# NULL CONTROLLABILITY OF 1-D HEAT EQUATION WITH SWITCHING CONTROLS

by

Ali Taghiyev

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# ABSTRACT


## NULL CONTROLLABILITY OF 1-D HEAT EQUATION WITH SWITCHING CONTROLS

In [1], Zuazua analyzed the problem of two switching controls for null controllability of the 1-d heat equation with Dirichlet's boundary conditions and obtained sufficient conditions for null controls satisfying switching conditions. In this thesis, we consider the same problem with arbitrary number of switching controls and obtain sufficient conditions for null controls satisfying switching conditions. Secondly, we consider the problem of switching controls for the null controllability of the 1-d heat equation with Robin's boundary condition and obtain sufficient conditions for switching controls.

# ÖZET

## ANAHTAR KONTROLLER ALTINDA BİR BOYUTLU ISI DENKLEMİNİN DENETLENEBİLİRLİĞİ

Bu çalışmada biz, Dirichlet tipi sınır şartlı bir boyutlu ısı denkleminin anahtar kontroller altında denetlenebilirliğini sağlayacak gerekli ve yeterli koşullar elde ettik ve aynı problemi bu sefere Robin tipi sınır şartları altında inceleyerek, anahtar kontroller ile bir boyutlu ısı denkleminin denetlenebilirliğini sağlayacak gerekli ve yeterli koşullar elde ettik.

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# LIST OF SYMBOLS

| | |
|---|---|
| $L^2$ | Hilbert space of square integrable functions |
| $L^\infty$ | Space of essentially bounded, Lebesgue measurable functions |
| $\mathbb{R}$ | Real numbers |
| $\mathbb{R}^n$ | n-dimensional Euclidean space |
| $T$ | Time |
| $\mathbb{Z}$ | Set of all integers |
| | |
| $\alpha$ | Real scalar |
| $\beta_k$ | Fourier coefficient |
| $\delta$ | Dirac delta function |
| $\mu$ | Lebesgue measure |
| $\mu_k$ | Positive eigenvalue of corresponding adjoint system |
| $\pi_E$ | Orthogonal projection over E |
| $\omega_k$ | Eigenfunction corresponding to positive eigenvalue |
| $1_S$ | Set function which is 1 on the set $S$ and 0 elsewhere |
| $\|\| \cdot \|\|$ | Norm defined on $L^2$ |
| $\| \cdot \|$ | Absolute value |
| $\equiv$ | Equivalence |
| $:=$ | Equal by definition |
| $\stackrel{\text{def}}{=}$ | Equality by definition |

# LIST OF ACRONYMS/ABBREVIATIONS

| | |
|---|---|
| 1-d | One Dimensional |
| a.e. | Almost everywhere |
| CS | Cauchy-Schwarz inequality |
| def | Definition |
| e.g. | For example |
| etc. | And so on |
| i.e. | In other words |
| ODE | Ordinary Differential Equations |
| PDE | Partial Differential Equations |
| RBC | Robin's Boundary Condition |
| UCP | Unique Continuation Property |

# 1. INTRODUCTION

Control theory is certainly, at present, one of the most interdisciplinary areas of research. Control theory arises in most modern applications. The same could be said about the very first technological discoveries of the industrial revolution. On the other hand, control theory has been a discipline where many mathematical ideas and methods have melt to produce a new body of important mathematics. Accordingly, it is nowadays a rich crossing point of engineering and mathematics.

Control problems for PDE arise in many different contexts and ways. A prototypical problem is that of controllability. Roughly speaking, it consists in analysing whether the solution of the PDE can be driven to a given final target by means of a control applied on the boundary or on a sub-domain of the domain in which the equation evolves. More precisely, the controllability problem may be formulated as follows. Consider an evolution system (either described in terms of partial or ordinary differential equations (PDE/ODE)). We are allowed to act on the trajectories of the system by means of a suitable control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0, T)$, and initial and final states we have to find a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$. This is a type of exact controllability problem. Here, "exact" refers to the fact that the target is achieved completely. This final condition can be relaxed in different ways leading to various weaker notions of controllability. For example, when the final target is achieved to zero, then the system is null controllable or when the set of reachable states (set of final targets) is dense in the space where the evolution system is satisfied, then the system is approximate controllable. However, in finite dimensions, these apparently weaker notions often coincide with the exact controllability one. For instance, when dealing with the problem of approximate controllability, as we know the system is said to be approximately controllable when the set of reachable states is dense in $\mathbb{R}^n$. But, in $\mathbb{R}^n$, the only close affine dense subspace is the whole space itself. Thus, in finite-dimension, approximate controllability and exact controllability are equivalent notions.

But this is no longer the case in the context of PDE because of the intrinsic infinite-dimensional nature of the state space. Indeed, in infinite-dimensional spaces there are strict dense subspaces, while in finite-dimension they do not exist. These are classical problems in control theory and there is a large literature on the topic. We refer for instance to the book by Lee and Marcus [15] for an introduction in the context of finite-dimensional systems. We also refer to the articles by Russell [8–12], the articles by Zuazua [3–7] and to the SIAM Review article and the book of Lions [13, 14] for an introduction to the controllability of PDE, also referred to as Distributed Parameter Systems.

In 1988, Lions [14] introduced the so-called Hilbert Uniqueness Method. Roughly speaking it is based on the principle that, whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system. Suitable variants of this functional allow building different types of controls: those of minimal $L^2$-norm turn out to be smooth while those of minimal $L^\infty$-norm are of bang-bang form. The main difficulty when minimizing these functionals is to show that they are coercive. This turns out to be equivalent to the so called observability property of the adjoint equation, a property which is equivalent to the original control property of the state equation.

Control systems in real applications are often endowed with several actuators. It is then desirable to design switching control strategies guaranteeing that, at each instant of time, only one control is activated. Hence, we analyse the problem of switching controls for control systems endowed with different actuators. The goal is to control the dynamics of the system by switching from an actuator to another in a systematic way so that, at each instant of time, only one actuator is active. In [1], Zuazua developed a first analysis of this problem of switching controls addressing some model cases. Under suitable conditions on the placement of actuators, Zuazua showed that his approach allows building switching controls.

In Chapter 2, we first consider the 1-d heat equation endowed with arbitrary number (finite) of pointwise controls and lumped controls respectively and under suitable conditions on the placement of actuators, we show that our approach allows building switching controls. The techniques we use in this case are inspired by those developed in [1].

In Chapter 3, we address the same issue for the 1-d heat equation endowed with two boundary controls, pointwise controls and lumped controls under Robin's boundary condition with positive eigenvalues respectively, and obtain sufficient conditions for building switching controls. To do this we introduce a new functional based on the adjoint system whose minimizers yield the switching controls. We show that, due to the time analyticity of solutions, under suitable conditions on the location of the controllers, switching control strategies exist in the 1-d heat equation under Robin's boundary condition with positive eigenvalues.

# 2. 1-D HEAT EQUATION WITH ARBITRARY NUMBER OF SWITCHING CONTROLS

In this chapter, we will consider the problem of null controllability of the 1-d heat equation under Dirichlet's boundary conditions with finite number of pointwise controls and lumped controls respectively and obtain sufficient conditions for switching controls. Firstly, we consider the problem in which three controllers act at three different points in the interval $(0, 1)$ and similarly, the same problem with lumped controls and then we will obtain sufficient conditions for switching controls. At the end, we will generalize the result obtained from each case.

## 2.1. Pointwise Controls

Consider the case in which three pointwise controllers act at three different points $a, b, c$, of the space interval $(0, 1)$ where the equation is satisfied:

$$\begin{cases} y_t - y_{xx} = u_a(t)\delta_a + u_b(t)\delta_b + u_c(t)\delta_c, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

We consider the problem of null controllability. More precisely, given an initial datum $y_0 \in L^2(0, 1)$ we look for controls $u_a(t), u_b(t), u_c(t) \in L^2(0, T)$ such that $y(x, T) = 0$ and the switching condition satisfies:

$$u_a(t)u_b(t) = 0, \quad u_a(t)u_c(t) = 0, \quad u_b(t)u_c(t) = 0, \qquad \text{a.e. } t \in (0, T). \tag{2.2}$$

We know that whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system (see e.g., [3–7], [14]).

For $\varphi^0$ in $L^2(0,1)$, we consider the solution $\varphi : [0,1]\times[0,T] \to C([0,T], L^2(0,1))$ of the following backward Cauchy linear problem:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases} \tag{2.3}$$

This linear system is called the adjoint system corresponding to the 1-d heat equation with Dirichlet's boundary condition. Let $\varphi^0$ has the following Fourier expansion:

$$\varphi^0 = \sum_{k\geq 1} \beta_k \omega_k(x), \qquad \text{where } \ \omega_k(x) = \sqrt{2}\sin(k\pi x)$$

then the solution $\varphi$ of adjoint system is of the form:

$$\varphi(x,t) = \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \omega_k(x). \tag{2.4}$$

In [1], we know that the null control of 1-d heat equation could be computed by minimizing the quadratic functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\Big] dt - \int_0^1 y^0(x)\varphi(x,0)dx$$

over the class $\mathcal{H}$ of initial data given by

$$\mathcal{H} = \{\varphi^0 : \int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\Big] dt < \infty\}$$

where $\varphi(x,t)$ is the solution of the adjoint system (2.3) associated to the final state $\varphi^0$. We will consider $\mathcal{H}$ space endowed with the canonical norm:

$$||\varphi^0||_{\mathcal{H}} = \Big[\int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2 dt\Big]^{\frac{1}{2}}$$

which constitutes a Hilbert space.

But, firstly, we will show that $||\varphi^0||_{\mathcal{H}}$ actually defines a norm on $\mathcal{H}$, i.e., for all $\varphi^0, \psi^0 \in \mathcal{H}$,

$$||\varphi^0||_{\mathcal{H}} > 0 \quad \text{and} \quad ||\varphi^0||_{\mathcal{H}} = 0 \Leftrightarrow \varphi^0 = 0, \tag{2.5}$$

$$||\lambda\varphi^0||_{\mathcal{H}} = |\lambda| ||\varphi^0||_{\mathcal{H}}, \qquad \forall \lambda \in \mathbb{R}, \tag{2.6}$$

$$||\varphi^0 + \psi^0||_{\mathcal{H}} \leq ||\varphi^0||_{\mathcal{H}} + ||\psi^0||_{\mathcal{H}}. \tag{2.7}$$

Let us analyse the positivity of the norm $|| \cdot ||_{\mathcal{H}}$ in $\mathcal{H}$ space. Here it is convenient to use the Fourier representation of solutions of the adjoint system (2.3). Therefore, we have:

$$\begin{aligned} ||\varphi^0||^2_{\mathcal{H}} &= \int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2 dt \\ &= \int_0^T \Big[ \Big| \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \omega_k(a) \Big|^2 + \Big| \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \omega_k(b) \Big|^2 \\ &\quad + \Big| \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \omega_k(c) \Big|^2 \Big] dt \end{aligned}$$

Before proving the positivity of the norm, we will give very important lemma on families of real exponentials. This lemma is known as estimates on families of real exponentials (see e.g., [1], [4], [12]).

