ROBERT COLLEGE GRADUATE SCHOOL
BEBEK, İSTANBUL

A SURVEY OF SUPERCONDUCTIVITY

by

Yusuf Ulug

Advisor

Prof. Mustafa Santur

DEPARTMENT OF ENGINEERING

School of Engineering

Robert College

Bebek, Istanbul, Turkey

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INTRODUCTION

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1.1. What is Superconductivity

Some metals and alloys when cooled below their critical temperature loose their electrical resistance and expel the applied static magnetic field if the magnetic field is of value less than Hc which is the criticed value. Where Hc depends on the material and the geometric properties of the sample.

Superconductive phenomena exists only at very low temperatures and solely in metals and metalic alloys. The temperature range for superconductivity is bounded above by approximately 20°K. To obtain temperatures below 20°K the specimen under investigation is immersed in liquid hydrogen or liquid helium evaporating at a given pressure which determines its temperature. Liquid helium is generally used and its boiling temperature at atmospheric pressure is 4.2°K. To lower the boiling temperature the pressure is decreased.

1.2. Discovery and Origin of the SuperConductive Phenomena

K. Onnes, the Leiden Physicist, was able to liquify helium in 1908 and obtained specimens that are cooled to temperatures in the neighbourhood of the boiling point of liquid helium (4.2°K).

First experiments by Onnes lead to the result that: The resistance of very pure platimum became constant instead of passing through a minimum or tending to vanish at absolute zero. Gold behaved the same way. K. Onnes expected that he would find the resistance of the sample will go down to zero when extrapolated to zero degree Kelvin. This was not the case and he concluded that the residual resistance was due to the impurities present in the specimen. His next try was to use highly purified mercury and his experiments with mercury showed that at 4.2°K the resistance had become 500 times less than that of the solid wire at the melting point of mercury. This experiment showed that at very low temperature metals can behave quite strangely, in particular exibiting the so called superconductive behaviour.

The origin of superconductivity has long been suspected to be a quantum mechanical mechanism which prevails at very low temperatures A model involving the condensation of pais of conduction electrons of opposite spin into a lower energy state was able to account for most of the known phenomena of the superconductive state. These paired electrons called "Cooper pairs" can travel through the lattice without being scattered. This accounts for the zero resistivity of the superconductors in the pure superconducting state. We shall have more to say about the mechanism which is responsible of the superciductive behaviour later on.

1.3. Magnetoresistance

K. Onnes in 1913 found out that a current exceeding the critical value will render a superconductor normal. The superconducting specimen can also be rendered normal by the application of an external magnetic field greater than Hc, where Hc in the critical field that would just accomplish the superconducting to normal transition. We can see straight away that there are two conditions for the persistance of superconductivity:

a) The current through the speciment must be less than i_c where i_c is the critical current.

b) The external field must be less than Hc.

In 1916 Silsbee suggested that the current induced transition could as well be a special case of the field induced transition, then the current induced transition can be explained on the grounds that when current through the wire reaches a value that would produce a magnetic field of strength Hc on the surface then the transition takes place. This condition is known as the Silsbee condition.

1.4. Diamagnetism

The application of a magnetic field to a material affects the motion of charges and this effect on the motion of electrons produces a negative induced magnetic moment. This magnetic moment opposes the applied field, and the effect is called diamagnetism.

Application of a magnetic field to non-ferromagnetic metal induces eddy currents at the surface so as to "shield" the interior. These currents are due to a change in the magnetic field and in the steady state they do not exist (they decay) and the magnetic field is uniform everywhere inside and outside.

In the case of a perfect conductor the eddy currents will not decay and the interior will always be "shielded". This is observed in superconductors with the proviso that at the surface the magnetic field does not exceed the value of Hc. Yet superconductors differ from the perfect conductor fundamentally. Since;

For a perfect conductor: if an applied external field is removed one would expect permanent eddy currents generated to keep the internal field constant.

For a super conductor: the magnetic field inside a body is expelled during the transition from normal-to-superconducting state. This effect was discovered by Meissner and Ochsenfeld in 1933, and is called the Meissner effect. What Meissner and Ochsenfeld actually found was that the field distribution around a homogeneous superconducting body of low demagnetising factor always correspond to a zero internal field, whatever its magnetic and thermal story. In other words, if a body in a magnetic field is cooled down to a temperature to attain superconductive properities it expels the magnetic flux abruptly as the normal-to-superconducting transition takes place.

1.5. Distribution of Super Conductivity Among the Elements

Superconductivity occurs in metallic elements and their alloys. Over 20 metallic elements can acquire the superconductive behaviour. These elements are non-ferromagnetic and have 2 to 8 valence electrons.

In general superconductors are divided into two categories as Class I superconductors and Class II superconductors.

Class I: These are known as the soft superconductors. Their chief characteristics are that they are not sensitive to strains and impurities because they can anneal at room temperatures. They also have low melting points as expected from the previous characteristics. These superconductors show sharp transitions and exhibit the Meissner Effect.

Class II: These are called the hard superconductors. In contrast to soft superconductors they have high melting points and their properities depend strongly on their impurity content. They are not perfectly diamagnetic and have broad magnetic transitions.

Table 6.1 Critical fields and temperatures of elemental and some compound superconductors

Element	T_c, °K	$\mathcal{H}_c(0)$, oe	Element	T_c, °K	$\mathcal{H}_c(0)$, oe	Compound	T_c, °K	$\mathcal{H}_c(0)$, oe
Al	1.196	99	Pb	7.175	802.6	$AlNb_3$	18	
Cd	0.56	30	Re	1.699	201	$BaBi_3$	5.69	740
Ga	1.091	51	Ru	0.49	66	Bi_2Pt	0.16	10
Hf	0.165		Sn	3.74	305	$CoSi_2$	1.40	105
Hg(α)	4.153	412	Ta	4.483	780	GeIr	4.70	
Hg(β)	3.949	339.3	Tc	11.2		$InLa_3$	10.4	
In	3.407	293	Th	1.37	162	Nb_3Sn	18.07	
Ir	0.14	19	Ti	0.39	100	SiV_3	16.8–17.1	
La(α)	5.0		Tl	2.36	171			
La(β)	5.95	1,600	U(α)	0.6–0.7	~2,000			
Mo	0.92	98	V	5.3	1,310			
Nb	9.25	1,944	Zn	0.91	53			
Os	0.655	65–82	Zr	0.55	47			

FIGURE 1.1. CRITICAL FIELDS AND TEMPERATURES OF ELEMENTAL AND SOME COMPOUND CUPERCONDUCTORS
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1.6. Phenomenological Theories of Superconductivity

In 1911 when K. Onnes first observed the phenomenon of superconductivity in mercury Quantum Mechanics was non-existent. The introduction of Quantum Mechanics into the scientific thought suggested the ways and means to deal with the odd behaviour of the metals at very low temperatures. Until 1957 a successful Quantum Mechanical theory of superconductivity was not established. During this period of 46 years physicists tried to account for the observed facts by phenomenological theories. These theories can explain neither the origin nor the mechanism of the superconductive state but occasionally have the power to predict related phenomena besides explaining the observed facts in terms of theoretical formalism.

Among these theories one by London Brothers is comparatively simple and covers a wide area of the field concerned.

In the following paragraphs Londons' theory of superconductivity will be very briefly stated. Most of the attention is paid to the fundamental equations of this theory. In the paragraphs following the Londons' theory two other theories which are really modifications on the London two fluid theory are mentioned. These are the Guizburg-Landau Theory of superconductivity and the Pippard's non-local theory of superconductivity.

Londons' theory of superconductivity was first published in the Pro. of Royal Soc. (London) 1935, under the title; "The Electromagnetic Equations of the Supraconductor". In this theory H. and F. London replaced

$$\frac{mc}{ne^2}\nabla\times\vec{\dot{J}}+\vec{\dot{H}}=0 \qquad 1.1$$

by

$$\frac{mc}{n_s e^2}\nabla\times\vec{J_s}+\vec{H}=0 \qquad 1.2$$

where n_s = density of superelectrons

J_s = current density due to superelectrons

B = magnetic flux density

m = effective moss of electrons

e = electronic charge

and they obtained the following set of relations to explain the phenomenon.

$$\vec{J} = \vec{J}_s + \vec{J}_n \quad 1.3$$

$$\vec{J}_n = \sigma \vec{E} \quad 1.4$$

$$\vec{E} = \Lambda \dot{\vec{J}}_s \quad 1.5$$

$$\Lambda \nabla \times \vec{J}_s + \vec{H}/c = 0 \quad 1.6$$

These (1.3 to 1.6) are the fundamental equations of the London theory of superconductivity and their derivations will be discussed later on,

where σ = conductivity (normal)

$\vec{J}_n$ = current density due to normal electrons

$\vec{E}$ = the applied electric field

We may also note the superelectrons are the electron participating in the conduction of the super current, i.e. the electrons that are not scattered by the lattice.

<u>The Ginzburg-Landau Theory</u>

The London theory is exact for $H \ll H_c$; $T \sim T_c$ and low frequencies. The following theory is valid for $H \sim Hc$ and it is a modified form of Londons' theory.

Gibbs free energy for a superconductor in magnetic field is given by

$$G_s(H) = G_s(0) + \frac{H^2}{8\pi}$$

where $G_s(H)$ is Gibbs free energy per unit volume.

Ginzburg and Landau assumed that $G_s(H)$ can be written of the form

$$G_s(H) = G_s(0) + \frac{H^2}{8\pi} + \frac{1}{2m}\left[-i\hbar\nabla\psi - \frac{e^*}{c}A\psi\right] \qquad 1.7$$

Of course there were some quantum mechanical arguments behind this assumption.

In 1.7

m = is the effective mass of superelectrons

e^* = is the effective charge of superelectrons

ψ = is the order parameter

$|\psi|^2$ = can be identified with n_s

$G_s(0)$ is taken as

$$G_s(0) = G_n(0) + \alpha(T)\psi^2 + \frac{1}{2}\beta(T)\psi^4$$

if ψ_0 is the equilibrium value of ψ

setting

$$\frac{\partial G_s(0)}{\partial\psi} = 0 \qquad \text{for } H = 0$$

ψ_0^2 can be obtained as

$$\psi_0^2 = -\frac{\alpha}{\beta}$$

and

$$G_s(0) - G_n(0) = -\frac{1}{2}\frac{\alpha^2}{\beta} = \frac{H_c^2}{8\pi}$$

therefore

$$\frac{H_c^2}{4\pi} = \frac{\alpha^2}{\beta}$$

where H_c is the thermodynamic critical field for a bulk superconductor 1.7.

Solution of Eq 1.7 gives the fundamental equations of the Ginzburg-Landau theory.

The Pippard Non-Local Theory

A.B. Pippard discovered that the addition of less than 3% of indium to pure tin produced a substantial increase in the penetration depth of the applied magnetic field which is given as the function of effective electron mass and density by the London 2 fluid theory. It is difficult to believe that the addition of small amounts of impurities can effect the effective mass and density of electrons as to cause a substantial change in the penetration depth. Pippard realizing that this cannot be explained in terms of the London 2 fluid theory modified the following London equation

$$c\Lambda(T)\nabla\times\vec{J_s}+\vec{H}=0 \quad \text{or} \quad c\Lambda(T)\vec{J_s}=-\vec{A}$$

where

$$\nabla\times\vec{A}=\vec{H} \qquad \nabla\cdot\vec{A}=0$$

by

$$J_s(0)=\text{const}\iiint r\frac{A(r)\cdot r}{r^4}\,dv \qquad 1.8$$

With (1.8) and the following empirical relation

$$\frac{1}{\xi}=\frac{1}{\xi_0}+\frac{1}{\alpha l}$$

where ξ = surrounding radius over which A is averaged

l = mean free path of electrons

ξ_0 = range of coherence for the pure metal

α = an empirical constant

$\vec{A}$ = vector potential

Pippard was able to account for the variation of the penetration depth for varying concentration of impurities.

1.7. B.C.S. (Bardeen, Cooper, Schrieffer) Theory: Microscopic Theory

A phenomenological theory basically aims to have a good fit with the experimental observations. This was about everything expected from the previously mentioned theories. For a microscopic theory the same requirements hold true and in addition the theory is to explain the origin, the mechanism of superconductivity. Furthermore, a microscopic theory must have the power to predict, related new phenomena.

The facts to be explained by a microscopic theory of superconductivity are given by Bardeen, Cooper and Schrieffer as:

a) A second-order phase transition at the critical temperature (T_c)

b) An electronic specific heat varying as exp $(-\frac{T_o}{T})$ near zero degrees Kelvin.

c) The Meissner effect

d) Infinite conductivity

e) The dependence of T_c on isotopic mass

J. Bardeen, L.N. Cooper, and J.R. Schrieffer presented a theory which can account for the above experimental observations and which gives good quantitative agreement for specific heats, penetration depth and their dependence on temperatures.

In their calculations Bardeen, Cooper and Schrieffer disregarded the band structure of allowed energies of electrons in metals. This is the case where no periodic potential exists inside of the bulk metal. It is a well known fact that without taking the periodicity of the lattice into account one cannot go very far when normal temperatures are concerned, yet the B.C.S. theory's success is prominant. The success of the theory can be attributed to what is called the "principle of similarity", or to the fact that the curves such as H v.s. T for all superconductors have the same shape.

In 1956 Cooper showed that pairs of electrons could condense into a lower energy phase provided that some attraction between them exists. Later Bardeen, Cooper and Schrieffer showed that such an attraction between electron pairs exist due to electron-phonon interactions. More specifically in the B.C.S. theory the attractive interaction is assumed to occur between two electrons of opposite spin and opposite moments ($\hbar k, -\hbar k$).

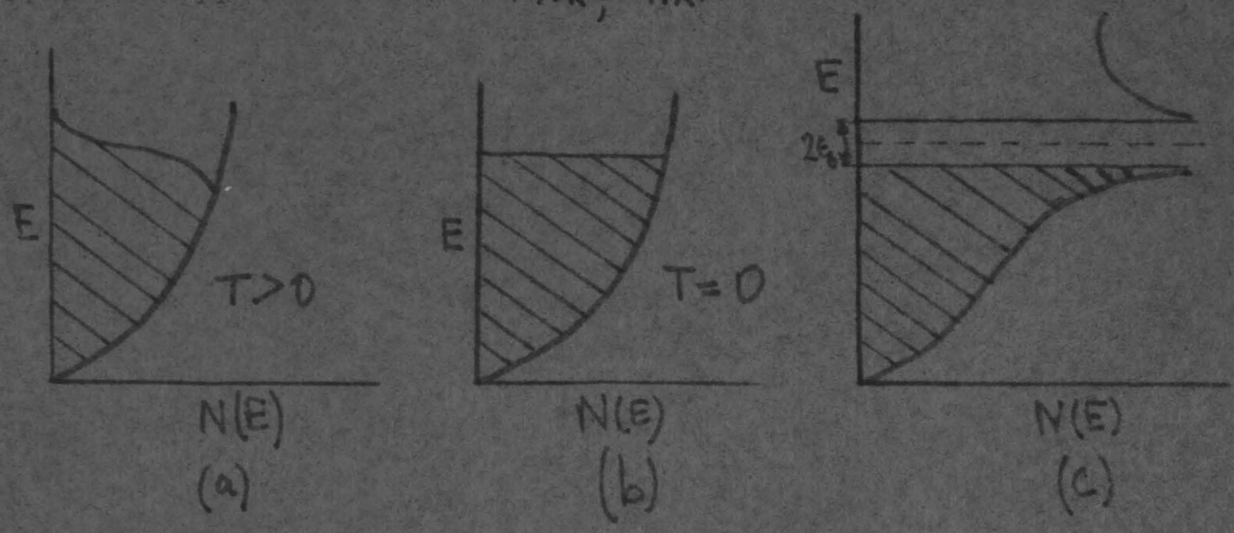

FIGURE 1.2. DENSITY OF STATES CURVES AND ELECTRON ENERGY DISTRIBUTION

a) Normal Metal at room temperature

b) Normal metal at absolute zero

c) B.C.S. model superconductor at absolute zero.

As shown in Figure 1.2(c) there is an energy gap of width ($2\epsilon_0$) centered around the Fermi level. At T = o all the electrons have energies equal or less than ($Ef - \epsilon_0$) and are coupled as virtual pairs. For higher temperatures some electrons are excited across the energy gap but the total number of states below Ef (Fermi level) remains unchanged. In supermonducting state the energy required to disrupt a Cooper pair and excite them to the normal band is equal to the gap energy ($2\epsilon_0$). Consequently, a Cooper pair to be scattered by the lattice requires an energy of $2\epsilon_0$. The probability of this to happen depends on the velocity or the kinetic energy of the pair and only at velocities higher than a certain value this probability becomes effective. Having this in mind the reason for the current induced transition can be readily seen since for low currents the pair velocities are low and the probability for a scatter interaction is negligible therefore the resistance appears to be zero. But as current increases pair velocities increase, so as the probability of scattering interactions and the superconducting-to-normal transition takes place.

