

INHOMOGENEOUS QUANTUM INVARIANCE GROUP OF COMMUTING
FERMIONS

by

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ABSTRACT

INHOMOGENEOUS QUANTUM INVARIANCE GROUP OF COMMUTING FERMIONS

Since the middle of the twentieth century, physicists have concentrated on finding quantum counterparts of classical systems. When a classical system is quantized its invariance group may still be a classical group. In the nineteen-eighties it was shown that when some classical systems are quantized, their classical group becomes a quantum group so that the system is invariant under a quantum group. So quantum groups play an important role in carrying physical properties to the quantum world.

In this thesis we will first review the fermionic and the bosonic algebras and their inhomogeneous invariance quantum groups. Then we will introduce the commuting fermion algebra and construct its inhomogeneous invariance quantum group.

ÖZET

KOMÜTATİF FERMİONLARIN HOMOJEN OLMAYAN KUANTUM DEĞİŞMEZLİK GRUBU

Yirminci yüzyılın ortalarından itibaren fizikçiler klasik sistemlerin kuantum karşılığının bulunmasına yoğunlaşmışlardır. Bir klasik sistem kuantize olduğu zaman onun değişmezlik grubu hala bir klasik grup olabilir. 1980'lerde gösterildi ki, bazı klasik sistemler kuantize olduğu zaman onların klasik grubu bir kuantum grubuna dönüşür. Hem de sistem kuantum grubu altında değişmezdir. Demek ki kuantum grupları fiziksel özellikleri kuantum dünyasına taşımakta önemli bir rol oynarlar.

Bu tezde öncelikle fermionik ve bosonik cebirleri ve onların değişmezlik kuantum gruplarını yeniden inceleyeceğiz. Son olarak da komütatif fermionik cebri takdim edip onun homojen olmayan değişmezlik kuantum grubunu inşa edeceğiz.

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1. INTRODUCTION

Quantum field theory, which describes the behavior of elementary particles and fields, basically depends on the bosonic oscillator which is described by the algebraic relation

$$cc^* - c^*c = 1 \quad (1.1)$$

and the fermionic oscillator is defined by

$$cc^* + c^*c = 1 \quad (1.2)$$

$$c^2 = 0. \quad (1.3)$$

The crucial property of the fermionic oscillator is the Pauli exclusion principle. According to the Pauli exclusion principle the two fermions can not be in the same state. this is the meaning of $c^2 = 0$.

Another important property is that unlike the bosonic oscillator there is no standard classical analogue for the fermionic oscillators. This means that, we can not talk about Bohr's correspondence principle for the fermionic oscillators.

As can be seen from these two obvious properties, quantum algebras are very important inexpressing physical phenomena in the quantum world. Some classical physical systems are invariant under a classical group. There should be quantum counterparts of classical groups. Until the early 80's, it was thought that, when a classical system is quantized, its invariance group remains a classical group. However, in the 1980's, it was recognized that when a completely integrable non-linear system is quantized, its classical group changes into a quantum group[1, 2, 3, 4]. In this thesis

we will address the question of quantum invariance group of fermions and bosons. We will particularly be interested in inhomogeneous quantum groups[5].

1.1. Quantum Invariance Group of Bosons

In this section we would like to review the quantum group invariance of d-bosons[6, 7].

First of all bosonic algebra for d-bosons is defined by

$$c_i c_j - c_j c_i = 0 \quad (1.4)$$

$$c_i c_j^* - c_j^* c_i = \delta_{ij}, \quad i, j = 1, 2, \dots, d. \quad (1.5)$$

If the creation and annihilation operators are transformed by

$$c'_i \rightarrow \alpha_{ij} \otimes c_j + \beta_{ij} \otimes c_j^* + \gamma_i \otimes 1, \quad (1.6)$$

$$c_i^* \rightarrow \alpha_{ij}^* \otimes c_j^* + \beta_{ij}^* \otimes c_j + \gamma_i^* \otimes 1. \quad (1.7)$$

The transformation matrix can be written as

$$M = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & \Gamma \\ 0 & 1 \end{bmatrix}. \quad (1.8)$$

In order to leave bosonic algebra invariant, algebraic properties, which are Equations (1.4) and (1.5) have to be satisfied by transformed creation and annihilation operators; (1.6) and (1.7). As we require invariance of the bosonic algebra, c' and c'^* are used

instead of c and c^* in (1.4) and (1.5) we get

$$[A_{ij}, A_{kl}] = 0 \quad (1.9)$$

$$[A_{ij}, \gamma_k] = 0 \quad (1.10)$$

$$[A_{ij}, \gamma_k^*] = 0 \quad (1.11)$$

and also the constraints

$$[\gamma_i, \gamma_j^*] = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* + \beta_{ik}\beta_{jk}^*, \quad (1.12)$$

$$[\gamma_i, \gamma_j] = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}. \quad (1.13)$$

The space of such matrices is a Hopf algebra, thus it is a quantum group. The coproduct and counit are defined as

$$\Delta(M) = M \otimes M \quad (1.14)$$

or more explicitly

$$\begin{aligned} \Delta(\alpha_{ij}) &= \alpha_{ik} \otimes \alpha_{kj} + \beta_{ik} \otimes \beta_{kj}^* \\ \Delta(\beta_{ij}) &= \alpha_{ik} \otimes \beta_{kj} + \beta_{ik} \otimes \alpha_{kj}^* \\ \Delta(\gamma_i) &= \alpha_{ik} \otimes \gamma_k + \beta_{ik} \otimes \gamma_k^* + \gamma_k \otimes 1, \end{aligned}$$

$$\varepsilon(M) = \mathcal{I} \quad (1.15)$$

or more explicitly

$$\begin{aligned} \varepsilon(\alpha_{ij}) &= \delta_{ij} \\ \varepsilon(\beta_{ij}) &= 0 \\ \varepsilon(\gamma_i) &= 0 \\ \varepsilon(1) &= 1. \end{aligned}$$

The antipode is

$$S(M) = M^{-1} \quad (1.16)$$

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{bmatrix}. \quad (1.17)$$

A^{-1} is an ordinary inverse of A , because all elements of A commute with each other. Also the coproduct, counit and antipode satisfy Hopf Algebra axioms. Note that Δ , ε and S are $*$ -algebra isomorphisms, i.e. for any element x belonging to the Hopf algebra

$$\begin{aligned} \Delta(x^*) &= (\Delta(x))^* \\ \varepsilon(x^*) &= (\varepsilon(x))^* \\ S(x^*) &= (S(x))^*. \end{aligned}$$

Instead of c and c^* , c' and c'^* have been used and it can be seen that the bosonic algebra remains invariant, satisfying the relations and the constraints. It gives us the

inhomogeneous quantum invariance group of bosons. The group is called the Bosonic Inhomogeneous Symplectic group $BISp(2d)$ [6, 7].

1.2. Quantum Invariance Group of Fermions

The fermionic algebra is defined by

$$c_i c_j + c_j c_i = 0 \quad (1.18)$$

$$c_i c_j^* + c_j^* c_i = \delta_{ij} \quad i = j = 1, 2, \dots, d. \quad (1.19)$$

The transformation matrix M is determined by (1.8). Also transformed creation and annihilation operators are defined as (1.6) and (1.7). In order to get the quantum invariance group of fermions, the same method in the previous section is used. Using that procedure we get:

$$[A_{ij}, A_{kl}] = 0 \quad (1.20)$$

$$\{A_{ij}, \gamma_k\} = 0 \quad (1.21)$$

$$\{A_{ij}, \gamma_k^*\} = 0 \quad (1.22)$$

and also constraints are

$$\{\gamma_i, \gamma_j^*\} = \delta_{ij} - \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*, \quad (1.23)$$

$$\{\gamma_i, \gamma_j\} = -\beta_{ik}\alpha_{jk} - \alpha_{ij}\beta_{jk}. \quad (1.24)$$

The coproduct, counit and antipode are defined like in bosonic part. Because of (1.20) to find A^{-1} ordinary inverse rule is used. In addition to these properties, Hopf algebra axioms hold.