**Lemma 2.1.** *In our case, it is guaranteed that*

$$\int_0^T \Big| \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \Big|^2 dt \geq c_1 \sum_{k\geq 1} e^{-2\pi^2 k^2 T} \beta_k^2$$

*for a suitable positive constant* $c_1 > 0$ *is independent of* $\{\beta_k\}_{k\geq 1}$.

By using this lemma, we will get weighted observability inequality:

$$||\varphi^0||^2_{\mathcal{H}} \geq c_1 \sum_{k\geq 1} e^{-2\pi^2 k^2 T} \Big[ |\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2 \Big] \beta_k^2. \tag{2.8}$$

Therefore, we get the property (2.5). Lastly, for obtaining (2.7), we first simplify our inequality and then use the property (2.5). At the end, we have:

$$\int_0^T \Big[|\varphi(a,t)-\psi(a,t)|^2+|\varphi(b,t)-\psi(b,t)|^2+|\varphi(c,t)-\psi(c,t)|^2\Big]dt>0 \tag{2.9}$$

where $\varphi(x,t), \psi(x,t)$ are solutions of (2.3) with initial data $\varphi^0, \psi^0$ respectively.[1] Hence $||\cdot||_{\mathcal{H}}$ defines a norm on $\mathcal{H}$. In addition, since the adjoint system is well posed, the functional $J(\varphi^0)$ is obviously continuous in $\mathcal{H}$, and strictly convex thanks to (2.9).

As we know that null controllability in time $T$ implies approximate controllability in time $T$. This comes form the fact that all the range of the semi-group generated by the heat equation is reachable (see e.g., [3]). Therefore, we first prove the approximate controllability of the heat system in time $T$ under some conditions. For this, we will consider new functional very similar with previous one: for any $\epsilon > 0$ and $y^1 \in L^2(0,1)$

$$\begin{aligned} J_\epsilon(\varphi^0) &= \frac{1}{2}\int_0^T \Big[|\varphi(a,t)|^2+|\varphi(b,t)|^2+|\varphi(c,t)|^2\Big]dt+\epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} \\ &+ \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx \end{aligned}$$

where $E$ is finite dimensional subspace of $L^2(0,1)$ and $\pi_E$ denotes the orthogonal projection from $L^2(0,1)$ over $E$.

Our aim is to build approximate pointwise control satisfying Dirichlet's boundary condition. In other words, given $\epsilon > 0$, we will try to find (finite) approximate controls $u_a^\epsilon, u_b^\epsilon, u_c^\epsilon$ such that the solution $y_\epsilon$ of heat equation satisfies the following condition:

$$||y_\epsilon(x,T)-y_1||_{L^2(0,1)} \leq \epsilon. \tag{2.10}$$

For this to be true, the following property suffices (see e.g., [3]):

$$\forall t\in(0,T),\quad \varphi(a,t)=\varphi(b,t)=\varphi(c,t)=0 \implies \varphi(x,t)\equiv 0 \tag{2.11}$$

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[1]Here we use the fact that $(\varphi-\psi)(x,t)$ is the unique solution of (2.3) with initial data $\varphi^0-\psi^0 \in \mathcal{H}$.

which is unique continuation property of the adjoint system.

**Lemma 2.2.** *Assume that unique continuation property (2.11) holds, then the heat system (2.1) is approximate controllable.*

*Proof.* For obtaining approximate controllability of (2.1), we should minimize $J_\epsilon(\varphi^0)$ over $\mathcal{H}$. We have already proved that $J_\epsilon$ is convex and continuous in $\mathcal{H}$. On the other hand, in view of (2.11) above, one can prove that

$$\lim_{||\varphi^0||_{L^2(0,1)}\to\infty} \frac{J_\epsilon(\varphi^0)}{||\varphi^0||_{L^2(0,1)}} \geq \epsilon. \tag{2.12}$$

Let us give the proof of this coercivity property. In order to prove above inequality, let $\{\varphi_j^0\} \subset L^2(0,1)$ be sequence of initial data for the adjoint system with $||\varphi_j^0||_{L^2(0,1)} \to \infty$. Now normalize them by

$$\tilde{\varphi}_j^0 = \frac{\varphi_j^0}{||\varphi_j^0||_{L^2(0,1)}}$$

so that $||\tilde{\varphi}_j^0||_{L^2(0,1)} = 1$. On the other hand, let $\tilde{\varphi}_j$ be the solution of adjoint system with initial data $\tilde{\varphi}_j^0$. Then we have

$$\begin{aligned} J_\epsilon(\varphi_j^0)/||\varphi_j^0||_{L^2(0,1)} &= \frac{1}{2}||\varphi_j^0||_{L^2(0,1)} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big] dt \\ &\quad + \epsilon||(I-\pi_E)\tilde{\varphi}_j^0||_{L^2(0,1)} + \int_0^1 \tilde{\varphi}_j^0 y^1 dx - \int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx. \end{aligned}$$

The following two cases may occur:

(i) $\liminf_{j\to\infty} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big] dt > 0$. In this case we have

$$J_\epsilon(\varphi_j^0)/||\varphi_j^0||_{L^2(0,1)} \to \infty.$$

(ii) $\liminf_{j\to\infty} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big] dt = 0.$

For the last case, since $\tilde{\varphi}_j^0$ is bounded in $L^2(0,1)$, by extracting a subsequence we can guarantee that $\tilde{\varphi}_j^0 \rightharpoonup \psi^0$ weakly in $L^2(0,1)$, moreover

$$\varphi^0 \longmapsto \int_0^T \left[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\right] dt$$

is lower semi-continuous in the weak topology of $L^2(0,1)$ (see e.g., [16]).

Therefore we obtain the following

$$\int_0^T \left[|\psi(a,t)|^2 + |\psi(b,t)|^2 + |\psi(b,t)|^2\right] dt \leq \liminf_{j\to\infty} \int_0^T \left[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\psi(c,t)|^2\right] dt$$

where $\psi$ is the solution of adjoint system with given initial data $\psi^0$. Therefore we have

$$\forall t \in (0,T), \quad \psi(a,t) = \psi(b,t) = \psi(c,t) = 0.$$

From (2.11) we conclude that $\psi \equiv 0$. Therefore $\psi^0 = 0$ and $\tilde{\varphi}_j^0 \rightharpoonup 0$ weakly in $L^2(0,1)$ and consequently

$$\int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx \to 0, \qquad \int_0^1 \tilde{\varphi}_j^0 y^1 dx \to 0.$$

Furthermore, $E$ is being finite-dimensional, $\pi_E$ would be compact operator and then $\pi_E \tilde{\varphi}_j^0 \to 0$ strongly in $L^2(0,1)$. Consequently,

$$||(I - \pi_E)\tilde{\varphi}_j^0||_{L^2(0,1)} \to 1$$

as $j \to \infty$. At the end, we obtain our coercivity property:

$$\liminf_{j\to\infty} \frac{J_\epsilon(\varphi_j^0)}{||\varphi_j^0||} \geq \liminf_{j\to\infty} \left[\epsilon + \int_0^1 \tilde{\varphi}_j^0 y^1 dx - \int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx\right] = \epsilon.$$

Therefore $J_\epsilon$ admits an unique minimizer $\hat{\varphi}^0 \in \mathcal{H}$. That means, for any $\psi^0 \in L^2(0,1)$

and $h \in \mathbb{R}$, we have $J_\epsilon(\hat{\varphi}^0) \leq J_\epsilon(\hat{\varphi}^0 + h\psi^0)$. Namely,

$$\begin{aligned} J_\epsilon(\hat{\varphi}^0 + h\psi^0) - J_\epsilon(\varphi^0) &= \int_0^T h\Big[\hat{\varphi}(a,t)\psi(a,t) + \hat{\varphi}(b,t)\psi(b,t) + \hat{\varphi}(c,t)\psi(c,t)\Big]dt \\ &+ \int_0^T h^2\Big[\psi(a,t)^2 + \psi(b,t)^2 + \psi(c,t)^2\Big]dt \\ &+ \epsilon\Big[||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)}\Big] \\ &+ \int_0^1 h\psi^0 y^1 dx - \int_0^1 h\psi(x,0)y^0(x)dx \geq 0. \end{aligned}$$

We know from triangular inequality that

$$||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)} \leq |h|||((I-\pi_E)\psi^0||_{L^2(0,1)}. \tag{2.13}$$

Now, let us define

$$\mathcal{A} \overset{\text{def}}{=} \int_0^T \Big[\hat{\varphi}(a,t)\psi(a,t) + \hat{\varphi}(b,t)\psi(b,t) + \hat{\varphi}(c,t)\psi(c,t)\Big]dt.$$

Then using above inequality and after considering the cases: $h > 0$, $h < 0$ and taking $h \to 0$ at the end, we will get the following relation:

$$\Big|\mathcal{A} + \int_0^1 \psi^0 y^1 dx - \int_0^1 \psi(x,0)y^0(x)dx\Big| \leq \epsilon\Big[||((I-\pi_E)\psi^0||_{L^2(0,1)}\Big]. \tag{2.14}$$

Now, if we take $u_a(t) = -\hat{\varphi}(a,t)$, $u_b(t) = -\hat{\varphi}(b,t)$, $u_c(t) = -\hat{\varphi}(c,t)$, and multiplying the heat equation (2.1) with initial data $y^0(x) \in L^2(0,1)$ by $\psi$ which is the solution of adjoint system (2.3) with initial data $\psi^0$ and integrating by parts we finally get

$$\mathcal{A} = \int_0^1 \psi(x,0)y^0(x)dx - \int_0^1 \psi^0 y(x,T)dx.$$

By combining these two and letting $E = 0$,[2] we finally get

$$\Big|\int_0^1 \psi^0(y(x,T) - y^1)dx\Big| \leq \epsilon||\psi^0||_{L^2(0,1)}$$

[2] In this case, finite approximate controllability turns out to be approximate controllability of (2.1)

for every $\psi^0 \in L^2(0,1)$ which is equivalent to (2.10) and letting $y^1 = 0$ we finally have

$$||y(x,T)||_{L^2(0,1)} \leq \epsilon. \tag{2.15}$$

Therefore for every $\epsilon > 0$, by using variational approach, we obtain approximate controls:

$$\begin{cases} u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t), \\ u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t), \\ u_c^\epsilon(t) = -\hat{\varphi}_\epsilon(c,t). \end{cases} \tag{2.16}$$

□

Now, to get null controls, we should prove that $u_a^\epsilon(t), u_b^\epsilon(t), u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.