1.8. Some Applications of Superconductivity

In the following paragraphs some application of superconductivity will be described or stated very briefly. It is not possible to discuss the full significance of these applications at this stage since very little theoretical background is provided up to now if not at all.

1.8a. Electromagnets

In the case of high field intensities ohmic resistance causes considerable amount of the total energy required to set up the magnetic field. Another problem with electromagnets using normal conductors is the removal of heat developed in the coil which usually requires a sophisticated system. Using superconducting coils these problems are eliminated provided that the critical field is not exceeded in any part of the coil. By use of hard superconducting alloys which have very high critical fields, field intensities of 100 Koersteds can be achieved.

1.8b. Suspension Systems

In order to obtain low viscosity suspension system Meissner effect can be utilized in the following ways. Refer to Figure 1.3.

a) A superconducting specimen can be balanced on a magnetic field.

b) A magnet can be made to float on a superconducting specimen.

1.8c. Electrical Switching Elements

We have previously noted the very high ratio between the resistance of a metal in the normal state and its superconducting state. This is in some cases conveniently used as a switching action which can switch the electric current between different paths. These switches can either be temperature controlled or field controlled.

1.8d. Persistent Current Storage

It was mentioned that a current set up in a superconducting

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Fig. 1.6 Superconducting suspension devices. (a) Magnet floating above superconducting lead cup (Arkadiev, 1945, 1947). (b) Lead-coated glass sphere supported by magnetic field trapped in two superconducting lead rings (Simon, 1953).

FIGURE 1.3

ring remains for a long time without decaying. This property of superconductors can be used in the memory of a computer. The advantages being a quicker response, smaller size and cheaper price.

1.8e. Bolometers

A bolometer is a radiation detector whose operation depends on the variation of the resistance with the temperature of a conductor exposed to the heating effect of incident radiation. Lower temperatures are preferred for sensitivity reasons since at low temperatures the thermal noise level which sets the limit of sensitivity is lower. Also the specific heat of materials decrease with decreasing temperatures which also helps for the sensitivity.

It may be worthwhile to note that using a small film of a superconductor as the detecting element it was possible to detect radiation power of 10^{-12} Watt.

1.8f. The Heat Valve

For some common superconductors the ratio of its heat resistance in the superconducting state to its heat resistance in its normal state has a value about 100. This characteristic suggests the use as a heat valve, with a distinguishing characteristic of no moving parts. Working with refrigeration systems designed to obtain temperatures below $0.3^{\circ}K$ a heat valve with moving parts can produce substantial heat by friction so as to limit the lowest temperature attainable by the system. The superconducting heat valve easily extends this limit.

CHAPTER II

THEORIES OF SUPERCONDUCTIVITY

In the introduction, the characteristic behaviour of materials in the superconducting state are laid down briefly. The fundamentals of theories of superconductivity are stated and some of the important applications of metals in the superconducting state are mentioned.

In this section the theories of superconductivity will be worked out somewhat in detail in the following order:

1. Thermodynamics of the Superconducting State
2. London Two Fluid Theory
3. B.C.S. Theory of Superconductivity

2.1. Thermodynamics of the Superconducting State

Before the discovery of the Meissner effect there was no evidence of the fact that the superconducting phase is a single phase and it is stable. Infinite conductivity shown by experiments implied that the electric field inside a superconductor is zero and the magnetic induction is constant in time. Now we may note that there are two different ways of preparing a sample, in an external field H_o ($H_o < H_c$) and at temperature T, where H_c and T_c are the critical field and the critical temperature respectively.

a) for $T > T_c$ the sample may be placed in a magnetic field H_o and temperature T is reduced to a value lower than T_c.

b) the sample may be cooled with $H_o = 0$ and H_o is switched on when sample is already at temperature T ($T < T_c$).

Now for the case (a), the magnetic induction inside the sample is H_o; for the case (b) the induction in the sample is zero. These results can be reasoned out as follows:

Case (a): Magnetic field H_o is set up in the material in the normal state and that remains to be the case after the transition.

Case (b): When H_o is switched on the sample is already in the superconducting state and the non-decaying eddies keep the magnetic field out of the material. We see that it is possible to prepare a superconducting specimen with a given value of H ($H < H_c$) present inside the specimen. Therefore, there is not a simple state of superconductivity. Hence the methods of equilibrium thermodynamics cannot be applied.

The discovery of the Meissner effect (1933) showed clearly that the superconductive state is a simple stable state (field inside is zero whatever the history of the specimen) and the methods of equilibrium thermodynamics can be used with confidence.

We must note that the remarks up to now strictly speaking applies for a slender cylinder with axis along the applied magnetic field.

The total free energy in fields greater than $H_c(T)$ is given by

$$G_n + E_m$$

G_n = energy of normal state

E_m = the energy with no specimen in the field.

For fields $H < H_c$ the specimen behaves like a perfect diamagnet and the magnetic moment of the specimen is

$$-V \frac{H}{4\pi}$$

where

V = volume of the specimen

Therefore for the magnetic energy we have

$$E_m + VH^2/8\pi$$

and for $H < H_c(T)$

$$g_s(H,T) = E_m + V\frac{H^2}{8\pi} + g_s$$

g_s = free energy of the superconductor at zero field.

or

$$g_n - g_s = V H_c^2/8\pi$$

In thermodynamics the following relations for entropy and specific heat are well known

$$S = \frac{-1}{V}\left(\frac{\partial g}{\partial T}\right)_{p,H} \qquad S = \text{entropy}$$

$$C = T\frac{\partial S}{\partial T} \qquad C = \text{specific heat per unit vol.}$$

$$S_n(T) = \frac{-1}{V}\left(\frac{\partial g}{\partial T}\right) = 0$$

$$S_s(T) = \frac{-1}{V}\frac{\partial}{\partial T}\left(V\frac{H_c^2}{8\pi}\right) = \frac{H_c}{4\pi}\frac{\partial H_c}{\partial T}$$

Similarly

$$C_n(T) - C_s(T) = -\frac{T}{8\pi}\frac{d^2}{dT^2}(H_c^2) \qquad 2.1$$

These expressions are exact for very small penetration depths which is the case for a slender cyliner oriented with its axis along the applied field.

The following is an empirical relation which relates H_c to T.

$$\frac{H_c}{H_0} = 1 - \left(\frac{T}{T_c}\right)^2 \qquad 2.2$$

Using 2.2 and 2.1 we obtain

$$C_n - C_s = \frac{H_0^2}{2\pi T_c}\left[\frac{T}{T_c} - 3\left(\frac{T}{T_c}\right)^3\right] \qquad 2.3$$

In order to account for the terms on the R.H.S. of relation 2.3 we reason as follows.

We know emprically that the specific heat of a superconductor decreases as fast as T^3. We also know that the lattice contribution is as T^3's; therefore C_{es} (of electrons) must decrease at least as fast as T^3. Consequently, the first term on the R.H.S. of 2.3 should be the specific heat of the electrons in the normal state

$$C_{en} = \frac{H_0^2 T}{2\pi T_c^2}$$

But for the specific heat of electrons in the normal state is γT

γ given by $$\frac{H_0^2}{2\pi T_c^2} = \gamma$$

Therefore, we can identify $\frac{H_0^2 T}{2\pi T_c^2}$ as the specific heat in the normal state.

If we assume that the lattice specific heats for the normal and superconducting state remains the same the the second term on the R.H.S of 2.3 represents C_{es} specific heat of electrons in the superconducting state viz :

$$C_{es} = \frac{3 H_c^2}{2\pi T_c} \left(\frac{T}{T_c} \right)^3$$

2.2. The Intermediate State

For the derivation of the thermodynamical results in the previous section a narrow cylinder placed in a magnetic field parallel to its axis is considered. The reason for this was to avoid the penetration depth and the intermediate state since for that particular geometry the penetration depth is negligible and the transition from normal-to-superconducting state and vice versa is complete. In other words we have assumed that the whole of the specimen is either in the superconducting or in the normal state, however, this is not the case always and some parts of the sample can be in the superconducting state while other parts in the normal state and this constitudes the Intermediate State. Intermediate State is clearly not a new state, but a combination of the two states in the same body.

In order to see how the intermediate state arises consider the following example:

A superconducting sphere at temperature T ($T < T_c$) and in H

($H > H_c$) as in Figure 2.1.

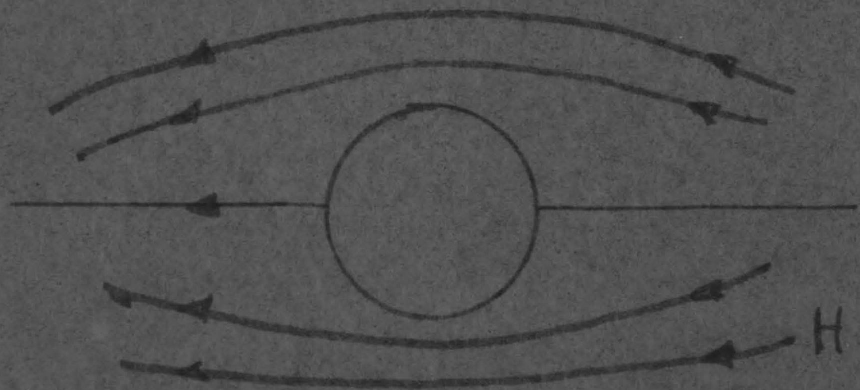

FIGURE 2.1

As H is lowered passed by H_c the magnetic flux is pushed out of the sphere as shown in the figure. This pushing out actions causes the field at the equator to intensify and due to this intensification the field there can exceed the critical value causing the metal at the equator to remain normal. Therefore, the metal at the equator turns to normal but then flux can penetrate in and this lowers the concentration of the force lines at the equator resulting in the action that the metal at the equator returns to the superconducting state. The following graph relates the magnetic moment of a sphere to the applied magnetic field.

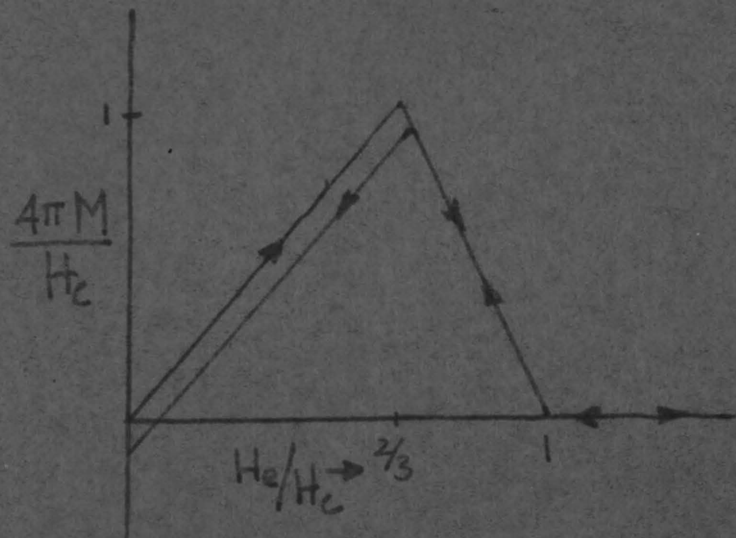

FIGURE 2.2

It is seen that as H increases beyond the value of $2/3H_c$ the penetration starts and with further increase in H the magnitude of the moment decreases until $H = H_c$ and is zero for $H \geqslant H_c$.

Landau suggested that in the intermediate state normal and superconducting regions should alternate in the form of thin laminae parallel to the applied magnetic field and branches out into finer laminae near the surface. The branching out into finer laminae near the surface is not observed but the laminae thickness is shown to be of the order of 1m.m.

For specimens of dimentions much greater than 1mm an average induction $\vec{B}$ can be used to describe the magnetic field inside the specimen. The actual laminae structure is not important here since we use the mean values for flux.

The following relation can be used to find the microscopic magnetic field $\vec{H}$ within the speciment

$$H(\vec{r}) = 4\pi \frac{\delta G}{\delta B(\vec{r})} \qquad 2.4$$

where

G = free energy

In order to utilize relation 2.4 we have to know the free energy density in the intermediate phase and that be calculated as follows let

a = the fraction of the specimen that is normal

(a-1) = the fraction of the speciment that is superconducting

Contribution of superconducting laminae to free energy density is given as

$$(1-a)\, g_s(0)$$

where $G_s(0)$ is the free energy density of a pure superconductive domain for zero field.

The induction in the normal laminae is H_c, and their contribution to the free energy density

$$a\left(g_n + \frac{H_c^2}{8\pi}\right) = a\left[g_s(0) + H_c^2/4\pi\right]$$

For the total free energy desnity of the speciment we have

$$G = \int d^3r \left(g_s(0) + a(\vec{r})\, H_c^2/4\pi\right)$$

where $a(\vec{r})$ is the fraction of the normal material at the macroscopic point $\vec{r}$. But $a(\vec{r})$ is related to the average $B(\vec{r})$ through

$$|B(\vec{r})| = a(\vec{r})\, H_c$$

given

$$g = g_s(0) + \frac{1}{4\pi}\int d^3r\, H_c |\vec{B}|$$

and the field within the specimen (macroscopic) is given as

$$\vec{H} = H_c \frac{\vec{B}}{|B|} \qquad 2.5$$

To find the magnetic moment of a specimen of any shape in a given magnetic field relation 2.5 can be solved with the Maxwells equations.

2.3. London Two Fluid Theory of Superconductivity

In the previous section the basic thermodynamical relations of the superconducting state are stated. In this section a theory which accounts for most of the electro-magnetic characteristics in the superconducting state is the topic. "The Electromagnetic Equations of the Superconductor" by F. and H. London appeared in the Proceedings of the Royal Society in 1935.

Londons' start with the following two statements:

1. The "acceleration equation" (to be modified)
2. Meissner and Ochsenfeld Effect

The acceleration equation is

$$\Lambda \dot{\vec{J}} = \vec{E} \quad \text{and} \quad \Lambda = \frac{m}{ne^2}$$

where m = mass of electron

n = number density of electrons

e = electronic charge

This equation implies that in superconductors there is no dissiputation or friction (scattering) for electrons is motion in the lattice, but, then electron can accelerate continuously which is not plausible at all. Using the acceleration equation as it stands in conjunction with the Maxwell's equations lead theorytitians into a number of difficulties. Londons' being aware of this, modified the acceleration equation so that; new acceleration equation, a relation for the Meissner effect and Maxwells equations together formed a consistent set of equations explaining the electromagnetic properties of the superconductors.

The development was as follows:

Taking the curl of both sides of the acceleration equation we get

$$\nabla \times \Lambda \vec{\dot{J}} = \nabla \times \vec{E}$$

But from Maxwells equations we have the relation

$$\nabla \times \vec{E} = -\frac{1}{c}\vec{\dot{H}}$$

therefore we can obtain

$$\nabla \times \Lambda \vec{\dot{J}} = -\frac{1}{c}\vec{\dot{H}} \qquad 2.6$$

where

J = current density $\qquad \dot{J} = \frac{d}{dt}J$

c = velocity of light $\qquad \dot{H} = \frac{d}{dt}H$

For low frequencies the displacement current can be neglected and we have the following relation from Maxwells Theory

$$\frac{1}{c}\vec{\dot{J}} = \nabla \times \vec{\dot{H}} \qquad 2.7$$

from 2.6 and 2.7 we have

$$\nabla \times \nabla \times \Lambda \vec{\dot{H}} = -\frac{1}{c^2}\vec{\dot{H}} \tag{2.8}$$

but

$$\nabla(\nabla \cdot \vec{\dot{H}}) - \nabla^2 \vec{\dot{H}} = -\frac{1}{c^2}\vec{\dot{H}} \tag{2.9}$$

and

$$\nabla \cdot \Lambda \vec{\dot{H}} = 0 \tag{2.10}$$

therefore using 2.9, 2.10 on 2.8 we get

$$\Lambda c^2 \nabla^2 \vec{\dot{H}} = \vec{\dot{H}} \tag{2.11}$$

Integrating 2.11 w.r.t time we have

$$\Lambda c^2 \nabla^2 (\vec{H} - \vec{H}_0) = \vec{H} - \vec{H}_0 \tag{2.12}$$

In this equation H_0 arising as an integration constant denotes H_0 magnetic field value at time (t = o).

