DISTRIBUTED GROUP CONSENSUS IN MULTI-AGENT NETWORKS

by

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ABSTRACT

DISTRIBUTED GROUP CONSENSUS IN MULTI-AGENT NETWORKS

Recent advances in computing and communication technologies have led to considerable progress in multi-agent networks. A variety of applications ranging from social networks to intelligent transportation systems makes the area promising in the sense that it seeks solutions to crucial questions in all realms of life. Among various issues studied in the context of multi-agent networks, the distributed consensus problem where a group of agents working collectively to achieve a common objective, has been one of the most popular. A multi-agent network utilizing a distributed linear consensus protocol may converge to different final values depending on the structure of the communication topology. The focus of this dissertation is to analyze the convergence properties of such networks where multi-equilibria consensus emerge. The problem is first examined for undirected networks represented by static or time-varying graphs. Joint connectivity and integral/sum connectivity conditions are presented that can be utilized to determine the number of equilibria as the interactions among the agents evolve over time. Subsequently, the analysis is extended to the study of multi-equilibria consensus in directed networks for which novel concepts of primary and secondary layer subgraphs are introduced. It is theoretically shown that the number of consensus equilibria of a network can be expressed as the total number of these subgraphs which can automatically be determined by a computer program. The convergence properties of multi-equilibria consensus in directed networks with bounded time-delays are also investigated and it is shown that communication time-delays do not affect the number of equilibria of a given network.

ÖZET

ÇOK ETMENLİ AĞLARDA DAĞITIK GRUP ONAYLAŞIMI

Bilgi işleme ve iletişim teknolojilerindeki son gelişmeler, çok etmenli ağlarda kayda değer ilerlemelere yol açmıştır. Bu ağların sosyal ağlardan akıllı ulaşım sistemlerine kadar çeşitlilik gösteren uygulamaları, hayatın her alanındaki önemli sorulara cevap araması nedeniyle, konuyu gelecek vaad edici bir hale getirmektedir. Bu araştırma alanındaki çeşitli konular arasında, ortak bir amacı gerçekleştirmek için birlikte çalışan bir grup etmenin yer aldığı dağıtık onaylaşım problemi en popülerlerinden biri olmuştur. Dağıtık ve doğrusal onaylaşım protokolü kullanan çok etmenli bir ağ, iletişim topolojisinin yapısına bağlı olarak farklı son değerlere yakınsayabilir. Bu tezin odağı, çoklu denge noktalarının ortaya çıktığı bu tür ağların yakınsama özelliklerinin araştırılmasıdır. Problem öncelikle statik veya zamanla değişen çizgelerle temsil edilen yönsüz ağlar için çalışılmıştır. Zamana bağlı değişen etkileşime sahip çok etmenli yönsüz ağlarda çoklu denge noktalarının sayısının belirlenmesi için kullanılabilecek ektentili bağlılık ve integral/toplam bağlılık koşulları sunulmuştur. Yapılan analizler, birincil ve ikincil katman alt çizge özgün kavramları kullanılarak yönlü ağların çoklu denge noktalı onaylaşımı için genişletilmiştir. Yönlü bir ağın onaylaşım denge sayısının, bir bilgisayar programı tarafından otomatik olarak belirlenebilen bu alt çizgelerin toplam sayısı olarak ifade edilebileceği kuramsal olarak gösterilmiştir. Ayrıca zaman gecikmesine sahip yönlü ağlarda çoklu denge noktalarının yakınsaklık özellikleri incelenmiş ve sınırlı zaman gecikmelerinin ağın denge noktası sayısını etkilemediği gösterilmiştir.

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LIST OF SYMBOLS

a_{ij}	The (i,j)-th element of the matrix A
A	The adjacency matrix
$\hat{B}$	The augmented input matrix
$\mathbf{e}$	The column vector with all entries equal to 1
$\mathcal{E}$	The set of edges
$\mathcal{G}$	A time-invariant graph
$\mathcal{G}(t)$	A time-varying graph for continuous-time networks
$\mathcal{G}(k)$	A time-varying graph for discrete-time networks
$\tilde{\mathcal{G}}_{[0,\infty)}$	The integral graph of $\mathcal{G}(t)$
$\check{\mathcal{G}}_{[0,\infty)}$	The sum graph of $\mathcal{G}(k)$
$\mathcal{I}$	The index set
l_p	The number of primary layer subgraphs
l_s	The number of secondary layer subgraphs
L	The Laplacian matrix
$\tilde{L}$	The Laplacian matrix of $\tilde{\mathcal{G}}_{[0,\infty)}$
n	The number of agents in the network
$\bar{n}_p$	The number vertices in the primary layer subgraphs
$\bar{n}_s$	The number vertices in the secondary layer subgraphs
$\mathcal{N}_i$	The neighbor set of agent i
q_i	The basis vector which has 1 as its i-th component and zeros elsewhere
R	The set of real numbers
S	The set of reachable vertices by agent i
$\mathcal{V}$	The set of vertices
w_{ij}	The (i,j)-th element of the matrix W
W	The network matrix of the discrete-time consensus protocol
$\hat{W}$	The augmented network matrix of the discrete-time consensus protocol
$\hat{W}_p$	The augmented matrix for the primary layer subgraphs

$\hat{W}_s$	The augmented matrix for the secondary layer subgraphs
x	The state vector
$\hat{x}$	The augmented state vector
δ	A positive constant
γ	A positive constant
$\lambda_i(W)$	The i-th largest eigenvalue of the matrix W in magnitude
Ψ	The infinite set of arbitrary positive numbers with the property of being closed under addition and multiplication by positive integers.
τ_{ij}	The amount of delay in communication between vertices j and i.
τ_{max}	The maximum amount of delay in communication

LIST OF ACRONYMS/ABBREVIATIONS

LMI	Linear Matrix Inequality
SLEM	Second Largest Eigenvalue Modulus

1. INTRODUCTION

In recent years, there has been a vast amount of research on the coordination control of multi-agent networks. Distributed coordination control problems in networks of dynamic agents are addressed in various forms such as consensus, optimization, task assignment and formation control. Among these problems, the problem of achieving a common value in a distributed context, namely consensus, has been extensively investigated [1–11]. Consensus related problems arise in a number of applications that include social networks [12], distributed sensing [13], unmanned air/ground/sea vehicles [14], biological oscillators [15], load balancing [16] and transportation networks [17]. All of these applications consist of multi-agent systems, for which it is important to develop robust and scalable distributed consensus protocols. Therefore, many results are obtained for the convergence properties of the consensus protocols by using tools from graph theory, set-valued Lyapunov theory, frequency domain analysis, switched-systems theory and non-negative matrix theory [2–5, 18].

Although there exist similar mathematical models in opinion dynamics [12] and distributed averaging [19], much of the work on multi-agent networks has been inspired by the model in [1]. In this study, the authors have proposed a discrete-time model for a network composed of self-propelled particles. These self-propelled particles (or autonomous agents) are assumed to move on a plane with constant speed but different orientations. The orientations of the particles are updated at each time step based on the average value of their neighbors. It is shown empirically that the particles can maintain the same orientation angle even in the absence of centralized coordination. Motivated by these observations, the convergence properties of the discrete-time consensus protocol have been investigated for switching network topologies with undirected information exchange [2]. The authors prove that there must be an infinite sequence of contiguous nonempty bounded time intervals across which all agents are linked together in order to achieve consensus. The proof relies on the convergence of stochastic matrix products established by Wolfowitz [20].

This result has later been extended to networks where the interaction topology among agents is represented by directed graphs [3]. It is shown that the existence of a spanning tree in the underlying graph is necessary and sufficient for the agents to reach consensus asymptotically.

A similar condition has also been derived for networks with time-dependent communication links in [4]. The condition states that the network should have at least one agent that can access (directly or indirectly) the rest of the agents over all time. The author utilizes the set-valued Lyapunov theory to prove the convergence of the protocol which is quite different from the graph theoretic approach presented in [3].

In [5], the authors propose a continuous-time consensus protocol for the solution of average-consensus problem where the agents converge to the average of their initial values. The convergence and performance properties of the consensus protocol are investigated for networks under fixed and switching topologies. Convergence of the protocol is established by using Lyapunov theory and spectral graph theory provided that the underlying graph topology is balanced and strongly connected. Furthermore, the authors present necessary and sufficient conditions to achieve average-consensus in networks that are subject to communication time-delays.

A probabilistic communication topology has been considered in [6], where the discrete-time consensus problem is expressed as the ergodicity of row stochastic matrix products. The authors utilize the properties of scrambling matrices to prove the convergence of the proposed protocol.

There have been some earlier convergence results obtained in the context of distributed computing in which asynchronous communication is considered [21]. The sufficient condition that requires the existence of a directed path from every agent to all other agents in the network has been presented to ensure the convergence of the protocol under time-delays. In [22], Blondel *et al.* have extended the results in [21] for the case of unbounded intercommunication intervals. They prove a convergence result on the delayed networks with switching topologies, assuming that the underlying graph is connected for all time and both communication intervals and delays are bounded.

The studies cited above mainly deal with multi-agent networks that reach a single equilibrium state, known also as classical consensus. However, as the networks get more complex with many interconnected components, the number of agreement values may differ according to the tasks, working space or time. Consensus on multiple equilibrium states finds applications in diverse areas ranging from social networks to science and engineering. Consider an electoral system, in which the voters' preferences vary according to the relationship they have with their friends, family, or colleagues. In this case, each individual will update his/her opinion based on the influences, thereby different opinions arise which lead to multi-equilibria consensus [23–26]. This phenomenon can also be observed in nature, especially in collective behavior of animal groups such as social aggregation or predator evasion [27]. In addition to the above examples, multiple equilibrium states exist in the problem of task allocation of swarm systems [28].

The aforementioned problem has been investigated under different names such as group consensus, cluster consensus/synchronization or multi-consensus [26, 29–37]. However, most of the work is subject to restrictive assumptions or requires prior knowledge/arrangement of the network topology.

In [29], the group consensus problem is considered based on the assumption that interaction exists not only among agents of individual subgroups but also between subgroups. Although some sufficient conditions for the convergence of the proposed algorithm are given in terms of linear matrix inequalities (LMI), network topologies for which these LMI conditions are feasible are not explicitly discussed. The model of [29] is revisited in [38] where the authors establish the convergence of the algorithm by relaxing the assumptions on the sum of adjacent weights between groups of agents.

In [31], cluster synchronization is studied and some conditions for the clusters having negative couplings with delays are derived. Nevertheless, the algebraic condition requiring that coupling within the same cluster to be sufficiently strong is difficult to be satisfied. More recently, the authors discuss the synchronization of multiple clusters composed of both linear agents and nonlinear oscillators [33]. They provide sufficient conditions in terms of coupling strengths and topologies based on the same assumptions given in [29].

In [30], the authors investigate the cluster consensus problem for discrete-time networks of multi-agents under fixed and switching topologies. A sufficient condition requiring that the number of clusters being equal to the period of agents is given to guarantee cluster consensus. However, the results are only valid for strongly connected graphs with the assumption that agents do not use their own information (self weights) to update the state value. In [26], the problem of cluster consensus with inter-cluster inputs is studied. Some sufficient conditions are provided for fixed and switching topology networks. It is shown that if every cluster in the graph has cluster spanning trees, then the system achieves cluster consensus.

One of the primary drawbacks of the above studies is that the network is artificially divided into multiple subnetworks which is in sharp contrast to the problem considered in this thesis. Moreover, all of the above work require the *in-degree balanced condition*, that is, the interaction between any two subnetworks (clusters) has to be balanced. However these restrictions on graph topology and/or coupling coefficients may not be suitable for real-life problems. On the other hand, we are interested in determining exactly how many consensus equilibria exist for a given graph, i.e., the network is not artificially divided into subnetworks a priori as opposed to [26, 29–31]. In this sense, the fundamental problem considered herein is different from what is already studied in the literature.

Another research direction of this thesis is to investigate convergence properties of multi-equilibria consensus problem in the presence of communication time-delays. Although there is a large number of studies on the consensus of delayed networks [4, 8, 21, 39–42], the effect of time-delays in networks with multi-equilibria consensus still remains a challenging topic which has not been investigated in the literature.

In [43], the group consensus problem for a network of dynamic agents employing a continuous-time protocol is addressed in the presence of communication delays. The authors introduce a double-tree form transformation which reduces the order of the system. In [31], the authors also investigate the cluster consensus problem for fixed topology networks under uniform time-delays in the presence of positive and negative coupling weights among agents in the network. They provide an algebraic condition, which is difficult to verify, to guarantee cluster consensus. Recently in [44], group consensus continuous-time multi-agent systems with time-delays is discussed. Some sufficient conditions have been provided and it is shown that the agents in nonzero in-degree groups converge to the convex hull of the groups with zero in-degrees.

1.1. Motivation of the Thesis

Recent advances in computing and communication technologies have led to considerable progress in multi-agent networks. A variety of applications ranging from social networks to intelligent transportation systems makes the area promising in the sense that it seeks solutions to crucial questions in all realms of life. Among various issues studied in the context of multi-agent networks, the distributed consensus problem where a group of agents working collectively to achieve a common objective, has been one of the most popular. Despite a vast amount of work on the study of consensus protocols, the existing results do not present a theoretical framework to analyze the case in which the agents in the network achieve multiple equilibrium states. Motivated by the lack of results as discussed above, we concentrate on understanding how and when multiple equilibrium states form in a network. To this end, we propose a novel approach to characterize the occurrence of multi-equilibria consensus in multi-agent networks. Furthermore, we present convergence analysis for the networks under time-delays as time-delays are one of the potential causes of instability and degraded performance in such systems. We believe that the insights provided in this thesis pave the way for a more systematic study of group formations in multi-agent networks.

1.2. Contributions of the Thesis

As mentioned in the previous section, this thesis focuses on the analysis of multi-equilibria consensus for networks of dynamic agents utilizing a distributed consensus protocol. The main contributions of this thesis can be summarized as follows:

- The stability properties in networks with undirected topologies converging on multi-equilibria states are examined. Necessary and sufficient conditions are established for networks that have static and time-dependent interactions based on the connectivity of the underlying graph.
- An alternative framework is proposed for analyzing networks with switching topologies as networks with static topologies. This framework utilizes the notions of integral and sum graph where a single static graph associated with the original network is constituted.
- Novel concepts of primary and secondary layer subgraphs that are instrumental in multi-equilibria consensus problem are introduced for the first time in the literature. Two algorithms are presented to determine the primary and secondary layer subgraphs of a given directed graph. Performance analysis of the proposed algorithms in terms of time-complexity and convergence is also carried out.
- For a multi-agent system employing the conventional distributed consensus protocol on a given directed graph, the number of consensus equilibria is obtained explicitly for the first time in the literature. Necessary and sufficient topological conditions are derived on the directed graph so that the network reaches two, three and K equilibria consensus.
- The effect of uniform and nonuniform delays for the multi-equilibria consensus problem is investigated. It is proved that when the delays are bounded, the network achieves the same number of equilibria consensus as the non-delayed case. It is stated in the literature that performance of distributed consensus protocols under time-delays depends on the underlying communication structure of the network. Therefore, insights are also provided on the convergence rate of the delayed networks with multi-equilibria consensus. Given a directed acyclic

graph, it is shown that time-delay does not adversely affect the convergence rate of the distributed consensus protocol.