As we require invariance of the fermionic algebra, transformed creation and annihilation operators c' and c'^* should be used in (1.18) and (1.19), then one can see that fermionic algebra stays invariant, satisfying the relations and the constraints. So, there is an inhomogeneous quantum invariance group of fermions which is the Fermionic Inhomogeneous Orthogonal group $FIO(2d)$ [6, 7, 8].

2. QUANTUM INVARIANCE GROUP OF TWO COMMUTING FERMIONS

For one fermion case, Equations (1.18) and (1.19) become only

$$c^2 = 0, \tag{2.1}$$

$$cc^* + c^*c = 1. \tag{2.2}$$

However for the case of two or more fermions, Equations (1.18) and (1.19) not only determine the relations of the fermions with themselves, but also determine the relations of the fermions with each other. For the latter case the equations have the form of anti-commutation. In this thesis we will investigate the consequences of defining these equations in the form of a commutation relation. Thus we will have the case the algebra is the commuting fermion algebra.

Commuting fermion algebra is defined by

$$c_i c_j + \sigma_{ij} c_j c_i = 0 \tag{2.3}$$

$$c_i c_j^* + \sigma_{ij} c_j^* c_i = \delta_{ij} \quad i = j = 1, 2, \dots, d. \tag{2.4}$$

$$\sigma_{ij} = \begin{cases} 1 & i = j, \\ -1 & i \neq j. \end{cases} \tag{2.5}$$

As in standard fermion algebra, the Pauli exclusion principle holds. However, relations between two fermions can be defined as a commutation relation. This is the

difference of commuting fermions from standard fermions. For the one fermion case there is no difference between the two fermionic algebras, because any commutation and anti-commutation relations between two fermions can not be defined.

2.1. One Fermion Case

For the one fermion case the algebra does not differ for the case of commuting or anticommuting fermions. However for the sake of completeness we would like to rederive the quantum group invariance of the particle algebra[8].

For one fermion transformed creation and annihilation operators can be defined

$$c' = \alpha \otimes c + \beta \otimes c^* + \gamma \otimes 1, \quad (2.6)$$

$$c^{*\prime} = \beta^* \otimes c + \alpha^* \otimes c^* + \gamma^* \otimes 1 \quad (2.7)$$

and the transformation matrix is

$$M = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.8)$$

The fermionic oscillator algebra is defined like

$$cc^* + c^*c = 1 \quad (2.9)$$

$$c^2 = 0. \quad (2.10)$$

When instead of c and c^* , c' and c'^* is used in (2.9) and (2.10), we get the relations:

$$[\alpha, \beta] = 0 \quad (2.11)$$

$$[\alpha, \alpha^*] = 0 \quad (2.12)$$

$$[\beta, \beta^*] = 0 \quad (2.13)$$

$$\{\alpha^*, \gamma\} = 0 \quad (2.14)$$

$$\{\beta^*, \gamma\} = 0 \quad (2.15)$$

also hermitian conjugates of the relations are valid.

The constraints are

$$\gamma^2 = -\alpha\beta, \quad (2.16)$$

$$\{\gamma, \gamma^*\} = 1 - \alpha\alpha^* - \beta\beta^*. \quad (2.17)$$

As it can be seen from the relations and the constraints for one fermion case, the quantum invariance group of commuting fermion is just $FIO(2)[6]$.

In order to get more information about the quantum invariance group of the commuting fermions, the two commuting fermion case has to be checked.

2.2. Two Fermion Case

For the two fermions, although the procedure is the same, calculations are a bit complicated. In this case transformed operators are

$$c'_1 = \alpha_{11} \otimes c_1 + \alpha_{12} \otimes c_2 + \beta_{11} \otimes c_1^* + \beta_{12} \otimes c_2^* + \gamma_1 \otimes 1 \quad (2.18)$$

$$c_1^{*'} = \beta_{11}^* \otimes c_1 + \beta_{12}^* \otimes c_2 + \alpha_{11}^* \otimes c_1^* + \alpha_{12}^* \otimes c_2^* + \gamma_1^* \otimes 1 \quad (2.19)$$

$$c'_2 = \alpha_{21} \otimes c_1 + \alpha_{22} \otimes c_2 + \beta_{21} \otimes c_1^* + \beta_{22} \otimes c_2^* + \gamma_2 \otimes 1 \quad (2.20)$$

$$c_2^{*'} = \beta_{21}^* \otimes c_1 + \beta_{22}^* \otimes c_2 + \alpha_{21}^* \otimes c_1^* + \alpha_{22}^* \otimes c_2^* + \gamma_2^* \otimes 1 \quad (2.21)$$

and the transformation matrix is

$$M = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} & \gamma_1 \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} & \gamma_2 \\ \beta_{11}^* & \beta_{12}^* & \alpha_{11}^* & \alpha_{12}^* & \gamma_1^* \\ \beta_{21}^* & \beta_{22}^* & \alpha_{21}^* & \alpha_{22}^* & \gamma_2^* \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.22)$$

For the two fermion case commuting fermion algebra is defined as

$$c_1^2 = 0 \quad (2.23)$$

$$c_2^2 = 0 \quad (2.24)$$

$$c_1 c_1^* + c_1^* c_1 = 1 \quad (2.25)$$

$$c_2 c_2^* + c_2^* c_2 = 1 \quad (2.26)$$

$$c_1 c_2 - c_2 c_1 = 0 \quad (2.27)$$

$$c_1 c_2^* - c_2^* c_1 = 0. \quad (2.28)$$

When the transformed operators are used in the equations of commuting fermion algebra, we get;

$$\begin{aligned}
[\alpha_{11}, \alpha_{11}^*] &= 0 & [\alpha_{11}, \beta_{11}^*] &= 0 & \{\alpha_{11}, \gamma_1^*\} &= 0 & \{\alpha_{12}, \beta_{11}\} &= 0 \\
\{\alpha_{11}, \alpha_{12}\} &= 0 & \{\alpha_{11}, \beta_{12}\} &= 0 & [\alpha_{11}, \gamma_2] &= 0 & \{\alpha_{12}, \beta_{11}^*\} &= 0 \\
\{\alpha_{11}, \alpha_{12}^*\} &= 0 & \{\alpha_{11}, \beta_{12}^*\} &= 0 & [\alpha_{11}, \gamma_2^*] &= 0 & [\alpha_{12}, \beta_{12}] &= 0 \\
\{\alpha_{11}, \alpha_{21}\} &= 0 & \{\alpha_{11}, \beta_{21}\} &= 0 & [\alpha_{12}, \alpha_{12}^*] &= 0 & [\alpha_{12}, \beta_{12}^*] &= 0 \\
\{\alpha_{11}, \alpha_{21}^*\} &= 0 & \{\alpha_{11}, \beta_{21}^*\} &= 0 & \{\alpha_{12}, \alpha_{21}\} &= 0 & [\alpha_{12}, \beta_{21}] &= 0 \\
[\alpha_{11}, \alpha_{22}] &= 0 & [\alpha_{11}, \beta_{22}] &= 0 & \{\alpha_{12}, \alpha_{21}^*\} &= 0 & [\alpha_{12}, \beta_{21}^*] &= 0 \\
[\alpha_{11}, \alpha_{22}^*] &= 0 & [\alpha_{11}, \beta_{22}^*] &= 0 & \{\alpha_{12}, \alpha_{22}\} &= 0 & \{\alpha_{12}, \beta_{22}\} &= 0 \\
[\alpha_{11}, \beta_{11}] &= 0 & \{\alpha_{11}, \gamma_1\} &= 0 & \{\alpha_{12}, \alpha_{22}^*\} &= 0 & \{\alpha_{12}, \beta_{22}^*\} &= 0
\end{aligned}$$