Solving 2.12 for a semi-infinite bodywith surface at $X = 0$, and internal field H_i at a distence X from the surface we get

$$\vec{H}_i = \vec{H}_0 + (\vec{H}_e - \vec{H}_0)\, e^{-x/\lambda} \tag{2.13}$$

where

$$\lambda^2 = \Lambda c^2$$

H_e = external field

H_0 = field present in the hypothetical body when its resistance fell to zero.

2.13 shows that a field H_o is trapped in the superconducting body when the resistance reduces to zero. The contradiction to this result comes from the Meissner and Onshenfeld effect which states that the field in a superconductor is expelled abrouptly during the transition from normal-to-superconducting state.

In order to rectify this contradiction we take the following homogeneous equation

$$\Lambda c^2 \nabla^2 \vec{H} = H$$

The solution of this equation is of the form

$$H_i = H_- e^{-x/\lambda}$$

This solution is in agreement with the Meissner and Onshenfeld effect since we don't have a residual value for the magnetic field inside the material in the superconducting state. Therefore we notice that the equations 2.11 or 2.12 is too general for our purposes. Now dealing with relations

$$\nabla \times \Lambda \vec{J} = \frac{1}{c} \vec{H}$$

and

$$-\frac{1}{c} \dot{\vec{H}} = \nabla \times \vec{E}$$

we obtain

$$\nabla \times \left(\Lambda \dot{\vec{J}} - \vec{E} \right) = 0 \qquad 2.13$$

and the London relations

$$\vec{J} = \vec{J}_s + \vec{J}_n \qquad 2.14$$

$$\vec{J}_n = \sigma(T) \vec{E} \qquad 2.15$$

$$E = \Lambda(T)\frac{dJ_s}{dt} \qquad 2.16$$

$$c\Lambda(T)\nabla \times J_s + H = 0 \qquad 2.17$$

where

J_n = normal current density

J_s = density of supercurrent

$$\Lambda = \frac{m}{n_s e^2} = \frac{4\pi\lambda^2}{c^2}$$

n_s = density of superelectrons

and if a vector potential $\vec{A}$ is defined with the conditions that

a) $\nabla \cdot \vec{A} = 0$

b) The compnent of $\vec{A}$ normal to the boundary of the superconductor is zero.

Then we have

$$\nabla \times \underline{A} = \vec{H} \qquad 2.18$$

$$c\Lambda\vec{J_s} = -\vec{A} \qquad 2.19$$

Since $c\Lambda\nabla \times \vec{J_s} + \nabla \times \vec{A} = 0$ by 2.18 and 2.17

and $\nabla \times [\Lambda c\vec{J_s} + \vec{A}] = 0$

therefore $c\Lambda\vec{J_s} + \vec{A} = 0$

Equations 2.14, 2.15, 2.16, 2.17, 2.19 constitute the fundamental equations of the London two fluid theory. As they are presented here they are not in their most useful form. The equations to follow constitude a set which can be applied to the practical problems with more ease.

Combining equation 2.14, 2.15, 2.16, 2.17

Since $c\Lambda\nabla\times\vec{J}_s + \vec{H} = 0$

$\vec{J} = \vec{J}_n + \vec{J}_s$ & $\vec{J}_n = \sigma\vec{E}$ therefore $\vec{J} = \sigma\vec{E} + \vec{J}_s$

and

$$\vec{J}_s = \vec{J} - \sigma\vec{E}$$

therefore

$$c\Lambda\nabla\times(\vec{J} - \sigma\vec{E}) + \vec{H} = 0$$
$$c\Lambda\nabla\times\vec{J} - c\Lambda\nabla\times\sigma\vec{E} + \vec{H} = 0$$
$$c\Lambda\nabla\times\vec{J} - c\Lambda\dot{\vec{H}} + \vec{H} = 0$$

and using $\nabla\times\vec{E} = -\frac{1}{c}\dot{\vec{H}}$

we obtain

$$-c\nabla\times\Lambda\vec{J} = \vec{H} + \sigma\Lambda\dot{\vec{H}} \quad 2.20$$

Also $$\frac{\partial}{\partial t}\Lambda\vec{J} = \Lambda\left(\frac{\partial}{\partial t}\sigma\vec{E} + \frac{\partial}{\partial t}\vec{J}_s\right)$$
$$= \Lambda\sigma\dot{\vec{E}} + \Lambda\dot{\vec{J}}_s$$
$$= \vec{E} + \sigma\Lambda\dot{\vec{E}} \quad 2.21$$

2.20 and 2.21 and Maxwells relations

1- $\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{\dot{\vec{E}}}{c}$

2- $\nabla\times\vec{E} = -\dot{\vec{H}}/c$ The Maxwell equations

3- $\nabla\cdot\vec{H} = 0$ 1 - 4

4- $\nabla\cdot\vec{E} = 4\pi\rho$ in Gaussian Units.

give

$$c^2\nabla\times\nabla\times\vec{H} + \frac{4\pi}{\Lambda}\vec{H} + 4\pi\sigma\dot{\vec{H}} + \ddot{\vec{H}} = 0 \quad 2.22$$

$$c^2 \nabla \times \nabla \times \vec{E} + \frac{4\pi}{\Lambda} \vec{E} + 4\pi\sigma \dot{\vec{E}} + \ddot{\vec{E}} = 0 \qquad 2.23$$

$$c^2 \nabla \times \nabla \times \vec{J} + \frac{4\pi}{\Lambda} \vec{J} + 4\pi\sigma \dot{\vec{J}} + \ddot{\vec{J}} = 0 \qquad 2.24$$

$$\frac{4\pi}{\Lambda} \rho + 4\pi\sigma \dot{\rho} + \ddot{\rho} = 0 \qquad 2.25$$

For the derivations look to the Appendix. Equation 2.25 can be solved straight away to give

$$\rho = A e^{-\gamma_1 t} + B e^{-\gamma_2 t}$$

A and B are constants of integration they are independent of time but arbitory functions of space. We also have

$$\gamma_1 = 2\pi\sigma \left(1 + \sqrt{1 - \frac{1}{\pi\Lambda\sigma^2}}\right) \cong 4\pi\sigma \simeq 10^{19}\, sec^{-1}$$

$$\gamma_2 = 2\pi\sigma \left(1 - \sqrt{1 - \frac{1}{\pi\Lambda\sigma^2}}\right) \cong \frac{1}{\Lambda\sigma} \simeq 10^{12}\, sec^{-1}$$

The relaxation time is 10^{12} sec^{-1} and

$$\tau = \frac{1}{\gamma_1} = 10^{-12}\, sec. \qquad \text{(time constant } \tau)$$

Since γ's are very small ρ is approximately 0 so as $\dot{\rho}$

Therefore the Maxwell equation 4 can be written as

$$4\pi\rho = \nabla \cdot \vec{E} \simeq 0 \qquad 2.26$$

and

$$\nabla \cdot \vec{J} = -\dot{\rho} \simeq 0$$

And for equations 2.22 – 2.24

$$c^2 \nabla^2 \vec{H} = \frac{4\pi}{\Lambda} \vec{H} + 4\pi\sigma \dot{\vec{H}} + \ddot{\vec{H}} \qquad 2.27$$

$$c^2 \nabla^2 \vec{E} = \frac{4\pi}{\Lambda} \vec{E} + 4\pi\sigma \dot{\vec{E}} + \ddot{\vec{E}} \qquad 2.28$$

$$c^2 \nabla^2 \vec{J} = \frac{4\pi \vec{J}}{\Lambda} + 4\pi\sigma \dot{\vec{J}} + \ddot{\vec{J}} \qquad 2.29$$

The three term on the R.H.S. in the set of equation 2.27 to 2.29 represents the contributers of the supercurrent $\vec{J}_s$, the normal current $\vec{J}_n$, and the displacement current $\vec{J}_d = \frac{\dot{E}}{4\pi}$ respectively. Note that

$$c \nabla \times \vec{J}_s = -\frac{1}{\Lambda} \vec{H} \; ; \; \frac{\partial}{\partial t} \vec{J}_s = \frac{1}{\Lambda} \vec{E} \qquad 2.30$$

$$c \nabla \times \vec{J}_n = -\sigma \dot{\vec{H}} \; ; \; \frac{\partial}{\partial t} \vec{J}_n = \sigma \dot{\vec{E}} \qquad 2.31$$

$$c \nabla \times \vec{J}_d = \frac{c}{4\pi} \nabla \cdot \vec{E} \; ; \; \frac{\partial}{\partial t} J_d = \frac{\ddot{\vec{E}}}{4\pi} \qquad 2.32$$

Therefore for alternating fields there is a phase difference of $\pi/2$ between J_s and J_n and a phase difference of π between J_s and J_d.

For frequency of $\omega/2\pi$ we have

$$|\vec{J}_s| : |\vec{J}_n| : |J_d| = 1 : \Lambda\sigma\omega : \frac{\Lambda\omega^2}{4\pi} \qquad 2.33$$

$$\text{if} \quad \omega \ll \frac{1}{\sigma\Lambda} \simeq 10^{12}\ \text{sec}^{-1}$$

then J_n and J_d are negligibly small c.f. J_s.

Therefore we have from 2.22 - 2.24.

$$\nabla \times \nabla \times \vec{H} + \frac{4\pi}{\Lambda c^2} \vec{H} = 0 \qquad 2.34$$

$$\nabla \times \nabla \times \vec{E} + 4\pi/\Lambda c^2 \vec{E} = 0 \qquad 2.35$$

$$\nabla \times \nabla \times \vec{J} + 4\pi/\Lambda c^2 \vec{J} = 0 \qquad 2.36$$

For stationary conditions we have

$$\nabla \times \vec{E} = -\frac{\dot{\vec{H}}}{c} = 0$$

$$E = 0 \qquad \text{by } 2.35$$

Therefore we see that an electric current in a superconductor may exist even when the electric field is zero accounting for the perfect conductivity of superconductors.

Again for stationary conditions we obtain from relations 2.27

$$\nabla^2 \vec{H} = \frac{4\pi}{\Lambda c^2} \vec{H} \qquad 2.37$$

The solution of 2.37 is a rapidly decreasing function for $\vec{H}$ therefore magnetic field decreases very rapidly as one receded from the surface toward the interior of the superconductor. (Proof given § 7 Pg. 51 F. London in Superfluids.)

The magnetic field reduces effectively to zero at a depth greater than $c\sqrt{\frac{\Lambda}{4\pi}}$ which is of the order of 10^{-5} cm. This is obviously what is known as the Meissner Effect.

If large bodies are considered we have

$$\vec{B} = \vec{H} = 0$$

ie for $\tilde{\chi} \gg c\sqrt{\frac{\Lambda}{4\pi}}$ where $\tilde{\chi}$ = average body dimention

For sufficiently small superconductors substantial amounts of flux can enter into the body in spite of the Meissner effect and this provides means of measuring $c\sqrt{\frac{\Lambda}{4\pi}}$ consequently Λ can be obtained.

Before considering some applications of the London 2 fluid theory here we state the boundary conditions in general as:

a) Fields $\vec{H}$, $\vec{E}$, $\vec{J}$ must remain finite

b) Therefore $\nabla\times\vec{H}$, $\nabla\times\vec{E}$, $\nabla\times\vec{J_s}$, $\nabla\cdot\vec{H}$ and $\nabla\cdot\vec{J}$ must be zero.

c) Due to surface charges

E perpendicular to the boundary can be discontinuous

B perpendicular to the boundary is continuous since μ is taken as unity.

d) H parallel to the boundary must be continuous.

E parallel to the boundary must be continuous.

These are the usual boundary conditions for an interphase in Maxwell's theory.

e) Also ΛJ_s parallel to the boundary must be continuous at the boundary between two different superconductors.

2.3a THE SUPERCONDUCTING SPHERE IN A MAGNETIC FIELD

Here we have a superconducting sphere of radius R in an applied magnetic field H_o at infinity. In Mathematical language we have the following boundary conditions:

1. for $r \gg R$ $\quad H \rightarrow H_o$

2. for $r \geq R$ we have

$$\nabla\cdot\vec{H} = 0$$
$$\nabla\times\vec{H} = 0$$

It seems plausable to try to describe the field of $r > R$ by the superposition of H_o and the field of a dipole both parallel to the polar axis of the sphere.

In spherical polar co-ordinates we have the following relation for H

$$H_r = \left(H_0 + \frac{2M}{r^3} \right) \cos\theta$$

$$H_\theta = \left(-H_0 + \frac{M}{r^3} \right) \sin\theta$$

$$H_\phi = 0$$

where M is the induced Magnetic moment of the sphere.

For the interior ($r \leq R$) the current density is given by

$$\nabla \times \nabla \times \vec{J} + \frac{4\pi}{\Lambda c^2} \vec{J} = 0 \qquad 2.38$$

Due to the symmetry of the system we assume that the currents in the sphere are parallel to the equator and is of the following form

$$J_\theta = 0 \qquad J_r = 0$$

$$J_\phi = f(r) \sin\theta \qquad 2.39$$

By 2.38 and 2.39 we have let $\alpha^2 = \frac{4\pi}{\Lambda c^2}$

$$\nabla \times \nabla \times J_\phi + \frac{4\pi}{\Lambda c^2} J_\phi = 0$$

$$\nabla \times \nabla \times f(r) \sin\theta + \frac{4\pi}{\Lambda c^2} f(r) \sin\theta = 0$$

we have

$$f'' + \frac{2}{r} f' - \left[\frac{2}{r^2} + \left(\frac{4\pi}{\Lambda c^2} \right)^2 \right] f = 0$$

for deviation look to Appendix.

Now the general solution of this equation is of the form

$$f = \frac{A}{r^2} \left(\sinh \alpha r - ar \cosh \alpha r \right) + \frac{B}{r^2} \left(\cosh ar - ar \sinh ar \right)$$

where A and B are constants of integration. To obtain a solution which remains finite at every point, the integration constant B must be equal to zero, otherwise at the centre (r = o) f becomes infinite. Therefore

$$J_\phi = \frac{A}{r^2}\left(\sinh \alpha r - \alpha r \cosh \alpha r\right)\sin\theta$$

we have $$-c\,\nabla\times\Lambda\vec{J} = \vec{H} + \sigma\Lambda\dot{\vec{H}}$$

but $$\dot{\vec{H}} = 0$$

Therefore $$\nabla\times\vec{J} = -\frac{H}{c}$$

$$H_r = \frac{4\pi}{\alpha^2 c}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, J_\phi\right)$$

where $$\alpha^2 = \frac{4\pi}{\Lambda c^2}$$

Therefore $$H_r = A'\frac{2}{\alpha^2 r^3}\left(\sinh\alpha r - \alpha r\cosh\alpha r\right)\cos\theta$$

and $$A' = \frac{4\pi A}{c} \quad \text{for} \quad r \leq R$$

$$H_\theta = -\frac{4\pi}{\alpha^2 c}\frac{1}{r}\frac{\partial}{\partial r}\left(r J_\phi\right)$$

$$= A'\frac{1}{\alpha^2 r^3}\left[\left(1+\alpha^2 r^2\right)\sinh\alpha r - \alpha r\cosh\alpha r\right]\sin\theta = 0$$

Another condition is that H_r and H_θ must be continuous at r=R.

$$M = -\frac{H_0 R^3}{2}\left(1 - \frac{3}{\alpha R}\coth\alpha R + \frac{3}{\alpha^2 R^2}\right) \qquad 2.40$$

$$A' = -\frac{3H_0}{2}\frac{R}{\sinh\alpha R}$$

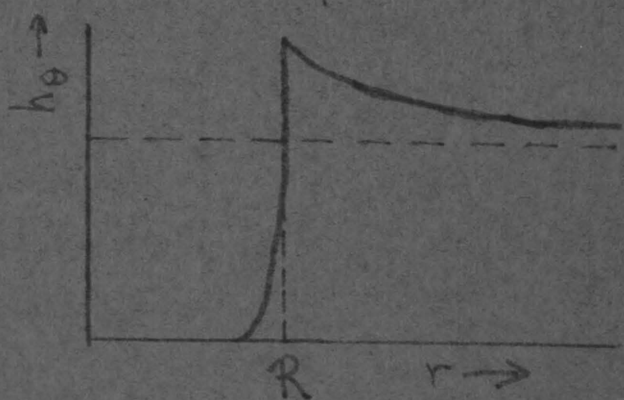

FIGURE 2.3

Figure 2.3 shows the variation of the magnetic field in an equatorial cross-section of a superconducting sphere of radius R. For large shperes i.e. $R \gg \alpha^{-1}$ from 2.40 we obtain

$$M = -\frac{H_0 R^3}{2}\left(1 - \frac{3}{\alpha R} + \frac{3}{\alpha^2 R^2} \cdots\right)$$

$$\simeq \frac{-H_0}{2}\left(R - \alpha^{-1}\right)^3$$

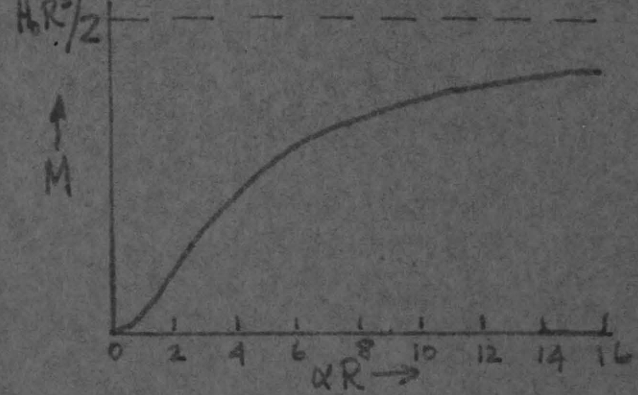

FIGURE 2.4

Above we have the variation of the induced magnetic moment M of a superconducting sphere with its radius and the penetration depth α^{-1}.