The main contributions have been presented in a number of publications [45–52].

1.3. Organization of the Thesis

The remainder of the thesis is organized as follows. Chapter 2 is an introductory chapter that outlines the mathematical framework of this thesis. Specifically, we review some of the preliminary concepts and definitions in graph theory that will be used throughout the thesis. Then, we present continuous-time and discrete-time consensus protocols along with the assumptions imposed on them. Chapter 2 also states the formal definition of multi-equilibria consensus.

Chapter 3 deals with the convergence analysis of undirected networks that achieve multi-equilibria consensus. The problem is considered for both continuous and discrete-time networks. In the first part of this chapter, we survey some of the important results concerning the classical consensus problem. The chapter continues with the framework provided for the stability analysis of networks under fixed interactions. The second part of the chapter is devoted to the analysis of networks under switching interactions. This chapter concludes with the illustration of theoretical results by numerical simulations.

In Chapter 4, we present an alternative approach in terms of graph connectivity to investigate the convergence properties of time-varying networks with multi-equilibria consensus. We propose two new notions of integral and sum connectivity that apply to undirected graphs. Using these conditions, we are able to convert the original time-varying network into a static network. Thus, the problem is reduced to the analysis of a network under fixed interactions. The proposed approach is then applied to establish necessary and sufficient conditions on networks with switching topology.

Chapter 5 is concerned with the multi-equilibria consensus problem in multi-agent networks with directed topologies. The chapter starts by introducing two novel concepts, namely, primary and secondary layer subgraphs which are used to determine the number of equilibria consensus in a given network. We then present detection algorithms for these subgraphs which are applicable to any network. This chapter also contains the theoretical analysis of the problem. We conclude this chapter with illustrative examples.

The focus of Chapter 6 is on convergence analysis of the networks that are subject to communication constraints. Specifically, we study the effects of uniform and nonuniform time-delays on the number of consensus equilibria. Furthermore, we discuss the impacts of time-delays on the convergence rate. The chapter is then closed with the demonstration of the theoretical results.

Finally, Chapter 7 offers concluding remarks and future research directions.

2. OVERVIEW OF DISTRIBUTED CONSENSUS PROTOCOLS

In this introductory chapter, we review some fundamental concepts of algebraic graph theory and provide definitions related to distributed consensus protocols that will be used throughout this thesis. This chapter also introduces the problem of multi-equilibria consensus in multi-agent networks which is the main focus of the thesis.

2.1. Fundamentals of Graph Theory

The information flow within a multi-agent network is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The graph consists a set of vertices $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ indexed by the set $\mathcal{I} = \{1, 2, \dots, n\}$; a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, representing the links between two vertices.

An undirected (directed) edge which shows bidirectional (unidirectional) link between vertices in $\mathcal{G}$ is denoted by (v_i, v_j). The graph is said to be undirected, if for all $v_i, v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$. In a directed graph the order of vertices is important, that is, (v_i, v_j) means that there is an edge from v_i to v_j. Two vertices linked by an edge are defined as the endpoints of the edge. Vertices v_i and v_j are said to be adjacent to each other, if they are the endpoints of the same edge. For undirected graphs, the neighbors of v_i is the vertices that are adjacent to v_i. For directed graphs, the neighbors of v_i are given by the set $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$.

Connectivity is one of the essential notions of graph theory as it is useful in studying the interactions among networked systems. A *directed path* is defined as a finite sequence of vertices $v_1, \dots, v_m$ such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i = 1, \dots, m-1$. An undirected (directed) graph $\mathcal{G}$ is called *connected* (*strongly connected*), if there exists a (directed) path between each pair of its vertices. A directed graph is said to have a *spanning tree* if there exists at least one vertex, which has no parent, such that there

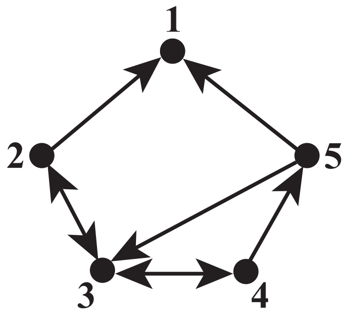

Figure 2.1. A directed graph with 5 vertices and 8 edges.

exist directed paths from the vertex to all other vertices in the graph.

The topological information of a graph is often described by means of adjacency and Laplacian matrices which will be defined below.

Definition 2.1. The adjacency matrix of a graph $\mathcal{G}$, denoted by $A = [a_{ij}]$ is a non-negative matrix where $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $i \neq j$. Otherwise, $a_{ij} = 0$.

Remark 2.1. A weighted graph $\mathcal{G}$ is a graph for which each edge has an associated numerical value. For a weighted graph $\mathcal{G}$, the adjacency matrix $A = [a_{ij}]$ is defined such that $a_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$, while $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$.

Definition 2.2. The Laplacian matrix of a graph $\mathcal{G}$, denoted by $L = [l_{ij}]$ where the elements are defined as

$$l_{ij} = \begin{cases} \sum\limits_{k=1, k\neq i}^{n} a_{ik}, & \text{if } j = i \\ -a_{ij}, & \text{if } j \neq i. \end{cases} \tag{2.1}$$

By definition, each row of a Laplacian matrix L, adds up to zero. Therefore, L has a zero eigenvalue associated with a right eigenvector $\mathbf{1} = [1 \dots 1]^T \in R^n$, i.e., $rank(L) \leq n - 1$.

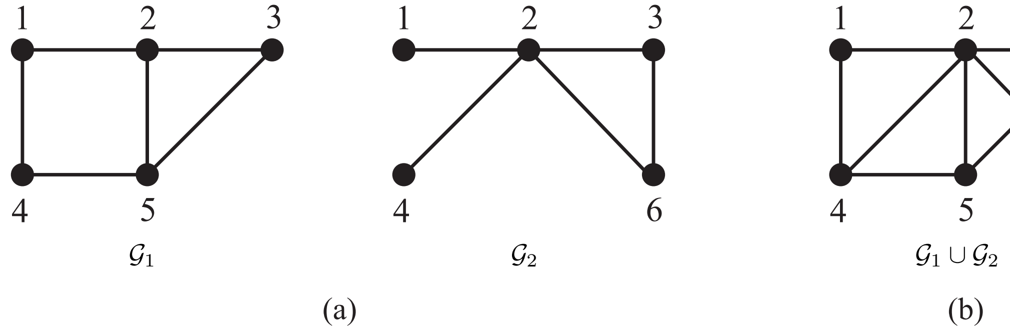

Figure 2.2. The union of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

Example 2.1. Consider the directed graph illustrated in Figure 2.1. The adjacency and Laplacian matrices associated with the graph are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

For the case when the topology is changing over time, the network can be described by a dynamic graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ where the set of edges $\mathcal{E}(t)$ varies with time while the set of vertices $\mathcal{V}$ remains same. For example, the graph topology may switch among a finite set of topologies given by $\mathcal{G}(t) = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N\}$. In such cases, we will utilize the following notion of union of graphs.

Definition 2.3. Given two graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, the *union* of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1 \cup \mathcal{E}_2)$.

Example 2.2. Consider the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ depicted in Figure 2.2(a). Based on Definition 2.3, the union of these two graphs is illustrated in Figure 2.2(b).

We also present some related concepts regarding the connectivity of an undirected graph.

Definition 2.4. (*Connected component*) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph. Suppose $(\mathcal{V}_c, \mathcal{E}_c)$ is connected where $\mathcal{V}_c \subseteq \mathcal{V}$ and $\mathcal{E}_c \subseteq \mathcal{E}$. If there exist no edges between the vertices of $\mathcal{V}_c$ and $\mathcal{V} \setminus \mathcal{V}_c$ then $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$ is said to be a *connected component* of the undirected graph $\mathcal{G}$.

Definition 2.5. (*K-Connected graph*) An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be *K-connected* if it has K connected components.

Lemma 2.1. Consider $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$. The number of connected components of $\mathcal{G}_1 \cup \mathcal{G}_2$ is less than or equal to the connected components of either $\mathcal{G}_1$ or $\mathcal{G}_2$.

Proof. From Definition 2.3, we have $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$. Then the result follows from Definition 2.4, and $\mathcal{E}_1 \cup \mathcal{E}_2 \supseteq \mathcal{E}_1$, $\mathcal{E}_1 \cup \mathcal{E}_2 \supseteq \mathcal{E}_2$. □

Definition 2.6. (*Equivalent K connected*) Given a graph $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ with K connected components $(\mathcal{V}_{c_1}, \mathcal{E}_{c_1}), \ldots, (\mathcal{V}_{c_K}, \mathcal{E}_{c_K})$, let $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ also have K connected components as $(\mathcal{V}_{c_1}, \overline{\mathcal{E}}_{c_1}), \ldots, (\mathcal{V}_{c_K}, \overline{\mathcal{E}}_{c_K})$. Then, the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *equivalent K connected.*

With a slight abuse of terminology, the concepts vertex and agent will be used interchangeably throughout this thesis.

2.1.1. Spectral Properties of the Laplacian Matrix

This section reviews some of the spectral properties of the graph Laplacian L. In general, these properties play a major role in the convergence analysis of distributed consensus protocols.

The Gershgorin's circle theorem can be utilized to conclude that all non-zero eigenvalues of L have positive real parts.

Lemma 2.2. (Gershgorin's Theorem, [53]) Let L be an $n \times n$ matrix. Then the eigenvalues of L lies within the union of Gershgorin discs, defined by

$$G(L) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - l_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |l_{ij}|\}.$$

Furthermore, if the graph has a spanning tree, then the following result states that L has a simple zero eigenvalue.

Lemma 2.3. ([3], Lemma 3.3) Given a matrix $L = [l_{ij}]$, where $l_{ii} \geq 0$, $l_{ij} \leq 0$, $\forall i \neq j$, and $\sum_{j=1}^{n} l_{ij} = 0$, $i, j \in \mathcal{I}$ for each j, L has at least one zero eigenvalue and the rest of the nonzero eigenvalues have positive real parts. Additionally, L has a simple zero eigenvalue if and only if the underlying directed graph has a spanning tree.

Moreover, we have the following results for undirected graphs.

Lemma 2.4. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be an undirected graph. Then the corresponding Laplacian matrix L is symmetric and positive semi-definite. Furthermore, for any $x \in R^n$ the following statements hold:

(i) [54] $x^T L x = \frac{1}{2} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}(x_i - x_j)^2$, and
(ii) [55] $Lx = 0$ if and only if $a_{ij}(x_i - x_j) = 0$ for $i, j \in \mathcal{I}$

where a_{ij}'s are the entries of the adjacency matrix A.

Remark 2.2. If an undirected graph $\mathcal{G}$ is connected, then its Laplacian matrix L has a simple zero eigenvalue and all its other eigenvalues are positive and real.

The following result characterizes the asymptotic behavior of Laplacian matrices where the corresponding graph is undirected and connected.

Lemma 2.5. [54] Suppose $\mathcal{G}$ is an undirected connected graph with Laplacian L satisfying $Ls = 0$, $r^T L = 0$ and $r^T s = 1$ where s and r are the right and left eigenvectors, respectively. Then

$$R = \lim_{k \to \infty} e^{-Lt} = sr^T.$$

2.2. Distributed Consensus Protocols

In this section, we first introduce the traditional consensus problem in multi-agent networks. We then present related mathematical models of continuous and discrete time protocols. The last part of the section is devoted to the statement of the multi-equilibria consensus problem.

2.2.1. The Consensus Problem

The consensus problem, where a group of agents agree upon certain quantities of interest while exchanging information among agents, is one of the most appealing topics in the control community as a result of its numerous applications. However, the problem in multi-agent networks may become complicated due to the uncertainties in the agents, and limited sensing and communication capabilities. As the networks are getting quite complex and large-scale, it is not possible to implement protocols in a centralized manner. Therefore, the need for distributed protocols arises, in which the information of each agent in the network is updated by relying on neighbor information.

Consider a network of n agents. Let $x_i \in R, i \in \mathcal{I}$ denote the state value associated with the i-th agent which is to follow a distributed control strategy. This iterative process for the agents consists of two steps. First, the state values of the agents are initialized. Then each agent updates its own state based only on the information coming from its neighbors.

The objective of the traditional consensus protocol is to ensure that the agents agree on a common value asymptotically, i.e.,

$$\lim_{t \to \infty} x(t) = c\mathbf{1} \tag{2.2}$$

for all initial conditions $x(0) = x_0$ where c denotes the common value of agents and $\mathbf{1}$ is the vector of all ones, i.e., $\mathbf{1} = [1, \ldots, 1]^T$.

The special cases of consensus protocols in which the final states of the agents are a function of initial states are average consensus ($Ave(x) = \frac{1}{n}(\sum_{i=1}^{n} x_i(0))$, max-consensus ($\max\limits_i x_i(0)$) and min-consensus ($\min\limits_i x_i(0)$).

2.2.2. Discrete-Time Consensus Protocol

In this section, we consider a multi-agent network consisting of n agents with linear discrete-time dynamics. The dynamics of the i-th agent are described by

$$x_i(k+1) = w_{ii}(k)x_i(k) + \sum_{v_j \in \mathcal{N}_i} w_{ij}(k)x_j(k) \tag{2.3}$$

where $x_i \in R$ is the state value of the i-th agent and $w_{ij}(k)$ is the non-negative averaging coefficient. Equation (2.3) can be expressed in matrix form as

$$x(k+1) = W(k)x(k), \tag{2.4}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in R^n$ and $W(k) \in R^{n \times n}$ is the network (system) matrix.

The following assumption is widely used in the analysis of multi-agent networks [1–4, 6, 8, 56, 57].

Assumption 2.1. The averaging coefficients are assumed to satisfy the conditions below:

(i) (Self weighting) $w_{ii}(k) \geq \delta, \forall i \in \mathcal{I}, \forall k \geq 0$, for some positive constant δ.

(ii) (Neighbor weighting) $w_{ij}(k) \in [\delta, 1)$ if $(v_j, v_i) \in \mathcal{E}(k)$ and $w_{ij}(k) = 0$ if $(v_j, v_i) \notin \mathcal{E}(k)$ for all $i, j \in \mathcal{I}$ and $i \neq j$, $\forall k \geq 0$.

(iii) (Total weights) $\sum_{j=1}^{n} w_{ij}(k) = 1, \forall i \in \mathcal{I}$, $\forall k \geq 0$.

