

DEFECTIVE RAMSEY NUMBERS AND DEFECTIVE COCOLORINGS

by

Ahu Akdemir

B.S., Industrial Engineering, Galatasaray University, 2009

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Industrial Engineering
Boğaziçi University

2012

Complete disorder is impossible.

Theodore S. Motzkin

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my thesis supervisor Assist. Prof. Tınaz Ekim Aşıcı for her invaluable support and encouragement throughout this study. Even though I was daunted in the beginning; her guidance, trust and willingness to help me were strong sources of motivation for me.

I owe thanks to Assist. Prof. Serdar Altok and Assist. Prof. Z. Caner Taşkın for their willingness to serve on my thesis committee and their insight, valuable advice, and suggestions throughout this process.

I wish to thank all my friends for being there whenever I needed. A special thanks belongs to Mehmet Hakan Akyüz, Arman Boyacı, Burak Kocuk and Özgür Emre Sivrikaya for their constant help and support.

I also wish to express my deepest gratitude to my parents and my grandmother for their never ending support, patience and endless love.

Last but not least, I would like to thank Ayberk Çalık for his never ending support, patience, optimism and for making me feel good in my difficult times for the last 10 years of my life.

ABSTRACT

DEFECTIVE RAMSEY NUMBERS AND DEFECTIVE COCOLORINGS

Ramsey number, $R(a, b)$, denotes the smallest positive integer n such that, among any n people, there exist a people who know each other or b people who do not know each other. $R(a, b)$ values are known for small a and b values and for unknown $R(a, b)$ values, the lower and upper bounds, namely the minimum and the maximum values that $R(a, b)$ can take, has been calculated. In this study, we study a generalization of Ramsey numbers which is called defective Ramsey numbers. Defective Ramsey number, $R_k(a, b)$, denotes the smallest positive integer n such that, among any n people, there exist a people who know each other or b people who do not know each other with at most k exceptions. We calculate some of the lower and upper bounds on some of the unknown $R_k(a, b)$ values and improve some of the lower bounds. We also study classical and defective Ramsey numbers in some graph classes and derive some formulas that either give the exact value of or an upper bound on Ramsey numbers for a specific graph class. In the second part, we study cocoloring of graphs. Cocoloring of a graph is an assignment of colors to the set of vertices such that each color class induces a clique or an independent set. We also studied defective cocoloring of graphs which considers k -densities, a defective clique in which the vertices can miss at most k vertices, and k -sparsities, a defective independent set in which the vertices can be adjacent to at most k vertices, as induced subgraphs. $c_k(m)$ is the parameter that gives the maximum graph order n such that all n -graphs can be k -defectively cocolored by using at most m colors. We propose the following results: $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 10$. We also prove a formula that gives the lower and upper bounds on the $c_k(m)$ parameter.

ÖZET

Kusurlu Ramsey Sayıları ve Kusurlu Tam Boyama

Ramsey sayısı, $R(a, b)$, herhangi n kişi arasında her zaman birbirini tanıyan a kişi ya da birbirini tanımayan b kişi olmasını sağlayacak en küçük pozitif n tamsayısını vermektedir. Küçük a ve b değerleri için Ramsey sayıları tam olarak bilinmektedir ve bilinmeyen Ramsey sayıları için alt ve üst sınırlar hesaplanmıştır. Bir Ramsey sayısının alt ve üst sınırları, bu Ramsey sayısının alabileceği sırasıyla en küçük ve büyük değerleri ifade etmektedir. Bu çalışmada Ramsey sayılarının bir genelleştirmesi olan kusurlu Ramsey sayıları üzerinde çalıştık. Kusurlu Ramsey sayısı, $R_k(a, b)$, n kişi arasında k istisnaıyla her zaman birbirini tanıyan a kişi ya da birbirini tanımayan b kişi olmasını sağlayacak en küçük pozitif n tamsayısını vermektedir. Bazı bilinmeyen $R_k(a, b)$ değerleri için alt ve üst sınırlar hesapladık ve bunların bazıları için alt sınırları geliştirdik. Ayrıca farklı çizge sınıflarında klasik ve kusurlu Ramsey sayılarını inceledik ve belirli bir çizge sınıfı için tam değerleri veya alt ve üst sınırları veren formüller geliştirdik. İkinci kısımda, çizgelerde tam boyama (cocoloring) problemi üzerine çalıştık. Bir çizgenin düğümlerini klik ya da bağımsız kümelerle ayırarak her birine bir renk atanması problemine tam boyama problemi denir. Klik ya da bağımsız kümeler yerine k -yoğun (k -dense), bir düğümün en fazla k düğüm kaçırıldığı klik, ve k -seyrek (k -sparse), bir düğümün en fazla k düğüme bağlanabileceği bağımsız küme, alt çizgelerle ilgilenen kusurlu tam boyama problemi de ele alınmıştır. $c_k(m)$ parametresi n düğümlü bütün çizgelerin k -kusurlu olarak en fazla m renkle tam boyanabileceği en büyük n sayısını vermektedir. Bu çalışmada şu sonuçlar elde edildi: $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 10$. Ayrıca $c_k(m)$ parametresi için alt ve üst sınırlar veren bir formül geliştirildi.

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LIST OF SYMBOLS

$c(m)$	The maximum graph order n such that all n -graphs can be cocolored by using at most m colors
$c_k(m)$	The maximum graph order n such that all n -graphs can be k -defectively cocolored by using at most m colors
C_n	A cycle on n vertices
EH	Any permutation graph contains a clique or an independent set of order at least EH
$ G $	The order of G
$hom(G)$	The maximum of the maximum clique number and the maximum independent set number of G
k	Defectiveness level
K_n	A clique on n vertices
$(m+1)$ - <i>extremal</i>	A graph G as $(m+1)$ - <i>extremal</i> for $c_k(m)$ if the order of G is $c_k(m) + 1$ and if G cannot be k -defectively cocolored by using m colors and requires $m + 1$ colors to k -defectively cocolor
$(m+1)$ - <i>cocritical</i>	A graph G as $(m+1)$ - <i>cocritical</i> if G has cochromatic number $m + 1$ but its induced subgraph G' , obtained by removing any vertex of G , has cochromatic number m
$R(a, b)$	The minimum graph order n such that all n -graphs contain a clique of order a or an independent set of order b
$R^{\mathcal{G}}(a, b)$	The minimum graph order n such that all n -graphs in graph class $\mathcal{G}$ contain a clique of order a or an independent set of order b
$R_k(a, b)$	The minimum graph order n such that all n -graphs contain a k -dense set of order a or a k -sparse set of order b
$R_k^{\mathcal{G}}(a, b)$	The minimum graph order n such that all n -graphs in graph class $\mathcal{G}$ contain a k -dense set of order a or a k -sparse set of order b
S_n	An independent set on n vertices
P_n	A path on n vertices

$V(G)$	Vertex set of G
$z(G)$	The cochromatic number of G
$z_k(G)$	The k -defective cochromatic number of G
$\alpha(G)$	The maximum independent set number of G
$\omega(G)$	The maximum clique number of G
$\chi(G)$	The chromatic number of G

1. INTRODUCTION

In the first decade of 20th century, one of the leading problems of mathematical logic was the problem of finding a regular procedure to determine the truth or falsity of any given logical formula. Researchers made intensive studies on this subject. In 1930, Frank Plumpton Ramsey published a paper named *On a Problem of Formal Logic* [1] which was the starting point of the *Ramsey theory*. He was studying on the problem in mathematical logic. He needed to show that the mathematical systems he was studying would always have a certain amount of order in them and this idea was the seed of the Ramsey theory. A few years later Ramsey's theorem was rediscovered by Paul Erdős and George Szekeres [2] while working on a problem in geometry. Problems in Ramsey theory typically ask a question of the following form: How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property? If we want to illustrate the problem with an example, we should consider the following question: How many people should we gather to guarantee that there are three people that know each other or three people that do not know each other? The answer of this question is, in a group of six people, there are always three people who know each other or three people who are complete strangers to each other. This problem is known as *The Party Problem* in graph theory [3, 4]. In general, the aim is to find the minimum number of people required in the group, say n , such that there are always a people that know each other or b people that are strangers to each other, where a and b are fixed given integers. When we translate this problem in graph-theoretical terms, a *Ramsey number* gives us the minimum graph order n such that all the graphs of order at least n contains a clique of order a or an independent set of order b and it is denoted by $R(a, b) = n$. For further reading on Ramsey numbers, see [5]. A *clique* in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subseteq V$, such that for every two vertices in C , there exists an edge connecting them. An *independent set* or *stable set* is a set of vertices in a graph such that no two vertices in this set are adjacent. The main objective is to find n values for larger a and b values. In the literature, Ramsey numbers are intensively utilized in different study

fields such as number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and theoretical computer science. For detailed information on these applications, see [6]. Because of the combinatorial structure of the problem, it is very difficult to show that the given order satisfies at least one of the requirements in every condition. Therefore, even though there are lots of studies in this field for years, there are only a few exactly known Ramsey numbers. As a result, researchers calculated lower and upper bounds for the unknown $R(a, b)$ values. We can see the exact values and the bounds represented so far in Table 2.1. Studies on classical Ramsey numbers in recent years are aimed to narrow the gaps between the bounds.

One may also want to give some tolerance to Ramsey numbers by relaxing the conditions on the subgraphs. In this version of the problem, we seek the answer for the following question: How many people should we gather to guarantee that there are three people such that each one must know at least another person or there are three people such that each one is allowed to know at most another person? In this way, it can be said that we have tolerated 1-*defectiveness* in the subgroups. In graph-theoretical terms, instead of cliques or independent sets, we consider *k-dense* or *k-sparse* sets in defective Ramsey numbers where k indicates the defectiveness level. A k -dense set is defined instead of the term k -defective clique and a k -sparse set is defined instead of the term k -defective independent set. *Defective Ramsey numbers*, $R_k(a, b)$, give us the minimum graph order n such that all the graphs of order at least n contains a k -dense set of order a or a k -sparse set of order b . Since the defective Ramsey numbers are the generalizations of Ramsey numbers, defective Ramsey numbers are also difficult to find. Defective Ramsey numbers is a relatively new generalization of Ramsey numbers which is studied in [7–9]. As mentioned, in the literature, lower and upper bounds have been represented when the exact values of classical Ramsey numbers are not known. A similar study for defective Ramsey numbers, to the best of our knowledge, has not been made. For this purpose, we have calculated some lower and upper bounds on the defective Ramsey numbers and improved most of the lower bounds by representing some counter examples. Best found bounds for defective Ramsey numbers are represented in Section 3.2.4.

We have also considered Ramsey numbers in some specific graph classes. For example, for P_4 -free graphs, graphs that do not contain induced paths on 4 vertices, we constructed a formula that gives the exact $R(a, b)$ values for P_4 -free graphs, which will be denoted as $R^{P_4\text{-free}}(a, b)$. Later, we have discovered that this formula has been represented in [2] for permutation graphs which is also valid for P_4 -free graphs since P_4 -free graphs is a subset of permutation graphs. But we have generalized this formula as an upper bound on Ramsey numbers for perfect graphs, which contains permutation graphs, and we propose the defective version of the formula again as an upper bound on defective Ramsey numbers for perfect graphs.

Another main subject of this study is cocoloring. Suppose that it is required to partition the people into groups such that every group consists of people that know every other person in the group or who do not know any other person in the group. In graph-theoretical terms, *cocoloring* is partitioning the vertex set of a graph into subgraphs, such that each subgraph is a clique or an independent set, and coloring each subgraph by a different color. The minimum number of groups such that every person is placed in a group satisfying this condition is known as *the cochromatic number* in graph theory. The cochromatic number is the minimum number of induced subgraphs to partition a graph G into cliques or independent sets and it is denoted by $z(G)$ and was first introduced in [10]. Cocoloring is extensively studied in [8, 10–14]. The cocoloring problem can also be generalized in the defective form. In this version of the problem, $z_k(G)$ gives the minimum number of induced subgraphs to partition the graph G into k -dense or k -sparse sets. $c(m)$ ($c_k(m)$) is another parameter that gives the maximum graph order n where all n -graphs can be partitioned into at most a total number of m cliques (k -dense sets) and independent sets (k -sparse sets). [11] gives the proof of $c_0(3) = 8$ and the following statements were left as open questions: $c_0(4) \in \{11, 12\}$ and $c_0(5) \in \{14, 15\}$. In [8], $c_1(3) \geq 11$ and $c_2(2) \geq 9$ were left as open questions. In this study, classical and defective Ramsey numbers are highly utilized to design algorithms to answer these open questions. We propose the following results: $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 10$. By the help of Ramsey numbers, the search space is narrowed by eliminations and the remaining graphs are searched

to check their cochromatic number. Since the number of graphs grows exponentially on the graph order, it gets harder and harder when the graph order is larger than 10. For example, when we check all 12-graphs whether they are all 4-cocolorable or not, we have to check 165091172592 different graphs 2.4. For each one of them, we have to check some subgraphs as we explain in Section 6.1.1. These subgraphs are a clique or an independent set of order 4 and the union of 4 subgraphs such that they are cliques or independent sets of order 3. Thus, for only one 12-graph, we have to check

$$\binom{12}{4} + \binom{12}{3} \binom{9}{3} \binom{6}{3} = 495 + 369600$$

different subgraph combinations if we do not make any eliminations or define any stopping conditions. We have defined several elimination and termination conditions to search these $c_k(m)$ parameters. At the end of the search algorithms, we observed that all 12-graphs can be cocolored by at most 4 colors and similarly, all 12-graphs can be 1-defectively 3-cocolorable and all 10-graphs can be 2-defectively 2-cocolorable. We also observed that all these orders are the maximum orders such that the corresponding number of colors are enough to cocolor all the graphs of that order. We represent the proofs showing why these values are the maximum graph orders. Also, counter examples are represented for each one. For example, for $c_0(4) = 12$, we represent a graph of order 13 such that it cannot be cocolored by 4 colors. In [11], a formula is represented to define lower and upper bounds for classical cochromatic numbers which also utilizes the Ramsey numbers. In the last chapter, Chapter 7, we prove the defective generalization of this formula.

2. RAMSEY NUMBERS

At a dinner table of six people, there are always three people who know each other or three people who are complete strangers to each others. This problem is known as ‘The Party Problem’ in graph theory [3, 4]. Let us represent these six people with A, B, C, D, E and F. There are two possible cases:

- Suppose A knows three (or more) people, say, B, C and D. If B and C or C and D or B and D are mutual acquaintances, then A and the acquainted pair know each other. Otherwise if these three people (B, C and D) do not know each other, they are mutual strangers.
- In other case, suppose A knows only two (or fewer) of the others, say, B and C. If D and F or E and F or D and E are strangers, then A and the unacquainted pair are strangers. Otherwise D, E and F are mutual acquaintances.

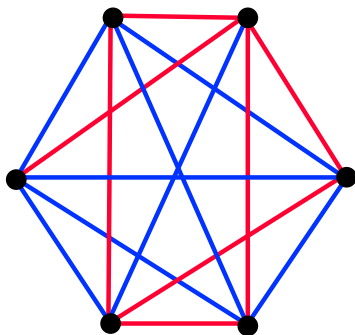


Figure 2.1. An example of relationships between 6 people by 2-edge coloring.

Let us represent people with vertices and the acquaintances between them with edges. Suppose that we construct a complete graph and color the edges according to the relationship between the people such that if two people know each other, we color the edge with blue, otherwise we color the edge with red. In this graph, we will always have a blue or red triangle which means there are three vertices connected with blue or red edges, namely, there are three people who know each other or do not know any of the others. The solution to the Party Problem is an example of Ramsey numbers. Ramsey

numbers give the minimum number of people there must be at the table to have a defined subgroup. For instance, the example explained above is denoted as $R(3, 3) = 6$ in graph theory, meaning that there must be at least six vertices to have a subgraph of three vertices which are all adjacent to each other with blue or red edges. This version can be defined as the *2-edge coloring* version of the Ramsey numbers which can be seen in Figure 2.1. $R(3, 3) = 6$ also means that there is at least one graph of order 5 that does not contain any blue or red triangle. Actually, there is only one 5-graph which neither contains a blue nor a red triangle: C_5 . C_5 is a cycle on 5 vertices and can be seen in Figure 2.2. We can also mention that in a 2-edge coloring of a complete graph on 6 vertices, there may be more than one blue or red triangle in the same graph. In Figure 2.3, we can see a 2-edge coloring of a complete graph of order 6 and we can see four different blue triangles in the same graph. We can deduce that in a graph of order $R(a, b)$, there must be at least one of the required subgraphs, which is a complete red subgraph of order a or a complete blue subgraph of order b , but there may exist more than one of the required subgraphs.

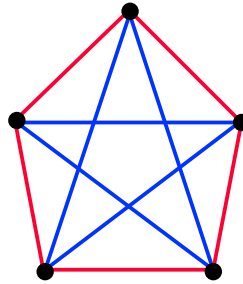


Figure 2.2. The only graph of order 5 that does not contain any blue or red triangle.

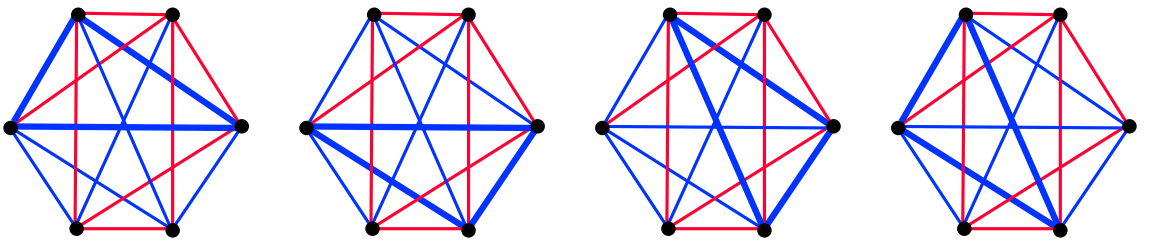


Figure 2.3. Different relationships between 6 people.

After this point, the graphs considered will not be complete graphs, we will have edges for only the blue ones. In Figure 2.4, we can see the transformed version of the 2-edge colored graph example in Figure 2.1. Since the blue subgraphs will be represented

with edges, a complete blue subgraph will be replaced with a complete subgraph and for the red subgraphs, the transformed subgraph will not contain any edges. The term *clique* will be used to define the complete blue subgraphs and *independent set* to define the complete red subgraphs.

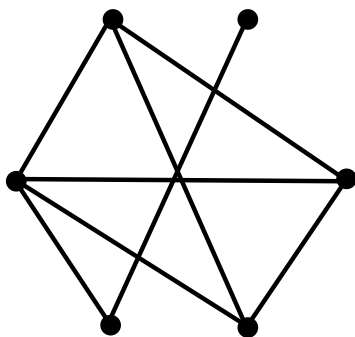
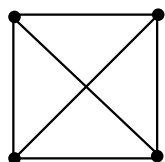


Figure 2.4. The same graph in Figure 2.1 by 1-edge coloring.

A *clique* in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subseteq V$, such that for every two vertices in C , there exists an edge connecting them. This is equivalent to saying that the subgraph induced by C is complete. A clique of order n is denoted by K_n . A *maximum clique* is a clique of the largest possible order in a given graph. The *clique number* $\omega(G)$ of a graph G is the order of the maximum clique in G .

An *independent set* or *stable set* is a set of vertices in a graph such that no two vertices in this set are adjacent. An independent set of order n is denoted by S_n . A *maximum independent set* is the largest independent set for a given graph G and its order is denoted $\alpha(G)$. In Figure 2.5 we can see a clique of order 4 and an independent set of order 6. For further information, see [15, 16].

Clique of order 4



Independent set of order 6

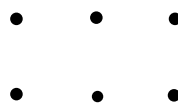


Figure 2.5. A clique of order 4 and an independent set of order 6.

For two colors, Ramsey's theorem states that for any pair of positive integers (a, b) , there exists a least positive integer $R(a, b)$ such that for any complete graph on $R(a, b)$ vertices, whose edges are colored *blue* or *red*, there exists a complete subgraph on a vertices which is entirely blue, or a complete subgraph on b vertices which is entirely red. Here, $R(a, b)$ signifies an integer that depends on both a and b . For further information, see [4, 5, 15, 17].

Another interpretation of the Ramsey number is that, it is the minimum number of vertices such that all undirected simple graphs of order $R(a, b)$ contain a clique of order a or an independent set of order b .

In Table 2.1, we can see some of the Ramsey numbers and in the 2.2 and 2.3, we can see the corresponding references. For some a and b pairs, the exact values are not known, thus, lower and upper bounds have been calculated for these pairs. The first value of the bounds corresponds to the best obtained lower bound and the second corresponds to the best obtained upper bound so far. For example, in order to have a clique of order 4 or an independent set of order 6 in every graph of order n , n must take a value between 36 and 41. By Remark 2.1 and 2.2, only $R(a, b)$ values or bounds where $a \geq 3$ and $b \geq 3$ are represented in Table 2.1. As seen in this table, $R(a, b)$ values are symmetric on diagonal with respect to a and b pairs by Remark 2.3.

Table 2.1. Ramsey Numbers.

$R(a, b)$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	[40, 43]
4	9	18	25	[36, 41]	[49, 61]	[56, 84]	[73, 115]	[92, 149]
5	14	25	[43, 49]	[58, 87]	[80, 143]	[101, 216]	[126, 316]	[144, 442]
6	18	[36, 41]	[58, 87]	[102, 165]	[113, 298]	[132, 495]	[169, 780]	[179, 1171]
7	23	[49, 61]	[80, 143]	[113, 298]	[205, 540]	[217, 1031]	[241, 1713]	[289, 2826]
8	28	[56, 84]	[101, 216]	[132, 495]	[217, 1031]	[282, 1870]	[317, 3583]	[331, 6090]
9	36	[73, 115]	[126, 316]	[169, 780]	[241, 1713]	[317, 3583]	[565, 6588]	[581, 12677]
10	[40, 43]	[92, 149]	[144, 442]	[179, 1171]	[289, 2826]	[331, 6090]	[581, 12677]	[798, 23556]

Remark 2.1. $R(a, 1) = R(1, a) = 1$.

Remark 2.2. $R(a, 2) = R(2, a) = a$.

Remark 2.3. $R(a, b) = R(b, a)$.

It can be observed that only the numbers for small graphs have been calculated exactly. Up from a level, it is very difficult to calculate the exact numbers in reasonable time, thus, lower and upper bounds are calculated. If the order of the graph increments more, the gap between the bounds increases. Studies on classical Ramsey numbers generally aim to narrow these gaps between the lower and the upper bounds. It is difficult to compute larger Ramsey numbers since the number of all possible non-isomorphic n -graphs grows exponentially with n as shown in Table 2.4. Let $G = (V_G, E_G)$, where V_G and E_G are vertex and edge sets of G , respectively, and $H = (V_H, E_H)$ be two graphs, then they are *isomorphic* if there is an invertible function f from V_G to V_H such that for all vertices u and v in V_G , we have

$$(u, v) \in E_G \text{ if and only if } (f(u), f(v)) \in E_H.$$

Here is a famous saying by Erdős that explains the difficulty of finding these exact values:

“Assume an evil spirit orders us to compute $R(5, 5)$, or else he will destroy all mankind. It may then be best if all mathematicians and computers start working on the answer. If, however, he orders us to compute $R(6, 6)$, then we had better think about how to destroy him before he destroys us.”

In this work, we consider a variation of Ramsey numbers, called defective Ramsey numbers, obtained by generalizing the classical notion of cliques and independent sets which will be explained in detail in the following chapter.

Table 2.2. References on Ramsey Numbers.

$R(a, b)$	Reference
$R(3, 3)$	Greenwood and Gleason, 1955 [18]
$R(3, 4)$	Greenwood and Gleason, 1955 [18]
$R(3, 5)$	Greenwood and Gleason, 1955 [18]
$R(3, 6)$	Kéry, 1964 [19]
$R(3, 7)$	Kalbfleisch, 1966 [20]
$R(3, 8)$	Grinstead and Roberts, 1982 [21]; McKay and Min, 1992 [22]
$R(3, 9)$	Grinstead and Roberts, 1982 [21]; Kalbfleisch, 1966 [20]
$R(3, 10)$	Exoo, 1989 [23]; Radziszowski and Kreher, 1988 [24]
$R(4, 4)$	Greenwood and Gleason, 1955 [18]
$R(4, 5)$	Kalbfleisch, 1965 [25]; McKay and Radziszowski, 1995 [26]
$R(4, 6)$	Exoo, 2012 [27]; McKay and Radziszowski, 1997 [28]
$R(4, 7)$	Exoo, 1989 [29]; Mackey, 1994 [30]
$R(4, 8)$	Exoo, 1998; 2002 [31, 32]; Mackey, 1994 [30]
$R(4, 9)$	Radziszowski, 1988 [33]; Mackey, 1994 [30]
$R(4, 10)$	Harborth and Krause, 2003 [34]; Mackey, 1994 [30]
$R(5, 5)$	Exoo, 1989 [35]; McKay and Radziszowski, 1997 [28]
$R(5, 6)$	Exoo, 1993 [36]; Huang and Zhang, 1998 [37]
$R(5, 7)$	Calkin et al., 1997 [38]; Spencer, 1994 [39]
$R(5, 8)$	Spencer, 1994 [39]; Harborth and Krause, 2003 [34]
$R(5, 9)$	Exoo, 1998 [31]; Haanpää, 2000 [40]; Mackey, 1994 [30]
$R(5, 10)$	Exoo, 1998 [31]; Mackey, 1994 [30]

Table 2.3. References on Ramsey Numbers - 2.

$R(a, b)$	Reference
$R(6, 6)$	Kalbfleisch, 1965 [25]; Mackey, 1994 [30]
$R(6, 7)$	Exoo, 1998 [31]; Xiaodong et al., 2002 [41]; Mackey, 1994 [30]
$R(6, 8)$	Xiaodong and Radziszowski, 2011 [42]; Xiaodong et al., 2002 [41]
$R(6, 9)$	Exoo, 1998 [31]; Mackey, 1994 [30]; Xiaodong et al., 2004 [43]
$R(6, 10)$	Exoo, 1998 [31]; Mackey, 1994 [30]
$R(7, 7)$	Shearer, 1986 [44]; Mackey, 1994 [30]
$R(7, 8)$	Xiaodong and Radziszowski, 2011 [42]; Mackey, 1994 [30]
$R(7, 9)$	Xiaodong and Radziszowski, 2011 [42]; Huang and Zhang, 1998 [37]
$R(7, 10)$	Xiaodong et al., 2004 [43]; Mackey, 1994 [30]
$R(8, 8)$	Burling and Reyner, 1972 [45]; Mackey, 1994 [30]
$R(8, 9)$	Xiaodong et al., 2004 [43]
$R(8, 10)$	Huang and Zhang, 1998 [37]
$R(9, 9)$	Shearer, 1986 [44]; Sheng and Zhang, 2001 [46]
$R(9, 10)$	Xiaodong and Radziszowski, 2011 [42]
$R(10, 10)$	Shearer, 1986 [44]; Shi, 2003 [47]

Table 2.4. Number of Non-Isomorphic Graphs on n Vertices.

Number of vertices	All graphs
2	2
3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592
13	50502031367952
14	29054155657235488
15	31426485969804308768
16	64001015704527557894928

2.2. Applications of Ramsey Numbers

As mentioned, Ramsey numbers are first discovered in 1930 by Ramsey while he was studying on the problem of finding a regular procedure to determine the truth or falsity of any given logical formula [1]. Later, Ramsey numbers are rediscovered by Paul Erdős and George Szekeres in 1935 [2]. After these studies, researches utilized Ramsey numbers for both theoretical and applied studies. For detailed information on the applications, see [6, 48–50]. In [49], some useful applications of Ramsey numbers are explained. The most important among them is on communication theory. The objective is to prevent making errors when receiving a signal in a noisy channel. The order of the maximum independent set is considered and Ramsey numbers are utilized to calculate an upper bound on the order of the independent set in this study. The author states that there is another study of Boyles and Exoo, which is not published as a paper, that tries to guarantee that in case of outages on some links, there will always remain at least one link joining every pair of vertices. They use Ramsey numbers to construct such a network. In [51], Ramsey numbers are used to design methodologies on information retrieval. For other applications on information exchange, see [52]. In [6], it is mentioned that Ramsey numbers are used in the proof of Cardinality Theorem, which is the reduced form of Beigel’s conjecture [53].

For applications of Ramsey numbers in logic (decidability, finite model theory), see [1, 53–62]. For applications of Ramsey numbers in concrete complexity theory (circuits, decision trees, complexity, branching programs, hashing), see [51, 52, 63–73]. For applications of Ramsey numbers in complexity classes, see [73–83]. For applications of Ramsey numbers in parallelism, see [84–90]. For applications of Ramsey numbers in algorithms, see [91–96]. For applications of Ramsey numbers in computational geometry, see [97–99].

3. DEFECTIVE RAMSEY NUMBERS

In Chapter 2, we mentioned that classical Ramsey numbers consider only a clique or an independent set as a required subgraph. One may also want to give some tolerance to Ramsey numbers by relaxing the conditions on the subgraphs. For example, we can agree on subgraphs such as the union of an edge and an independent set or the induced subgraph when we remove an edge from a clique. In this way, it can be said that we have tolerated 1-*defectiveness* in the subgroups. In graph-theoretical terms, instead of cliques or independent sets, we consider *k-dense* or *k-sparse* sets in defective Ramsey numbers where *k* indicates the defectiveness level. A *k-dense* *n*-set is defined instead of the term *k-defective* clique of order *n*, which can be described as a subgraph of order *n* such that every vertex in that subgraph misses, is not adjacent to, at most *k* other vertices in that subgraph. Similarly, a *k-sparse* *n*-set is defined instead of the term *k-defective* independent set of order *n*, which can be described as a subgraph of order *n* such that every vertex in that subgraph can be adjacent to at most *k* other vertices in that subgraph. In Figure 3.1, we can see a 1-dense 4-set on the left and a 1-sparse 4-set on the right. *Defective Ramsey numbers*, $R_k(a, b)$, give us the minimum graph order *n* such that all graphs of order at least *n* contains a *k-dense* set of order *a* or a *k-sparse* set of order *b*. Since the defective Ramsey numbers are the generalizations of Ramsey numbers, defective Ramsey numbers are also difficult to find. As mentioned, in literature, lower and upper bounds have been represented when the exact values of classical Ramsey numbers are not known. A similar study for defective Ramsey numbers, to the best of our knowledge, has not been made. For this purpose, we have calculated the lower and upper bounds of defective Ramsey numbers and improved most of the lower bounds by representing the counter examples which are some graphs which do not contain a *k-dense* *a*-set or a *k-sparse* *b*-set even though the order of that graph is larger than the theoretical lower found. Best found bounds for defective Ramsey numbers are represented in Section 3.2.4.

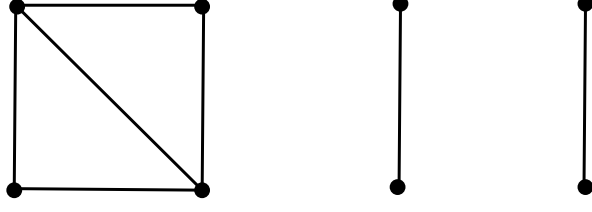


Figure 3.1. An example of a 1-dense 4-set and a 1-sparse 4-set.

Firstly, 1-defectiveness has been studied as *1-dependent sets* in [7]. Later, in [8], the defectiveness level has been generalized. In Table 3.1 we can see the known 1-defective Ramsey numbers which are represented in [7] and [8]. In Table 3.2, Table 3.3, Table 3.4, Table 3.5, Table 3.6 and 3.7 we can see all the numbers found so far corresponding to different defectiveness levels. The numbers written by bold characters represent the latest calculated defective Ramsey numbers in [9]. Some of these values can be see in Table 3.8 with their *extremal graph* numbers. A graph G is *extremal* for $R_k(a, b)$ if the order of G is $R_k(a, b) - 1$ and G contains no k -dense set of order a and no k -sparse set of order b .

Table 3.1. 1-defective Ramsey Numbers.

$R_1(a, b)$	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	2	3	4	5	6	7	8	9	10
4	2	4	6	9	11	16	17		
5	2	5	9	15					
6	2	6	11						
7	2	7	16						
8	2	8	17						
9	2	9							
10	2	10							

Table 3.2. 2-defective Ramsey Numbers.

$R_2(a, b)$	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3
4	3	4	5	6	7	8	9	10
5	3	5	7	8	10	12		
6	3	6	8	12				
7	3	7	10					
8	3	8	12					
9	3	9						
10	3	10						

Table 3.3. 3-defective Ramsey Numbers.

$R_3(a, b)$	4	5	6	7	8	9	10	11
4	4	4	4	4	4	4	4	4
5	4	5	6	7	8	9	10	11
6	4	6	8	9	10	12		
7	4	7	9	11	13			
8	4	8	10	13				
9	4	9	12					
10	4	10						
11	4	11						

Table 3.4. 4-defective Ramsey Numbers.

$R_4(a, b)$	5	6	7	8	9	10	11	12
5	5	5	5	5	5	5	5	5
6	5	6	7	8	9	10	11	12
7	5	7	9	10	11	13		
8	5	8	10	12				
9	5	9	11					
10	5	10	13					
11	5	11						
12	5	12						

Table 3.5. 5-defective Ramsey Numbers.

$R_5(a, b)$	6	7	8	9	10	11	12	13
6	6	6	6	6	6	6	6	6
7	6	7	8	9	10	11	12	13
8	6	8	10	11	12			
9	6	9	11	13				
10	6	10	12					
11	6	11						
12	6	12						
13	6	13						

Table 3.6. 6-defective Ramsey Numbers.

$R_6(a, b)$	7	8	9	10	11	12	13	14
7	7	7	7	7	7	7	7	7
8	7	8	9	10	11	12	13	14
9	7	9	11	12				
10	7	10	12	14				
11	7	11						
12	7	12						
13	7	13						
14	7	14						

Table 3.7. 7-defective Ramsey Numbers.

$R_7(a, b)$	8	9	10	11	12	13	14	15
8	8	8	8	8	8	8	8	8
9	8	9	10	11	12	13	14	15
10	8	10	12					
11	8	11		15				
12	8	12						
13	8	13						
14	8	14						
15	8	15						

Table 3.8. New Results on Defective Ramsey Numbers.

$R_k(a, b)$	Graph Order	Number of Extremal Graphs
$R_2(5, 6)$	8	13
$R_2(5, 7)$	10	16
$R_2(5, 8)$	12	8
$R_2(6, 6)$	12	2
$R_3(6, 7)$	9	28
$R_3(6, 8)$	10	159
$R_3(6, 9)$	12	4
$R_3(7, 7)$	11	4
$R_3(7, 8)$	13	43
$R_4(7, 8)$	10	84
$R_4(7, 9)$	11	550
$R_4(7, 10)$	13	4
$R_4(8, 8)$	12	8
$R_5(8, 9)$	11	316
$R_5(8, 10)$	12	2430
$R_5(9, 9)$	13	22
$R_6(9, 10)$	12	1712

3.1. Properties On Defective Ramsey Numbers

This section explains the basic properties of defective Ramsey numbers by explaining the sketch of proofs represented in [9]. Some of these properties will be considered to find the lower and upper bounds of defective Ramsey numbers in the following sections.

Definition 3.1. *The complement of a graph G is a graph H on the same vertices such that two vertices of H are adjacent if and only if they are not adjacent in G . That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there. The complement graph of G is denoted by $\overline{G}$.*

Let k , a , b and c be nonnegative integers for all the following propositions.

Proposition 3.2. *Let G be a graph, and let $T \subseteq V(G)$. T is a k -sparse set in G if and only if T is a k -dense set in $\overline{G}$.*

Proof. Suppose T is a k -sparse set in G . That means, the vertex with the highest degree, v , can be adjacent to at most k other vertices in T . Then, by Definition 3.1, these edges that are incident to vertex v become non-edges in $\overline{G}$, thus, vertex v can miss at most k other vertices in $\overline{G}$. Since the degrees of other vertices are equal to or less than v in G , they cannot miss more than k vertices in $\overline{G}$. Therefore, T becomes a k -dense set in $\overline{G}$.

Similarly, suppose T is a k -dense set in $\overline{G}$. That means, the vertex with the lowest degree, v , can miss at most k other vertices in T . Then, by Definition 3.1, the non-edges that are incident to vertex v become edges in G , thus, vertex v can be adjacent to at most k other vertices in G . Since the degrees of other vertices are equal to or more than v in $\overline{G}$, they cannot be adjacent to more than k vertices in G . Therefore, T becomes k -sparse in G . $\square$

Proposition 3.3. *Let G be a graph, and let $T \subseteq V(G)$ with $|T| \leq k + 1$. Then T is k -sparse in G .*

Proof. If the order of the subgraph T is at most $k + 1$, then even if it is a complete subgraph, the vertex with the highest degree would have degree k . Then, T will always be a k -sparse set since it cannot contain a vertex that has degree more than k . $\square$

Proposition 3.4. *Let G be a graph, and let $T \subseteq V(G)$ with $|T| = k + 2$. Then, either T is k -sparse in G or k -dense in G , in other words, T is k -sparse in $\overline{G}$.*

Proof. If T is not k -sparse in G , that means there is a vertex v in T such that it has degree $k + 1$. Because there must be a vertex that has degree more than k since T is not k -sparse and the vertex cannot be adjacent to $k + 2$ other vertices since there are only $k + 1$ remaining vertices other than v . As a result, v is adjacent to all the vertices in G . In $\overline{G}$, v will have 0 degree, and since it is an *isolated vertex*, which is not adjacent to any other vertex, the remaining vertices can be adjacent to at most k other vertices in $\overline{G}$ and no vertex has degree more than k . Thus, T will be k -sparse in $\overline{G}$. $\square$

Proposition 3.5. *Let G be a graph, and let $T \subseteq V(G)$. Then T is k -sparse in G if and only if every $(k + 2)$ -subset of T is k -sparse in G .*

Proof. If every $(k + 2)$ -subset of T is k -sparse, then no vertex in T will have degree more than k in T . Since otherwise, a vertex of degree at least $k + 1$, say v , with its neighbors form a $(k + 2)$ -subset of T which is clearly not k -sparse since v has degree at least $k + 1$. Thus, T will be a k -sparse set. If T is not k -sparse, then a vertex in T will have degree at least $(k + 1)$. When the subgraph, that is induced by that vertex and its $(k + 1)$ neighbors, is considered, this $(k + 2)$ -subset is not k -sparse neither. $\square$

Proposition 3.6. $R_{k+1}(a, b) \leq R_k(a, b)$.

Proof. If a graph G contains a k -sparse set T as a subgraph, then it is known that there is no vertex with degree $k + 1$ in T . Then, T is also a $(k + 1)$ -sparse set in G . But a $(k + 1)$ -sparse set T' may have a vertex which has degree $k + 1$, then it will not be a k -sparse set in G . $\square$

Proposition 3.7. $R_k(a, b) = R_k(b, a)$.

Proof. If every $R_k(a, b)$ -graph contains a k -dense set of order a or a k -sparse set of order b , their complements will obviously contain a k -dense set of order b or a k -sparse set of order a by Proposition 3.2. $\square$

Proposition 3.8. If $a \leq k + 1$ or $b \leq k + 1$, then $R_k(a, b) = \min \{a, b\}$.

Proof. Any graph G of order $R_k(a, b)$ contains a k -dense a -set or a k -sparse b -set. By Proposition 3.3, without loss of generality, if $a \leq k + 1$, every a -set in G will be k -sparse. Since Ramsey number denotes the smallest cardinality number which satisfies one of the desired conditions, every $(\min \{a, b\})$ -set is k -sparse in G , thus, $R_k(a, b) = \min \{a, b\}$. $\square$

Proposition 3.9. $R_k(b, k + 2) = b$

Proof. If $b \leq k + 1$, then it follows from Proposition 3.8, $R_k(k + 2, b) = \min \{k + 2, b\} = b$. Suppose $b \geq k + 2$. K_{b-1} does not contain a k -dense or a k -sparse set of order $k + 2$. Thus, $R_k(b, k + 2) \geq b$. On the other hand, let G be a graph of order b which does not contain a k -sparse set of order $k + 2$. By Proposition 3.4, since there is not a k -sparse $(k + 2)$ -set in G , then every $(k + 2)$ -subset of G is a k -dense $(k + 2)$ -set in G , therefore, every $(k + 2)$ -subset of G is a k -sparse $(k + 2)$ -set in $\overline{G}$ by Proposition 3.2. Thus, by Proposition 3.5, since every $(k + 2)$ -subset is k -sparse in $\overline{G}$, there is a k -sparse set of

order b in $\overline{G}$, thus, a k -dense set of order b in G . Hence, $R_k(b, k+2) \leq b$. When all the cases combined together, it follows that $R_k(k+2, b) = b$. $\square$

Proposition 3.10. *If $a \geq 1$ and $b \geq k+2$, then $R_k(a, b) \geq R_k(a-1, b) + 1$.*

Proof. If $a = 1$, then from Proposition 3.8, $R_k(a, b) = R_k(a-1, b) + 1 = \min\{a, b\} = a - 1 + 1 = a$. If $b \geq 2$, let G be a graph of order $R_k(a-1, b) - 1$ such that it neither contains a k -dense $(a-1)$ -set nor a k -sparse b -set. Let G^* be a graph of order $R_k(a-1, b)$ constructed by adding a universal vertex v to G . A *universal vertex* v is a vertex such that it is adjacent to all the other vertices in the vertex set. G^* does not contain a k -dense a -set. We need to show that G^* also does not contain a k -sparse b -set. If $|G| = R_k(a-1, b) - 1 < k+1$, the order of G^* is less than $k+2$, thus, since $b \geq k+2$, G^* does not contain a k -sparse b -set. If $|G| \geq k+1$, since G does not contain any k -sparse b -set, we have to search a k -sparse b -set composed of $b-1$ vertices of G and v . In every subgraph like that, v has degree $b-1$ which is greater than $k+1$, because v is a universal vertex. Thus, G^* does not contain any k -sparse b -set. Therefore, $R_k(a, b)$ must be greater than the order of G^* , $R_k(a, b) > R_k(a-1, b)$. $\square$

Proposition 3.11. *If $a \geq 2k+1$ and $c, d \geq 1$, then $R_k(a, c+d-1) \geq R_k(a, c) + R_k(a, d) + 1$.*

Proof. Let G_1 and G_2 be two graphs of order $R_k(a, c) - 1$ and $R_k(a, d) - 1$, respectively. G_1 does not contain any k -dense set on a vertices or any k -sparse set on c vertices. In the same manner, G_2 does not contain any k -dense set on a vertices or any k -sparse set on d vertices. Let G be a graph constructed by taking the edge-disjoint union of graphs G_1 and G_2 . The cardinality of G is $R_k(a, c) + R_k(a, d) - 2$. In order to see that G does not contain any k -dense set on a vertices, let $S \subseteq V(G)$ with $|S| = a$. If S is a subset of G_1 or G_2 , then it is not k -dense. Otherwise, these two graphs would have contained a k -dense set on a vertices. Since $a \geq 2k+1$, S must contain at least $k+1$ vertices from one of the graphs, G_1 and G_2 , and at least 1 vertex, v , from the

other one. As we do not add any edges between these graphs, v will miss at least $k + 1$ vertices. Then, S cannot be a k -dense set.

Let G' be a graph constructed by taking the disjoint union of graphs G_1 and G_2 and adding all the edges between them. The cardinality of G is $R_k(a, c) + R_k(a, d) - 2$. In order to see that G does not contain any k -sparse set on $c + d - 1$ vertices, let $S \subseteq V(G)$ with $|S| = c + d - 1$. The maximum k -sparse size that G_1 can contain is $c - 1$ and similarly G_2 can contain a k -sparse set of order at most $d - 1$. In order to construct a k -sparse $c + d - 1$ -set, we must combine $c - 1$ vertices of G_1 with at least $(c + d - 1) - (c - 1) = d$ vertices of G_2 . Since G_2 does not contain any k -sparse d -set and it is restricted to construct larger k -sparse sets by adding all the edges between G_1 and G_2 , we cannot form a k -sparse $c + d - 1$ -set by taking $c - 1$ vertices from G_1 . Similarly, if we take the largest k -sparse set of G_2 , which has order $d - 1$, then we have to take $(c + d - 1) - (d - 1) = c$ vertices from G_1 which does not form a k -sparse $c + d - 1$ -set by the same reasons as before. $\square$

Proposition 3.12. *If $a, b \geq 1$, then $R_k(a, b) \leq R_k(a - 1, b) + R_k(a, b - 1)$.*

Proof. Let G be a graph of order $R_k(a - 1, b) + R_k(a, b - 1)$. There is a vertex v such that it has either $R_k(a - 1, b)$ neighbors or $R_k(a, b - 1)$ non-neighbors in G . Considering the first case, let T be the set of neighbors of v . If T has a k -sparse b -set in G , then we are done. Otherwise, there is a k -dense $(a - 1)$ -set in T which will form a k -dense a -set with v . Considering the second case, let T be the set of non-neighbors of v . If T has a k -dense a -set, then we are done. Otherwise, there is a k -sparse $(b - 1)$ -set in T which will form a k -sparse b -set with v . $\square$

Proposition 3.13. *If $a, b \geq 2$, then $R_k(k + a, k + b) \geq k + a + b - 2$.*

Proof. If $a = 2$, Proposition 3.9, $R_k(k+2, k+b) = k+b$. If $a > 2$, by Proposition 3.10,

$$\begin{aligned} R_k(k+a, k+b) &\geq R_k(k+a-1, k+b) + 1 \\ R_k(k+a, k+b) &\geq k+a-1+b-2+1 = k+a+b-2. \end{aligned}$$

In order to simplify the notation, we replace $k+a$ by a' and $k+b$ by b' , then, replace a' and b' by a and b , respectively.

$$\begin{aligned} R_k(a', b') &\geq k + (a' - k) + (b' - k) - 2 \\ R_k(a, b) &\geq a + b - k - 2. \end{aligned}$$

□

Proposition 3.14. *If $a, b \geq 2$, then*

$$R_k(k+a, k+b) \leq \binom{a+b-4}{a-2}k + \binom{a+b-2}{a-1}.$$

Proof. If $a = 2$, the right hand side become $k+b$, thus, $R_k(k+2, k+b) \leq k+b$. We need to show that $R_k(k+2, k+b) \geq k+b$, it can be shown by Proposition 3.9, therefore, $R_k(k+2, k+b) = k+b$. Similarly, the equation holds when $b = 2$. Let $a, b \geq 3$, we apply Proposition 3.12:

$$\begin{aligned} R_k(k+a, k+b) &\leq R_k(k+a-1, k+b) + R_k(k+a, k+b-1) \\ R_k(k+a, k+b) &\leq \left[\binom{a+b-5}{a-3}k + \binom{a+b-3}{a-3} \right] + \left[\binom{a+b-5}{a-2}k + \binom{a+b-3}{a-1} \right] \\ R_k(k+a, k+b) &\leq \binom{a+b-4}{a-2}k + \binom{a+b-2}{a-1}. \end{aligned}$$

In order to simplify the notation, we replace $k+a$ by a' and $k+b$ by b' , then, replace

a' and b' by a and b , respectively.

$$\begin{aligned} R_k(a', b') &\leq \binom{(a' - k) + (b' - k) - 4}{(a' - k) - 2} k + \binom{(a' - k) + (b' - k) - 2}{(a' - k) - 1} \\ R_k(a, b) &\leq \binom{a + b - 2k - 4}{a - k - 2} k + \binom{a + b - 2k - 2}{a - k - 1}. \end{aligned}$$

□

Proposition 3.15. *If $k \geq a - 3$, then $R_k(k + a, k + a) \geq k + 3a - 4$.*

Proof. If $a = 0, 1$, it follows by Proposition 3.8 $R_k(k + a, k + a) = \min \{k + a, k + a\} = k + a$ and if $a = 2$, it follows by Proposition 3.9 $R_k(k + a, k + a) = k + a$. Now suppose $a \geq 3$ and $k \geq a - 3$. Define a graph G of order $k + 3a - 5$. Let P , Q and R be the disjoint sets of vertices that construct G , such that $|P| = |Q| = 2a - 4$ and $|R| = k - (a - 3) = k - a + 3$. Let the edges between sets P and Q form a regular bipartite graph with degree $a - 2$ and Q be a complete subgraph. Also assume that every vertex of Q is adjacent to every other vertex of R . Note that $P \cup R$ is an independent set in G . Since the order of G is $k + 3a - 5$, we have to remove $2a - 5$ vertices to have a set of order $k + a$.