$$\begin{aligned}
\{\alpha_{12}, \gamma_1\} &= 0 & \{\alpha_{21}, \beta_{11}^*\} &= 0 & [\alpha_{21}, \gamma_1^*] &= 0 & \{\alpha_{22}, \beta_{21}\} &= 0 \\
\{\alpha_{12}, \gamma_1^*\} &= 0 & [\alpha_{21}, \beta_{12}] &= 0 & \{\alpha_{21}, \gamma_2\} &= 0 & \{\alpha_{22}, \beta_{21}^*\} &= 0 \\
[\alpha_{12}, \gamma_2] &= 0 & [\alpha_{21}, \beta_{12}^*] &= 0 & \{\alpha_{21}, \gamma_2^*\} &= 0 & [\alpha_{22}, \beta_{22}] &= 0 \\
[\alpha_{12}, \gamma_2^*] &= 0 & [\alpha_{21}, \beta_{21}] &= 0 & [\alpha_{22}, \alpha_{22}^*] &= 0 & [\alpha_{22}, \beta_{22}^*] &= 0 \\
[\alpha_{21}, \alpha_{21}^*] &= 0 & [\alpha_{21}, \beta_{21}^*] &= 0 & [\alpha_{22}, \beta_{11}] &= 0 & [\alpha_{22}, \gamma_1] &= 0 \\
\{\alpha_{21}, \alpha_{22}\} &= 0 & \{\alpha_{21}, \beta_{22}\} &= 0 & [\alpha_{22}, \beta_{11}^*] &= 0 & [\alpha_{22}, \gamma_1^*] &= 0 \\
\{\alpha_{21}, \alpha_{22}^*\} &= 0 & \{\alpha_{21}, \beta_{22}^*\} &= 0 & \{\alpha_{22}, \beta_{12}\} &= 0 & \{\alpha_{22}, \gamma_2\} &= 0 \\
\{\alpha_{21}, \beta_{11}\} &= 0 & [\alpha_{21}, \gamma_1] &= 0 & \{\alpha_{22}, \beta_{12}^*\} &= 0 & \{\alpha_{22}, \gamma_2^*\} &= 0
\end{aligned}$$

$$\begin{aligned}
[\beta_{11}, \beta_{11}^*] &= 0 & \{\beta_{11}, \gamma_1^*\} &= 0 & \{\beta_{12}, \gamma_1\} &= 0 & [\beta_{21}, \gamma_1^*] &= 0 \\
\{\beta_{11}, \beta_{12}\} &= 0 & [\beta_{11}, \gamma_2] &= 0 & \{\beta_{12}, \gamma_1^*\} &= 0 & \{\beta_{21}, \gamma_2\} &= 0 \\
\{\beta_{11}^*, \beta_{12}\} &= 0 & [\beta_{11}, \gamma_2^*] &= 0 & [\beta_{12}, \gamma_2] &= 0 & \{\beta_{21}, \gamma_2^*\} &= 0 \\
\{\beta_{11}, \beta_{21}\} &= 0 & [\beta_{12}, \beta_{12}^*] &= 0 & [\beta_{12}, \gamma_2^*] &= 0 & [\beta_{22}, \beta_{22}^*] &= 0 \\
\{\beta_{11}^*, \beta_{21}\} &= 0 & [\beta_{12}, \beta_{21}] &= 0 & [\beta_{21}, \beta_{21}^*] &= 0 & [\beta_{22}, \gamma_1] &= 0 \\
[\beta_{11}, \beta_{22}] &= 0 & [\beta_{12}, \beta_{21}^*] &= 0 & \{\beta_{21}, \beta_{22}\} &= 0 & [\beta_{22}, \gamma_1^*] &= 0 \\
[\beta_{11}, \beta_{22}^*] &= 0 & \{\beta_{12}, \beta_{22}\} &= 0 & \{\beta_{21}, \beta_{22}^*\} &= 0 & \{\beta_{22}, \gamma_2\} &= 0 \\
\{\beta_{11}, \gamma_1\} &= 0 & \{\beta_{12}, \beta_{22}^*\} &= 0 & [\beta_{21}, \gamma_1] &= 0 & \{\beta_{22}, \gamma_2^*\} &= 0
\end{aligned}$$

also hermitian conjugates of the relations are valid. The constraints are

$$\begin{aligned}
\gamma_1^2 &= -\alpha_{11}\beta_{11} - \alpha_{12}\beta_{12} \\
\gamma_2^2 &= -\alpha_{21}\beta_{21} - \alpha_{22}\beta_{22} \\
[\gamma_1, \gamma_2] &= -\alpha_{11}\beta_{21} - \alpha_{12}\beta_{22} - \beta_{11}\alpha_{21} - \beta_{12}\alpha_{22} \\
[\gamma_1, \gamma_2^*] &= -\alpha_{11}\alpha_{21}^* - \alpha_{12}\alpha_{22}^* - \beta_{11}\beta_{21}^* - \beta_{12}\beta_{22}^* \\
\{\gamma_1, \gamma_1^*\} &= 1 - \alpha_{11}\alpha_{11}^* - \alpha_{12}\alpha_{12}^* - \beta_{11}\beta_{11}^* - \beta_{12}\beta_{12}^* \\
\{\gamma_2, \gamma_2^*\} &= 1 - \alpha_{21}\alpha_{21}^* - \alpha_{22}\alpha_{22}^* - \beta_{21}\beta_{21}^* - \beta_{22}\beta_{22}^*
\end{aligned}$$

also hermitian conjugates of the constraints can be found from above. The coproduct and counit are defined as;

$$\Delta(M) = M \dot{\otimes} M \quad (2.29)$$

$$\varepsilon(M) = \mathcal{I} \quad (2.30)$$

and the antipode is

$$S(M) = M^{-1} \quad (2.31)$$

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{bmatrix}. \quad (2.32)$$

However as all elements of matrix A do not commute with each other, ordinary inverse and determinant rules can not be used. First of all determinant of matrix A has to be determined somehow. In order to find $Det(A)$, the block expansion is used.

We modify the block expansion[9]

$$Det(A) = \sum s(-1)^{i+j+k+l} \Delta_{kl}^{ij} \Delta^{other}. \quad (2.33)$$

Where Δ_{kl}^{ij} is the determinant of the matrix whose elements come from i^{th} row, k^{th} column and j^{th} row, l^{th} column. Δ^{other} is the determinant of the matrix whose elements come from rows and columns except i, j and k, l . s is a sign factor which must be included since our matrix elements do not commute.