Assumption 2.1(i) requires that the agent uses its own data in its update. Assumption 2.1(i) is a reasonable assumption unless the agent intentionally disrupts the update process. Assumption 2.1(ii) states that the information coming from a neighbor should be used with strictly positive weighting. In other words, each agent in the network is positively influenced by its neighbors. Assumption 2.1(iii) requires that the sum of the weighting coefficients for each agent to be equal to one. Furthermore, Assumption 2.1(iii) ensures that state of an agent will be a convex combination of the neighbor agent states at each time step. This assumption together with Assumption 2.1(i) and Assumption 2.1(ii) implies that the resulting network matrix is row-stochastic. Row-stochastic matrices arise in many applications, e.g., social networks [12], Markov chains [58], Google PageRank [59].

Definition 2.7. A matrix $W \in R^{n \times n}$ is said to be *row-stochastic* if $w_{ij} \geq 0$ for all $i, j \in \mathcal{I}$ and $\sum\limits_{j=1}^{n} w_{ij} = 1$, for all $1 \leq i \leq n$.

Definition 2.8. A stochastic matrix $W \in R^{n \times n}$ is said to be *indecomposable* and *aperiodic* if $\lim_{k \to \infty} W^k = \mathbf{1} y^T$ for some column vector y.

In words, we say that the powers of a stochastic, indecomposable and aperiodic matrix converge to a matrix with identical rows, i.e. a rank one matrix, as k approaches to infinity. This property plays an important role in convergence analysis of consensus protocols.

2.2.3. Continuous-Time Consensus Protocol

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let $x_i(t) \in R$ be the state value of the i-th agent in the network at time t. The state value of each agent is updated in continuous-time according to the following distributed consensus protocol

$$\dot{x}_i(t) = \sum_{v_j \in \mathcal{N}_i} a_{ij}(t)(x_j(t) - x_i(t)), \tag{2.5}$$

where $\mathcal{N}_i$ is the set of neighbors and $a_{ij}(t)$ is the (i, j)-th entry of the corresponding network (system) matrix at time t that satisfies Assumption 2.2 below. Note that the update rule depends only on the neighbor relationships which may be illustrated as a social network where an individual forms her opinion based on the information she encounters in her social circle.

Assumption 2.2. If there exists a directed communication link from agent v_j and v_i at time t, then $a_{ij}(t) \geq \delta$ for all $i, j \in \mathcal{I}$ and for some positive parameter δ. Otherwise, $a_{ij}(t) = 0$.

Intuitively, Assumption 2.2 imposes two features: (i) Each agent's state will contribute to its own state and the final state of the network, and (ii) by using a positive coefficient in the update rule, all agents in the network agree not to destabilize the final state. Given protocol (2.5), the state of agents, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, evolves according to

$$\dot{x}(t) = -L(t)x(t), \quad x(0) = x_0 \tag{2.6}$$

where $L(t)$ denotes the graph Laplacian at each time t.

2.3. Problem Statement

Next, we clarify the problem of multi–equilibria consensus as used in this thesis.

Definition 2.9. (*Multi-Equilibria Consensus*) We say that the network represented by the discrete-time system (2.3) (the continuous-time system (2.5)) converges to K consensus equilibria states if there exist K distinct constants c_l, and K non-empty sets S_l, $l = 1, \ldots, K$, such that

$$\bigcup_{l=1}^{K} S_l = \mathcal{V}, \quad S_l \cap S_m = \emptyset, \quad \text{for } l \neq m, \text{ and } l, m = 1, \ldots, K$$

and for the set S_l we have

$$\lim_{k \to \infty} x_i(k) = c_l, \quad \forall v_i \in S_l, \quad i = 1, \ldots, n$$

$$(\lim_{t \to \infty} x_i(t) = c_l, \quad \forall v_i \in S_l, \quad i = 1, \ldots, n)$$

and for arbitrary initial condition $[x_1(0), \ldots, x_n(0)]^T \in R^n$ and arbitrary choice of weighting coefficients w_{ij} (a_{ij}).

Remark 2.3. In Definition 2.9, S_l, $l = 1, \ldots, K$ denotes the groups of agents and $\lim_{k \to \infty} x_i(k) = c_l$ $(\lim_{t \to \infty} x_i(t) = c_l)$, $l = 1, \ldots, K$ denotes the final values of these groups. We define the sets S_l in such a way that the agents in the same group converge to same limit value as $k \to \infty$ $(t \to \infty)$. Furthermore, the agents belonging to different groups converge to different distinct final values. Note that the union of these groups constitutes the vertex set whereas their intersection is empty which guarantees the uniqueness of each S_l, $l = 1, \ldots, K$. Given a graph, the main focus of this thesis is to determine the groups of agents S_l by exploiting graph structures.

2.4. Chapter Summary

In this chapter, we have first reviewed the graph theoretical concepts and definitions that will be used in the analysis of the proposed problem. We have discussed the spectral properties of the Laplacian matrix L which is associated with the graph. The second section has been devoted to the mathematical models of continuous and discrete time consensus protocols, where the assumptions imposed on the protocols are also presented. Finally, we have defined the multi-equilibria consensus problem which will be studied for undirected networks in Chapters 3-4 and for directed networks in Chapter 5.

3. MULTI-EQUILIBRIA CONSENSUS IN UNDIRECTED NETWORKS

The emergence of new technologies such as the Internet of Things and the Cloud transforms the way we interact. Whether it be human to human interaction or machine to machine interaction, the size of the networks keeps growing which imposes new challenges. Motivated by these recent developments, in this chapter, we consider a network of n agents whose interaction graph is given by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, i.e., there exist mutual information exchange among neighbors. Armed with the concepts and definitions given in Chapter 2, one can be interested in investigating how the groups form in an undirected network with fixed and time-varying communication topologies. This chapter presents a comprehensive convergence analysis for the number of consensus equilibria in such networks where the agents employ an averaging based distributed consensus protocol. The contribution of this chapter is to derive some necessary and sufficient conditions in terms of graph connectivity such that the network reaches multi-equilibria consensus.

3.1. Existing Results on The Convergence of Linear Consensus Protocols

As pointed out in the former chapters, agents in a network do not generally agree on a single certain quantity of interest. Even when there exists mutual information exchange among the agents, the network may settle on different equilibria. Our goal in this chapter is to analyze such behavior of the multi-agent networks. Before discussing the main contributions of this chapter, we summarize some of the pioneering works related to classical consensus in multi-agent networks.

For the case of undirected networks, Olfati-Saber and Murray have shown that average consensus can be achieved asymptotically if and only if the underlying graph is connected [54]. The convergence analysis of the distributed protocol relies on spectral graph theory and matrix theory. In the same work, the conditions for a time-varying network to achieve average-consensus, that is the graph being strongly connected and balanced, have also been obtained.

In [2], Jadbabaie *et al.* have discussed the consensus problem in undirected networks with both fixed and time-varying topologies. Periodical connectivity notion has been proposed to investigate the convergence properties of time-varying networks assuming that the network matrices are primitive and row-stochastic. A sufficient condition requiring the associated graph of the network to be periodically connected is proved by utilizing the properties of products of indecomposable, aperiodic and stochastic matrices.

Ren and Beard have extended the results in [2] to the case of directed graphs. In order to solve the consensus problem in networks with static communication topologies, the authors have presented a necessary and sufficient condition based on the spanning tree concept. In the presence of a spanning tree in a graph, there exists a root vertex from which the rest of the vertices are reachable. Under this condition, it is proved that the states of the agents converge to the state of the leader (root) agent. The authors also derive a sufficient condition in terms of graph connectivity for the networks with time-varying communication topologies. Analogous to the result in [2], the agents achieve consensus if the union of the graphs across certain time intervals has a spanning tree [3]. In [4], a theoretical analysis for the convergence of time-varying network is presented. Similar to the condition established in [3], the author prove that convergence is ensured if the graph of the infinite product of network matrices are connected [4].

3.2. Analysis of Networks with Fixed Topology

In this section, we investigate the case of static communication topology, that is, the interaction among the agents in the network remains unchanged as time evolves. This case may be interpreted as unnatural since it assumes that information exchange is not a function of time and the agents have the same set of neighbors all the time. However, not only such cases may be observed in biology and statistical physics, especially in collective behavior of animal groups, but also it paves the way for the analysis of networks with time-varying interactions.

The problem of achieving asymptotical consensus which is stated in the following lemma corresponds to the $K = 1$ case in our framework.

Lemma 3.1. ([3], Theorem 3.8) For a network with fixed topology, the discrete-time protocol (2.3) (or the continuous-time protocol (2.5)) achieves consensus asymptotically if and only if the associated graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has a spanning tree.

Remark 3.1. From Lemma 3.1, we note that the graph of the network should have a spanning tree for the agents to reach a common goal. This condition requires the existence of an agent which directly or indirectly influences the rest of the network. This result has also been extended to time-varying networks by requiring the union of the collection of associated graphs across bounded time intervals to have a spanning tree frequently enough [3]. While this result is important for forming a single consensus value, the objective of this thesis is to determine the number of groups for a given network in the sense of Definition 2.9.

We first present the following result on undirected time-invariant graphs with multi-equilibria.

Theorem 3.1. Consider a network of n agents represented by a fixed topology graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Under Assumption 2.1, the network converges to K consensus equilibria, where $K \in Z, (n \geqslant K \geqslant 1)$, for the protocol (2.3) if and only if the network has K connected components.

Proof. (*Sufficiency*): Suppose the network has K connected components. Without loss of generality, the averaging matrix W can be written in the form

$$W = \text{blkdiag}\{\overline{W}_1, \ldots, \overline{W}_K\}$$

where each $\overline{W}_i, i = 1, \ldots, K$ has exactly one λ which is 1 and the rest of its eigenvalues (if any) satisfying $|\lambda| < 1$. If W is not in this form, it can be transformed into this form with a proper permutation matrix. Note also that W is a row stochastic matrix.

By applying Lemmas 2.5 and 3.1, and using Definition 2.5, the solution of the linear system converges to $\overline{W_i}^k x_{0i} \to c_i \mathbf{1}_{n_i}$ where $c_i = r_i^\top x_{0i}, i = 1, \ldots, K$. Here, r_i denotes left eigenvector corresponding to the eigenvalue $\lambda = 1$ of $\overline{W}_i$, i.e. $r_i^\top \overline{W}_i = 1$ and $x_0 = [x_{01} x_{02} \ldots x_{0K}]^\top$, $x_{0i} \in R^{n_i}$. By selecting arbitrary x_{0i}, we have $c_i \neq c_j$ for any i, j, $i \neq j$. Note that c_i becomes equal to c_j only when $r_i^\top = r_j^\top = 0$. But, from the definition of an eigenvector, we have $r_i^\top \neq 0$ which implies $c_i \neq c_j$. Therefore, the network achieves K equilibrium consensus states.

(*Necessity*): Suppose that the network does not have K connected components. We have two cases:

(i) The network has *at least* $K + 1$ connected components,
(ii) or the network has *at most* $K - 1$ connected components.

Case (i) There are *at least* $K + 1$ connected components in the network. Following the same logic in the sufficiency part; we can represent W as

$$W = \text{blkdiag}\{\overline{W}_1, \ldots, \overline{W}_{K+1}\}$$

which leads to $K + 1$ equilibrium consensus states in the network. Similar conclusion holds for a network with $K + 2$ connected components. Thus, it implies there are at least $K + 1$ equilibrium consensus states.
Case (ii) Suppose that there are *at most* $K - 1$ connected components in the network. Similar to case (i), it can be seen that there are at most $K - 1$ equilibrium consensus states.

Combining both cases lead to a contradiction that the network has K equilibrium consensus states, thereby completing the proof. □

In analogy with the discrete-time case, we present the following necessary and sufficient condition for undirected networks utilizing the continuous-time protocol (2.5).

Theorem 3.2. Consider a network of n agents represented by a fixed topology graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Under Assumption 2.2, the network converges to K equilibria consensus, where $K \in Z, (n \geqslant K \geqslant 1)$, for the protocol (2.5) if and only if the network has K connected components.

Proof. The proof is similar to the proof of Theorem 3.1. For details, please see [60]. ☐

Theorems 3.1 and 3.2 state that the number of equilibria depends on the number of connected components of the network. In other words, a multi-agent network with static interaction devolves into multiple opinions if and only if it has agent groups that have the same opinion within the group. Notably, the final states of the network depends only on the communication structure. While the result seems intuitive for the case of fixed networks, it cannot be immediately extended to time-varying networks. In the following, we will focus on networks with varying interactions and neighbors.

3.3. Analysis of Networks with Time-Varying Topology

We have so far discussed networks where the interaction between the agents is static, i.e., neighbor sets $\mathcal{N}_i$ are fixed and weighting coefficients $w_{ij}(k)$ (or $a_{ij}(t)$) are constant. Yet, it is more suitable to represent a network by means of a dynamic graph $\mathcal{G}(k)$ (or $\mathcal{G}(t)$) due to the fact that agents cooperate, conflict, participate and exchange information over time. These attributes of agents define the strength of influences and the structure of the communication topology. As an example, consider a social network where the neighbors of each individual change over time, e.g., a person may have new friends or break up with existing ones. In another scenario, the value that is given to others' opinion might also vary over time, e.g., trust levels might differ which would result in a decrease or an increase on the state value of certain neighbor. Such examples motivate the study of time-varying networks where the neighbor set of each agent varies over time, i.e. $\mathcal{N}_i(k)$ (or $\mathcal{N}_i(t)$), is a function of k (or t).

We now present the following results which will be used in the convergence analysis of networks with time-varying topology.

Lemma 3.2. [20] Let $S = \{W_i\}_{i=1}^{k}$ be a finite set stochastic matrices. If any finite products $P = \prod_{j=1}^{m} W_{i_j}$ with $1 \leq i_j \leq k$ and $m \geq 1$ are indecomposable and aperiodic, then any infinite product of matrices in S is convergent.

Lemma 3.3. [61] Let $W_1, \ldots, W_j$ be a finite set of row stochastic matrices that correspond to equivalent K connected graphs. Suppose that for each sequence $W_{i_1}, \ldots, W_{i_j}$ of finite length, the graph associated with the product is also equivalent K connected. Then, for each infinite sequence $W_{i_1}, W_{i_2}, \ldots$, there exist vectors $c_1, \ldots, c_K$ such that

$$\lim_{j \to \infty} W_{i_j} W_{i_{j-1}} \ldots W_{i_1} = C_K \tag{3.1}$$

where C_K is in the form

$$C_K \triangleq diag\{\mathbf{1}_{n_1} c_1^{\top}, \ldots, \mathbf{1}_{n_K} c_K^{\top}\} \tag{3.2}$$

Proof. Since the product $W_{i_j} W_{i_{j-1}} \ldots W_{i_1}$ is equivalent K connected and in the block diagonal form, Lemma 3.2 can be applied to the diagonal blocks independently. Therefore, (3.1) holds. □

Specifically, Lemma 3.3 extends the theorem given in [20] which will be instrumental in proving the main contributions of this section.

3.3.1. Discrete-Time Case

In this part, we consider the discrete system given in (2.3) and present necessary and sufficient conditions for time-varying networks to reach multi-equilibria consensus.