Let $S \subseteq G$ and $|S| = k + a$. P and R have $(2a - 4) + (k - a + 3) = k + a - 1$ vertices together, thus, S contains at least one vertex v from Q . Every vertex in Q is incident to $a - 2$ vertices of P , $2a - 5$ other vertices of Q and $k - a + 3$ vertices of R , therefore, every vertex in Q has degree $(a - 2) + (2a - 5) + (k - a + 3) = k + 2a - 4$. Then, the subgraph induced by S contains v that has degree at least $(k + 2a - 4) - (2a - 5) = k + 1$ when we remove the edges between v and vertices of Q . Therefore, S cannot be a k -sparse set.

Similarly, let $S \subseteq G$ and $|S| = k + a$. This time, S contains at least one vertex v from P since Q and R have $(2a - 4) + (k - a + 3) = k + a - 1$ vertices together. Every vertex in P is only incident to $a - 2$ vertices of Q . Since there are $k + a - 1$ vertices in

S other than v and v has degree $a - 2$, v misses $(k + a - 1) - (a - 2) = k + 1$ vertices. Thus, S is not a k -dense set.

It is possible to construct a graph of order $k + 3a - 5$ that neither contains a k -dense nor a k -sparse $(k + a)$ -set. Thus, $R_k(k + a, k + a) \geq k + 3a - 4$. $\square$

Proposition 3.16. *If $a \geq 2$ and $k \geq 3a - 6$, then $R_k(k + a, k + a) = k + 3a - 4$.*

Proof. Since $a \geq 2$ and $k \geq 3a - 6$, $k \geq a - 3$. In that case, Proposition 3.15 can be applied and it only remains to show that $R_k(k + a, k + a) \leq k + 3a - 4$. For a contradiction, assume $R_k(k + a, k + a) > k + 3a - 4$ which means there is a graph G of order $k + 3a - 4$ such that it neither contains a k -dense nor k -sparse subgraph of order $k + a$. Thus, in every $k + a$ -subset of G , there must be a vertex x of degree at least $k + 1$ and a vertex y that misses $k + 1$ other vertices, therefore, has degree at most $(k + a - 1) - (k + 1) = a - 2$. Because there are $k + a - 1$ vertices other than y and y misses at least $k + 1$ vertices.

Define a vertex v as *strong*, such as, if there is a subgraph of order $k + a$ in which v has degree at least $k + 1$ then, v is strong. On the other hand, if there is a subgraph of order $k + a$ in which v has degree at most $a - 2$, we define the vertex v as *weak*. Thus, v is weak in G if it misses $(k + a) - (a - 2) = k + 2$ vertices and has degree $(k + 3a - 4) - (k - 2) = 3a - 6$. Since $k \geq 3a - 6$; $k + 1 > 3a - 6$, therefore, no vertex can be strong and weak at the same time.

In G , there are at least $2a - 3$ weak vertices, otherwise, we can remove $2a - 4$ vertices and that leaves $(k + 3a - 4) - (2a - 4) = k + a$ vertices none of which is weak, thus, these vertices form a k -dense in G .

A weak vertex v is *special*, if v is adjacent to at most $a - 2$ strong vertices in G . When we remove $2a - 4$ weak vertices from G , the remaining $k + a$ vertices must contain a vertex x of degree at most $a - 2$. Since only weak vertices are removed, remaining

set of vertices contain every strong vertex of the graph. Therefore, x is a special weak vertex. When we remove $2a - 4$ weak vertices, we find a special weak vertex, thus, G must contain at least $2a - 3$ special weak vertices. Let S be the set of these $2a - 3$ special weak vertices and T be the set of all strong vertices of G which are adjacent to more than $a - 2$ vertices of S . S and T are disjoint sets. Each vertex in S is adjacent to at most $a - 2$ strong vertices, $|T| < |S| = 2a - 3$ and $|V(G) - T| \geq k + a$.

Let U be a set of vertices of order $k + a$, such that $S \subseteq U \subseteq V(G) - T$. Since $a - 3 \leq k$ and $|S| = 2a - 3 \leq k + a = |U|$, the subgraph U exists. Let z be a vertex in U . If z is not strong in G , then it must have degree at most k . If it is strong, T does not contain z and $z \notin T$, z is adjacent to at most $a - 2$ vertices in S . U has $(k + a) - (2a - 3) - 1 = k - a + 2$ vertices which are not in S , except z . In the graph induced by U , z has degree at most $(a - 2) + (k - a + 2) = k$, thus U is k -sparse.

By contradiction, $R_k(k + a, k + a) = k + 3a - 4$. □

3.2. Bounds on Defective Ramsey Numbers

It is already mentioned that finding the exact values for Ramsey numbers gets very hard when the graph order increases. Since the defective Ramsey numbers are the generalization of Ramsey numbers, this issue is also valid for defective Ramsey numbers. For classical Ramsey numbers, as in Table 2.1, some bounds are calculated for the Ramsey numbers which are not known exactly. A similar study for defective Ramsey numbers, to the best of our knowledge, has not been made for the unknown values even though there are some formulas on the bounds. In the first part of this study, bounds of defective Ramsey numbers are calculated by the help of propositions represented in [9]. In the second part, some counter example graphs will be represented to improve the lower bounds.

3.2.1. Theoretical Bounds

Some theoretical bounds have been represented in [9]. All the known values will be calculated by the help of these bounds.

- (i) Lower bounds have been calculated by Proposition 3.10, 3.11 and 3.13 which are respectively listed below.

Let k, a, b and c be nonnegative integers.

- If $a \geq 1$ and $b \geq k + 2$, then $R_k(a, b) \geq R_k(a - 1, b) + 1$.
- If $a \geq 2k + 1$ and $c, d \geq 1$, then $R_k(a, c + d - 1) \geq R_k(a, c) + R_k(a, d) - 1$.
- If $a, b \geq 2$, then $R_k(a, b) \geq a + b - k - 2$.

- (ii) Upper bounds have been calculated by Proposition 3.12 and 3.14 which are respectively listed below.

Let k, a, b and c be nonnegative integers.

- If $a, b \geq 1$, then $R_k(a, b) \leq R_k(a - 1, b) + R_k(a, b - 1)$.
- If $a, b \geq 2$, then

$$R_k(a, b) \leq \binom{a + b - 2k - 4}{a - k - 2} k + \binom{a + b - 2k - 2}{a - k - 1}.$$

In order to be able to calculate the tightest values, the maximum value among the lower bounds and the minimum value between the upper bounds will be chosen. In order to calculate all these bounds, we will need some other defective Ramsey numbers. Since there are so many undiscovered values, we can only calculate all the bounds for the defective Ramsey numbers $R_k(a, b)$, if the values of $R_k(a - 1, b)$ and $R_k(a, b - 1)$ are known. If these values are not known, we calculate the bounds on $R_k(a, b)$ using not the exact values but the bounds on $R_k(a - 1, b)$ and $R_k(a, b - 1)$.

3.2.2. Calculation Examples

Let us calculate all the bounds for $R_2(6, 7)$ for which we know all the defective Ramsey numbers that we require. Later, we will calculate the bounds of $R_2(7, 7)$ by

the help of the bounds of $R_2(6, 7)$.

3.2.2.1. Calculation of bounds of $R_2(6, 7)$.

(i) Lower Bounds on $R_2(6, 7)$

- By Proposition 3.10, $R_2(6, 7) \geq R_2(5, 7) + 1$. Since $R_2(5, 7) = 10$, $R_2(6, 7) \geq 11$. But also, since defective Ramsey numbers are symmetric on a and b values, $R_2(7, 6) \geq R_2(6, 6) + 1$. In this case, since $R_2(6, 6) = 12$, $R_2(6, 7) \geq 13$ which gives us a better bound. Thus, this bound will be calculated as follows:

$$R_k(a, b) \geq \max \{R_k(a - 1, b), R_k(a, b - 1)\} + 1.$$

- Since defective Ramsey numbers are symmetric on a and b values, similar to Proposition 3.10, we consider $R_2(6, 7)$ and $R_2(7, 6)$ to calculate the lower bounds by Proposition 3.11. We do not only consider different a values, but also consider different combinations of c and d values.

First, let us consider the case where $a = 6$, $b = 7$, thus, $c + d = 8$. Since the defective Ramsey number that we are calculating the bounds is 2-defective, the order of the subgraphs c and d must be greater than the defectiveness level 2. Different combinations of c and d values are investigated as follows:

Calculation of the lower bound where $a = 6$, $c = 3$ and $d = 5$,

$$\begin{aligned} R_2(6, 7) &\geq R_2(6, 3) + R_2(6, 5) - 1 \\ &\geq 3 + 8 - 1 = 10 \end{aligned}$$

Calculation of the lower bound where $a = 6$, $c = 4$ and $d = 4$,

$$\begin{aligned} R_2(6, 7) &\geq R_2(6, 4) + R_2(6, 4) - 1 \\ &\geq 6 + 6 - 1 = 11 \end{aligned}$$

When we increment c by 1, we will obtain the first case, thus, we stop here for $a = 6$ and $b = 7$.

Let us consider the case where $a = 7$, $b = 6$, thus $c + d = 7$. Different combinations of c and d values are investigated as follows:

Calculation of the lower bound where $a = 7$, $c = 3$ and $d = 4$,

$$\begin{aligned} R_2(7, 6) &\geq R_2(7, 3) + R_2(7, 4) - 1 \\ &\geq 3 + 7 - 1 = 9 \end{aligned}$$

Since the maximum number between the two different cases is 11, the lower bound of $R_2(6, 7)$ obtained by Proposition 3.11 is 11.

- By Proposition 3.13, a lower bound will be calculated by using the a , b values and the defectiveness level k .

$$R_2(6, 7) \geq 6 + 7 - 2 - 2 = 9$$

The maximum one among these values will give us the lower bound of $R_2(6, 7)$ which is 13.

(ii) Upper Bounds on $R_2(6, 7)$

- By Proposition 3.12, an upper bound for $R_2(6, 7)$ will be calculated by using $R_2(5, 7)$ and $R_2(6, 6)$. Since we know the exact values of these defective

Ramsey numbers, this bound can be calculated easily.

$$\begin{aligned} R_2(6, 7) &\leq R_2(5, 7) + R_2(6, 6) \\ &\leq 10 + 12 = 22 \end{aligned}$$

- By Proposition 3.14, another upper bound will be calculated using the corresponding a , b values and the defectiveness level k .

$$R_2(6, 7) \leq \binom{6+7-4-4}{6-2-2} 2 + \binom{6+7-4-2}{6-2-1} = 55$$

The minimum one among these values will give us the upper bound of $R_2(6, 7)$ which is 22. Therefore,

$$13 \leq R_2(6, 7) \leq 22.$$

3.2.2.2. Calculation of bounds of $R_2(7, 7)$.

(i) Lower Bounds on $R_2(7, 7)$

- By Proposition 3.10, $R_2(7, 7) \geq R_2(6, 7) + 1$. But we do not know the exact value of $R_2(6, 7)$. Since we are trying to calculate a lower bound for $R_2(7, 7)$, we can use the lower bound of $R_2(6, 7)$ to be sure that $R_2(7, 7)$ is greater than the lower bound that we have calculated. $R_2(6, 7) \geq 13$, then, $R_2(7, 7) \geq 14$.

Since a and b are equal in that case, it is trivial to check the symmetric case.

- Since $R_2(7, 7)$ is symmetric on a and b , we calculate a lower bound by Proposition 3.11 for $a = 7$, $b = 7$ and $c + d = 8$. It is known that defective Ramsey numbers are symmetric on a and b values. Similar to Proposition 3.10, we consider $R_2(6, 7)$ and $R_2(7, 6)$ to calculate the lower bounds by Proposition

3.11. We do not only consider different a values, but also consider different combinations of c and d values.

Calculation of the lower bound where $a = 7$, $c = 3$ and $d = 5$,

$$\begin{aligned} R_2(7, 7) &\geq R_2(7, 3) + R_2(7, 5) - 1 \\ &\geq 3 + 10 - 1 = 12 \end{aligned}$$

Calculation of the lower bound where $a = 7$, $c = 4$ and $d = 4$,

$$\begin{aligned} R_2(7, 7) &\geq R_2(7, 4) + R_2(7, 4) - 1 \\ &\geq 7 + 7 - 1 = 13 \end{aligned}$$

When we increment c by 1, we will obtain the first case, thus, we stop here for $a = 7$ and $b = 7$.

Since the maximum number between the two different cases is 13, the lower bound of $R_2(7, 7)$ obtained by Proposition 3.11 is 13.

- By Proposition 3.13, a lower bound will be calculated by using the a , b values and the defectiveness level k .

$$\begin{aligned} R_2(7, 7) &\geq 7 + 7 - 2 - 2 \\ R_2(7, 7) &\geq 10 \end{aligned}$$

The maximum one among these values will give us the lower bound of $R_2(7, 7)$ which is 14. If we did not use the bounds of $R_2(6, 7)$, the lower bound of $R_2(7, 7)$ would be 13.

(ii) Upper Bounds on $R_2(7, 7)$

- By Proposition 3.12, an upper bound for $R_2(6, 7)$ will be calculated by using the exact value of $R_2(6, 7)$. Since we do not know this value, we will be using the bounds of $R_2(6, 7)$. This time, since we are calculating an upper bound for $R_2(7, 7)$, the upper bound of $R_2(6, 7)$, which is 22, will be used.

$$\begin{aligned} R_2(7, 7) &\leq R_2(6, 7) + R_2(7, 6) \\ &\leq 22 + 22 = 44 \end{aligned}$$

- By Proposition 3.14, another upper bound will be calculated by using the corresponding a, b values and the defectiveness level k .

$$R_2(7, 7) \leq \binom{7+7-4-4}{7-2-2} \cdot 2 + \binom{7+7-4-2}{7-2-1} = 110$$

The minimum one among these values will give us the upper bound of $R_2(7, 7)$ which is 44. If we did not use the bounds of $R_2(6, 7)$, the upper bound of $R_2(7, 7)$ would be 110. As a conclusion,

$$14 \leq R_2(7, 7) \leq 44.$$

3.2.3. Known Values for Bounds on Defective Ramsey Numbers

In this part of this chapter, bounds of defective Ramsey numbers will be calculated. In order to inform the reader, the values will be studied in two parts: the bounds calculated by using the exact values and the bounds calculated by using other bounds. These bounds are calculated by a recursive function written in R programming language. As mentioned in Section 3.2.2, if we know the exact values of all the other defective Ramsey numbers required to calculate the bounds of a defective Ramsey number, the function uses these exact values and returns the lower and the upper

bounds by taking the maximum value among three lower bounds and by taking the minimum value between two upper bounds. But as in the second example in Section 3.2.2.2, sometimes we do not know the exact values of some defective Ramsey numbers, therefore, first the bounds of these numbers are calculated. In such cases, when we do not know the exact value of a defective Ramsey number, instead of its exact value, the function uses the lower bound (upper bound) to calculate the investigated lower bound (upper bound). Suppose that the number that we are calculating the bounds is labeled as Level 0 and the required defective Ramsey numbers to make the calculation are labeled as Level 1. In some cases, calculating only the bounds of the numbers in Level 1 is not sufficient. For example, in order to calculate the bounds for $R_2(10, 10)$ (Level 0), we need the bounds of $R_2(9, 10)$ (Level 1) since we do not know its exact value. But, we also do not know all the exact values, i.e. $R_2(8, 10)$ (Level 2), to make the bound calculation of $R_2(9, 10)$. In such cases, the function continues to calculate the bound until it reaches Level i in which we use only the exact values to calculate its bounds. Then, it uses recursively these bounds of Level i to calculate to bounds of Level $i - 1$ until it reaches Level 0 which is the defective Ramsey number of which we are investigating the bounds. The main algorithm can be seen in Figure 3.2.

In Table 3.9, we can see the bounds of some defective Ramsey numbers which are calculated by using some exactly known defective Ramsey numbers. In Table 3.10, we can see the bounds of 1-defective Ramsey numbers, which are calculated by using the bounds of other defective Ramsey numbers, in Table 3.9. The bounds of 2-defective Ramsey numbers, which are calculated by using the bounds of other defective Ramsey numbers, are represented in Table 3.11. Similarly, the bounds of 3-defective, 4-defective, 5-defective, 6-defective and 7-defective Ramsey numbers are represented respectively in Table 3.12, 3.13, 3.14, 3.15 and 3.16. These tables can be enlarged and also can be calculated for higher defectiveness levels, but we only calculated the values that are necessary to complete the tables in [9].

```

if All the required exact defective Ramsey numbers are known then
    Calculate the lower bound by taking the maximum of values calculated by
    Proposition 3.10, 3.11 and 3.13
    Calculate the upper bound by taking the minimum of values calculated by
    Proposition 3.12 and 3.14
else
    repeat
        Calculate the bounds for unknown defective Ramsey numbers
    until All the bounds are calculated by using exact values
    repeat
        Use the lower bounds of defective Ramsey numbers
        Calculate the lower bound by taking the maximum of values calculated by
        Proposition 3.10, 3.11 and 3.13
        Use the upper bounds of defective Ramsey numbers
        Calculate the upper bound by taking the minimum of values calculated by
        Proposition 3.12 and 3.14
    until The bounds of the defective Ramsey number investigated are calculated
end if

```

Figure 3.2. Bound Calculation Algorithm.

Table 3.9. Bounds Calculated by Using the Exact Values.

Lower Bound	Defective Ramsey Number	Upper Bound
19	$R_1(4, 9)$	26
16	$R_1(5, 6)$	26
13	$R_2(5, 9)$	21
13	$R_2(6, 7)$	22
13	$R_3(6, 10)$	22
14	$R_3(7, 9)$	25
14	$R_3(8, 8)$	26
14	$R_4(7, 11)$	24
13	$R_4(8, 9)$	23
13	$R_5(8, 11)$	23
14	$R_5(9, 10)$	25
13	$R_6(9, 11)$	23
13	$R_7(10, 11)$	23

Table 3.10. Bounds on 1 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
21	$R_1(4, 10)$	36
19	$R_1(5, 7)$	42
23	$R_1(5, 8)$	59
29	$R_1(5, 9)$	85
30	$R_1(5, 10)$	121
17	$R_1(6, 6)$	52
22	$R_1(6, 7)$	94
24	$R_1(6, 8)$	153
29	$R_1(6, 9)$	238
30	$R_1(6, 10)$	359
31	$R_1(7, 7)$	188
33	$R_1(7, 8)$	341
34	$R_1(7, 9)$	579
35	$R_1(7, 10)$	938
34	$R_1(8, 8)$	682
35	$R_1(8, 9)$	1261
36	$R_1(8, 10)$	2199
35	$R_1(9, 9)$	2522
36	$R_1(9, 10)$	4721
36	$R_1(10, 10)$	9442

Table 3.11. Bounds on 2 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
14	$R_2(5, 10)$	31
14	$R_2(6, 8)$	34
17	$R_2(6, 9)$	55
19	$R_2(6, 10)$	86
14	$R_2(7, 7)$	44
16	$R_2(7, 8)$	78
19	$R_2(7, 9)$	133
20	$R_2(7, 10)$	219
19	$R_2(8, 8)$	156
23	$R_2(8, 9)$	289
24	$R_2(8, 10)$	508
24	$R_2(9, 9)$	578
25	$R_2(9, 10)$	1086
25	$R_2(10, 10)$	2172

Table 3.12. Bounds on 3 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
15	$R_3(6, 11)$	33
15	$R_3(7, 10)$	47
17	$R_3(7, 11)$	80
15	$R_3(8, 9)$	51
17	$R_3(8, 10)$	98
20	$R_3(8, 11)$	178
17	$R_3(9, 9)$	102
20	$R_3(9, 10)$	200
23	$R_3(9, 11)$	378
21	$R_3(10, 10)$	400
23	$R_3(10, 11)$	778
25	$R_3(11, 11)$	1556

Table 3.13. Bounds on 4 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
15	$R_4(7, 12)$	36
14	$R_4(8, 10)$	36
15	$R_4(8, 11)$	60
17	$R_4(8, 12)$	96
14	$R_4(9, 9)$	46
15	$R_4(9, 10)$	82
17	$R_4(9, 11)$	142
19	$R_4(9, 12)$	238
16	$R_4(10, 10)$	164
19	$R_4(10, 11)$	306
22	$R_4(10, 12)$	544
21	$R_4(11, 11)$	612
23	$R_4(11, 12)$	1156
24	$R_4(12, 12)$	2312

Table 3.14. Bounds on 5 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
14	$R_5(8, 12)$	35
15	$R_5(8, 13)$	48
15	$R_5(9, 11)$	48
16	$R_5(9, 12)$	83
17	$R_5(9, 13)$	131
15	$R_5(10, 10)$	50
16	$R_5(10, 11)$	98
17	$R_5(10, 12)$	181
19	$R_5(10, 13)$	312
16	$R_5(11, 11)$	196
17	$R_5(11, 12)$	377
21	$R_5(11, 13)$	689
17	$R_5(12, 12)$	754
23	$R_5(12, 13)$	1443
25	$R_5(13, 13)$	2886

Table 3.15. Bounds on 6 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
14	$R_6(9, 12)$	35
15	$R_6(9, 13)$	48
16	$R_6(9, 14)$	62
15	$R_6(10, 11)$	37
16	$R_6(10, 12)$	72
17	$R_6(10, 13)$	120
18	$R_6(10, 14)$	182
15	$R_6(11, 11)$	74
16	$R_6(11, 12)$	146
17	$R_6(11, 13)$	266
18	$R_6(11, 14)$	448
16	$R_6(12, 12)$	292
17	$R_6(12, 13)$	558
18	$R_6(12, 14)$	1006
18	$R_6(13, 13)$	1116
19	$R_6(13, 14)$	2122
20	$R_6(14, 14)$	4244

Table 3.16. Bounds on 7 - defective Ramsey Numbers Calculated by Using Bounds of Other Defective Ramsey Numbers.

Lower Bound	Defective Ramsey Number	Upper Bound
14	$R_7(10, 12)$	35
15	$R_7(10, 13)$	48
16	$R_7(10, 14)$	62
17	$R_7(10, 15)$	77
16	$R_7(11, 12)$	50
17	$R_7(11, 13)$	98
18	$R_7(11, 14)$	160
19	$R_7(11, 15)$	237
16	$R_7(12, 12)$	100
17	$R_7(12, 13)$	198
18	$R_7(12, 14)$	358
19	$R_7(12, 15)$	595
17	$R_7(13, 13)$	396
18	$R_7(13, 14)$	754
19	$R_7(13, 15)$	1349
19	$R_7(14, 14)$	1508
20	$R_7(14, 15)$	2857
21	$R_7(15, 15)$	5714

3.2.4. Improvements on Lower Bounds on Defective Ramsey Numbers

In this section, we improve some of the lower bounds represented in Section 3.2.3 found by the recursive function mentioned in the previous part. We search for counter examples to improve the lower bounds.

A generalized search function has been implemented in R programming language. It takes k , a and b values to begin the search. The search function randomly generates a graph G of order $n = \max \{a, b\}$, in other words, it creates a matrix of size $n \times n$ and assigns either 0 or 1 with equal probability to the cells of the matrix such that the matrix is symmetric on diagonal. Later, it searches G for the forbidden subgraphs: k -dense a -set and k -sparse b -set. If it finds a forbidden subgraph, it generates a new random graph. This search continues until it finds a graph of order n that does not contain any of the forbidden subgraphs. If it finds a graph G that satisfies the conditions, that is, it neither contains a k -dense a -set nor a k -sparse b -set, it stores G and starts trying to enlarge G . Actually, the algorithm may start by constructing a graph of order n' which is equal to the lower bound of $R_k(a, b)$ for given k , a and b values. But it gets harder to find a graph of order n' , such that it neither contains a k -dense a -set nor a k -sparse b -set, when $n' \geq n$ and it gets harder when the difference between them gets larger. Because the number of non-isomorphic graphs grows exponentially on the graph order, thus, search space gets larger and it is more likely to encounter a k -dense a -set or a k -sparse b -set in a graph of order n' than in a graph of order n .

After the algorithm finds a graph G of order n , such that it neither contains a k -dense a -set nor a k -sparse b -set, it creates a graph G' of order $n + 1$ which contains G as an induced subgraph, namely, G' is obtained by adding a vertex v to G . The algorithm randomly generates edges between the vertex v and G and checks whether G' contains any of the subgraphs or not. The algorithm only checks subgraphs containing v , because we know that all the subgraphs without vertex v are the subgraphs of graph G which does not contain a k -dense a -set or a k -sparse b -set. When it finds a graph of order $n + 1$ without a k -dense a -set or a k -sparse b -set in it, it stores this graph

and searches a new graph of order $n + 2$. This search continues until the algorithm cannot enlarge the graph in 1800 seconds and then, terminates. Since the algorithm generates random graphs and continues by taking them as subgraphs, it is possible to generate a graph G of order n in the beginning such that every case of adding a vertex to G results in graphs that contain a k -dense a -set or a k -sparse b -set. Therefore, in order to prevent this issue, the algorithm has been run five times for each k , a and b combination. The main algorithm can be seen in Figure 3.3

```

Require:  $k$ ,  $a$  and  $b$ 

repeat
    Generate a graph  $G$  of order  $n = \max \{a, b\}$ 
until A graph of order  $n$  is found such that it neither contains a  $k$ -dense  $a$ -set
    nor a  $k$ -sparse  $b$ -set
     $G_{new} := G$ 
repeat
    Add a vertex  $v$  to graph  $G_{new}$ 
repeat
    Randomly generate edges between  $G_{new}$  and the vertex  $v$ 
until A graph of order  $|G_{new} + 1|$  is found such that it neither contains a
     $k$ -dense  $a$ -set nor a  $k$ -sparse  $b$ -set
    Update  $G_{new}$ 
until  $|G_{new}|$  is equal to the upper bound of  $R_k(a, b)$  minus 1 or the run time
    reaches 1800 seconds
return Lower bound =  $|G_{new}| + 1$ 

```

Figure 3.3. Counter Example Searcher.

The improved lower bounds for 1-defective Ramsey numbers are represented in italic characters in Table 3.17. Similarly, the improved lower bounds for 2-defective, 3-defective, 4-defective, 5-defective, 6-defective and 7-defective Ramsey numbers are represented in bold characters in Table 3.18, 3.19, 3.20, 3.21, 3.22 and Table 3.23, respectively. The counter examples for improved lower bounds are represented according

to their defectiveness levels in Appendix A, B, C, D, E, F and G.

Table 3.17. 1-defective Ramsey Numbers with Bounds.

$R_1(a, b)$	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	2	3	4	5	6	7	8	9	10
4	2	4	6	9	11	16	17	[19, 26]	[21, 36]
5	2	5	9	15	[18, 26]	[21, 42]	[25, 59]	[29, 85]	[31, 121]
6	2	6	11	[18, 26]	[21, 52]	[26, 94]	[30, 153]	[34, 238]	[34, 359]
7	2	7	16	[21, 42]	[26, 94]	[31, 188]	[33, 341]	[34, 579]	[35, 938]
8	2	8	17	[25, 59]	[30, 153]	[33, 341]	[40, 682]	[40, 1261]	[40, 2199]
9	2	9	[19, 26]	[29, 85]	[34, 238]	[34, 579]	[40, 1261]	[40, 2522]	[40, 4721]
10	2	10	[21, 36]	[31, 121]	[34, 359]	[35, 938]	[40, 2199]	[40, 4721]	[40, 9442]

Table 3.18. 2-defective Ramsey Numbers with Bounds.

$R_2(a, b)$	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3
4	3	4	5	6	7	8	9	10
5	3	5	7	8	10	12	[13, 21]	[14, 31]
6	3	6	8	12	[13, 22]	[16, 34]	[18, 55]	[20, 86]
7	3	7	10	[13, 22]	[16, 44]	[19, 78]	[22, 133]	[26, 219]
8	3	8	12	[16, 34]	[19, 78]	[23, 156]	[26, 289]	[30, 508]
9	3	9	[13, 21]	[18, 55]	[22, 133]	[26, 289]	[29, 578]	[34, 1086]
10	3	10	[14, 31]	[20, 86]	[26, 219]	[30, 508]	[34, 1086]	[36, 2172]

For these defectiveness levels from 1 to 7, the largest improvement is made for $R_2(10, 10)$ by increasing the lower bound by 9. There are 55 improved lower bounds. We can see the breakdown of the total number of the improved bounds on different defectiveness levels in Table 3.24. We can observe that for smaller defectiveness levels, there are more improved lower bounds. But the number of improved lower bounds starts to increase again when the defectiveness level reaches 6. Also another interesting observation is that, generally we obtained a more significant improvement for the lower bounds on $R_k(a, b)$ values when a is equal to b . In some cases, when n is greater than 30 or when the number of a -subsets of n or b -subsets of n gets very large etc., we did not run the search algorithm represented in Figure 3.3 because of the insufficient memory problem.

Table 3.19. 3-defective Ramsey Numbers with Bounds.

$R_3(a, b)$	4	5	6	7	8	9	10	11
4	4	4	4	4	4	4	4	4
5	4	5	6	7	8	9	10	11
6	4	6	8	9	10	12	[13, 22]	[15, 33]
7	4	7	9	11	13	[14, 25]	[15, 47]	[17, 80]
8	4	8	10	13	[14, 26]	[16 , 51]	[19 , 98]	[22 , 178]
9	4	9	12	[14, 25]	[16 , 51]	[19 , 102]	[23 , 200]	[24 , 378]
10	4	10	[13, 22]	[15, 47]	[19 , 98]	[23 , 200]	[26 , 400]	[29 , 778]
11	4	11	[15, 33]	[17, 80]	[22 , 178]	[24 , 378]	[29 , 778]	[29 , 1556]

Table 3.20. 4-defective Ramsey Numbers with Bounds.

$R_4(a, b)$	5	6	7	8	9	10	11	12
5	5	5	5	5	5	5	5	5
6	5	6	7	8	9	10	11	12
7	5	7	9	10	11	13	[14, 24]	[15, 36]
8	5	8	10	12	[13, 23]	[14, 36]	[15, 60]	[17, 96]
9	5	9	11	[13, 23]	[14, 46]	[15, 82]	[17, 142]	[19, 238]
10	5	10	13	[14, 36]	[15, 82]	[17 , 164]	[19, 306]	[22, 544]
11	5	11	[14, 24]	[15, 60]	[17, 142]	[19, 306]	[21, 612]	[24 , 1156]
12	5	12	[15, 36]	[17, 96]	[19, 238]	[22, 544]	[24 , 1156]	[28 , 2312]

Table 3.21. 5-defective Ramsey Numbers with Bounds.

$R_5(a, b)$	6	7	8	9	10	11	12	13
6	6	6	6	6	6	6	6	6
7	6	7	8	9	10	11	12	13
8	6	8	10	11	12	[13, 23]	[14, 35]	[15, 48]
9	6	9	11	13	[14, 25]	[15, 48]	[16, 83]	[17, 131]
10	6	10	12	[14, 25]	[15, 50]	[16, 98]	[17, 181]	[19, 312]
11	6	11	[13, 23]	[15, 48]	[16, 98]	[17, 196]	[18, 377]	[21, 689]
12	6	12	[14, 35]	[16, 83]	[17, 181]	[18, 377]	[20, 754]	[23, 1443]
13	6	13	[15, 48]	[17, 131]	[19, 312]	[21, 689]	[23, 1443]	[26, 2886]

Table 3.22. 6-defective Ramsey Numbers with Bounds.

$R_6(a, b)$	7	8	9	10	11	12	13	14
7	7	7	7	7	7	7	7	7
8	7	8	9	10	11	12	13	14
9	7	9	11	12	[13, 23]	[14, 35]	[15, 48]	[16, 62]
10	7	10	12	14	[15, 37]	[16, 72]	[17, 120]	[18, 182]
11	7	11	[13, 23]	[15, 37]	[15, 74]	[16, 146]	[17, 266]	[18, 448]
12	7	12	[14, 35]	[16, 72]	[16, 146]	[17, 292]	[18, 558]	[19, 1006]
13	7	13	[15, 48]	[17, 120]	[17, 266]	[18, 558]	[20, 1116]	[20, 2122]
14	7	14	[16, 62]	[18, 182]	[18, 448]	[19, 1006]	[20, 2122]	[23, 4244]

Table 3.23. 7-defective Ramsey Numbers with Bounds.

$R_7(a, b)$	8	9	10	11	12	13	14	15
8	8	8	8	8	8	8	8	8
9	8	9	10	11	12	13	14	15
10	8	10	12	[13, 23]	[14, 35]	[15, 48]	[16, 62]	[17, 77]
11	8	11	[13, 23]	15	[16, 50]	[17, 98]	[18, 160]	[19, 237]
12	8	12	[14, 35]	[16, 50]	[18, 100]	[19, 198]	[18, 358]	[19, 595]
13	8	13	[15, 48]	[17, 98]	[19, 198]	[19, 396]	[18, 754]	[19, 1349]
14	8	14	[16, 62]	[18, 160]	[18, 358]	[18, 754]	[20, 1508]	[20, 2857]
15	8	15	[17, 77]	[19, 237]	[19, 595]	[19, 1349]	[20, 2857]	[22, 5714]

Table 3.24. Number of Improved Lower Bounds.

Defectiveness Levels	Total Number of Improved Lower Bounds
1-defective	15
2-defective	13
3-defective	9
4-defective	3
5-defective	4
6-defective	6
7-defective	5

4. DEFECTIVE RAMSEY NUMBERS IN RESTRICTED GRAPH CLASSES

In this chapter, Ramsey numbers will be studied in restricted graph classes. We define Ramsey numbers in restricted graph classes as follows:

Definition 4.1. $R_k^{\mathcal{G}}(a, b) = \min n$ such that all n -graphs in graph class $\mathcal{G}$ contain a k -sparse a -set or a k -dense b -set.

Remark 4.2. For any graph class $\mathcal{G}$, we have $R_k^{\mathcal{G}}(a, b) \leq R_k(a, b)$.

In the first part of the study, we study defective Ramsey numbers for P_4 -free graphs which do not contain induced paths on 4 vertices. We have proposed a formula and constructed the proof for classical Ramsey numbers in P_4 -free graphs such that $R^{P_4\text{-free}}(a, b) = (a - 1)(b - 1) + 1$. Afterwards, we have discovered that this formula has been found and proven in [2] for permutation graphs and it is known as the Erdős-Szekeres theorem. A permutation graph G_p is a graph whose edges $\{v_i, v_j\}$ correspond exactly to (i, j) being a permutation inversion in some permutation p , i.e., $i < j$ but j occurs before i in p [100]. Since P_4 -free graphs is a subset of permutation graphs [101], we propose only the proof for P_4 -free graphs to propose another point of view. Next, we generalize the formula for perfect graphs which gives an upper bound and not the exact value for $R^{Perfect}(a, b)$ values. Also we propose a defective version of this formula which also appears to give an upper bound for permutation graphs, therefore, for P_4 -free graphs.

4.1. Ramsey Numbers for Cographs

In this section, we utilize the tree representation of cographs, called *cotrees*. A graph G is a *cograph* if and only if G does not contain a P_4 as an induced subgraph. In cotrees, the internal nodes are labeled with the numbers 0 and 1. A subtree consisting of a single leaf node corresponds to an induced subgraph with a single vertex. A subtree

rooted at a node labeled 0 corresponds to the union of the subgraphs defined by the children of that node. A subtree rooted at a node labeled 1 corresponds to the join of the subgraphs defined by the children of that node; that is, we form the union and add an edge between every two vertices corresponding to leaves in different subtrees. It is known that every cograph has a cotree representation [102].

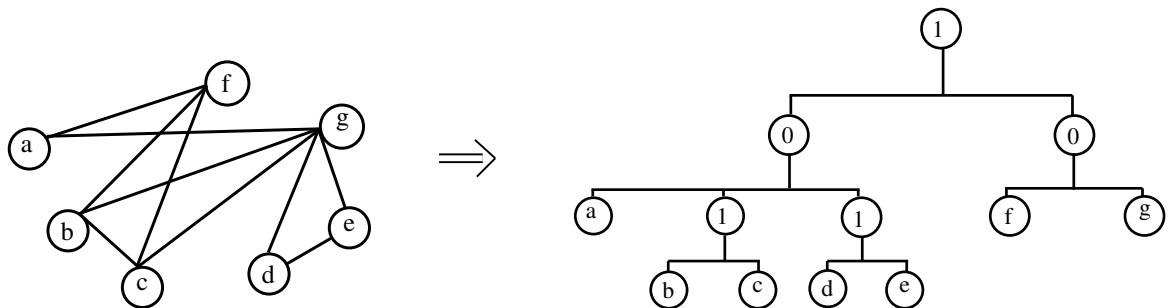


Figure 4.1. Cotree representation of a P_4 -free graph.

We calculate Ramsey numbers in cographs by the help of their cotree representation of P_4 -free graphs. Since P_4 is self-complementary, that is, the complement of a P_4 is again a P_4 , cographs are also self-complementary. A *self-complementary* graph is a graph which is isomorphic to its complement. Therefore, it is enough to consider only the cotrees with root labeled 1. In this way, when we observe all possible n -graphs, the complement graphs of any n -graph will also be considered.

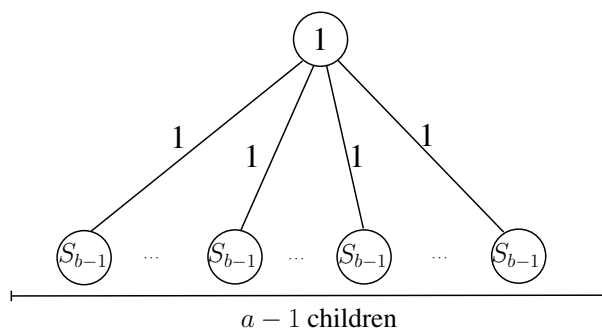


Figure 4.2. $(a-1)(b-1)$ -graph.

The graph in Figure 4.2 is an $(a-1)(b-1)$ -graph which contains neither a clique of order a nor an independent set of order b . This graph has the largest cardinality without having a clique of order a or an independent set of order b . Since we join the children of the node labeled 1, the clique number of the graph is determined by the sum

of the clique numbers of children. All the children are independent sets of order $b - 1$, thus, the clique number of the children is equal to 1. Since there are $a - 1$ children, which are completely linked to each other, the clique number of the graph is $a - 1$. Thus, in a $(a - 1)(b - 1)$ -graph as in the Figure 4.2, we neither have a clique of order a nor an independent set of order b . When another vertex is added, namely, when the graph becomes a $(a - 1)(b - 1) + 1$ -graph, it will always contain a clique of order a or an independent set of order b .

We claim that $R^{P_4-free}(a, b) = (a - 1)(b - 1) + 1$. In order to prove this proposition, the following statements are needed:

Definition 4.3. $M^{P_4-free}(a, b)$ is the maximum order of a P_4 -free graph G such that $\omega(G) < a$ and $\alpha(G) < b$.

Lemma 4.4. $M^{P_4-free}(a, b) = (a - 1)(b - 1)$

Proof. Let G be a P_4 -free graph with maximum order such that $\omega(G) < a$ and $\alpha(G) < b$. Without loss of generality, the root is labeled by 1 and has i children.

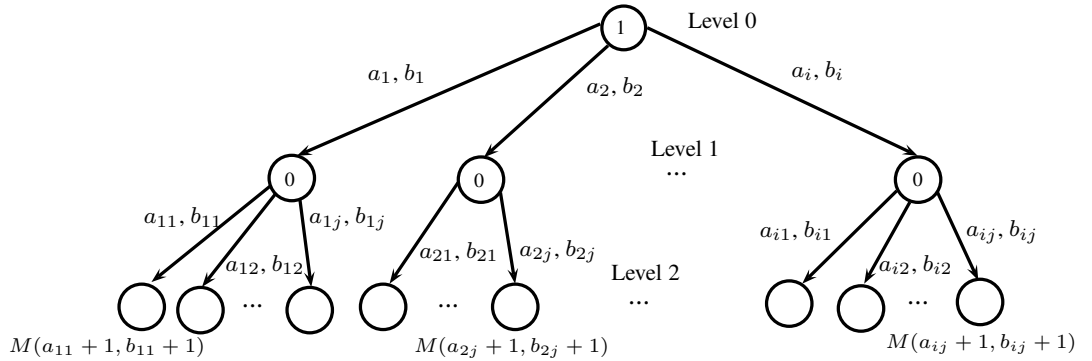


Figure 4.3. The cotree representing the graph G .

In Figure 4.3, the values on the edges, denoted by a_i, b_i , show the maximum clique size and the maximum independent set size of the branch i respectively. Similarly, a_{ij}, b_{ij} show the maximum clique size and the maximum independent set size of the leaf j under the branch i respectively. Since the leaves of branches of nodes labeled by 0 are disjoint, a_i will be equal to $\max_j a_{ij}$. The graph is constructed by connecting every

vertex below the node labeled with 1, thus, every vertex of subgraphs at level 1 will be connected to each other. That will create a graph with clique size $\sum_i a_i$. Since we want to obtain a graph G such that $\omega(G) < a$, the following must hold:

$$\sum_i a_i < a \quad \text{where} \quad 0 < a_i < a - 1.$$

Similarly, we want to obtain a graph G such that $\alpha(G) < b$. If any subgraph constructed at level 1 has an independent set of order b , the graph will contain an independent set of order b . Thus, the maximum independent set order, b_i , of these subgraphs is at most $b - 1$. Since we do not connect any vertex below the node labeled with 0, the maximum independent set order of the branch i will be the sum of the maximum independent set order of each leaf j as follows:

$$b_i = \sum_j b_{ij}$$

The maximum independent set order of graph G will be $\max_i b_i$, thus, $\sum_j b_{ij}$ must be smaller than $b - 1$ for each i . Any additional vertex will create a clique of order a or an independent set of order b .

Clearly, by Definition 4.3, $M^{P_4\text{-free}}(2, 2) = 1$. Assuming that $M^{P_4\text{-free}}(a - 1, b - 1) = (a - 2).(b - 2)$ is true, we must show that $M^{P_4\text{-free}}(a, b) = (a - 1).(b - 1)$ holds. Consider the leaf j of the branch i : $M(a_{ij} + 1, b_{ij} + 1) = a_{ij}.b_{ij}$. This is the maximum order that the subgraph of the leaf j of the branch i can take without having an independent set of order a or a clique of order b , since $a_{ij} \leq a - 1$ and $b_{ij} \leq b - 1$. By induction, when this operation is applied to all the nodes in the tree, the maximum order of G , such that it neither contains a clique of order a nor an independent set of order b , becomes

$$\sum_i \sum_j a_{ij} b_{ij} = (a - 1)(b - 1).$$

For any values of i and j on different branches, this equality holds. $\square$

Proposition 4.5. $R^{P_4-free}(a, b) = (a - 1)(b - 1) + 1$

Proof. By Definition 4.3, it is mentioned that the maximum graph order of a P_4 -free graph G such that it neither contains a clique of order a nor an independent set of order b is denoted as $M^{P_4-free}(a, b)$. Since this is the maximum order that satisfies these constraints, when we add a vertex to graph G , we obtain a P_4 -free graph such that it contains a clique of order a or an independent set of order b . The minimum order that guarantees to contain a clique of order a or an independent set of order b becomes $(a - 1)(b - 1) + 1$ by Lemma 4.4.

$$R^{P_4-free}(a, b) = M^{P_4-free}(a, b) + 1$$

$$R^{P_4-free}(a, b) = (a - 1)(b - 1) + 1$$

$\square$

Table 4.1. Ramsey Numbers on P_4 -free Graphs.

a / b	3	4	5	6	7	8	9
3	5	7	9	11	13	15	17
4	7	10	13	16	19	22	25
5	9	13	17	21	25	29	33
6	11	16	21	26	31	36	41
7	13	19	25	31	37	43	49
8	15	22	29	36	43	50	57
9	17	25	33	41	49	57	65

In Table 4.1, we can see the results of the formula $R^{P_4-free}(a, b) = (a - 1)(b - 1) + 1$. This formula give the exact values of the Ramsey numbers on P_4 -free graphs. As mentioned in the beginning of this chapter, this formula is proposed for permutation graphs

in [2]. Since P_4 -free graphs is a subset of permutation graphs [101], this proposition is not contributing to literature but this study proposes a new proof for P_4 -free graphs. Ramsey numbers for permutation graphs can be found for any given a and b by Proposition 4.6.

Proposition 4.6.

$$R^{Permutation}(a, b) = (a - 1)(b - 1) + 1$$

4.2. Ramsey Numbers for Perfect Graphs

In this section, we generalize Proposition 4.5 for perfect graphs. Proposition 4.8 gives an upper bound on Ramsey numbers for perfect graphs and it will be proven by Theorem 4.7.