We know that the space of null controllable initial data is the dual one $\mathcal{H}'$. Therefore, to get null controllability of (2.1), we should put some conditions on the Fourier coefficients $\{y_k^0\}_{k\geq 1}$ of initial datum $y^0$.

**Lemma 2.3.** *Assume that Fourier coefficients $\{y_k^0\}_{k\geq 1}$ of initial datum $y^0$ of (2.1) satisfy the finiteness property:*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{|\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2} |y_k^0|^2 < \infty. \tag{2.17}$$

*Then $y^0 \in \mathcal{H}'$ and our approximate controls $u_a^\epsilon(t), u_b^\epsilon(t), u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.*

*Proof.* Using (2.8) and Cauchy-Schwarz inequality (see e.g., [17]), we will get the following:

$$\begin{aligned} \frac{\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|}{||\varphi^0||_{\mathcal{H}}} &\leq C\frac{\Big|\sum_{k\geq 1} y_k^0\upsilon_k\beta_k\upsilon_k^{-1}\Big|}{\Big|\sum_{k\geq1}(\beta_k\upsilon_k^{-1})^2\Big|^{\frac{1}{2}}} \overset{\text{CS}}{\leq} C\frac{\Big|\sum_{k\geq1}(y_k^0\upsilon_k)^2\Big|^{\frac{1}{2}}\Big|\sum_{k\geq1}(\beta_k\upsilon_k^{-1})^2\Big|^{\frac{1}{2}}}{\Big|\sum_{k\geq1}(\beta_k\upsilon_k^{-1})^2\Big|^{\frac{1}{2}}} \\ &\leq C\Big[\sum_{k\geq1}\frac{e^{2\pi^2k^2T}}{|\omega_k(a)|^2+|\omega_k(b)|^2+|\omega_k(c)|^2}|y_k^0|^2\Big]^{\frac{1}{2}} < \infty \end{aligned}$$

where

$$\upsilon_k = \Big|\frac{e^{2\pi^2k^2T}}{|\omega_k(a)|^2+|\omega_k(b)|^2+|\omega_k(c)|^2}\Big|^{\frac{1}{2}}.$$

As a result we get $y^0 \in \mathcal{H}'$.

Now, we will prove that our approximate controls $u_a^\epsilon(t)$, $u_b^\epsilon(t)$, $u_b^\epsilon(t)$ satisfy uniform boundedness in $L^2(0,T)$. Note that $u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t)$, $u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t)$, $u_c^\epsilon(t) = -\hat{\varphi}_\epsilon(c,t)$ where $\hat{\varphi}_\epsilon(x,t)$ solves adjoint system (2.3) with initial data $\hat{\varphi}_\epsilon^0$ at time $t=T$ obtained by minimizing the functional $J_\epsilon$ when $E=0$ and $y^1=0$. At the minimizer $\hat{\varphi}_\epsilon^0$, we have

$$J_\epsilon(\hat{\varphi}_\epsilon^0) \leq J_\epsilon(0) = 0.$$

This implies that

$$\frac{1}{2}\int_0^T \Big[|\hat{\varphi}_\epsilon(a,t)|^2+|\hat{\varphi}_\epsilon(b,t)|^2+|\hat{\varphi}_\epsilon(c,t)|^2\Big]dt \leq \Big|\int_0^1 y_0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|.$$

From (2.8), we have

$$\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2+|\hat{\varphi}_\epsilon(b,t)|^2+|\hat{\varphi}_\epsilon(a,t)|^2dt \leq \frac{\hat{C}\Big|\int_0^1 y_0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|^2}{\Big|\sum_{k\geq1}\beta_k^2\upsilon_k^{-2}\Big|}$$

for suitable $\hat{C}>0$ is independent of $\{\beta_k\}_{k\geq1}$. Since $\{\omega_k(x)\}_{k\geq1}$ form orthogonal basis in $L^2(0,1)$ after some simplification, we will have

$$\Big|\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt\Big| \leq \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2\upsilon_k^{-2}\Big|}.$$

But applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2\upsilon_k^{-2}\Big|} &= \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\upsilon_k\beta_k\upsilon_k^{-1}\Big|^2}{\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|} \\ &\overset{\text{CS}}{\leq} \frac{\hat{C}\Big|\sum_{k\geq 1}(y_k^0\upsilon_k)^2\Big|\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|}{\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|} \\ &= \hat{C}\sum_{k\geq 1}(y_k^0\upsilon_k)^2.\end{aligned}$$

Hence, at the end we obtain:

$$\Big|\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt\Big| \leq \tilde{C}\sum_{k\geq 1}|\upsilon_k|^2|y_k^0|^2 < \infty \qquad (2.18)$$

where $\tilde{C} > 0$ is independent of $\{\beta_k\}_{k\geq 1}$. Since we know that

$$||u_a^\epsilon(t)||^2_{L^2(0,T)} \leq \int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt,$$

and similarly, the above inequality is valid for $u_b^\epsilon$ and $u_c^\epsilon$. Now, using (2.18) we conclude that $\forall\epsilon > 0$, $u_a^\epsilon(t), u_b^\epsilon(t)$ and $u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. □

Therefore, by using Lemma 2.3, we conclude that under the assumption of finiteness property, $u_a^\epsilon(t), u_b^\epsilon(t), u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$ and so, by extracting subsequences, we have $u_a^\epsilon \rightharpoonup u_a$, $u_b^\epsilon \rightharpoonup u_b$ and $u_c^\epsilon \rightharpoonup u_c$ weakly in $L^2(0,T)$. Using the continuous dependence of the solution of the heat equation, we can show that $y_\epsilon(x,T)$ converges to $y(x,T)$ weakly in $L^2(0,T)$ which implies that $y(x,T) = 0$, i.e., the limit controls $u_a$, $u_b$ and $u_c$ fulfil the null controllability requirement. Hence, from Lemma

2.2 and Lemma 2.3, we understand that under the assumption of (2.11) and (2.17), we obtain null pointwise controls by minimizing $J(\varphi^0)$ which is the same as $J_\epsilon(\varphi^0)$ when $y^1 = 0$ and $\epsilon = 0$ over the Hilbert space $\mathcal{H}$.

However, our null controls (2.16) do not fulfil the switching condition (2.2) in general. Therefore, we realize that minimizing $J$ over $\mathcal{H}$ just solves the problem of null controllability of heat system, but we still have no switching controls. For getting switching controls, we will consider the following functional $J_s$, which is a variant of our functional $J$, with the same coercivity properties, allows building switching controllers:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\Big\{|\varphi(a,t)|^2, |\varphi(b,t)|^2, |\varphi(c,t)|^2\Big\}dt - \int_0^1 y^0(x)\varphi(x,0)dx. \quad (2.19)$$

The functional $J_s : \mathcal{H} \longrightarrow \mathbb{R}$ is well defined, continuous thanks to well-posedness of adjoint system (2.3) and convexity comes from the following inequality:
For given $a_1, a_2, b_1, b_2 \in \mathbb{R}$,

$$\max((a_1+a_2)^2, (b_1+b_2)^2) \leq \max(a_1^2, b_1^2) + 2\max(a_1a_2, b_1b_2) + \max(a_2^2, b_2^2). \quad (2.20)$$

Same as before, we will consider the problem of approximate controllability, i.e., for all $\epsilon > 0$ we could find (finite) approximate controls $u_a^\epsilon, u_b^\epsilon, u_c^\epsilon$ such that the solution $y_\epsilon$ of heat equation satisfies (2.10).

For obtaining approximate controls, we should consider the following new functional very similar with (2.19): for any $\epsilon > 0$ and any $y^1 \in L^2(0,1)$

$$\begin{aligned} J_s^\epsilon(\varphi^0) &= \frac{1}{2}\int_0^T \max\Big\{|\varphi(a,t)|^2, |\varphi(b,t)|^2, |\varphi(c,t)|^2\Big\}dt + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} \\ &+ \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx \end{aligned}$$

where $E$ is finite dimensional subspace of $L^2(0,1)$ and $\pi_E$ denotes the orthogonal projection from $L^2(0,1)$ over $E$. Observe that when $y^1 = 0$ and $E = 0$ we obtain our previous functional $J_s$.

**Lemma 2.4.** *Assume that the following unique continuation property holds:*

$$\mu(I) = \mu\{t \in (0,T) : |\varphi(a,t)| = |\varphi(b,t)| = |\varphi(c,t)|\} > 0 \Rightarrow \varphi \equiv 0. \tag{2.21}$$

*Then (2.1) system is approximate controllable.*

We will skip the proof of that lemma which is closely related with Lemma 2.2 (see e.g., [1]). Hence, from Lemma 2.4, we know that for getting approximate controllability of (2.1), we need to have (2.21). Since we know that

$$\begin{aligned}
\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} &\subset \{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)|\},\\
\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} &\subset \{t \in (0,T) : |\psi(b,t)| = |\psi(c,t)|\},\\
\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} &\subset \{t \in (0,T) : |\psi(c,t)| = |\psi(a,t)|\}.
\end{aligned}$$

Hence, we will have that

$$\begin{aligned}
I_a^b &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)|\},\\
I_b^c &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(b,t)| = |\psi(c,t)|\},\\
I_c^a &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(c,t)| = |\psi(a,t)|\},
\end{aligned}$$

are of positive measure. Now using again the Fourier representation of solution of (2.3) we have

$$\varphi(a,t) \pm \varphi(b,t) = \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} (\omega_k(a) \pm \omega_k(b)).$$

The function $\varphi(a,t) \pm \varphi(b,t)$ are time analytic for $t \leq T$. Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by the real exponentials $e^{-\eta^2(t-T)}$

successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(\omega_k(a) \pm \omega_k(b)) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$ for all $k \geq 1$, it is sufficient to show that

$$\omega_k(a) \pm \omega_k(b) = \sin(k\pi a) \pm \sin(k\pi b) \neq 0, \qquad \forall k \geq 1.$$

This holds if and only if

$$a \pm b \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.22}$$

Similarly, we have:

$$b \pm c \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.23}$$

$$c \pm a \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.24}$$

As a result, under irrationality conditions (2.22), (2.23) and (2.24), we have $I_a^b, I_b^c, I_c^a$ and $I$ are of measure zero, i.e., (2.21) satisfies.[3] Now define

$$S_a \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(a,t)| > \max(|\varphi(b,t)|, |\varphi(c,t)|)\},$$
$$S_b \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(b,t)| > \max(|\varphi(a,t)|, |\varphi(c,t)|)\},$$
$$S_c \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(c,t)| > \max(|\varphi(b,t)|, |\varphi(a,t)|)\}.$$