Lemma 3.4. Let W_{i_j}, $j = 1, \ldots, m$ be the corresponding network matrix for the undirected graphs $\mathcal{G}_{i_1}, \ldots, \mathcal{G}_{i_m}$. If the union of a set of undirected graphs $\mathcal{G}_{i_1}, \ldots, \mathcal{G}_{i_m}$

has K connected components, then the matrix product $W_{i_m}\dots W_{i_1}$ satisfies

$$\lim_{k\to\infty}(W_{i_m}\dots W_{i_1})^k = C_K \tag{3.3}$$

where C_K is in the form (3.2).

Proof. Suppose the union $\mathcal{G}_{i_1},\dots,\mathcal{G}_{i_m}$ has K connected components. Then, without loss of generality, the corresponding network matrix W matrix can be expressed as a block diagonal matrix

$$W = \text{diag}\{\overline{W}_1,\dots,\overline{W}_K\}$$

From the properties of stochastic matrices, it can be deduced that the product of $n\times n$ row stochastic matrices with positive diagonal elements is a row stochastic matrix with positive elements. By mimicking the proof for Theorem 3.2, one could conclude that $\overline{W_l}^k x_{0i} \to c_l\mathbf{1}_{n_i}$ where $c_l = r_l^\top x_{0i},\ i,l=1,\dots,K$ and $x_0=[x_{01}x_{02}\dots x_{0K}]^\top$, $x_{0i}\in R^{n_i}$, it follows that (3.3) holds. □

Remark 3.2. Lemma 3.4 indicates that the system with network matrix $W_{i_m}\dots W_{i_1}$ will have K equilibrium consensus states as $k\to\infty$.

Theorem 3.3. Consider a network of n agents represented by the switching topology graph $\mathcal{G}(k)=(\mathcal{V},\mathcal{E}(k))$. Under Assumption 2.1, consensus protocol (2.3) achieves K equilibrium consensus states if there exist infinite sequence of non-overlapping time intervals $[k_i,k_{i+1})$ starting at $i=0$, with the property that $[k_0,k_1)$ is bounded and $[k_i,k_{i+1})$ $i=1,2,\dots$ is uniformly bounded and for each $i,j\geq 1$ the union of the graphs over the intervals $[k_i,k_{i+1})$ and $[k_j,k_{j+1})$ are equivalent K connected.

Proof. Suppose that T is the least upper bound on the lengths of intervals $[k_i,k_{i+1})$, $i=0,1,2,\dots$ and satisfies $T<\infty$. Let $\Phi(k,k)=I, k\geq 0$ and $\Phi(k,\tau)\triangleq W_{k*}\dots W_{(\tau+1)\tau}$,

$k > \tau \geq 0$. It is obvious that $x(k) = \Phi(k, 0)x_0$. We will show that

$$\lim_{k\to\infty} \Phi(k, 0) = C_K \tag{3.4}$$

where C_K is as in (3.2). Lemma 3.4 states that each matrix product $\Phi(t_{j+1}, t_j)$, $j \geq 0$ has K equilibrium consensus states. Furthermore, since each $\Phi(k_{j+1}, k_j)$ is a product of at most T matrices taking values from a finite set $\{W_{i_1}, \ldots, W_{i_j}\}$, the set of possible matrix product $\Phi(k_{j+1}, k_j)$, $j \geq 0$, is finite. However, $\Phi(k_j, 0) = \Phi(k_j, k_{j-1}) \ldots \Phi(k_1, k_0)$. By using Lemma 3.3, it can be concluded

$$\lim_{j\to\infty} \Phi(k_j, 0) = C_K \tag{3.5}$$

Suppose j^* denotes the largest non-negative integer satisfying $0 \leq k_{j^*} \leq k$. Then, $\Phi(k, 0) = \Phi(k, k_{j^*})\Phi(k_{j^*}, 0)$. By Lemma 2.1, it can be concluded that $\Phi(k, k_{j^*})$ corresponds to a graph that is at least $K + 1$ connected. Therefore we have,

$$\Phi(k, k_{j^*})C_K = C_K \tag{3.6}$$

and

$$\Phi(k, 0) - C_K = \Phi(k, k_{j^*})(\Phi(k_{j^*}, 0) - C_K) \tag{3.7}$$

Since $\Phi(k, k_{j^*})$ is the product of at most $T - 1$ matrices coming from a bounded set, $\Phi(k, k_{j^*})$ is a bounded function. Following from (3.5) as $k \to \infty$, $\Phi(k_{j^*}, 0) \to C_K$. Combining $\Phi(k_{j^*}, 0) \to C_K$ and (3.7) implies

$$\lim_{k\to\infty} \Phi(k, 0) = C_K$$

The solution of the linear switching system (2.4) converges to

$$C_K x_0 = [\mathbf{1}_{n_1} c_1^\top x_{01} \ldots \mathbf{1}_{n_K} c_K^\top x_{0K}]^\top$$

where $x_0 = [x_{01}x_{02}\dots x_{0K}]^\top$, $x_{0i} \in R^{n_i}$, implying that the network reaches K consensus equilibria. $\square$

3.3.2. Continuous-Time Case

In time-varying networks, continuous-time protocols of the form (2.5) can cause chattering due to the abrupt changes in the agent's states. For instance, in social networks, if the individual wavers between opinions, fluctuations occur. Therefore, we do not allow infinitely small switching to prevent this phenomenon.

By employing a positive dwell time τ_i, (2.6) can be rewritten as

$$\dot{x}(t) = -L(t_i)x(t), \quad t \in [t_i, t_i + \tau_i) \tag{3.8}$$

where $t_0, t_1, \dots$ is an infinite time sequence such that $\tau_i = t_{i+1} - t_i$, $i \geq 0$. The solution of (3.8) is given by

$$x(t) = e^{-L(t_k)(t-t_k)}e^{-L(t_{k-1})(\tau_{k-1})}\dots e^{-L(t_1)(\tau_1)}e^{-L(t_0)(\tau_0)}x(0)$$

where k denotes the largest nonnegative integer, $t_k \leq t$. Let $\bar{\tau}$ denote a set of arbitrary positive numbers with finite length, and let Ψ be an infinite set of arbitrary positive numbers with the property of being closed under addition and multiplication by positive integers. By assumption $\tau_i \in \Psi$, $i \geq 0$.

Lemma 3.5. [60] If the union of a set of undirected graphs $\mathcal{G}_{t_1}, \dots, \mathcal{G}_{t_m}$ has K connected components and L_{t_j}, $j = 1, \dots, m$ is the Laplacian matrix corresponding to each graph $\mathcal{G}_{t_j}$ in (2.6), then we have

$$\lim_{k\to\infty} (e^{-L_{t_m}\Delta t_m} \dots e^{-L_{t_1}\Delta t_1})^k = C_K \tag{3.9}$$

where C_K is as in (3.2) and $\Delta t_j > 0$ are bounded.

Proof. Suppose that the union of the graphs $\mathcal{G}_{t_1}, \ldots, \mathcal{G}_{t_m}$ has K connected components. Note that, $-L_{t_j}, j = 1, \ldots, m$ can be expressed as $M_{t_j} - \eta_{t_j} I_n$ where η_{t_j} is the maximum value of diagonal entries of L_{t_j} and M_{t_j} is a matrix with non-negative elements. It can be written $e^{-L_{t_j}\Delta t_j} = e^{M_{t_j}\Delta t_j} e^{-\eta_{t_j}\Delta t_j}$, which implies that $e^{-L_{t_j}}$ is a stochastic matrix with positive diagonal elements. Then following the proof of Lemma 3.4, it can be shown that (3.9) holds. □

Theorem 3.4. Consider a network of n agents represented by a time-varying undirected graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ whose switching times $t_1, t_2, \ldots$ satisfy $\tau_i = t_{i+1} - t_i$, $i \geq 0$. Let $\mathcal{G}(t_i)$ be a time-varying graph at time $t = t_i$. Under Assumption 2.2, consensus protocol (2.5) achieves K distinct equilibrium consensus states if there exist infinite sequence of non-overlapping time intervals $[t_{i_j}, t_{i_j+l_j})$ starting at $t_{i_1} = t_0$, with the property that $[t_{i_j+l_j}, t_{i_j+1})$, $j \geq 1$ is uniformly bounded and the union of the graphs across each interval $[t_{i_j+l_j}, t_{i_j+1})$ is equivalently K connected.

Proof. The set of all possible matrices $e^{-L(t_i)}$ can be chosen from the finite set $\bar{F} = \{e^{-L(t_i)}, \tau_i \in \bar{\tau}\}$. Suppose $[t_{i_j+l_j}, t_{i_j+1})$ is uniformly bounded, i.e., there exists a constant B such that $|t_{i_j+1} - t_{i_j+l_j}| < B$ for all i, j. Consider the j-th time interval $[t_{i_j}, t_{i_j+1})$ which is bounded due to fact that both $[t_{i_j}, t_{i_j+l_j})$ and $[t_{i_j+l_j}, t_{i_j+1})$ are bounded. The union of the graphs across $[t_{i_j}, t_{i_j+1})$, denoted as $\bar{\mathcal{G}}(t_{i_j})$, has K connected components since the union of graphs across $[t_{i_j}, t_{i_j+l_j})$ has K connected components. Let $\{L(t_{i_j}), L(t_{i_j+1}), \ldots, L(t_{i_{j+1}-1})\}$ be the set of Laplacian matrices corresponding to each graph in the union $\bar{\mathcal{G}}(t_{i_j})$.

From Lemma 3.5, the matrix product $e^{-L(t_{i_{j+1}-1})(\tau_{i_{j+1}-1})} \ldots e^{-L(t_{i_j+1})(\tau_{i_j+1})} e^{-L(t_{i_j})(\tau_{i_j})}$, $j \geq 1$ is K connected. Then by Lemma 3.3, we conclude that a network of agents utilizing consensus protocol (2.5) achieves K consensus equilibria. □

Theorem 3.4 characterizes the conditions for convergence to K distinct states in terms of the time-varying network structure. The result shows that if the agents

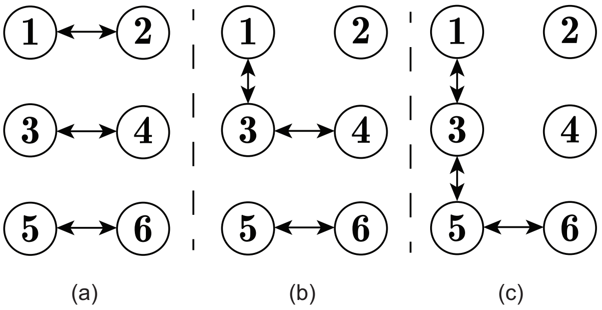

Figure 3.1. (a), (b), (c) topology graphs for a network of $n = 6$ agents.

communicate frequently enough, they tend to adopt the same state within the group (set) in the long-run. It should be noted that the agents do not have to be in interaction at all times. It is enough to interact only across certain time intervals.

3.4. Numerical Examples

In this section, we provide some examples to show the effectiveness of the proposed results. In the first example, we illustrate long-time behavior of a network of six agents with a time-varying communication topology. The second example considers a social network of individuals participating in a political discussion. Both of the simulations demonstrate how the interactions play role in the convergence of these networks.

Example 3.1. (Illustration of Theorem 3.3) Consider a network of six agents represented by three topologies shown in Figure 3.1. Let the corresponding network matrices W_a and W_b be

$$W_a = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 1/7 & 6/7 \end{bmatrix}, W_b = \begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 4/6 & 1/6 & 0 & 0 \\ 2/5 & 0 & 1/5 & 2/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 2/4 & 2/4 \end{bmatrix}.$$

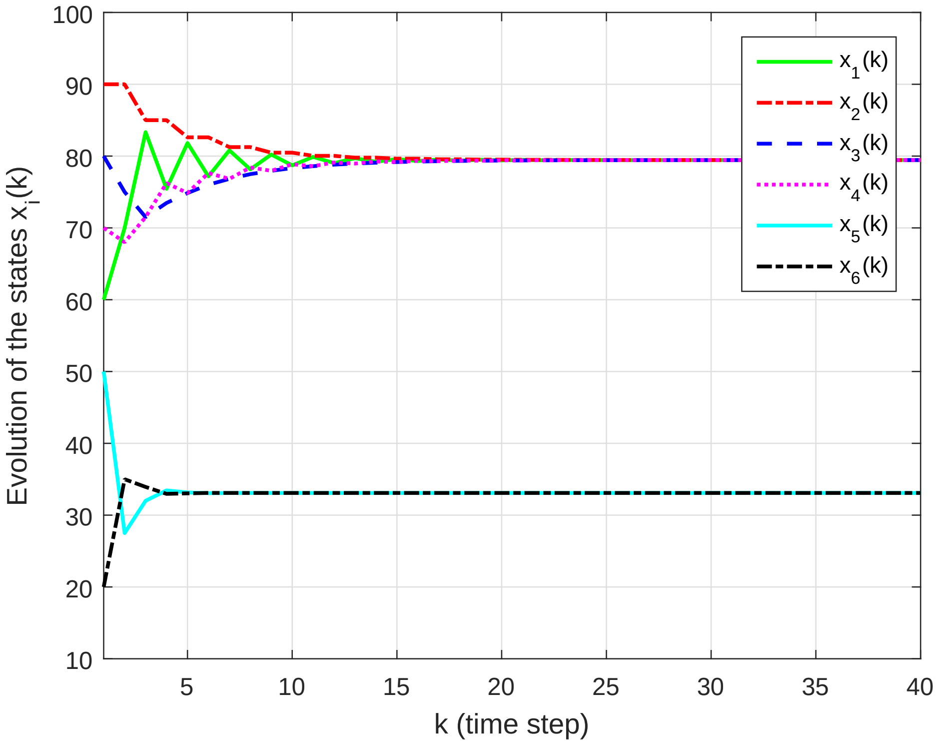

Figure 3.2. Multi-equilibria consensus for the network with switching topologies in Figures 3.1 (a) and (b).

In order to demonstrate our result on discrete-time switching networks, we will first consider the case in which the communication topology is periodically varying between the topologies W_a and W_b. From Theorem 3.3, we expect the network to converge to two equilibria consensus. The state trajectories of the agents are depicted in Figure 3.2 with the initial state $x_0 = [60\ 90\ 80\ 70\ 50\ 20]^\top$. Note that when networks with the underlying system matrices W_a and W_b are evaluated independently, it can be inferred from Definition 2.5 and Theorem 3.1 that they converge to three distinct consensus equilibria.