As Δ_{kl}^{ij} and Δ^{other} look like a two parameter quantum group, determinant rule of Δ_{kl}^{ij} and Δ^{other} are written by the determinant rule of $GL_{p,q}(d)$ [10]. As Δ_{kl}^{ij} and Δ^{other} are two dimensional matrices, we should write the commutation and determinant rules for $GL_{p,q}(2)$.

$$\begin{aligned} ab &= pba \quad ; \quad ac = qca \\ cd &= pdc \quad ; \quad bd = qdc \\ Det(T) &= ad - pbc, \end{aligned} \quad (2.34)$$

where

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (2.35)$$

Using Equation (2.33), we can write the $Det(A)$ as,

$$Det(A) = s [\Delta_{12}^{12} \Delta_{34}^{34} - \Delta_{13}^{12} \Delta_{24}^{34} + \Delta_{14}^{12} \Delta_{23}^{34} + \Delta_{23}^{12} \Delta_{14}^{34} - \Delta_{24}^{12} \Delta_{13}^{34} + \Delta_{34}^{12} \Delta_{12}^{34}]. \quad (2.36)$$

As Δ_{kl}^{ij} and Δ^{other} look like a $GL_{p,q}(2)$, we should define the parameters p and q . For

every Δ_{kl}^{ij} and Δ^{other} pairs we can write the parameters like

$$\begin{aligned}
\Delta_{12}^{12}\Delta_{34}^{34} &\Rightarrow p = -1, q = -1 \\
\Delta_{13}^{12}\Delta_{24}^{34} &\Rightarrow p = 1, q = -1 \\
\Delta_{14}^{12}\Delta_{23}^{34} &\Rightarrow p = -1, q = -1 \\
\Delta_{23}^{12}\Delta_{14}^{34} &\Rightarrow p = -1, q = -1 \\
\Delta_{24}^{12}\Delta_{13}^{34} &\Rightarrow p = 1, q = -1 \\
\Delta_{34}^{12}\Delta_{12}^{34} &\Rightarrow p = -1, q = -1.
\end{aligned} \tag{2.37}$$

Then we can write the terms which are in the square bracket in (2.36)

$$\begin{aligned}
\Delta_{12}^{12}\Delta_{34}^{34} &= (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) \\
\Delta_{13}^{12}\Delta_{24}^{34} &= (\alpha_{11}\beta_{21} - \beta_{11}\alpha_{21})(\beta_{12}^*\alpha_{22}^* - \alpha_{12}^*\beta_{22}^*) \\
\Delta_{14}^{12}\Delta_{23}^{34} &= (\alpha_{11}\beta_{22} + \beta_{12}\alpha_{21})(\beta_{12}^*\alpha_{21}^* + \alpha_{11}^*\beta_{22}^*) \\
\Delta_{23}^{12}\Delta_{14}^{34} &= (\alpha_{12}\beta_{21} + \beta_{11}\alpha_{22})(\beta_{11}^*\alpha_{22}^* + \alpha_{12}^*\beta_{21}^*) \\
\Delta_{24}^{12}\Delta_{13}^{34} &= (\alpha_{12}\beta_{22} - \beta_{12}\alpha_{22})(\beta_{11}^*\alpha_{21}^* - \alpha_{12}^*\beta_{21}^*) \\
\Delta_{34}^{12}\Delta_{12}^{34} &= (\beta_{11}\beta_{22} + \beta_{12}\beta_{21})(\beta_{11}^*\beta_{22}^* + \beta_{12}^*\beta_{21}^*).
\end{aligned} \tag{2.38}$$

Then the sign factor s is defined like

$$s = \begin{cases} -1 & \text{if } \Delta_{kl}^{ij}\Delta^{other} \text{ is not the form of pure } \alpha \text{ or } \beta \\ 1 & \text{if } \Delta_{kl}^{ij}\Delta^{other} \text{ is the form of pure } \alpha \text{ or } \beta \end{cases} \tag{2.39}$$

Using (2.38) and (2.39) in (2.36), $Det(A)$ can be written;

$$\begin{aligned}
Det(A) &= (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) \\
&+ (\alpha_{11}\beta_{21} - \beta_{11}\alpha_{21})(\beta_{12}^*\alpha_{22}^* - \alpha_{12}^*\beta_{22}^*) \\
&- (\alpha_{11}\beta_{22} + \beta_{12}\alpha_{21})(\beta_{12}^*\alpha_{21}^* + \alpha_{11}^*\beta_{22}^*)
\end{aligned}$$

$$\begin{aligned}
& - (\alpha_{12}\beta_{21} + \beta_{11}\alpha_{22})(\beta_{11}^*\alpha_{22}^* + \alpha_{12}^*\beta_{21}^*) \\
& + (\alpha_{12}\beta_{22} - \beta_{12}\alpha_{22})(\beta_{11}^*\alpha_{21}^* - \alpha_{11}^*\beta_{21}^*) \\
& + (\beta_{11}\beta_{22} + \beta_{12}\beta_{21})(\beta_{11}^*\beta_{22}^* + \beta_{12}^*\beta_{21}^*).
\end{aligned} \tag{2.40}$$

Also it should be noted that $Det(A)$ is central, which means $Det(A)$ commutes with all elements of matrix A .

The inverse matrix is defined by using (2.40)

$$A^{-1} = \frac{1}{Det(A)} \begin{bmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} & \Delta_{33} \\ \Delta_{14} & \Delta_{24} & \Delta_{34} & \Delta_{34} \end{bmatrix}. \tag{2.41}$$

$$\begin{aligned}
\Delta_{11} &= \alpha_{22}(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) + \beta_{21}(\beta_{12}^*\alpha_{22}^* - \alpha_{12}^*\beta_{22}^*) \\
&\quad - \beta_{22}(\beta_{12}^*\alpha_{21}^* + \alpha_{11}^*\beta_{22}^*), \\
\Delta_{12} &= \alpha_{21}(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) - \beta_{21}(\beta_{11}^*\alpha_{22}^* + \alpha_{12}^*\beta_{21}^*) \\
&\quad + \beta_{22}(\beta_{11}^*\alpha_{21}^* - \alpha_{11}^*\beta_{21}^*), \\
\Delta_{13} &= -\alpha_{21}(\beta_{12}^*\alpha_{22}^* - \alpha_{12}^*\beta_{22}^*) - \alpha_{22}(\beta_{11}^*\alpha_{22}^* + \alpha_{12}^*\beta_{21}^*) \\
&\quad + \beta_{22}(\beta_{11}^*\beta_{22}^* + \beta_{12}^*\beta_{21}^*), \\
\Delta_{14} &= -\alpha_{21}(\beta_{12}^*\alpha_{21}^* + \alpha_{11}^*\beta_{22}^*) - \alpha_{22}(\beta_{11}^*\alpha_{21}^* - \alpha_{11}^*\beta_{21}^*) \\
&\quad + \beta_{21}(\beta_{11}^*\beta_{22}^* + \beta_{12}^*\beta_{21}^*), \\
\Delta_{21} &= \alpha_{12}(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) + \beta_{11}(\beta_{12}^*\alpha_{22}^* - \alpha_{12}^*\beta_{22}^*) \\
&\quad - \beta_{12}(\beta_{12}^*\alpha_{21}^* + \alpha_{11}^*\beta_{22}^*), \\
\Delta_{22} &= \alpha_{11}(\alpha_{11}^*\alpha_{22}^* + \alpha_{12}^*\alpha_{21}^*) - \beta_{11}(\beta_{11}^*\alpha_{22}^* + \alpha_{12}^*\beta_{21}^*) \\
&\quad + \beta_{12}(\beta_{11}^*\alpha_{21}^* - \alpha_{11}^*\beta_{21}^*),
\end{aligned}$$