Theorem 4.7. [103] *A graph G is perfect if and only if $\omega(G)\alpha(G) \geq |G|$.*

Proposition 4.8. $R^{Perfect}(a, b) \leq (a - 1)(b - 1) + 1$

Proof. Let $R^{Perfect}(a, b) = n$ and let G be a perfect graph of order $n - 1$ such that it neither contains a clique of order a nor an independent set of order b . The maximum clique size of G , $\omega(G)$, can be at most $a - 1$ and similarly the maximum independent set size of G , $\alpha(G)$, can be at most $b - 1$. By Theorem 4.7, an upper bound for $R^{Perfect}(a, b)$ can be derived as follows:

$$\omega(G) \alpha(G) \geq n - 1$$

$$(a - 1)(b - 1) \geq n - 1$$

$$(a - 1)(b - 1) + 1 \geq n$$

□

4.3. Defective Ramsey Numbers for Restricted Graph Classes

In this section, we propose an upper bound on Ramsey number for permutation graphs. We use the same proof construction as the proof of Proposition 4.6 by the pigeonhole principle; it states that if n items are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item. Therefore, firstly the proof of Proposition 4.6 by pigeonhole principle will be explained.

Proof. (Proposition 4.6) Given a permutation Π of length $(a-1)(b-1)$ that neither contains a monotonically decreasing subsequence of length a nor a monotonically increasing subsequence of order b . This will be the representation of a permutation graph G that neither contains a clique of order a nor an independent set of order b . We can label the i^{th} number, say n_i , of the permutation Π with the pair (r_i, s_i) , where r_i denotes the longest monotonically decreasing subsequence and s_i denotes the longest monotonically increasing subsequence that ends at number n_i . We note that each number n_i will be labeled by a different pair; if $i < j$ and $n_i < n_j$ then $r_i < r_j$, else $n_j < n_i$ then $s_i < s_j$. We may have a labeling such that neither a monotonically decreasing subsequence of length at least a nor a monotonically increasing subsequence of length at least b exist as follows:

$$\bar{1} \quad \underline{(a-1)(b-1)} \quad \bar{2} \quad \underline{(a-2)(b-1)} \quad \bar{3} \quad \dots \quad \overline{b-2} \quad \underline{2(b-1)} \quad \overline{b-1}$$

We underline the decreasing subsequence and overline the increasing subsequence. We have a monotonically decreasing subsequence of length $(a-2)(b-1)+1$ and a monotonically increasing subsequence of length $b-1$ both with the last element which is $b-1$. Thus, the total length of the sequence is $(a-2)(b-1)+1+(b-1)-1 = (a-1)(b-1)$, since we count the last element in both subsequences, we subtract one from the sum of the lengths of the subsequences. This permutation of length $(a-1)(b-1)$ neither contains a monotonically decreasing subsequence of order a nor a monotonically increasing subsequence of order b . For this sequence, we have only $(a-1)(b-1)$ different (r_i, s_i) labels. Thus, when we add another number t to Π and obtain a new permutation Π'

of length $(a - 1)(b - 1) + 1$, the length of the numbers will be larger than the number of different labels. By pigeonhole principle, Π' , the number t will have a label outside that range, therefore, r_t is equal to a or s_t is equal to b . Because it is not possible to label $(a - 1)(b - 1) + 1$ numbers by using only $(a - 1)(b - 1)$ labels. We have the following result: in a permutation Π' of length $(a - 1)(b - 1) + 1$, there is a monotonically decreasing subsequence of length at least a or a monotonically increasing subsequence of length at least b . Therefore, in the graph representation of Π' , we have a clique of order a or an independent set of order b . $\square$

We propose the following inequality as an upper bound on defective Ramsey number for permutation graphs:

Proposition 4.9. $R_k^{Permutation}(a, b) \leq (a - k - 1)(b - k - 1) + k + 1$

Proof. For a permutation Π of length $(a - k - 1)(b - k - 1)$, label each number n_i of Π with the pair r_i, s_i , where r_i denotes the length of the longest k -defectively decreasing subsequence that ends with n_i and s_i denotes the length of the longest k -defectively increasing subsequence ending with n_i . For the first $k + 1$ numbers, r_i and s_i both increase and at number n_{k+1} , both will be equal to $k + 1$. After these $k + 1$ numbers, like the classical version, each number n_i will be labeled by a different pair. In order to prevent a k -defectively decreasing subsequence of length a , r_i can take values between $k + 1$ and $a - 1$ after the first $k + 1$ numbers. That means there are $a - 1 - (k + 1) + 1 = a - k - 1$ different values that r_i can take after the first $k + 1$ numbers. Similarly, in order to prevent a k -defectively increasing subsequence of length b , s_i can take values between $k + 1$ and $b - 1$ after the first $k + 1$ numbers. That means there are $b - 1 - (k + 1) + 1 = b - k - 1$ different values that s_i can take after the first $k + 1$ numbers. That creates $(a - k - 1)(b - k - 1)$ different labels. But in these $(a - k - 1)(b - k - 1)$ labels, we have also counted the $(k + 1), (k + 1)$ pair which is already used at the $(k + 1)^{\text{th}}$ number. Therefore, there are actually $(a - k - 1)(b - k - 1) - 1$ different labels that we can use after the first $k + 1$ numbers. Thus, when we have a permutation of length $(a - k - 1)(b - k - 1) - 1 + k + 1 = (a - k - 1)(b - k - 1) + k$, we may have a

sequence which contains neither a k -defectively decreasing subsequence of length a nor a k -defectively increasing subsequence of length b . Since we have used all the possible pairs to label Π , when we add another number t to Π and obtain a new permutation Π' , r_t will be equal to a or s_t will be equal to b . We have the following result: in a permutation Π' of length $(a - k - 1)(b - k - 1) + k + 1$, there is a k -defectively decreasing subsequence of length at least a or a k -defectively increasing subsequence of length at least b . Therefore, in the graph representation of Π' , we have a k -dense set of order a or a k -sparse set of order b . $\square$

For small graphs, it is observed that the formula gives the exact values. For example, $R_1(4, 4) = (4 - 1 - 1)(4 - 1 - 1) + 1 + 1 = 6$ which gives the exact value of the defective Ramsey numbers for both all simple graphs and permutation graphs. But, when we calculate the value of $R_3(6, 7)$, we find an upper bound such that $R_3(6, 7) \leq (6 - 3 - 1)(7 - 3 - 1) + 3 + 1 = 10$. Because even though we consider all the simple graphs, instead of only permutation graphs, we can see that $R_3(6, 7) = 9$ by Table 3.3, meaning that, for all 9-graphs, there are always a 3-dense 6-set or 3-sparse 7-set which is also true for permutation graphs of order 9.

5. ERDÖS - HAJNAL CONJECTURE AND RAMSEY NUMBERS

In this chapter, we will study the famous conjecture of Erdős and Hajnal. We will also bring out a new approach to show that the conjecture holds for permutation graphs.

5.1. Erdős-Hajnal Conjecture

In order to state the conjecture, we firstly give some definitions.

Definition 5.1. *The homogeneous number of a graph G , denoted $\text{hom}(G)$, is the maximum of the clique number and the independence number.*

$$\text{hom}(G) = \max(\alpha(G), \omega(G))$$

Definition 5.2. *A graph is H -free if it does not have H as an induced subgraph.*

Conjecture 5.3. *[104] For every graph G , there is a non-zero constant ϵ such that when G is an H -free graph with n vertices, $\text{hom}(G) \geq n^\epsilon$.*

It is known that $\epsilon = \frac{1}{2}$ for P_4 -free graphs and it is mentioned in [104] that if the Strong Perfect Graph Conjecture is true, then, $\epsilon = \frac{1}{2}$ also holds for perfect graphs. Since the Strong Perfect Graph Theorem is proven [105], it results in $\epsilon = \frac{1}{2}$ holds for perfect graphs.

5.2. Erdős-Hajnal Conjecture Through Ramsey Numbers

We propose a new formula for permutation graphs which appears to be a transformation of Erdős-Hajnal conjecture.

Proposition 5.4. *If G is a graph of order n , then we have*

$$hom(G) \geq \max_{R(a,b) \leq n} \{\min \{a, b\}\}.$$

Proof. Recall that $R(a, b) = n$ denotes that every graph of order at least n contains a clique of order a or an independent set of order b . Thus, for each graph G of order n the following statement is true for a and b values such that $R(a, b) = n$:

$$hom(G) = \max(\alpha(G), \omega(G)) \geq \min \{a, b\}.$$

Without loss of generality, suppose $a < b$ and G contains a maximum clique of order a and does not contain an independent set of order a (since a is smaller than b , G does not contain an independent set of order b obviously). Then, $hom(G) = a$. Since Ramsey numbers only define a lower bound on the order of subgraphs and does not give an upper bound on the order of that subgraph, $hom(G)$ can be greater than the minimum of these numbers. Since the graph G must satisfy all the Ramsey number equalities which hold for values equal to n or smaller than n , then, we can generalize the statement above by taking the maximum value among different (a, b) pairs such that $R(a, b) \leq n$. $\square$

For example, let G be a graph of order 19. The Ramsey numbers that are smaller than 19 are listed below. We can observe that since every 14-graph should contain a complete or an empty subgraph of order 3, that will also be the case for all 19-graphs. Since $R(3, 3) = 6$ and $\min \{3, 3\} = 3$, graphs of order 6 always contain a complete or an empty subgraph of order 3.

Since $R(3, 4) = 9$ and $\min \{3, 4\} = 3$, graphs of order 9 always contain a complete or an empty subgraph of order 3.

Since $R(3, 5) = 14$ and $\min \{3, 5\} = 3$, graphs of order 14 always contain a complete or an empty subgraph of order 3.

Since $R(3, 6) = 18$ and $\min \{3, 6\} = 3$, graphs of order 18 always contain a complete

or an empty subgraph of order 3.

Since $R(4, 4) = 18$ and $\min \{4, 4\} = 4$, graphs of order 18 always contain a complete or an empty subgraph of order 4.

Then, $\text{hom}(G) \geq \max \{3, 3, 3, 3, 4\} = 4$, because G always contains a complete or an empty subgraph of order at least 4, but it may contain a larger subgraph which will give a larger homogeneous number.

In this manner, when we observe all the Ramsey numbers that are smaller than or equal to the order of the graph, we can find a lower bound on the homogeneous number of the graph.

It can be seen in Table 2.1 that $\min \{a, b\}$ increases when a and b get closer for $R(a, b) \leq n$. Therefore,

$$\max_{R(a,b) \leq n} \{\min \{a, b\}\}$$

gets the maximum value when a and b are equal. This value can be obtained from the values on the diagonal of Table 2.1. If a and b are not equal to each other, this value will be equal to the number of the largest diagonal value which is smaller than n . Thus, the inequality becomes:

$$\text{hom}(G) \geq a \quad \text{where} \quad a = \max_{R(t,t) \leq n} t.$$

Claim 5.5. *Let $n \geq 1$,*

$$a = \lfloor 1 + \sqrt{n-1} \rfloor \text{ for a permutation graph of order } n \text{ such that } a = \max_{R(t,t) \leq n} t.$$

Proof. If we want to find the value of a for permutation graphs, we should find the value of $a = \max_{R(t,t) \leq n} t$ where $R(a, a) = (a-1)(a-1) + 1 = a^2 - 2a + 2$ by Proposition

4.6.

$$a^2 - 2a + 2 \leq n$$

$$a^2 - 2a + 2 - n \leq 0$$

$$a_{1,2} = 1 \mp \sqrt{n-1}$$

$$\text{Since } a \geq 0, a = 1 + \sqrt{n-1}.$$

This is the maximum value that a can take. Since that operation can violate the integrality condition, we must apply floor operation to get the integer value of a . $\square$

We claim that every permutation graph of order n should contain at least a complete or an empty subgraph of order $1 + \sqrt{n-1}$. It can be checked that this value is greater than or equal to $n^{1/2}$ according to the Erdős-Hajnal conjecture for $n \geq 1$.

Claim 5.6. *For every integer $n \geq 1$, we have*

$$EH = \lfloor 1 + \sqrt{n-1} \rfloor = \lceil \sqrt{n} \rceil$$

Proof. We have to show that $\sqrt{n} \leq 1 + \sqrt{n-1} \leq \sqrt{n} + 1$

$$\sqrt{n} \leq 1 + \sqrt{n-1}$$

$$(\sqrt{n})^2 \leq (1 + \sqrt{n-1})^2$$

$$n \leq 1 + n - 1 + 2\sqrt{n-1}$$

$$0 \leq \sqrt{n-1}$$

Since $n \geq 1$, $0 \leq \sqrt{n-1}$, therefore, it shows that $\sqrt{n} \leq 1 + \sqrt{n-1}$. The second part

of the proof is obvious:

$$1 + \sqrt{n-1} \leq \sqrt{n} + 1$$

$$\sqrt{n-1} \leq \sqrt{n}$$

$$n-1 \leq n$$

□

EH gives the minimum subgraph (a clique or an independent set) order that any permutation graph of order n always contains. $hom(G) \geq EH$, since there always exists a clique or an independent set of order at least EH but the graph may contain a clique or an independent set of order larger than EH . As it can be seen in Figure 5.1, $1 + \sqrt{n-1}$ gives an upper bound on the homogeneous number whereas $\sqrt{n}$ gives a lower bound which shows that Proposition 5.4 gives another version of Erdős-Hajnal conjecture for permutation graphs. For a given graph order n , we find the corresponding $1 + \sqrt{n-1}$ and $\sqrt{n}$ values. When we apply the floor operation on $1 + \sqrt{n-1}$ and the ceiling on $\sqrt{n}$ in order to find the corresponding integer EH value, both functions give the same result.

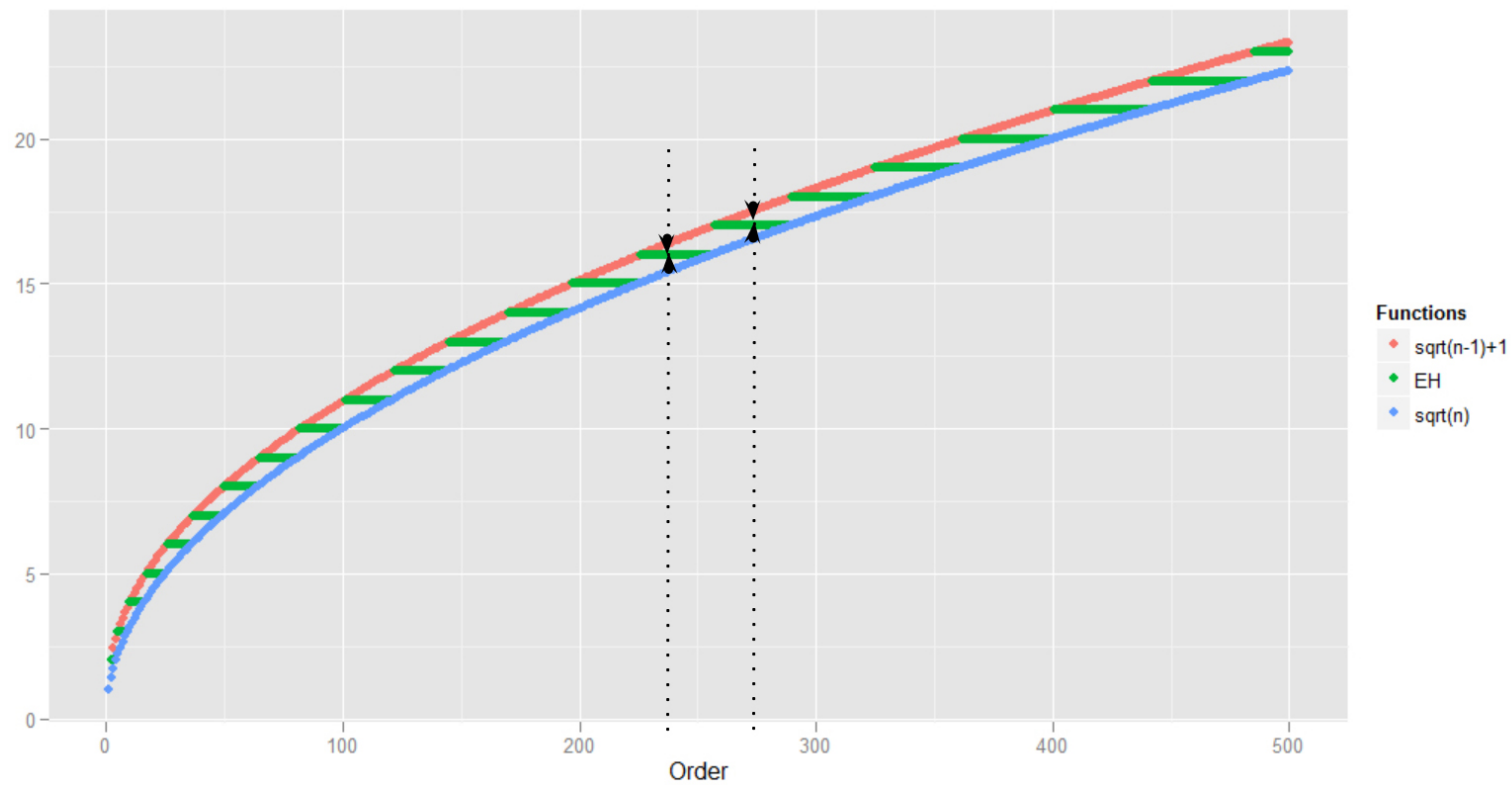


Figure 5.1. Erdős-Hajnal Conjecture for Permutation Graphs Through Ramsey Numbers.

6. COCOLORING ON GRAPHS

Since cliques and stable sets are closely related to the vertex coloring problem, we will be also interested in the generalization of the vertex coloring problem where defective sets are used. A *vertex coloring* of G is an assignment of colors to the vertices of G so that adjacent vertices do not get the same color. A vertex coloring with k colors is called a proper *k-coloring*. Graph G is *k-colorable* if it has a proper *k-coloring*. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is *k-colorable* [15, 16].

A well studied generalization of the graph coloring problem is *cocoloring* where the color classes consist of not only independent sets but also cliques. The *cochromatic number* of G , denoted $z(G)$, is the minimum number of colors to color the vertices such that each color class is an independent set or a clique [10].

Given a natural number k , a partition of vertices of G is a *k-defective coloring* of G if each part induces a graph of maximum degree at most k . A partition of vertices of G is a *k-defective cocoloring* of G if each part is *k-sparse* or *k-dense*. The minimum order of all *k-defective cocolorings* of G is the *k-defective cochromatic number* of G and denoted by $z_k(G)$ [8].

The parameter $c_k(m)$ is defined as the maximum order n such that all n -graphs have a *k-defective m-cocoloring*. It is obvious that $c_0(1) = 2$, $c_1(1) = 3$ and $c_2(1) = 4$ by Proposition 3.4. We know $c_0(2) = 4$, $c_0(3) = 8$ and $c_0(4) \in \{11, 12\}$ [11]. We also know that $c_1(2) = 7$, $c_1(3) \geq 11$ and $c_2(2) \geq 9$ [8]. Thus, the next questions are to answer whether $c_1(3) = 11$ is true or not, namely, whether there exists a 12-graph G which has $z_0(G) = 4$. Similarly, it is important to be able to show whether the bound $c_2(2) \geq 9$ is tight. To show $c_k(m) = n$, it requires to show two things:

- All n -graphs can be *k-defectively m-cocolored*
- There exists an $(n + 1)$ -graph G such that $z_k(G) = m + 1$.

Our aim is to change the inequalities into equalities whenever possible. If there do not exist such graphs, the next steps of the study will be to find the exact numbers that satisfy the equalities. For that part of the study, computer search results will guide us. The exact numbers will be computed by a computer program, then it is aimed to support these results with theoretical proofs.

6.1. Computation of $c_0(4)$, $c_1(3)$ and $c_2(2)$

Recall that we have defined $c_k(m)$ as the maximum order n such that all n -graphs has a k -defective m -cocoloring. The results of $c_0(4)$, $c_1(3)$ and $c_2(2)$ were left in the literature. In this section, the results of these parameters and the methodology will be represented.

We generate and search the graphs by considering their *adjacency matrices*. *Adjacency matrix* is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position $[v_i, v_j]$ according to whether vertices v_i and v_j are adjacent or not. Information on the edges between a specific vertex to all the other vertices is hold by a row (also by a column, but without loss of generality we can make the calculations on rows) in the matrix. The sum of the rows will give us the degree of the corresponding vertex. Thus, taking sum over the rows will give us all the degrees. We can easily observe defectiveness violations by looking at the degrees. In most of the cases, existence of a subgraph directly gives information on the cochromatic number of the graph. For example, we know that $c_0(3) = 8$ [11]. Consider a graph G constructed by the edge-disjoint union of a C_5 and a clique of order 4, in other words, we combine a C_5 and a clique of order 4 without adding any edge between them to construct G . We can see that G is a 9-graph and it cannot be cocolored by 3 colors. Thus, the maximum graph order n such that all n -graphs can be cocolored by at most 3 colors must be less than 9. The proof of $c_0(3) = 8$ can be found in [11]. Since $c_0(3) = 8$, we can easily say that an 11-graph G has always $z_0(G) = 4$. Because we know that $R(3, 3) = 6$ and that means every 11-graph always contains a clique or an independent set of order 3, since every 6-graph always contains a clique or an independent set of order 3. Thus,

when we give a color to that subgraph of order 3, there remains 8 vertices which can be always cocolored by 3 colors. Hence, every 11-graph can be cocolored by at most 4 colors. That results in $c_0(4) \geq 11$. Thus, firstly we try to find a 12-graph which has a cochromatic number 5. If we find such a graph that will lead us to the following result: there is at least one 12-graph such that it cannot be cocolored by 4 colors, thus, the maximum graph order n such that every n -graph can be cocolored by 4 colors is 11. To find such a graph, we must forbid cliques and independent sets of order 4. We will denote such subgraphs as *forbidden subgraphs*. Because if the graph contains a forbidden subgraph, we assign a color to that subgraph and the remaining vertices can be cocolored by 3 colors, since $c_0(8) = 3$ and this 12-graph has cochromatic number of 4. But we want to find a 12-graph which can be cocolored by at least 5 colors, thus, if we encounter a forbidden subgraph in a graph, then it is eliminated. At each step, if a graph passes all the forbidden subgraph checks, namely if it does not contain any of the forbidden subgraphs, isomorphism check is applied. If we did not find an isomorphic graph before, we store the graph. Otherwise, we stop its generation.

In order to emphasize the importance of the eliminations, we can consider two examples. In Section 6.1.2, we forbid 1-defective 2-cocolorable 8-subgraphs which will only leave 160 among all 12346 8-graphs which is the number of graphs on 8 unlabeled vertices that can be seen in Table 2.4. If we did not make eliminations, we would have $12346 \times 2^8 = 3160576$ 9-graphs instead of 40960 9-graphs. When the order of the graph becomes 10, the number of graphs without elimination will be 1618214912 as it is 20971520 with eliminations. Actually we have to check 77 times as many graphs as when we do not make eliminations. We can see that the difference between two methods are remarkable especially when the order of the graph gets larger. Again in Section 6.1.2, the concept of authorized lines will be considered. By this strategy, we consider 105148867 graphs instead of 369098752 graphs which is almost one-third.

6.1.1. Computing $c_0(4)$

For the first part of the study, we will search a 12-graph G which has $z_0(G) = 5$. Since $R(3, 4) = 9$, every 12-graph contains a clique of order 3 or an independent set of order 4. This brings out two cases:

- Case 1. If there exists an independent set of order 4, 8 vertices remain. Since $c_0(3) = 8$, in any case the remaining 8 vertices can be cocolored by at most 3 colors. Then, we will assign another color to the independent set of order 4 and the cochromatic number of the graph will be 4, it is trivial to investigate that case.
- Case 2. If there is not any independent set of order 4, then there exists a clique of order 3. Again, $R(3, 4) = 9$ is being used.
 - Case 2.1. Remaining 9 vertices contain an independent set of order 4, 5 vertices remain. Since $c_0(2) = 4$, there are 5-graphs that require 3 colors to cocolor, this case must be investigated.
 - Case 2.2. Remaining 9 vertices do not contain an independent set of order 4 but contain a clique of order 3. 6 vertices remain. Since $R(3, 3) = 6$, two sub cases rise.
 - Case 2.1.1 Remaining 6 vertices contain a clique of order 3, 3 vertices remain. There are 3-graphs which cannot be cocolored by 1 color, thus, this case must be investigated.
 - Case 2.2.2 Remaining 6 vertices do not contain a clique of order 3 but contain an independent set of order 3, 3 vertices remain and the case must be investigated again.

We will search a 12-graph G , which has $z_0(G) = 5$, by studying upwardly the cases above. We eliminate 6-graphs that contain an independent set of order 4 to prevent construction of Case 1. Also, since the cocoloring considers the cliques and independent sets equally, the cases are also valid when we consider the complements, in other words, the cases are also valid when we replace cliques with independent sets and vice versa. Therefore, we also eliminate 6-graphs that contain a clique of order 4.

Let us denote the set of 6-graphs that neither contains a complete subgraph nor an empty subgraph of order 4 by A . We combine the remaining graphs with 3-graphs to obtain 9-graphs. We consider all the edge combinations between these two subgraphs. But as soon as we construct a 9-graph which contains a complete or empty subgraph of order 4, we eliminate that graph. After these eliminations, again a combination of 3-graphs is made. The resulting graphs are graphs of order 12, after the elimination of complete or empty 4-graphs, we investigate all the remaining graphs. Since we have eliminated all the complete or empty subgraphs of order 4, we do not need to check larger complete or empty subgraphs by Proposition 6.1.

Proposition 6.1. *For every integer $k \geq 0$, if there exists a k -defective n -graph then all its proper subsets are also k -defective.*

Proof. Assume that there exists a graph which is k -sparse n -graph. That means, we have n vertices all of which have a degree of at most k . Thus, removal of any vertex does not change the degrees or decrease them. Namely, removal of any vertex does not reveal a vertex of degree of $k + 1$. As a result, all the subgraphs of k -sparse n -graphs which has order smaller than n will be k -sparse too.

Assume that there exists a graph which is k -dense n -graph. That means, we have n vertices all of which misses at most k other vertices. Thus, when we remove any vertex, the remaining vertices will be still missing at most k other remaining vertices. In other words, removal of any vertex does not cause any remaining vertex to miss more than k vertices. As a result, all the subgraphs of k -dense n -graphs which has order smaller than n will be k -dense too. $\square$

There is only one other case such that the graph can be cocolored by 4 colors: combination of 4 complete or empty subgraphs of order 3. After all the graphs that can be partitioned into 4 complete or empty subgraphs of order 3 are eliminated, if we encounter a remaining graph, that means 5 colors are required to cocolor this graph.

Even though the main logic is to combine subgraphs, since the number of graphs containing complete or empty subgraphs are not too many, to increase the number of eliminations, after obtaining the 6-graphs that do not contain complete or empty subgraphs of order 4, we continue by adding only one vertex at a time. We consider all the possible edges between the older vertices and the new vertex. If we encounter a complete or empty 4-subgraph consisting of 3 old vertices and the new vertex, we do not consider the other edge configurations of that graph since it will contain a complete or empty subgraph of order 4 in any generation. We also check whether a generated graph is isomorphic to another graph generated before or not. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs, then they are *isomorphic* if there is an invertible function f from V_G to V_H such that for all vertices u and v in V_G ,

$$(u, v) \in E_G \text{ if and only if } (f(u), f(v)) \in E_H.$$

More informally, G and H are isomorphic to each other if the vertices of one of them can be rearranged, without any changes in the edge set, such that when the labels of vertices are ignored, two graphs are identical. If a new generated graph is isomorphic to a graph generated before, that means, these two graphs are the same since the vertex label is not important. Thus, we also eliminate these isomorphic graphs.

At the end of the computer generations for this part by the algorithm in Figure 6.1, we have seen that there does not exist a 12-graph which has cochromatic number of 5. In order to conclude that $c_0(4) = 12$, we must show that there is at least one 13-graph that cannot be cocolored by 4 colors.

Claim 6.2. $c_0(4) < 13$

Proof. We know that $R(4, 4) = 18$ by Table 2.1. Therefore, there are some 17-graph that neither contain a clique of order 4 nor an independent set of order 4. Thus, it is obvious that there are some 13-graph that neither contain a clique of order 4 nor an independent set of order 4, for example, any subgraphs on 13 vertices of an extremal graph for $R(4, 4)$ of order 17. A graph G is *extremal* for $R_k(a, b)$ if the order of G is

```

for all Graph  $i$  in Set A do
  repeat
    Add a vertex then eliminate the graph if it contains a complete or empty
    subgraph of order 4.
    Eliminate the graph if it is isomorphic to a graph generated before.
  until The order of the graph is equal to 12
  for all The remaining graphs do
    Eliminate the graph if it can be partitioned into 4 complete or empty sub-
    graphs of order 3.
  end for
end for
if There remains a graph then
  The graph is a 12-graph which has cochromatic number of 5.
end if

```

Figure 6.1. Algorithm for $c_0(4)$: Find a 12-graph which has cochromatic number of 5 if it exists.

$R_k(a, b) - 1$ and G contains no k -dense set of order a and no k -sparse set of order b .

Suppose G is a graph of order 13 that does not contain any clique or independent set of order 4. The maximum clique or independent set, that G can contain, is at most of degree 3. Even though we partition G into four subgraphs such that each subgraph is a clique or an independent set of order 3, we can partition 12 vertices into four color classes but one vertex remain unassigned. Thus, there is at least one 13-graph which cannot be cocolored by 4 colors. $\square$

By Claim 6.2, the maximum order n such that all n -graphs can be 0-defectively cocolored by 4 colors is 12, $c_0(4) = 12$. In order to represent a 13-graph that cannot be cocolored by 4 colors, we generate random graphs of order 13 by constructing adjacency matrices of size 13×13 and we put 0 or 1 to the cells of the matrix with equal probability. Then, we eliminate the graphs that contains any combination of cliques and independent sets that will make the graph 4-cocolorable. For example, if the graph even contains one clique or one independent set of order 5, there will remain 8 vertices that can be cocolorable by 3 colors and that means this 13-graph can be cocolored by at most 4 colors. We generate 13-graphs until we find a 13-graph that can be cocolored by at least 5 colors. In Figure 6.2, we can see a 13-graph which cannot be cocolored by using 4 colors. Vertices that are colored by the same color except red, construct a K_3 , but we cannot include the red vertex into any of these subgraphs. It can be checked that similar cases arise for other subgraph combinations.

6.1.2. Computing $c_1(3)$

In this part of the study, we will search a 12-graph G which has $z_1(G) = 4$.

Since $R_1(4, 6) = 11$, every 12-graph contains a 1-dense 4-subgraph or a 1-sparse 6-subgraph. This brings out two cases:

Case 1. If there exists a 1-sparse 6-subgraph, 6 vertices remain and since $c_1(2) = 7$, they

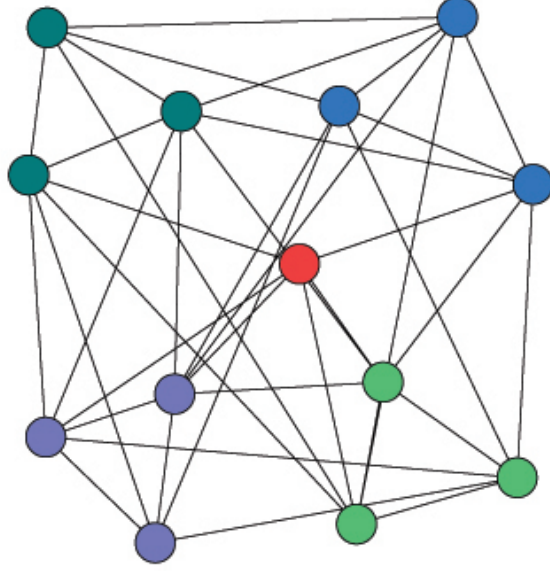


Figure 6.2. Counter example: A 13-graph which has chromatic number of 5.

can be 1-defectively cocolored with at most 2 colors. Then, we will assign a new color to the 1-sparse 6-subgraph and the chromatic number of this graph will be 3.

Case 2. If there is not any 1-sparse 6-subgraph, then there exists a 1-dense 4-subgraph.

It is also known that $R_1(4, 4) = 6$.

Case 2.1. Remaining 8 vertices contain a 1-dense 4-subgraph. 4 vertices remain. Since there are 4-graphs which cannot be 1-defective 1-cocolorable, this case should be investigated.

Case 2.2. Remaining 8 vertices do not contain a 1-dense 4-subgraph but contain a 1-sparse 4-subgraph. 4 vertices remain again and this case should be investigated too.

We will search a 12-graph G , which has $z_1(G) = 4$, by studying upwardly the steps of the above methodology. We will represent the set of 1-dense 4-subgraphs as set A and the set of 1-sparse 4-subgraphs as set B . We have two cases to observe: $A + A + 4$ vertices and $A + B + 4$ vertices. $A + A + 4$ notation shows that we will have two induced subgraphs isomorphic to a graph in A and 4 other vertices.

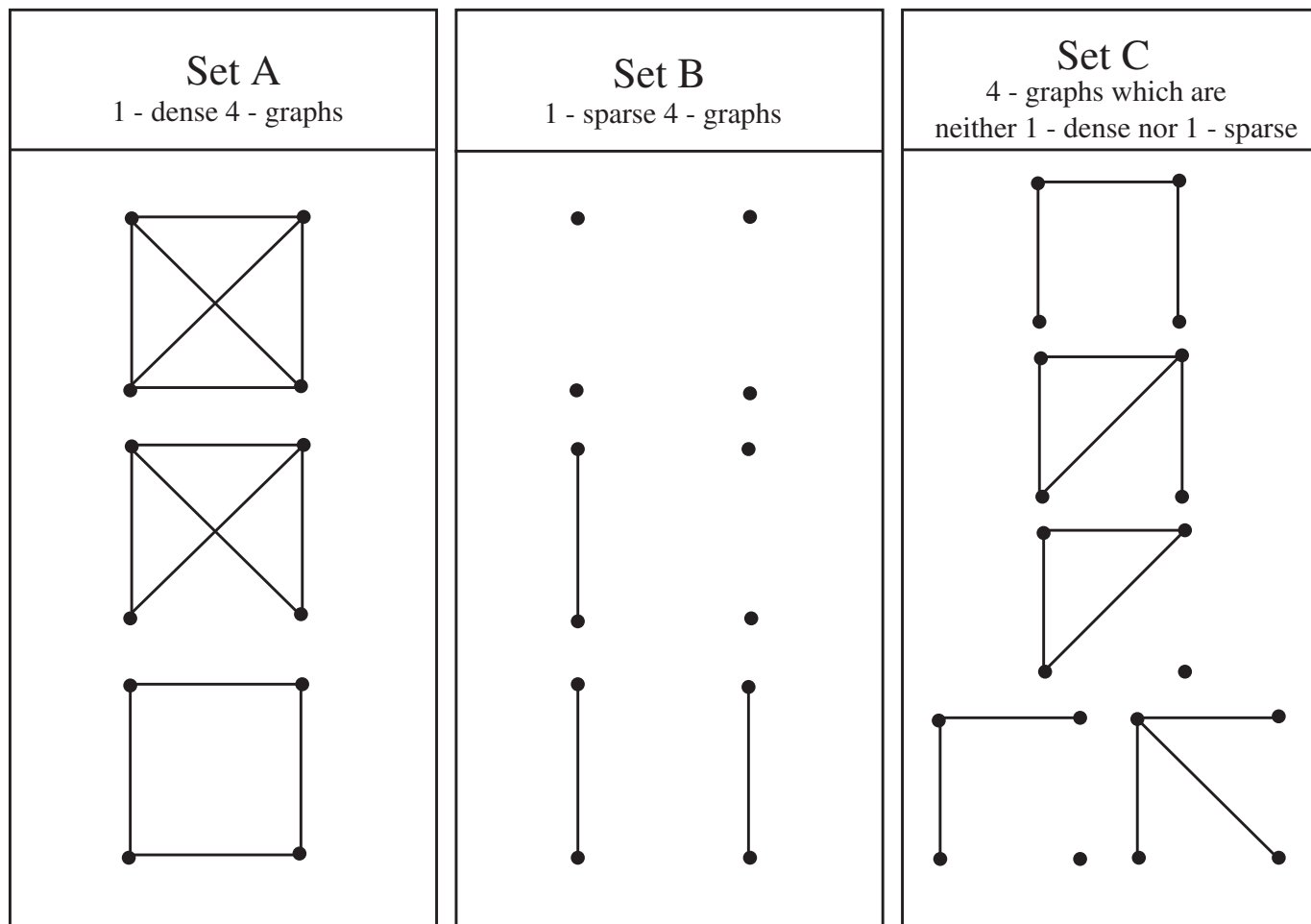


Figure 6.3. The sets A, B and C.

Without changing the induced subgraphs, we consider all the edge combinations between these subgraphs and 4 vertices. For both cases, if the remaining 4 vertices form a 1-defective set, then the graph under consideration is trivially 1-defectively 3-cocolorable. Hereby, we should also define a set C which contains 4-graphs which are not 1-defective. Thus, the two cases become such as: $A + A + C$ and $A + B + C$. The graphs in each set can be seen in Figure 6.3. To make the generations easier we will first generate the combinations of sets A and C which are common for both cases. Since the graphs are generated by considering their adjacency matrices, based on two 4×4 matrices belonging to the graphs of sets A and C, by combining these two graphs, an 8-subgraph and an 8×8 matrix are being constructed. We will combine these 8-subgraphs with sets A and B, which will create 12×12 matrices. Since A and B are 1-defective, when an 8-subgraph, obtained by combining a graph in A and a graph in C, is 1-defective 2-cocolorable, the last graph will become 1-defective 3-cocolorable and the combinations will be eliminated again. Thus, we should forbid 1-defective 2-cocolorable 8-subgraphs when we combine sets A and C. For this restriction, we try to prevent generations of some 1-defective 5-subgraphs; when we have a 1-defective 5-subgraph as an induced subgraph of the 8-subgraph, since all 3-subgraphs are 1-defective, the 8-subgraph will be 1-defective 2-cocolorable. To prevent that, authorized lines are defined for each graph in A and C and it is only authorized to have these lines in the adjacency matrices. Using this method, we eliminate the 8-subgraphs which contain a 1-defective 5-subgraph or which can be partitioned into two 1-defective 4-subgraphs. After these eliminations, we have to generate 12-graphs which contain these 8-subgraphs as induced subgraphs. In order to reduce computational time, instead of adding sets A and B as a whole, we add one vertex at a time. Since it is known that $c_1(2) = 7$, if the last graph contains a 1-defective 5-subgraph as a subgraph, that will result in a 1-defective 3-cocolorable 12-graph. Therefore, we have to eliminate all the graphs that contain a 1-defective 5-subgraph. It is also important to note that when we make the eliminations according to 5-subgraphs, we do not need to check for 1-defective 6-subgraphs or 7-subgraphs by Proposition 6.1. At each generation step, we check every new graph whether it is isomorphic to another graph generated before or not. If the new generated graph is isomorphic to a graph generated before, that

means, these two graphs are the same since the vertex label is not important. Thus, we also eliminate the isomorphic graphs at each generation step.

We have only one other case to eliminate, the combination of three 1-defective 4-subgraphs which will be also 1-defective 3-cocolorable. To be able to eliminate these graphs, we have to wait until we reach the order of 12. In Figure 6.4, the main algorithm can be seen.

At the end of the computer generations for this part by the algorithm in Figure 6.4, we have seen that there does not exist a 12-graph which has a 1-defective cochromatic number of 4. In order to conclude that $c_1(3) = 12$, we must show that there is at least one 13-graph that cannot be 1-defectively cocolored by 3 colors.

Claim 6.3. $c_1(3) < 13$

Proof. We know that $R_1(5, 5) = 15$ by Table 3.1. Therefore, there are some 14-graphs that neither contain a 1-dense 5-set nor a 1-sparse 5-set. Thus, it is obvious that there are some 13-graphs that neither contain a 1-dense 5-set nor a 1-sparse 5-set, for example, any subgraphs on 13 vertices of an extremal graph for $R_1(5, 5)$ of order 14.

Suppose G is a graph of order 13 that does not contain any 1-dense or 1-sparse 5-set. The maximum 1-dense or 1-sparse set order of graph G can be at most 4. Even though we partition G into three subgraphs each of which is a 1-dense or 1-sparse 4-set, we can partition 12 vertices into three color classes but one vertex remain unassigned. Thus, there is at least one 13-graph which cannot be 1-defectively cocolored by 3 colors. $\square$

By Claim 6.3, the maximum order n such that all n -graphs can be 1-defectively cocolored by 3 colors is 12, $c_1(3) = 12$. In order to represent a 13-graph that cannot be 1-defectively cocolored by 3 colors, we again generate random graphs of order 13. We eliminate the graphs that contains any combination of 1-dense sets and 1-sparse sets

```

for all Graph  $i$  in Set A do
  for all Graph  $j$  in Set C do
    Combine graph  $i$  with graph  $j$  without generating a 1-defective 5-subgraph
    consisting of a vertex from graph  $j$  and four vertices from graph  $i$ .
    Eliminate the generated 8-graphs which contains a 1-defective 5-subgraph.
    Eliminate the generated 8-graphs which can be partitioned into two 1-
    defective 4-subgraphs.
    Eliminate the graph if it is isomorphic to a graph generated before.
  for all The remaining graphs do
    repeat
      Add a vertex then eliminate the graph if it contains a 1-defective 5-
      subgraph
    until The order of the graph is equal to 12
    Eliminate the graph if it can be partitioned into three 1-defective 4-
    subgraphs
    Eliminate the graph if it is isomorphic to a graph generated before.
  end for
end for
if There remains a graph then
  The graph is a 12-graph which has 1-defective cochromatic number of 4.
end if

```

Figure 6.4. Algorithm for $c_1(3)$: Find a 12-graph which has 1-defective cochromatic number of 4 if it exists.

that will make the graph 1-defectively 3-cocolorable. We continue generations until we find a 13-graph that can be 1-defectively cocolored by at least 5 colors. In Figure 6.5, we can see a 13-graph which cannot be 1-defectively cocolored by using 3 colors. Vertices that are colored by the same color except red, construct 1-dense 4-sets, but we cannot include the red vertex into any of these subgraphs. It can be checked that similar cases arise for other subgraph combinations.

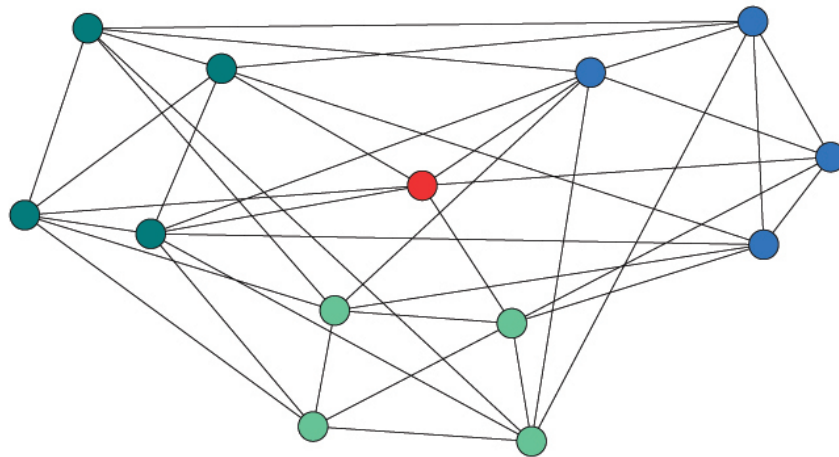


Figure 6.5. Counter example: An 13-graph which has 1-defective cochromatic number of 4.

6.1.3. Computing $c_2(2)$

In this part of the study, we will search a 2-defective 10-graph such as its 2-defective cochromatic number will be 3.

We construct our algorithm based on $R_2(5, 7) = 10$ [9]. That will bring out two cases:

Case 1 If there exists a 2-sparse 7-subgraph, 3 vertices remain. Since all 3-graphs are 2-defective, all the combinations of these subgraphs will be 2-defectively 2-cocolorable. Thus, for this case it is unnecessary to search a 2-defective 10-graph G such that $z_2(G) = 10$.

Case 2 If there exists 2-dense 5-graph, 5 vertices remain. $R_2(4, 5) = 5$ is also known.

- Case 2.1 If there exists a 2-sparse 5-subgraph, no vertex remains. The graph will be obviously 2-defectively 2-cocolorable.
- Case 2.2 If there exists a 2-dense 4-subgraph, a vertex remain. This case should be investigated. For that, we first combine all 2-dense 4-subgraphs with a vertex such that the generated graph should not be 2-defective 5-subgraph (namely a 2-dense 5-subgraph or a 2-sparse 5-subgraph). Because if we combine a 2-dense 5-subgraph with a 2-defective 5-subgraph, all the combinations will be 2-defective 2-cocolorable. Therefore, the only combination that we can look for a 10-graph with 2-defective cochromatic number of 3 is the combinations of a 2-dense 5-subgraph and a 5-subgraph which is neither 2-dense nor 2-sparse.