In a similar manner, consider the finite set of network matrices $\mathbb{W} = \{W_a, W_b, W_c\}$. Let the network matrix W_c for the topology Figure 3.1-(c) be given by

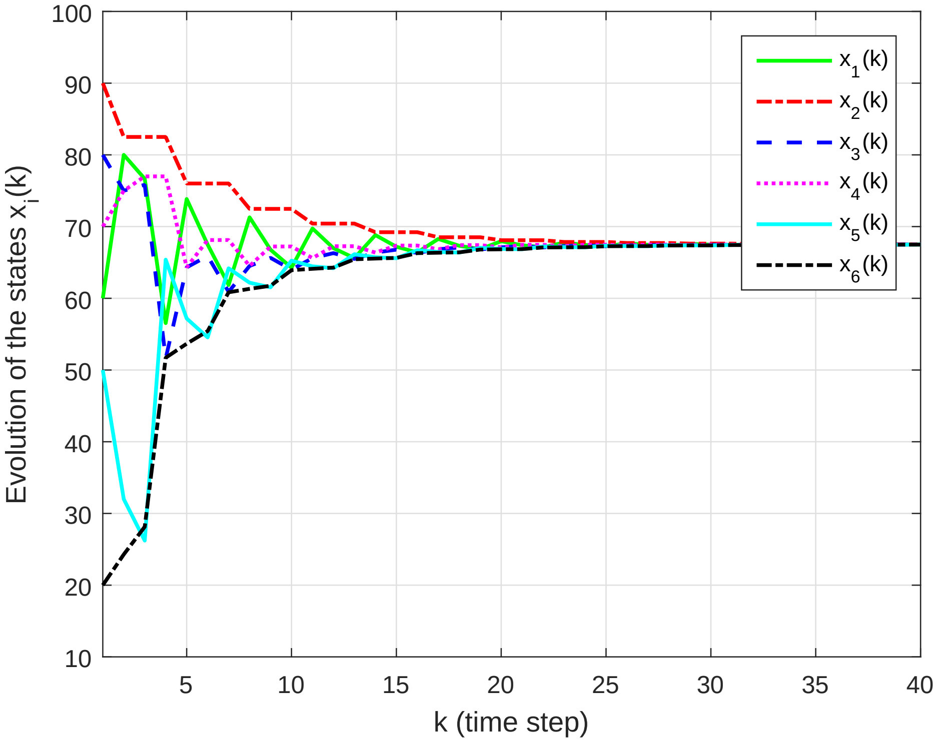

Figure 3.3. Consensus of the network with switching topologies in Figures 3.1 (a),(b) and (c).

$$W_c = \begin{bmatrix} 1/5 & 0 & 2/5 & 0 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2/6 & 0 & 1/6 & 0 & 2/6 & 1/6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4/9 & 0 & 3/9 & 0 & 1/9 & 1/9 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \end{bmatrix}.$$

Suppose now that the topologies are switching between W_a, W_b, W_c respectively. According to Theorem 3.3, the network is expected to achieve a *single* equilibrium state, therefore reaching classical consensus. The state values of the agents are shown in Figure 3.3 with the initial state $x_0 = [60\ \ 90\ \ 80\ \ 70\ \ 50\ \ 20]^T$ for the distributed consensus protocol (2.3). It is worth mentioning that we do not require the weighting coefficients $w_{ij}(k)$ and $w_{ji}(k)$ to be equal. The results are valid for all network matrices that satisfy Assumption 2.1.

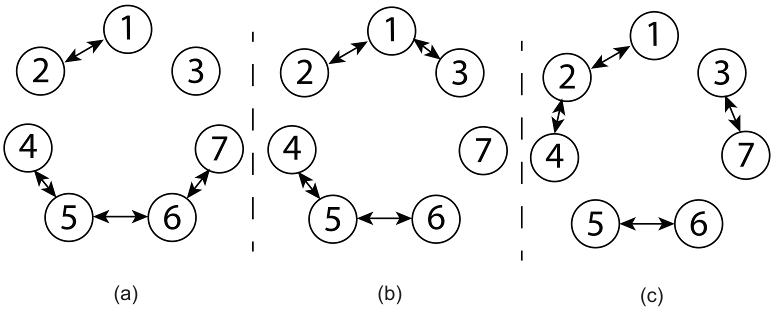

Figure 3.4. (a), (b), (c) topology graphs with 7 vertices.

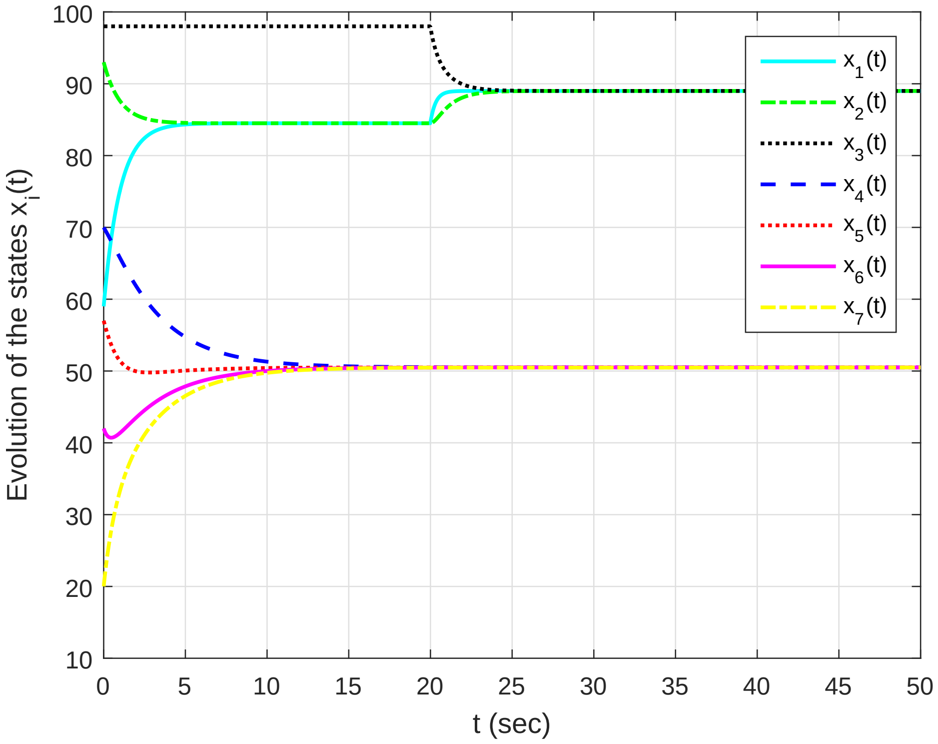

Figure 3.5. Formation of opinions when the network switches between the topologies given in Figures 3.4 (a) and (b) with switching period $T = 20$ s.

Example 3.2. (Illustration of Theorem 3.4) In this example, we focus on a social network where there exists mutual information exchange among the individuals. In particular, consider a network of 7 individuals in an online chat room having a conversation on the candidates of upcoming elections. We are interested in the political decisions of seven people in the long-run for the given communication topology illustrated in Figure 3.4.

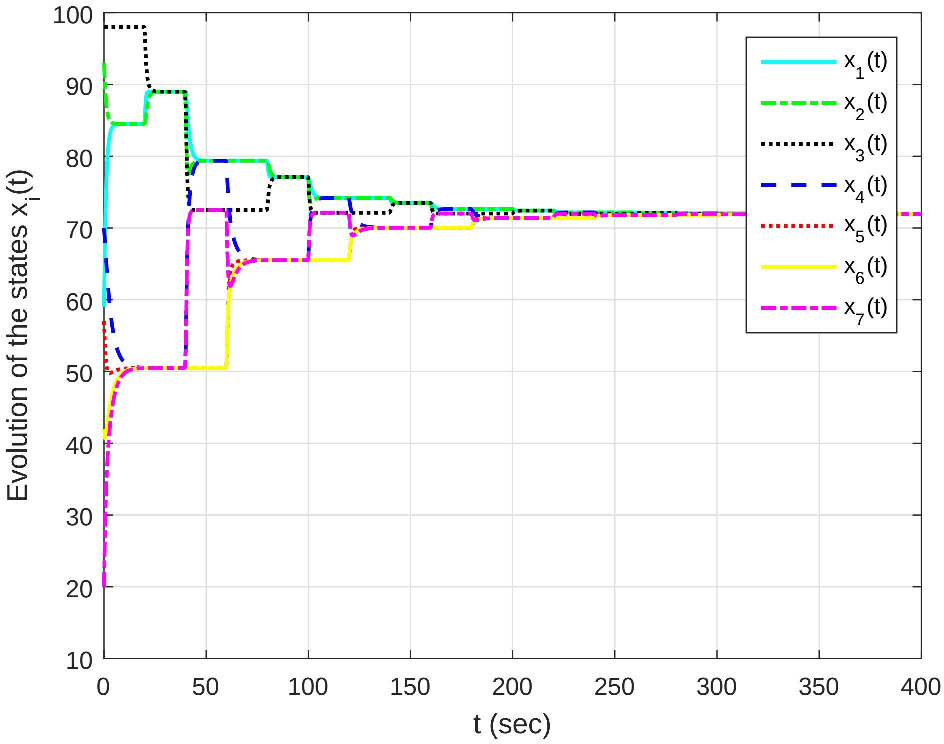

Figure 3.6. Formation of opinion when the network switches between the topologies given in Figures 3.4 (a),(b) and (c) with switching period $T = 20$ s.

The initial preferences of the individuals are selected randomly and the interactions (weighting coefficients) depend on the confidence levels between the people in the network that satisfy Assumption 2.2. It is worth mentioning that when the underlying communication networks are evaluated independently, by Definition 2.5 and Theorem 3.2, it can be deduced that each of the networks is three connected which leads to three opinion groups. Suppose that the communication topology among the individuals switches every 20 s in the sequence of (a), (b).

In line with Theorem 3.4, the time-varying network is expected to converge to two opinion groups. The change in the opinion (states) of the agents are depicted in Figure 3.5. While the agents 1, 2 and 3 tend to agree on the same candidate, the agents 4, 5, 6, and 7 are more likely to select the opponent candidate. This is due to the formation of two disconnected subgroups, namely 2 consensus equilibria.

Suppose now that the communication network switches between (a),(b) and (c) sequentially with switching period $T = 20$ s. The evolution of the opinions is illustrated in Figure 3.6 where the individuals are assumed to have arbitrary confidence levels. From Theorem 3.4, we can deduce that the network converges to a *single* equilibrium state as expected, i.e., all agents in the society agree on the same candidate.

3.5. Chapter Summary & Concluding Remarks

In this chapter, we have addressed the multi-equilibria consensus problem for an undirected network of n agents interacting with fixed or time-varying communication links. The problem is handled both in continuous and discrete-time settings. We have presented necessary and sufficient conditions on the network topology such that the network converges more than one equilibrium consensus state. For the convergence analysis of networks with time-varying topologies, we have introduced a condition based on joint connectivity notion which requires the union of graphs to be equivalently K connected. At the end of the chapter, we have presented extensive simulation studies to validate the results of the chapter. Particularly, we have studied opinion formation in a social network by examining the interaction pattern among individuals. While this chapter presents a joint connectivity condition for undirected networks, we investigate an integral connectivity condition in Chapter 4.

4. INTEGRAL AND SUM CONNECTIVITY CRITERIA

In this chapter, we propose an alternative framework to study the multi-equilibria consensus problem in networks with switching topologies. The interactions among agents are assumed to be undirected and the network matrix is symmetric. We present two graph based conditions for continuous and discrete-time networks, namely integral and sum K connectivity, which are superior than the similar conditions in the literature in the sense that it does not impose any constraint on the time axis. These conditions enable checking the convergence characteristics of networks with switching topologies at ease since the analysis is carried out only on a single constant graph which is constructed from the original networks.

4.1. Analysis of Networks under Integral Connectivity Condition

Recall that Assumption 2.2 requires the existence of some positive parameter δ such that $a_{ij}(t) \geq \delta$ if $(v_j, v_i) \in \mathcal{E}(t)$. Another way to interpret Assumption 2.2 is the following: whenever there is an information exchange among the agents in the network, the agent must positively consider the information gathered. However, there may be networks with time-varying coefficients where such an assumption does not hold. Consider the weight $a_{ij}(t) = e^{-t}$ which is always positive (this corresponds to decaying importance). Nonetheless, there does not exist a positive parameter δ such that $a_{ij}(t) \geq \delta$ holds for all $t \geq 0$. In this section, we relax this restriction and assume the following condition.

Assumption 4.1. $L(t)$ is symmetric, i.e., $l_{ij}(t) = l_{ji}(t)$.

Basically, Assumption 4.1 imposes the information exchange among the agents to be balanced, that is, the strength of the interaction are assumed to be equally worthy.

Definition 4.1. [55] Consider a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$ where $A(t)$ is the adjacency matrix at time t. The integral graph of $\mathcal{G}(t)$ on the interval $[0, \infty)$ can be defined as a constant graph $\widetilde{\mathcal{G}}_{[0,\infty)} := (\mathcal{V}, \widetilde{\mathcal{E}}, \widetilde{A})$ with the same vertex set $\mathcal{V}$, and the elements of adjacency matrix $\widetilde{A} = (\widetilde{a}_{ij})$ are given by

$$\widetilde{a}_{ij} = \begin{cases} 1, & \text{if } \int_0^\infty a_{ij}(t)dt = \infty, \\ 0, & \text{if } \int_0^\infty a_{ij}(t)dt < \infty. \end{cases} \tag{4.1}$$

Let $\widetilde{L}$ denote the graph Laplacian of $\widetilde{\mathcal{G}}_{[0,\infty)}$ which is formed from the elements of the adjacency matrix $\widetilde{A} = (\widetilde{a}_{ij})$.

A necessary and sufficient integral connectivity condition to solve the consensus problem in time-varying networks is stated in the following lemma:

Lemma 4.1. ([55], Theorem 4.1) For the undirected network with the dynamics (2.5), the time-varying topology $\mathcal{G}(t)$ achieves consensus if and only if $\widetilde{\mathcal{G}}_{[0,\infty)}$ is connected.

We will now extend Definition 4.1 to define an integrally K connected graph.

Definition 4.2. (*Integrally K connected graph*) A time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$ is said to be integrally K connected over the interval $[0, \infty)$, if its corresponding integral graph $\widetilde{\mathcal{G}}_{[0,\infty)}$ is K connected.

Using Definition 4.2 and Lemma 4.1, we present an important condition to determine the number of consensus equilibria on time-varying graphs.

Theorem 4.1. [49] Consider a network of n agents represented by a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$. Under Assumption 4.1, consensus protocol (2.5) achieves K equilibria consensus if and only if the corresponding integral graph $\widetilde{\mathcal{G}}_{[0,\infty)}$ is K connected.

Proof. (*Sufficiency*): Suppose that the corresponding integral graph $\widetilde{\mathcal{G}}_{[0,\infty)}$ is K connected. Then, the Laplacian matrix $\widetilde{L}$ can be represented in the form

$$\widetilde{L} = \text{blkdiag}\{\widetilde{L}_1, \widetilde{L}_2, \dots, \widetilde{L}_K\}$$

where each $\widetilde{L}_l, l = 1, \ldots, K$ has exactly one eigenvalue at zero and the rest of its eigenvalues (if any) are positive. Note that, if $\widetilde{L}_l$ is not in this form, it can be transformed into this form with a proper permutation matrix. By Lemma 4.1, it can be deduced that each subgraph with the corresponding $\widetilde{L}_l, l = 1, \ldots, K$ achieves consensus. Therefore, K equilibria consensus is achieved in the sense of Definition 2.9.
(*Necessity*): Suppose now that the corresponding integral graph $\widetilde{\mathcal{G}}_{[0,\infty)}$ is not K connected. Then it has to be either at least $K+1$ connected or at most $K-1$ connected. Thus, by using the arguments in the sufficiency part, we conclude that the system achieves either $K+1$ or $K-1$ equilibria consensus and we arrive at a contradiction. □

Remark 4.1. The integral connectivity condition in Theorem 4.1 provides ease of analysis for a given time-varying graph by forming a new *static* communication structure based upon the long-run behavior of the information exchange between the agents.