We know that $J_s^\epsilon$ admits an unique minimizer $\hat{\varphi}^0 \in \mathcal{H}$. Namely, for any $\psi^0 \in L^2(0,1)$ and $h \in \mathbb{R}$ sufficiently small, we will have $J_s^\epsilon(\hat{\varphi}^0) \leq J_s^\epsilon(\hat{\varphi}^0 + h\psi^0)$. Hence by using variational approach, for all $\epsilon > 0$, we obtain our approximate switching controls:

$$u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t)1_{S_a}, \qquad u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t)1_{S_b}, \qquad u_c^\epsilon(t) = -\hat{\varphi}_\epsilon(c,t)1_{S_a}.$$

---

[3]In other words, (2.21) means, $\varphi^0 \neq 0$ implies $I$ is of measure zero.

and final state satisfies (2.15). Using Lemma 2.3, as $\epsilon \to 0$, we will obtain our null switching controls. Consequently, we obtain the following result:

**Theorem 2.5.** *Assume that points $a, b, c$ in the interval (0,1) are such that the irrationality conditions hold (i.e., $a \pm b, b \pm c, c \pm a \neq m/k$). Assume that the Fourier coefficients of $y_0$ satisfying (2.17). Then, for all $T > 0$, there exist switching controls*

$$u_a(t) = -\hat{\varphi}(a,t)1_{S_a}, \qquad u_b(t) = -\hat{\varphi}(b,t)1_{S_b}, \qquad u_c(t) = -\hat{\varphi}(c,t)1_{S_a},$$

*satisfying switching condition (2.2) and that the solution of heat equation (2.1) satisfies*

$$y(x,T) = 0$$

*i.e, null controllability is satisfied. These switching controls obtained by minimizing the functional (2.19) over $\mathcal{H}$.*

In general, we could examine the case in which $n \in \mathbb{N}$, pointwise controllers act at $n$ different points $(a_i)_{i=1}^{i=n}$ of the space interval $(0,1)$. Consider the heat system:

$$\begin{cases} y_t - y_{xx} = \sum_{i=1}^{i=n} u_{a_i}(t)\delta_{a_i}, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.25}$$

Here now, given an initial datum $y_0 \in L^2(0,1)$ we are looking for controls $\{u_{a_i}(t)\}_{i=1}^{i=n} \in L^2(0,T)$ such that null controllability of heat equation holds, i.e, $y(x,T) = 0$ and switching condition satisfies:

$$u_{a_i}(t)u_{a_j}(t) = 0, \quad \forall i \neq j, \qquad \text{a.e. } t \in (0,T). \tag{2.26}$$

At first, we will consider the approximate controllability problem. To obtain approximate switching controls, one should minimize an appropriate quadratic functional over suitable Hilbert space, and under some conditions on the Fourier coefficients of $y_0$, we will get our desired null switching controls satisfying switching property. Consequently, we obtain following general result for switching controls:

**Theorem 2.6.** *Assume that points* $\{a_i\}_{i=0}^{i=n}$ *in the interval (0,1) are such that the irrationality conditions hold (i.e,* $a_i \pm a_j$*, are irrationals* $\forall i \neq j$*). Let the initial datum* $y^0$ *be in* $H_n'$ *which is the dual space of class of initial data of (2.3):*

$$H_n = \{\varphi^0 : \int_0^T \sum_{i=1}^{n} |\varphi(a_i,t)|^2 dt < \infty\}.$$

*More precisely, let* $y^0$ *be of the form*

$$y^0 = \sum_{k\geq 1} y_k^0 \omega_k(x) \quad with \quad \sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{\sum_{i=1}^{i=n} |\omega(a_i)|^2} |y_k^0|^2 < \infty.$$

*Then, for all* $T > 0$*, there exist switching controls* $\{u_{a_i}(t)\}_{i=1}^{i=n} \in L^2(0,T)$ *satisfying (2.26) and that the solution of heat equation satisfies null controllability condition. These switching controls are*

$$u_{a_i}(t) = -\hat{\varphi}(a_i,t),\ u_{a_j}(t) = 0 \ \ for\ j \neq i, \quad in\ S_{a_i} \quad \forall i \in \{1,2,...,n\}$$

*where*

$$S_{a_i} = \left\{ t \in (0,T) : |\varphi(a_i,t)| > \max_{\substack{1\leq j\leq n \\ j\neq i}} \{|\varphi(a_j,t)|\} \right\}$$

*and* $\hat{\varphi}^0 = \hat{\varphi}(x,T)$ *is the minimizer of the functional*

$$J_s^n(\varphi^0) = \frac{1}{2}\int_0^T \max_{1\leq i\leq n} \{|\varphi(a_i,t)|^2\} dt - \int_0^1 y^0(x)\varphi(x,0)dx$$

*and* $\hat{\varphi}(x,t)$ *is the solution of adjoint system with initial data* $\hat{\varphi}^0$*.*

## 2.2. Lumped Controls

Similar results hold in the case of lumped controls, in which the pointwise Dirac controls of the previous section are replaced by controls distributed by means of give control functions. More precisely, let $f_0 = f_0(x)$, $f_1 = f_1(x)$ and $f_2 = f_2(x)$ be three control profiles in $L^2(0,1)$. Consider the heat equation:

$$\begin{cases} y_t - y_{xx} = u_0(t)f_0 + u_1(t)f_1 + u_2(t)f_2, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.27}$$

Consider the problem of null controllability. More precisely, given an initial datum $y_0 \in L^2(0,1)$ we look for controls $u_0(t), u_1(t), u_2(t) \in L^2(0,T)$ such that the switching condition satisfies:

$$u_0(t)u_1(t) = 0, \quad u_0(t)u_2(t) = 0, \quad u_1(t)u_2(t) = 0, \qquad \text{a.e.} \quad t \in (0,T). \tag{2.28}$$

Here, for this problem we will consider the adjoint system (2.3). As discussed before, we may compute the null control of (2.27) by minimizing the quadratic functional,

$$\begin{aligned} \hat{J}(\varphi^0) &= \frac{1}{2}\int_0^T \max\left[\left|\int_0^1 f_0(x)\varphi dx\right|^2, \left|\int_0^1 f_1(x)\varphi dx\right|^2, \left|\int_0^1 f_2(x)\varphi dx\right|^2\right] dt \\ &\quad - \int_0^1 y^0(x)\varphi(x,0)dx \end{aligned}$$

over the class $\hat{\mathcal{H}}$ of initial data given by

$$\hat{\mathcal{H}} = \{\varphi^0 : \int_0^T \left[\left|\int_0^1 f_0(x)\varphi dx\right|^2 + \left|\int_0^1 f_1(x)\varphi dx\right|^2 + \left|\int_0^1 f_2(x)\varphi dx\right|^2\right] dt < \infty\}$$

where $\varphi(x,t)$ is the solution of the adjoint system (2.3) associated to the final state $\varphi^0$. We will consider $\mathcal{H}$ space endowed with the canonical norm

$$||\varphi^0||^2_{\hat{\mathcal{H}}} = \int_0^T \Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_1(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt$$

which constitutes a Hilbert space. But, firstly, we will show that $||\varphi^0||_{\mathcal{H}}$ actually defines a norm on $\mathcal{H}$. As we discussed before, the property (2.6) comes directly, and we could obtain the property (2.7) by using the positivity of $||\cdot||_{\hat{\mathcal{H}}}$ and the inequality (2.20). Therefore let us analyse the positivity of the norm $||\cdot||_{\hat{\mathcal{H}}}$ in $\hat{\mathcal{H}}$ space.

Here we will use (2.4) as a Fourier representation of the solution of (2.3). Therefore, we have:

$$\begin{aligned} ||\varphi^0||^2_{\hat{\mathcal{H}}} &= \int_0^T \Big|\int_0^1 \sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_0(x)\omega_k(x)dx\Big|^2 dt \\ &+ \int_0^T \Big|\int_0^1 \sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_1(x)\omega_k(x)dx\Big|^2 dt \\ &+ \int_0^T \Big|\int_0^1 \sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_2(x)\omega_k(x)dx\Big|^2 dt. \end{aligned}$$

Assume that the controls $f_0$, $f_1$ and $f_2$ have Fourier series expansions of the form

$$f_0(x) = \sum_{k\geq 1} f_{0,k}\omega_k(x), \quad f_1(x) = \sum_{k\geq 1} f_{1,k}\omega_k(x), \quad f_2(x) = \sum_{k\geq 1} f_{2,k}\omega_k(x). \tag{2.29}$$

Hence after some calculation, we have

$$||\varphi^0||^2_{\hat{\mathcal{H}}} = \int_0^T \Big|\sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_{0,k}\Big|^2 + \Big|\sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_{1,k}\Big|^2 + \Big|\sum_{k\geq 1} \beta_k e^{\pi^2k^2(t-T)} f_{2,k}\Big|^2 dt.$$

Now using Lemma 2.1, we then get following weighted observability inequality:

$$||\varphi^0||^2_{\hat{\mathcal{H}}} \geq c_1 \sum_{k\geq 1} e^{-2\pi^2k^2T}\Big[|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2\Big]\beta_k^2 \tag{2.30}$$

where positive constant $c_1$ is independent from $\{\beta_k\}_{k\geq 1}$. Hence the property (2.5) satisfied and at the end, $||\cdot||_{\hat{\mathcal{H}}}$ defines a norm in $\hat{\mathcal{H}}$.

In addition, since the adjoint system (2.3) is well posed, the functional $\hat{J}(\varphi^0)$ is obviously continuous in $\hat{\mathcal{H}}$, and the convexity (strictly) of $\hat{J}(\varphi^0)$ comes from the inequality (2.9).

Firstly, we should get approximate controllability of (2.27), i.e., for $\epsilon > 0$ we could find approximate controls $u_0^\epsilon, u_1^\epsilon$, $u_2^\epsilon$ such that the solution $y_\epsilon$ of heat equation satisfies the condition (2.10). For this, we consider new functional very similar with $\hat{J}$: for any $\epsilon > 0$ and $y^1 \in L^2(0,1)$

$$\begin{aligned}\hat{J}_\epsilon(\varphi^0) &= \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2, \Big|\int_0^1 f_1(x)\varphi dx\Big|^2, \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt \\ &\quad + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} + \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx\end{aligned}$$

where $E$ is finite dimensional subspace of $L^2(0,1)$ and $\pi_E$ denotes the orthogonal projection from $L^2(0,1)$ over $E$.