For this part of the study, we consider all 2-dense 5-graphs. If we combine these subgraphs with a 2-sparse 5-graph or another 2-dense 5-graph, then the generated graphs will be trivially 2-defectively 2-cocolorable. Thus, we should combine all 2-dense 5-graphs with a 5-graph, generated by adding a vertex to a 2-dense 4-graph, which is neither 2-dense nor 2-sparse (which will be the set A). Since $R_2(4, 5) = 5$, the set of all the 5-graphs that are not 2-dense or 2-sparse will be the same with the set of all the 5-graphs generated by adding a vertex to a 2-dense 4-graph. Hereby, we have two sets of 5-graphs. One, represented as set A, holds all the 5-graphs which are neither 2-dense nor 2-sparse. The other one, represented as set B, holds all the 2-dense 5-graphs as shown in the Figure 6.6. All the combinations of these two sets must be searched one by one. But as the first part of the study, to reduce computational time, vertices will be added one by one and at each step some eliminations will be applied. Since all the 3-graphs and 4-graphs are 2-defectively 1-cocolorable, any occurrence of a 2-defective 6-subgraph or 7-subgraph will result a 10-graph which has a 2-defective cochromatic number of 2. In order to decrease the number of graphs generated, we forbid 2-defective 6-subgraphs as much as possible. In order to do that, we forbid some of the connections from a vertex from A to vertices of B which will create a 2-defective 6-subgraph. By this method, we eliminate some of the 2-defective 6-subgraphs before generating the graphs. But there can still be 2-defective 6-subgraphs with two or more vertices from A and the remaining vertices from B, such graphs should be eliminated

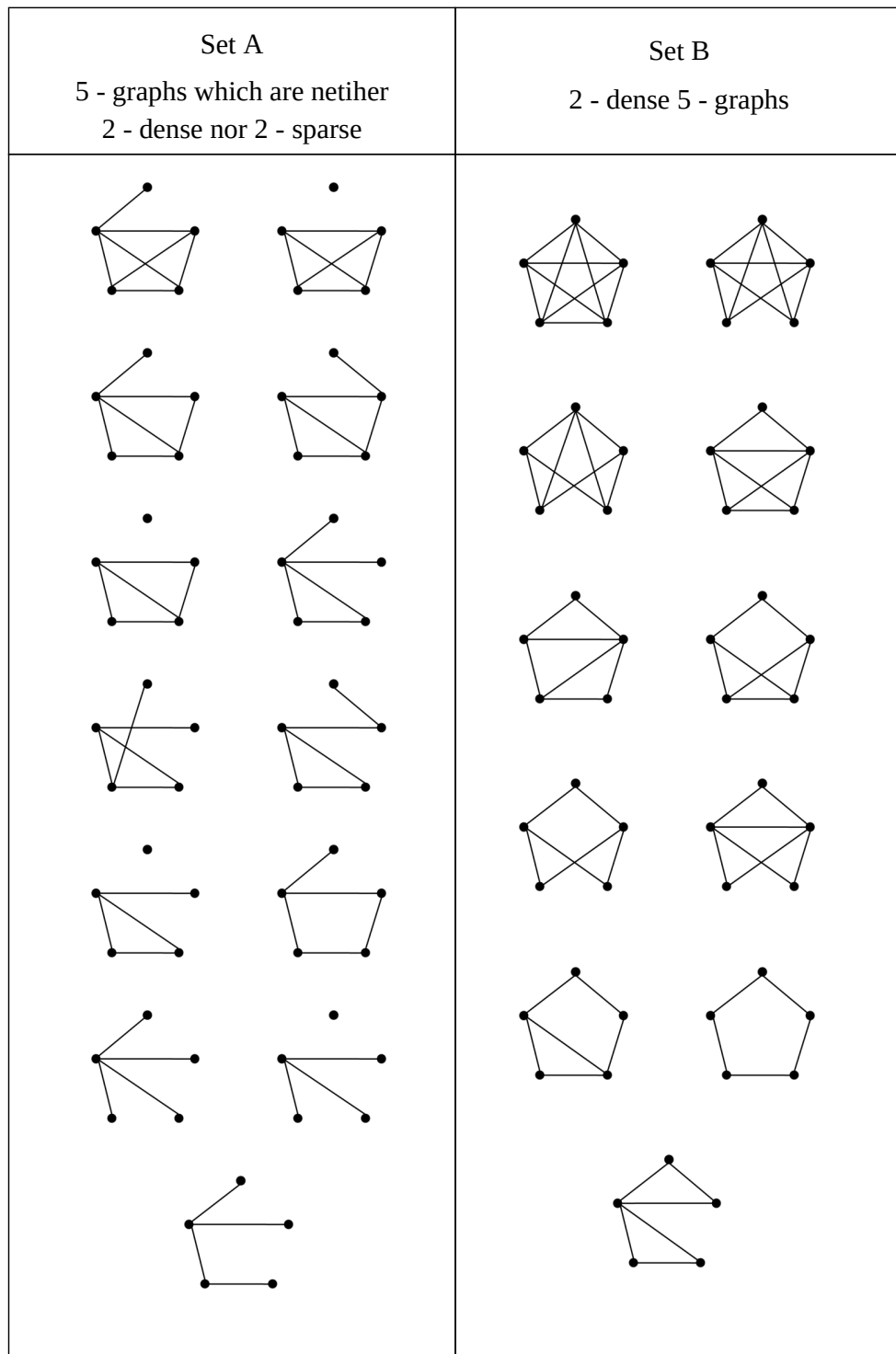


Figure 6.6. The sets A and B.

after generation. Even so, we consider 105148867 graphs instead of 369098752 graphs through the forbidding strategy. At each generation step, we check every new graph whether it is isomorphic to another graph generated before or not. If the new generated graph is isomorphic to a graph generated before, that means, these two graphs are the same since the vertex label is not important. Thus, we also eliminate the isomorphic graphs at each generation step.

For the generated graphs, it is unnecessary to look for a 2-defective 7-subgraphs, because when we eliminate 2-defective 6-subgraphs, we are also handling these cases by Proposition 6.1.

The 10-graphs that are generated do not contain any 2-defective 6-subgraph but another check has to be done. We have to look for a combination of two 2-defectively 5-subgraphs which will yield a 2-defective 2-cocolorable case. Again we also eliminate these cases. After all the processes, if there is any 10-graph left, that will be a 10-graph which has a 2-defective cochromatic number of 3. The algorithm can be found in Figure 6.7.

At the end of the computer generations for this part by the algorithm in Figure 6.7, we have seen that there does not exist a 10-graph which has a 2-defective cochromatic number of 3. In order to conclude that $c_2(2) = 10$, we must show that there is at least one 11-graph that cannot be 2-defectively cocolored by 2 colors.

Claim 6.4. $c_2(2) < 11$

Proof. We know that $R_2(6, 6) = 12$ by Table 3.2. There are exactly two different 11-graphs that neither contain a 2-dense 6-set nor a 2-sparse 6-set by Table 3.8. Suppose G is one of these graphs. The maximum 2-dense or 2-sparse set order of graph G can be at most 5. Even though we partition G into two subgraphs such that each subgraph is a 2-dense or 2-sparse 5-set, we can partition 10 vertices into two color classes but one vertex remain unassigned. Thus, there are at least two 11-graphs which cannot be 2-defectively cocolored by 2 colors. $\square$

```

for all Graph  $i$  in Set B do
  for all Graph  $j$  in Set A do
    Combine graph  $i$  with graph  $j$  without generating a 2-defective 6-subgraph
    consisting of a vertex from graph  $j$  and five vertices from graph  $i$ .
    Eliminate the generated 10-graphs which contains a 2-defective 6-subgraph.
    Eliminate the generated 10-graphs which can be partitioned into two 2-
    defective 5-subgraphs.
    Eliminate the graph if it is isomorphic to a graph generated before.
  end for
end for
if There remains a graph then
  The graph is a 10-graph which has 2-defective cochromatic number of 3.
  Eliminate the graph if it is isomorphic to a graph generated before.
end if

```

Figure 6.7. Algorithm for $c_2(2)$: Find a 10-graph which has 2-defective cochromatic number of 3 if it exists.

By Claim 6.4, the maximum order n such that all n -graphs can be 2-defectively cocolored by 2 colors is 10, $c_2(2) = 10$. In order to represent an 11-graph that cannot be 2-defectively cocolored by 2 colors, we generate random graphs of order 11. We eliminate the graphs that contains any combination of 2-dense sets and 2-sparse sets that will make the graph 2-defectively 2-cocolorable. We continue generations until we find an 11-graph that can be 2-defectively cocolored by at least 3 colors. In Figure 6.8, we can see an 11-graph which cannot be 2-defectively cocolored by using 2 colors. Green colored vertices construct a 2-defective 6-dense and the blue vertices construct a 2-defective 4-sparse. But we cannot include the red vertex into any of these subgraphs. It can be checked that similar cases arise for other subgraph combinations.

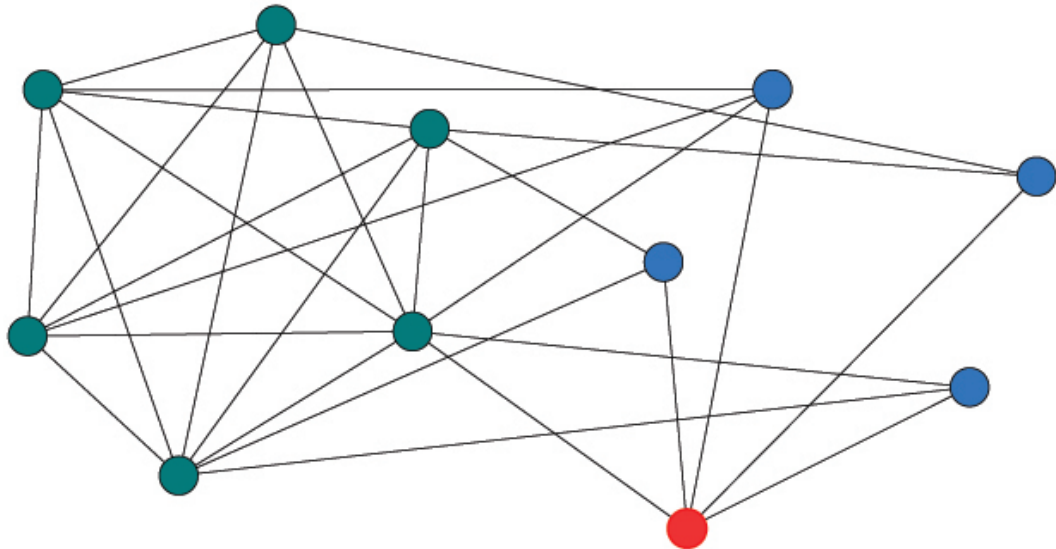


Figure 6.8. Counter example: An 11-graph which has 2-defective cochromatic number of 3.

6.2. Number of Extremal and Critical Graphs

The total number of graphs that disrupts an order was always an interesting problem. To have an idea on this problem, we represent some numbers according to their cochromatic numbers for different graph orders and defectiveness levels. Recall that we consider a graph G as *extremal* for $R_k(a, b)$ if the order of G is $R_k(a, b) - 1$ and G contains no k -dense set of order a and no k -sparse set of order b . Also we consider a graph G as $(m+1)$ -*extremal* for $c_k(m)$ if the order of G is $c_k(m) + 1$ and

if G cannot be k -defectively cocolored by using m colors and requires $m + 1$ colors to k -defectively cocolored. In this section, we represent the number of all the graphs that cannot be cocolored by using only m colors and have order between $c_k(m) + 1$ and 10 to be able to see the change through orders. Between these extremal graphs, we count the number of cocritical graphs. A graph G is $(m+1)$ -cocritical if the induced subgraph G' , obtained by removing any vertex of G , has cochromatic number m whereas G has cochromatic number $m + 1$. In Table 6.1, the number of 2-extremal graphs that can be cocolored by using at least 2 colors, in other words they cannot be cocolored by only one color, can be seen from 0-defectiveness level to 7-defectiveness level. For instance, there are 156 non-isomorphic 6-graphs as it is seen in Table 2.4. Among all these graphs, there are 154 2-extremal graphs that cannot be 0-defectively cocolored by using only one color. In other words, there are only 2 graphs that can be 0-defectively cocolored by using only one color which are K_6 and S_6 . Similarly, from 12346 8-graphs, there are 7284 2-extremal graphs that cannot be 4-defectively cocolored by using only one color. But since all the graphs of order less than or equal to $k + 2$ can be k -defectively cocolored by using only one color by Proposition 3.4, for all defectiveness levels k , the number of 2-extremal graphs are zero for orders that are less than or equal to $k + 2$. In Table 6.2, the number of 2-cocritical graphs are represented again from 0-defectiveness level to 7-defectiveness level. These are the graphs which cannot be cocolored by using only one color, but when we remove any one of the vertices, all the remaining induced subgraphs can be cocolored by using only one color. Since the original graph must be cocolored by using at least 2 colors, we have to search for 2-cocritical graphs among the 2-extremal graphs since the graphs that are not 2-extremal can already be cocolored by using only one color. For example, among 1044 7-graphs, there are 744 2-extremal graphs that cannot be 3-defectively cocolored by using only one color. But among these 744 graphs, there are only 12 2-cocritical graphs that can be 3-defectively cocolored by using only one color when we remove any one of its vertices.

Proposition 6.5. *The number of k -defective $(m+1)$ -extremal graphs of order $c_k(m)+1$ are equal to the number of k -defective $(m+1)$ -cocritical graphs of order $c_k(m) + 1$.*

Proof. Let G be a graph of order $c_k(m) + 1$. If G can be k -defectively cocolored by m

colors, then G is neither a k -defective m -extremal graph nor a k -defective m -cocritical graph directly. But if G cannot be k -defectively cocolored by m colors, then, G is a k -defective $(m + 1)$ -extremal graph. Let G' be an induced subgraph of G obtained by removing any vertex from G . Since the order of G' is $c_k(m)$, G' can be k -defectively cocolored by using m colors whereas G has k -defective cochromatic number $m + 1$. Therefore, G is both a k -defective $(m + 1)$ -extremal graph and a k -defective $(m + 1)$ -cocritical graph. $\square$

As mentioned in Proposition 6.5, there are some cases for which the number of extremal graphs and cocritical graphs are the same. These cases are represented by bold characters in Table 6.1, 6.2 6.3, 6.4 and 6.5. We can see that number of extremal graphs monotonically increases with n for all defectiveness levels. However, the number of critical graphs does not have regular behaviour affected by n .

Table 6.1. Number of 2-Extremal Graphs.

[illegible]

Table 6.2. Number of 2-Cocritical Graphs.

[illegible]

Table 6.3. Number of 3-Extremal Graphs.

3-Extremal Graphs	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
0 - defective	1	50	730	11217	270249	12983496
1 - defective	-	-	-	160	81236	8480804

Table 6.4. Number of 3-Cocritical Graphs.

Cocritical Graphs	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
0 - defective	1	42	2	0	2	0
1 - defective	-	-	-	160	55416	77772

Table 6.5. Number of 4-Extremal and 4-Cocritical Graphs.

0 - defective	$n=9$	$n=10$
4-Extremal	2026	2126488
4-Cocritical	2026	1490072

Another study on cocritical graphs has been made for only 3-cocritical graphs in [106] and the following results have been proposed: A 3-cocritical graph is either an odd cycle or the complement of an odd cycle or one of the 42 graphs on 6 vertices. We have made a similar study to define a common structure in defective cocritical graphs but none has been discovered yet.

6.3. Computational Complexity

All the algorithms in this chapter are written in R programming languages like the previous chapters. In the algorithms represented in Figure 6.1, 6.4 and 6.7 we eliminate the isomorphic graphs. For this elimination, we used the *igraph* package. Except this check, we have written all the generations and eliminations for this study.

The longest computational time for these algorithms were spent for the algorithm in Figure 6.1 which was approximately 6 days. But if we did not make the eliminations and define the termination conditions on the forbidden subgraphs, we would have to check all 165091172592 graphs of order 12. For each one of them, we would make

$$\binom{12}{4} + \binom{12}{3} \binom{9}{3} \binom{6}{3} = 495 + 369600$$

checks and that would take approximately 283253651 days. Similarly, the algorithm in Figure 6.4 and Figure 6.7 took approximately 3 days and 1 day, respectively.

For the extremal and cocritical graphs, the running time of each order is in Table 6.6.

Table 6.6. Running Time for Counting the Extremal and Cocritical Graphs.

Order	Extremal	Cocritical
3	0 sec	0 sec
4	0 sec	0 sec
5	0 sec	0.02 sec
6	0.02 sec	0.04 sec
7	0.12 sec	0.30 sec
8	2.08 sec	4.63 sec
9	65.07 sec	153.87 sec
10	3300 sec	7140 sec

7. GENERALIZATION OF STRAIGHT'S FORMULA

The main objective of this chapter is to generalize the theorem of Straight, which gives bounds for the cochromatic number of graphs, for the defective version.

7.1. Definitions

Recall that the *cochromatic number* of a graph G , denoted by $z(G)$, is the minimum number of subsets into which the vertex set of G can be partitioned such that each subset induces an empty or a complete subgraph of G . If such a partition contains m complete or empty subgraphs, it is called an m -cocoloring of G [11]. This parameter was first studied in [10]. Several bounds for the $z(G)$ parameter were introduced. One of them is

$$z(G) \leq \chi(G),$$

where $\chi(G)$ denotes the chromatic number of G . Recall that the chromatic number of a graph G , $\chi(G)$, is the smallest number of colors needed to color G such that no adjacent vertices take the same color. It can be seen that the graph is partitioned into $\chi(G)$ independent sets in a $\chi(G)$ -coloring. Since a $\chi(G)$ -coloring is a feasible cocoloring, $z(G)$ is less than or equal to $\chi(G)$. It is also useful to observe that a graph G and its complement $\overline{G}$ have the same cochromatic number. Suppose there is a graph G that has an n -cocoloring. It means that the graph G can be partitioned into n induced subsets which are an independent set or a clique. When we consider $\overline{G}$, every independent set will become a clique and similarly every clique will become an independent set which lead us to the result that the graph $\overline{G}$ also has an n -cocoloring.

$C(m)$ is another parameter defined in [11] which is the smallest n such that there exists a graph G of order n having $z(G) = m$. $C(m)$ and $z(G)$ parameters can be

related such as

$$C(m) \leq n < C(m+1) \quad \text{if and only if} \quad \max_{|V(G)|=n} z(G) = m.$$

Recall that we have defined another parameter $c(m)$ as the maximum graph order n such that all the n -graphs has an m -cocoloring. In other words, there is at least one $(n+1)$ -graph which has a $(m+1)$ -cocoloring. $c(m)$ is the parameter which tells the maximum graph order n where m colors will be enough to partition all the n -graphs into induced subgraphs such that all the subgraphs related to the colors will be an independent set or a clique. The $C(m)$ parameter approaches the same information from the other side and it gives us the minimum graph order n which means there is at least one graph of order n that cannot be partitioned into $(m-1)$ induced subsets (independent set or clique) and will require m colors to color each subset by a different color. For example, we can partition all the 4-graphs into 2 induced subsets (independent set or clique). But there is a 5-graph such that 2 colors will not be sufficient. Thus, $c(m)$ tells that the maximum order for which m colors are enough, which is $c(2) = 4$ in that case. On the other hand, $C(m)$ shows the minimum order of a graph that needs at least m colors to cocolor, which is $C(3) = 5$ in that case. If we relate these two parameters, we obtain a transformation as follows:

Remark 7.1. $c(m-1)+1=C(m)$

7.2. Transformation of Straight's Formula

We transform Straight's formula with the $c(m)$ parameter that we have defined and we obtain a similar formula. Then, the proof of the transformed formula will be represented to provide a better understanding of the generalization.

Theorem 7.2. [11] $C(m-1) + t \leq C(m) \leq C(m-1) + m$, where t is the largest positive integer satisfying $C(m-1) + t \geq R(t, t) + 1$.

By Remark 7.1,

$$\begin{aligned} c(m-2) + 1 + t &\leq c(m-1) + 1 \leq c(m-2) + 1 + m \\ c(m-2) + t &\leq c(m-1) \leq c(m-2) + m \\ c(m-1) + t &\leq c(m) \leq c(m-1) + m + 1 \end{aligned}$$

where t is the largest positive integer satisfying $c(m-1) + t \geq R(t, t)$.

Theorem 7.3. $c(m-1) + t \leq c(m) \leq c(m-1) + m + 1$, where t is the largest positive integer satisfying $c(m-1) + t \geq R(t, t)$.

Proof. The proof is composed of two parts: proof of the upper bound and the lower bound.

$$c(m) \leq c(m-1) + m + 1.$$

In order to prove that, we need to show that there is at least one graph of order $c(m-1) + m + 2$ which cannot be cocolored by using at most m colors.

Since $c(m-1)$ is the largest order such that all $c(m-1)$ -graphs can be cocolored using $m-1$ colors, there exists a graph H of order $c(m-1) + 1$ such that $z(H) = m$. Construct a graph G such that it is the edge-disjoint union of H and K_{m+1} . There are only two possible cases which both results in $z(G) = m + 1$. First one is the case where the each vertex of K_{m+1} takes place in different independent sets, thus, takes different colors in the cocoloring of G . In this case, $z(G)$ will require at least $m + 1$ different colors to partition K_{m+1} in different independent sets. The other one is the cocoloring in which we color K_{m+1} (or at least 1 vertex of it) as a clique using one color. Since the remaining of G , H , requires m colors none of which can be combined with the clique used to partition K_{m+1} , $z(G) = z(H) + 1 = m + 1$. Since G has order $c(m-1) + 1 + m + 1$, we can construct a graph of order $c(m-1) + 1 + m + 1$ which will require $m+1$ colors. Therefore, the maximum graph order that m colors will be enough,

$c(m)$ is smaller than or equal to $c(m-1) + 1 + m + 1$; we have $c(m) \leq c(m-1) + m + 1$.

$$c(m-1) + t \leq c(m)$$

where t is the largest positive integer satisfying $c(m-1) + t \geq R(t, t)$.

This time, let us consider a graph G of order $c(m-1) + t$. Since we want to show that $c(m-1) + t \leq c(m)$, it is enough to show that $z(G) \leq m$. Since the order of G is $c(m-1) + t$ and $c(m-1) + t \geq R(t, t)$, G contains K_t or $\overline{K}_t$ as an induced subgraph. When this induced subgraph (K_t or $\overline{K}_t$) is removed, we obtain a subgraph H of order $c(m-1)$ which can be cocolored by at most $m-1$ colors by definition of $c(m-1)$. Since K_t or $\overline{K}_t$ can be cocolored by one color, $z(G) \leq m$. We know that a graph of order $c(m-1) + t$ requires m or less colors, thus, the maximum graph order such that m colors are enough to cocolored is greater than or equal to $c(m-1) + t$, $c(m-1) + t \leq c(m)$, where t is the largest positive integer satisfying $c(m-1) + t \geq R(t, t)$. $\square$

7.2.1. Generalization

The main objective of this section is to generalize Straight's formula to defective version.

Theorem 7.4. $c_k(m-1) + t \leq c_k(m) \leq c_k(m-1) + (m+1)(k+1)$, where t is the largest positive integer satisfying $c_k(m-1) + t \geq R_k(t, t)$.

Proof. The proof is composed by two parts: proof of the upper bound and the lower bound.

$$c_k(m) \leq c_k(m-1) + (m+1)(k+1).$$

In order to prove that, we need to show that there is at least one graph of order $c_k(m-1) + (m+1)(k+1) + 1$ which cannot be k -defectively cocolored by using at most

m colors. Since $c_k(m-1)$ is the largest order such that all $c_k(m-1)$ -graphs can be k -defectively cocolored using $m-1$ colors, there exists a graph H of order $c_k(m-1)+1$ such that $z_k(H) = m$. Construct a graph G such that it is the edge-disjoint union of H and $K_{(m+1)(k+1)}$. There are only two possible cases which both results in $z_k(G) = m+1$. First one is the case where each induced subgraph K_{k+1} of $K_{(m+1)(k+1)}$ takes place in a different k -sparse set, thus, takes different colors in k -defective cocoloring of G . Note that since the vertices in K_{k+1} have degree k , they can take place in a k -sparse set, possibly with some vertices in H but we cannot place a K_{k+2} in a k -sparse set. In that case, $z_k(G)$ will require at least $m+1$ different colors to partition $K_{(m+1)(k+1)}$ into different k -sparse sets.

The other case is the k -defective cocoloring in which we color $K_{(m+1)(k+1)}$ (or at least 1 vertex of it) as a k -dense set using one color. Consider an induced set S of $K_{(m+1)(k+1)}$. If the order of S is smaller than or equal to $k+1$, since the maximum degree in S is k , S and some vertices of H may form a k -sparse set. But in that case, we need to partition $K_{(m+1)(k+1)}$ into at least $m+1$ sets, thus, the k -defective cocoloring of G will require at least m colors. Therefore, the order of S must be larger than $k+1$. Even though the order of S becomes $k+2$, we cannot combine this set with any vertex v of H . Because the vertices of S have degree $k+1$, therefore we cannot construct a k -sparse set. Also v will miss all $k+2$ vertices of S since H and $K_{(m+1)(k+1)}$ are edge-disjoint sets and this time, we cannot construct a k -dense set. As a result, we cannot combine any induced sets of order larger than or equal to $k+2$ with some vertices of H . In these both partitions, we have to use at least $m+1$ colors to k -defectively cocolor G .

We constructed the graph G of order $c_k(m-1)+1+(m+1)(k+1)$ such that it cannot be k -defectively cocolored by using at most m colors, $z_k(G) = m+1$. Therefore, the maximum graph order that m colors will be enough, $c_k(m)$ is smaller than $c_k(m-1)+1+(m+1)(k+1)$; we have $c_k(m) \leq c_k(m-1)+(m+1)(k+1)$.

$$c_k(m-1) + t \leq c_k(m)$$

where t is the largest positive integer satisfying $c_k(m-1) + t \geq R_k(t, t)$.

This time, let us consider a graph G of order $c_k(m-1) + t$. Since we want to show that $c_k(m-1) + t \leq c_k(m)$, it is enough to show that $z_k(G) \leq m$. Since the order of G is $c_k(m-1) + t$ and $c_k(m-1) + t \geq R_k(t, t)$, G contains a k -dense t -set or a k -sparse t -set as an induced subgraph. When this induced subgraph (a k -dense or a k -sparse t -set) is removed, we obtain a subgraph H of order $c_k(m-1)$ which can be k -defectively cocolored by $m-1$ colors by definition of $c_k(m-1)$. Since a k -dense or a k -sparse t -set can be k -defectively cocolored by one color, $z_k(G) \leq m$. We know that a graph of order $c_k(m-1) + t$ requires m or less colors, thus, the maximum graph order such that m colors are enough to k -defectively cocolored is greater than or equal to $c_k(m-1) + t$; $c_k(m-1) + t \leq c_k(m)$, where t is the largest positive integer satisfying $c_k(m-1) + t \geq R_k(t, t)$.

□

8. CONCLUSION

In this study, classical Ramsey numbers, defective Ramsey numbers, cocoloring and defective cocoloring are considered. In the first part of the study, we focused on defective Ramsey numbers. There are still many defective Ramsey numbers which are not known exactly. For these numbers, we have calculated some lower and upper bounds and improved some of the lower bounds. As a future work, we aim to improve these bounds and find some exact values.

In the second part of the study, classical and defective Ramsey numbers have been studied for restricted graph classes. Another proof of Erdős - Szekeres theorem is proposed for only P_4 -free graphs and also we have proven that Erdős - Szekeres theorem also holds for perfect graphs as an upper bound for Ramsey numbers. Also, a defective formula is represented as an upper bound for Ramsey numbers in permutation graphs. We also constructed another proof of Erdős - Hajnal conjecture for permutation graphs using Erdős - Szekeres theorem.

In the next part of this study is the classical and defective cocoloring. The exact values of $c_0(4)$, $c_1(3)$ and $c_2(2)$ were left open in the literature. We have represented search algorithms that gives us $c_0(4) \geq 12$, $c_1(3) \geq 12$ and $c_2(2) \geq 11$. We also proved and presented counter examples on the upper bounds and obtain the following exact values for these open questions: $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 11$. Constructing theoretical proofs on these values is aimed as a future study. These search algorithms are also used to evaluate the number of extremal and cocritical graphs for various defectiveness levels.

In the last part of the study, we have proved a formula that gives the upper bounds and lower bounds of defective cochromatic numbers through defective Ramsey numbers, using the formula, that gives the upper bounds and lower bounds of the cochromatic numbers through Ramsey numbers, proposed by Straight.

APPENDIX A: 1 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_1(5, 6) \geq 18$

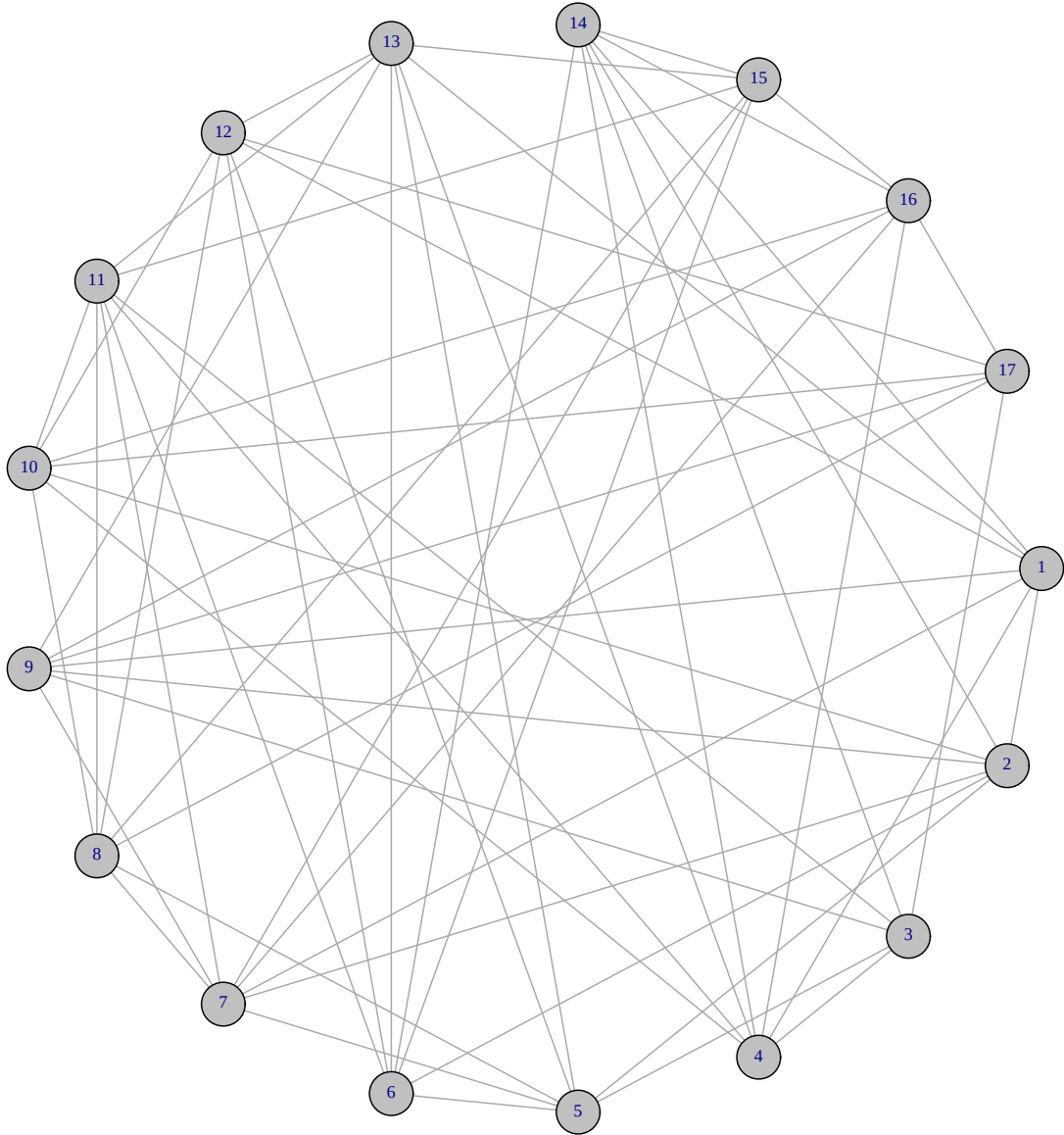


Figure A.1. Counter example: A 17-graph which neither contains 1-dense 5-set nor 1-sparse 6-set.

Counter example: $R_1(5, 7) \geq 21$

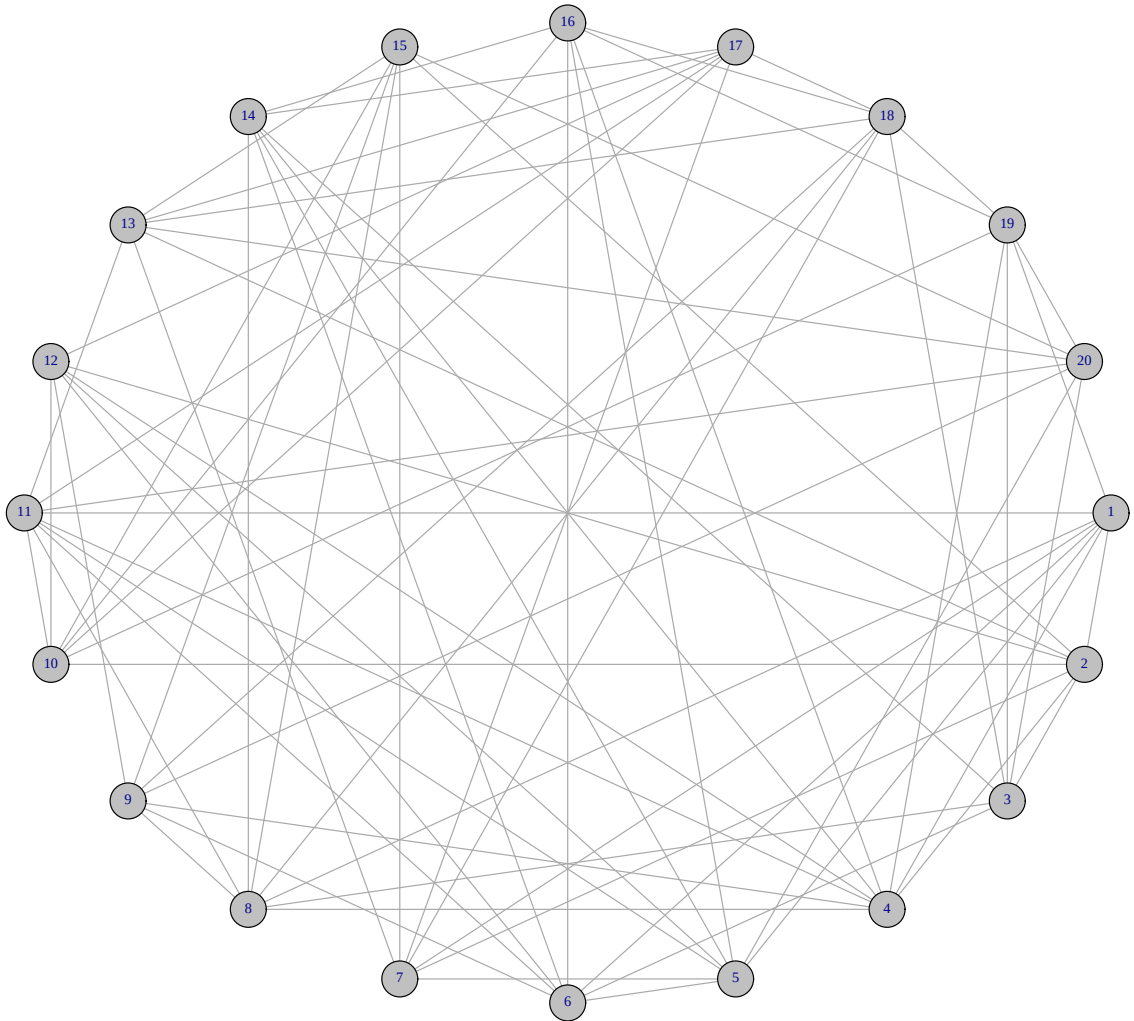


Figure A.2. Counter example: A 20-graph which neither contains 1-dense 5-set nor 1-sparse 7-set.

Counter example: $R_1(5, 8) \geq 25$

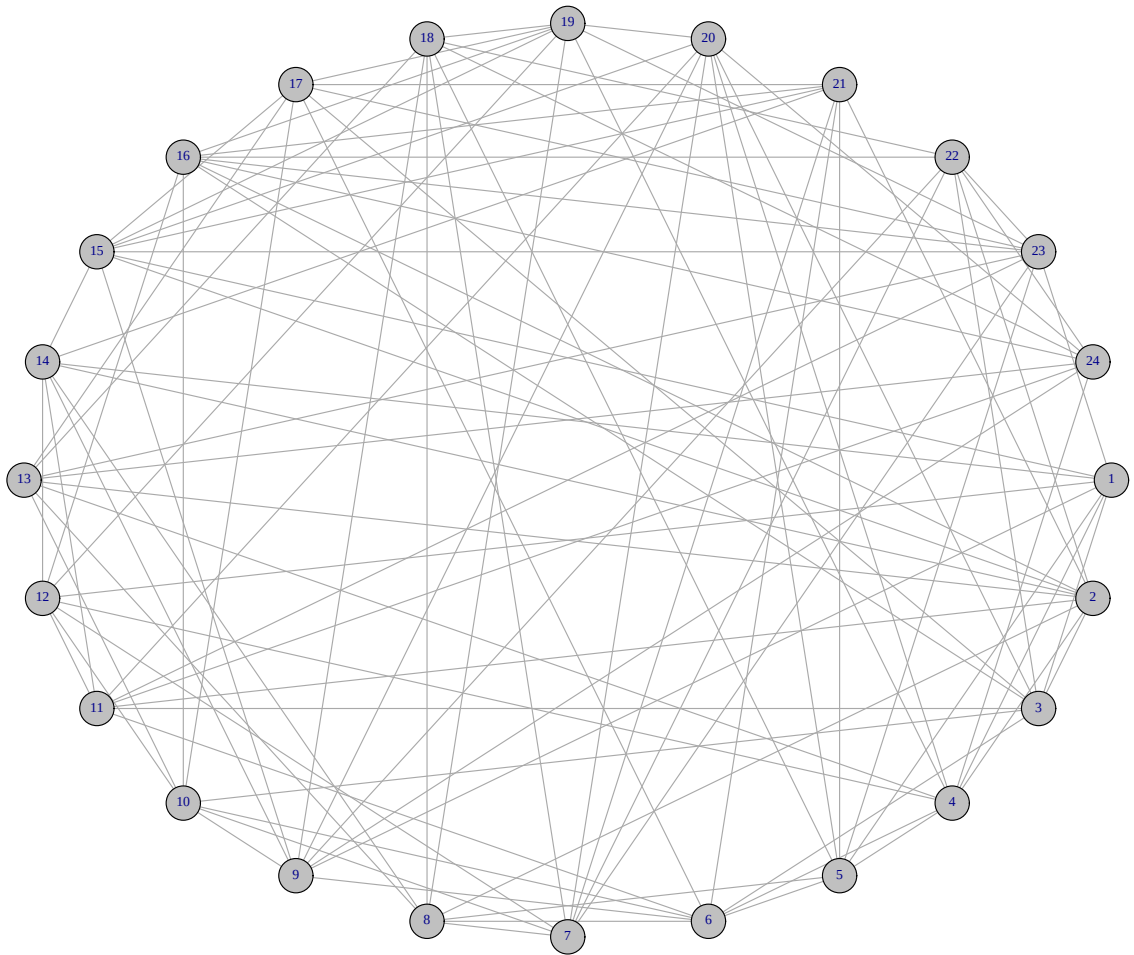


Figure A.3. Counter example: A 24-graph which neither contains 1-dense 5-set nor 1-sparse 8-set.

Counter example: $R_1(5, 10) \geq 31$

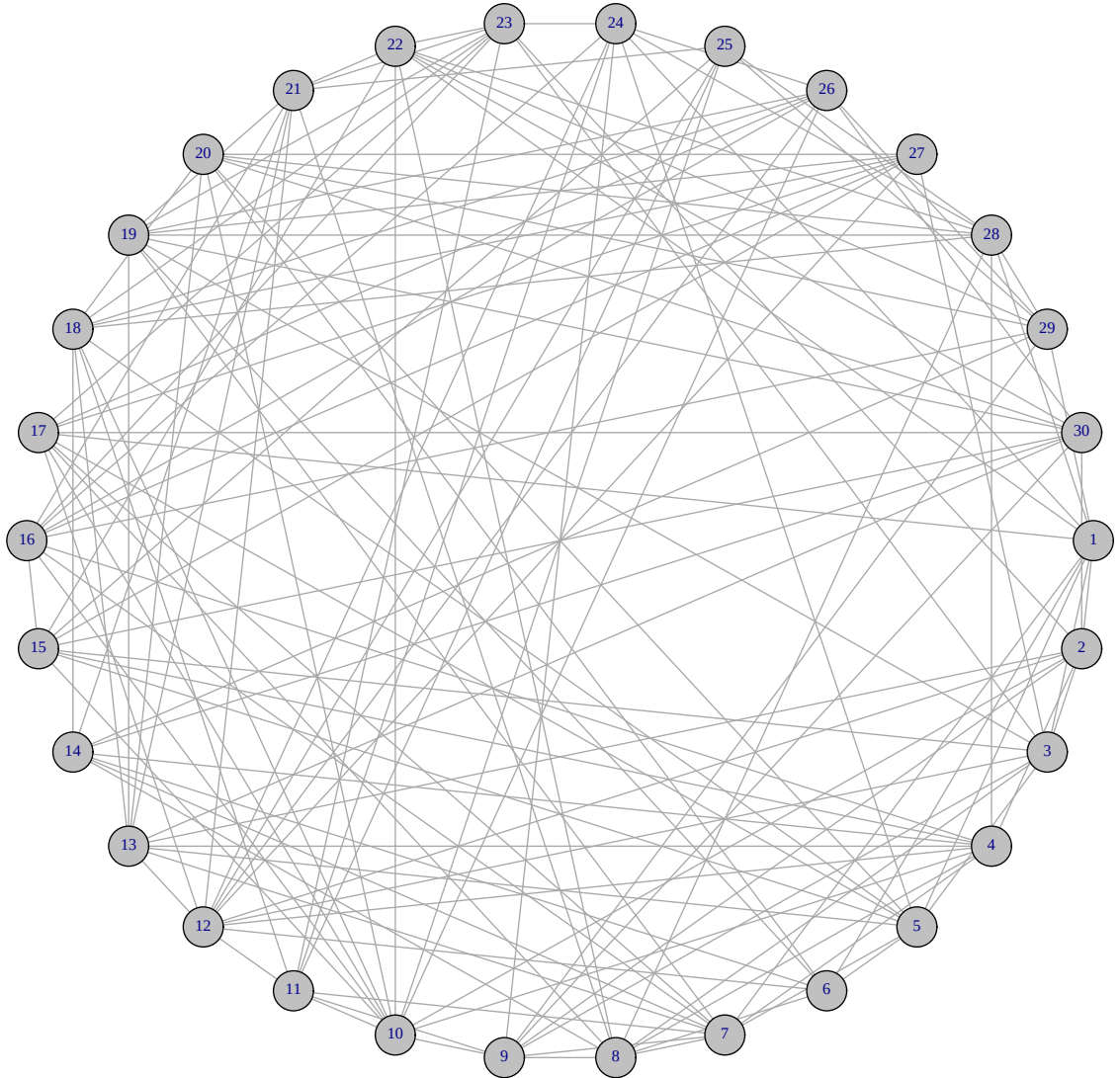


Figure A.4. Counter example: A 30-graph which neither contains 1-dense 5-set nor 1-sparse 10-set.

Counter example: $R_1(6,6) \geq 21$

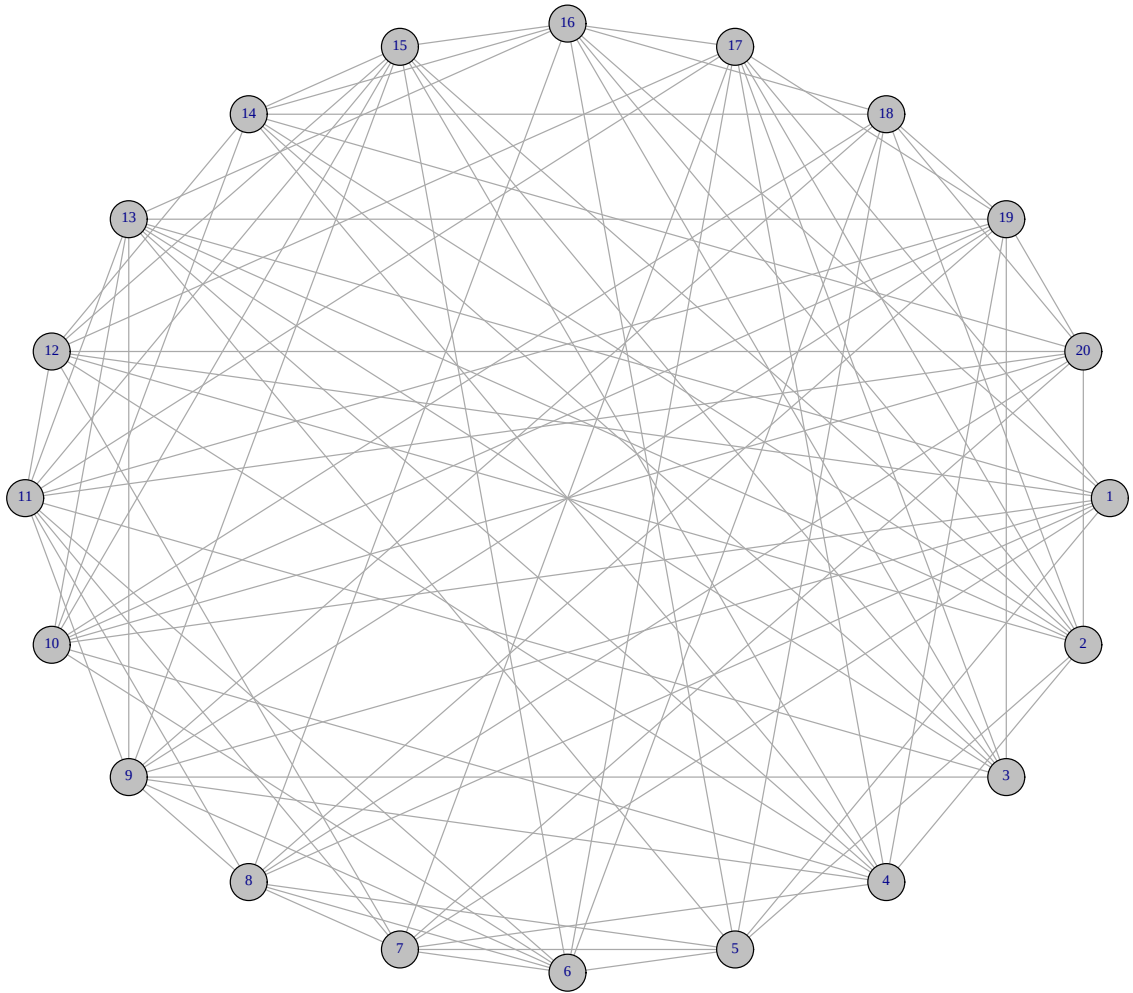


Figure A.5. Counter example: A 20-graph which neither contains 1-dense 6-set nor 1-sparse 6-set.