Remark 4.2. One obvious question is whether it is possible extend Theorem 4.1 to the networks modeled with directed graphs. Although connectivity notion can be defined similarly, the agents may not achieve K equilibria consensus in the case that the corresponding integral graph is K connected. Therefore, it is not possible to extend the result to the case of directed graphs.

Remark 4.3. Theorem 3.4 presents a joint K connectivity condition to check the multi-equilibria consensus under Assumption 2.2. Although Theorem 3.4 is applicable to a broader class of networks, Assumption 2.2 is restricted in the sense that the proposed result is only valid for graphs with $a_{ij}(t) \geq \delta$ weighting coefficients. However, Theorem 4.1 does not suffer from this restriction. Based on Assumption 4.1, Theorem 4.1 provides an easier condition which relies on investigating the convergence properties of a single fixed interaction graph. Nevertheless, Theorem 4.1 is only valid for symmetric networks.

4.2. Analysis of Networks under Sum Connectivity Condition

In this section, we address the same problem for time-varying discrete-time networks. Consider a network with n agents characterized by a switching graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k), A(k))$. The state value of each agent is updated according to (2.3) where $w_{ij}(k)$ is the non-negative averaging coefficient for the information coming from agent j to agent i. We assume that the following condition holds for the weighting matrix $W(k)$.

Assumption 4.2. $W(k)$ is symmetric, i.e. $w_{ij}(k) = w_{ji}(k)$.

4.2.1. Dynamics of The Time-Varying Network

In this part, we aim to show that the states of the agents in the network converge as k goes to ∞.

Lemma 4.2. Let $x(k)$ denote the trajectory of (2.3) with an undirected time-varying graph $\mathcal{G}(k)$. For the Lyapunov function candidate $V(k) = x^\top(k)x(k)$, we have $\Delta V(k) \leq 0$, $\forall k \geq 0$.

Proof.

$$\begin{aligned}\Delta V(k) &= V(k+1) - V(k) \\ &= x^\top(k+1)x(k+1) - x^\top(k)x(k) \\ &= x^\top(k)W^\top(k)W(k)x(k) - x^\top(k)x(k) \\ &= x^\top(k)[W^2(k) - I]x(k).\end{aligned} \tag{4.2}$$

Let $L(k) := I - W^2(k)$ where $W(k) = [w_{ij}(k)]$ is an $n \times n$ matrix and the entries l_{ij} are given by

$$\begin{aligned} l_{ij}(k) &= -\sum_{l=1}^{n} w_{il}(k)w_{lj}(k),\ i \neq j \\ l_{ii}(k) &= 1 - \sum_{l=1}^{n} w_{il}(k)w_{lj}(k). \end{aligned} \tag{4.3}$$

Using Lemma 2.4 (i), (4.2) can be written as

$$\begin{aligned} \Delta V(k) &= -x^{\top}(t)L(k)x(k) \\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}(k)(x_j(k) - x_i(k))^2. \end{aligned} \tag{4.4}$$

Substituting (4.3) into (4.4), we have

$$\Delta V(k) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k)(x_j(k) - x_i(k))^2 \leq 0. \tag{4.5}$$

□

Corollary 4.1. $\|x(k)\|$ is a non-increasing function of k.

Proof.

$$\|x(k+1)\|^2 - \|x(k)\|^2 = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k)(x_j(k) - x_i(k))^2 \tag{4.6}$$

Under Assumption 2.1, the right hand side of the equality becomes non-positive, then $\|x(k+1)\|^2 \leq \|x(k)\|^2$ implies $\|x(k+1)\| \leq \|x(k)\|$. □

Corollary 4.2. $|x_i(k)| \leq \|x(0)\|$ for any $i \in \mathcal{I}$ and $k \geq 0$.

Proof. From Corollary 4.1, we have $\|x(k)\| \leq \|x(0)\|$ which can be manipulated as follows to obtain the desired result:

$$\sqrt{|x_i(k)|} \leq \sqrt{\sum_{i=1}^{n} |x_i(k)|^2} \leq \|x(0)\|. \tag{4.7}$$

□

Lemma 4.3. For each $i, j \in \mathcal{I}$

$$\|x(0)\|^2 \geq \frac{1}{2} \sum_{k=0}^{\infty} \sum_{l=1}^{n} w_{il}(k) w_{lj}(k) (x_j(k) - x_i(k))^2. \tag{4.8}$$

Proof. From (4.5),

$$\|x(1)\|^2 - \|x(0)\|^2 = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} w_{il}(0) w_{lj}(0) (x_j(0) - x_i(0))^2$$

$$\vdots$$

$$\|x(k)\|^2 - \|x(k-1)\|^2 = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} w_{il}(k-1) w_{lj}(k-1) (x_j(k-1) - x_i(k-1))^2$$

Adding the above set of equations side by side leads to

$$\|x(0)\|^2 \geq \frac{1}{2} \sum_{m=0}^{k-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} w_{il}(m) w_{lj}(m) (x_j(m) - x_i(m))^2 \tag{4.9}$$

Since (4.9) holds for any k, it also is true for $k = \infty$:

$$\|x(0)\|^2 \geq \frac{1}{2} \sum_{k=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} w_{il}(k) w_{lj}(k) (x_j(k) - x_i(k))^2 \tag{4.10}$$

Using the above result, we obtain the statement of the lemma:

$$\|x(0)\|^2 \geq \frac{1}{2}\sum_{k=0}^{\infty}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k)(x_j(k)-x_i(k))^2 \tag{4.11}$$

$\square$

To be able to discuss the conditions for multi-equilibria consensus, the following lemma is needed.

Lemma 4.4. Let $\mathcal{I}_1 \subseteq \mathcal{I}$, $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$. Then, we have

$$\sum_{i\in\mathcal{I}_1}[x_i(k+1)-x_i(k)] = \sum_{i\in\mathcal{I}_1}\sum_{j\in\mathcal{I}_2} w_{ij}(k)x_j(k). \tag{4.12}$$

Proof.

$$\begin{aligned}\sum_{i\in\mathcal{I}_1}[x_i(k+1)-x_i(k)] =& \sum_{i\in\mathcal{I}_1}\left[\sum_{j=1}^{n} w_{ij}(k)x_j(k) - \sum_{j=1}^{n} w_{ji}(k)x_i(k)\right] \\ =& \sum_{i\in\mathcal{I}_1}\left[\sum_{j\in\mathcal{I}_1\cup\mathcal{I}_2} w_{ij}(k)(x_j(k)-x_i(k))\right] \\ =& \sum_{i\in\mathcal{I}_1}\left[\sum_{j\in\mathcal{I}_1} w_{ij}(k)(x_j(k)-x_i(k)) + \sum_{j\in\mathcal{I}_2} w_{ij}(k)(x_j(k)-x_i(k))\right].\end{aligned} \tag{4.13}$$

Multiplying out the right hand side gives us

$$\sum_{i\in\mathcal{I}_1}[x_i(k+1)-x_i(k)] = \sum_{i\in\mathcal{I}_1}\sum_{j\in\mathcal{I}_1} w_{ij}(k)(x_j(k)-x_i(k)) + \sum_{i\in\mathcal{I}_1}\sum_{j\in\mathcal{I}_2} w_{ij}(k)(x_j(k)-x_i(k)). \tag{4.14}$$

For simplicity, denote the first part of (4.14) as M:

$$M = \sum_{i\in\mathcal{I}_1}\sum_{j\in\mathcal{I}_1} w_{ij}(k)(x_j(k)-x_i(k)).$$

Since is $W(k)$ symmetric, i.e., $w_{ij}(k) = w_{ji}(k)$, and index is preserved, M becomes

$$\sum_{i,j \in \mathcal{I}_1} w_{ij}(k) x_j(k) - \sum_{i,j \in \mathcal{I}_1} w_{ji}(k) x_i(k) = 0$$

Thus (4.14) reduces to,

$$\sum_{i \in \mathcal{I}_1} [x_i(k+1) - x_i(k)] = \sum_{i \in \mathcal{I}_1} \sum_{j \in \mathcal{I}_2} w_{ij}(k)(x_j(k) - x_i(k)). \tag{4.15}$$

□

4.2.2. Consensus under Sum Connectivity

In this section, we address the consensus problem of discrete-time multi-agent networks under time-varying topologies. Analogous to the integral connectivity concept, we first present the sum graph definition.

Definition 4.3. (*Sum Graph*) Given a time-varying graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k), A(k))$, the sum graph of $\mathcal{G}(k)$ on $[0, \infty)$ is a constant graph $\check{\mathcal{G}}_{[0,\infty)} := (\mathcal{V}, \check{\mathcal{E}}, \check{A})$ where $\check{A} = [\check{a}_{ij}]$ is defined as

$$\check{a}_{ij} = \begin{cases} 1, & \text{if } \sum\limits_{k=0}^{\infty} \sum\limits_{l=1}^{n} w_{il}(k) w_{lj}(k) = \infty \\ 0, & \text{if } \sum\limits_{k=0}^{\infty} \sum\limits_{l=1}^{n} w_{il}(k) w_{lj}(k) < \infty. \end{cases} \tag{4.16}$$

Definition 4.4. A time-varying graph $\mathcal{G}(k)$ is said to be sum connected over the interval $[0, \infty)$ if its sum graph $\check{\mathcal{G}}_{[0,\infty)}$ is connected.

Suppose that $x^* := \lim_{k \to \infty} x(k)$ exists. Let $\check{L}$ be the corresponding Laplacian matrix of the sum graph which is constructed from $\check{A}$. The following result presents conditions for a time-varying network to achieve consensus under a sum connectivity condition.

Theorem 4.2. Suppose that $x^* := \lim_{k\to\infty} x(k)$ exists. The network represented by system (2.3) achieves consensus if and only if $\check{\mathcal{G}}_{[0,\infty)}$ is connected.

Proof. (*Sufficiency*): Suppose $\check{\mathcal{G}}_{[0,\infty)}$ is connected. Then, $\check{L}$ is connected which corresponds to $\check{L}\mathbf{1} = 0$, i.e., $\mathrm{Ker}\check{L} = span\{\mathbf{1}\}$. In order to complete the sufficiency part, we will show that $x^* \in \mathrm{Ker}\check{L}$. Let x_i^* and x_j^* be the two components of x^*, and suppose that these components are not equal, i.e., $x_i^* \neq x_j^*$. Let $d{:=}|x_j^* - x_i^*|$. Since $\lim\limits_{k\to\infty} x(k)$ exists, for any $\gamma > 0$ there is $T \geq 0$ such that $|x_i(k) - x_i^*|< \gamma$ and $|x_j(k) - x_j^*|< \gamma$ at any time step $k \geq T$. Rearranging the inequalities, we obtain $|x_i(k) - x_j(k)|\geq d - 2\gamma$. Then, we can write

$$\frac{1}{2}\sum_{k=T}^{\infty}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k)(x_i(k) - x_j(k))^2 \geq \frac{1}{2}\sum_{k=T}^{\infty}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k)(d - 2\gamma)^2 \tag{4.17}$$

Utilizing Lemma 4.3, we obtain

$$\sum_{k=T}^{\infty}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k) \leq 2\frac{\|x(0)\|^2}{(d-2\gamma)^2}$$

Since the right-hand side of the above inequality is finite, we obtain

$$\sum_{k=T}^{\infty}\sum_{l=1}^{n} w_{il}(k)w_{lj}(k) < \infty.$$

Definition 4.3 implies that $\check{a}_{ij} = 0$ when $x_i^* \neq x_j^*$. Furthermore, if $x_i^* = x_j^*$ then $x_i^* - x_j^* = 0$. When both statements are combined, one concludes that $\check{a}_{ij}(x_i^* - x_j^*) = 0$ for all $i, j \in \mathcal{I}$. The statement then follows from Lemma 2.4 (ii) which implies $\check{L}x^* = 0$.

(*Necessity*): We proceed by contradiction. Suppose $\check{\mathcal{G}}_{[0,\infty)}$ is not connected. Then, it can be expressed as the union of two disjoint networks, $\check{\mathcal{G}}_{[0,\infty)} = \check{\mathcal{G}}_a \bigcup \check{\mathcal{G}}_b$. Let $\mathcal{I}_1 = \{i \in \mathcal{I} | v_i$ is a vertex of $\check{\mathcal{G}}_a\}$ and $\mathcal{I}_2 = \{j \in \mathcal{I} | v_j$ is a vertex of $\check{\mathcal{G}}_b\}$ which means $\check{a}_{ij} = 0$ for $i \in \mathcal{I}_1$ and $j \in \mathcal{I}_2$. By Definition 4.3, it implies that $\sum\limits_{k=0}^{\infty}\sum\limits_{l=1}^{n} w_{il}(k)w_{lj}(k) < \infty$.

Furthermore, for any positive γ, there is $T_0 \geq 0$ satisfying $T > T_0$

$$\sum_{k=T_0}^{T} \sum_{l=1}^{n} w_{il}(k) w_{lj}(k) < \gamma \text{ for all } i \in \mathcal{I}_1 \text{ and } j \in \mathcal{I}_2. \tag{4.18}$$

Under Assumption 2.1, (4.18) can be written as follows

$$\sum_{k=T_0}^{T} w_{ij}(k) < \gamma \text{ for all } i \in \mathcal{I}_1 \text{ and } j \in \mathcal{I}_2 \tag{4.19}$$

Consider a sequence $x(k)$ with

$$x_i(T_0) = \begin{cases} 0, & \text{if } i \in \mathcal{I}_1 \\ 1, & \text{if } i \in \mathcal{I}_2. \end{cases} \tag{4.20}$$

Then for this special $x(k)$, check if it converges to the average vector $\bar{x}^* \in R^n$. When $\sum_{i=1}^{n}(x_i(k+1) - x_i(k)) = 0$, the average vector $\bar{x}^*$ becomes a constant function, therefore we have

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i(T_0) = \frac{|\mathcal{I}_2|}{n}.$$

We can express $\sum_{i\in\mathcal{I}_1}(x_i(T) - x_i(T_0))$ as $\sum_{k=T_0}^{T-1} \sum_{i\in\mathcal{I}_1}(x_i(k+1) - x_i(k))$.