From Figure 2.4 we can extract the following information:

For large spheres ($R \gg \alpha^{-1}$) induced Magnetic moment M is as it would be for a perfectly diamagnetic sphere of radius ($R - \alpha^{-1}$).

In the other extreme for "very small" spheres ($R \ll \alpha^{-1}$) developing M in powers of αR as

$$M = \frac{-H_0 R^3}{2}\left(\frac{\alpha^2 R^2}{15} - \frac{2}{315}\alpha^4 R^4 - - -\right)$$

Therefore for spheres of $R \ll \alpha^{-1}$ the diamagnetism becomes very small.

2.3b. The Quantized Fluxoid

We have previously noted that the flux in a body, when a transition from normal-to-superconducting state takes place, is abruply expelled provided that the body is in a magnetic field H_o ($H_o < H_c$). Having this in mind consider the case of a body which has the slope of a donut ( ) in a magnetic field H_o ($H_o < H_c$). Evidently when the normal-to-superconducting transition takes place in such a body due to the Meissner effect some flux will be trapped in the hole which is not superconducting. In the paragraphs to follow we shall try to find a quantity that is conserved throughout the normal-to-superconducting transition called "fluxoid" the name, implying its similarity to the flux.

Consider the following

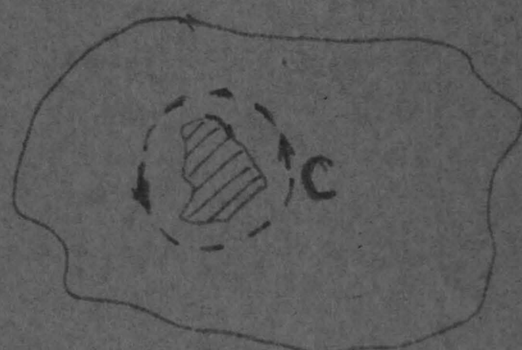

FIGURE 2.5

The hole enclosed by C is surrounded entirely by a superconductor. The surface S bounded by the curve C encloses both superconducting and non-superconducting region. We have the following relation valid for all cases on the surface S.

$$-c\nabla\times\vec{E} = \dot{H}$$

Integrating this equation over the surface S bounded by the curve C we have

$$\iint \dot{\vec{H}} \cdot \vec{ds} + \iint c\, \nabla \times \vec{E} \cdot \vec{ds} = 0$$

Using Stokes theorem we have

$$\iint_S c\, \nabla \times \vec{E} \cdot \vec{ds} = \int_C c\vec{E} \cdot \vec{dl}$$

$$\iint \dot{\vec{H}} \cdot \vec{ds} + c \oint \vec{E} \cdot \vec{dl} = 0$$

$$\frac{d}{dt} \iint \vec{H} \cdot \vec{ds} + c \oint_C \frac{\partial}{\partial t} (\Lambda \vec{J}_s) \cdot \vec{dl}$$

Since

$$\frac{\partial}{\partial t} (\Lambda \vec{J}_s) = \vec{E}$$

we have

$$\frac{d}{dt} \left\{ \iint \vec{H} \cdot \vec{ds} + c \oint_C \Lambda \vec{J}_s \cdot \vec{dl} \right\} = 0$$

Integrating with respect to time

$$\phi_c = \iint_S \vec{H} \cdot \vec{ds} + c \oint_C \Lambda \vec{J}_s \cdot \vec{dl} \qquad 2.41$$

We see that the quantity ϕ_c is constant in time therefore conserved. ϕ_c is called the "fluxoid" because of its close relation to the magnetic flux through the curve C.

To illustrate this similarity let us consider a sufficiently thick superconductor in which we can push the curve C deep into the interior of the body. The current density in the deep interior is small and can be neglected reducing expression 2.41 to

$$\phi \simeq \iint \vec{H} \cdot \vec{ds}$$

Therefore for sufficiently thick superconductors the fluxoid Φ_c becomes the flux ϕ.

For rapidly varying fields using the proper expression viz.

$$\frac{\partial}{\partial t}(\Lambda \vec{J}) = \vec{E} - \sigma\Lambda\dot{\vec{E}}$$

$$\frac{d}{dt}\iint_S \vec{H}\cdot\vec{dS} = -c\oint_G \vec{E}\cdot\vec{dl} = -c\frac{d}{dt}\oint_G (\Lambda\vec{J} - \sigma\Lambda\dot{\vec{E}})\cdot\vec{dl}$$

Integration with respect to time gives

$$\iint_S \vec{H}\cdot\vec{dS} + c\oint_G (\Lambda\vec{J} - \sigma\Lambda\dot{\vec{E}})\cdot\vec{dl} = \text{Constant}$$

Now if initially and finally stationary conditions prevail $E \rightarrow 0$

Consequently
$$\oint_G \vec{E}\cdot\vec{dl} \rightarrow 0$$

In non stationary condition we can again neglect the value of the above integral by placing G well into the deep superconductor where fields are negligibly small.

We can summarize the results in the section up to now as

$$\frac{d}{dt}\Phi_c = 0 \qquad \text{or} \qquad \Phi_c = \underset{(\text{in time})}{\text{constant}} \qquad 2.42$$

Relation 2.42 is valid whether there is a hole enclosed by the curve G or not so far as the surrounding body is superconducting. This result is obviously true for hole or no hole enclosed by G yet we can ask ourselves the question; is 2.42 a specific enough result for the case of no hole enclosed by G, if not so could a more specific result be obtained?

For the case of No hole: Maxwell's equation $c\,\nabla\times\vec{E} = -\dot{\vec{H}}$ is of no use for this case using the superconductive relation

$$c\,\nabla\times\Lambda\vec{J} = -\vec{H}$$

we obtain by Stokes theorem

$$\iint_S \vec{H}\cdot d\vec{S} = -c\oint_{C_1} \Lambda\vec{J}\cdot d\vec{l}$$

or

$$\Phi_c = \iint_S \vec{H}\cdot d\vec{S} + c\oint_{C_1} \Lambda\vec{J}\cdot d\vec{l} = 0$$

therefore for curves such as C_1 enclosing no holes the fluxoid Φ_c vanishes

therefore for no holes enclosed by C_1 we have $\Phi_c = 0$

<u>The General Relation for the Fluxoid</u>

In order to obtain a general relation for the fluxoid that is conserved consider the following

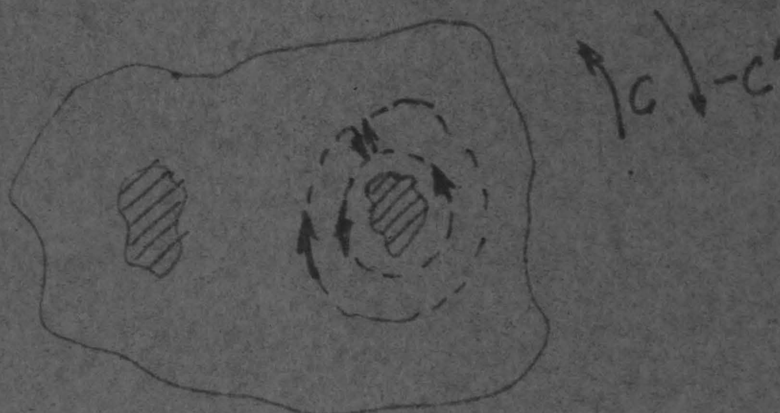

FIGURE 2.6

The curve C' can be obtained by continuously deforming C with the restrictions that C' remains in the superconductor.

Now consider the integral expression for the fluxoid Φ_c for the surface $(S - S')$

where

S = is the surface bounded by C

S^1 = is the surface bounded by C'

$$\iint_{S-S'} \vec{H} \cdot \vec{dS} + c\oint_{C} \Lambda \vec{J} \cdot \vec{dl} + c\oint_{-C'} \Lambda \vec{J} \cdot \vec{dl} = 0$$

We also have the condition

$$\oint_{-C'} F(x)\,dx = -\oint_{C'} F(x)\,dx$$

Therefore

$$\Phi_C = \iint_S \vec{H} \cdot \vec{dS} + c\oint_C \Lambda \vec{J} \cdot \vec{dl}$$

$$= \iint_{S'} \vec{H} \cdot \vec{dS} + c\oint_{C'} \Lambda \vec{J} \cdot \vec{dS} = \Phi_{C'}$$

Therefore $\Phi_{C'}$ is independent of the curve C', $\Phi_{C'}$ has the same value for all closed curves C' provided that C' like curves embrace the same hole once . Consequently we see that the "fluxoid" is a property of the hole in question and has nothing to do with the path of the curve used for integration purposes.

We can briefly express the results obtained in this section as follows"

$$\phi = \iint_S \vec{H} \cdot \vec{dS} + c\oint_C \Lambda \vec{J} \cdot \vec{dl}$$

1) ϕ is given by the above relation

2) ϕ is zero for any closed curve situated entirely in a superconductor and does not embrace a hole.

3) if the curve C embraces a hole than the value of ϕ is constant in time and does not depend on the particular

choice of curve C.

The Quantrization of the Fluxoid

From the previous results we had

$$\Phi_f = \text{const} = \iint_S \vec{B}\cdot d\vec{s} + \frac{4\pi\lambda^2}{c}\oint \vec{J}\cdot d\vec{l}$$

where $\lambda^2 = \dfrac{mc^2}{4\pi ne^2}$

Using the well known relation of the vector potential for the magnetic field H. $\vec{H} = \nabla\times\vec{A}$ and applying Stokes theorem we obtain

$$\mu\oint \vec{A}\cdot d\vec{l} + \frac{4\pi}{c}\lambda^2\oint \vec{J_s}\cdot d\vec{l} = \Phi_f$$

and

$$\vec{J_s} = n\vec{v}e$$

we obtain

$$\frac{c}{e}\oint \vec{p}\cdot d\vec{l} = \Phi_f$$

where $\vec{p} = m\vec{v} + \dfrac{\mu e}{c}\vec{A}$

m = mass of the electron

v = velocity of electron

e = charge of electron

c = velocity of light in free space

μ = magnetic susceptibility

$\vec{A}$ = magnetic vector potential

Alas! the expression for $\vec{p}$ is the expression for the momentum of an electron moving with velocity $\vec{v}$ in a magnetic field whose vector potential is $\vec{A}$.

A body moving in a closed path and having a linear momentum $\vec{p}$

has an angular momentum given by

$$|\Omega| = \oint \vec{P} \cdot \vec{dl}$$

$|\Omega|$ is the angular momentum (magnitude of).

We know from the quantum theory of matter that the angular momentum of circulating electrons are quantized, consequently, it is not unreasonable to expect that the angular momentum of electrons circulating in a superconductor is quantized.

The angular momentum of the circulating electrons being quantized, the "fluxoid" Φ_f is also quantized. Therefore

$$|\Omega| = nh$$

$$\frac{e}{c}\Phi_f = nh$$

n = integer

h = Planck's constant

therefore

$$\Phi_f = \iint \vec{B} \cdot \vec{ds} + \frac{4\pi\lambda^2}{c} \oint \vec{J} \cdot \vec{dl} = \frac{nhc}{e} \qquad 2.43$$

The result 2.43 predicted by F. London is replaced by the following in 1959 in view of the condensation or pairing of electrons according to the B.C.S. theory.

$$\Phi_f' = \frac{nhc}{2e} \qquad 2.44$$

In the new relation we have $2e$ instead of e of the older relation. This is because in the B.C.S. theory of superconductivity the supercurrent is transported by pairs of electrons, called the Cooper pairs, rather than single electrons.

APPLICATIONS

Lets investigate the values of current and the trapped flux for the simple geometry of a superconducting cylinder of wall thickness "d" and inner radius "r".

$$\Phi_f = \pi r^2 \bar{H} + \frac{4\pi\lambda^2}{c}\frac{i}{d}2\pi r \qquad \text{by 2.41}$$

where $\bar{H}$ is the average field intensity trapped in the cylinder.
i - the current circulating in unit lenght of the walls.

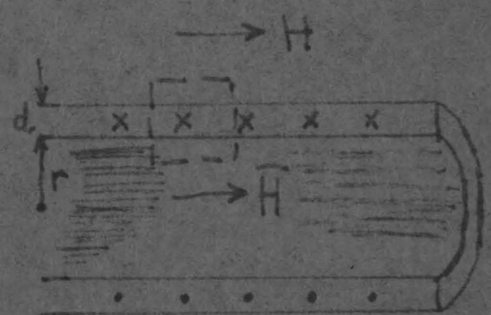

FIGURE 2.7

To find the magnitude of the trapped fluxoid in the cylinder we have to have an expression relating i to H or vice versa. Using the well known Maxwell relation

$$\frac{4\pi}{c} J = \int \vec{H} \cdot \vec{dl}$$

we obtain $\frac{4\pi}{c} i \simeq \bar{H}$

for $r \gg d$

$$\phi_f = \pi r^2 \bar{H} + \frac{4\pi\lambda^2}{c}\frac{i}{d}2\pi r \simeq \pi r^2 \bar{H}$$

but $\quad \bar{H} = \frac{4\pi}{c} i$

therefore $\quad \pi r^2 \bar{H} = \frac{nhc}{2e}\left(1 + \frac{2\lambda^2}{rd}\right)^{-1}$ follows

for a thick walled cylinder $\quad \lambda^2 \ll rd$

Then

$$\pi r^2 \bar{H} \simeq nhc/2e \sim n\,(2\times 10^{-7})\ \text{gauss cm}^2$$

In the light of the above results we see that for a cylinder of radius $r = 10\mu$, the trapped field is about 0.063 oersted. This is a detectable value. The experiments by Deaver and Fairbank and others verified this.*

These experiments directly establishes the correctness of the factor "2" added to the expression 2.43 on the grounds of the B.C.S. model of superconductivity.

2.3c. PENETRATION OF FIELD IN THE SUPERCONDUCTING STATE

We have previously mentioned that the London two fluid theory predicts the quick decay of magnetic fields after entering into the superconducting material. In other words the magnitude of the applied field is reduced to a negligible value. This is in accordance with the Meissner effect since the field penetration is quite small.

To see the orders of the penetration depth let us consider two simple geometries:

1. An infinite slab of thickness 2a subjected to a magnetic field H_o parallel to its surface.

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* 1. Deaver & Fairbank 1961 2. Doll & Näbauer 1961.