$$\begin{aligned}
\Delta_{23} &= -\alpha_{11}(\beta_{12}^* \alpha_{22}^* - \alpha_{12}^* \beta_{22}^*) - \alpha_{12}(\beta_{11}^* \alpha_{22}^* + \alpha_{12}^* \beta_{21}^*) \\
&\quad + \beta_{12}(\beta_{11}^* \beta_{22}^* + \beta_{12}^* \beta_{21}^*), \\
\Delta_{24} &= -\alpha_{11}(\beta_{12}^* \alpha_{21}^* + \alpha_{11}^* \beta_{22}^*) - \alpha_{12}(\beta_{11}^* \alpha_{21}^* - \alpha_{11}^* \beta_{21}^*) \\
&\quad + \beta_{11}(\beta_{11}^* \beta_{22}^* + \beta_{12}^* \beta_{21}^*), \\
\Delta_{31} &= -\alpha_{12}(\beta_{21} \alpha_{22}^* + \beta_{22} \alpha_{21}^*) - \beta_{11}(\alpha_{22} \alpha_{22}^* - \beta_{22} \beta_{22}^*) \\
&\quad + \beta_{12}(\alpha_{22} \alpha_{21}^* + \beta_{21} \beta_{22}^*), \\
\Delta_{32} &= -\alpha_{11}(\beta_{21} \alpha_{22}^* + \beta_{22} \alpha_{21}^*) + \beta_{11}(\alpha_{21} \alpha_{22}^* + \beta_{22} \beta_{21}^*) \\
&\quad - \beta_{12}(\alpha_{21} \alpha_{21}^* - \beta_{21} \beta_{21}^*), \\
\Delta_{33} &= \alpha_{11}(\alpha_{22} \alpha_{22}^* - \beta_{22} \beta_{22}^*) + \alpha_{12}(\alpha_{21} \alpha_{22}^* + \beta_{22} \beta_{21}^*) \\
&\quad - \beta_{12}(\alpha_{21} \beta_{22}^* + \alpha_{22} \beta_{12}^*), \\
\Delta_{34} &= \alpha_{11}(\alpha_{22} \alpha_{21}^* + \beta_{21} \beta_{22}^*) + \alpha_{12}(\alpha_{21} \alpha_{21}^* - \beta_{21} \beta_{21}^*) \\
&\quad - \beta_{11}(\alpha_{21} \beta_{22}^* + \alpha_{22} \beta_{21}^*), \\
\Delta_{41} &= -\alpha_{12}(\beta_{21} \alpha_{12}^* + \beta_{22} \alpha_{11}^*) - \beta_{11}(\alpha_{22} \alpha_{12}^* - \beta_{22} \beta_{12}^*) \\
&\quad + \beta_{12}(\alpha_{22} \alpha_{11}^* + \beta_{21} \beta_{12}^*), \\
\Delta_{42} &= -\alpha_{11}(\beta_{21} \alpha_{12}^* + \beta_{22} \alpha_{11}^*) + \beta_{11}(\alpha_{21} \alpha_{12}^* + \beta_{22} \beta_{11}^*) \\
&\quad - \beta_{12}(\alpha_{21} \alpha_{11}^* - \beta_{21} \beta_{12}^*), \\
\Delta_{43} &= \alpha_{11}(\alpha_{22} \alpha_{12}^* - \beta_{22} \beta_{12}^*) + \alpha_{12}(\alpha_{21} \alpha_{12}^* + \beta_{22} \beta_{11}^*) \\
&\quad - \beta_{12}(\alpha_{21} \beta_{12}^* + \alpha_{22} \beta_{11}^*), \\
\Delta_{44} &= \alpha_{11}(\alpha_{22} \alpha_{11}^* + \beta_{21} \beta_{12}^*) + \alpha_{12}(\alpha_{21} \alpha_{11}^* - \beta_{21} \beta_{11}^*) \\
&\quad - \beta_{11}(\alpha_{21} \beta_{12}^* + \alpha_{22} \beta_{11}^*).
\end{aligned}$$

From the definitions antipode can be found. Moreover antipode, counit and comultiplication satisfy Hopf Algebra axioms.

3. GENERALIZATION TO d-FERMION CASE

In the previous chapter we have studied the one and two fermion case. As it can be seen from the two fermion case, the inhomogeneous quantum invariance group of commuting fermions is not $FIO(4)$. However, in order to say that there is a new quantum group, there should be a generalization.

The transformed creation and annihilation operators have been defined before. The transformation matrix M is written in a shorthand notation

$$M = \begin{bmatrix} A & \Gamma \\ 0 & 1 \end{bmatrix}. \quad (3.1)$$

We want the elements of the matrix M to belong to a Hopf algebra where co-product is given by matrix multiplication

$$\Delta(M) = M \dot{\otimes} M. \quad (3.2)$$

The counit is

$$\varepsilon(M) = \mathcal{I} \quad (3.3)$$

and the antipode is

$$S(M) = M^{-1} \quad (3.4)$$

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{bmatrix}. \quad (3.5)$$

As all the elements of A do not commute with each other, we can not use the ordinary inverse rule. We should first find the determinant rule. For the two fermion case we have found that

$$Det(A) = \sum s(-1)^{i+j+k+l} \Delta_{kl}^{ij} \Delta^{other}. \quad (3.6)$$

However, this formula is not true. When we try it for the 3 fermion case, it has been seen that we can not get the proper determinant of A . So there should be another formalism which is true for the d -fermion case. Starting from a determinant rule which has been constructed to calculate the determinant in $GL_{p,q}(d)[10]$, the determinant rule has been found as

$$Det(A) = \sum \left[\prod (-\sigma_{i_\alpha i_\beta})(-\sigma_{j_\alpha j_\beta}) \right] A_{i_1}^{j_1} A_{i_2}^{j_2} \dots A_{i_{2d}}^{j_{2d}}. \quad (3.7)$$

The summation is over all permutations i_1, i_2, \dots, i_{2d} of j_1, j_2, \dots, j_{2d} , $i_\alpha i_\beta$ and $j_\alpha j_\beta$ should be written modulo d . The σ -symbol is the same as in (2.5). Let us explain the term which is in the square bracket. For example set $i_1 = 1, i_2 = 2, i_3 = 3$ and $i_4 = 4$. As there is not any permutation the term which is in the square bracket is 1. If we take $i_1 = 3, i_2 = 2, i_3 = 1$ and $i_4 = 4$, in order to get this permutation there are three transpositions with respect to the case $i_1 = 1, i_2 = 2, i_3 = 3$ and $i_4 = 4$. As 1 is transposed with 2 and 3, the factors $(-\sigma_{12})$ and $(-\sigma_{11})$ appear. Moreover as 2 is transposed with 3, the factor $(-\sigma_{21})$ appears. We have to be careful about the indices of σ , because indices should be written modulo d .

Using the determinant rule (3.7) antipode can be found.

$$A^{-1} = \frac{1}{\text{Det}(A)} \begin{bmatrix} \Delta_{11} & \Delta_{21} & \dots & \Delta_{2d1} \\ \Delta_{12} & \Delta_{22} & \dots & \Delta_{2d2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \Delta_{12d} & \dots & \dots & \Delta_{2d2d} \end{bmatrix}. \quad (3.8)$$

The relations can be defined

$$A_i^j A_k^j = \sigma_{ik} A_k^j A_i^j, \quad (3.9)$$

$$A_i^j A_i^l = \sigma_{jl} A_i^l A_i^j, \quad (3.10)$$

$$A_i^j A_k^l = \frac{\sigma_{jl}}{\sigma_{ik}} A_k^l A_i^j, \quad (3.11)$$

$$A_i^j \Gamma^l = -\sigma_{jl} \Gamma^l A_i^j. \quad (3.12)$$

In general the indices of the σ -symbol refer to the indices i and k modulo d . More explicitly

$$\begin{aligned} \sigma_{ij} &= \sigma_{i+d,j} \\ &= \sigma_{i,j+d} \\ &= \sigma_{i+d,j+d} \\ &= \begin{cases} 1 & i = j, \\ -1 & i \neq j. \end{cases} \end{aligned}$$

Also the constraints are

$$\gamma_i \gamma_j + \sigma_{ij} \gamma_j \gamma_i = -\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}, \quad (3.13)$$

$$\gamma_i \gamma_j^* + \sigma_{ij} \gamma_j^* \gamma_i = \delta_{ij} - \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*. \quad (3.14)$$

In the d -fermion case, counit, comultiplication and the antipode satisfy Hopf alge-

bra axioms. We have shown that the inhomogeneous quantum invariance group of commuting fermions is different from FIO(2d). We will call this new quantum group CoFIO(2d).