**Lemma 2.7.** *Assume that the following unique continuation property holds:*

$$\mu\Big\{t\in(0,T) : \Big|\int_0^1 f_0\varphi dx\Big| = \Big|\int_0^1 f_1\varphi dx\Big| = \Big|\int_0^1 f_2\varphi dx\Big|\Big\} > 0 \Longrightarrow \varphi\equiv 0. \qquad (2.31)$$

*Then the heat system (2.27) is approximate controllable.*

*Proof.* For obtaining approximate controllability of (2.27), we should minimize $\hat{J}_\epsilon(\varphi^0)$ over $\mathcal{H}$. We have already known that $\hat{J}_\epsilon$ is strictly convex and continuous in $\hat{\mathcal{H}}$. Also, in view of the unique continuation property above, one can prove the following property of $\hat{J}_\epsilon$:

$$\lim_{||\varphi^0||_{L^2(0,1)}\to\infty} \frac{\hat{J}_\epsilon(\varphi^0)}{||\varphi^0||} \geq \epsilon.$$

Hence, we have proved that $\hat{J}_\epsilon$ is convex, continuous and coercive in $\hat{\mathcal{H}}$. Therefore $\hat{J}_\epsilon$ admits an unique minimizer $\hat{\varphi}^0 \in \hat{\mathcal{H}}$, i.e., for any $\psi^0 \in L^2(0,1)$ and $h \in \mathbb{R}$ sufficiently small, we will have $\hat{J}_\epsilon(\hat{\varphi^0}) \leq \hat{J}_\epsilon(\hat{\varphi^0} + h\psi^0)$. More precisely,

$$
\begin{aligned}
\Delta \hat{J}_\epsilon &= \int_{I_0} h \int_0^1 f_0(x)\hat{\varphi}(x,t)dx \int_0^1 f_0(x)\psi(x,t)dxdt \\
&+ \int_{I_1} h \int_0^1 f_1(x)\hat{\varphi}(x,t)dx \int_0^1 f_1(x)\psi(x,t)dxdt \\
&+ \int_{I_2} h \int_0^1 f_2(x)\hat{\varphi}(x,t)dx \int_0^1 f_2(x)\psi(x,t)dxdt \\
&+ \int_{I_0} h^2 \Big| \int_0^1 f_0(x)\psi(x,t)dx \Big|^2 dt + \int_{I_1} h^2 \Big| \int_0^1 f_1(x)\psi(x,t)dx \Big|^2 dt \\
&+ \int_{I_2} h^2 \Big| \int_0^1 f_2(x)\psi(x,t)dx \Big|^2 dt - \int_0^1 hy^0(x)\psi(x,0)dx + \int_0^1 h\psi^0 y^1 dx \\
&+ \epsilon \Big[ ||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)} \Big] \geq 0
\end{aligned}
$$

where

$$
\begin{aligned}
I_0 &\stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^1 f_0(x)\varphi dx \Big| > \max\Big( \Big| \int_0^1 f_1(x)\varphi dx \Big|, \Big| \int_0^1 f_2(x)\varphi dx \Big| \Big) \Big\}, \\
I_1 &\stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^1 f_1(x)\varphi dx \Big| > \max\Big( \Big| \int_0^1 f_0(x)\varphi dx \Big|, \Big| \int_0^1 f_2(x)\varphi dx \Big| \Big) \Big\}, \\
I_2 &\stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^2 f_2(x)\varphi dx \Big| > \max\Big( \Big| \int_0^1 f_0(x)\varphi dx \Big|, \Big| \int_0^1 f_1(x)\varphi dx \Big| \Big) \Big\}.
\end{aligned}
$$

Let us define

$$
\begin{aligned}
\hat{\mathcal{A}} \stackrel{\text{def}}{=} & \int_{I_0} \int_0^1 f_0(x)\hat{\varphi} dx \int_0^1 f_0(x)\psi dxdt + \int_{I_1} \int_0^1 f_1(x)\hat{\varphi} dx \int_0^1 f_1(x)\psi dxdt \\
&+ \int_{I_2} \int_0^1 f_2(x)\hat{\varphi} dx \int_0^1 f_2(x)\psi dxdt.
\end{aligned}
$$

After using (2.13) and considering cases: $h > 0$, $h < 0$ and taking $h \to 0$, at the end, we have:

$$
\Big| \hat{\mathcal{A}} + \int_0^1 \psi^0 y^1 dx - \int_0^1 \psi(x,0)y^0(x)dx \Big| \leq \epsilon \Big[ ||((I-\pi_E)\psi^0||_{L^2(0,1)} \Big]. \tag{2.32}
$$

Now, if we take

$$u_0^\epsilon(t) = -1_{I_0} \int_0^1 f_0(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.33}$$

$$u_1^\epsilon(t) = -1_{I_1} \int_0^1 f_1(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.34}$$

$$u_2^\epsilon(t) = -1_{I_2} \int_0^1 f_0(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.35}$$

and multiplying the heat equation (2.27) with initial data $y^0(x) \in L^2(0,1)$ by $\psi$ which is the solution of adjoint system (2.3) with initial data $\psi^0$ and integrating by parts we finally get

$$\hat{\mathcal{A}} = \int_0^1 \psi(x,0)y^0(x)dx - \int_0^1 \psi^0 y(x,T)dx$$

and putting this identity into (2.32), and letting $E = 0$, we finally get

$$\Big| \int_0^1 \psi^0(y(x,T) - y^1)dx \Big| \leq \epsilon ||\psi^0||_{L^2(0,1)}$$

for every $\psi^0 \in L^2(0,1)$ which is equivalent to (2.10) and letting $y^1 = 0$, we get (2.15), i.e., (2.27) is approximate controllable. □

From Lemma 2.7, we understand that, for approximate controllability of (2.27) it suffices to obtain (2.31). Observe that

$$I_0^1 \stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^1 f_0(x)\varphi(x,t)dx \Big| = \Big| \int_0^1 f_1(x)\varphi(x,t)dx \Big| \Big\},$$

$$I_1^2 \stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^1 f_1(x)\varphi(x,t)dx \Big| = \Big| \int_0^1 f_2(x)\varphi(x,t)dx \Big| \Big\},$$

$$I_2^0 \stackrel{\text{def}}{=} \Big\{ t \in (0,T) : \Big| \int_0^1 f_2(x)\varphi(x,t)dx \Big| = \Big| \int_0^1 f_0(x)\varphi(x,t)dx \Big| \Big\},$$

are of positive measure. Now using (2.29), we have

$$\int_0^1 f_0(x)\varphi(x,t)dx \pm \int_0^1 f_1(x)\varphi(x,t)dx = \sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)} (f_{1,k} \pm f_{0,k}).$$

The function $\int_0^1 \varphi(x,t)(f_0(x) \pm f_1(x))dx$ are time analytic for $t \leq T$. Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(f_{1,k} \pm f_{0,k}) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$, for all $k \geq 1$, it is sufficient to assume that

$$f_{1,k} \pm f_{0,k} \neq 0, \qquad \forall k \geq 1. \tag{2.36}$$

Similarly, we will have:

$$f_{2,k} \pm f_{1,k} \neq 0, \qquad \forall k \geq 1. \tag{2.37}$$

$$f_{0,k} \pm f_{2,k} \neq 0, \qquad \forall k \geq 1. \tag{2.38}$$

As a result, under the assumption of (2.36), (2.37) and (2.38), we prove that (2.31) satisfies. Now, we would like to say that for each $\epsilon > 0$ we must have the fact that $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.

**Lemma 2.8.** *Assume that the Fourier coefficient of the initial datum $y^0$ of (2.27) satisfying*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2} |y_k^0|^2 < \infty. \tag{2.39}$$

*Then $y^0 \in \hat{\mathcal{H}}'$ and our approximate controls $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.*

*Proof.* We will skip the proof of the first part, i.e., $y^0 \in \hat{\mathcal{H}}'$ which comes from direct application of Cauchy-Schwarz inequality. Now, let us prove that our approximate controls $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. Observe that at the minimizer $\hat{\varphi}_\epsilon^0$ we have $\hat{J}_\epsilon(\hat{\varphi}_\epsilon^0) \leq \hat{J}_\epsilon(0) = 0$. This implies that

$$\begin{aligned}
&\frac{1}{6}\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big] dt \\
&\leq \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2, \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2, \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big] dt \\
&\leq \Big|\int_0^1 y_0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|.
\end{aligned}$$

From (2.30), we have

$$\begin{aligned}
&\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big] dt \\
&\qquad \leq \frac{\hat{C}\Big|\int_0^1 y_0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2 k^2 T}\{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2\}\Big|}
\end{aligned}$$

for suitable $\hat{C} > 0$ is independent of $\{\beta_k\}_{k\geq 1}$. Since $\{\omega_k(x)\}_{k\geq 1}$ form orthonormal basis in $L^2(0,1)$ after some simplification, we have

$$\begin{aligned}
&\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon(x,t) dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon(x,t) dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon(x,t) dx\Big|^2 dt\Big] \\
&\qquad \leq \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2 k^2 T}\{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2\}\Big|}.
\end{aligned}$$

But applying Cauchy-Schwarz inequality, we will obtain:

$$\begin{aligned}
\frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2 k^2 T}\{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2\}\Big|} &= \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\hat{\upsilon}_k\beta_k\hat{\upsilon}_k^{-1}\Big|^2}{\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|} \\
&\overset{\text{CS}}{\leq} \frac{\hat{C}\Big|\sum_{k\geq 1}(y_k^0\hat{\upsilon}_k)^2\Big|\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|}{\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|} \\
&= \hat{C}\sum_{k\geq 1}(y_k^0\hat{\upsilon}_k)^2 < \infty.
\end{aligned}$$

where

$$\hat{\upsilon}_k = \left| \frac{e^{2\pi^2 k^2 T}}{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2} \right|^{\frac{1}{2}}$$

and $\hat{C} > 0$ is independent of $\{\beta_k\}_{k\geq 1}$.

Therefore, we conclude that $\forall \epsilon > 0$, $u_0^\epsilon(t), u_1^\epsilon(t)$ and $u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. □

Since $\{u_0^\epsilon\}_{\epsilon>0}$, $\{u_1^\epsilon\}_{\epsilon>0}$ and $\{u_2^\epsilon\}_{\epsilon>0}$ are uniformly bounded, by extracting subsequences, we have $u_0^\epsilon \rightharpoonup u_0$, $u_1^\epsilon \rightharpoonup u_1$ and $u_2^\epsilon \rightharpoonup u_2$ weakly in $L^2(0,T)$. Hence using the continuous dependence of the solution of the heat equation, we can show that $y_\epsilon(x,T)$ converges to $y(x,T)$ weakly in $L^2(0,T)$ which implies that $y(x,T) = 0$ , i.e., the limit controls $u_0$, $u_1$ and $u_2$ fulfil the null controllability requirement.