Counter example: $R_1(6, 7) \geq 26$

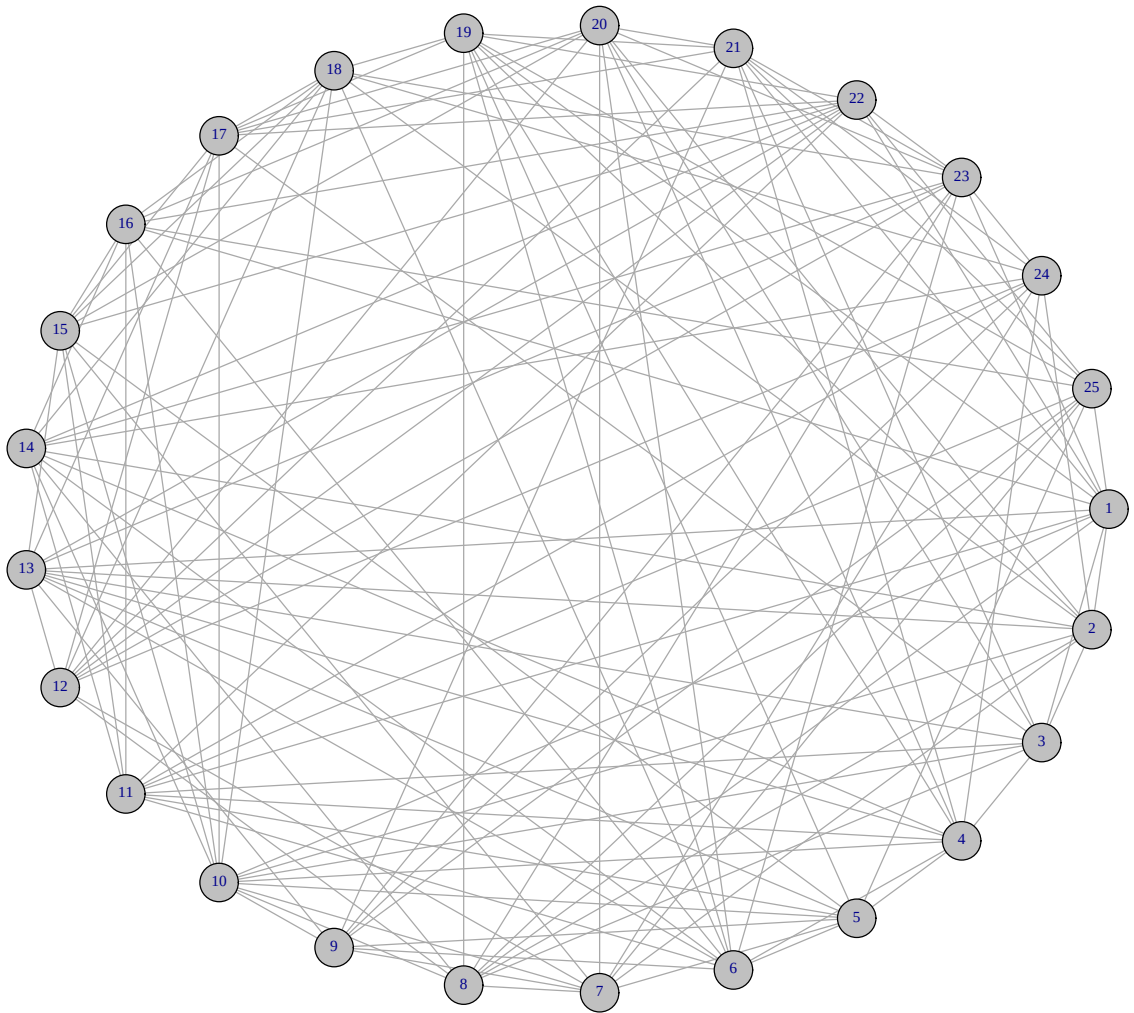


Figure A.6. Counter example: A 25-graph which neither contains 1-dense 6-set nor 1-sparse 7-set.

Counter example: $R_1(6, 8) \geq 30$

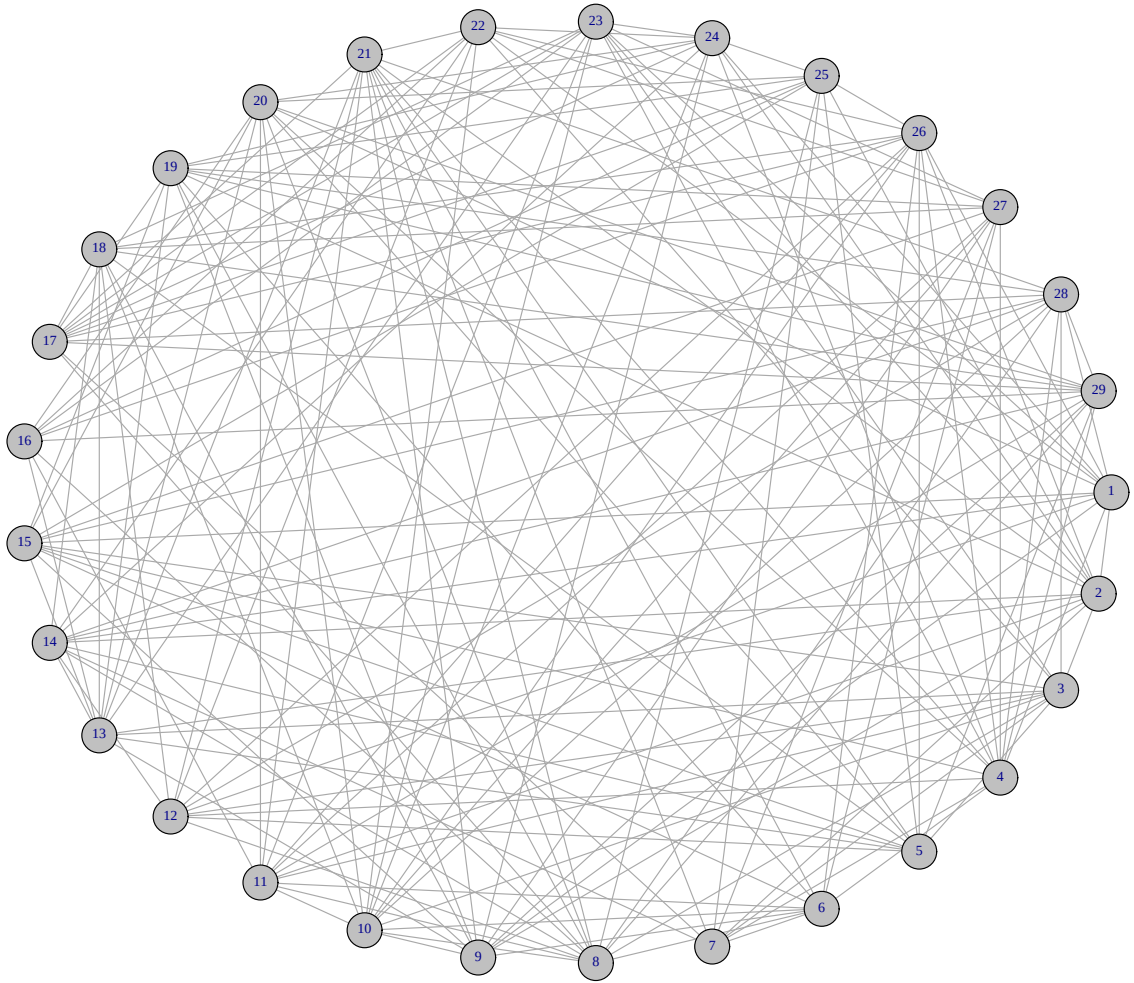


Figure A.7. Counter example: A 29-graph which neither contains 1-dense 6-set nor 1-sparse 8-set.

Counter example: $R_1(6, 9) \geq 34$

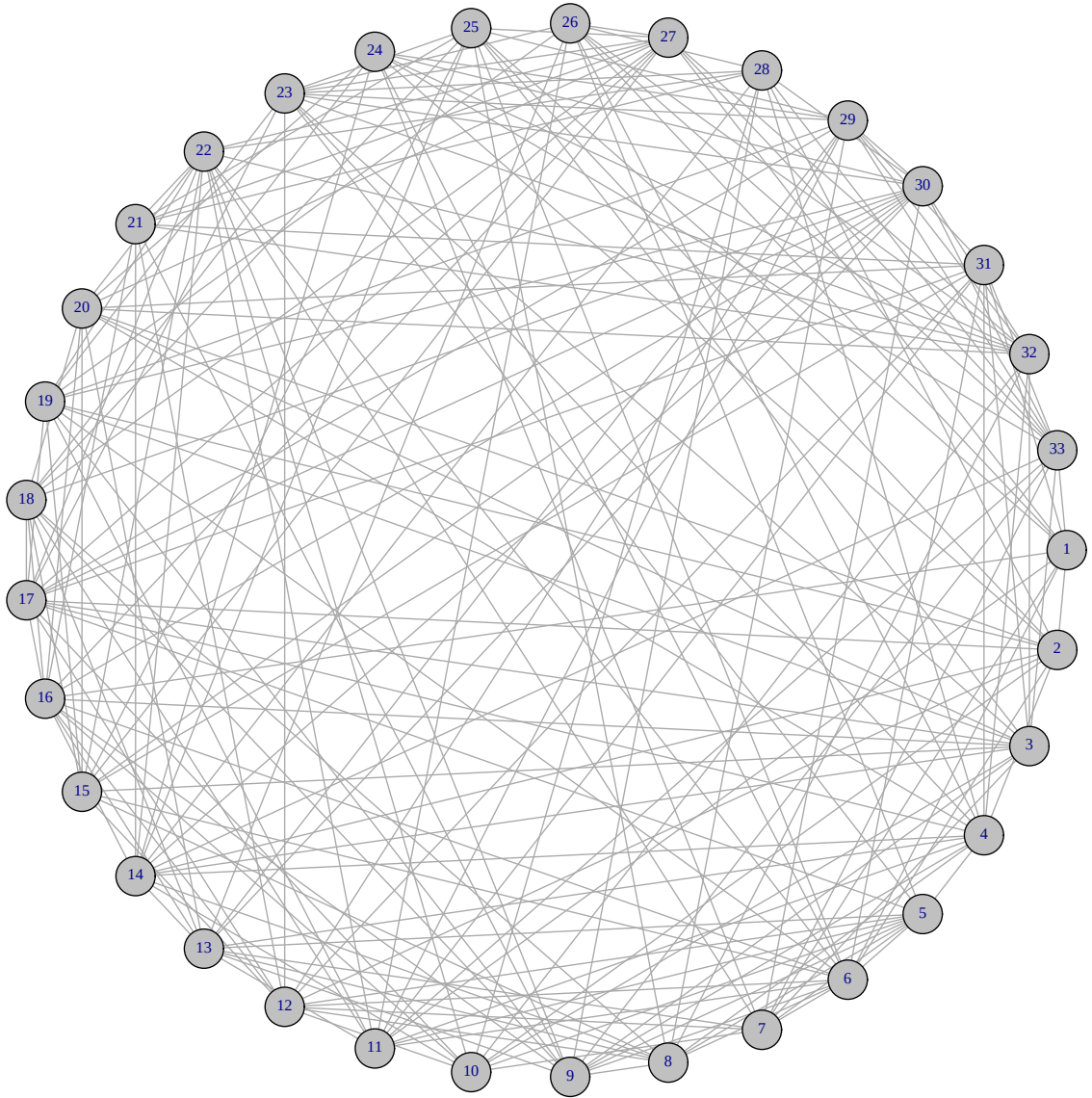


Figure A.8. Counter example: A 33-graph which neither contains 1-dense 6-set nor 1-sparse 9-set.

Counter example: $R_1(6, 10) \geq 34$, same with the counter example of $R_1(6, 9)$.

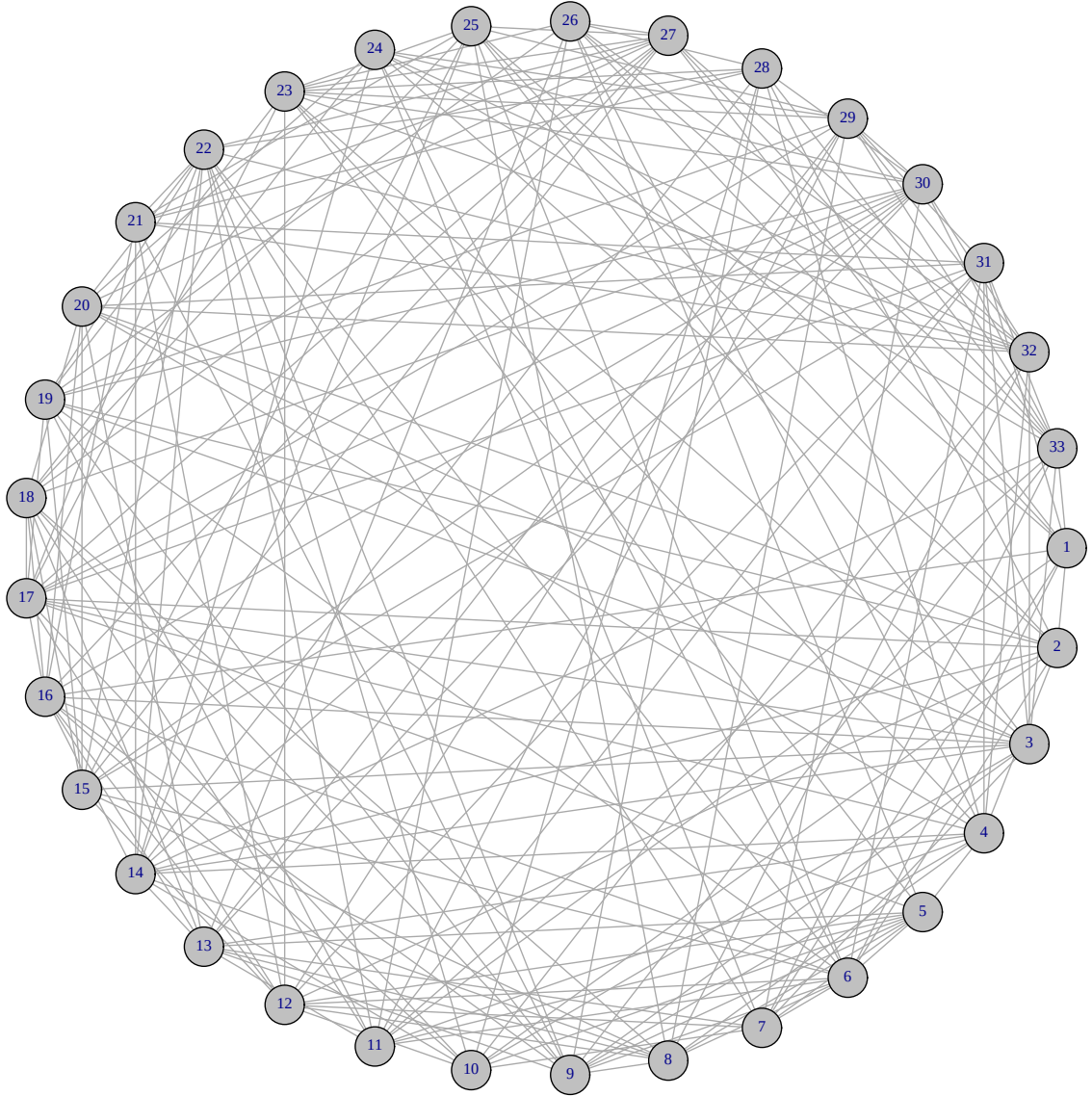


Figure A.9. Counter example: A 33-graph which neither contains 1-dense 6-set nor 1-sparse 10-set.

Counter example: $R_1(8, 8) \geq 40$

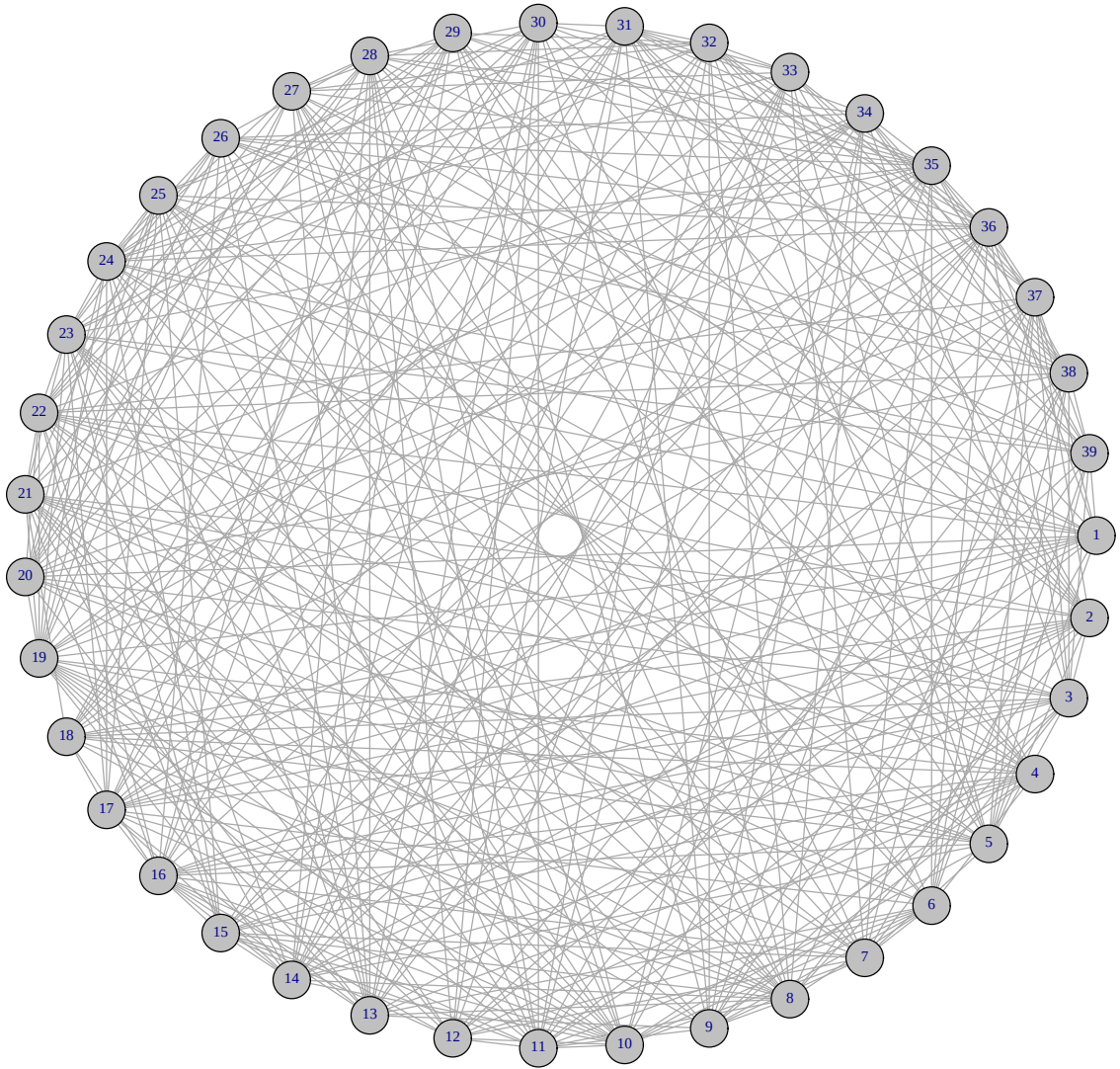


Figure A.10. Counter example: A 39-graph which neither contains 1-dense 8-set nor 1-sparse 8-set.

Counter example: $R_1(8, 9) \geq 40$, same with the counter example of $R_1(8, 8)$.

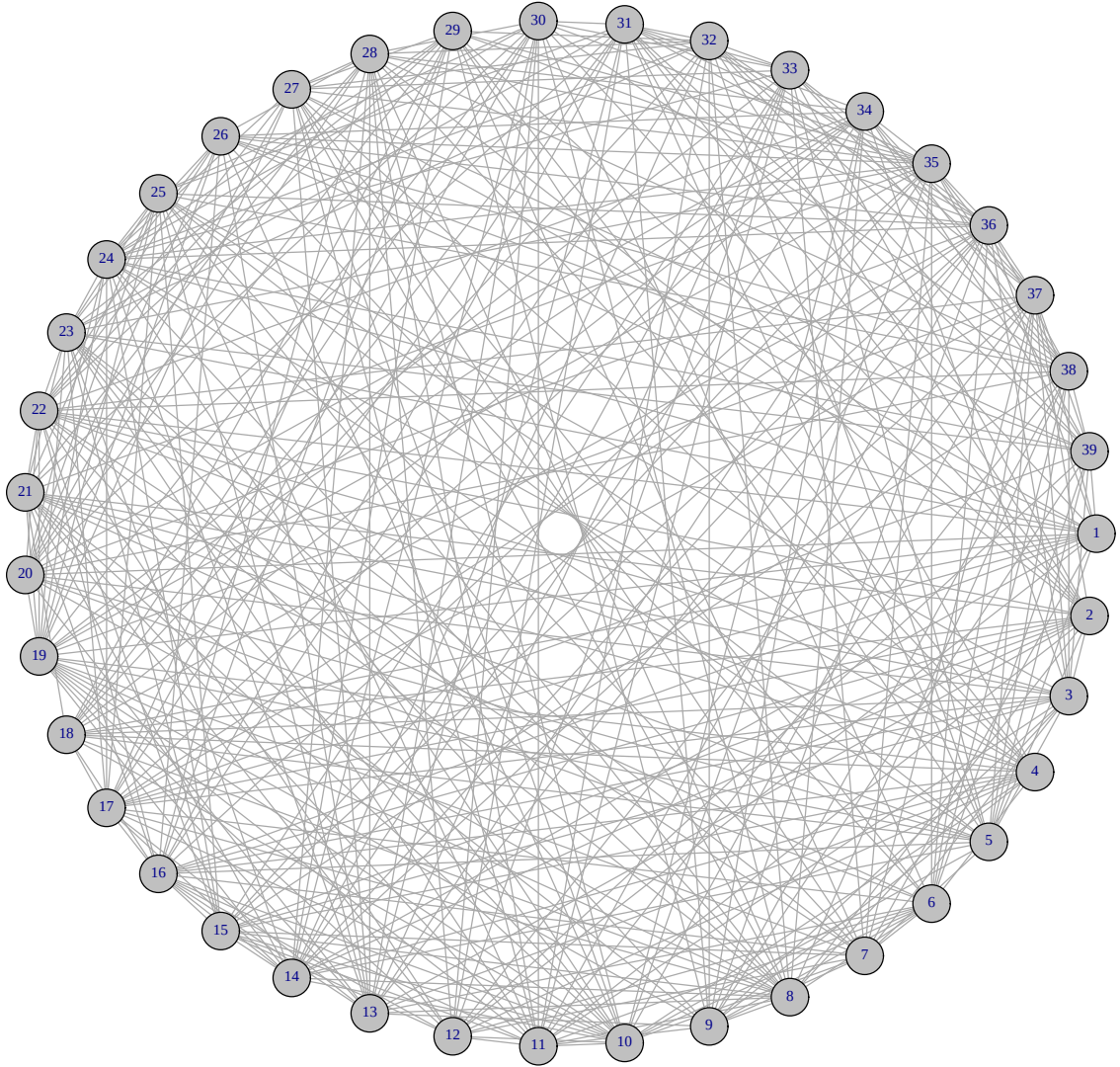


Figure A.11. Counter example: A 39-graph which neither contains 1-dense 8-set nor 1-sparse 9-set.

Counter example: $R_1(8, 10) \geq 40$, same with the counter example of $R_1(8, 8)$.

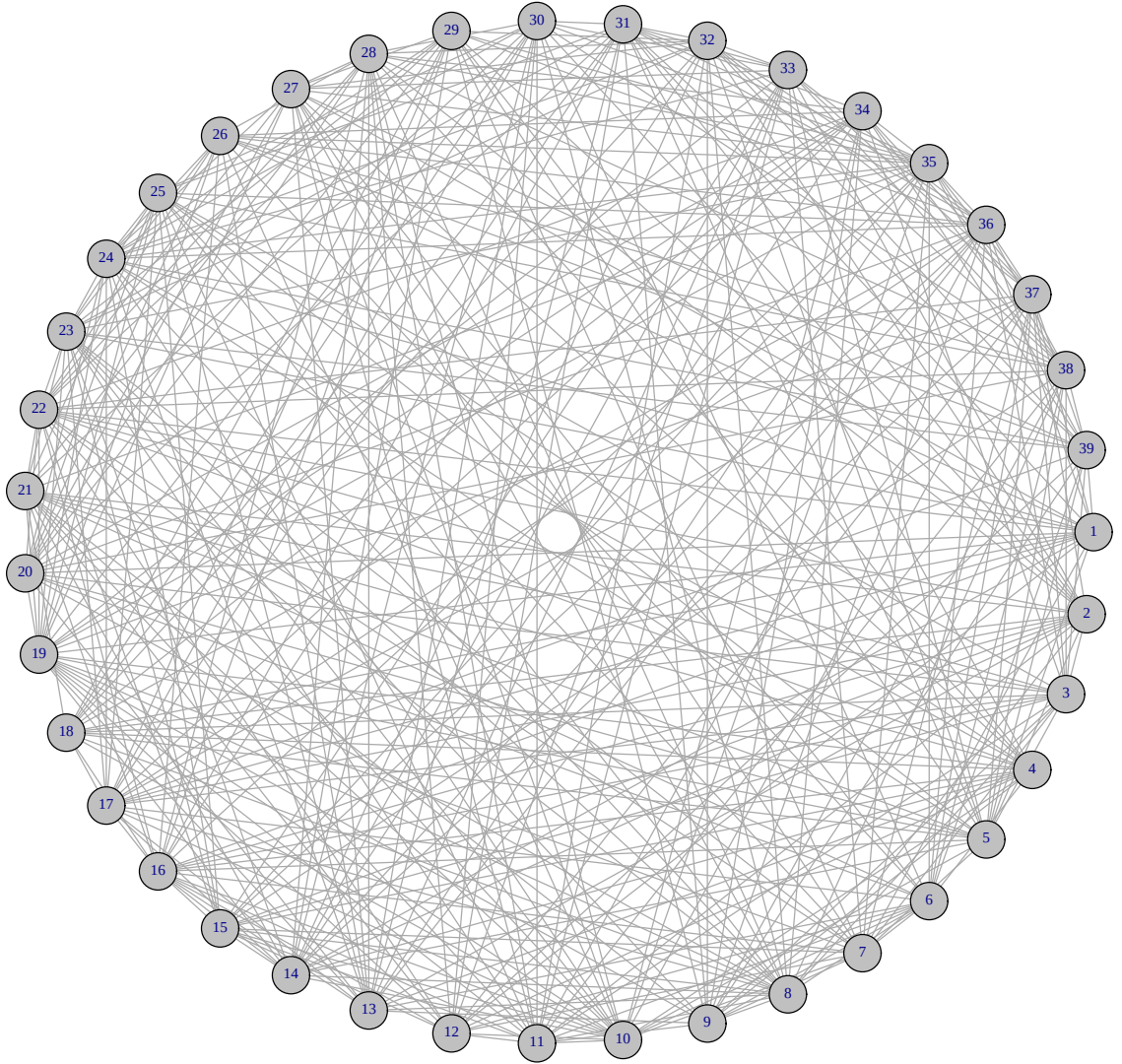


Figure A.12. Counter example: A 39-graph which neither contains 1-dense 8-set nor 1-sparse 10-set.

Counter example: $R_1(9, 9) \geq 40$, same with the counter example of $R_1(8, 8)$.

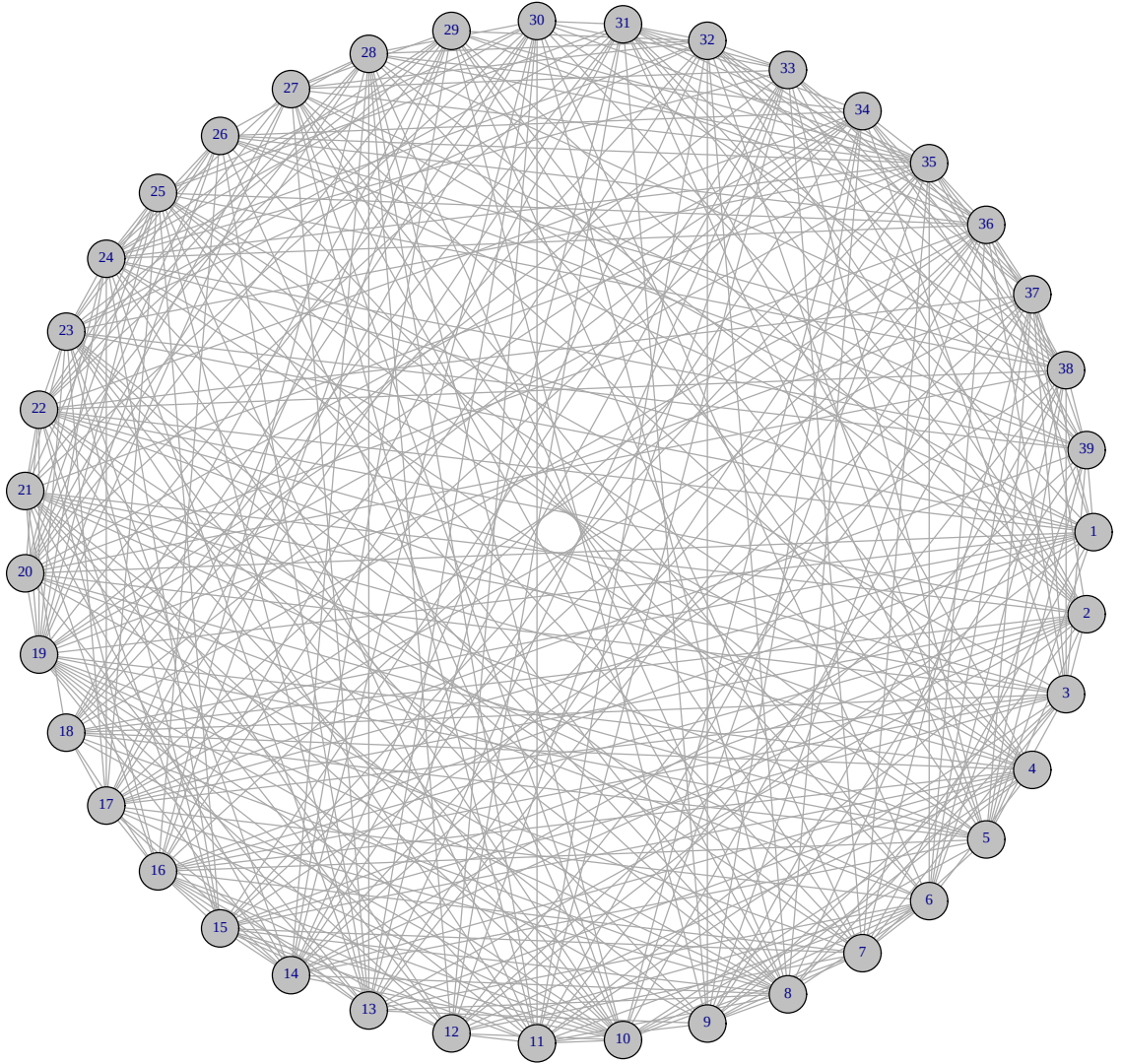


Figure A.13. Counter example: A 39-graph which neither contains 1-dense 9-set nor 1-sparse 9-set.

Counter example: $R_1(9, 10) \geq 40$, same with the counter example of $R_1(8, 8)$.

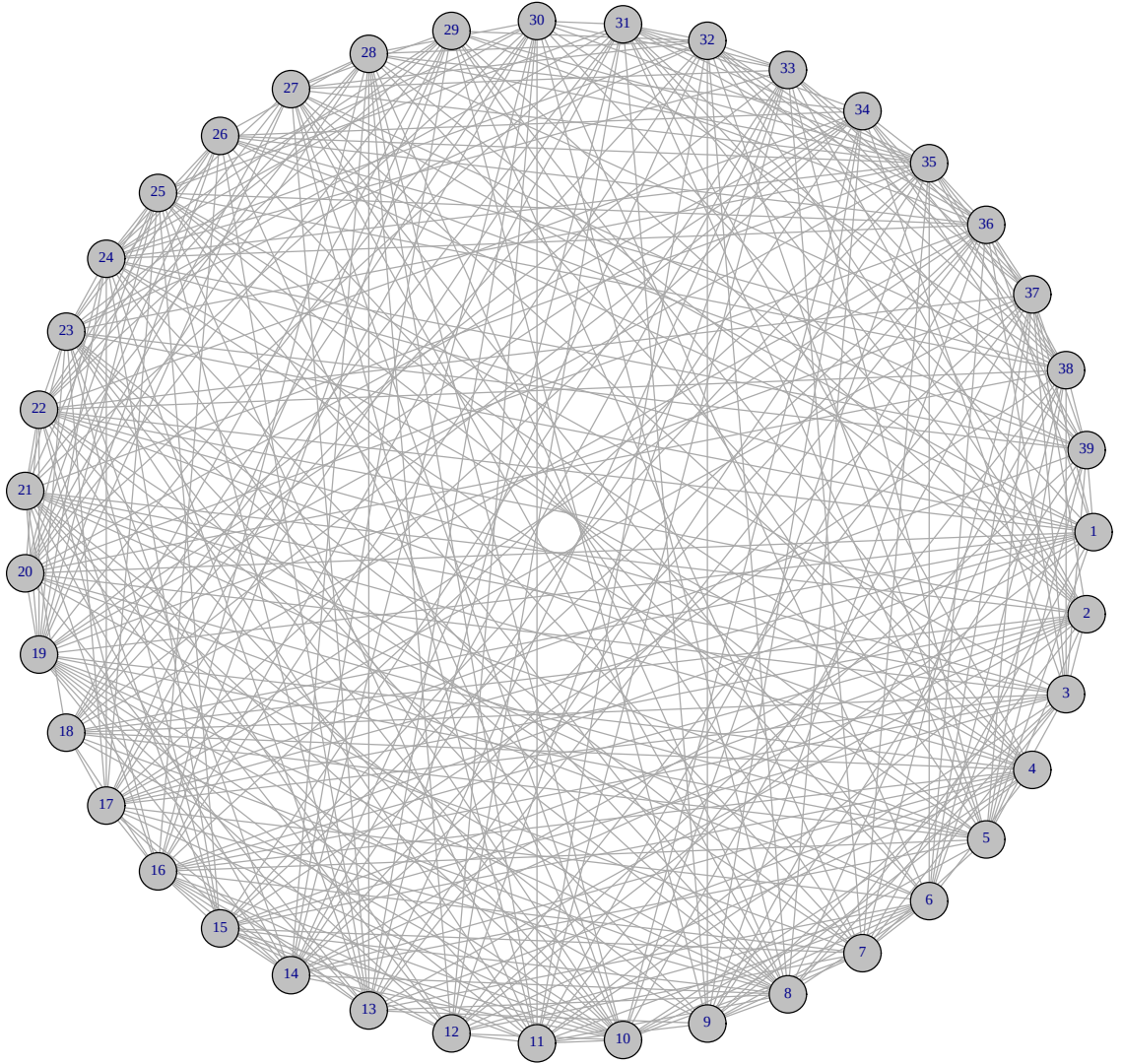


Figure A.14. Counter example: A 39-graph which neither contains 1-dense 9-set nor 1-sparse 10-set.

Counter example: $R_1(10, 10) \geq 40$, same with the counter example of $R_1(8, 8)$.

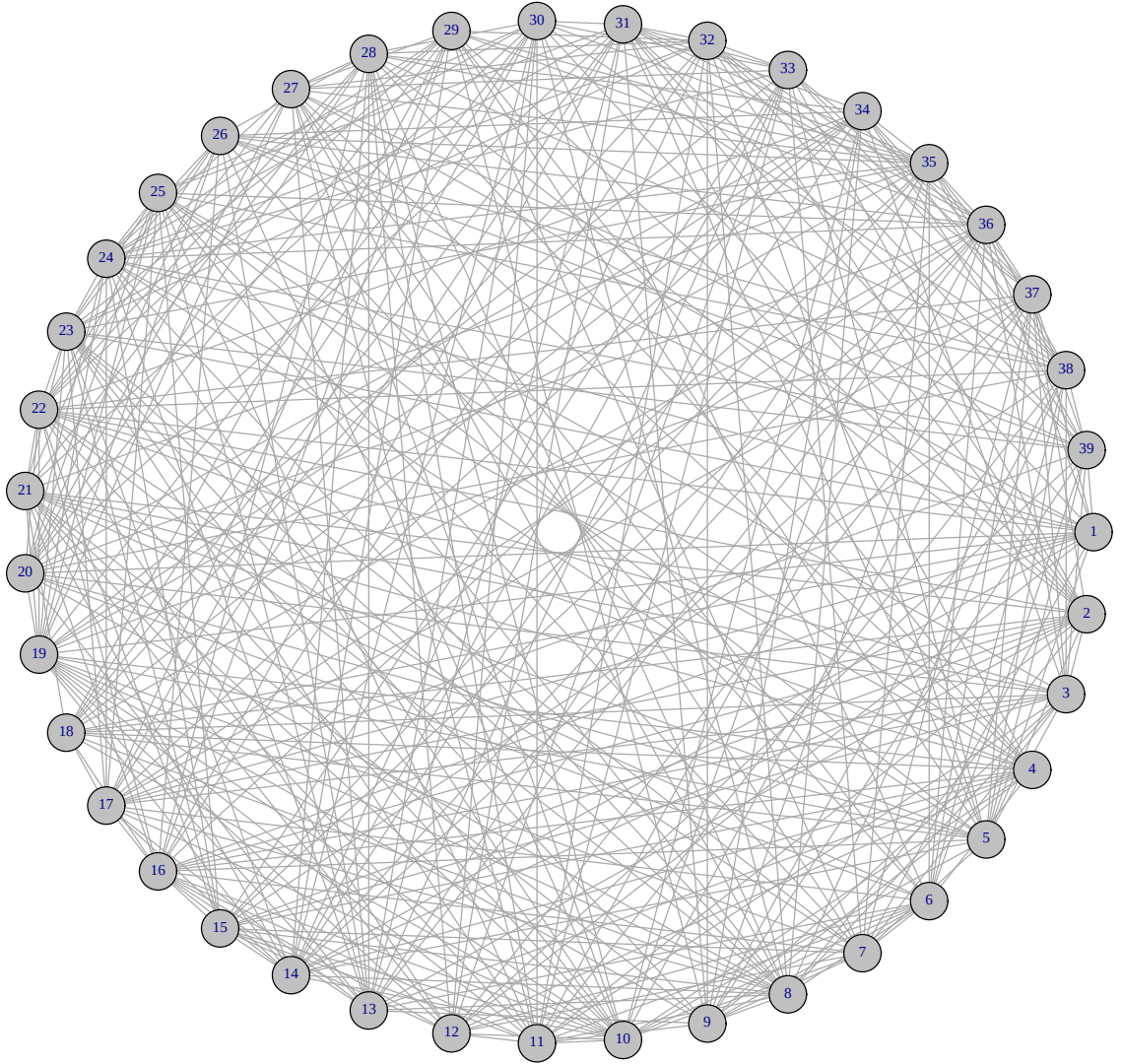


Figure A.15. Counter example: A 39-graph which neither contains 1-dense 10-set nor 1-sparse 10-set.

APPENDIX B: 2 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_2(6, 8) \geq 16$

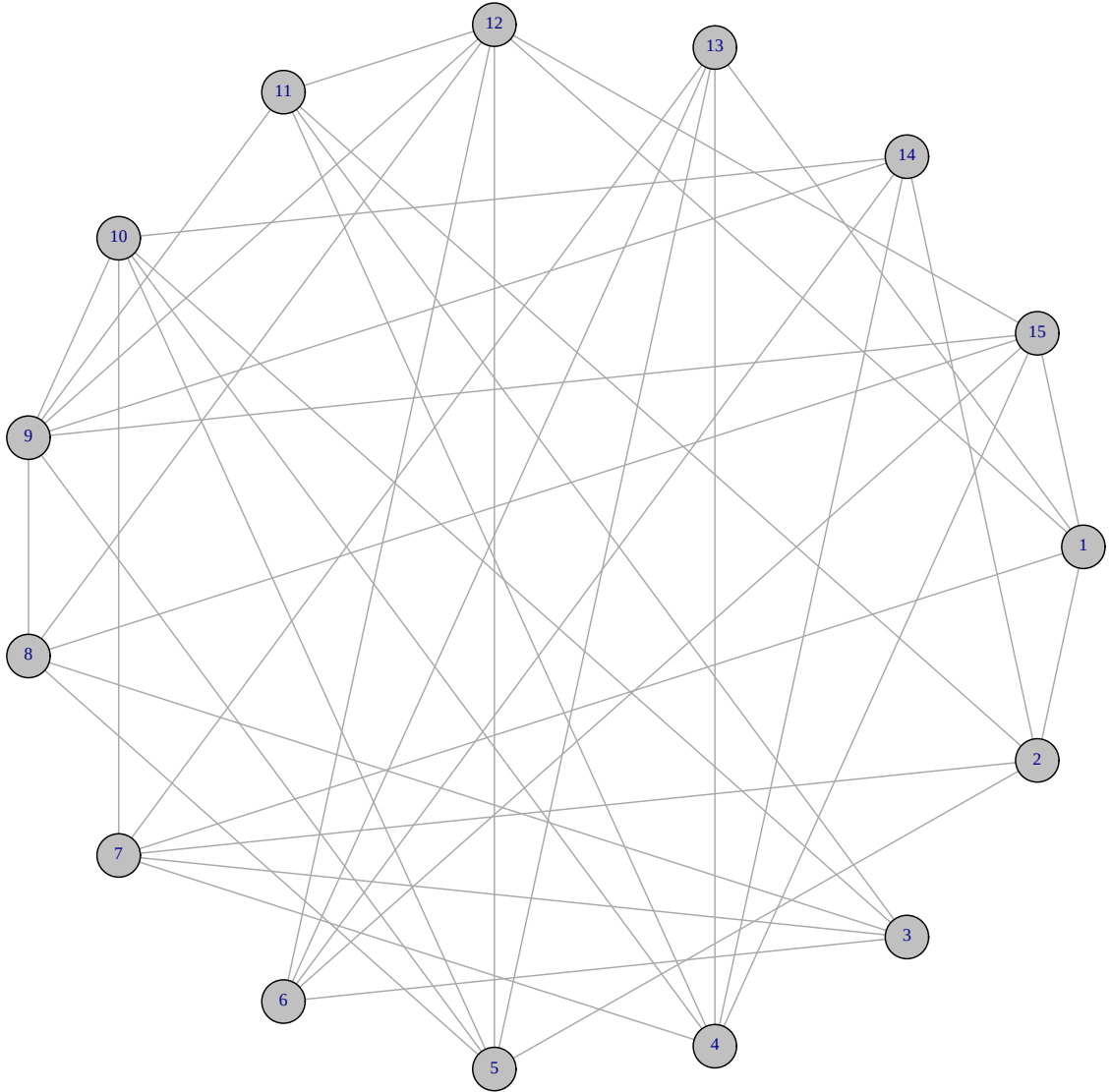


Figure B.1. Counter example: A 15-graph which neither contains 2-dense 6-set nor 2-sparse 8-set.

Counter example: $R_2(6, 9) \geq 18$

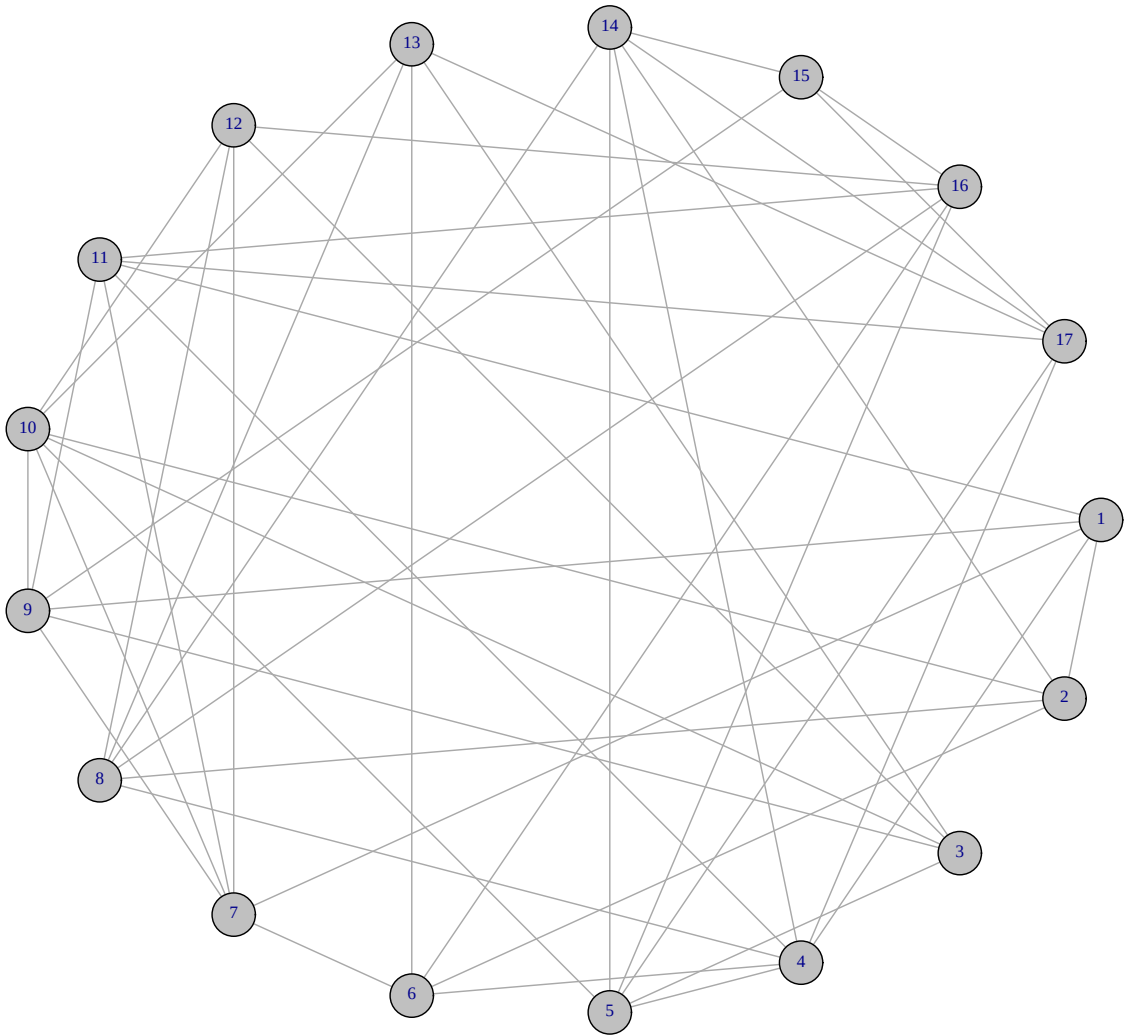


Figure B.2. Counter example: A 17-graph which neither contains 2-dense 6-set nor 2-sparse 9-set.