As an explicit example we explicitly show our results for CoFIO(4) and show that they are the same as in Section (2.2). First of all, we will start to check the constraints which are defined by Equations (3.13) and (3.14).

Taking $i=1$ and $j=1$ in equation (3.13) we get

$$\gamma_1^2 + \sigma_{11}\gamma_1^2 = -\alpha_{11}\beta_{11} - \beta_{11}\alpha_{11} - \alpha_{12}\beta_{12} - \beta_{12}\alpha_{12}, \quad (3.15)$$

as $\sigma_{11} = 1$ the result is

$$\gamma_1^2 = -\alpha_{11}\beta_{11} - \beta_{12}\alpha_{12} .$$

Taking $i=2$ and $j=2$ in Equation (3.13) we get

$$\gamma_2^2 + \sigma_{22}\gamma_2^2 = -\alpha_{21}\beta_{21} - \beta_{21}\alpha_{21} - \alpha_{22}\beta_{22} - \beta_{22}\alpha_{22}, \quad (3.16)$$

as $\sigma_{22} = 1$ the result is

$$\gamma_2^2 = -\alpha_{21}\beta_{21} - \beta_{22}\alpha_{22} .$$

Taking $i = 1$ and $j = 2$ in Equation (3.13) we get

$$\gamma_1\gamma_2 + \sigma_{12}\gamma_2\gamma_1 = -\alpha_{11}\beta_{21} - \alpha_{12}\beta_{22} - \beta_{11}\alpha_{21} - \beta_{12}\alpha_{22}, \quad (3.17)$$

as $\sigma_{12} = -1$ the result is

$$[\gamma_1, \gamma_2] = -\alpha_{11}\beta_{21} - \alpha_{12}\beta_{22} - \beta_{11}\alpha_{21} - \beta_{12}\alpha_{22} .$$

When $i = 1$ and $j = 1$ in Equation (3.14) we get

$$\gamma_1 \gamma_1^* + \sigma_{11} \gamma_1^* \gamma_1 = \delta_{11} - \alpha_{11} \alpha_{11}^* - \alpha_{12} \alpha_{12}^* - \beta_{11} \beta_{11}^* - \beta_{12} \beta_{12}^* . \quad (3.18)$$

It can be easily seen that

$$\{\gamma_1, \gamma_1^*\} = 1 - \alpha_{11} \alpha_{11}^* - \alpha_{12} \alpha_{12}^* - \beta_{11} \beta_{11}^* - \beta_{12} \beta_{12}^* .$$

When $i=2$ and $j=2$ in Equation (3.14) we get

$$\gamma_2 \gamma_2^* + \sigma_{22} \gamma_2^* \gamma_2 = \delta_{22} - \alpha_{21} \alpha_{21}^* - \alpha_{22} \alpha_{22}^* - \beta_{21} \beta_{21}^* - \beta_{22} \beta_{22}^* . \quad (3.19)$$

It can be easily seen that

$$\{\gamma_2, \gamma_2^*\} = 1 - \alpha_{21} \alpha_{21}^* - \alpha_{22} \alpha_{22}^* - \beta_{21} \beta_{21}^* - \beta_{22} \beta_{22}^* .$$

When $i = 1$ and $j = 2$ in Equation (3.14) we get

$$\gamma_1 \gamma_2^* + \sigma_{12} \gamma_2^* \gamma_1 = \delta_{12} - \alpha_{11} \alpha_{21}^* - \alpha_{12} \alpha_{22}^* - \beta_{11} \beta_{21}^* - \beta_{12} \beta_{22}^* . \quad (3.20)$$

It can be easily seen that

$$[\gamma_1, \gamma_2^*] = -\alpha_{11} \alpha_{21}^* - \alpha_{12} \alpha_{22}^* - \beta_{11} \beta_{21}^* - \beta_{12} \beta_{22}^* .$$

One can see that the constraints are the same constraints with the ones in the Section (2.2). We will check the relations. However as the relations are too many to check step by step, only some of them will be shown here,

$$A_1^1 A_1^2 = \sigma_{jl} A_1^2 A_1^1$$

$$j = 1, l = 2 \Rightarrow \sigma_{12} = -1$$

$$A_1^1 A_1^2 = -A_1^2 A_1^1 \Rightarrow \alpha_{11} \alpha_{12} = -\alpha_{12} \alpha_{11}$$

$$A_2^3 A_1^4 = \frac{\sigma_{jl}}{\sigma_{ik}} A_2^3 A_1^4$$

$$i = 2, \text{ as } 3 = 1 \pmod{2} \quad j = 1, k = 1, l = 2 \Rightarrow \sigma_{12} = -1, \sigma_{21} = -1$$

$$A_2^3 A_1^4 = A_2^3 A_1^4 \Rightarrow \beta_{12}^* \beta_{21}^* = \beta_{21}^* \beta_{12}^*$$

$$A_1^3 A_4^3 = \sigma_{ik} A_1^3 A_4^3$$

$$i = 1, \text{ as } 3 = 1 \pmod{2} \quad k = 1 \Rightarrow \sigma_{12} = -1$$

$$A_1^3 A_4^3 = A_1^3 A_4^3 \Rightarrow \beta_{11}^* \alpha_{12}^* = -\alpha_{12}^* \beta_{11}^*$$

$$A_4^3 \Gamma^1 = -\sigma_{jl} \Gamma^1 A_4^3$$

$$j = 1, l = 1 \Rightarrow \sigma_{11} = 1$$

$$A_4^3 \Gamma^1 = -\Gamma^1 A_4^3 \Rightarrow \alpha_{12}^* \gamma_1 = -\gamma_1 \alpha_{12}^*$$

$$A_3^2 \Gamma^3 = -\sigma_{jl} \Gamma^3 A_3^2$$

$$j = 2, l = 1 \Rightarrow \sigma_{12} = -1$$

$$A_3^2 \Gamma^3 = \Gamma^3 A_3^2 \Rightarrow \beta_{21} \gamma_1^* = \gamma_1^* \beta_{21}$$

As in the two fermion case number of the relations is 192, we will not check all of them in here. One can find all the relations using Equations (3.9), (3.10), (3.11) and (3.12). However these five examples of the relations cover all situations.

We have already checked Equations (3.9), (3.10), (3.11), (3.12), (3.13) and (3.14). We see that these equations satisfy the relations and the constraints. The last check is the determinant rule which is defined by (3.7). The determinant rule is

$$Det(A) = \sum \left[\prod (-\sigma_{i_\alpha i_\beta}) (-\sigma_{j_\alpha j_\beta}) \right] A_{i_1}^{j_1} A_{i_2}^{j_2} \dots A_{i_{2d}}^{j_{2d}}.$$

