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. Clearly, $\mathcal{Q}_p^\pm$ are LMI sets and it can be easily verified that they are symmetric reflection of each other with respect to the point $x_c := -A^+ b$, which completes the proof for the case considered.

Lastly, assume $b \notin \mathrm{Im}(A)$. This implies $\bar{b} \neq 0$, and $A^T \bar{b} = 0$. As a result, it can be verified that the vectors

$$\begin{bmatrix} (\bar{c}/2 \pm \sqrt{\bar{c}^2/4 + \bar{b}^T \bar{b}})\bar{b} \\ -\bar{b}^T \bar{b} \end{bmatrix},$$

are eigenvectors of $\bar{P}$, the corresponding eigenvalues of which are

$$-\frac{\bar{b}^T \bar{b}}{\bar{c}/2 \pm \sqrt{\bar{c}^2/4 + \bar{b}^T \bar{b}}}.$$

It is clear that one of these eigenvalues has to be negative while the other is positive.

Therefore, $\bar{P}$ cannot have another positive eigenvalue because $\pi(P) = 1$ and inertia is preserved under congruence. Based on this fact, it can be shown that $A \preceq 0$. In order to see this, assume A has a positive eigenvalue α with the associated eigenvector u. Because $u \in \mathrm{Im}(A)$, the condition $\bar{b}^T u = 0$ must hold true. This leads to the fact that the vector $[u^T \ 0]^T$ must be another eigenvector of $\bar{P}$ with the associated positive eigenvalue α, which is a contradiction.

Now, because A is negative semidefinite, we can rewrite (B.1) as

$$\bar{c} + 2\bar{b}^T \bar{x} - \bar{x}^T V V^T \bar{x} \geq 0$$

using the decomposition $A = -VV^T$. This leads to the representation

$$\begin{bmatrix} I & V^T \bar{x} \\ \bar{x}^T V & \bar{c} + 2\bar{b}^T \bar{x} \end{bmatrix} = \begin{bmatrix} I & V^T (x + A^+ b) \\ (x + A^+ b)^T V & \bar{c} + 2\bar{b}^T (x + A^+ b) \end{bmatrix} \succeq 0. \tag{B.6}$$

Hence, $\mathcal{Q}_p$ is an LMI set. $\square$

Proof of Proposition 7.3.2. We will use the linear transformation introduced in the proof of Proposition 7.3.1. Since $\mathcal{Q}_p$ is composed of two LMI components, we have $\bar{c} \leq 0$, $\pi(A) = 1$, and hence, (B.1) becomes

$$\bar{p}(\bar{x}) = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^T \begin{bmatrix} A & 0 \\ 0 & \bar{c} \end{bmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0$$

as described before. Consider an arbitrary halfspace defined by the inequality

$$q(\bar{x}) = \alpha^T \bar{x} + \beta = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^T \begin{bmatrix} 0 & \alpha/2 \\ \alpha^T/2 & \beta \end{bmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0.$$

Now, assume this halfspace contains $\mathcal{Q}_p$. This means $q(\bar{x}) \geq 0$ for every $\bar{x}$ satisfying $\bar{p}(\bar{x}) \geq 0$. Because $q(\bar{x})$ can be made strictly positive, Lemma 4.5.1 can be applied and one can infer that there exists a $\lambda \geq 0$ such that

$$\begin{bmatrix} 0 & \alpha/2 \\ \alpha^T/2 & \beta \end{bmatrix} - \lambda \begin{bmatrix} A & 0 \\ 0 & \bar{c} \end{bmatrix} \succeq 0.$$

However, because $\pi(A) = 1$, the above given inequality cannot be satisfied for any $\lambda \geq 0$, which is a contradiction. Hence, there does not exist a halfspace containing $\mathcal{Q}_p$. □