Then after applying Lemma 4.4, we get

$$\sum_{i\in\mathcal{I}_1}(x_i(T) - x_i(T_0)) = \sum_{k=T_0}^{T-1} \sum_{i\in\mathcal{I}_1} \sum_{j\in\mathcal{I}_2} w_{ij}(k)(x_j(k) - x_i(k))$$

Given (4.20), $\sum_{i\in\mathcal{I}_1} x_i(T_0)$ becomes zero. Hence,

$$\left| \sum_{i\in\mathcal{I}_1} x_i(T) \right| \leq \sum_{i\in\mathcal{I}_1} \sum_{j\in\mathcal{I}_2} \sum_{k=T_0}^{T-1} w_{ij}(k) |(x_j(k) - x_i(k))|. \tag{4.21}$$

Note that,

$$|x_j(k) - x_i(k)| \leq |x_i(k)| + |x_j(k)| \leq 2\ ||x(k)|| \leq 2||x(T_0)|| = 2\sqrt{|\mathcal{I}_2|}$$

whenever $k \geq T_0$. Since (4.19) holds, the right hand side of (4.21) is less than $\sum_{i\in\mathcal{I}_1}\sum_{j\in\mathcal{I}_2} 2\sqrt{|\mathcal{I}_2|}\gamma = 2|\mathcal{I}_1||\mathcal{I}_2|\sqrt{|\mathcal{I}_2|}\gamma$. We can choose a sufficiently small constant γ such that $2|\mathcal{I}_1||\mathcal{I}_2|\sqrt{|\mathcal{I}_2|}\gamma = 1/n^2$, thus

$$\left|\sum_{i\in\mathcal{I}_1} x_i(T)\right| \leq \frac{1}{n^2}. \tag{4.22}$$

Nevertheless, the discrete-time network (2.3) can achieve consensus only if the following condition is satisfied:

$$\lim_{T\to\infty}\sum_{i\in\mathcal{I}_1} x_i(T) = |\mathcal{I}_1|\frac{|\mathcal{I}_2|}{n}.$$

This leads to a contradiction which completes the proof. □

Similar to the continuous-time counterpart, the sum graph definition can be extended to the sum K connected graph.

Definition 4.5. (*Sum K connected graph*) A time-varying graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k), A(k))$ is said to be sum K connected over $[0,\infty)$ if its corresponding sum graph $\check{\mathcal{G}}_{[0,\infty)}$ is K connected.

We state now the main result on the discrete-time networks with switching topology where multi-equilibria consensus states are achieved.

Theorem 4.3. Consider a network of n agents represented by a time-varying graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$. Suppose that $x^* := \lim_{k\to\infty} x(k)$ exists for (2.3). Under Assumption 4.2, consensus protocol (2.3) achieves K equilibria consensus if and only if the corresponding sum graph $\check{\mathcal{G}}_{[0,\infty)}$ is K connected.

Proof. (*Sufficiency*): Let $\check{L}$ be defined as $\check{L} = I - W^2$ where $W = [w_{ij}]$ is an $n \times n$ network matrix. Suppose that the corresponding sum graph $\check{\mathcal{G}}_{[0,\infty)}$ is K connected. Then, the Laplacian matrix $\check{L}$ can be represented in the form

$$\check{L} = \text{blkdiag}\{\check{L}_1, \check{L}_2, \ldots, \check{L}_K\}$$

where each $\check{L}_l, l = 1, \ldots, K$ has exactly one eigenvalue at zero and the rest of its eigenvalues (if any) are positive. We note that, if $\check{L}_l$ is not in this form, it can be transformed into this form with a proper permutation matrix. It follows from Theorem 4.2 that each subgraph with the corresponding $\check{L}_l, l = 1, \ldots, K$ reaches a single equilibrium consensus state. Therefore, K equilibria consensus is achieved in the sense of Definition 2.9.

(*Necessity*): Suppose now that the corresponding sum graph $\check{\mathcal{G}}_{[0,\infty)}$ is not K connected. Then it has to be either at least $K+1$ connected or at most $K-1$ connected. Thus, by using the arguments in the sufficiency part, we conclude that the system achieves either $K+1$ or $K-1$ equilibria consensus states which is a contradiction. ☐

Remark 4.4. Note that the conditions presented in this section only rely on the sum graph which is constructed based on the time-varying network topology. The proposed technique is easier to implement than the similar approaches in the literature since the problem is analyzed as in static networks.

4.3. Numerical Examples

This section presents the demonstration of the theoretical results via numerical examples. In the first example, we perform simulation for a network of six agents where the interaction among agents are assumed to be varying with respect to time. The second example not only illustrates the sum connectivity condition for a five agent network with switching topology but also shows the necessity of Assumption 2.1(i).

Example 4.1. We consider an undirected network with $n = 6$ agents associated with the communication topologies given in Figure 4.1 to illustrate the results based on the notion of integral connectivity. The Laplacian matrices for the network are given as

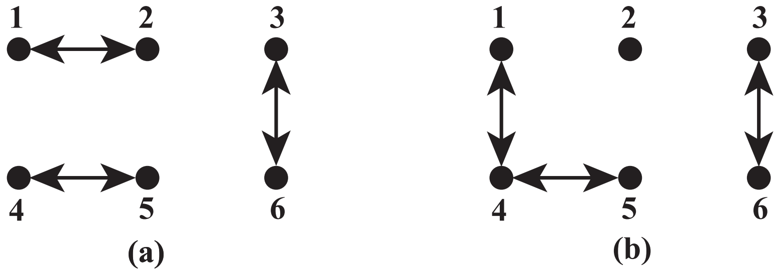

Figure 4.1. (a) and (b) topology graphs for a network of 6 agents.

$$L_a(t) = \begin{bmatrix} e^{-2t} & -e^{-2t} & 0 & 0 & 0 & 0 \\ -e^{-2t} & e^{-2t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-t/3} & 0 & 0 & -e^{-t/3} \\ 0 & 0 & 0 & e^{-t} & -e^{-t} & 0 \\ 0 & 0 & 0 & -e^{-t} & e^{-t} & 0 \\ 0 & 0 & -e^{-t/3} & 0 & 0 & e^{-t/3} \end{bmatrix},$$

$$L_b = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -3 \\ -1/2 & 0 & 0 & 5/4 & -3/4 & 0 \\ 0 & 0 & 0 & -3/4 & 3/4 & 0 \\ 0 & 0 & -3 & 0 & 0 & 3 \end{bmatrix}.$$

To investigate the effect of time-dependent weighting coefficients a_{ij} on the multi-equilibria consensus, we model one of the weight (network) matrices with time-varying coefficients. Note that, under Assumption 2.2, we can not choose δ such that $a_{ij}(t) \geq \delta$. Therefore, Theorem 3.4 stated in Chapter 3 is not applicable in this case. We suppose that the communication topology varies between topologies given in Figures 4.1 (a) and (b) with the period of $T = 5$ seconds in the sequence of $L_a(t), L_b$.

In line with Theorem 4.1, the time-varying network is expected to converge to three consensus equilibria. The evolution of states for the agents is depicted in Figure 4.2 with the initial state $x_0 = [40\ 95\ 30\ 20\ 55\ 80]^T$. Note that, even if agent 2 is influenced by some of the agents in the first topology, it can not adopt the group's

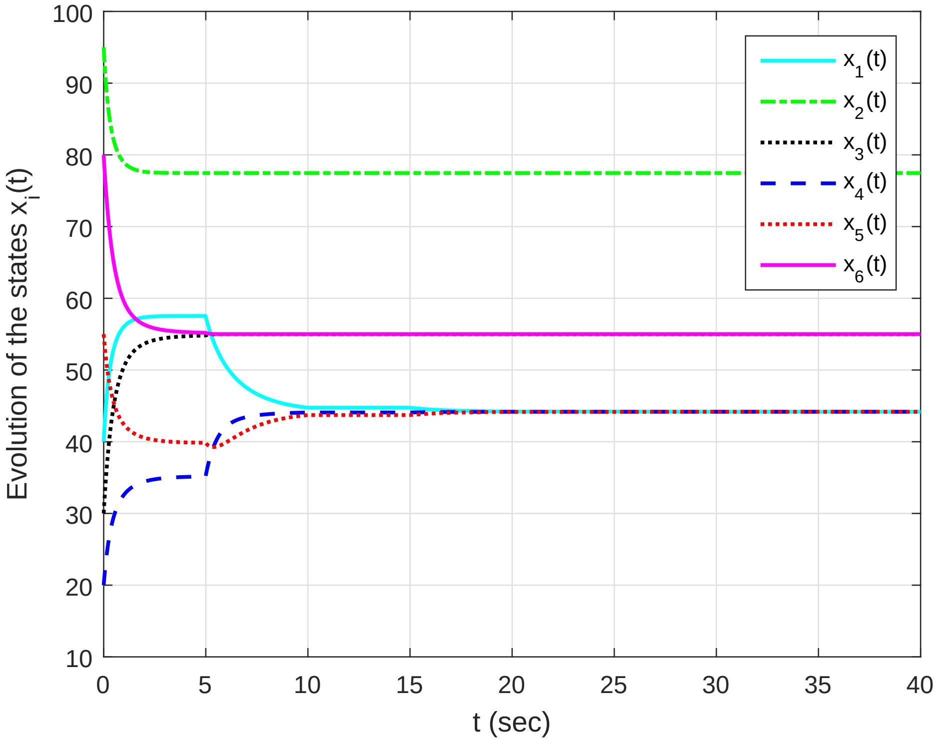

Figure 4.2. State trajectories of the agents in Example 4.1.

ultimate state in the long-run due to decaying weighting coefficient.

Example 4.2. In this example, we study the multi-equilibria consensus problem for discrete-time networks with time-varying topologies under the sum connectivity condition. For this case, we consider a five agent network with the corresponding weight matrices

$$W_a(k) = \begin{bmatrix} 3^{-k}+\delta & 1-3^{-k}-\delta & 0 & 0 & 0 \\ 1-3^{-k}-\delta & 3^{-k}+\delta & 0 & 0 & 0 \\ 0 & 0 & k^{-2}+\delta & 1-k^{-2}-\delta & 0 \\ 0 & 0 & 1-k^{-2}-\delta & k^{-2}+\delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

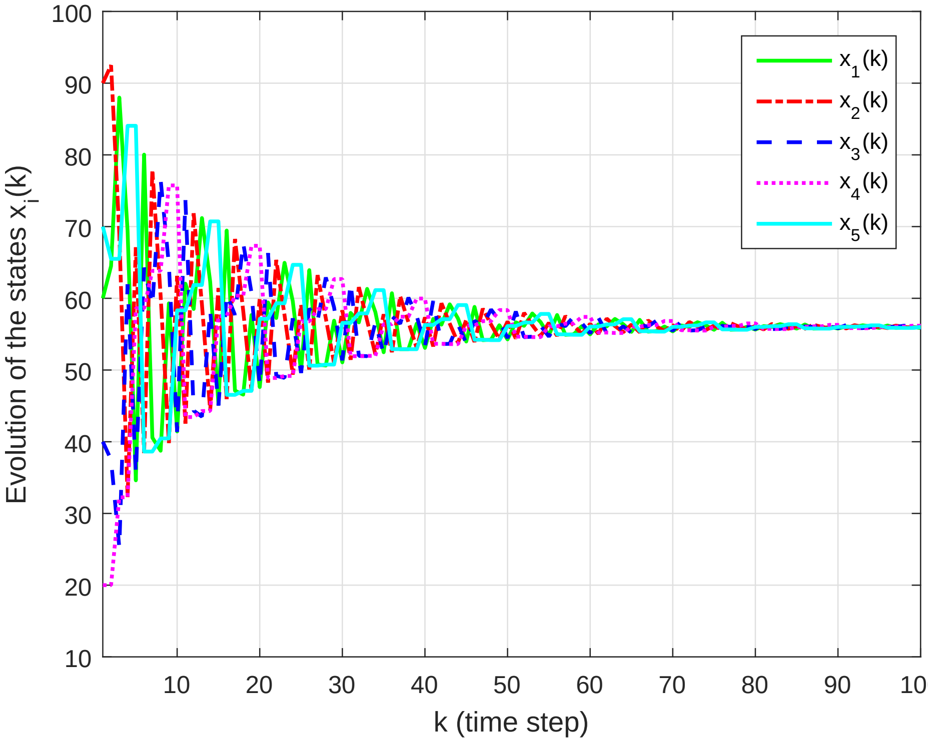

Figure 4.3. State trajectories of the agents in Example 4.2.

$$W_b(k) = \begin{bmatrix} 2^{-k}+\delta & 0 & 0 & 0 & 1-2^{-k}-\delta \\ 0 & k^{-2}+\delta & 1-k^{-2}-\delta & 0 & 0 \\ 0 & 1-k^{-2}-\delta & k^{-2}+\delta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1-2^{-k}-\delta & 0 & 0 & 0 & 2^{-k}+\delta \end{bmatrix}$$

where δ is a positive constant given in Assumption 2.1. By Theorem 4.2, the agents are expected to convergence to a single equilibrium state due to the fact that the sum graph of the switching network becomes a connected graph. The evolution of the states of agents under periodic switching with the initial state $x_0 = [60\ 90\ 40\ 20\ 70]^\top$ is depicted in Figure 4.3. Note that, this example also demonstrates the necessity of Assumption 2.1(i) since in the absence of positive constant δ, the states of the agents in the considered network oscillates rather than achieving multi-equilibria consensus.

4.4. Chapter Summary & Concluding Remarks

In this chapter, multi-equilibria consensus problem for undirected networks with time-varying communication topologies has been investigated. A novel condition based on the notion of integral connectivity is proposed to examine the convergence properties of continuous-time networks. Furthermore, the result is extended to the networks of agents with discrete-time dynamics. We prove that the time-varying network achieves K equilibria consensus if only if the associated integral (or sum) graph is K connected. The analysis of the networks is shown by using tools from linear system theory, algebraic graph theory and matrix theory. Finally, numerical examples are presented to illustrate both of the conditions. The simulations also provide a comparison between jointed connectivity and integral connectivity conditions which are presented to solve the multi-equilibria consensus problem in varying continuous-time networks.

Most of the approaches in multi-agent coordination problems suggest that bidirectional communication among agents in a network will eventually result in a common decision. However, this may be unrealistic for some cases since the researchers assume the whole network to be interacting all the time. Interaction patterns among the agents play a major role when building a large scale network working in a distributed manner. We thus believe that the insights provided in this chapter can serve the basis to create and design a networks of larger size for real-world applications.

5. MULTI-EQUILIBRIA CONSENSUS IN DIRECTED NETWORKS

In this chapter, we analyze the multi-equilibria consensus problem for a given multi–agent network with an associated directed graph. Many studies in the literature have attempted to relate the number of connected component in an undirected graph with the number of consensus equilibria. However, this correlation does not apply for the networks modeled with directed graphs. A supportive example is presented in the first section, where we illustrate that vertices in a strongly connected component of a graph do not necessarily reach the same final value. For this reason, we propose two novel definitions of primary and secondary layer subgraphs to analyze the consensus equilibria of a given network. We also introduce two algorithms which are used to determine primary and secondary layer subgraphs in a directed graph. Based on the definitions, we then analyze the number of equilibria in directed networks of agents that utilize a discrete-time consensus protocol. We prove that the number of consensus equilibria is related to the total number of primary and secondary layer subgraphs, which is one of the most important contributions of this thesis.