Counter example: $R_2(6, 10) \geq 20$

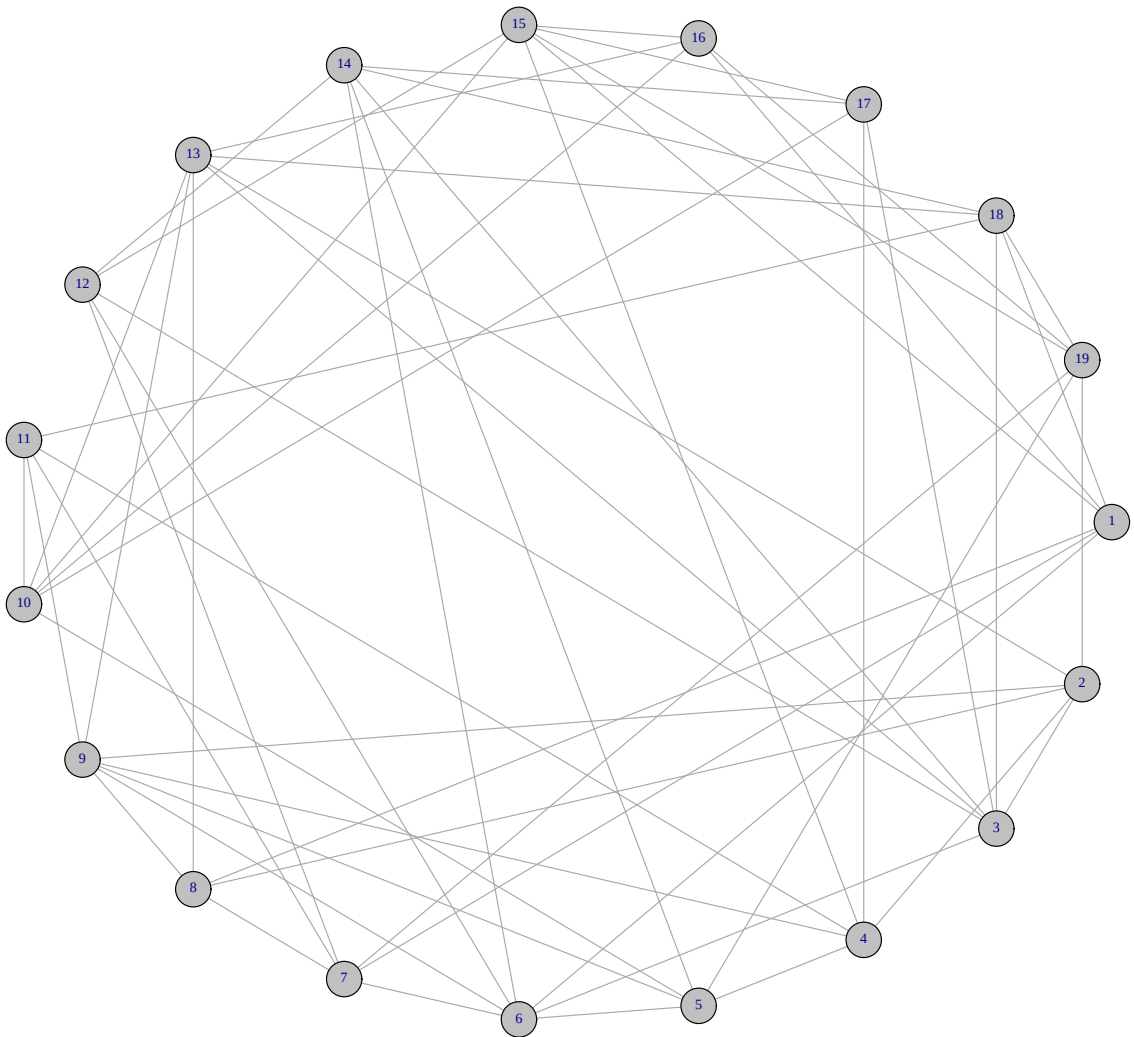


Figure B.3. Counter example: A 19-graph which neither contains 2-dense 6-set nor 2-sparse 10-set.

Counter example: $R_2(7, 7) \geq 16$

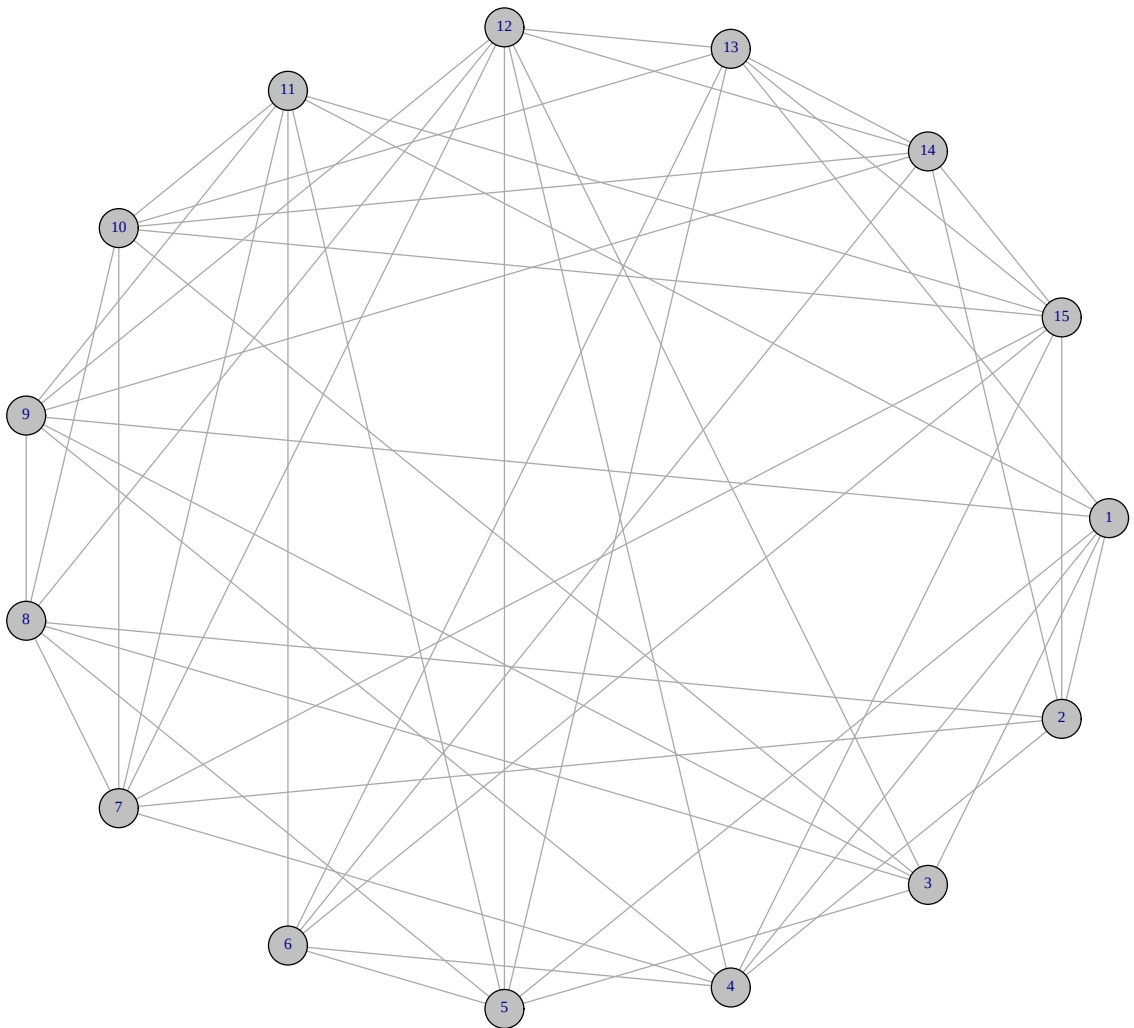


Figure B.4. Counter example: A 15-graph which neither contains 2-dense 7-set nor 2-sparse 7-set.

Counter example: $R_2(7,8) \geq 19$

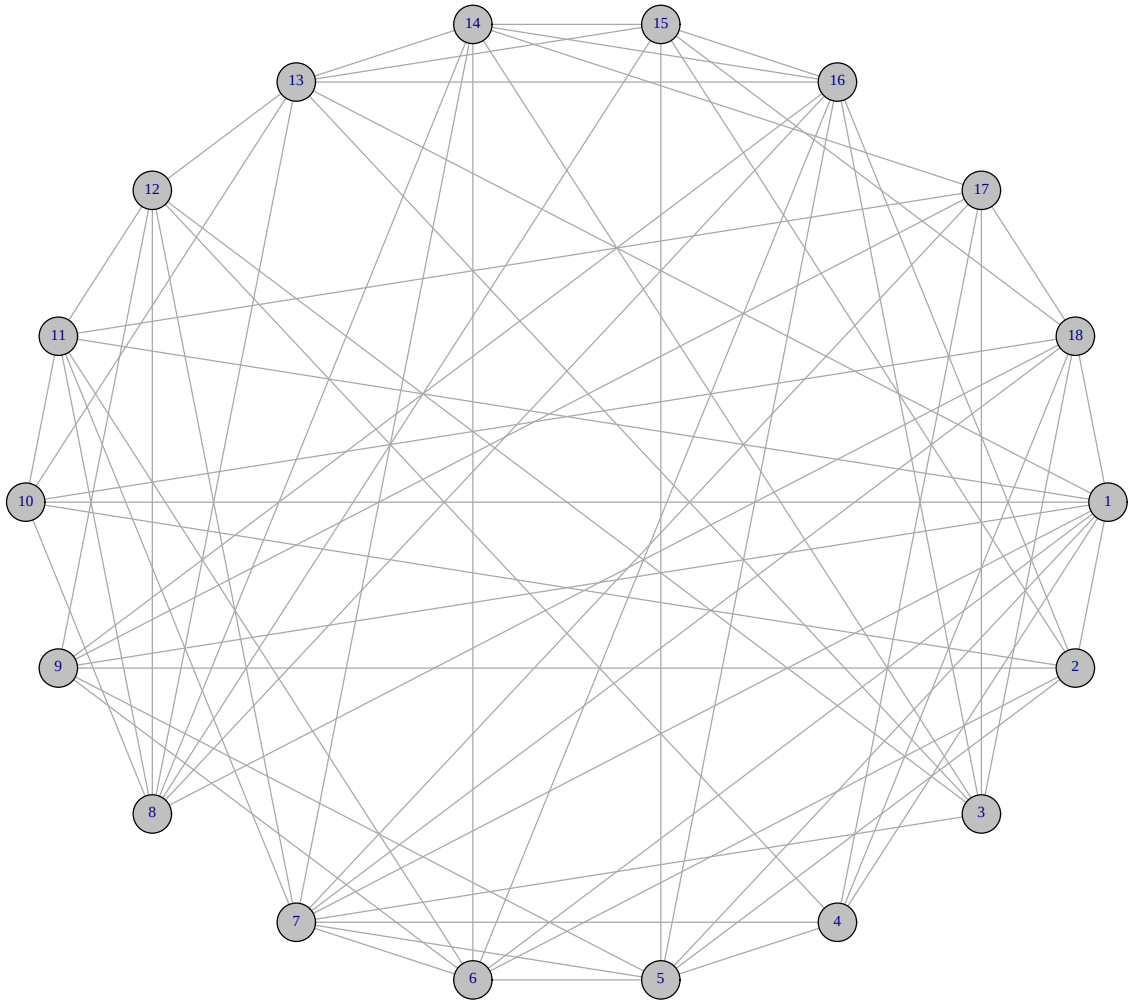


Figure B.5. Counter example: A 18-graph which neither contains 2-dense 7-set nor 2-sparse 8-set.

Counter example: $R_2(7, 9) \geq 22$

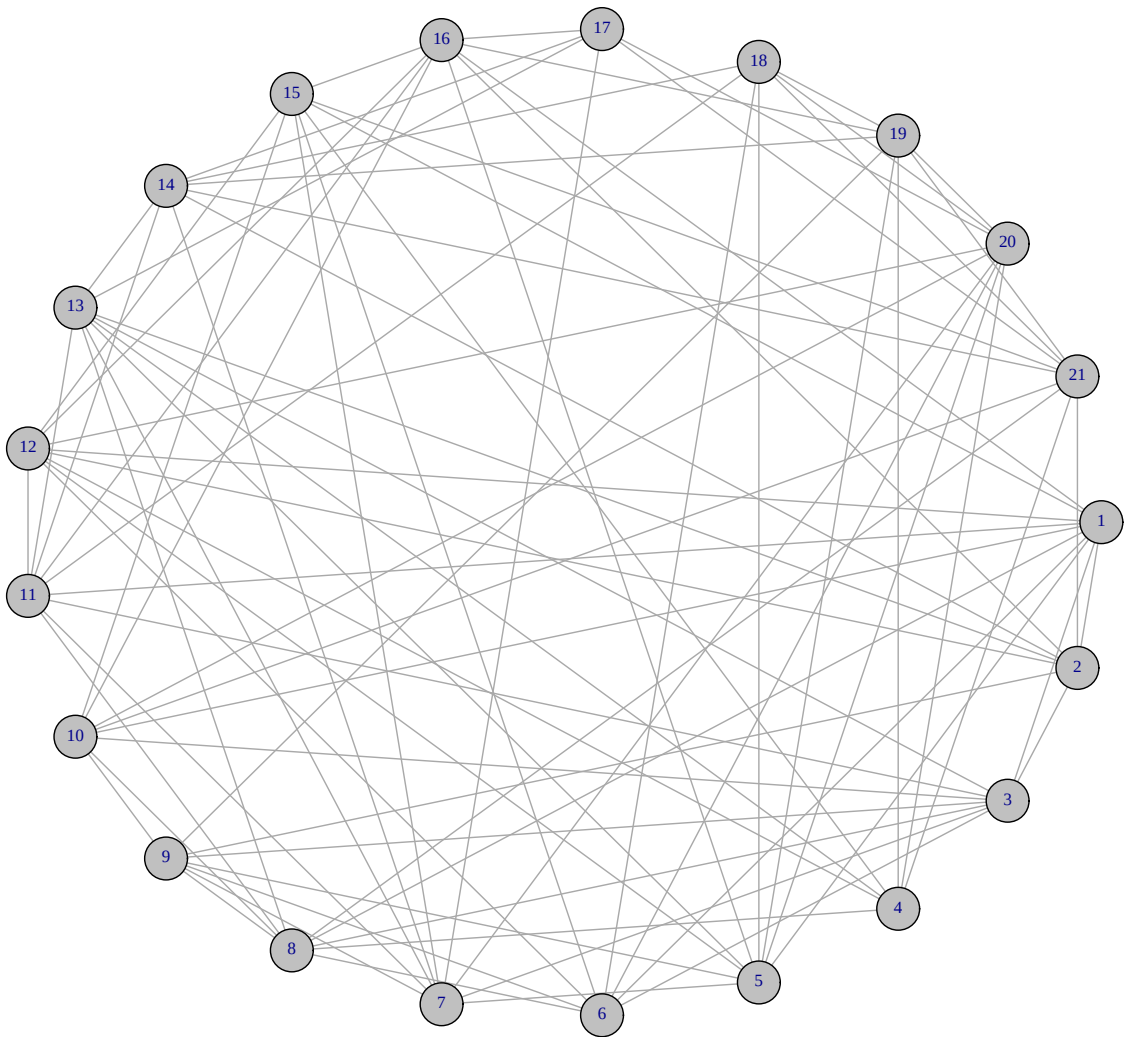


Figure B.6. Counter example: A 21-graph which neither contains 2-dense 7-set nor 2-sparse 9-set.

Counter example: $R_2(7, 10) \geq 26$

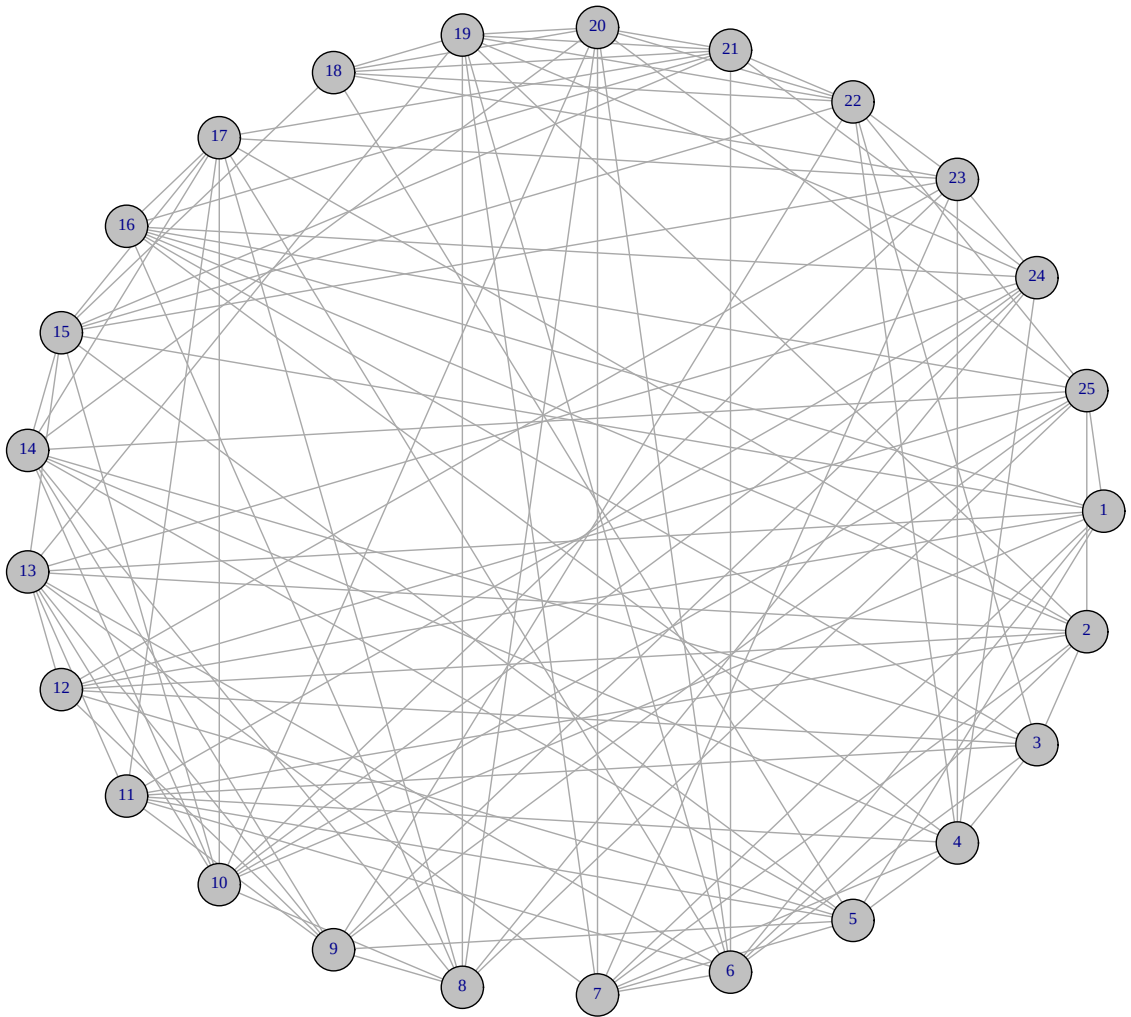


Figure B.7. Counter example: A 25-graph which neither contains 2-dense 7-set nor 2-sparse 10-set.

Counter example: $R_2(8,8) \geq 23$

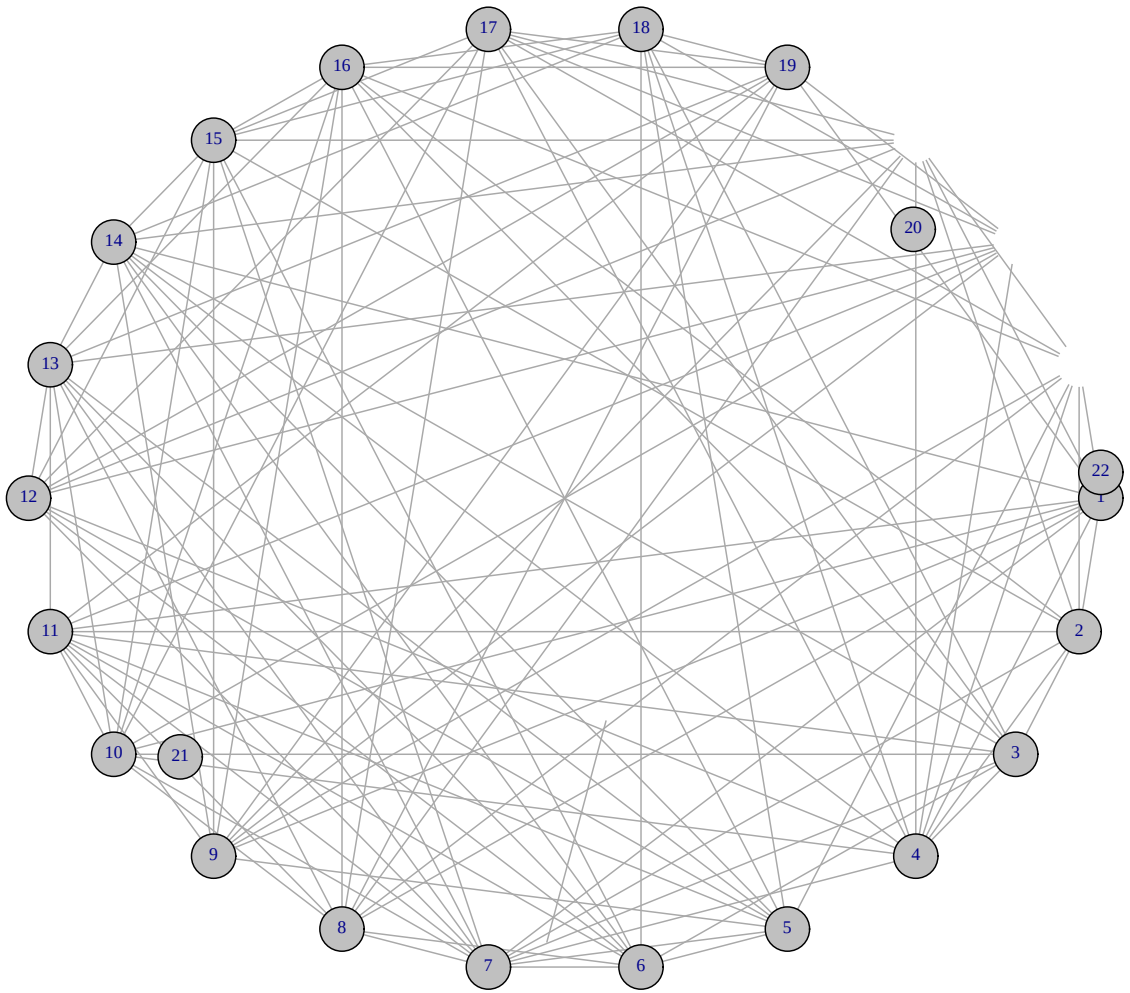


Figure B.8. Counter example: A 22-graph which neither contains 2-dense 8-set nor 2-sparse 8-set.

Counter example: $R_2(8, 9) \geq 26$

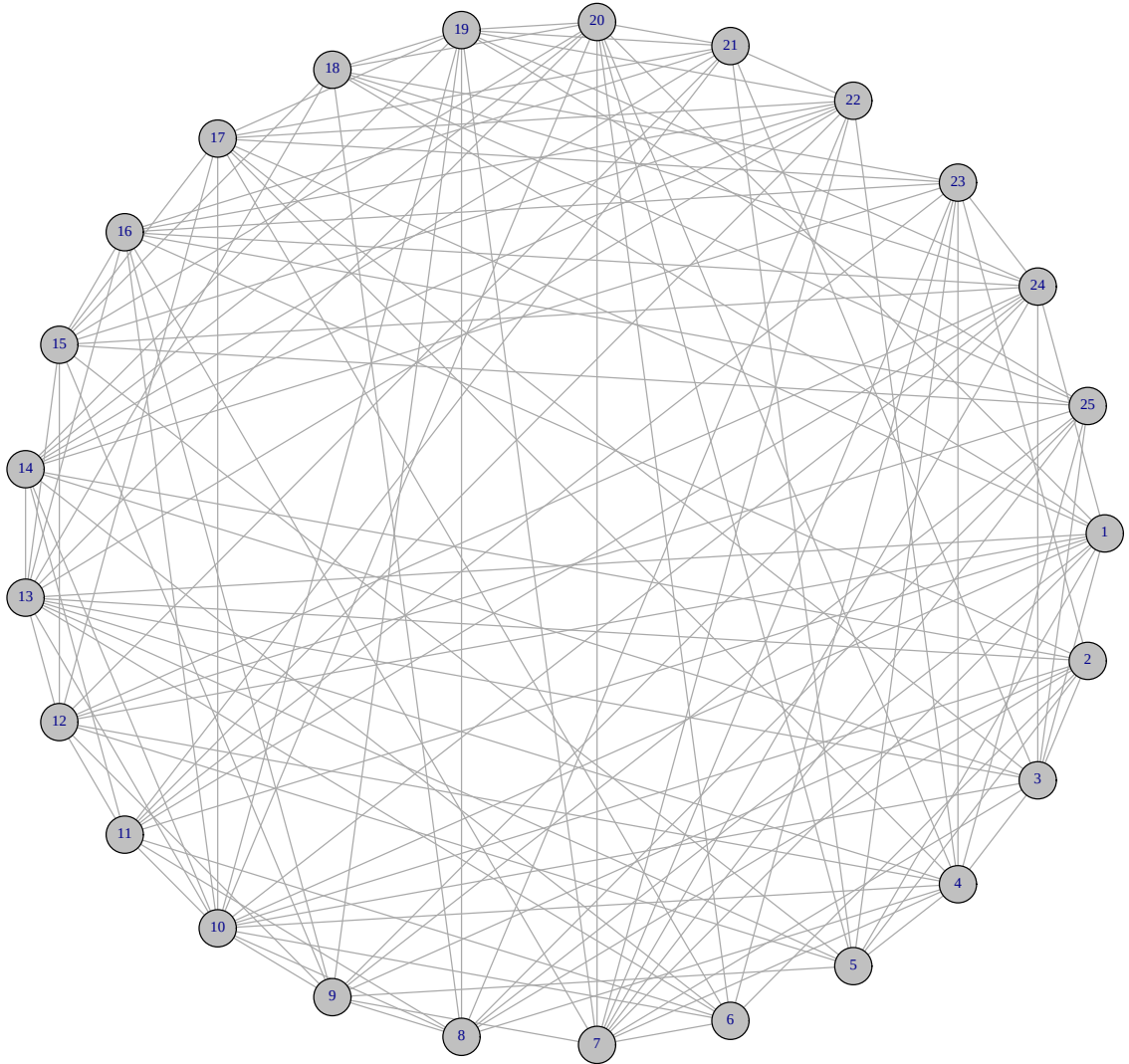


Figure B.9. Counter example: A 25-graph which neither contains 2-dense 8-set nor 2-sparse 9-set.

Counter example: $R_2(8, 10) \geq 30$

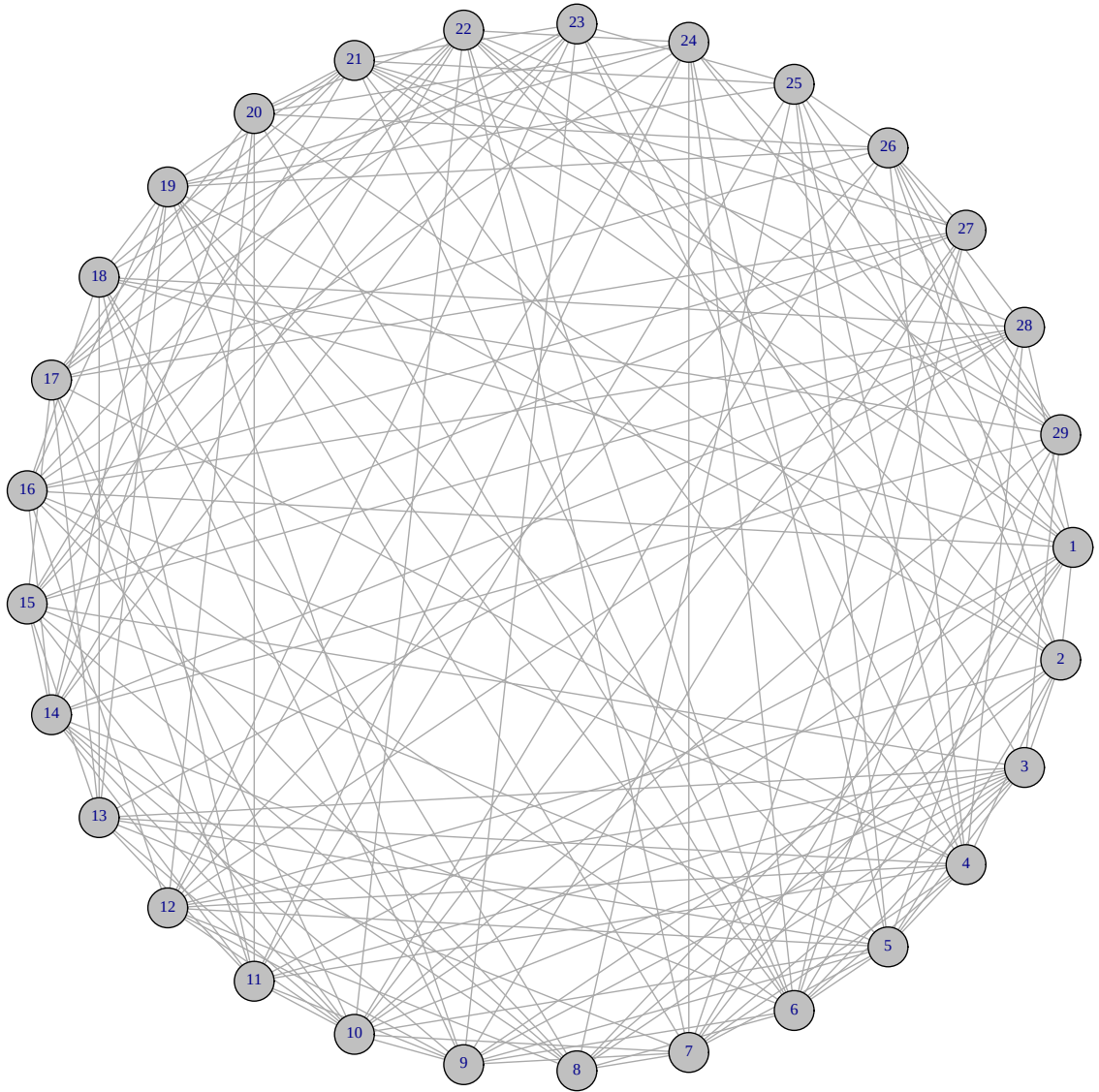


Figure B.10. Counter example: A 29-graph which neither contains 2-dense 8-set nor 2-sparse 10-set.

Counter example: $R_2(9, 9) \geq 29$

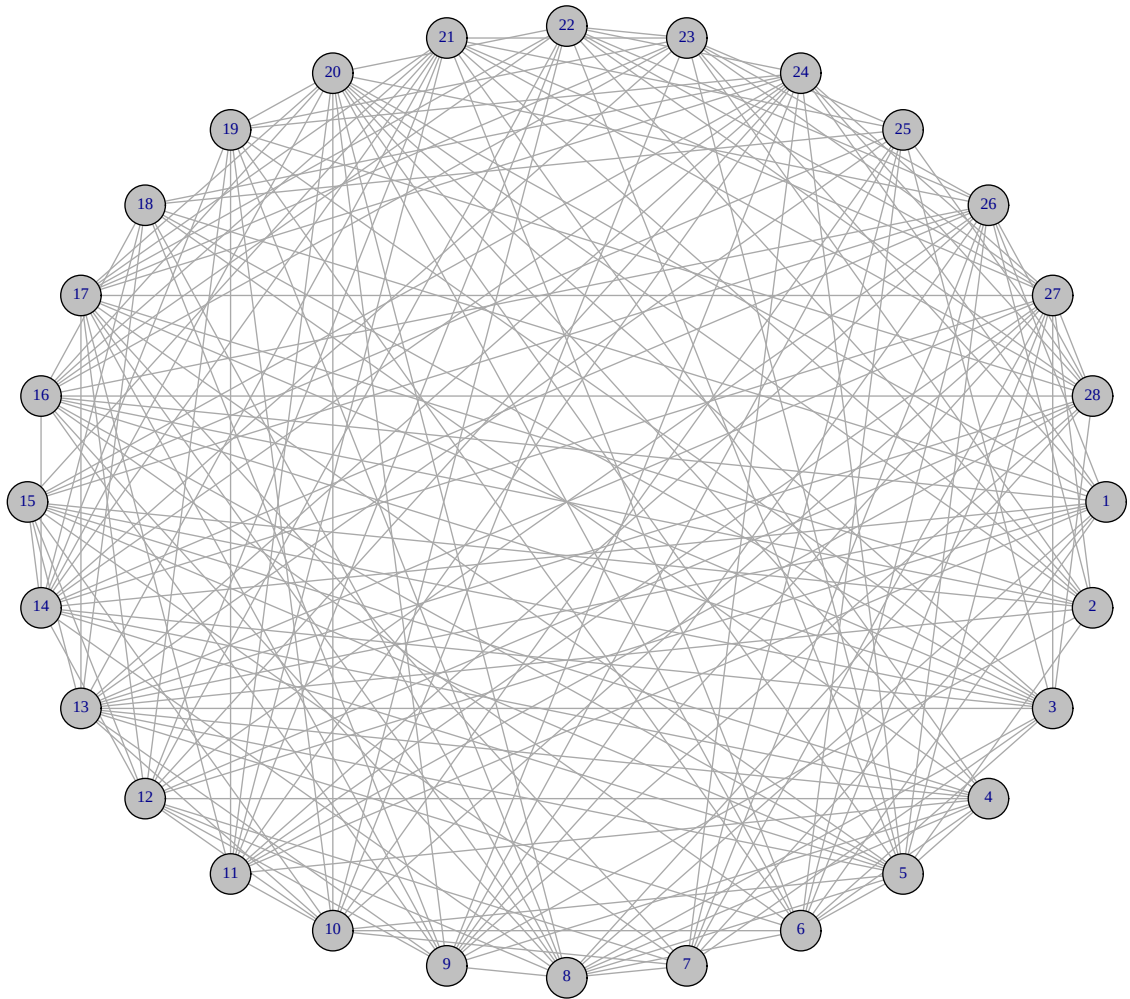


Figure B.11. Counter example: A 28-graph which neither contains 2-dense 9-set nor 2-sparse 9-set.

Counter example: $R_2(9, 10) \geq 34$

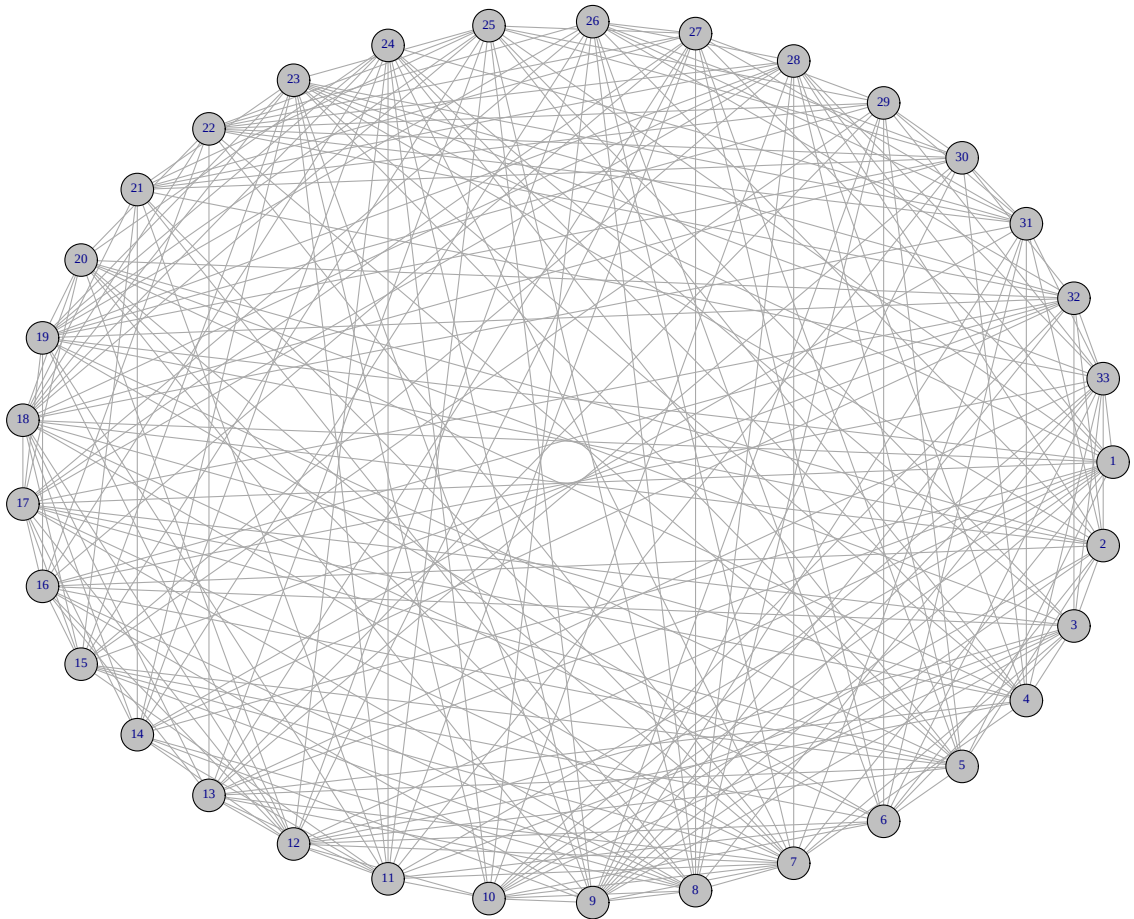


Figure B.12. Counter example: A 33-graph which neither contains 2-dense 9-set nor 2-sparse 10-set.

Counter example: $R_2(10, 10) \geq 36$

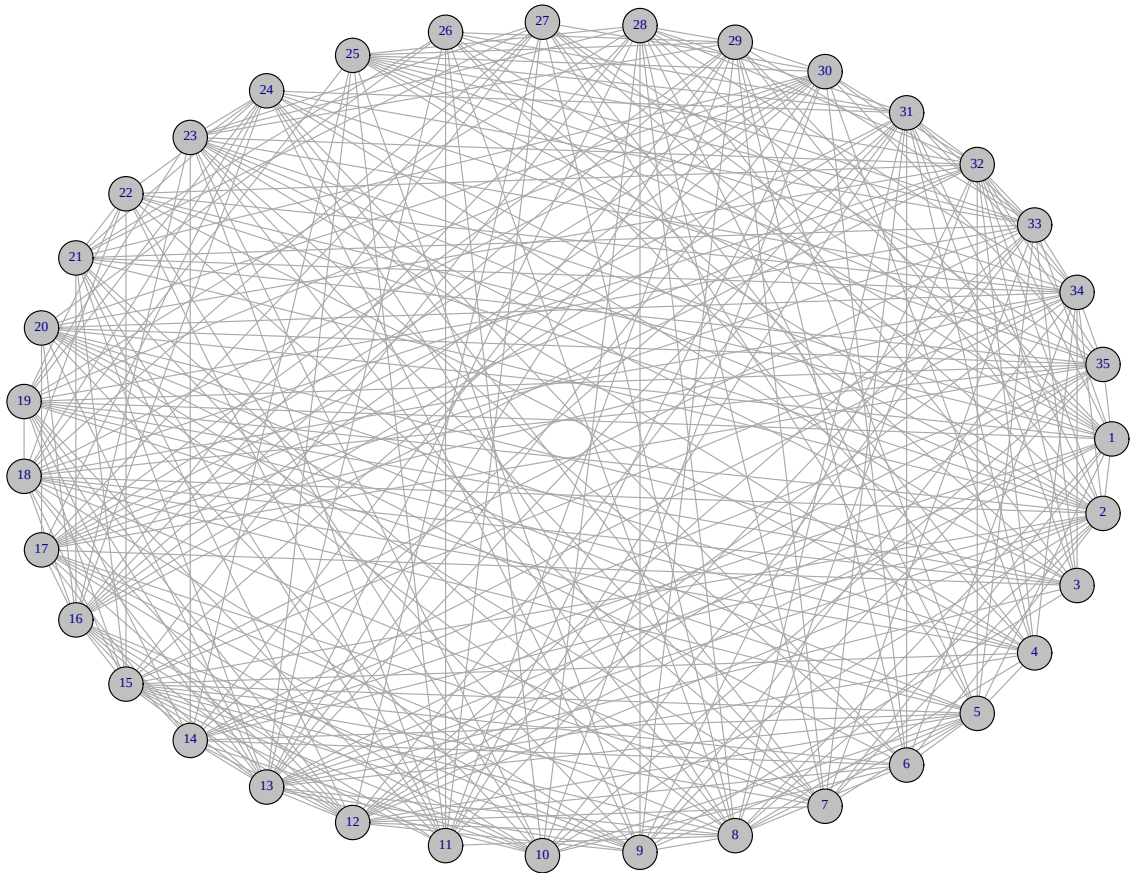


Figure B.13. Counter example: A 35-graph which neither contains 2-dense 10-set nor 2-sparse 10-set.

APPENDIX C: 3 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_3(8, 9) \geq 16$

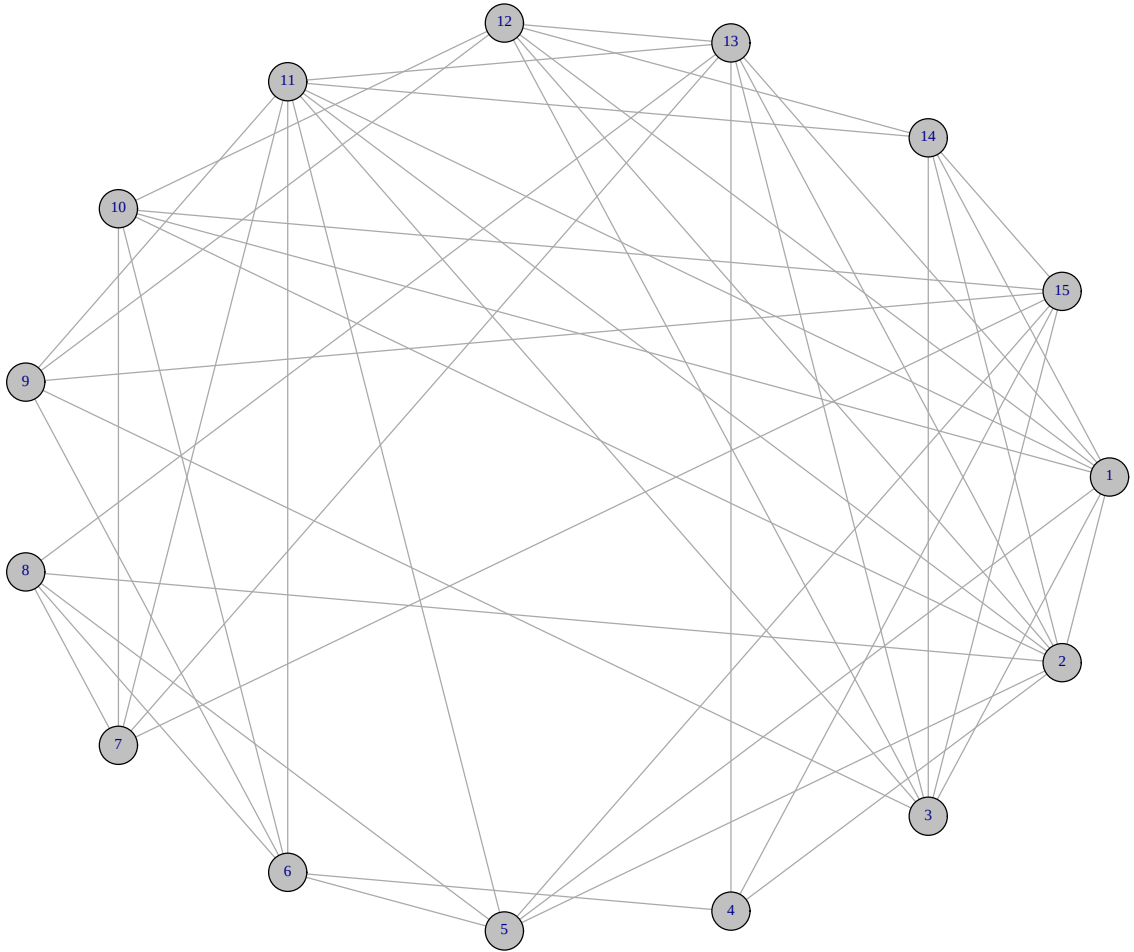


Figure C.1. Counter example: A 15-graph which neither contains 3-dense 8-set nor 3-sparse 9-set.

Counter example: $R_3(8, 10) \geq 19$

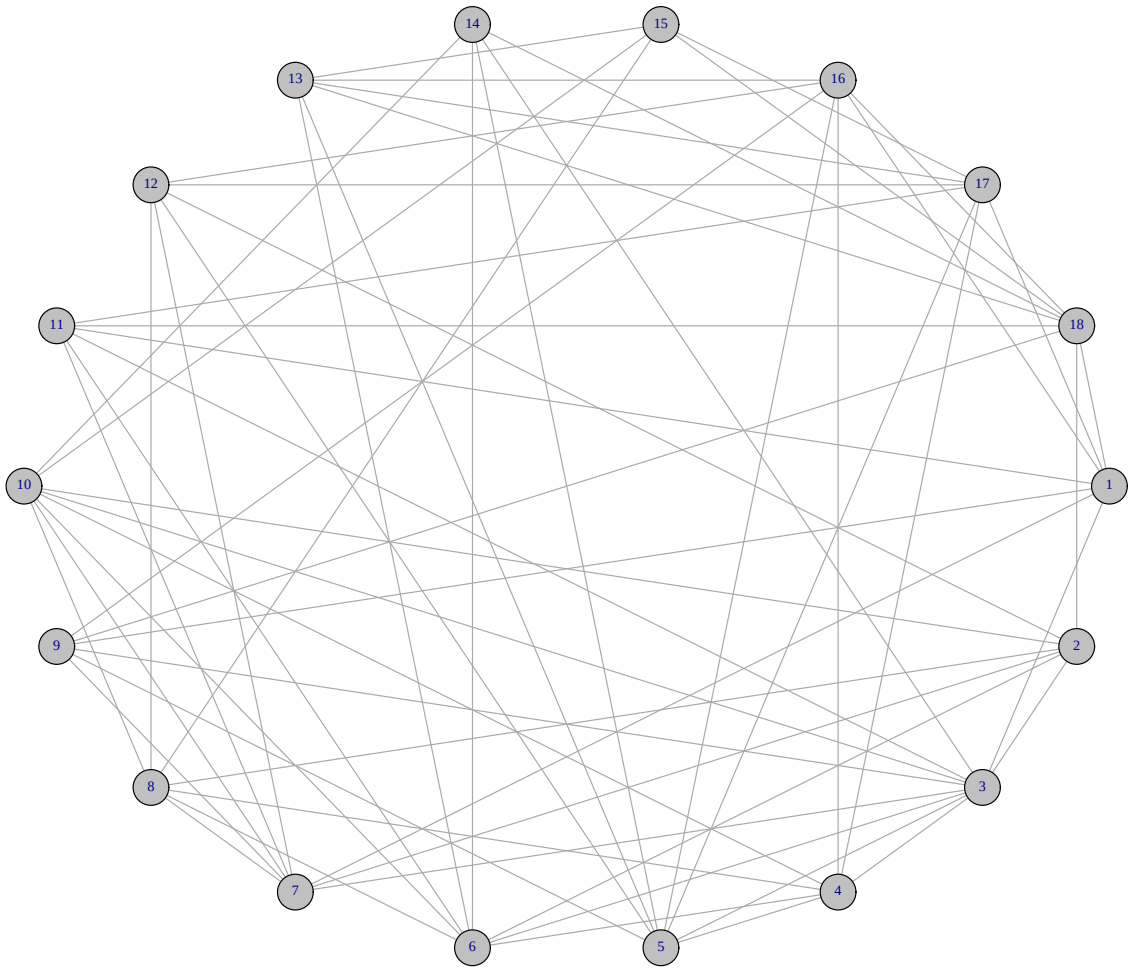


Figure C.2. Counter example: A 18-graph which neither contains 3-dense 8-set nor 3-sparse 10-set.