B.2. Proof of Lemma 7.3.3

Proof of Lemma 7.3.3. Consider parametrization of the hyperplane $\mathcal{T}$ given by

$$x = Uw + y,$$

where $w \in \mathbb{R}^{n-1}$ and $U \in \mathbb{R}^{n\times(n-1)}$ has full-column rank. By substituting this parametrization into polynomial p, one obtains

$$p(Uw + y) = \begin{pmatrix} w \\ 1 \end{pmatrix}^T \tilde{P} \begin{pmatrix} w \\ 1 \end{pmatrix}, \tag{B.7}$$

where

$$\tilde{P} := \begin{bmatrix} U & y \\ 0 & 1 \end{bmatrix}^T P \begin{bmatrix} U & y \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}.$$

Because $y \in \mathcal{B}$, it satisfies $p(y) = 0$ and $\tilde{P} \in \mathbb{S}^n$ simplifies to

$$\tilde{P} = \begin{bmatrix} U^T A U & U^T(Ay + b) \\ (Ay + b)^T U & 0 \end{bmatrix}. \tag{B.8}$$

We will show that $\xi := U^T(Ay + b) = 0$. To this end, assume $\xi \neq 0$. This means

$\xi_i \neq 0$ for an $i \in 1 : n-1$. Let's substitute the vector $w := \alpha e_i$ into (B.7), where e_i denotes the ith standard basis vector. We obtain

$$p(\alpha U e_i + y) = \bar{a}_{ii}\alpha^2 + 2\xi_i \alpha,$$

where $\bar{a}_{ii} := (U^T A U)_{ii}$. This expression can be made both positive and negative for values of α having arbitrarily small absolute value. This means that the hyperplane $\mathcal{T}$ does not stay on the same side of the surface $\mathcal{Z}_p$ within the set $\mathcal{S}$. Hence, it cannot be a supporting hyperplane of $\mathcal{U}$, which is a contradiction. As a result, we must have $U^T(Ay+b) = 0$. (From a geometric point of view, this conditions means the hyperplane $\mathcal{T}$ is either tangent to the surface $\mathcal{Z}_p$ or the tangent of $\mathcal{Z}_p$ at y is not well defined.)

For later use, we note that employing the Poincaré separation theorem [40] and the fact that $U^T(Ay+b) = 0$, one can obtain the inertia inequalities

$$\pi(P) \geq \pi(\tilde{P}) = \pi(U^T A U) \geq \pi(P) - 1 \tag{B.9a}$$

$$\nu(P) \geq \nu(\tilde{P}) = \nu(U^T A U) \geq \nu(P) - 1. \tag{B.9b}$$

Now consider inequality and equality constraints separately and begin with inequality constraints first. Consider a neighborhood of the origin $N_0 \subseteq \mathbb{R}^{n-1}$ such that $Uw + y \in \mathbf{int}(\mathcal{S})$ for every $w \in N_0$. Note that such a neighborhood exists because $y \in \mathbf{int}(\mathcal{S})$. Since $\mathcal{T}$ is a supporting hyperplane, $p(Uw+y) \leq 0$ for all $w \in N_0$. Otherwise, $\mathcal{T} \bigcap \mathbf{int}(\mathcal{U})$ would be non-empty and $\mathcal{T}$ would not be a supporting hyperplane. Since $U^T(Ay+b) = 0$, we infer $p(Uw+y) = w^T U^T A U w$. Hence, we obtain $w^T U^T A U w \leq 0$ for every $w \in N_0$. This can be satisfied if and only if $U^T A U \preceq 0$, which means $p(Uw+y) \leq 0$ for every $w \in \mathbb{R}^{n-1}$. Therefore, $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$. Since $U^T A U \preceq 0$, we conclude that $\pi(P) \leq 1$ by (B.9a). Due to Assumption 7.1.2, $\pi(P) = 1$.