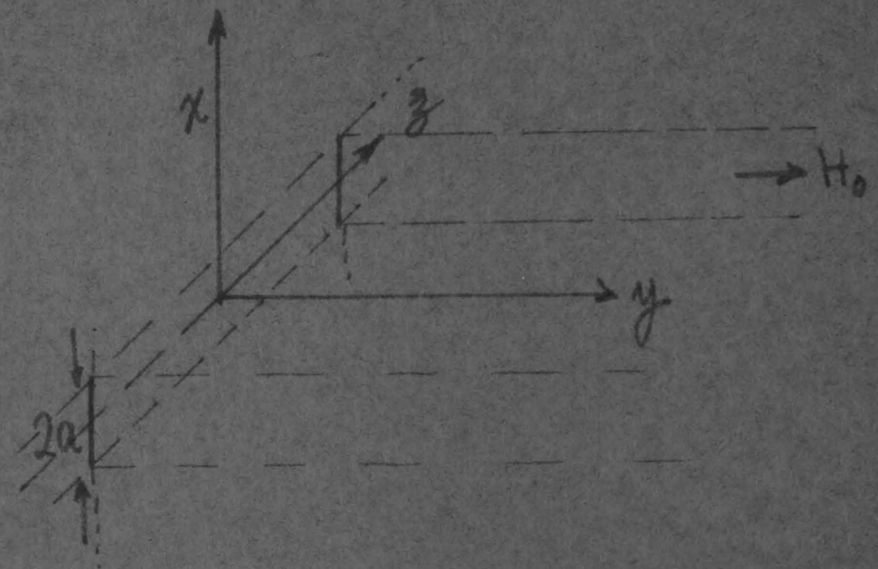


FIGURE 2.8

Equation $\lambda^2 \nabla^2 \vec{H} = \vec{H}$

reduces to $\frac{d^2 H_y}{dx^2} = \frac{H_y}{\lambda^2}$

and the Boundary conditions are $H_y = H_o$ at $x = \pm a$

The general solution is $H_y = A e^{-x/\lambda} + B e^{x/\lambda}$

Inserting the boundary conditions we obtain

$$H_y = H_o \frac{\cosh x/\lambda}{\cosh a/\lambda}$$

2. A thin walled cylinder with Magnetic field H applied along its axis.

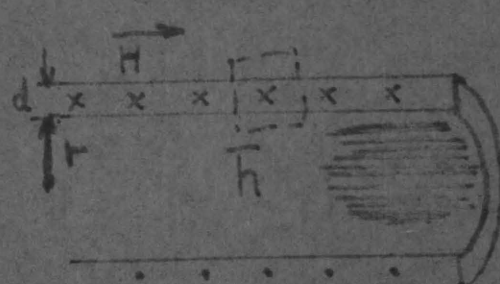


FIGURE 2.9

When H is applied along the axis of the cylinder currents are induced in the walls of the cylinder so as to prevent the penetration of the magnetic field into the walls and the internal core. However, for a think walled cylinder (i.e., wall thickness comparable to the penetration depth) some field will appear in the core.

For a superconducting cylinder of radius r and wall thickness d subjected to an external field $\vec{H}$ we assume that the field appearing in the core is in the same direction with the externally applied field and of magnitude $\bar{h}$. We denote the average screening current density by $\vec{J}$.

By relation 2.44 we have

$$\Phi_f = \int \vec{B}\cdot d\vec{s} + \frac{4\pi\lambda^2}{c}\oint \vec{J}\cdot d\vec{l} = \frac{nhc}{2e}$$

$$\Phi_f = \bar{h}\pi r^2 + \frac{4\pi}{c} 2\pi r J = 0 \qquad 2.45$$

From Ampere's law

$$\oint_{\mathcal{L}} \vec{H}\cdot d\vec{l} = \frac{4\pi}{c}\iint \vec{J}\cdot d\vec{s}$$

we integrate H around the curve $\mathcal{L}$ and obtain

$$H - \bar{h} = \frac{4\pi J d}{c} \qquad 2.46$$

When $d \ll r$ and $d \leq \lambda$ is the case, then the variation of the screening current can be ignored and by elliminating J; using 2.45 and 2.46 we obtain

$$\frac{\bar{h}}{H} = \frac{2\lambda^2}{2\lambda^2 - rd}$$

We notice that for $r \gg \frac{\lambda^2}{d}$

$\frac{\bar{h}}{H}$ reduces to zero.

and for an "infinitely thin" film $\frac{\bar{h}}{H}$ approaches unity.

2.3d. THE ENERGY THEOREM

We state the well known energy theorem of the classical Electro-Magnetic theory as

$$\nabla \cdot \frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right] + \frac{\partial}{\partial t}\frac{1}{8\pi}\left(H^2+E^2\right) = -\left(\vec{J}\cdot\vec{E}\right)$$

where $\frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]$ represents the energy current

$\frac{1}{8\pi}\left(E^2+H^2\right)$ represents the energy density of the Electro-Magnetic field.

$\vec{J}\cdot\vec{E}$ represents the work done by the field on the moving electric charges.

In superconductors the term $(J\cdot E)$ which represents the dissipation (joule Heat) is zero for the supercurrent.

Therefore $\vec{J}\cdot\vec{E} = \left(\vec{J}_s + \vec{J}_n\right)\cdot\vec{E}$

Using $\frac{\partial}{\partial t}\Lambda\vec{J}_s = \vec{E}$ for the supercurrent

and $\vec{J}_n = \sigma\vec{E}$ for the normal current

we have

$$\vec{J}\cdot\vec{E} = \frac{\partial}{\partial t}\left(\frac{1}{2}\Lambda J_s^2\right) + \frac{1}{\sigma}J_n^2$$

therefore the energy equation can be written as

$$\nabla\cdot\left(\frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]\right) + \frac{\partial}{\partial t}\left(\frac{1}{8\pi}\left(H^2+E^2\right) + \frac{1}{2}\Lambda J_s^2\right) + \frac{1}{\sigma}J_n^2 = 0 \qquad 2.47$$

Here $\frac{1}{2}\Lambda J_s^2$ represents the kinetic energy of the supercurrent.

$\frac{1}{2}\Lambda J_s^2$ is positive when supercurrent is generated and negative when the supercurrent is switched off. We may further note that this term does appear for superconductors only, it is also there for the normal conductors in principle, however, for normal conductors it is negligible compared to the magnetic energy density.

The term on the R.H.S. of 2.47 viz. $-1/\sigma\, J_n^2$ is either negative or zero. This clearly represents the energy dissipation by the formation of heat due to the normal current. In superconductors at low frequencies this term is zero but as we go up to higher frequencies it cannot be neglected.

2.3f. THE INTERMEDIATE STATE

Previously we have defined the intermediate state as the state in which both superconducting and normal states coexist in a single body. This was about all that was said about the intermediate state up to now except that we have investigated the superconducting sphere in a magnetic field H_0 and found out that the field or the equator is different from other regions on the sphere. This suggests that a superconducting sphere subjected to a magnetic field can be in the intermediate state for a range of values of the applied field. First, we try to figure out how does the intermediate state come to being in a superconducting sphere subjected to a magnetic field H_0.

Let's apply a magnetic field that is just enough to produce a magnetic field of H_c at the equator say H_∞. Now an increase in H_∞ will destroy the superconductivity at the equator (just behind it) as a result of this the magnetic field will penetrate into the sphere releaving the concentration of the force lines which in turn result in the establishment of the superconductivity again. If we consider a further increase in the value of the applied field the same set of events will take place and the superconducting region will advance deeper into the sphere. This behaviour suggests that the result of the set of events described above will lead to the formation of alternating domains of normal and superconducting states.

In order to analyize this state one encounters a differential equation with very complex boundary conditions to be solved. The boundary conditions constitude of the shape of the boundaries and their number (normal-superconducting).

This is a hopeless situation and under these circumstances the line of attack to the problem is shifted into looking for the possibility of developing a theory on the mean values neglecting the complexities of the micro-structure.

In this theory the following is the natural assumption:

1. The details of the sizes and structure of these domains will be of no importance when average values are taken.

THE MAGNETO-STATICS OF THE INTERMEDIATE STATE

To simplify matters the penetration of the field into the superconducting region is assumed to be negligible, i.e, the domains are large enough compared to the penetration depth. So far as the boundary condition we see that the perpendicular magnetic field is continuous but that may not be the case for the parallel component of H. The field in the superconducting domain is zero and the perpendicular component of the magnetic field is continuous, therefore the perpendicular component of the magnetic field in the normal side of the interphase is also zero.

We have the general relation for the magnetic fields viz.

$$\nabla \cdot \vec{H} = 0$$

This means that the magnetic lines cannot end anywhere or no magnetic charges can exist separately. The fields in the normal domains must be parallel to the normal-to-superconducting boundaries near the interphase yet magnetic lines cannot end anywhere. This determines the shape of the domains to be either channel or lamina type. Here we pose for a moment to see if everything fits to its place, and we see that everything fits to its place well but the very first assumption might ruin everything, (if not true).

In fact there is nothing to worry about since Shalnikov and Meshkovsky showed that the domain sizes are of the order of 1m.m. diameter which is much greater than the penetration depth. Therefore we proceed by letting $\vec{B}$ the magnetic induction, then

$$\vec{B} = \frac{1}{V}\int_V H\,dV = \overline{H}$$

where V is the volume just large enough to contain a large number of superconducting and normal domains. If a is the volume fraction which is normal conducting, then

$$|\vec{B}| = a H_c \qquad (0 < a \leq 1)$$

and $H = 0$ for superconducting regions

$H = H_c$ for normal regions.

The free-energy density of a system having magnetic induction $\vec{B}$ has two parts:

a) Contributions of the superconducting region.

b) Contribution of the normal region.

a) We have $\Phi_0(T) - \frac{1}{8\pi} H_c^2(T)$ for g_s

$(1-a)$ fraction of the superconducting regions

$$f_s \text{ per cc} = (1-a)\left[\Phi_0(T) - \frac{1}{8\pi} H_c^2(T)\right]$$

b) $$\Phi_0(T) + \frac{1}{8\pi} H^2$$

a fraction of normal regions

$$f_n \text{ per cc} = a\left[\phi_0(T) + \frac{1}{8\pi} H_c^2(T)\right]$$

Therefore the total free energy density

$$f = \phi_0(T) + (2a-1) H_c^2/8\pi$$

but we also have

$$|\vec{B}| = a H_c$$

therefore

$$f = \phi_0(T) - \frac{H_c^2}{8\pi} + \frac{H_c|B|}{4\pi}$$

where f = valid for $|B| \leq H_c$

This relates the free energy density of the superconductor in the intermediate state as a function of B and T. In the classical theory of Electricity and Magnetism, the Magnetic field $\vec{H}$ is defined as follows:

$$\left.\begin{array}{l}\text{The work per cc required for}\\ \text{changing B to B + dB.}\end{array}\right\} = \frac{1}{4\pi}(\vec{H}\cdot d\vec{B})$$

if the work is reversible and isotermal

$$df = \frac{\vec{H}\cdot d\vec{B}}{4\pi}$$

therefore we have $H = H_c \dfrac{\vec{B}}{|\vec{B}|}$ for $0 < |\vec{B}| \leq H_c$

since $f = \phi_0(T) - \dfrac{H_c^2}{8\pi} + H_c \dfrac{|\vec{B}|}{4\pi}$ for $|B| \leq H_c$

and

$$\vec{H} = H_x + H_y + H_z$$

$$H_x = 4\pi \frac{\partial f}{\partial B_x} = 4\pi \frac{H_c}{4\pi} \frac{\partial}{\partial B_x}|\vec{B}|$$

therefore

$$\vec{H} = H_c\left\{\frac{\partial}{\partial B_x} + \frac{\partial}{\partial B_y} + \frac{\partial}{\partial B_z}\right\}|\vec{B}|$$

but

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

therefore

$$\frac{\partial|\vec{B}|}{\partial B_x} = \frac{B_x}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{B_x}{|\vec{B}|}$$

therefore $$\vec{H} = H_c \left\{ \frac{\vec{B_x + B_y + B_z}}{|B|} \right\} = H_c \frac{\vec{B}}{|B|}$$

Therefore in the intermediate state $\vec{H}$ is in the same direction as $\vec{B}$ and has magnitude H_c independent of $\vec{B}$. For the case of $\vec{B} = 0$ $\vec{H}$ looses its significance because for $\vec{B} = 0$ superconducting regions become macroscopically very large and variables $\vec{H}$ and $\vec{B}$ are not legitimate anymore.

Below we have H as a function of B in the intermediate state:

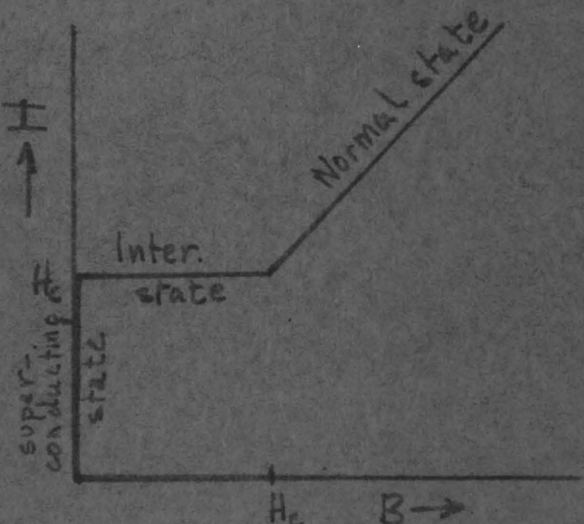


FIGURE 2.10

$\vec{H} = H_c \frac{\vec{B}}{|B|}$ is a non linear relation therefore the principle of superposition cannot be used; this introduces considerable mathematical diffculties. However the fact that $|\vec{H}| = |H_c|$ introduces some simplifications.

Let's consider the following Magneto-static case ($\vec{J} = 0$).

We have $$\vec{H} = H_c \frac{\vec{B}}{|B|} \qquad \text{for} \quad (0 < |B| \leq H_c)$$

$H^2 = H_c^2 =$ constant

therefore $\nabla \vec{H}^2 = 0$

but $\nabla \vec{H}^2 = 2\vec{H} \cdot \nabla \vec{H} + 2\left[\vec{H} \times \nabla \times \vec{H}\right]$

and $\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} = 0$

therefore we obtain

$$(\vec{H} \cdot \nabla)\vec{H} = 0 \qquad 2.48$$

Now $k(\vec{H} \cdot \nabla)H_x$ where k - small constant represents the change in the x component of the magnetic field $\vec{H}$ for an infinitely small displacement in the direction of $\vec{H}$. According to 2.48 if vector $\vec{H}$ is followed along its own lines of force, its cartesian components does not change. Hence $\vec{H}$ lines must be straight lines.

2.3g. THE TRANSITION OF A SUPERCONDUCTING SPHERE IN A MAGNETIC FIELD

Consider a superconducting sphere of radius R in a homogeneous magnetic field of strength H_0 at infinity. We have previously calculated the field and current for the pure superconducting case (i.e. for $H_0 < 2/3\ H_c$).

Now we shall investigate the field for the case $(2/3\ H_c \leq H_0 \leq H_c)$. For this we retain the previous expressions for the field

$$\left.\begin{aligned} H_r = B_r &= \left(H_0 + \frac{2M}{r^3}\right)\cos\theta \\ H_\theta = B_\theta &= \left(-H_0 + \frac{M}{r^3}\right)\sin\theta \end{aligned}\right\} \text{for } r \geq R$$

We have to impose the following restrictions on these equations

a. The lines of force of $\vec{H}$ inside the sphere must be straight lines.

b. $|\vec{H}| = H_c$

c. $\vec{H}$ and $\vec{B}$ must be in the same direction.

In view of these restrictions it is reasonable to try a solution where $\vec{H}$ and $\vec{B}$ are parallel to the axis of the external field and in the solution $\vec{B}$ must be homogeneous in the sphere parallel to the z-axis (ie. B_z = const).

Therefore we try

$$H_r = H_c \frac{B_z}{|B_z|}\cos\theta \qquad B_r = B_z\cos\theta$$

$$H_\theta = -H_c \frac{B_z}{|B_z|}\sin\theta \qquad B_\theta = -B_z\sin\theta$$

for $r \leq R$.

The boundary conditions require that B_r and H_o to be continuous at $r = R$.

$$B_r = B_z\cos\theta = \left(H_o + \frac{2M}{R^3}\right)\cos\theta$$

Therefore

$$B_z = H_o + \frac{2M}{R^3} \qquad 2.49$$

$$H_\theta = \left(-H_o + \frac{M}{r^3}\right)\sin\theta = -H_c\frac{B_z}{|B_z|}\sin\theta$$

Therefore

$$-H_o + \frac{M}{R^3} = -H_c\frac{B_z}{|B_z|} \qquad 2.50$$

Eliminating M by using 2.49 and 2.50 we obtain

$$B_z = 3H_o - 2H_c\frac{B_z}{|B_z|}$$

$$= \begin{cases} 3H_o - 2H_c & \text{if } B_z > 0 \text{ or for } \tfrac{2}{3}H_c < H_o < H_c \\ 3H_o + 2H_c & \text{if } B_z < 0 \text{ or for } -\tfrac{2}{3}H_c > H_o > -H_c \end{cases}$$

which give for M

$$M = \frac{R^3}{2}\left(B_z - H_o\right)$$

$$= \begin{cases} -R^3\left(H_c - H_o\right) & \text{for } \tfrac{2}{3}H_c < H_o < H_c \\ R^3\left(H_c + H_o\right) & \text{for } -\tfrac{2}{3}H_o > H_o > -H_c \end{cases}$$

Equations

$$H_0 + \frac{2M}{R^3} = B_z$$

and

$$-H_0 + \frac{M}{R^3} = -H_c \frac{B_z}{|B_z|}$$

have no solutions unless $\frac{2}{3}H_0 \leq |H_0| \leq H_c$

at $\frac{2}{3}H_c = |H_0|$ B_z vanishes.