The determinant can be written

$$\begin{aligned}
\text{Det}(A) = & A_1^1 A_2^2 A_3^3 A_4^4 + (-\sigma_{12}) A_1^1 A_2^2 A_4^3 A_3^4 + (-\sigma_{21}) A_1^1 A_3^2 A_2^3 A_4^4 \\
& + (-\sigma_{21})(-\sigma_{22}) A_1^1 A_3^2 A_4^3 A_2^4 + (-\sigma_{21})(-\sigma_{22})(-\sigma_{12}) A_1^1 A_4^2 A_3^3 A_2^4 \\
& + (-\sigma_{22})(-\sigma_{12}) A_1^1 A_4^2 A_2^3 A_3^4 + (-\sigma_{12}) A_2^1 A_1^2 A_3^3 A_4^4 + (-\sigma_{12})(-\sigma_{12}) A_2^1 A_1^2 A_4^3 A_3^4 \\
& + (-\sigma_{12})(-\sigma_{11}) A_2^1 A_3^2 A_1^3 A_4^4 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12}) A_2^1 A_3^2 A_4^3 A_1^4 \\
& + (-\sigma_{12})(-\sigma_{12})(-\sigma_{12}) A_2^1 A_4^2 A_1^3 A_3^4 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{12}) A_2^1 A_4^2 A_3^3 A_1^4 \\
& + (-\sigma_{11})(-\sigma_{21}) A_3^1 A_1^2 A_2^3 A_4^4 + (-\sigma_{11})(-\sigma_{21})(-\sigma_{22}) A_3^1 A_1^2 A_4^3 A_2^4 \\
& + (-\sigma_{12})(-\sigma_{11})(-\sigma_{21}) A_3^1 A_2^2 A_1^3 A_4^4 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{21}) A_3^1 A_2^2 A_4^3 A_1^4 \\
& + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22}) A_3^1 A_4^2 A_2^3 A_1^4 \\
& + (-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22}) A_3^1 A_4^2 A_1^3 A_2^4 \\
& + (-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22})(-\sigma_{12}) A_4^1 A_3^2 A_1^3 A_2^4 \\
& + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22})(-\sigma_{12}) A_4^1 A_3^2 A_2^3 A_1^4 \\
& + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{22})(-\sigma_{12}) A_4^1 A_2^2 A_3^3 A_1^4 \\
& + (-\sigma_{21})(-\sigma_{12})(-\sigma_{12})(-\sigma_{12}) A_4^1 A_2^2 A_1^3 A_3^4 + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12}) A_4^1 A_1^2 A_2^3 A_3^4 \\
& + (-\sigma_{12})(-\sigma_{21})(-\sigma_{22})(-\sigma_{12}) A_4^1 A_1^2 A_3^3 A_2^4 .
\end{aligned}$$

Using the property of σ_{ij} , we finally write

$$\begin{aligned}
\text{Det}(A) = & \alpha_{11} \alpha_{22} \alpha_{11}^* \alpha_{22}^* + \alpha_{11} \alpha_{22} \alpha_{12}^* \alpha_{21}^* + \alpha_{11} \beta_{21} \beta_{12}^* \alpha_{22}^* - \alpha_{11} \beta_{21} \alpha_{12}^* \beta_{22}^* \\
& - \alpha_{11} \beta_{22} \alpha_{11}^* \beta_{22}^* - \alpha_{11} \beta_{22} \beta_{12}^* \alpha_{21}^* + \alpha_{12} \alpha_{21} \alpha_{11}^* \alpha_{22}^* + \alpha_{12} \alpha_{21} \alpha_{12}^* \alpha_{21}^* \\
& - \alpha_{12} \beta_{21} \beta_{11}^* \alpha_{22}^* - \alpha_{12} \beta_{21} \alpha_{12}^* \beta_{21}^* + \alpha_{12} \beta_{22} \beta_{11}^* \alpha_{21}^* - \alpha_{12} \beta_{22} \alpha_{11}^* \beta_{21}^* \\
& - \beta_{11} \alpha_{21} \beta_{12}^* \alpha_{22}^* + \beta_{11} \alpha_{21} \alpha_{12}^* \beta_{22}^* - \beta_{11} \alpha_{22} \beta_{11}^* \alpha_{22}^* - \beta_{11} \alpha_{22} \alpha_{12}^* \beta_{21}^* \\
& + \beta_{11} \beta_{22} \beta_{12}^* \beta_{21}^* + \beta_{11} \beta_{22} \beta_{11}^* \beta_{22}^* + \beta_{12} \beta_{21} \beta_{11}^* \beta_{22}^* + \beta_{12} \beta_{21} \beta_{12}^* \beta_{21}^* \\
& + \beta_{12} \alpha_{22} \alpha_{11}^* \beta_{21}^* - \beta_{12} \alpha_{22} \beta_{11}^* \alpha_{21}^* - \beta_{12} \alpha_{21} \beta_{12}^* \alpha_{21}^* - \beta_{12} \alpha_{12} \alpha_{11}^* \beta_{22}^* .
\end{aligned}$$

We can write the determinant in the form of $\text{Det}(A) = \alpha_{11} \Delta_{11} + \alpha_{12} \Delta_{12} + \beta_{11} \Delta_{13} + \beta_{12} \Delta_{14}$. As an example let us find Δ_{11} . Set $j_1 = 1, j_2 = 2, j_3 = 3, j_4 = 4$ and $i_1 = 1$.

So the determinant rule becomes

$$Det(A) = \sum \left[\prod (-\sigma_{i_\alpha i_\beta})(-\sigma_{j_\alpha j_\beta}) \right] A_1^1 A_2^2 A_3^3 A_4^4. \quad (3.21)$$

Using the properties of $(-\sigma_{i_\alpha i_\beta})(-\sigma_{j_\alpha j_\beta})$, it can be found that

$$\begin{aligned} \Delta_{11} &= \alpha_{22}(\alpha_{11}^* \alpha_{22}^* + \alpha_{12}^* \alpha_{21}^*) + \beta_{21}(\beta_{12}^* \alpha_{22}^* - \alpha_{12}^* \beta_{22}^*) \\ &\quad - \beta_{22}(\beta_{12}^* \alpha_{21}^* + \alpha_{11}^* \beta_{22}^*). \end{aligned}$$

Another example is Δ_{23} . Set $j_1 = 2, j_2 = 1, j_3 = 3, j_4 = 4$ and $i_1 = 3$. As there is one transposition in j $(-\sigma_{j_1 j_2}) = -\sigma_{21} = 1$ and we start the permutation $i_1 = 3, i_2 = 1, i_3 = 2, i_4 = 4$. So we can write that

$$\Delta_{23} = \sum \left[\prod (-\sigma_{i_\alpha i_\beta}) \right] A_{i_2}^1 A_{i_3}^3 A_{i_4}^4. \quad (3.22)$$

We write

$$\begin{aligned} \Delta_{23} &= (-\sigma_{12})(-\sigma_{11})A_1^1 A_2^3 A_4^4 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{22})A_1^1 A_4^3 A_2^4 \\ &\quad + (-\sigma_{12})(-\sigma_{11})(-\sigma_{21})A_2^1 A_1^3 A_4^4 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{21})(-\sigma_{12})A_2^1 A_4^3 A_1^4 \\ &\quad + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22})A_4^1 A_2^3 A_1^4 \\ &\quad + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{21})(-\sigma_{22})(-\sigma_{12})A_4^1 A_1^3 A_2^4. \end{aligned}$$

Finally, using the property of $(-\sigma_{i_\alpha i_\beta})$ we can write

$$\begin{aligned} \Delta_{23} &= -\alpha_{11}(\beta_{12}^* \alpha_{22}^* - \alpha_{12}^* \beta_{22}^*) - \alpha_{12}(\beta_{11}^* \alpha_{22}^* + \alpha_{12}^* \beta_{21}^*) \\ &\quad + \beta_{12}(\beta_{11}^* \beta_{22}^* + \beta_{12}^* \beta_{21}^*). \end{aligned}$$

As it can be seen from these examples, the determinant rule gives us the same results in Section(2.2). If the Equation (3.7) is true, one can find the determinant by expanding along any row. For example, set $j_1 = 2, j_2 = 1, j_3 = 4$ and $j_4 = 3$. As there are two transpositions with respect to the case $j_1 = 1, j_2 = 2, j_3 = 3$ and $j_4 = 4$,