Now consider the equality constraint. There exists a neighborhood $N_0 \subseteq \mathbb{R}^{n-1}$ such that the polynomial $p(Uw+y)$ is sign-definite over N_0. Otherwise, there would be points of $\mathcal{B}$, which is a subset of $\mathcal{U}$, on both sides of $\mathcal{T}$, and hence, it would not be a supporting hyperplane. Consequently, $U^T AU$ must be sign-definite. In order to show that it is in fact negative semi-definite, note the following. Assumptions 7.1.1 and 7.1.2 imply $\nu(P) \geq \pi(P) \geq 1$. Together with sign-definiteness of $U^T AU$ and (B.9), this implies that $\pi(P) = 1$. Moreover, $U^T AU \preceq 0$ if $\nu(P) > 1$. Hence, it only remains to show negative semi-definiteness when $\pi(P) = \nu(P) = 1$. For this case, if $U^T AU \succeq 0$, we can multiply both sides of the corresponding constraint by a negative number without loss of generality. This is because p appears in an equality constraint and Assumption 7.1.1 is not violated. Hence it can be ensured that $U^T AU \preceq 0$. Consequently, $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$. □

B.3. Proof of Lemma 7.3.4

Before proving Lemma 7.3.4, we need the following proposition.

Proposition B.3.1. *Consider two distinct hyperplanes*

$$\mathcal{T}_i = \{x \in \mathbb{R}^n | \quad \alpha_i^T x + \beta_i = 0\}, \quad i = 1, 2.$$

Assume $\pi(P) = 1$ and $\mathcal{Q}_p$ is union of two LMI sets, $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. If the conditions

$$\alpha_i^T x + \beta_i \geq 0, \quad \forall x \in \mathcal{Q}_p^+, \quad i = 1, 2 \tag{B.10a}$$

$$\alpha_i^T x + \beta_i \leq 0, \quad \forall x \in \mathcal{Q}_p^-, \quad i = 1, 2 \tag{B.10b}$$

are satisfied, then

$$\mathcal{Q}_p^+ = \{x \in \mathcal{Q}_p | \quad \alpha_i^T x + \beta_i \geq 0, \quad i = 1, 2\}. \tag{B.11}$$

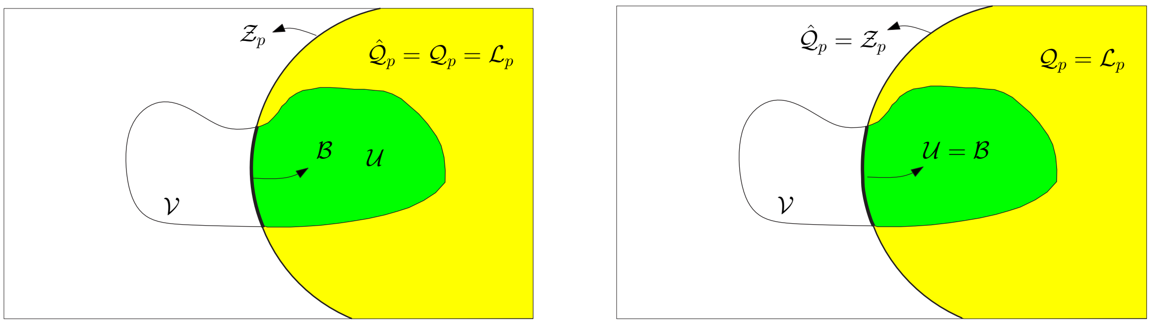
Proof. Suppose that (B.11) does not hold true. Then, there exists a point $y \in \mathcal{Q}_p^- \setminus \mathcal{Q}_p^+$ which is an element of the set appearing on the left-hand side of (B.11). Hence, taking into account (B.10b), one can infer that $\alpha_i^T y + \beta_i = 0$, $i = 1, 2$. This implies $y \notin \mathbf{int}(\mathcal{Q}_p)$.

Otherwise, at least one of (B.10a) and (B.10b) is violated. Hence, one can see that $y \in \mathcal{Z}_p$. Without loss of generality, assume that the coordinate system is translated so that $p(x) = c + x^T Ax \geq 0$ as in the proof of Proposition 7.3.1. The normal of the surface $\mathcal{Z}_p$ at y can be found as Ay. When $Ay \neq 0$, the normal is well-defined and $\mathcal{T}_1$ and $\mathcal{T}_2$ have to be tangents of $\mathcal{Z}_p$ at y. This is because they become supporting hyperplane of $\mathcal{Q}_p^+$ or $\mathcal{Q}_p^-$ at y. However, the tangent is unique, which is a contradiction. Hence, we must have $Ay = 0$.