Counter example: $R_3(8, 11) \geq 22$

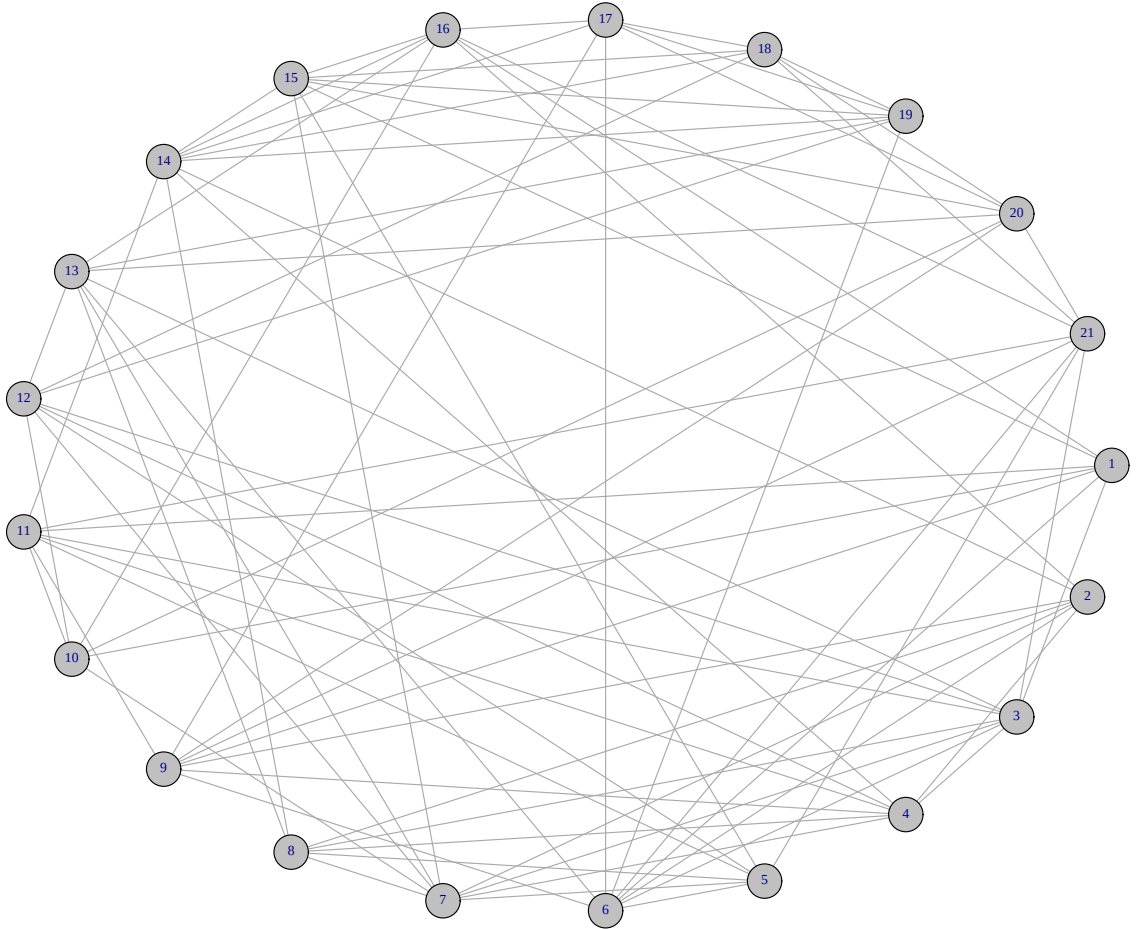


Figure C.3. Counter example: A 21-graph which neither contains 3-dense 8-set nor 3-sparse 11-set.

Counter example: $R_3(9, 9) \geq 19$

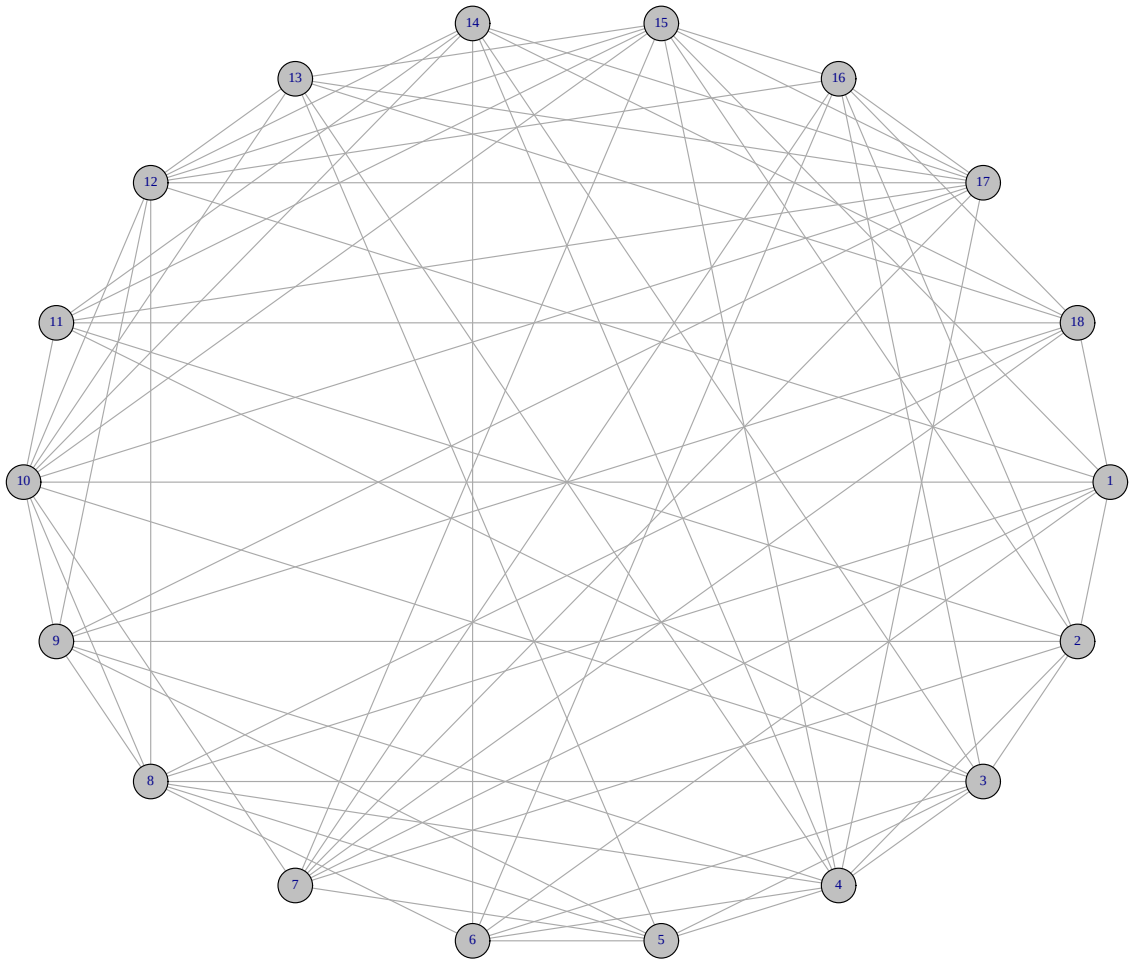


Figure C.4. Counter example: A 18-graph which neither contains 3-dense 9-set nor 3-sparse 9-set.

Counter example: $R_3(9, 10) \geq 23$

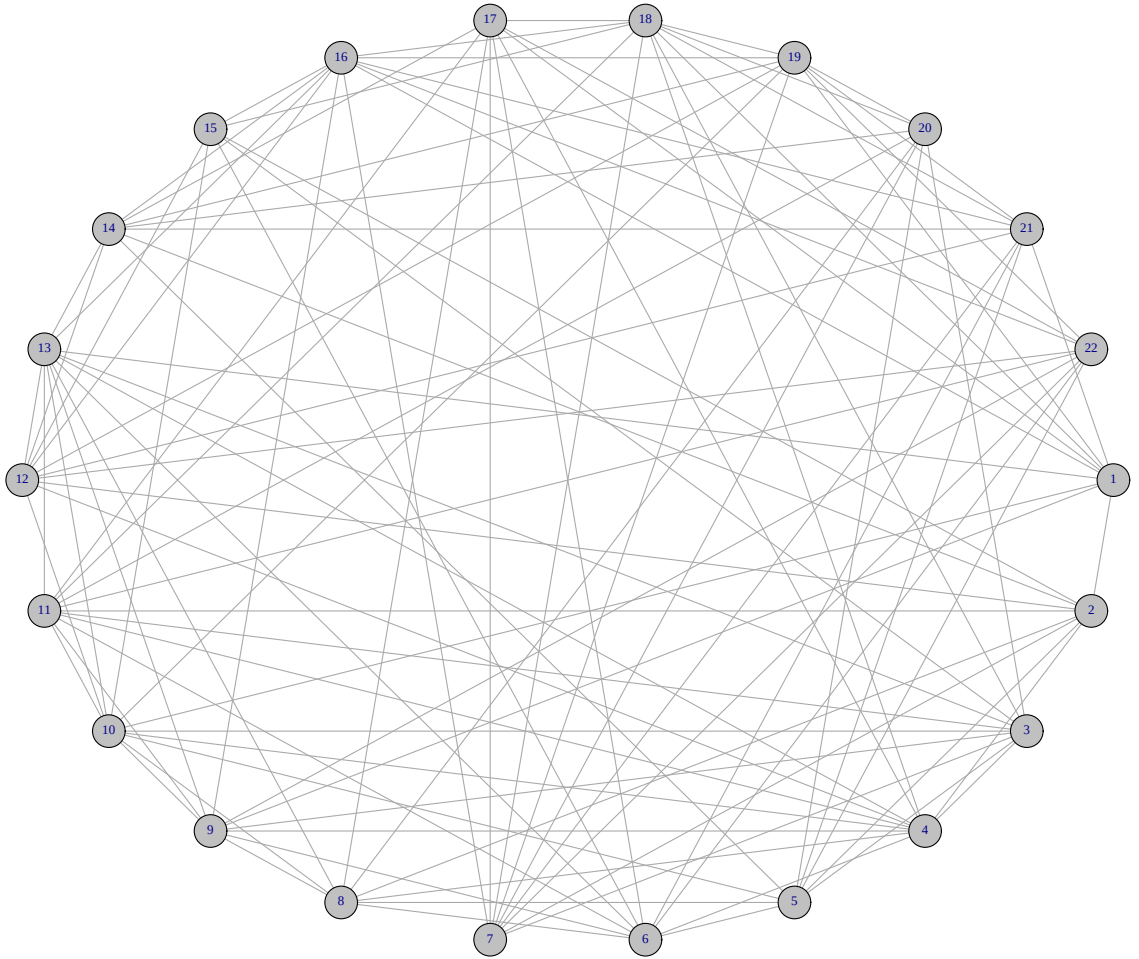


Figure C.5. Counter example: A 22-graph which neither contains 3-dense 9-set nor 3-sparse 10-set.

Counter example: $R_3(9, 11) \geq 24$

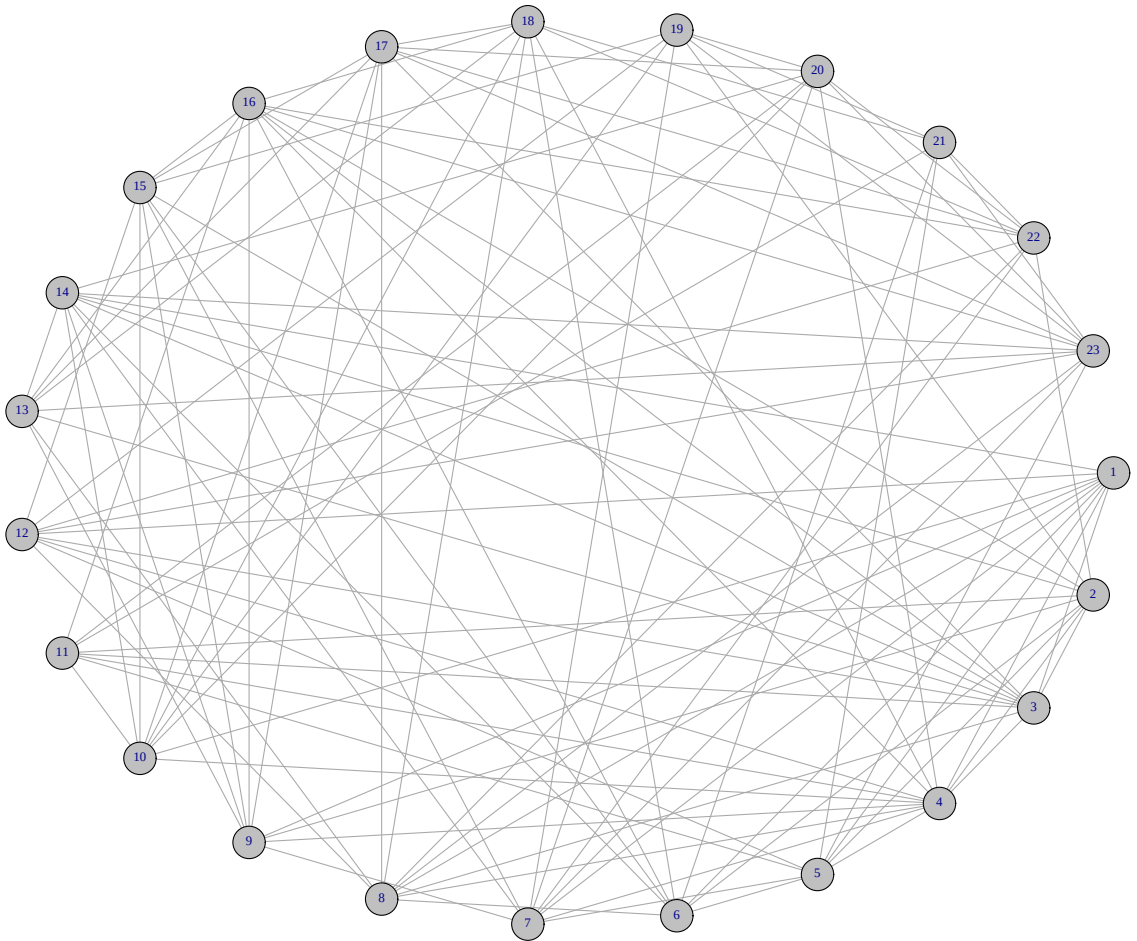


Figure C.6. Counter example: A 23-graph which neither contains 3-dense 9-set nor 3-sparse 11-set.

Counter example: $R_3(10, 10) \geq 26$

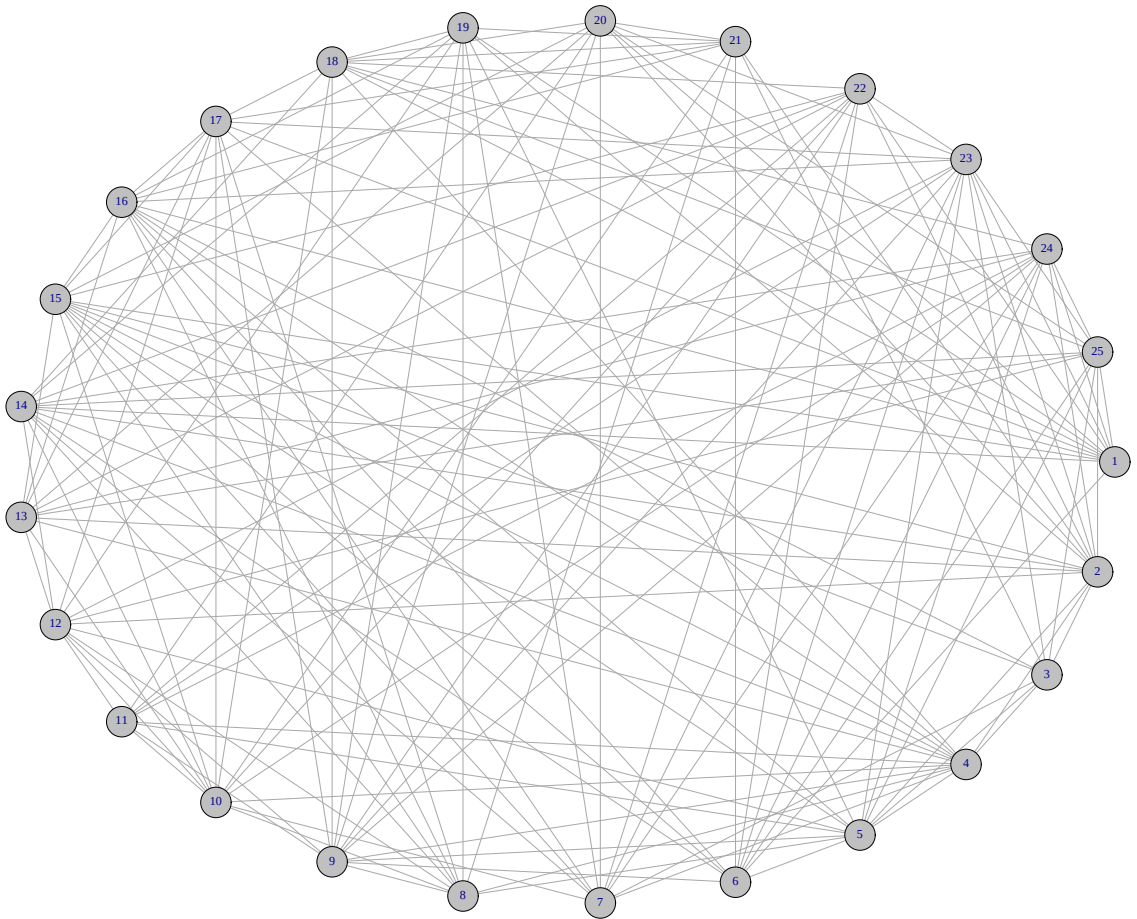


Figure C.7. Counter example: A 25-graph which neither contains 3-dense 10-set nor 3-sparse 10-set.

Counter example: $R_3(10, 11) \geq 29$

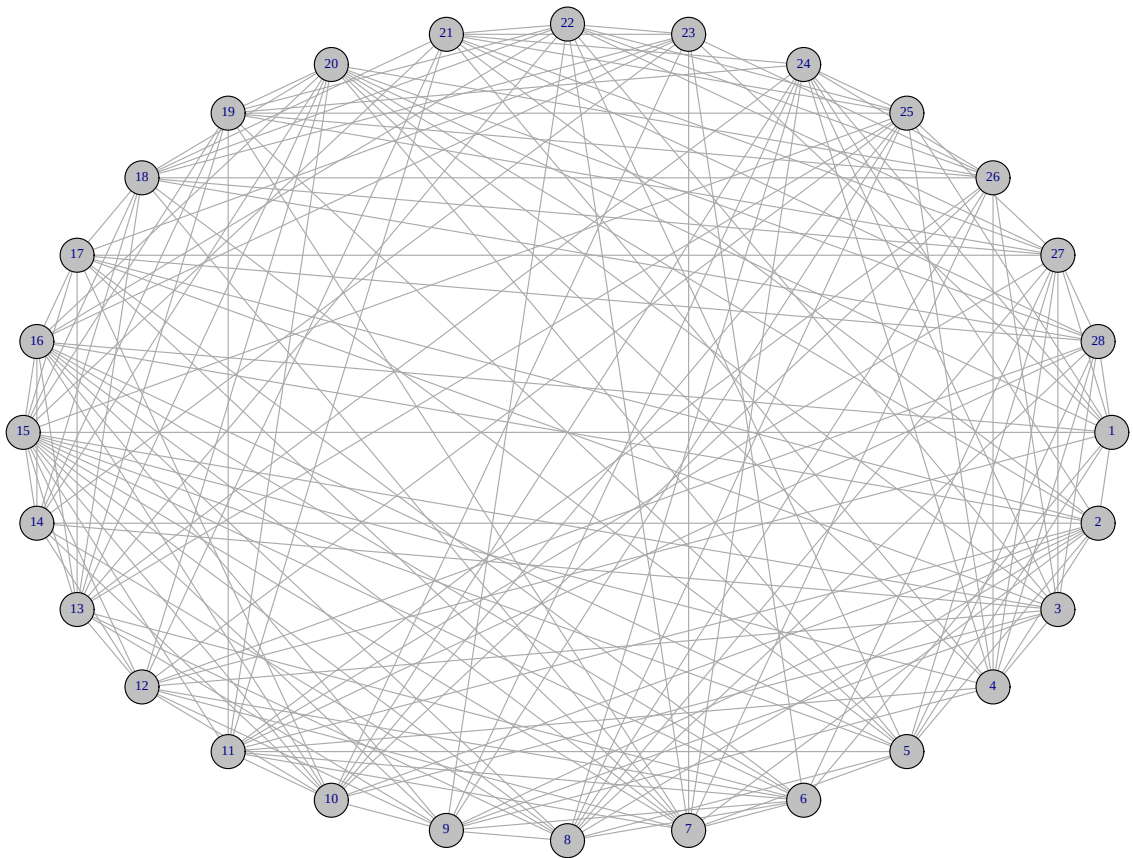


Figure C.8. Counter example: A 28-graph which neither contains 3-dense 10-set nor 3-sparse 11-set.

Counter example: $R_3(11, 11) \geq 29$

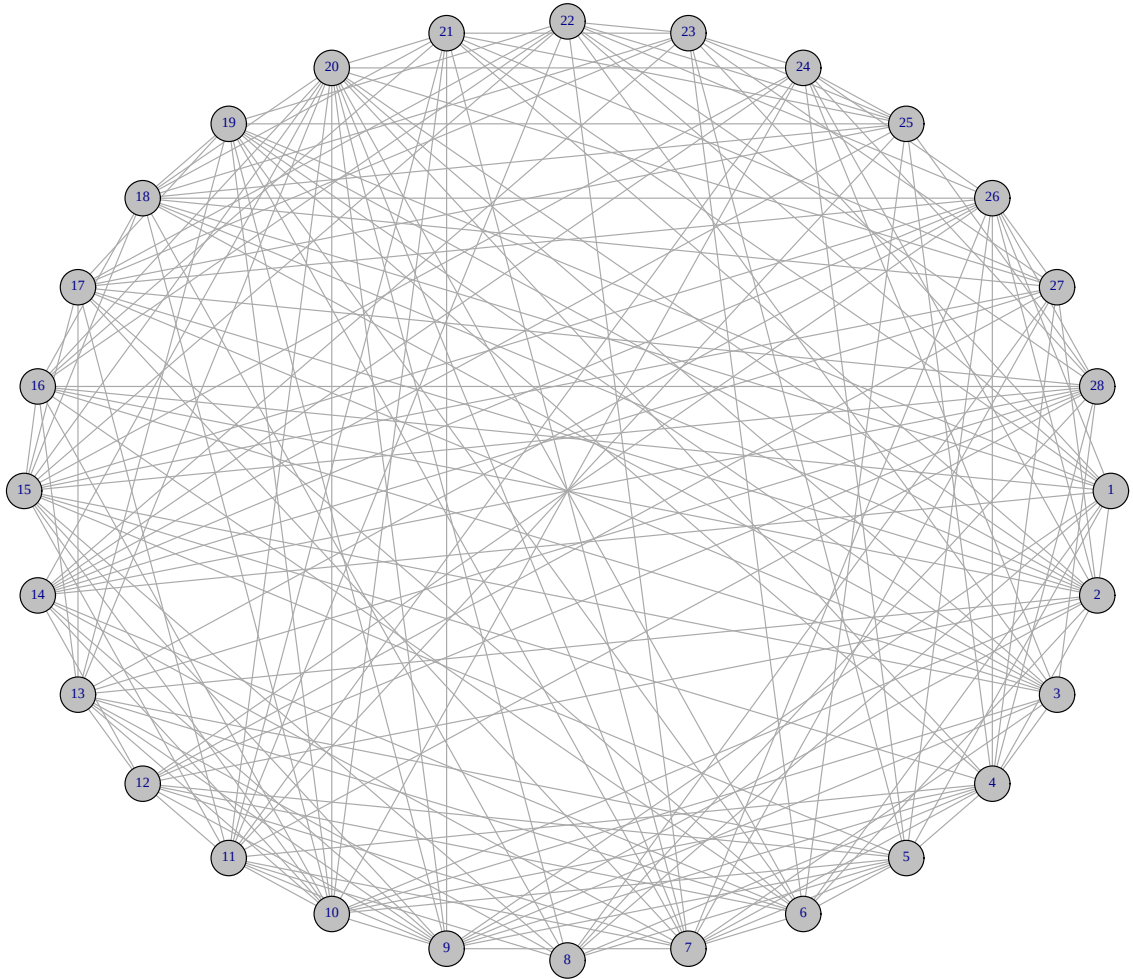


Figure C.9. Counter example: A 28-graph which neither contains 3-dense 11-set nor 3-sparse 11-set.

APPENDIX D: 4 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_4(10, 10) \geq 17$

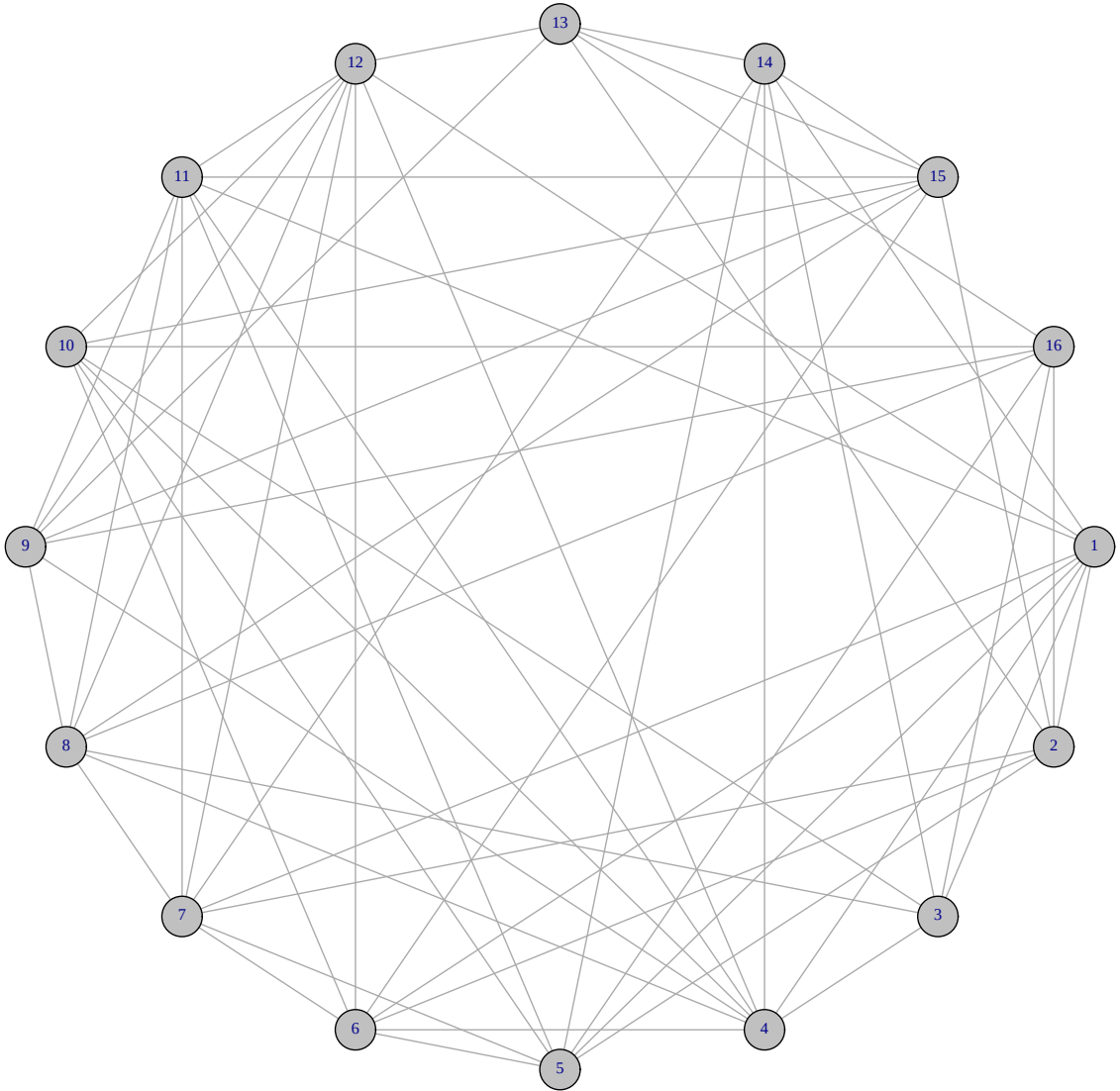


Figure D.1. Counter example: A 16-graph which neither contains 4-dense 10-set nor 4-sparse 10-set.

Counter example: $R_4(11, 12) \geq 24$

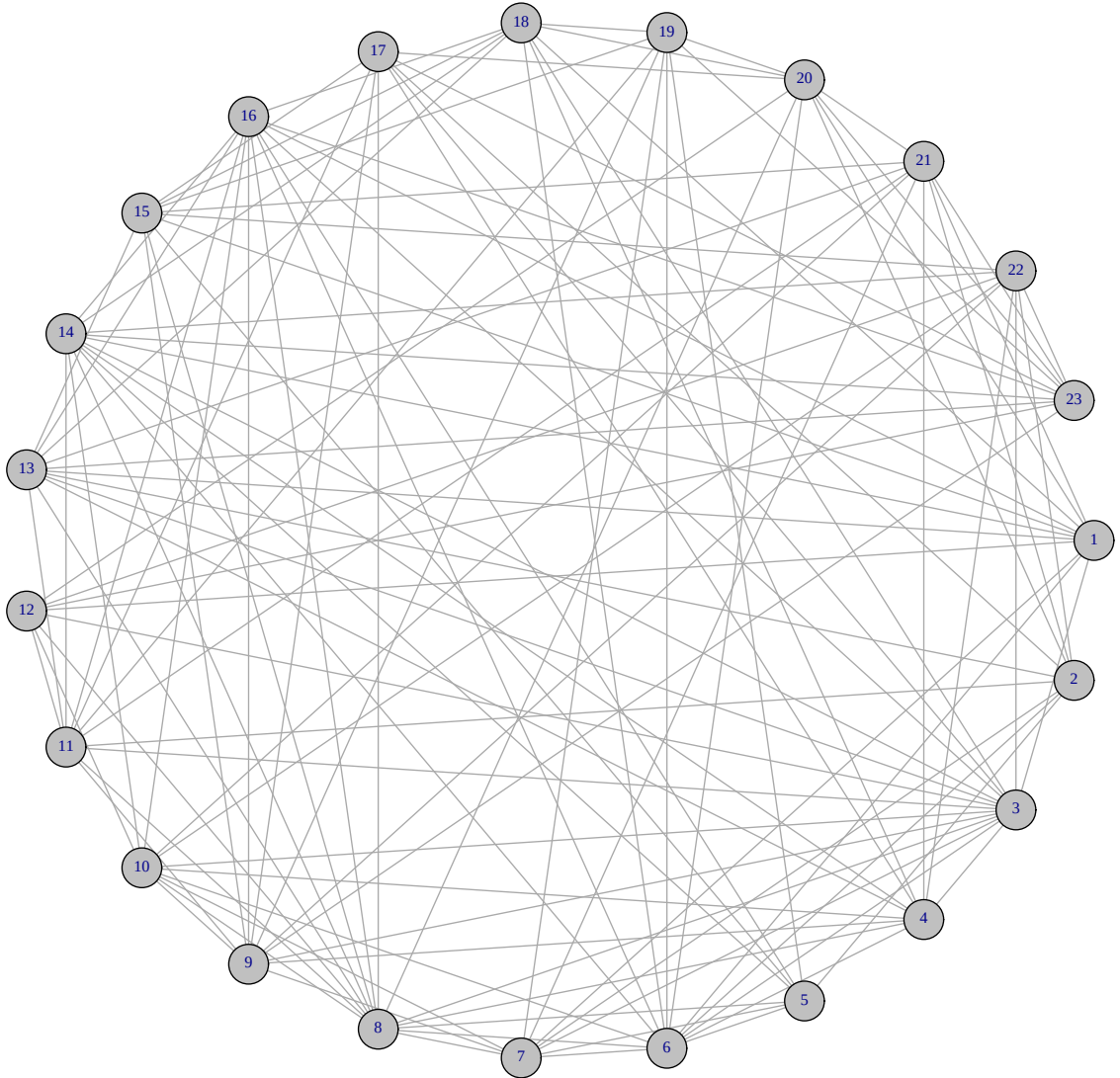


Figure D.2. Counter example: A 23-graph which neither contains 4-dense 11-set nor 4-sparse 12-set.

Counter example: $R_4(12, 12) \geq 28$

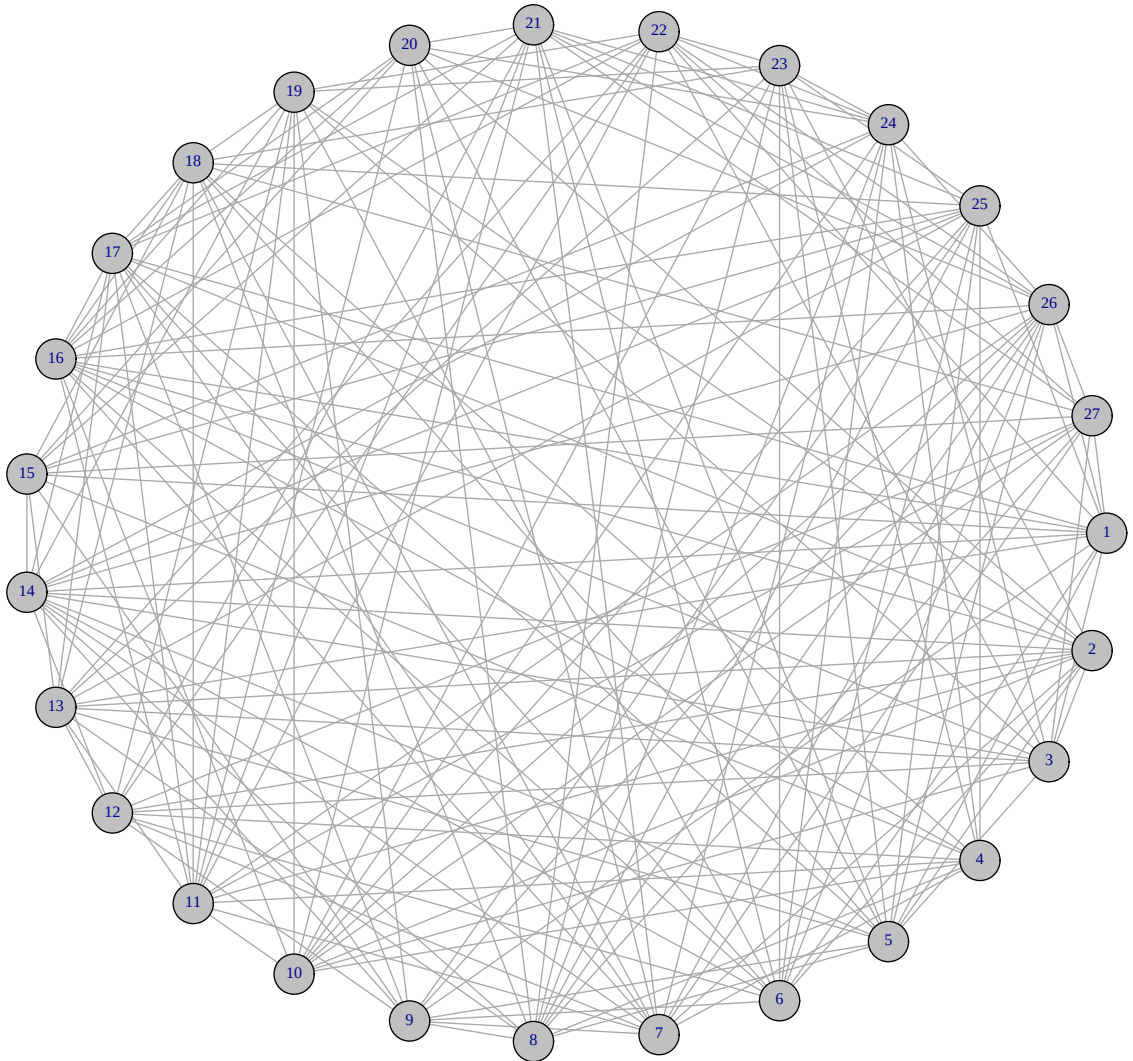


Figure D.3. Counter example: A 27-graph which neither contains 4-dense 12-set nor 4-sparse 12-set.

APPENDIX E: 5 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_5(11, 11) \geq 17$

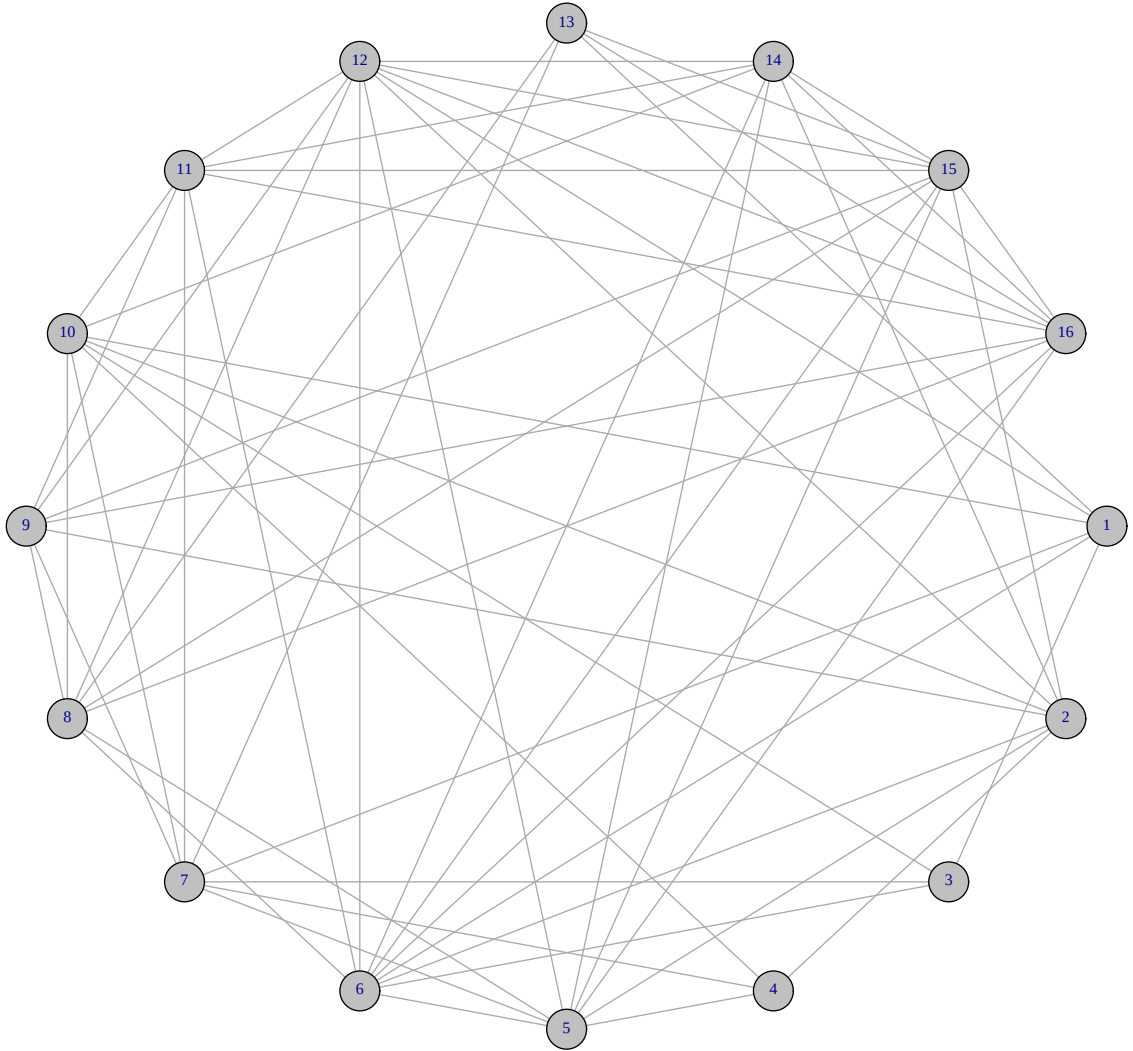


Figure E.1. Counter example: A 16-graph which neither contains 5-dense 11-set nor 5-sparse 11-set.

Counter example: $R_5(11, 12) \geq 18$

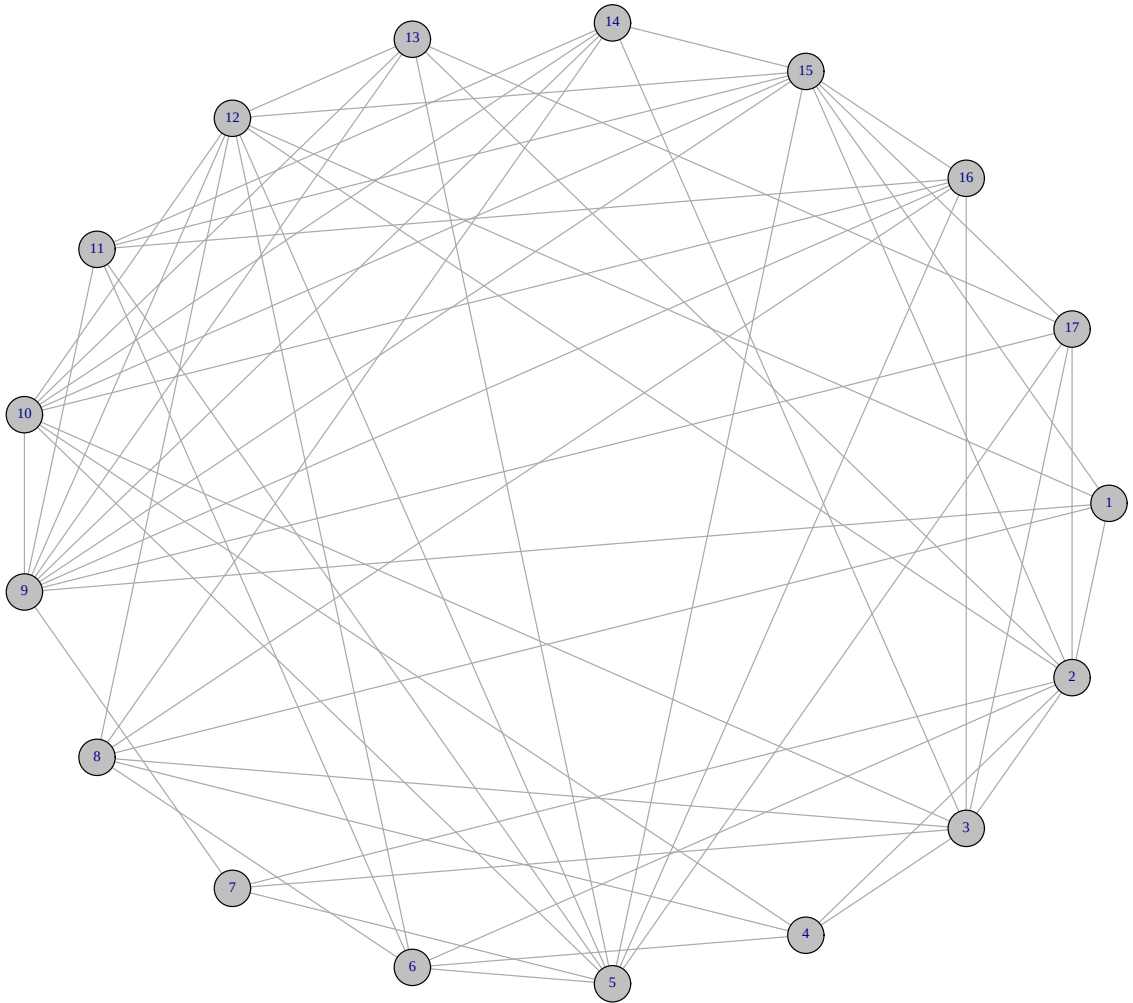


Figure E.2. Counter example: A 17-graph which neither contains 5-dense 11-set nor 5-sparse 12-set.

Counter example: $R_5(12, 12) \geq 20$

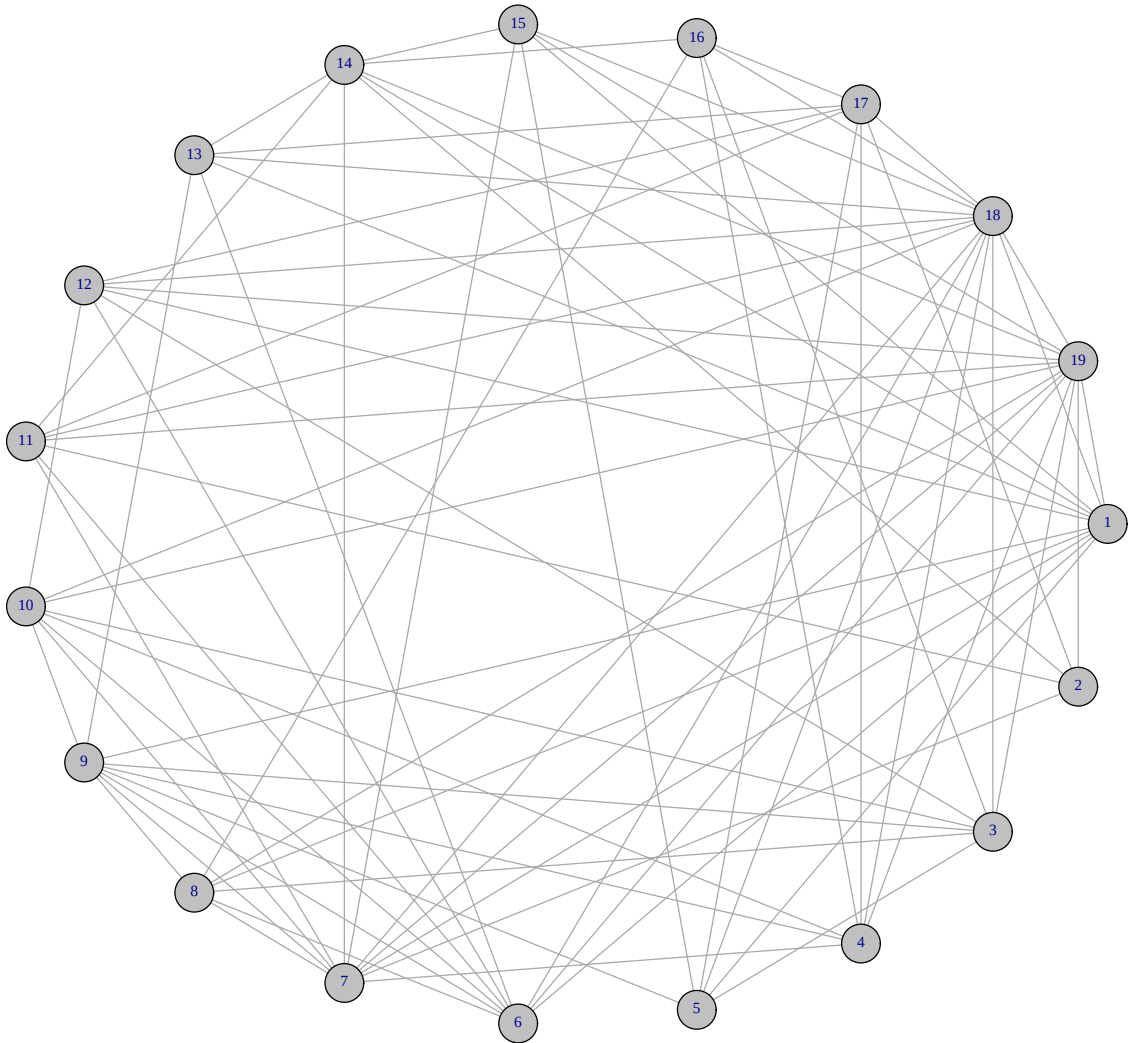


Figure E.3. Counter example: A 19-graph which neither contains 5-dense 12-set nor 5-sparse 12-set.