For $|H_0| < \frac{2}{3}H_c$ theory of the pure superconducting state is applicable.

For $|H_0| < \frac{2}{3}H_c$

$$M = -\frac{H_0 R^3}{2}$$

Since penetration depth is taken to be negligible and $B_z = 0$.

The magnetic moment per cc

$$I = \frac{M}{\frac{4\pi}{3}R^3} = \begin{cases} -\frac{3}{4\pi}(H_c - H_0) & \text{for } \frac{2}{3}H_c \leq H_0 \leq H_c \\ -\frac{3}{8\pi} & \text{for } |H_0| < \frac{2}{3}H_c \end{cases}$$

and H pole = 0

H equator = $\frac{2}{3}H_0$

We can summarize our results in the following diagram

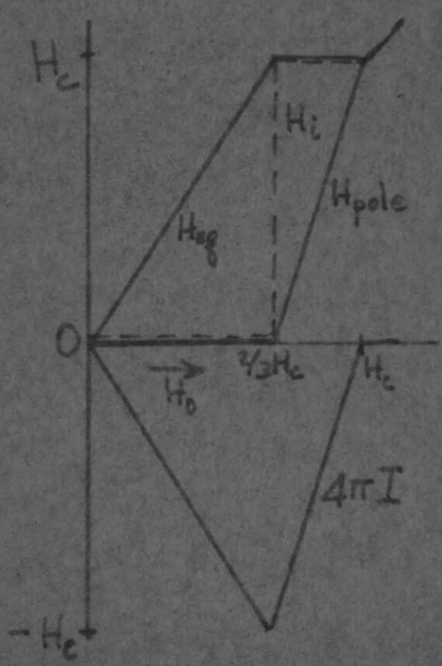


FIGURE 2.11

where H_i = field inside the sphere

H_{eq} = field near the equator

H_{pole} = field near the pole

I = the magnetization

a) $H_i = 0$ for $|H_o| < 2/3\, H_c$

b) $H_i = H_c$ for $|H_o| > 2/3\, H_c$

c) $H_i = H^1$ for $|H_o| > 4/3\, H_c$

where H^1 is the field outside the sphere but in the close proximity of the equator. The reason for this is that the field for H_o can be described by the macroscopic vectors $\hat{H}$ and $\hat{E}$ and the boundary conditions that H parallel is continuous.

For $|H_o| < 2/3\, H_c$ (pure superconducting state) $H_{eq} = 3/2\, H_o$ just outside the equator and the field inside is zero. In this case $\nabla \times \vec{H} \neq 0$ and the macroscopic current is present implying that the field can no longer be described by H, B and μ alone.

2.3h. ELECTRIC FIELDS IN THE INTERMEDIATE STATE

In this section we shall consider the particular case where the electric field is stationary, perpendicular to the local magnetic induction $\vec{B}$ and parallel to the macroscopic current $\vec{J}$.

We start with the assumptions:

a) Diameter of the domains is large in comparison to the penetration depth and in the superconducting region we have H = 0 = E.

b) Surface currents and surface charges keep the electric and magnetic fields outside of these regions.

Therefore we have the continuity of the perpendicular component of

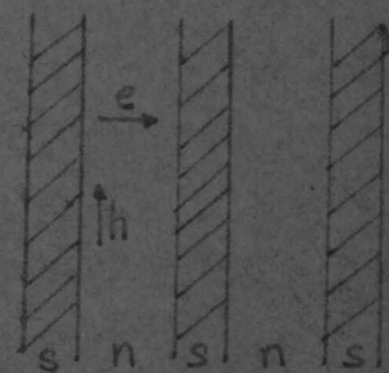
$\vec{E}$ to the boundary at the surface of the domains.

In superconducting regions where H = O = E, the normal component of $\vec{H}$ and the parallel component of $\vec{E}$ are zero at the surface of the normal conducting regions next to superconducting region. Therefore in normal regions neighbouring superconducting regions we have

H = parallel to the boundaries

E = normal to the boundaries

c) Finally we assume that the boundaries exist in an approximately parallel orientation as shown in the figure. Then the two field vectors are perpendicular to each other as in the figure.



We proceed with $E = a\bar{E}$

and $\beta = a\bar{H}$

Figure 2.12

where a is the volume fraction of the normal conducting lamine Therefore (1- a) is the volume fraction of the superconducting regions.

First we investigate the stresses on the boundaries separating the two phases:

The pressure due to Maxwell stresses, i.e., the push due to the magnetic and the pull due to the electric fields given by

$$\frac{1}{8\pi}\left(\bar{H}^2+\bar{E}^2\right)$$

But to cause the expulsion of flux we must have another pressure called the "Meissner Pressure". Therefore we can say that whenever the Meissner pressure is greater than Maxwell pressure we have the Meissner effect.

The Meissner pressure is given by $1/8\pi \; H_c^2$

This is implied by the free energy term $-\left(H_c^2/8\pi\right)V$

Since energy or work done = (pressure) x V (change in volume).

Introducing the macroscopic quantities B and E and equating the pressures

$$\left|\sqrt{B^2-E^2}\right| = a H_c$$

but $0 \leq a \leq 1$

therefore $0 \leq B^2 - E^2 \leq H_c^2$

Lets now investigate the free energy.

a) Free energy filling a of unit volume of the metal and localized in normal conducting regions is

$$f_1 = a\left[\phi_0(T) + 1/8\pi\,(H^2 - E^2)\right]$$

b) The contribution to free energy by the superconducting regions

$$f_2 = (1-a)\left[\phi_0(T) - \frac{1}{8\pi} H_c^2\right]$$

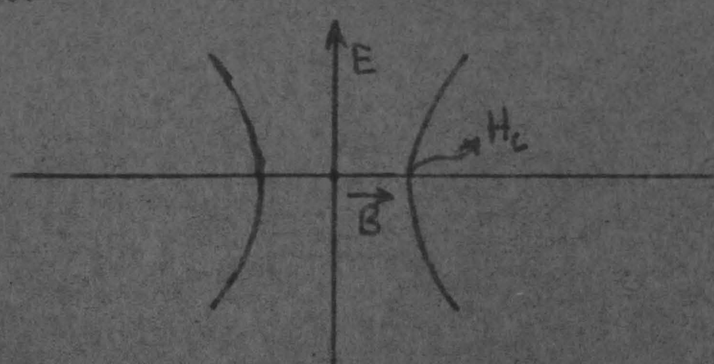
Therefore the total free energy per unit volume given by

$$f = \phi_0 - \frac{1}{8\pi} H^2 + \frac{1}{4\pi}\frac{H_c B^2}{\sqrt{B^2+E^2}} \qquad 2.51$$

for

$$0 \leq B^2 - E^2 \leq H_c^2$$

normal sate intermediate state normal state



In the classical theory we have

$$H_k = 4\pi\left(\frac{\partial f}{\partial B_k}\right)_D \quad \text{and} \quad E_k = 4\pi\left(\frac{\partial f}{\partial D_k}\right)_B \qquad 2.52$$

since $$f(B, D, T) = \frac{1}{4\pi}\int_0^D \vec{E}\cdot d\vec{D} + \frac{1}{4}\int_0^B \vec{H}\cdot d\vec{B} + \phi(T)$$

$$\left.\frac{\partial f}{\partial B_k}\right|_D = \frac{1}{4\pi} H_k \quad \text{and} \quad \left.\frac{\partial f}{\partial D_k}\right|_B = \frac{1}{4\pi} E_k$$

Here we have supposed that f is given as a function of $\vec{D}$ and $\vec{B}$ however f is obtained as a function of $\vec{E}$ and $\vec{B}$ only. Therefore we cannot employ 2.51 to obtain $\vec{H}$ by means of 2.52 because $\vec{D}$ must be kept constant when differentiating f. To introduce the desired variables the usuall Legendre transformations are employed.

The Lagrange function L (E,B) is defined as

$$f = L + \frac{1}{4\pi}(\vec{E}\cdot\vec{D}) \qquad 2.53$$

First calculating $$E_i = 4\pi \left.\frac{\partial f}{\partial D_i}\right|_B$$

we have $$E_i = 4\pi \sum_k \left(\frac{\partial L}{\partial E_k}\right)_B \left(\frac{\partial E_k}{\partial D_i}\right)_B + E_i + \sum_k \left(\frac{\partial E_k}{\partial D_i}\right)_B D_k$$

therefore

$$4\pi \frac{\partial L}{\partial E_k} = -D_k \qquad 2.54$$

Secondly, $$H_i = 4\pi\left(\frac{\partial f}{\partial B_i}\right)_D$$

$$= 4\pi\left(\frac{\partial L}{\partial B_i}\right)_E + 4\pi \sum_k \left(\frac{\partial L}{\partial E_k}\right)_B \left(\frac{\partial E_k}{\partial B_i}\right)_D + \sum_k \left(\frac{\partial E_k}{\partial B_i}\right)_D D_k$$

Using 2.54 we obtain

$$H_i = 4\pi\left(\frac{\partial L}{\partial B_i}\right)_E \qquad 2.55$$

since $\sum\left(\frac{\partial L}{\partial E_k}\right)_B\left(\frac{\partial E_k}{\partial B_i}\right)_D + \sum_k\left(\frac{\partial E_k}{\partial B_i}\right)D_k$

is equal to $\sum_k - D_k\left(\frac{\partial E_k}{\partial B_i}\right)_D + \sum_k\left(\frac{\partial E_k}{\partial B_i}\right)_D D_k = 0$

and 2.55 follows.

According to 2.54 and 2.53.

$$f = L - \sum_k E_k\left(\frac{\partial L}{\partial E_k}\right)_B$$

and

$$L = f + \sum_k E_k\left(\frac{\partial L}{\partial E_k}\right)_B$$

and

$$L = \Phi_0 - \frac{1}{8\pi}H_c^2 + \frac{1}{4\pi}H_c\sqrt{B^2-E^2} + \text{const}\,|\vec{E}|$$

Putting constant equals zero to have $D = E = 0$ we obtain

$$\left.\begin{aligned} D_k &= -4\pi\frac{\partial L}{\partial E_k} = \frac{E_k}{\sqrt{B^2-E^2}}H_c \\ H_k &= 4\pi\frac{\partial L}{\partial B_k} = \frac{B_k}{\sqrt{B^2-E^2}}H_c \end{aligned}\right\} \quad ...\ 2.56$$

and these transformations imply the identity

$$\left(H^2 - D^2 - H_c^2\right)\left(B^2 - E^2\right) = 0 \qquad 2.57$$

When $B^2 - E^2 = 0$ in 2.57 we have the pure superconducting state. Free energy given by 2.51 goes to infinity if $\vec{E}$ and $\vec{B}$ are not separately zero. Under these conditions $\vec{H}$ and $\vec{D}$ are not defined by relations 2.56.

Now consider relation 2.57 for the intermediate state

$$B^2 - E^2 \neq 0$$

therefore

$$H^2 - D^2 - H_c^2 = 0$$

or

$$H^2 - D^2 = H_c^2 \qquad 2.58$$

Due to the existance of this identity transformations 2.56 cannot be inverted, ie. ($\vec{E}$ and $\vec{B}$ cannot be expressed as functions of $\vec{H}$ and $\vec{D}$). Now as implied by Equation 2.58 $|\vec{H}|$ and $|\vec{D}|$ cover a one dimentional region where as $|\vec{E}|$ and $|\vec{B}|$ extend over a two dimentional field. Therefore, there is no way to establish a one to one correspondence between ($\vec{E}$, $\vec{B}$) and ($\vec{H}$,$\vec{D}$).

The normal state is reached if $a = 1$ or for these conditions the free energy function is

$$f_n = \Phi_0 + \frac{1}{8\pi}(B^2 + E^2)$$

where f_n - is the free energy of the normal state.

$$L_n = \Phi_0 + \frac{1}{8\pi}(B^2 - E^2)$$

where L_n is the Lagrange function with continuous derivatives.

On the transition line to the normal state we obtain H = B and D = E, but

$$f = L - E \cdot \frac{\partial L}{\partial E}$$

Expressing f as a function of B and D using transformations 2.56 we get

$$f = \text{constant} + \frac{1}{4\pi}|B|\sqrt{H_c^2 + D^2}$$

The transformation defined by f.

$$H_k = 4\pi \frac{\partial f}{\partial B_k} = \sqrt{H_c^2 + D^2}\, \frac{B_k}{|B|}$$

$$E_k = 4\pi \frac{\partial f}{\partial D_k} = \frac{1}{\sqrt{H_c^2 + D^2}}|B|\, D_k$$

This is equivalent to transformation 2.79. We can also express free energy in terms of $\vec{H}$ and $\vec{E}$.

$$f = \text{const} + \frac{1}{4\pi}\vec{B}\cdot\vec{H}$$

Here we can mention that a close analogy between the thermodynamical and electromagnetic variables exist as illustrated by thermodynamical pairs of variables.

(V,S) ; (V,T) ; (p, S) ; (p, T)

where each of these form pairs can be considered as independent variables

V = volume

S = entropy

T = temperature

p = pressure

and the transformations

$dU = TdS - pdV$ $\qquad$ $dH = TdS + Vdp$

$dF = SdT - pdV$ $\qquad$ $dG = SdT + Vdp$

where

U is the energy

F is the Helmholtz free energy

H is the entholpy

G is the Gibbs thermodynamical potential

Correspondingly in the Electromagnetic theory we have the pairs

(D,B) ; (E,B) ; (D,H) ; (E,H)

$4\pi\, df = E\cdot dD + H\cdot dB$

$4\pi\, dl = -D\cdot dE + H\cdot dB$

where

f = free energy = $f\,(D\ B)$.

$L\,(E,B) = f - \frac{1}{4\pi}\vec{D}\cdot\vec{E}$ Lagrangian the other two potentials are $g = f - \frac{1}{4\pi}\vec{H}\cdot\vec{B}$

Consequently

$$4\pi\, dg = \vec{E}\cdot d\vec{D} - \vec{B}\cdot d\vec{H} \qquad 2.59$$

but we had $f = \text{constant} + \frac{1}{4\pi}\vec{B}\cdot\vec{H}$

and $g = f - \frac{1}{4\pi}\vec{H}\cdot\vec{B}$

therefore $g = \text{constant}$ and 2.59 reduces to

$$\vec{E}\cdot d\vec{D} - \vec{B}\cdot d\vec{H} = 0 \qquad 2.60$$

but now $\vec{D}$ and $\vec{H}$ are not independent variables because $H^2 - D^2 = H_c^2$

and we have

$$\vec{D}\cdot d\vec{D} - \vec{H}\cdot d\vec{H} = 0 \qquad 2.61$$

2.60 and 2.61 yields $\frac{E_k}{B_l} = \frac{D_k}{H_l}$

or $\frac{D_k}{E_k} = \frac{H_l}{B_l} = K$ (independent of k and l)

by $H^2 - D^2 = H_c^2$

K is determined as $K = \frac{H_c}{\sqrt{B^2 - E^2}}$

We still need a relation between the current and the field. For that we have $\vec{J}_c = \sigma_m \vec{E}$

where σ_m represents the conductivity in a magnetic field which destroys superconductivity. (Notice that this has nothing to do with the normal current in the pure superconducting state.)

$$\vec{J}_c = \sigma_m \vec{E} = \frac{\sigma_m \vec{E}}{\sqrt{B^2 - E^2}} H_c$$

This gives the Joule heat as

$$\vec{E}\cdot\vec{J} = \sigma_m \frac{E^2}{\sqrt{B^2 - E^2}}$$

Where J_c is the component of J parallel to E.

$$\sigma' = \sigma_m \frac{H_c}{\sqrt{B^2 - E^2}}$$

σ' is the anisotropic conductivity in the intermediate state.

Lastly we have the boundary conditions which are the usual Maxwellian ones.