Because y is in the null space of A, it has to be orthogonal to eigenvectors of A. This means $u^T y = 0$, where u is defined as in (B.4). Hence, from (B.4) and (B.5), $y \in \mathcal{Q}_p^+ \bigcap \mathcal{Q}_p^-$. This contradicts the fact that $y \in \mathcal{Q}_p^- \setminus \mathcal{Q}_p^+$, which completes the proof. $\square$

Proof of Lemma 7.3.4. Because there exists a hyperplane supporting $\mathcal{U}$ at y, Lemma 7.3.3 implies that $\pi(P) = 1$. Due to Proposition 7.3.1, this means $\mathcal{Q}_p$ is either an LMI set or a union of two LMI sets having disjoint interiors. If the former holds true, we simply choose $\mathcal{L}_p$ as $\mathcal{Q}_p$, which ensures $\mathcal{U} \subseteq \hat{\mathcal{Q}}_p \subseteq \mathcal{Q}_p = \mathcal{L}_p$.

Now, consider an arbitrary hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$. Using Lemma 7.3.3 again, we can infer that it does not intersect $\mathbf{int}(\mathcal{Q}_p)$. Because $\mathcal{Q}_p$ is a convex set having a nonempty interior, this leads to the fact that the hyperplane supports $\mathcal{L}_p$ (See Figure B.1).



(a) (b)

Figure B.1. Example for Lemma 7.3.4 with $\pi(P) = 1$ and $\mathcal{Q}_p$ has a single component. (a) Inequality constraint, (b) equality constraint

Assume $\mathcal{Q}_p$ is the union of two LMI sets $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. If $\mathcal{U}$ is a subset of only one of these components, the one containing $\mathcal{U}$ can be chosen as $\mathcal{L}_p$. The preceding argument can be directly utilized to show that hyperplanes supporting $\mathcal{U}$ along $\mathcal{B}$ also supports $\mathcal{L}_p$. (See Figure B.2.)

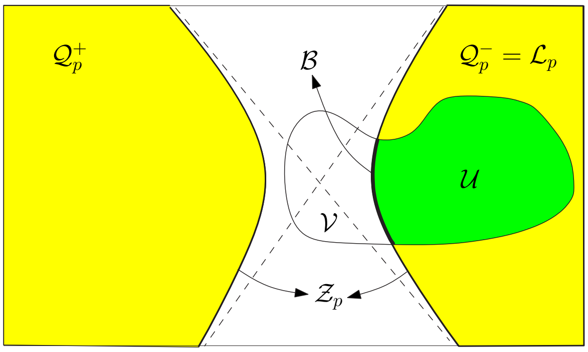


Figure B.2. Example for an inequality constraint in Lemma 7.3.4, for which $\pi(P) = 1$ and $\mathcal{U}$ is a subset of a convex component of $\mathcal{Q}_p$

In the rest of the proof, we validate our claim when $\mathcal{Q}_p$ is composed of two components under the condition $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$ and $\mathcal{U} \nsubseteq \mathcal{Q}_p^-$. First, we show that if $\nu(P) > 1$ or we have an inequality constraint, there exists at most one hyperplane supporting $\mathcal{U}$ at points of $\mathcal{B}$. To this end, assume there are two such distinct hyperplanes given by

$$\mathcal{T}_i = \{x \in \mathbb{R}^n | \quad \alpha_i^T x + \beta_i = 0\}, \quad i = 1, 2. \tag{B.12}$$

We know from Lemma 7.3.3 that they cannot intersect the nonempty disjoint sets $\mathbf{int}(\mathcal{Q}_p^+)$ and $\mathbf{int}(\mathcal{Q}_p^-)$ (nonemptiness of these sets is due to Assumption 7.1.2). Moreover, due to Proposition 7.3.2, the halfspaces induced by $\mathcal{T}_1$ and $\mathcal{T}_2$ cannot contain $\mathcal{Q}_p$. Consequently, one can infer that these hyperplanes separate $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. Therefore, multiplying the equations used in the definition (B.12) by a negative number if necessary, one can always guarantee

$$\begin{aligned} \alpha_i^T x + \beta_i \geq 0, &\quad \forall x \in \mathcal{Q}_p^+, \quad i = 1, 2 \\ \alpha_i^T x + \beta_i \leq 0, &\quad \forall x \in \mathcal{Q}_p^-, \quad i = 1, 2. \end{aligned} \tag{B.13}$$