Consequently, we obtain the following result:

**Theorem 2.9.** *Assume that $f_0(x), f_1(x), f_2(x)$ are three control profiles in $L^2(0,1)$ in which their Fourier coefficients satisfy (2.36), (2.37) and (2.38). Let the initial datum $y^0$ be in $\hat{\mathcal{H}}'$. More precisely, let Fourier coefficients of $y^0$ satisfy (2.39). Then, for all $T > 0$, there exist switching controls*

$$\begin{aligned}
u_0(t) &= -1_{I_0} \int_0^1 f_0(x)\hat{\varphi}(x,t)dx, \\
u_1(t) &= -1_{I_1} \int_0^1 f_1(x)\hat{\varphi}(x,t)dx, \\
u_2(t) &= -1_{I_2} \int_0^1 f_0(x)\hat{\varphi}(x,t)dx,
\end{aligned}$$

*satisfying (2.28) and solution of heat equation (2.27) satisfies*

$$y(x,T) = 0.$$

*These switching controls can be obtained by minimizing the functional $\hat{J}$ over $\hat{\mathcal{H}}$.*

In general, we could examine the case in which $n \in \mathbb{N}$ control profiles given in $L^2(0,1)$. Consider the heat equation:

$$\begin{cases} y_t - y_{xx} = \sum_{i=1}^{i=n} u_i(t) f_i(x), & 0 < x < 1, \ 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.40}$$

Here now, given an initial datum $y_0 \in L^2(0,1)$ we are looking for controls $\{u_i\}_{i=1}^{i=n} \in L^2(0,T)$ such that null controllability of heat equation holds, i.e, $y(x,T) = 0$ and switching condition satisfies:

$$u_i(t)u_j(t) = 0, \quad \forall i \neq j, \qquad \text{a.e. } t \in (0,T). \tag{2.41}$$

At first, we will consider the approximate controllability problem. To obtain approximate switching controls, one should minimize an appropriate quadratic functional over suitable Hilbert space, and under some conditions on the Fourier coefficients of $y_0$, we will get our desired null switching controls satisfying switching property.

In conclusion, we obtain following general result for switching controls:

**Theorem 2.10.** *Assume that $\{f_i(x)\}_{i=1}^{i=n}$ are $n$ control profiles in $L^2(0,1)$ and their Fourier expansions are*

$$f_i(x) = \sum_{k \geq 1} f_{i,k} \omega_k(x), \qquad i \in \{1,2,..,n\}, \ \forall k \geq 1$$

*and satisfying*

$$(f_{i,k} \pm f_{j,k}) \neq 0, \quad i \neq j, \quad i,j \in \{1,2,..,n\}, \ \forall k \geq 1.$$

*Now, let the initial datum $y^0$ be in $H_n'$ which is the dual space of the class of initial* *data given by*

$$H_n = \{\varphi^0 : \int_0^T \Big[\sum_{i=1}^{n} \Big| \int_0^1 f_i(x)\varphi(x,t)dx\Big|^2\Big]dt < \infty\}.$$

*More precisely, let the Fourier coefficients of* $y^0$ *satisfy*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{\sum_{i=1}^{i=n} |f_{i,k}|^2} |y_k^0|^2 < \infty.$$

*Then, for all* $T > 0$*, there exist switching controls* $\{u_i(t)\}_{i=1}^{i=n} \in L^2(0,T)$ *satisfying (2.41) and solution of heat equation with* $\{f_i(x)\}_{i=1}^{i=n}$ *control profiles satisfies null controllability condition, i.e.,* $y(x,T) = 0$.

*These controls are*

$$u_i(t) = -\int_0^1 f_i(x)\hat{\varphi}(x,t)dx, \quad u_j(t) = 0, \ \ \forall j \neq i, \quad in\ S_i, \quad \forall i \in \{1,2,..,n\}$$

*where* $\{S_i\}_{i=1}^{i=n}$ *defined by*

$$S_i = \Big\{t \in (0,T) : \Big| \int_0^1 f_i(x)\hat{\varphi}(x,t)\Big| > \max_{\substack{1\leq j\leq n \\ j\neq i}} \Big(\Big| \int_0^1 f_j(x)\hat{\varphi}(x,t)\Big|\Big)\Big\}.$$

*These switching controls can be obtained by minimizing the functional*

$$\hat{J}_s^n(\varphi^0) = \frac{1}{2}\int_0^T \max_{1\leq i\leq n} \Big\{\Big| \int_0^1 f_i(x)\varphi(x,t)dx\Big|^2\Big\}dt - \int_0^1 y^0(x)\varphi(x,0)dx$$

*over the Hilbert space* $H_n$.

# 3. SWITCHING CONTROLS FOR THE 1-D HEAT EQUATION WITH RBC

In this part, we will consider the null controllability of the 1-d heat equation under Robin's boundary condition with positive eigenvalues, with two boundary controls, pointwise controls and lumped controls respectively and obtain sufficient conditions for switching controls.

## 3.1. Boundary Controls

Consider the heat equation in the space interval $(0,1)$ with two controls located at the extremes $x = 0$,1 and satisfying RBC:

$$\begin{cases} y_t - y_{xx} = 0, & 0 < x < 1, \ 0 < t < T, \\ y_x(0,t) - a_0 y(0,t) = u_0(t), & 0 < t < T, \\ y_x(1,t) + a_1 y(1,t) = u_1(t), & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{3.1}$$

We consider the problem of null controllability. More precisely, given an initial datum $y_0 \in L^2(0,1)$ we look for controls $u_0(t), u_1(t) \in L^2(0,T)$ such that $y(x,T) = 0$ and satisfying switching property:

$$u_0(t)u_1(t) = 0, \qquad \text{a.e.} \quad t \in (0,T). \tag{3.2}$$

As we discussed in Chapter 2, whenever a system is controllable, the control can be constructed by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system.

For $\varphi^0$ in $L^2(0,1)$, we consider the solution $\varphi : [0,1] \times [0,T] \to C([0,T], L^2(0,1))$

of the following backward Cauchy linear problem:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1, \ 0 < t < T, \\ \varphi_x(0,t) - a_0\varphi(0,t) = 0, & 0 < t < T, \\ \varphi_x(1,t) + a_1\varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases} \tag{3.3}$$

This linear system is called the adjoint system corresponding to the 1-d heat equation with Robin's boundary condition. In [2], we know that the Fourier representation of solutions of the adjoint system with positive eigenvalues are of the form:

$$\varphi(x,t) = \sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)}\omega_k(x) \tag{3.4}$$

where

$$\omega_k(x) = \cos\mu_k x + \frac{a_0}{\mu_k}\sin\mu_k x. \tag{3.5}$$

Now, for obtaining switching controls, we will consider the quadratic functional:

$$J_s^\alpha(\varphi^0) = \frac{1}{2}\int_0^T \max\Big\{|\varphi(0,t)|^2, |\alpha\varphi(1,t)|^2\Big\}dt + \int_0^1 y^0(x)\varphi(x,0)dx \tag{3.6}$$

where $\alpha \in \mathbb{R}$ and minimize (3.6) over the class $\mathcal{H}$ of initial data given by

$$\mathcal{H} = \{\varphi^0 : \int_0^T \Big[|\varphi(0,t)|^2 + |\varphi(1,t)|^2\Big]dt < \infty\}.$$

where $\varphi(x,t)$ is the solution of the adjoint system (3.3) associated to the final state $\varphi^0$. This space endowed with the canonical norm:

$$||\varphi^0||_{\mathcal{H}} = \Big[\int_0^T |\varphi(0,t)|^2 + |\varphi(1,t)|^2 dt\Big]^{\frac{1}{2}}$$

constitutes a Hilbert space. At first, we will show that $||\varphi^0||_{\mathcal{H}}$ actually defines a norm on $\mathcal{H}$, i.e., satisfying the properties (2.5), (2.6) and (2.7). As we discussed in Chapter 2, we could obtain the property (2.7) by using the positivity of $||\cdot||_{\mathcal{H}}$. Therefore let us analyse the positivity of the norm $||\cdot||_{\mathcal{H}}$ in $\mathcal{H}$ space. Here we will use (3.4), (3.5) as a Fourier representation of the solution of (3.3). Therefore we have:

$$\int_0^T |\varphi(0,t)|^2 + |\alpha\varphi(1,t)|^2 dt \geq \int_0^T |\varphi(0,t)|^2 dt = \int_0^T \Big| \sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)} \Big|^2 dt. \tag{3.7}$$

Observe that

$$\begin{aligned} ||\varphi(x,0)||^2_{L^2(0,1)} &= \int_0^1 \varphi^2(x,0)dx = \int_0^1 \Big| \sum_{k\geq 1} \beta_k e^{-\mu_k^2 T} \omega_k(x) \Big|^2 dx \\ &\leq C \sum_{k\geq 1} \beta_k^2 e^{-2\mu_k^2 T} \end{aligned} \tag{3.8}$$

where $C > 0$ is independent of $\{\beta_k\}_{k\geq 1}$. Also, by using Lemma 2.1 on (3.7) and comparing with inequality (3.8), we will have the following observability inequality:

$$||\varphi(x,0)||^2_{L^2(0,1)} \leq \hat{C} \int_0^T |\varphi(0,t)|^2 + |\alpha\varphi(1,t)|^2 dt \tag{3.9}$$

for positive constant $\hat{C} > 0$ is independent of $\{\beta_k\}_{k\geq 1}$.

Therefore, we get the property (2.5). Hence $||\cdot||_{\mathcal{H}}$ defines norm on $\mathcal{H}$. The functional $J_s^\alpha : \mathcal{H} \longrightarrow \mathbb{R}$ is well defined, continuous, and strictly convex. For checking the coercivity property, one should prove (2.12). For this to be true, the following unique continuation property of the adjoint system (3.3) suffices:

$$\mu\{t \in (0,T) : |\varphi(0,t)| = |\alpha\varphi(1,t)|\} > 0 \Rightarrow \varphi \equiv 0. \tag{3.10}$$

**Lemma 3.1.** *Assume that*

$$|\alpha| \neq \left[\frac{\mu_k^2 + a_1^2}{\mu_k^2 + a_0^2}\right]^{\frac{1}{2}}, \qquad \forall k \geq 1, \tag{3.11}$$

*holds. Then, (3.10) will satisfy for the adjoint system (3.3).*

*Proof.* First, assume that $\mu\{t \in (0,T) : |\varphi(0,t)| = |\alpha\varphi(1,t)|\} > 0$. We will show that under the assumption of (3.11), we have $\varphi \equiv 0$. In [2], we know that positive eigenvalues $\{\mu_k\}_{k\geq 1}$ of adjoint system (3.3) satisfy eigenvalue equation:

$$\tan(\mu_k) = \frac{(a_0 + a_1)\mu_k}{\mu_k^2 - a_0 a_1} \tag{3.12}$$

and also we have

$$(k-1)\pi < \mu_k < k\pi \quad \text{and} \quad \lim_{k\to\infty} \mu_k - (k-1)\pi = 0, \quad (k = 1, 2, 3, 4, ...).$$

Now assume that $\mu(I) > 0$, using again the Fourier representation of solution (3.4) of (3.3), we have

$$\varphi(0,t) \pm \alpha\varphi(1,t) = \sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)}(1 \pm \alpha(\cos\mu_k + \frac{a_0}{\mu_k}\sin\mu_k)).$$