$(-\sigma_{12})(-\sigma_{12}) = 1$. We can write that

$$Det(A) = \sum \left[\prod (-\sigma_{i\alpha i\beta}) \right] A_{i_1}^2 A_{i_2}^1 A_{i_3}^4 A_{i_4}^3.$$

$$\begin{aligned} Det(A) = & A_1^2 A_2^1 A_3^4 A_4^3 + (-\sigma_{12}) A_1^2 A_2^1 A_4^4 A_3^3 + (-\sigma_{12}) A_1^2 A_3^1 A_2^4 A_4^3 \\ & + (-\sigma_{12})(-\sigma_{22}) A_1^2 A_3^1 A_4^4 A_2^3 + (-\sigma_{12})(-\sigma_{22}) A_1^2 A_4^1 A_2^4 A_3^3 \\ & + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12}) A_1^2 A_4^1 A_3^4 A_2^3 + (-\sigma_{12}) A_2^2 A_1^1 A_3^4 A_4^3 \\ & + (-\sigma_{12})(-\sigma_{12}) A_2^2 A_1^1 A_4^4 A_3^3 + (-\sigma_{12})(-\sigma_{11}) A_2^2 A_3^1 A_1^4 A_4^3 \\ & + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12}) A_2^2 A_3^1 A_4^4 A_1^3 + (-\sigma_{12})(-\sigma_{12})(-\sigma_{12}) A_2^2 A_4^1 A_1^4 A_3^3 \\ & + (-\sigma_{12})(-\sigma_{12})(-\sigma_{12})(-\sigma_{11}) A_2^2 A_4^1 A_3^4 A_1^3 + (-\sigma_{12})(-\sigma_{11}) A_3^2 A_1^1 A_2^4 A_4^3 \\ & + (-\sigma_{12})(-\sigma_{11})(-\sigma_{22}) A_3^2 A_1^1 A_4^4 A_2^3 + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12}) A_3^2 A_2^1 A_1^4 A_4^3 \\ & + (-\sigma_{12})(-\sigma_{11})(-\sigma_{12})(-\sigma_{12}) A_3^2 A_2^1 A_4^4 A_1^3 \\ & + (-\sigma_{12})(-\sigma_{11})(-\sigma_{22})(-\sigma_{12}) A_3^2 A_4^1 A_1^4 A_2^3 \\ & + (-\sigma_{12})(-\sigma_{11})(-\sigma_{22})(-\sigma_{12})(-\sigma_{12}) A_3^2 A_4^1 A_2^4 A_1^3 \\ & + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12}) A_4^2 A_1^1 A_2^4 A_3^3 + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12})(-\sigma_{12}) A_4^2 A_1^1 A_3^4 A_2^3 \\ & + (-\sigma_{12})(-\sigma_{12})(-\sigma_{12})(-\sigma_{22}) A_4^2 A_2^1 A_1^4 A_3^3 \\ & + (-\sigma_{12})(-\sigma_{12})(-\sigma_{12})(-\sigma_{22})(-\sigma_{11}) A_4^2 A_2^1 A_3^4 A_1^3 \\ & + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12})(-\sigma_{12})(-\sigma_{11}) A_4^2 A_3^1 A_1^4 A_2^3 \\ & + (-\sigma_{12})(-\sigma_{22})(-\sigma_{12})(-\sigma_{12})(-\sigma_{11})(-\sigma_{12}) A_4^2 A_3^1 A_2^4 A_1^3. \end{aligned}$$

Finally we can write the determinant

$$\begin{aligned} Det(A) = & \alpha_{21} \alpha_{12} \alpha_{21}^* \alpha_{12}^* + \alpha_{21} \alpha_{12} \alpha_{22}^* \alpha_{11}^* + \alpha_{21} \beta_{11} \beta_{22}^* \alpha_{12}^* - \alpha_{21} \beta_{11} \alpha_{22}^* \beta_{12}^* \\ & - \alpha_{21} \beta_{12} \alpha_{21}^* \beta_{12}^* - \alpha_{21} \beta_{12} \beta_{22}^* \alpha_{11}^* + \alpha_{22} \alpha_{11} \alpha_{21}^* \alpha_{12}^* + \alpha_{22} \alpha_{11} \alpha_{22}^* \alpha_{11}^* \\ & - \alpha_{22} \beta_{11} \beta_{21}^* \alpha_{12}^* - \alpha_{22} \beta_{11} \alpha_{22}^* \beta_{11}^* - \alpha_{22} \beta_{12} \alpha_{21}^* \beta_{11}^* + \alpha_{22} \beta_{12} \beta_{21}^* \alpha_{11}^* \\ & - \beta_{21} \alpha_{11} \beta_{22}^* \alpha_{12}^* + \beta_{21} \alpha_{11} \alpha_{22}^* \beta_{12}^* - \beta_{21} \alpha_{12} \beta_{21}^* \alpha_{12}^* - \beta_{21} \alpha_{12} \alpha_{22}^* \beta_{11}^* \\ & + \beta_{21} \beta_{12} \beta_{21}^* \beta_{12}^* + \beta_{21} \beta_{12} \beta_{22}^* \beta_{11}^* - \beta_{22} \alpha_{11} \alpha_{21}^* \beta_{12}^* - \beta_{22} \alpha_{11} \beta_{22}^* \alpha_{11}^* \end{aligned}$$

$$-\beta_{22}\alpha_{12}\beta_{21}^*\alpha_{11}^* + \beta_{22}\alpha_{12}\alpha_{21}^*\beta_{11}^* + \beta_{22}\beta_{11}\beta_{22}^*\beta_{11}^* + \beta_{22}\beta_{11}\beta_{21}^*\beta_{12}^* .$$

When we compare this with the determinant (2.40), we can see that these two determinants are the same.

4. CONCLUSION

As can be seen from two fermion case there is a new inhomogeneous quantum group. For the one particle case there is no difference between $FIO(2)$ and the new quantum inhomogeneous group, $CoFIO(2)$. However for two particles, the relations are not the same as $FIO(4)$. This means that there is a new quantum group and there should be a generalization to d -particles. In this thesis we have built up the general formulae of the quantum group and proved that the new quantum inhomogeneous group works for more particles. So the inhomogeneous quantum group of commuting fermions is called $CoFIO(2d)$.

APPENDIX A: HOPF ALGEBRA

As a quantum group is defined to be a Hopf algebra, we would like to review Hopf algebra. Since a Hopf algebra is a bi-algebra with an antipode, we should define a bi-algebra[11, 12]. A bi-algebra satisfies the conditions of an algebra and a co-algebra. First of all, we build up the notation

$$\begin{aligned}
 \mu & : A \otimes A \rightarrow A && \text{(multiplication)} \\
 \eta & : \mathbb{K} \rightarrow A && \text{(unit map)} \\
 \Delta & : A \rightarrow A \otimes A && \text{(comultiplication)} \\
 \epsilon & : A \rightarrow \mathbb{K} && \text{(counit map)} \\
 S & : A \rightarrow A && \text{(antipode)}
 \end{aligned} \tag{A.1}$$

In (A.1), A refers an algebra and \mathbb{K} refers a field. Algebra axioms are

$$\begin{aligned}
 \mu \circ (id \otimes \mu) &= \mu \circ (\mu \otimes id) && \text{(associativity)} \\
 \mu \circ (id \otimes \eta) &= \mu \circ (\eta \otimes id) && \text{(existence of unit),}
 \end{aligned} \tag{A.2}$$

and co-algebra axioms are

$$\begin{aligned}
 (\Delta \otimes id) \circ \Delta &= (id \otimes \Delta) \circ \Delta && \text{(coassociativity)} \\
 (\epsilon \otimes id) \circ \Delta &= (id \otimes \epsilon) \circ \Delta && \text{(existence of counit).}
 \end{aligned} \tag{A.3}$$

An algebra can be a Hopf algebra, if it satisfies these four axioms and the antipode axioms, which are

$$\begin{aligned}
 \mu \circ (id \otimes S) \circ \Delta &= \mu \circ (S \otimes id) \circ \Delta \\
 \Delta \circ \mu &= \mu \otimes \mu \circ (\Delta \otimes \Delta) && \text{(connecting axiom).}
 \end{aligned} \tag{A.4}$$

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