Based on this fact, in what follows, we show that elements of $\mathcal{U}$ must lie on the set defined by the constraint

$$(\alpha_1^T x + \beta_1)(\alpha_2^T x + \beta_2) = 0. \tag{B.14}$$

In order to see this, assume the contrary. Then, there exists a point $y \in \mathcal{U}$ for which $\alpha_i^T y + \beta_i \neq 0$, $i = 1, 2$. Let's investigate different possibilities.

- If $\alpha_i^T y + \beta_i > 0$, $i = 1, 2$, we have

$$\alpha_i^T x + \beta_i \geq 0, \quad \forall x \in \mathcal{U}, \quad i = 1, 2. \tag{B.15}$$

  Otherwise, there would exist points of $\mathcal{U}$ on both sides of one of its supporting hyperplanes, which is a contradiction. However, due to Proposition B.3.1, (B.15) implies $\mathcal{U} \subseteq \mathcal{Q}_p^+$, which clearly contradicts our hypothesis that $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$. Hence, the condition $\alpha_i^T y + \beta_i > 0$, $i = 1, 2$ cannot be satisfied. Similarly, it can also be shown that these linear functions cannot be made strictly negative by y.
- If $\alpha_1^T y + \beta_1 > 0$ and $\alpha_2^T y + \beta_2 < 0$, then y cannot be an element of $\mathcal{Q}_p$ due to (B.13), which means it is not also an element $\mathcal{U}$. Therefore, these inequalities cannot be satisfied. Similarly, y cannot satisfy $\alpha_1^T y + \beta_1 < 0$ and $\alpha_2^T y + \beta_2 > 0$. As a result, we have shown that (B.14) must be satisfied for every element of $\mathcal{U}$.

Now, consider a point $x \in \mathcal{B}$. If $\nu(P) > 1$, $\mathcal{Z}_p$ exhibits a nonplanar characteristics around x (*i.e.*, it does not lie on a hyperplane) as opposed to the set defined by (B.14). Therefore, this condition cannot be satisfied. If $\mathcal{U}$ is defined by an inequality constraint, its interior is nonempty. Hence, $\mathcal{U}$ cannot be a subset of the set induced by (B.14). Consequently, we have shown that when $\nu(P) > 1$ or we have an inequality constraint, there do not exist two distinct supporting halfspaces $\mathcal{T}_1$ and $\mathcal{T}_2$, which means there exists at most one hyperplane supporting $\mathcal{U}$ along $\mathcal{B}$. This also shows that there exists one such halfspace because $\mathcal{U}$ does not lie on a hyperplane.

Lastly, let's consider the case in which $\pi(P) = \nu(P) = 1$ and $\mathcal{U}$ is defined by an equality constraint. For this case, the constraint defining $\mathcal{Z}_p$ takes the form given in

(B.14). Hence, $\mathcal{Z}_p$ becomes the union of two hyperplanes. Because $\mathcal{B}$ is smooth, any hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$ has to be tangent to $\mathcal{Z}_p$. This means there exists at most two distinct supporting hyperplanes (hyperplanes of $\mathcal{Z}_p$ themselves). This also implies that there exists at most two supporting halfspaces.

When $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$ and $\mathcal{U} \nsubseteq \mathcal{Q}_p^-$, we have shown that there exists finitely many halfspaces (at most two) supporting $\mathcal{U}$ along $\mathcal{B}$. We choose the intersection of these halfspaces as $\mathcal{L}_p$ when they exist. Conceptual pictures of this case are given in Figure B.3.