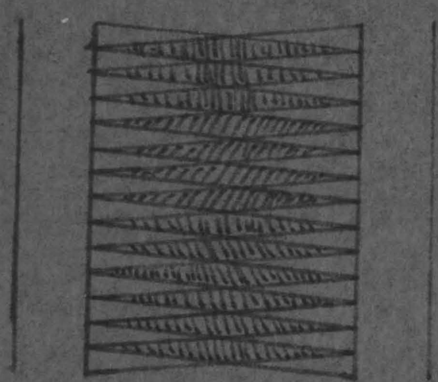
In general $E \ll B$ and E^2 can be neglected in comparison to B^2.

Therefore we can describe the intermediate state for $(0 < |B| < H_c)$ by

$$\left.\begin{aligned} \vec{H} &= \frac{\vec{B}}{|B|} H_c \\ \vec{D} &= \frac{\vec{E}}{|B|} H_c \\ \vec{J}_c &= \frac{\sigma \vec{E}}{|B|} H_c \end{aligned}\right\} \qquad \text{2.62}$$

In the theory of the intermediate state the current component was parallel to the electric field which was normal to the lamine structure. These are also supercurrents along the superconducting layers. These are shown by Shubnikov and Nakhutin.*

2.31. TRANSITION OF A WIRE IN WHICH A CURRENT FLOWS



Transition in a superconducting wire in which a current flows.

Figure 2.13

* Shubnikov and Nakhutin Nature 139, 589 (1937).

The solution for this problem consists of the application of the theory to an infinitely long cylindrical wire (radius a) in which a current of i is flowing. As we have mentioned in the opening lines of the section on the intermediate state, the magnetic field established by the current destroys superconductivity gradually. This leads to the plausible inference that the lamine must be oriented approximately normal to the axis of the wire.

The natural choice of co-ordinates is evidently the cylindrical polar co-ordinates (z, r, ϕ). Now we try to calcualte the electric field corresponding to the current. There are two cases to be considered:

a. Field produced on the surface of the wire is less than the critical field H_c.

b. Field produced on the surface of the wire is larger than the critical field H_c.

a) For $i < \frac{ac}{2} H_c$

We have the pure superconducting state everywhere, therefore, $E = 0$ and the critical current $i_c = \frac{ac}{2} H_c$ is defined as the current which produces H_c on the surface of the wire of radius a.

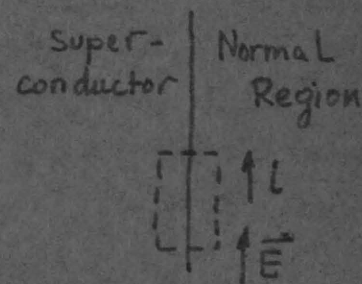
b) For $i > i_c$

For this case the field on the surface of the wire is greater than H_c, consequently, this region consists of the normal state only. This region is an annulus of a cylinder having radii $r = a$ to $R < a$.

Now, some current must flow through this normal region since otherwise the field will exceed the critical value inside this hollow cylindrical region. The electric field in the normal region given by ohm's law and $\nabla \times E = 0$ under stationary conditions in this region. Relation $\nabla \times E = 0$ implies that an electric field inside the annulus must exist (ie for $r < R$).

This can be shown by taking the integral of electric field We know that $\nabla \times E = 0$, therefore

$$\oint \vec{E} \cdot \vec{dl} = 0$$



and electric field $\vec{E}$ exists in the normal region as shown. Therefore, we see that an electric field must exist on the superconducting side so that the above conditions are satisfied.

We also note that due to the symmetry we have no dependance on z. and

$$\left(\nabla \times \vec{E}\right)_\phi = \frac{\partial E_z}{\partial r} = 0$$

which shows that the electric field has the z component ($E = E_z$) only which is constant provided that a stationary solution exists.

For $r < R$ we see that the pure superconducting state cannot exist since $E \neq 0$ consequently $\frac{dJ}{dt} \neq 0$ therefore the state inside hasnon-stationary conditions, and the state is the mixed state.

Using the symmetry considerations for the magnetic field we see that the magnetic field has the component H_ϕ only and we further have $H_\phi = H_c$ = constant in the mixed state.

This condition together with Maxwell's equation $\frac{4\pi}{c}\vec{J} = \nabla \times \vec{H}$ gives the current density $J = J_z$ as

$$J = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}\left(r H_\phi\right) = \pm \frac{c H_c}{4\pi r}$$

To obtain R we have

$$J = \frac{\sigma E H_c}{B} = \pm \frac{c}{4\pi}\frac{H_c}{r}$$

$$B = \frac{\sigma E}{J} H_c = \sigma E \frac{4\pi r}{c}$$

hence $\frac{E}{B} = \frac{c}{4\pi\sigma r} < \frac{10^{-7}}{r}$

the smallness of E/B justifies neglecting E^2 in comparison to B^2.

We also have $|B| < H_c$ or $r < R$, therefore

$$R = \frac{cH_c}{4\pi\sigma E} \qquad 2.63$$

Current i_1, in the intermediate state $0 < r < R$ is given by

$$i_1 = \int_0^R \frac{cH_c}{4\pi r} 2\pi r\, dr = \frac{1}{2} c H_c R = \frac{c^2 H_c^2}{8\pi\sigma E}$$

Current i_2 in the normal state for $(R < r < a_1)$ is given by

$$i_2 = \int_R^{a_1} \sigma E\, 2\pi r\, dr = \sigma E \pi a_1^2 - \frac{c^2 H_c^2}{16\pi\sigma E}$$

therefore the total current i given by

$$i = i_1 + i_2 = \frac{E}{\varrho_0} + \frac{\varrho_0}{E}\frac{i_c^2}{4} \qquad \text{for} \quad i \geq i_c$$

Solving for E

$$E = \frac{\varrho_0 i}{2}\left[1 \pm \sqrt{1 - (i_c/i)^2}\right] \qquad 2.64$$

but R cannot be larger than a and $q_o = (\sigma \pi a_1^2)^{-1}$

by 2.63 we have $E \geq cH_c / 4\pi\sigma a_1 = i_c \frac{\varrho_0}{2}$

therefore only the positive sign for 2.64 can have a physical significance.

For, $E < i_c\, q_o/2$ there is no solution and for; $i < i_c$ we have the pure superconducting state with $E = 0$.

Therefore E jumps from zero to $E = i_c \frac{\varrho_0}{2}$ at the point where $i = i_c$.

Therefore the resistance $q = \frac{E}{i}$ is given by

$$\varrho = \frac{\varrho_0}{2}\left[1 + \sqrt{1 - (i_c/i)^2}\right] \qquad \text{for} \quad i \geq i_c$$

This adds up to the following:

For $i < i_c$ we have the pure superconducting state where $E = 0$ and $q = 0$.

For $i = i_c$ the resistance jumps from $q = 0$ to $q = \frac{\rho_0}{2}$ and increase continuously with increasing current towards the value q_0.

In another attempt keeping i constant and considering i_c as a function of temperature (T). Thus for q as a function of T for constant i we have

$$\rho = \begin{cases} \frac{\rho_0}{2}\left[1+\sqrt{1-k^2(T_0-T)^2/i^2}\right] & \text{for } i > i_c \\ 0 & \text{for } i < i_c \end{cases}$$

In the neighbourhood of the transition temperature T_c, H_c varies approximately linearly with T. Therefore,

$i_c = \frac{ac}{2} H_c$ is also a linear function of T.

2.4 B.C.S. THEORY

Superconductivity occurs in metals or alloys which have very complex structures yet there is remarkable simplicity the way it occurs in all metals. Therefore for a qualitative understanding we may ignore the complexities of structure.

At normal temperatures the Bloch solution of Schrödingers equation for periodic potential produced by fixed ions successfully explains the observed data. At low temperatures in metals and alloys that become superconductors, the properties of the valance electrons deviate from their normal behaviour radically. The lattice seems to affect the transition temperature but it remains unchanged in the transition.

A number of experiments showed that:

1. The electronic wave functions are highly correlated.
2. The energy of condensation is small compared to the Fermi energy of electrons. Ratio $\sim 10^{-8}$.

NORMAL METAL

In the normal state the wave function is given by modulated plane waves

$$\varphi_K(\xi) = U_K(\xi)\, e^{i\mathbf{k}\cdot\bar{r}}$$

where $K = k, \sigma$

k = wave vector of the electron

σ = spin state of the electron

ξ = r, s (space and spin co-ordinates)

$U_K(\xi)$ = is a spinor with the lattice periodicity

By Pauli exclusion principle no two electrons can be in the same Bloch State $\varphi_K(\xi)$. Therefore many electron wave function can be written as

$$\Phi_N = \frac{1}{\sqrt{N!}} \sum_{\substack{\text{permutations} \\ \text{of } \xi_1 \cdots \xi_n}} (-1)^P \varphi_{K_1}(\xi_1) \ldots \varphi_{K_N}(\xi_N)$$

The energy of the entire system is then $W = \sum_{i=1}^{N} \mathcal{E}_i$

where $\mathcal{E}_i$ - is the Bloch energy of the i^{th} single electron state.

The resistence phenomena in the normal state is expained by deviations from perfect periodicity of the crystal and the lattice vibrations. Hence the contribution due to the impurities is not temperature dependent. But the contribution of the lattice due to its vibrations is obviously temperature dependent. These phenomena cause scattering of electrons from one momentum state to another resulting in randomization of electron motion in the absence of the applied field. In the superconducting state these single particle processes are restricted and the ordered current carrying state can persist in the absense of driving fields.

ELECTRON CORRELATIONS

For a normal metal the probability of finding two electrons with opposite spins and separated by a distance r, we have

$$\rho^{\uparrow\downarrow}(r) = \frac{1}{4}n^2$$

$\rho^{\uparrow\downarrow}(r)$ - Probability of finding an electron with spin down at a distance r from another having a spin up, (irrespective where the electrons 3 to N may be.)

n - density of electrons

Therefore $\rho^{\uparrow\downarrow}(r)$ depends on the density of electrons with spin up only.

It is in this respect that the superconductor differs from the normal metal. We may inquire what produces the correlation between the electrons. Coulomb repulsion between electrons prevents 2 electrons coming very close together and at normal temperatures the correlation between electrons are negligible but at low temperatures the system prefers to go into a correlated state.

What interactions are to be considered? The inspiration comes from the isotope effect; it is observed that

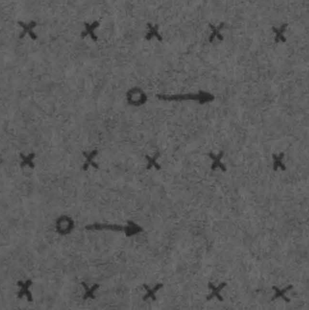
$$T_c\sqrt{M} = \text{Const}$$

T_c = critical (transition temperature)

M = mass of the ions that make up the lattice.

If the ions are considered to be stationary in the lattice, then their effective mass becomes infinite and T_c approaches zero degree Kelvin. Therefore we strongly suspect that the transition temperature T_c is related to the motion of the ions in the lattice and the origin of the correlation lies in this. In actual fact, the interaction between two electrons via the lattice vibrations gives rise to superconducting phenomena. But how does the electrons interact via the lattice vibrations. This is a quantum mechanized phenomena, in terms of classical methods we can visualize the phenomenon as:

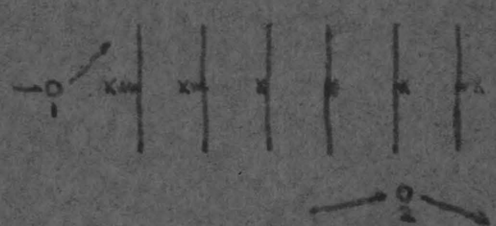
1. For a perfect lattice with rigidly fixed ions.



Crystal lattice   x - ions

O - electrons

2. Electrons move undisturbed through the lattice



Lattice wave produced by electron 1 effects electron 2 if the ions are not fixed.

without polarization   with polarization (a)   with polarization (b)

The interaction between 2-electrons due to the polarization of the lattice can be attractive or repulsive depending on the relative phase of the two electrons.

In case (a) the interaction is repulsive and in case (b) it is attractive.

From the figures we can observe that the periodic potential assumed by Bloch is disturbed by the electrons and the lattice polarized. Now the disturbance of the potential is a function of position and time and shows up as a retarded interaction between two electrons.

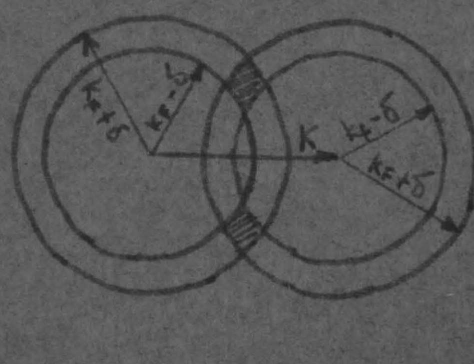
2. The short-range coulomb interaction: This is neglected in the zero order picture.

Long - range Coulomb interaction is screened by other electrons in the lattice.

3. We limit ourselves by taking the correlations of two electrons ignoring correlations among the number of electrons more than 2. Take

$$\rho^{\uparrow\downarrow} = \frac{1}{4} n^2 + f(r,K)$$

of (r,K) - extra correlation produced by the interactions between electrons. r - relative co-ordinate of the two electrons involved, and K is the total momentum of the two electrons.



The individual moments restricted to the shells defined by radii

$$k_f - \delta \leq k \leq k_f + \delta$$

The shaded area represents the cross section of the phase space available for scattering. If the total momentum K is to be conserved the volume of the available phase space has a maximum for K = 0.

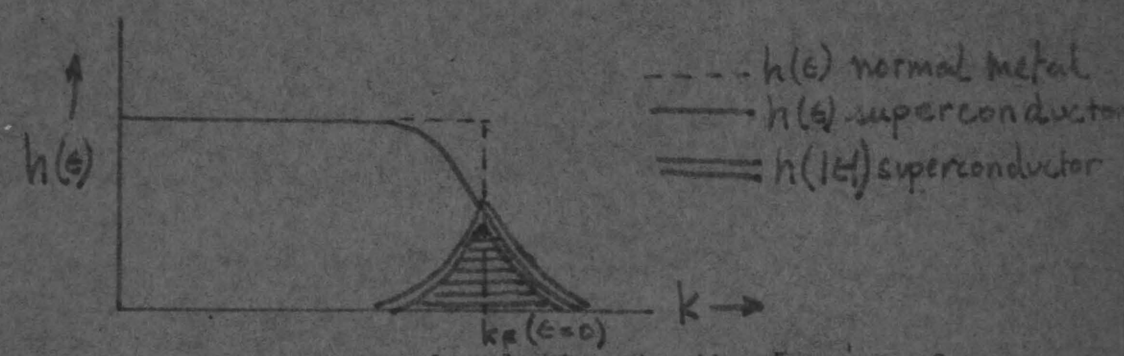
The fundamental approximation of the B.C.S. Theory is that only the two body correlations are taken into account and the two body correlations have a strong preference for singlet zero momentum pairs.

GROUND STATE OF THE SUPERCONDUCTOR

With the above expression the extra energy due to the pair correlations is given by

$$W_c = 2\sum_{-\hbar\omega<\epsilon<0} |\epsilon|\left(1-h(\epsilon)\right) + 2\sum_{\hbar\omega>\epsilon>0} \epsilon h(\epsilon) - V \sum_{|\epsilon|,|\epsilon'|\leq\hbar\omega} \left[h(\epsilon)\left(1-h(\epsilon')\right)h(\epsilon')\left(1-h(\epsilon)\right)\right]^{1/2}$$

where; $h(\epsilon)$ is the probability that the pair state ϵ is occupied.



2ϵ - Energy of the pair measured relative to the Fermi surface

V - matrix element between the Bloch pair state of relative momentum k and k^1.

Here interaction matrix $(k^1 \; |H_1| \; k)$ is assumed to be constant - V.

for $|\epsilon|$ and for $|\epsilon'| \leq \hbar\omega$ (an average photon energy) zero elsewhere.

There is a third solution for $W_c = 0$, i.e.,

$$h = 0 \qquad \epsilon > 0$$

$$h = 1 \qquad \epsilon < 0$$

this is the normal state.