Counter example: $R_5(13, 13) \geq 26$

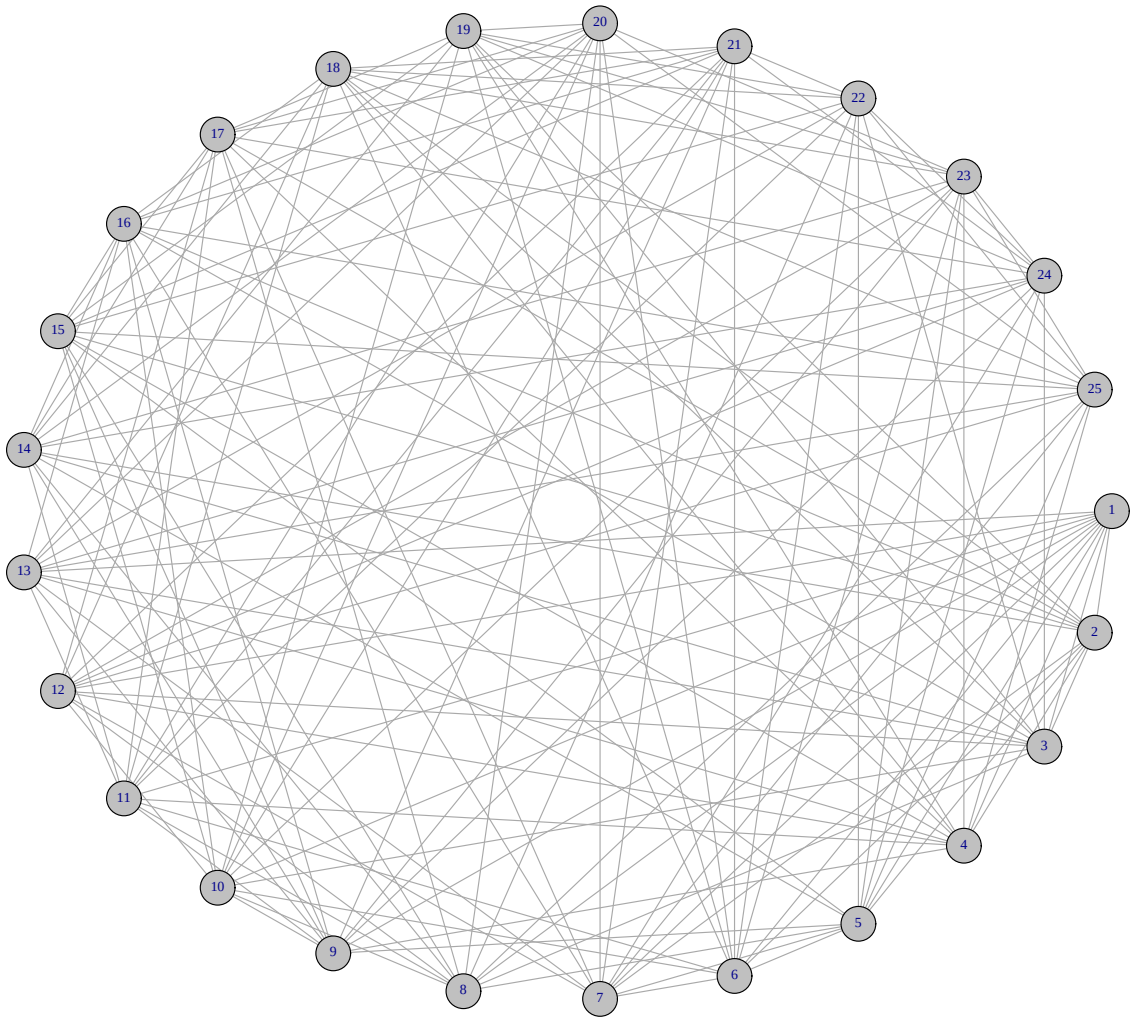


Figure E.4. Counter example: A 25-graph which neither contains 5-dense 13-set nor 5-sparse 13-set.

APPENDIX F: 6 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_6(12, 12) \geq 17$

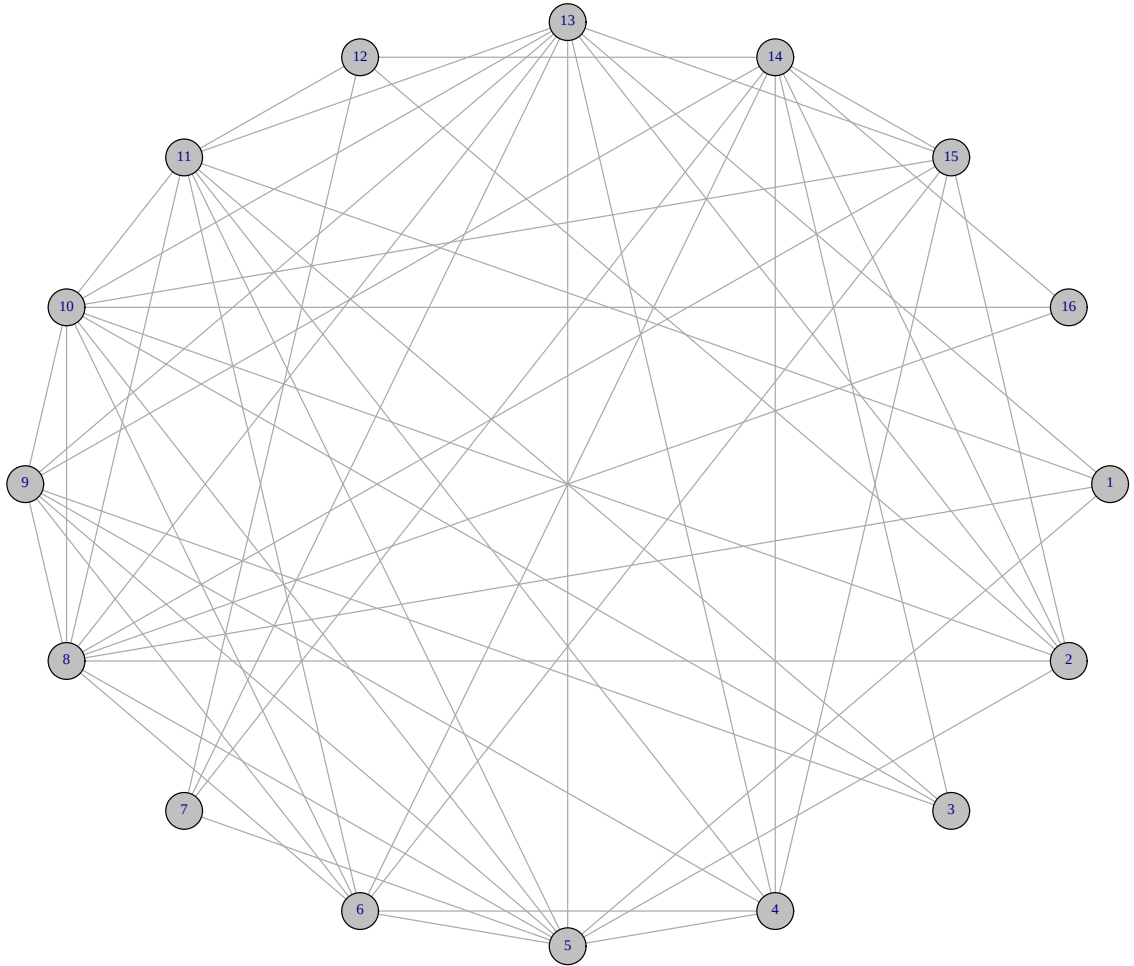


Figure F.1. Counter example: A 16-graph which neither contains 6-dense 12-set nor 6-sparse 12-set.

Counter example: $R_6(12, 13) \geq 18$

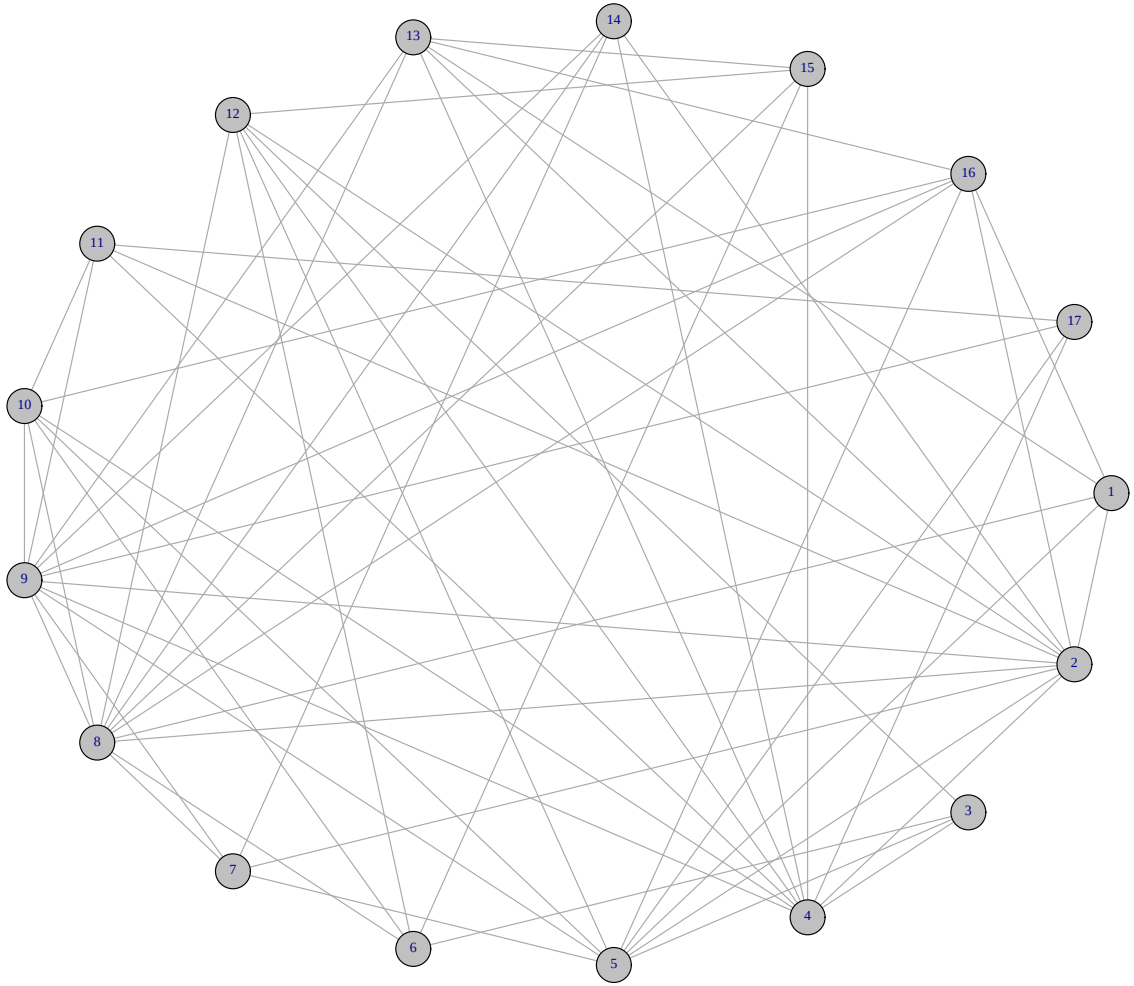


Figure F.2. Counter example: A 17-graph which neither contains 6-dense 12-set nor 6-sparse 13-set.

Counter example: $R_6(12, 14) \geq 19$

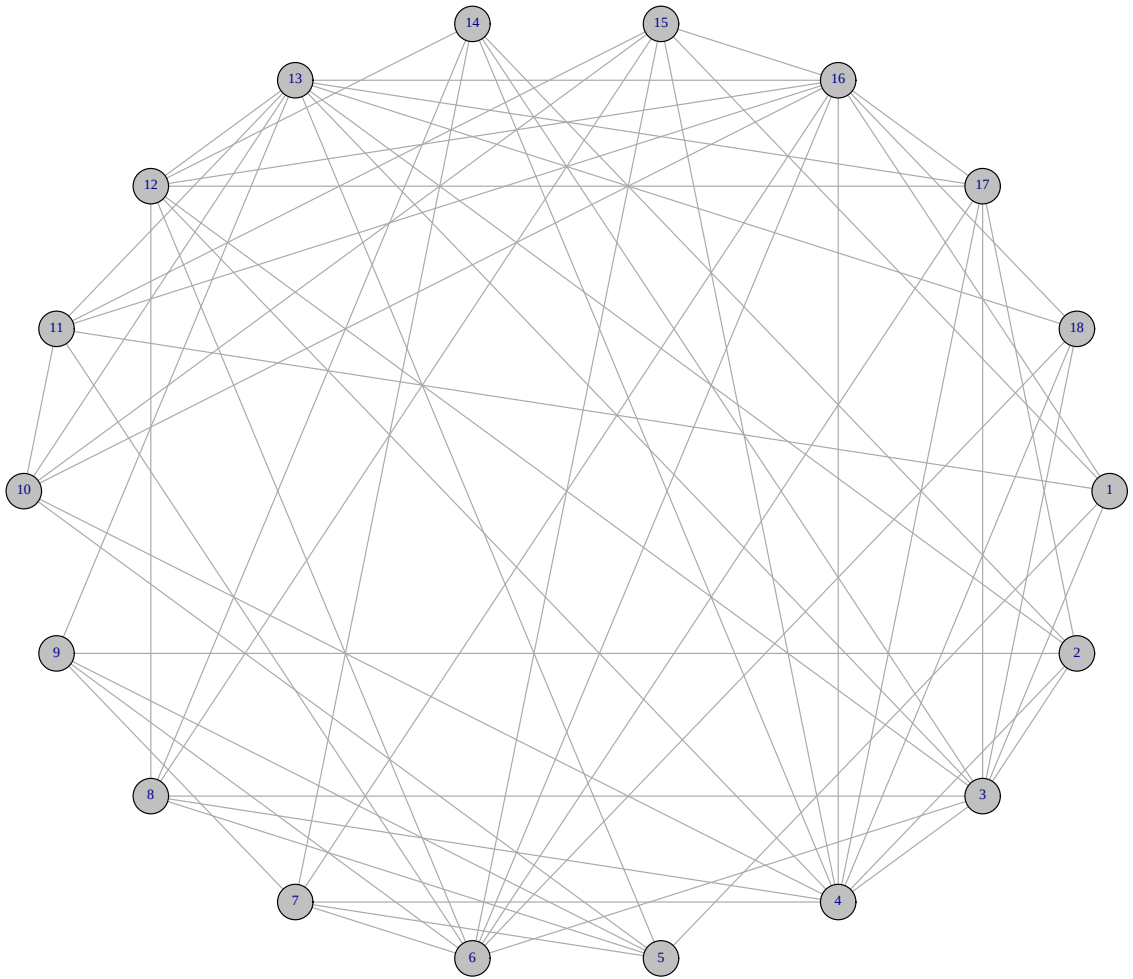


Figure F.3. Counter example: A 18-graph which neither contains 6-dense 12-set nor 6-sparse 14-set.

Counter example: $R_6(13, 13) \geq 20$

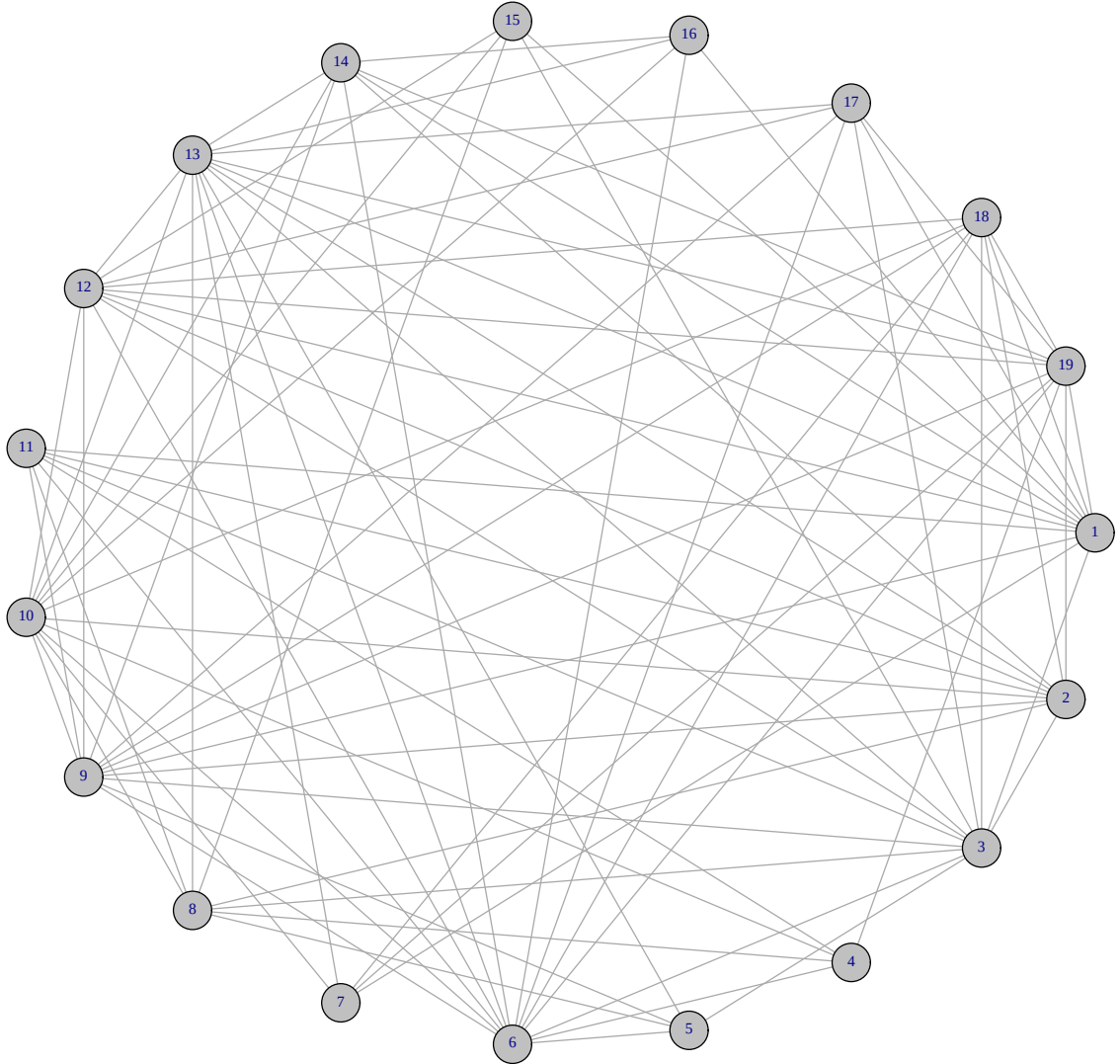


Figure F.4. Counter example: A 19-graph which neither contains 6-dense 13-set nor 6-sparse 13-set.

Counter example: $R_6(13, 14) \geq 20$

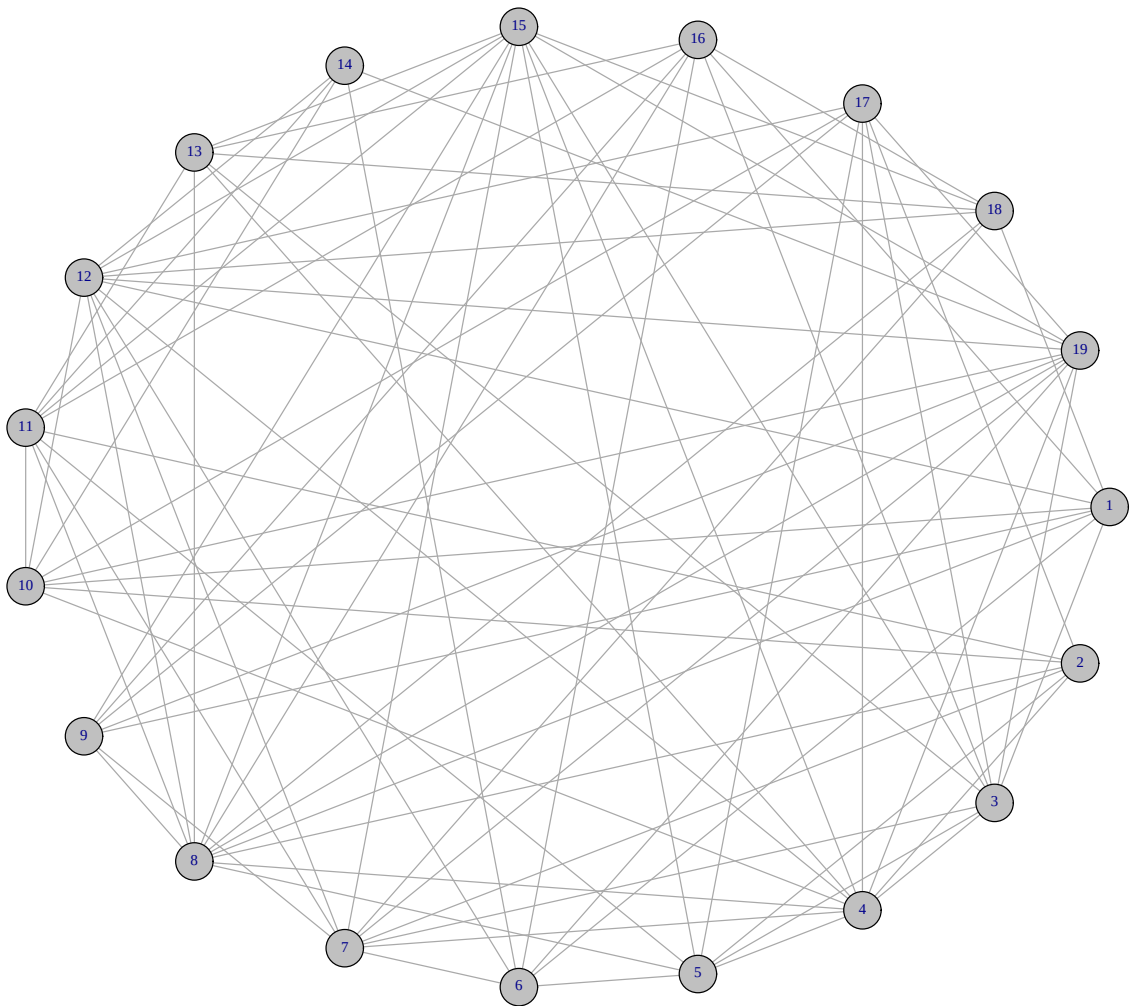


Figure F.5. Counter example: A 19-graph which neither contains 6-dense 13-set nor 6-sparse 14-set.

Counter example: $R_6(14, 14) \geq 23$

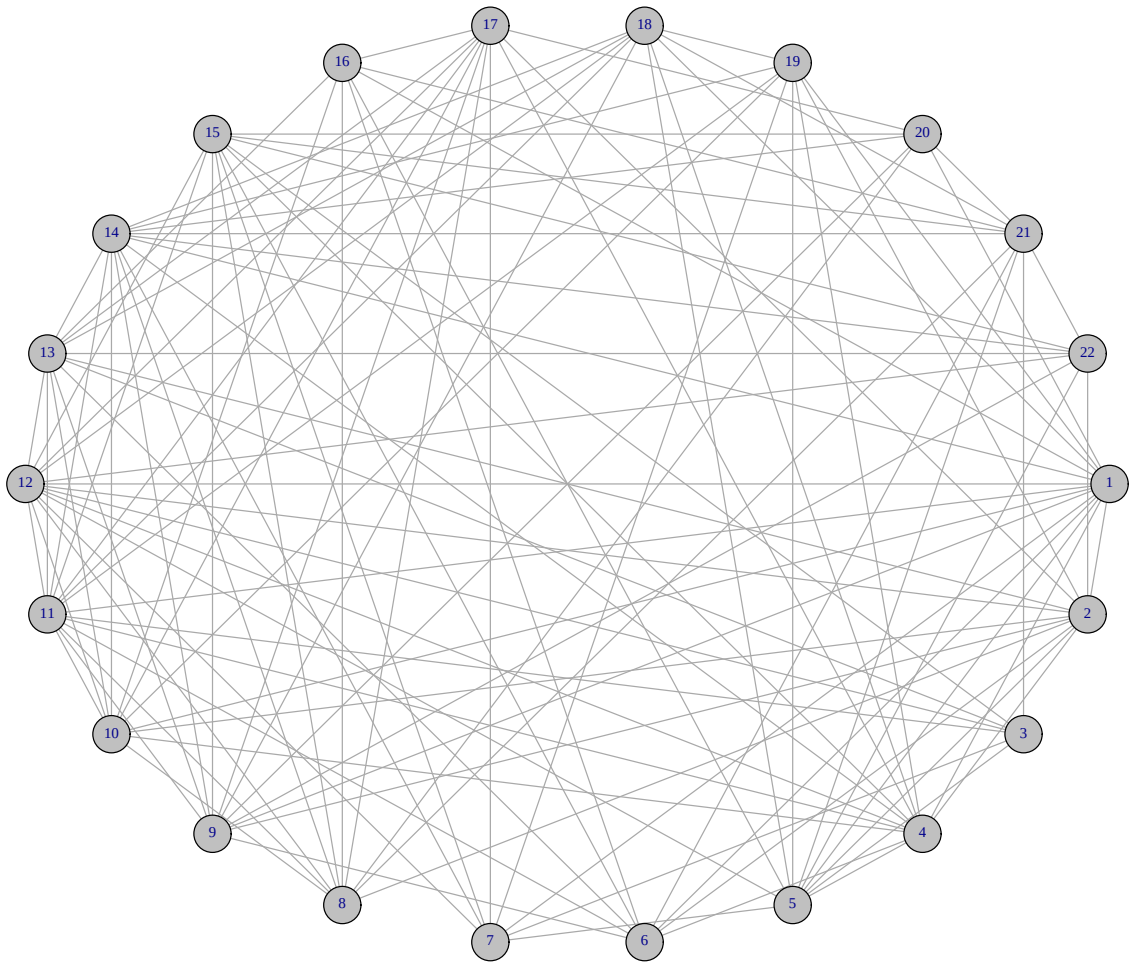


Figure F.6. Counter example: A 22-graph which neither contains 6-dense 14-set nor 6-sparse 14-set.

APPENDIX G: 7 - DEFECTIVE COUNTER EXAMPLE GRAPHS

Counter example: $R_7(12, 12) \geq 18$

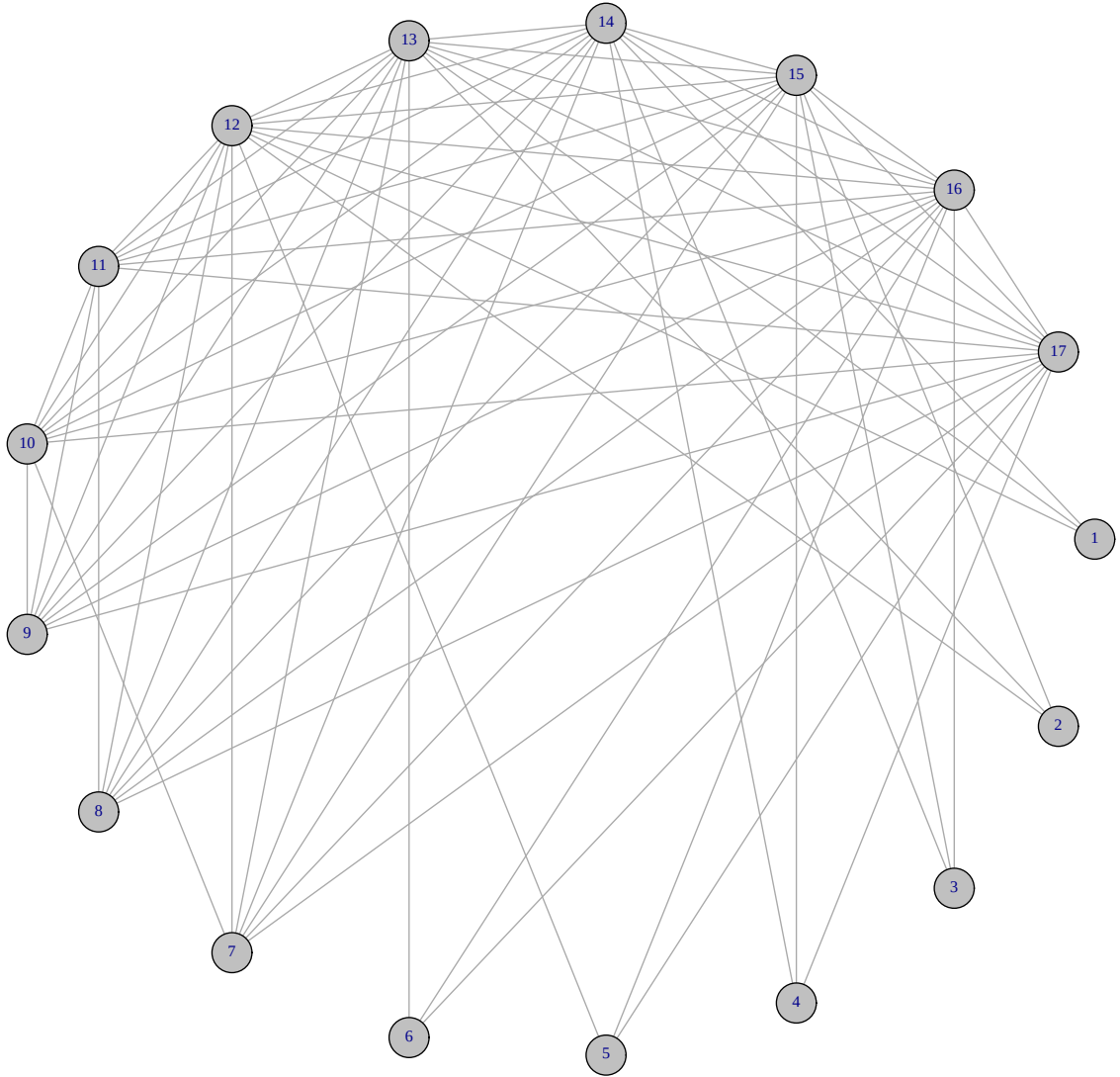


Figure G.1. Counter example: A 17-graph which neither contains 7-dense 12-set nor 7-sparse 12-set.

Counter example: $R_7(12, 13) \geq 19$

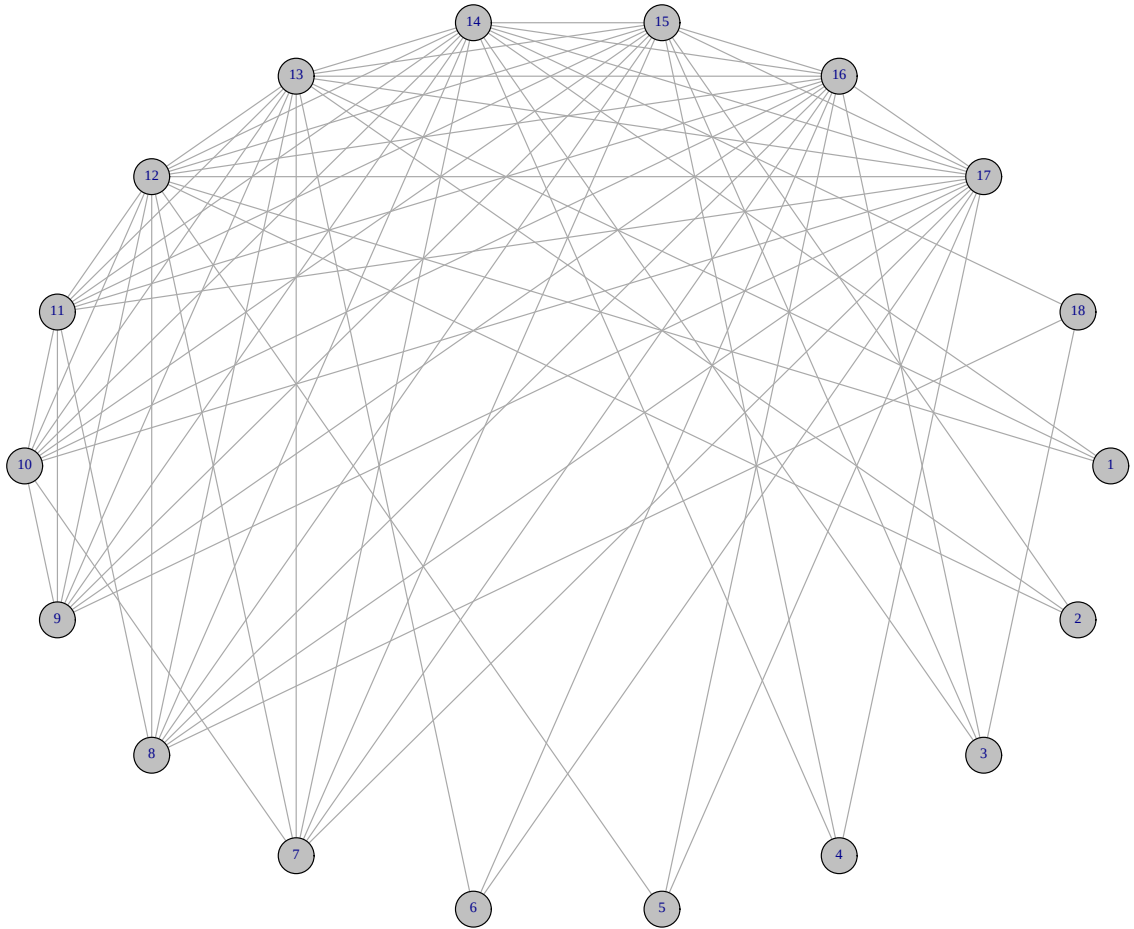


Figure G.2. Counter example: A 18-graph which neither contains 7-dense 12-set nor 7-sparse 13-set.

Counter example: $R_7(13, 13) \geq 19$

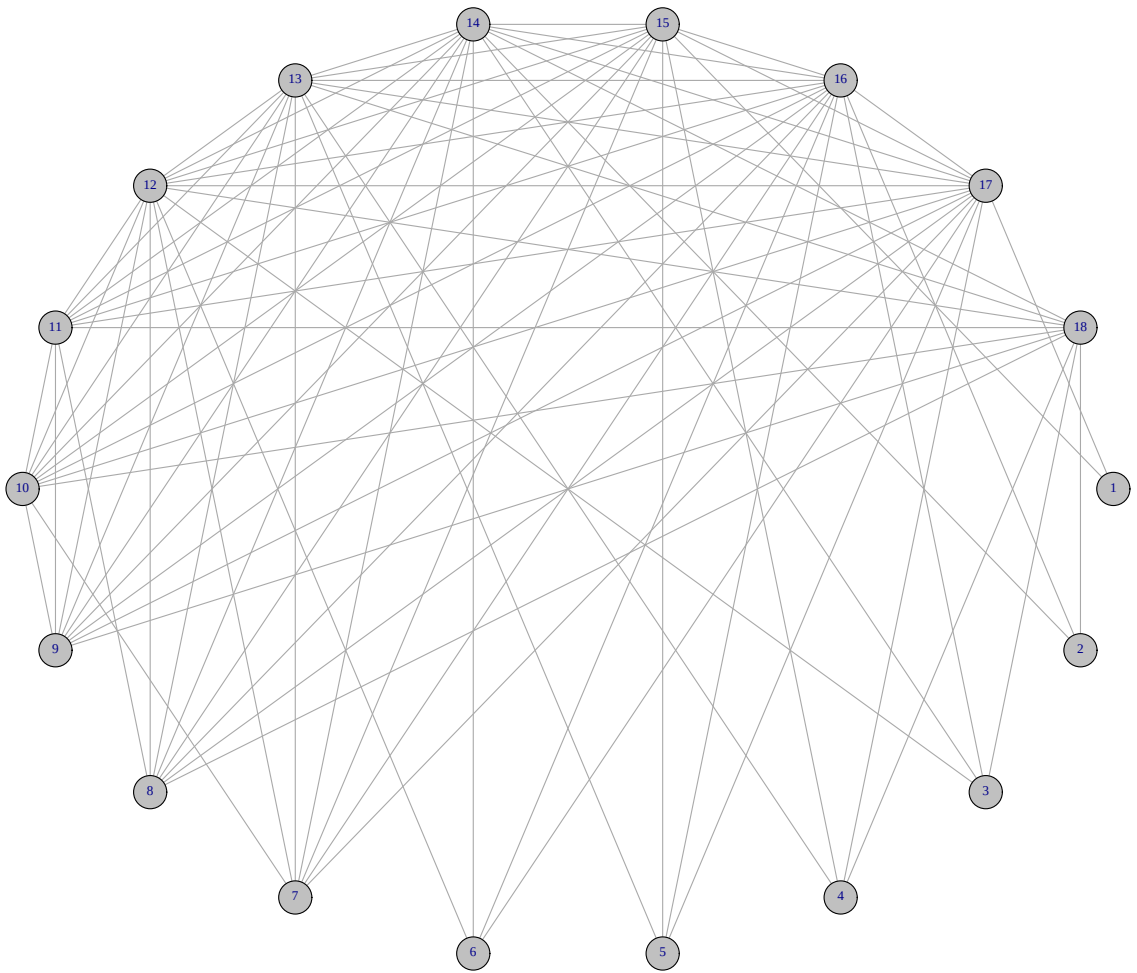


Figure G.3. Counter example: A 18-graph which neither contains 7-dense 13-set nor 7-sparse 13-set.

Counter example: $R_7(14, 14) \geq 20$

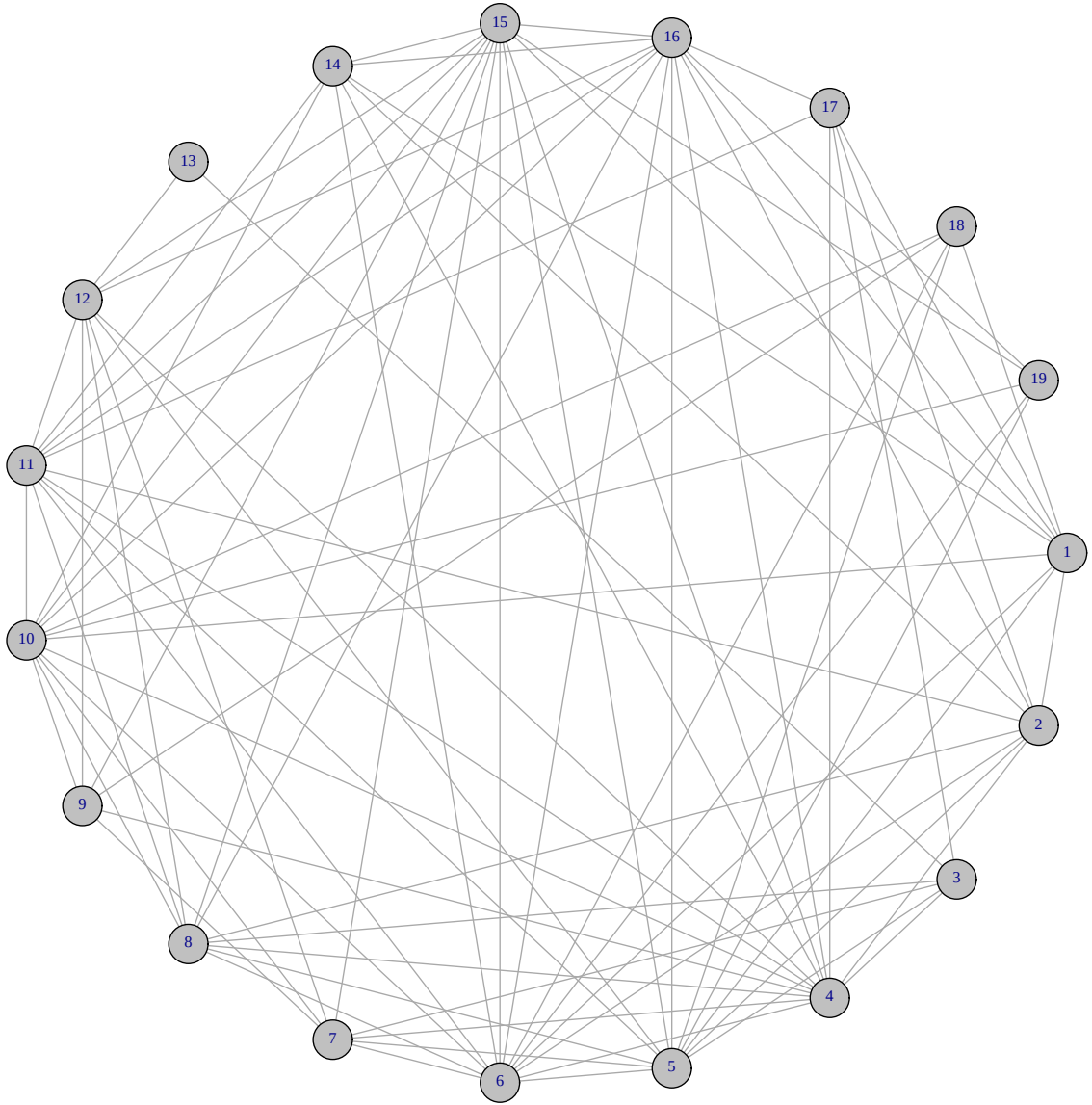


Figure G.4. Counter example: A 19-graph which neither contains 7-dense 14-set nor 7-sparse 14-set.

Counter example: $R_7(15, 15) \geq 22$

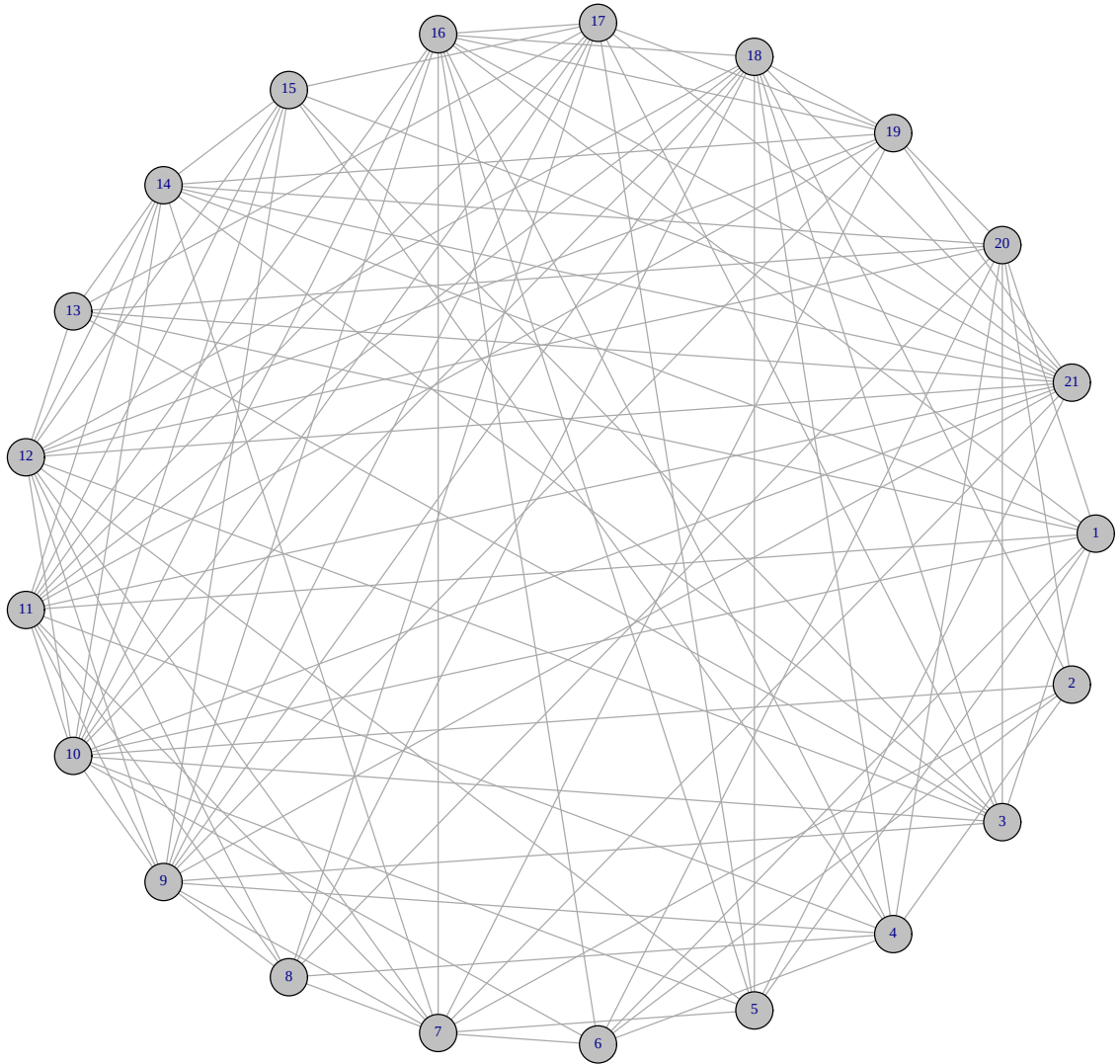


Figure G.5. Counter example: A 21-graph which neither contains 7-dense 15-set nor 7-sparse 15-set.

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