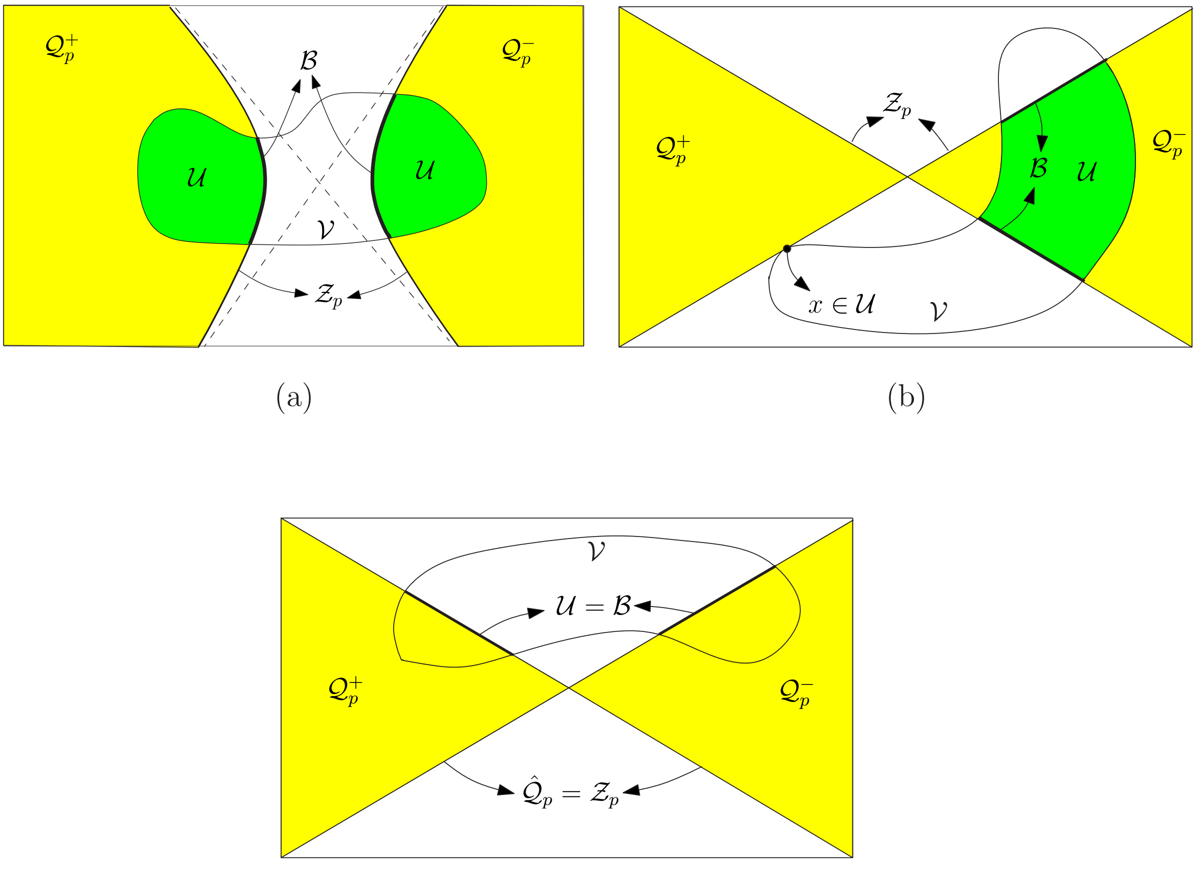


Figure B.3. An illustration of Lemma 7.3.4 for $\pi(P) = 1$ and two components under the condition $\mathcal{U} \nsubseteq \mathcal{Q}_p^+, \mathcal{U} \nsubseteq \mathcal{Q}_p^-$: (a) for an inequality constraint, $\mathcal{U}$ does not have a supporting halfspace at a point of $\mathcal{B}$, (b) for an inequality constraint, it has a single supporting halfspace; (c) for an equality constraint, it has two supporting halfspaces

For the example depicted in Figure B.3.a, there does not exist any supporting halfspace. In Figure B.3.b, an example with one supporting halfspace is given. The last figure shows a case with two supporting halfspaces.

□

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