Setting $\frac{d}{dh} W_c = 0$ we obtain

$$h = \frac{1}{2}\left(1 - \frac{\epsilon}{E}\right)$$

$$E = \left(\epsilon^2 + \epsilon_0^2\right)^{1/2}$$

and

$$\epsilon_0 = \frac{V}{2} \sum_{|\epsilon| \geq \hbar\omega} \frac{\epsilon_0}{\left(\epsilon^2 + \epsilon_0^2\right)^{1/2}}$$

Eqn. has no non-zero solution for ϵ_0 if $V < 0$. Therefore

$V = -(k'|H_1|k) > 0$ gives the criterion for superconductivity.

if $V > 0$ we have

$$1 = N(0)\, V \int_0^{\hbar\omega} \frac{d\epsilon}{\left(\epsilon^2 + \epsilon_0^2\right)^{1/2}}$$

or

$$\epsilon_0 = \hbar\omega \Big/ \sinh \frac{1}{N(0)V}$$

where $N(0)$ - density per unit energy of electrons of one spin at the Fermi-surface. In the limit of weak coupling, i.e., $N(0)V \ll 1$

we have $\epsilon_0 \simeq 2\hbar\omega\, e^{-1/N(0)V}$

$$W_S - W_N = W_c = -2N(0)(\hbar\omega)^2 e^{-2/N(0)V}$$

$(W_s - W_n)$ - energy differences between the normal and the superconducting state.

The dependence of the correlation energy on $(\hbar\omega)^2$ gives the isotope effect.

Therefore

$$\rho^{\uparrow\downarrow}(r) = \frac{1}{2}n^2 + \left| \left(\frac{1}{2\pi}\right)^3 \int dk\, h^{1/2}(1-h)^{1/2} e^{i\vec{k}\cdot\vec{r}}\right.$$

It turns out that in the superconducting ground state wave function, there are strong corelations between pairs of electrons with opposite spins and zero total momentum. The reason that these correlations can be constructed is that some of the non interacting electrons have large wave numbers due to the Pauli exclusion principle. Therefore with a small additional expenditure of kinetic energy given by first 2 terms of W_c, there can be a great gain in the potential energy term.

It is also possible to build a similar correlated wave function giving the singlet pairs the same non-zero total momentum. This corresponds to a current carrying state. The energy of this current carrying state is higher than the energy of the ground state.

A typical property of these correlated wave functions is that not a single pair can be broken up nor can a single element of phase space be removed without a finite expenditure of energy. In short the correlated wave function has a high degree of coherence and a great resistance to any kind of change, whereas, in the normal state, it is very easy through normal perturbations to change the electronic part of the wave function.

EXCITED STATES

There are 2 cases to be considered

a. single particle excitation

b. Collective excitations (Plasmons)

The single particle excitations are responsible for the super-field properties, the collective excitations can remain to be the same in both states.

A single particle excitation is obtained by occupying a single particle state K_1 above K_F, becoming a vacancy K_2 below K_F.

The energy of the excited state w.r.t. the ground state is

$$\mathcal{E}_1 - \mathcal{E}_2 = (\mathcal{E}_1 - \mathcal{E}_F) - (\mathcal{E}_2 - \mathcal{E}_F)$$
$$= \epsilon_1 - \epsilon_2 = |\epsilon_1| + |\epsilon_2|$$

This can be made as small as desired for a macroscopic sample.

If we define the excited states of a superconductor by one to one correspondence with the excited states of the normal metal we get

$$E_1 + E_2 = (\epsilon_1^2 + \epsilon_0^2)^{1/2} + (\epsilon_2^2 + \epsilon_0^2)^{1/2}$$

corresponding to (N) for ϵ_1 & ϵ_2 equal to zero.

$$E_1 + E_2 = 2\epsilon_0$$

The lowest excited state and the ground state has the gap of $2\epsilon_0$. This gap inhibits the single particle processes giving rise to the superconductive phenomena.

The excited electrons behave like normal electrons, they can be scattered or excited further. The correlated electrons do not enjoy this freedom and the fluid consists of 2 parts:

1. Super fluid
2. Normal fluid

THERMAL PROPERTIES

Free energy F is given by

$$F = W_e(T) - TS$$

T = absolute temperature

S = entropy

Free energy is function of f (k) and h (k) where:

f(k) is the probability that the state of momentum k is occupied by an excitation or a quasi-particle and h(k) is the relative probablity that the state k is occupied by a pair given that it not occupied by a quasi-particle.

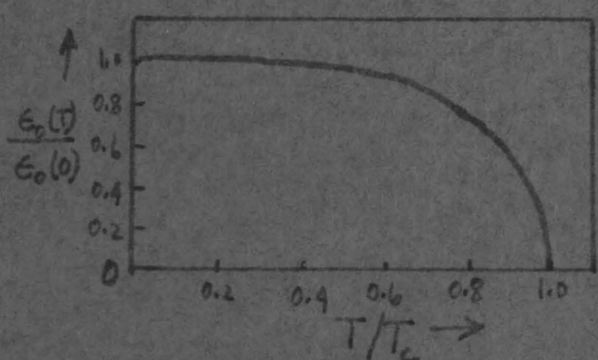
Setting $\dfrac{\partial F}{\partial f} = 0 \qquad \dfrac{\partial F}{\partial h} = 0$

we have $h = 1/2\,(1 - \epsilon/E)$

but $f = \dfrac{1}{1 + e^{\beta E}} \qquad E = (\epsilon^2 + \epsilon_0^2)^{1/2}$

and
$$\epsilon_0(T) = N(0)\, V \epsilon_0(T) \int_0^{\hbar\omega} \frac{d\epsilon}{E} \tanh\left(\frac{E}{2kT}\right).$$

ϵ_0 - gap energy.



Therefore: for $T < T_c$ (1) singlet sign zero momentum electrons are strongly correlated. (2) there is an energy gap modified by $f = (1+e^{\beta E})^{-1}$. In region $T > T_c$ $\epsilon_0 = 0$ and we have in every respect the normal solution.

ELECTRODYNAMIC PROPERTIES AND SUPERFLUIDITY

For a qualitative picture consider an impure material at absolute zero. Apply some field to the specimen, then electrons have some total momentum and there is a current. When the field is turned off the current ceases to exist due to the randomization of momentum by scattering effect of the impurity atoms.

In the superconducting state the electron wave function is highly correlated. To break up a correlated pair requires a finite amount of energy. This single particle scattering can only happen if this finite energy is received by the pair and this requires electrons to attain the depairing velocity. At this velocity the extra kinetic energy of the electron due to current flow is larger than the gap energy.

This can be calculated as follows:

CALCULATION OF THE ELECTRON DEPAIRING VELOCITY

The binding energy of the Cooper pair is $2\epsilon_0$. Depairing results in an increase of $2\epsilon_0$ in the potential energy of the electrons. For simplicity consider a thin film much thinner than the penetration depth so that the current carried by the film can be assumed to be uniformuly distributed throught the entire thickness.

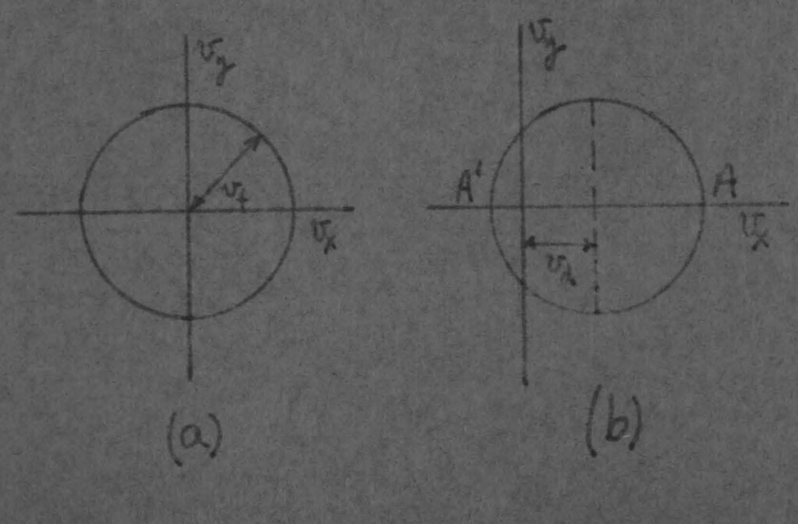


Fig. Shows Distribution of the conduction electrons among velocity states

a. Zero net current

b. Current density $J = nev_d$.

where v_f is the corresponding velocity for E_f, i.e.,

$$E_f = \frac{1}{2} m v_f^2$$

m = effective electron mass

In the superconducting state at absolute zero the max. energy of electrons is given by

$$E_f' = E_f - \epsilon_0$$

when $\epsilon_0 \ll E_f'$

E_f = Fermi level

E_f^1 = max energy of electrons in the superconducting state.

Now the total energy per electron E is

$$E = \frac{1}{2} m v^2 - \epsilon_0$$

for electron velocity v.

If the mean electron drift velocity is v_d then

$$J = env_d$$

If J is along the x - axis the Fermi sphere will be displaced by v_d above this axis as shown in Figure .

The greatest loss of energy to the lattice occurs when an electron at point A, vel($v_d + v_f$) is scattered to A^1 vel ($v_d - v_f$). In the normal metal this process gives a change in energy ΔE_n

$$\Delta E_n = \frac{1}{2} m (v_d - v_f)^2 - \frac{1}{2} m (v_d - v_f)^2$$

$$= -2m v_d v_f$$

E_n, E_s for normal and superconducting state.

$$\delta E_s = \delta E_n + 2\epsilon_0$$
$$= 2\epsilon_0 - 2mv_d v_f$$

Since the finding energy of electrons at points A and A' is $-2\epsilon_0$ where δE_s = change in the scattered electron in the superconducting state.

By the second law of thermodynamics such a scattering process between electrons and the lattice can occur only if the energy of electrons reduced and the energy of the lattice is increased. In other words, when δE_s must be negative. Therefore

$$0 > 2\epsilon_0 - 2mv_d v_f$$

and the critical value given by (v_c)

$$v_c = \frac{\epsilon_0}{m v_f}$$

for velocities smaller than v_c the electrons cannot give energy to the lattice.

Since we have

$$J = nev_d$$

then the critical current density at absolute zero is

$$J_c = -\frac{ne\epsilon_0}{v_f m}$$

where

J_c = critical current density at absolute zero.

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APPENDIX:

I) MAXWELL'S EQUATIONS:

IN M.K.S UNITS	IN GAUSSIAN UNITS
1) $\nabla\cdot\vec{D}=\rho$	$\nabla\cdot\vec{E}=4\pi\rho$
2) $\nabla\cdot\vec{B}=0$	$\nabla\cdot\vec{B}=0$
3) $\nabla\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}$	$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial H}{\partial t}$
4) $\nabla\times\vec{H}=\vec{J}+\frac{\partial}{\partial t}\vec{D}$	$\nabla\times\vec{H}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

II DERIVATIONS OF EQUATIONS

2.22 TO 2.25

a) Taking Curl of Maxwell 1.

$$\nabla\times\nabla\times\vec{H}=-\frac{4\pi}{c}\nabla\times\vec{J}+\frac{\partial}{\partial t}\nabla\times\frac{\vec{E}}{c}$$

$$=-\frac{4\pi}{c}\left(\frac{\vec{H}}{\Lambda c}+\sigma\frac{\dot{\vec{H}}}{c}\right)+\frac{\partial}{\partial t}\frac{1}{c}\left(-\frac{\dot{\vec{H}}}{c}\right)$$ By 2 & 2.17

$$=-\left(\frac{4\pi\vec{H}}{\Lambda c^2}+\frac{4\pi\sigma\dot{\vec{H}}}{c^2}\right)-\frac{1}{c^2}\ddot{\vec{H}}$$

$$=-\frac{4\pi}{c^2}\left(\frac{\vec{H}}{\Lambda}+\sigma\dot{\vec{H}}\right)-\frac{1}{c^2}\ddot{\vec{H}}$$

b) Taking Curl of Maxwell 2.

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \times \frac{H}{c}$$

$$= -\left(\frac{4\pi}{c^2} \dot{J} + \frac{\ddot{E}}{c^2} \right) \text{ by Maxwell 1}$$

$$= -\left[\frac{4\pi}{c^2} \left(\frac{E + \sigma \Lambda \dot{E}}{\Lambda} \right) + \frac{\ddot{E}}{c^2} \right]$$

by 2.14, 2.15 & 2.16

$$c^2 \nabla \times \nabla \times \vec{E} + \frac{4\pi}{\Lambda} \vec{E} + 4\pi\sigma \dot{E} + \ddot{E} = 0$$

c) Curl of 2.17

$$-c\Lambda \nabla \times \nabla \times \vec{J} = \nabla \times \vec{H} + \sigma \Lambda \nabla \times \dot{\vec{H}}$$

$$= \frac{4\pi}{c} \vec{J} + \frac{\dot{E}}{c} + \frac{\partial}{\partial t} \sigma \Lambda \left(\frac{4\pi}{c} J + \frac{\dot{E}}{c} \right)$$

by 1

$$= \frac{4\pi}{c} \vec{J} + \frac{\sigma \Lambda 4\pi}{c} \dot{\vec{J}} + \frac{1}{c} (\dot{\vec{E}} + \ddot{\vec{E}})$$

$$= \frac{4\pi}{c} \vec{J} + \frac{\sigma \Lambda 4\pi}{c} \dot{\vec{J}} + \frac{1}{c} \frac{\partial}{\partial t} (E + \dot{E})$$

but $\frac{\partial}{\partial t} \Lambda \vec{J} = E + \sigma \Lambda \dot{E}$

Therefore

$$-c^2 \nabla \times \nabla \times \vec{J} = \frac{4\pi}{\Lambda} \vec{J} + 4\pi\sigma \dot{\vec{J}} + \ddot{\vec{J}}$$

d) We have

$$\frac{\partial}{\partial t}(\Lambda \vec{J}) = \vec{E} + \sigma \Lambda \dot{\vec{E}}$$

$$\frac{\partial}{\partial t}(\nabla \cdot \Lambda \vec{J}) = \nabla \cdot \vec{E} + \sigma \Lambda \, \nabla \cdot \dot{\vec{E}}$$

Using $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

& $\nabla \cdot \vec{E} = 4\pi\rho$

We obtain

$$\frac{\partial}{\partial t}\left(\Lambda - \frac{\partial \rho}{\partial t}\right) = 4\pi\rho + \sigma \Lambda 4\pi \dot{\rho}$$

Therefore $\Lambda \ddot{\rho} + \sigma \Lambda 4\pi \dot{\rho} + 4\pi\rho = 0$

III.. As refered in section 2.3a

We have $\nabla \times \nabla \times \vec{J} + \alpha^2 \vec{J} = 0$

where $J_\theta = 0 \quad J_r = 0 \quad J_\phi = f(r)\sin\theta$

But $\nabla \times \nabla \times \vec{J} = \nabla(\nabla \cdot \vec{J}) - \nabla^2 \vec{J}$

$$\nabla \cdot \vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta J_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} J_\phi$$

Therefore $\nabla \cdot \vec{J} = 0$

$$\nabla(\nabla \cdot \vec{J}) = 0$$

$$\nabla \times \nabla \times \vec{J} = -\nabla^2 \vec{J}$$

$$\nabla^2 \vec{J} = \vec{a}_r \left[\nabla^2 J_r - \frac{2}{r^2}\left(J_r + \cot\theta\, J_\theta + \csc\theta \frac{\partial}{\partial\phi} J_\phi + \frac{\partial}{\partial\theta} J_\theta \right)\right]$$
$$+ \vec{a}_\theta \left[\nabla^2 J_\theta - \frac{1}{r^2}\left(\csc^2\theta\, J_\theta - 2\frac{\partial J_r}{\partial\theta} + 2\cot\theta\csc\theta \frac{\partial}{\partial\phi} J_\phi \right)\right]$$
$$+ \vec{a}_\phi \left[\nabla^2 J_\phi - \frac{1}{r^2}\left(\csc^2\theta\, J_\theta - 2\csc\frac{\partial J_r}{\partial\phi} - 2\cot\theta\csc\theta \frac{\partial}{\partial\phi} J_\phi \right)\right]$$

Therefore

$$\nabla^2 \vec{J} = \nabla^2 J_\phi - \frac{1}{r^2}\csc^2\theta\, J_\phi$$

$$\nabla^2 J_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial J_\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial J_\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 J_\phi}{\partial\phi^2}$$

Therefore

$$\nabla^2 J_\phi - \frac{1}{r^2}\csc^2\theta\, J_\phi = \frac{1}{r^2}\sin\theta\left[2rf' + r^2 f'' \right]$$
$$+ \frac{1}{r^2\sin\theta} f\cos 2\theta - \frac{1}{r^2}\csc^2\theta\, f\sin\theta$$

and

$$\nabla^2 \vec{J} + \alpha^2 \vec{J} = f'' + \frac{2}{r} f' - \left(\frac{2}{r^2} + \alpha^2 \right) f$$

Therefore

$$\nabla \times \nabla \times \vec{J} + \alpha^2 \vec{J} = f'' + \frac{2}{r} f' - \left(\frac{2}{r^2} + \alpha^2 \right) f$$