The function $\varphi(0,t) \pm \alpha\varphi(1,t)$ are time analytic for $t \leq T$. Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by the real exponentials $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(1 \pm \alpha(\cos\mu_k + \frac{a_0}{\mu_k}\sin\mu_k)) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$ for all $k \geq 1$, it is sufficient to have that

$$1 \pm \alpha(\cos\mu_k + \frac{a_0}{\mu_k}\sin\mu_k) \neq 0. \tag{3.13}$$

Assume the converse of (3.13), then we will have

$$\begin{aligned}\alpha(\cos\mu_k + \frac{a_0}{\mu_k}\sin\mu_k) \pm 1 = 0 \qquad &\Longleftrightarrow [\alpha(\cos\mu_k + \frac{a_0}{\mu_k}\sin\mu_k)]^2 = 1\\ &\Longleftrightarrow \alpha^2(1 + 2\frac{a_0}{\mu_k}\tan\mu_k + \frac{a_0^2}{\mu_k^2}\tan^2\mu_k) = 1 + \tan^2\mu_k.\end{aligned}$$

Now using eigenvalue equation (3.12) and after some simplification, finally we will obtain the following:

$$\alpha^2(\mu_k^2 + a_0^2) = (\mu_k^2 + a_1^2)$$

which is a contradiction with the assumption. □

Therefore, by using Lemma 3.1, we have that $J_s^\alpha$ admits an unique minimizer $\hat{\varphi}^0 \in \mathcal{H}$. As a result, by using variational approach, we find our switching controls:

$$u_0(t) = -\hat{\varphi}(0,t)1_{S_0}, \qquad u_1(t) = \alpha^2\hat{\varphi}(1,t)1_{S_1} \qquad \text{for } t \in (0,T) \tag{3.14}$$

where

$$\begin{aligned}S_0 &= \{t \in (0,T) : |\varphi(0,t)| > |\alpha\varphi(1,t)|\},\\ S_1 &= \{t \in (0,T) : |\alpha\varphi(1,t)| > |\varphi(0,t)|\}.\end{aligned}$$

**Theorem 3.2.** *Consider the null controllability problem for 1-d heat equation with boundary controls satisfying RBC. By minimizing the functional (3.6) under the condition (3.11) over $\mathcal{H}$, for all $T > 0$, we obtain switching controls (3.14) satisfying switching condition (3.2) and that the solution of (3.1) satisfies*

$$y(x,T) = 0$$

*i.e, null controllability is satisfied.*

### 3.2. Pointwise Controls

Now in this part, we will consider the case in which two pointwise controllers act at two different points $a$ and $b$ of the space interval $(0,1)$ where the equation is satisfied. Consider the heat equation:

$$\begin{cases} y_t - y_{xx} = u_a(t)\delta_a + u_b(t)\delta_b, & 0 < x < 1,\ 0 < t < T, \\ y_x(0,t) - a_0 y(0,t) = 0, & 0 < t < T, \\ y_x(1,t) + a_1 y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{3.15}$$

We consider the problem of null controllability. More precisely, given an initial datum $y_0 \in L^2(0,1)$ we look for controls $u_a(t)$ and $u_b(t) \in L^2(0,T)$ such that $y(x,T) = 0$ and the switching condition satisfies:

$$u_a(t)u_b(t) = 0, \qquad \text{a.e. } t \in (0,T). \tag{3.16}$$

Now, we consider the quadratic functional:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\Big\{|\varphi(a,t)|^2, |\varphi(b,t)|^2\Big\}dt - \int_0^1 y^0(x)\varphi(x,0)dx \tag{3.17}$$

over the class $\hat{\mathcal{H}}$ of initial data given by

$$\hat{\mathcal{H}} = \{\varphi^0 : \int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2\Big]dt < \infty\} \tag{3.18}$$

where $\varphi(x,t)$ is the solution of the adjoint system (3.3) associated to the final state $\varphi^0$. This space endowed with the canonical norm:

$$||\varphi^0||_{\hat{\mathcal{H}}} = \Big[\int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 dt\Big]^{\frac{1}{2}}$$

constitutes a Hilbert space. First of all, we will show that $||\cdot||_{\hat{\mathcal{H}}}$ actually defines norm on $\hat{\mathcal{H}}$, i.e., the properties (2.5), (2.6) and (2.7) must be satisfied. As we noted before, we could directly obtain the property (2.7) by using the positivity of $||\cdot||_{\hat{\mathcal{H}}}$ and the inequality (2.20). Therefore let us analyse the positivity of the norm $||\cdot||_{\hat{\mathcal{H}}}$ in $\hat{\mathcal{H}}$ space. Here it is convenient to use the Fourier representation of solutions (3.4) of the adjoint system (3.3). Hence, we have:

$$\begin{aligned}
||\varphi^0||^2_{\hat{\mathcal{H}}} &= \int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 dt \\
&= \int_0^T \Big[\Big|\sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)}\omega_k(a)\Big|^2 dt + \Big|\sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)}\omega_k(b)\Big|^2 dt.
\end{aligned}$$

Using Lemma 2.1, we finally get weighted observability inequality:

$$||\varphi^0||^2_{\hat{\mathcal{H}}} \geq c_1 \sum_{k\geq 1} e^{-2\mu_k^2 T}\Big[|\omega_k(a)|^2 + |\omega(b_k)|^2\Big]\beta_k^2. \tag{3.19}$$

Hence $||\cdot||_{\hat{\mathcal{H}}}$ defines a norm on $\hat{\mathcal{H}}$. Observe that the functional $J_s : \mathcal{H} \longrightarrow \mathbb{R}$ is well defined, continuous thanks to well-posedness of (3.3) and strictly convex due to (3.19). As same before, we will consider the problem of approximate controllability. For this, we will construct the new functional very similar with previous one (3.17) and with the same coercivity property, allows building approximate controllers: for any $\epsilon > 0$ and $y^1 \in L^2(0,1)$,

$$\begin{aligned}
J_s^\epsilon(\varphi^0) &= \frac{1}{2}\int_0^T \max\Big\{|\varphi(a,t)|^2, |\varphi(b,t)|^2\Big\}dt + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} \\
&+ \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx
\end{aligned}$$

where $E$ is finite dimensional subspace of $L^2(0,1)$ and $\pi_E$ denotes the orthogonal projection from $L^2(0,1)$ over $E$. When $y^1 = 0$ and $E = 0$, we will obtain (3.17).

**Lemma 3.3.** *Assume that the adjoint system (3.3) satisfies the following unique continuation property:*

$$\mu\{t \in (0,T) : |\varphi(a,t)| = |\varphi(b,t)|\} > 0 \Rightarrow \varphi \equiv 0. \tag{3.20}$$

*Then, the heat system (3.15) is approximate controllable.*

We will skip the proof in which one should firstly obtain (2.12) under the assumption of (3.20) and then by using variational approach on the functional $J_s^\epsilon$, one could easily get approximate controls for (3.15) (see e.g., [3]). Now let us obtain sufficient conditions for getting (3.20). Using again the Fourier representation of solution (3.4) of adjoint system (3.3), we have

$$\varphi(a,t) \pm \varphi(b,t) = \sum_{k\geq 1} \beta_k e^{\mu_k^2(t-T)}(\omega_k(a) \pm \omega_k(b)).$$

The function $\varphi(a,t) \pm \varphi(b,t)$ are time analytic for $t \leq T$. Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by the real exponentials $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(\omega_k(a) \pm \omega_k(b)) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$ for all $k \geq 1$, it is sufficient to show that

$$\omega_k(a) \pm \omega_k(b) \neq 0 \qquad \forall k \geq 1.$$

But

$$\begin{aligned}
\omega_k(a) \pm \omega_k(b) = 0 \qquad &\Longleftrightarrow \quad [\cos\mu_k a + \frac{a_0}{\mu_k}\sin\mu_k a]^2 = [\cos\mu_k b + \frac{a_0}{\mu_k}\sin\mu_k b]^2 \\
&\Longleftrightarrow \quad (\cos^2\mu_k a - \cos^2\mu_k b) + \frac{a_0^2}{\mu_k^2}(\sin^2\mu_k a - \sin^2\mu_k b) \\
&\qquad + \ 2\frac{a_0}{\mu_k}(\cos\mu_k a\sin\mu_k a - \cos\mu_k b\sin\mu_k b) = 0.
\end{aligned}$$

Observe that

$$\cos^2\mu_k a - \cos^2\mu_k b = \sin^2\mu_k b - \sin^2\mu_k a = -\sin\mu_k(a+b)\sin\mu_k(a-b)$$

and finally,

$$\begin{aligned}\cos\mu_k a\sin\mu_k a &= \cos^2\mu_k\frac{a-b}{2}\cos\mu_k\frac{a+b}{2}\sin\mu_k\frac{a+b}{2}\\&- \sin^2\mu_k\frac{a-b}{2}\cos\mu_k\frac{a+b}{2}\sin\mu_k\frac{a+b}{2}\\&- \sin^2\mu_k\frac{a+b}{2}\cos\mu_k\frac{a-b}{2}\sin\mu_k\frac{a-b}{2}\\&+ \cos^2\mu_k\frac{a+b}{2}\cos\mu_k\frac{a-b}{2}\sin\mu_k\frac{a-b}{2},\end{aligned}$$

$$\begin{aligned}\cos\mu_k b\sin\mu_k b &= \cos^2\mu_k\frac{a-b}{2}\cos\mu_k\frac{a+b}{2}\sin\mu_k\frac{a+b}{2}\\&- \sin^2\mu_k\frac{a-b}{2}\cos\mu_k\frac{a+b}{2}\sin\mu_k\frac{a+b}{2}\\&+ \sin^2\mu_k\frac{a+b}{2}\cos\mu_k\frac{a-b}{2}\sin\mu_k\frac{a-b}{2}\\&- \cos^2\mu_k\frac{a+b}{2}\cos\mu_k\frac{a-b}{2}\sin\mu_k\frac{a-b}{2}.\end{aligned}$$

Hence using above identities, we get the following

$$\begin{aligned}(1-\frac{a_0^2}{\mu_k^2})\sin\mu_k(a-b)\sin\mu_k(a+b) &= 2\frac{a_0}{\mu_k}\sin\mu_k(a-b)[\cos^2\mu_k\frac{a+b}{2}-\sin^2\mu_k\frac{a+b}{2}]\\&= 2\frac{a_0}{\mu_k}\sin\mu_k(a-b)\cos\mu_k(a+b).\end{aligned}$$

Assume first

$$\sin\mu_k(a-b)\neq 0, \qquad \text{for } k\geq 1 \tag{3.21}$$

and after some simplification, we will obtain the following identity:

$$\frac{\mu_k^2-a_0^2}{\mu_k^2}\sin\mu_k(a+b)\neq 2\frac{a_0}{\mu_k}\cos\mu_k(a+b), \qquad \text{for } k\geq 1. \tag{3.22}$$

Hence, if we assume that $a, b\in(0,1)$ such that satisfying (3.21) and (3.22), then (3.20) will be satisfied, i.e., $\beta_k = 0$ for all $k\geq 1$.

Under the assumption of (3.20), by using variational approach we will obtain our approximate controls:

$$u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t), \qquad u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t),$$

where

$$S_a = \{t \in (0,T) : |\varphi(a,t)| > |\varphi(b,t)|\},$$
$$S_b = \{t \in (0,T) : |\varphi(b,t)| > |\varphi(a,t)|\}.$$

Under the assumption on Fourier coefficients of initial datum $y^0(x)$:

$$\sum_{k\geq 1} \frac{e^{2\mu_k^2 T}}{|\omega_k(a)|^2 + |\omega_k(b)|^2} |y_k^0|^2 < \infty, \tag{3.23}$$

one could show that approximate internal controls $u_a^\epsilon(t)$ and $u_b^\epsilon(t)$ satisfy uniform boundedness property in $L^2(0,T)$. Then passing to the limit, we will have null switching controls:

$$u_a(t) = -\hat{\varphi}(a,t), \qquad u_b(t) = -\hat{\varphi}(b,t). \tag{3.24}$$

In conclusion, we obtain the following result:

**Theorem 3.4.** *Consider the null controllability problem for 1-d heat equation with pointwise controls satisfying RBC. Assume that points $a, b \in (0,1)$ satisfying (3.21) and (3.22) and Fourier coefficients of initial datum $y^0(x)$ satisfy (3.23). Then, for all $T > 0$, there exist switching controls (3.24) satisfying switching condition (3.16) and that the solution of (3.15) satisfies*

$$y(x,T) = 0$$

*i.e, null controllability is satisfied.*

### 3.3. Lumped Controls

Similar results hold in the case of lumped controls, in which the pointwise Dirac controls of the previous section are replaced by controls distributed by means of give control functions. More precisely, let $f_0(x)$ and $f_1(x)$ be control profiles in $L^2(0,1)$. Consider the heat equation:

$$\begin{cases} y_t - y_{xx} = u_0(t)f_0(x) + u_1(t)f_1(x), & 0 < x < 1,\ 0 < t < T, \\ y_x(0,t) - a_0 y(0,t) = 0, & 0 < t < T, \\ y_x(1,t) + a_1 y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{3.25}$$

Here, we will consider the same problem, i.e., given an initial datum $y_0 \in L^2(0,1)$ we are looking for controls $u_0(t), u_1(t) \in L^2(0,T)$ such that null controllability of heat equation holds, i.e, $y(x,T) = 0$ and switching condition satisfies:

$$u_0(t)u_1(t) = 0, \qquad \text{a.e.} \quad t \in (0,T). \tag{3.26}$$

The null control of 1-d heat equation may be computed by minimizing the quadratic functional,

$$\begin{aligned} \hat{J}_s(\varphi^0) \;=\; & \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big|^2, \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big|^2\Big]dt \\ & - \int_0^1 y^0(x)\varphi(x,0)dx \end{aligned}$$

over the class $\tilde{\mathcal{H}}$ of initial data given by

$$\tilde{\mathcal{H}} = \{\varphi^0 : \int_0^T \Big[\Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big|^2 + \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big|^2\Big]dt < \infty\}.$$

This space endowed with the canonical norm:

$$||\varphi^0||^2_{\tilde{\mathcal{H}}} = \int_0^T \Big[\Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big|^2 + \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big|^2\Big]dt$$

will constitute a Hilbert space. At first, we will show that $||\cdot||_{\tilde{\mathcal{H}}}$ actually defines norm on $\tilde{\mathcal{H}}$. For this, it is sufficient to show the positivity of $||\cdot||_{\tilde{\mathcal{H}}}$. Observe that

$$||\varphi^0||^2_{\tilde{\mathcal{H}}} = \int_0^T \Big[\Big|\sum_{k\geq 1}\beta_k e^{\mu_k^2(t-T)} f_{0,k}\Big|^2 + \Big|\sum_{k\geq 1}\beta_k e^{\mu_k^2(t-T)} f_{1,k}\Big|^2\Big]dt$$

where

$$f_0(x) = \sum_{k\geq 1} f_{0,k}\omega_k(x), \qquad f_1(x) = \sum_{k\geq 1} f_{1,k}\omega_k(x),$$

and using Lemma 2.1, we then get weighted observability inequality:

$$||\varphi^0||^2_{\tilde{\mathcal{H}}} \geq c_1 \sum_{k\geq 1} e^{-2\mu_k^2 T}\Big[|f_{0,k}|^2 + |f_{1,k}|^2\Big]\beta_k^2 \tag{3.27}$$

where positive constant $c_1$ is independent from $\{\beta_k\}_{k\geq 1}$. Hence $||\cdot||_{\tilde{\mathcal{H}}}$ defines a norm in $\tilde{\mathcal{H}}$.

In addition, since (3.3) is well posed, the functional $\hat{J}(\varphi^0)$ is obviously continuous in $\tilde{\mathcal{H}}$, the convexity (strictly) of $\hat{J}(\varphi^0)$ comes from the weighted observability inequality (3.27). Now, consider new functional very similar with previous one: for any $\epsilon > 0$ and any $y^1 \in L^2(0,1)$

$$\begin{aligned}\hat{J}_\epsilon(\varphi^0) &= \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2, \Big|\int_0^1 f_1(x)\varphi dx\Big|^2\Big]dt \\ &\quad + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} + \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx\end{aligned}$$

where $E$ is finite dimensional subspace of $L^2(0,1)$ and $\pi_E$ denotes the orthogonal projection from $L^2(0,1)$ over $E$.

First of all, we will consider the problem of approximate controllability of (3.25).

**Lemma 3.5.** *Assume the following unique continuation property for (3.3):*

$$\mu\Big\{t \in (0,T) : \Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big| = \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big|\Big\} > 0 \Rightarrow \varphi \equiv 0. \qquad (3.28)$$

*Then the heat system (3.25) is approximate controllable.*

For the proof of Lemma 3.5, one should first prove that the functional $\hat{J}_\epsilon$ is coercive in $\tilde{\mathcal{H}}$ which directly comes from the assumption (3.28) and at the end, by using variational approach, one could easily get approximate controls for (3.25) (see e.g., [3]). Therefore, from Lemma 3.5, to get approximate controls, we should prove (3.28). Using (3.4), we have

$$\int_0^1 f_0(x)\varphi(x,t)dx \pm \int_0^1 f_1(x)\varphi(x,t)dx = \sum_{k\geq 1}\beta_k e^{\mu_k^2(t-T)}(f_{1,k} \pm f_{0,k}).$$

The function $\int_0^1 \varphi(x,t)(f_0(x) \pm f_1(x))dx$ are time analytic for $t \leq T$. Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by the real exponentials $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(f_{1,k} \pm f_{0,k}) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$ for all $k \geq 1$, it is sufficient to assume that

$$f_{1,k} \pm f_{0,k} \neq 0 \qquad \forall k \geq 1. \qquad (3.29)$$

Therefore, under the condition (3.29), we have approximate controllability of (3.25). Hence, for every $\epsilon > 0$, by using variational approach, we will obtain approximate switching controls. We would like to say that for each $\epsilon > 0$ we must have the fact that $u_0^\epsilon(t), u_1^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. But under the condition on Fourier coefficients of the initial datum $y^0$ :

$$\sum_{k\geq 1} \frac{e^{2\mu_k^2 T}}{|f_{0,k}|^2 + |f_{1,k}|^2} |y_k^0|^2 < \infty \tag{3.30}$$

being satisfied, by using weighted observability inequality (3.27), one could easily prove that $u_0^\epsilon(t)$ and $u_1^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. Hence at the end, by using variational approach we obtain the following switching controls

$$u_0(t) = -\int_0^1 f_0(x)\hat{\varphi}(x,t)dx, \qquad u_1(t) = 0, \qquad \text{in } S_0, \tag{3.31}$$

$$u_1(t) = -\int_0^1 f_1(x)\hat{\varphi}(x,t)dx, \qquad u_0(t) = 0, \qquad \text{in } S_1, \tag{3.32}$$

where

$$S_0 = \Big\{ t \in (0,T) : \Big| \int_0^1 f_0(x)\hat{\varphi}(x,t)dx \Big| > \Big| \int_0^1 f_1(x)\hat{\varphi}(x,t)dx \Big| \Big\},$$
$$S_1 = \Big\{ t \in (0,T) : \Big| \int_0^1 f_1(x)\hat{\varphi}(x,t)dx \Big| > \Big| \int_0^1 f_0(x)\hat{\varphi}(x,t)dx \Big| \Big\}.$$

In conclusion, we obtain the following result:

**Theorem 3.6.** *Consider the null controllability problem for 1-d heat equation with lumped controls satisfying RBC. Assume that $f_0(x), f_1(x)$ are two control profiles in $L^2(0,1)$ and their Fourier coefficients satisfying (3.29). Let Fourier coefficients of the initial datum $y^0$ satisfy (3.30). Then, for all $T > 0$, there exist switching controls (3.31) and (3.32) satisfying our switching condition (3.26) and that the solution of (3.25) satisfies*

$$y(x,T) = 0$$

*i.e, null controllability is satisfied.*

# 4. CONCLUSION

In this thesis, we examined the problem of switching controls of 1-d heat equation under Dirichlet's boundary condition and Robin's boundary condition respectively and obtained the sufficient conditions for switching controls. Also, it could be generalized for arbitrary (finite) number of switching controls act on 1-d heat equation under RBC with positive eigenvalues. Also, it would be interesting to analyse the same problem under RBC with zero or negative eigenvalue. In [2], we know that 1-d heat system has just one zero or negative eigenvalue if it exists and of course, when dealing with that problem, the sufficient conditions for switching controls would change. The null controls for 1-d heat equation with zero or negative eigenvalue would be computed by minimizing the appropriate quadratic functional over the solutions of adjoint system. However, the main difficulty is to show that these null controls are satisfying switching conditions.

The problem of controllability is by now only well understood in one space dimension. There is still to be done in this field to address controllability in several space dimensions and then for covering the non-linear free boundary problems. In [1], the techniques, Zuazua have developed in the case of one dimensional space can also be applied in the multidimensional case. However, the non-degeneracy conditions that need to be imposed are this time less explicit because they depend on the spectrum of the underlying elliptic operator. Besides, it would be interesting to see if the methodology we have applied and the results on switching controls can be adapted to more general equations involving, for instance, potentials depending both on space and time. There is a rich literature on the null control of those equations (see [3]). But the problem of switching controls has not been addressed so far.

Consequently, it must be underlined that most ideas, methods and results presented here do extend to this more general setting.

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