5.1. The Number of Consensus Equilibria in Directed Networks

In this section, we are interested in determining the conditions on the directed network topology such that the network reaches $K \geq 2$ consensus equilibrium states. To do so, we first investigate the role of strongly connected components in a graph. Then, we introduce our graph based tools which will be used in the analysis of the proposed problem.

5.1.1. The Role of Connectivity in the Underlying Graph

As a first step towards the solution of the problem, one might be tempted to believe that strongly connected components of the graph play a major role in the

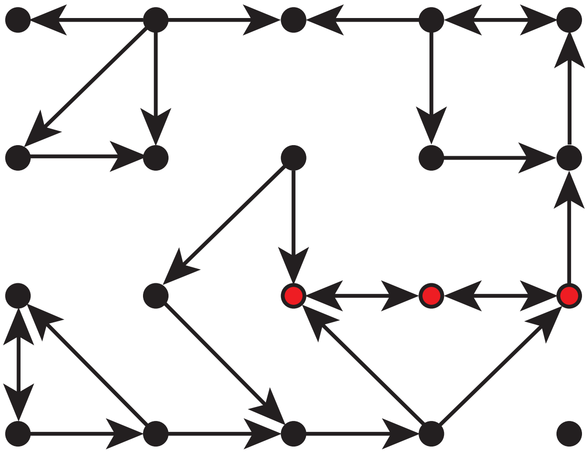

Figure 5.1. A directed graph with 20 vertices and 27 edges.

analysis. However, this is not so, as the vertices of a strongly connected component in a directed graph do not necessarily converge to the same equilibrium value as illustrated by the following example.

Example 5.1. Consider the network of 20 agents shown in Figure 5.1. Note that the vertices which are depicted in red form a strongly connected component together. For a random choice of initial state $x(0)$ and weighting coefficients w_{ij} which satisfy Assumption 2.1, the evolution of the states is depicted in Figure 5.2. It is seen that the network converges to 9 distinct consensus values. However, the vertices which form a strongly connected component do not converge to the same value.

5.1.2. Graph Related Definitions

In this part, we introduce the novel concepts of primary and secondary layer subgraphs that will be instrumental in the analysis of the multi–equilibria consensus problem. We first start with the subgraph and primary layer subgraph definitions.

Definition 5.1. A subgraph of a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph, $\mathcal{G}_{sg} = (\mathcal{V}_{sg}, \mathcal{E}_{sg})$, such that $\mathcal{V}_{sg} \subseteq \mathcal{V}$ and $\mathcal{E}_{sg} \subseteq \mathcal{E} \cap (\mathcal{V}_{sg} \times \mathcal{V}_{sg})$.

Definition 5.2. (*Primary layer subgraphs*) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph under consideration. There exist l_p ($l_p \geq 1$) subsets in the vertex set $\mathcal{V}$ such that each subset $V_{p,i}$, $i = 1, \ldots, l_p$, is the largest possible subset that has a spanning tree for its subgraph $\mathcal{G}_{p,i}$ and for all $v_a \in V_{p,i}$ and $v_b \notin V_{p,i}$, we have $(v_b, v_a) \notin \mathcal{E}$. We say

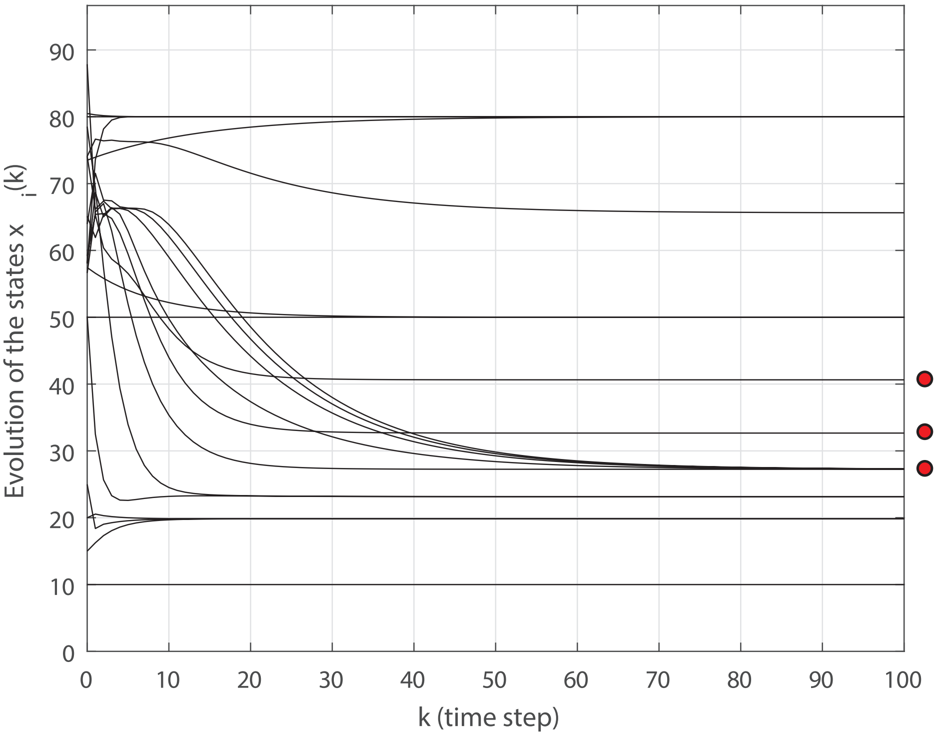

Figure 5.2. The simulation results for the network given in Figure 5.1.

$\mathcal{G}_{p,i}$ $(i = 1, \ldots, l_p)$ are the $p$rimary layer subgraphs of $\mathcal{G}$ where the number of primary layer subgraphs is denoted by l_p.

Now that we have defined the primary layer subgraphs, we are ready to define the notion of a secondary layer subgraph.

Definition 5.3. (*Secondary layer subgraphs*) Let $\bar{V}$ be the set which consists of the vertices that are not in the primary layer subgraphs, i.e., $\bar{V} = V \setminus \bigcup_{i=1}^{l_p} V_{p,i}$. Then there exist l_s subsets in $\bar{V}$ such that each subset $V_{s,i}$, $i = 1, \ldots, l_s$, has a spanning tree for its subgraph and there exists exactly one vertex $v_a \in V_{s,i}$ which satisfies the following

(i) For all $v_b \in V_{s,i} \setminus v_a$ and $v_c \in V \setminus V_{s,i}$, we have $(v_c, v_b) \notin \mathcal{E}$.

(ii) There exist at least two vertices in two different subgraphs (either primary or secondary layer) v_d and v_e such that $(v_d, v_a) \in \mathcal{E}$ and $(v_e, v_a) \in \mathcal{E}$.

(iii) v_a is the root of a spanning tree in $V_{s,i}$.

We define the subsets $V_{s,i}$, $i = 1, \ldots, l_s$ as the secondary layer subgraphs of $\mathcal{G}$.

5.2. Primary and Secondary Layer Subgraph Detection Algorithms

This section is devoted to our two algorithms which detect primary and secondary layer subgraphs in a directed graph. Furthermore, we discuss the performance of the algorithms in terms of convergence and time complexity.

Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the following algorithm can be used to determine the primary layer subgraphs.

```
procedure $[\bar{S}_1, \ldots, \bar{S}_{l_p}]$ = PrimaryLayer($\mathcal{G}(\mathcal{V}, \mathcal{E})$)
    for $i := 1$ to $n$ do
        Compute $S_i$, the set of reachable vertices by $v_i$
    end for
    Initialize the set $L_p := \{v_1, \ldots, v_n\}$
    for $i := 1$ to $n$ do
        for $j := 1$ to $n$ do
            if $i \neq j$ and $S_i \subseteq S_j$ then
                Update $L_p := L_p \setminus \{v_i\}$
            end if
        end for                                    ▷ The set $L_p$ consists of all vertices
    end for                                        ▷ in the primary layer subgraphs
    for all $v_i \in L_p$ do
        Compute the set $\bar{S}_i := S_i \setminus \bigcup_{j \neq i} S_j$
    end for
    return $[\bar{S}_1, \ldots, \bar{S}_{l_p}]$    ▷ $\bar{S}_i$ consists of the vertices in the
end procedure                                      ▷ $i$-th primary layer subgraph
```

Figure 5.3. Primary layer subgraph detection algorithm.

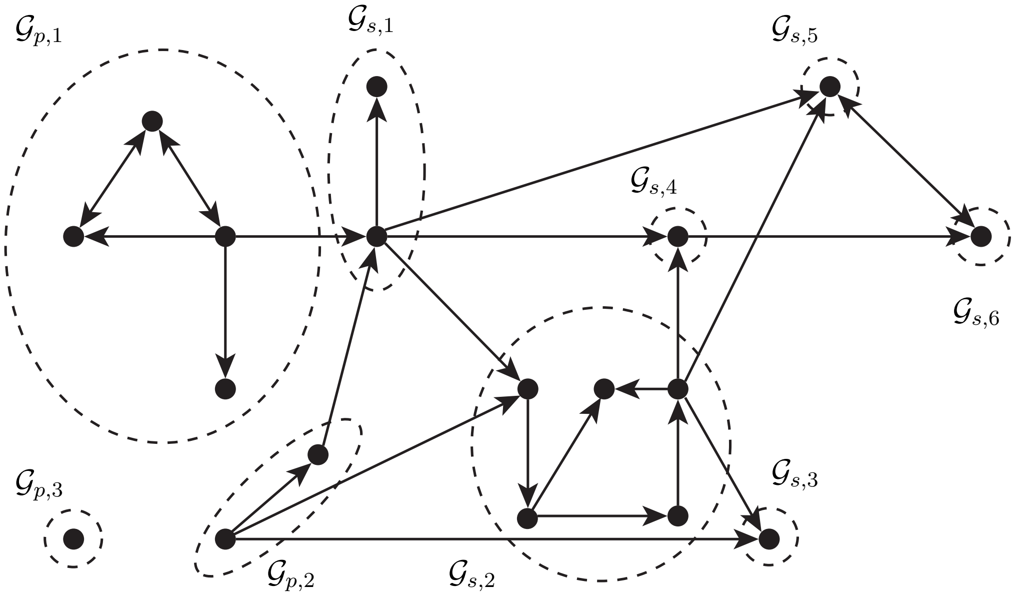

Figure 5.4. A directed graph with 18 vertices and 26 edges.

Example 5.2. Consider the network shown in Figure 5.4. By applying the primary layer detection algorithm, it can be determined that the primary layer subgraphs of the graph are $\mathcal{G}_{p,1}$, $\mathcal{G}_{p,2}$, $\mathcal{G}_{p,3}$. Note that despite having an isolated vertex, $\mathcal{G}_{p,3}$ is a primary layer subgraph since we assume every vertex has a self loop (Assumption 2.1(i)) and is a spanning tree itself.

Once the primary layer subgraphs of a network are determined, the algorithm given in Figure 5.2 can be used to extract the secondary layer subgraphs.

Example 5.3. Recall the network in Figure 5.4 where the primary layer subgraphs have been determined as $\mathcal{G}_{p,1}$, $\mathcal{G}_{p,2}$ and $\mathcal{G}_{p,3}$. By applying the secondary layer detection algorithm, the six secondary layer subgraphs of the network are extracted as $\mathcal{G}_{s,j}$, $j = 1, \ldots, 6$. Note that each secondary layer subgraph receives information from at least two other subgraphs (which may be either primary or secondary layer).

Remark 5.1. We remark here that our work is a generalization of the results of traditional consensus with a single equilibrium, which corresponds to a network with single primary layer, i.e., $K = 1$. The case of $K = 1$ requires the existence of a spanning tree to achieve consensus whereas we utilize novel concepts of primary and secondary layer subgraphs to analyze multi-equilibria consensus. Our analysis relies on two algorithms that determine primary and secondary layer subgraphs in a given

```
procedure $[\bar{S}_{l_p+1}, \dots, \bar{S}_{l_p+l_s}] = \textsc{SecondaryLayer}(\mathcal{G}(\mathcal{V}, \mathcal{E}), \bar{S}_1, \dots, \bar{S}_{l_p})$
    Initialize $l := l_p + 1$
    while $\bigcup \bar{S}_i \neq \mathcal{V}$ do
        Compute $L_s := \{v_k : (v_j, v_k) \in \mathcal{E}, \forall v_j \in \bigcup \bar{S}_i$ and $\forall v_k \notin \bigcup \bar{S}_i\}$
        for all $v_i \in L_s$ do
            Compute $\tilde{\mathcal{G}}_i$ and $\tilde{\mathcal{E}}_i$, the graph and the edge set obtained by removing all edges associated with the vertices $L_s \setminus v_i$, respectively
        end for
        for all $v_i \in L_s$ do
            Compute $\tilde{S}_i$, the set of reachable vertices by $v_i$ in the graph $\tilde{\mathcal{G}}_i$
        end for
        for all $v_i \in L_s$ do
            Compute $\bar{S}_l := \tilde{S}_i \setminus \bigcup_{i \neq j} \tilde{S}_j$
            Update $l := l + 1$
        end for
    end while                                        ▷ $\bar{S}_i$, $i = l_p + 1, \dots l_p + l_s$, consists of
    return $[\bar{S}_{l_p+1}, \dots, \bar{S}_{l_p+l_s}]$               ▷ the vertices in the $(i - l_p)$-th
end procedure                                        ▷ secondary layer subgraph
```

Figure 5.5. Secondary layer subgraph detection algorithm.

network. Note that along with the transitive closure computation algorithms which are used to find spanning trees in a network [62], the first three lines of primary layer subgraph detection algorithm may also be used for detection of the spanning trees.

5.2.1. Convergence Analysis of The Algorithms

Convergence of algorithm in Figure 5.2 is ensured since $L_p \neq \emptyset$ holds after line 12. This corresponds to the fact that there exists at least one primary layer subgraph in any given graph by Definition 5.2. The convergence of secondary layer subgraph detection algorithm in Figure 5.5 follows since the while loop will be executed unless all vertices are assigned into a subgraph and at each iteration, at least one secondary layer subgraph (and its vertices) will be determined. Hence, this ensures the while loop to be finite and the algorithm to be convergent.

5.2.2. Complexity Analysis of The Algorithms

In order to analyze the time complexity of the algorithm in Figure 5.2, we consider the following sub-operations:

(i) Computation of transitive closure: Transitive closure of a graph can be computed by algorithms such as Depth-First Search (DFS), Breadth-First Search (BFS) and Floyd-Warshall Algorithm with a time complexity of $O(n^3)$ [62], [63].

(ii) Set sorting: The elements of an unordered set can be sorted by algorithms such as Heapsort or Mergesort with a time complexity of $O(nlog(n))$ [63], [64].

(iii) Set difference: Given two sorted sets, the set difference can be computed by linear scan which brings a time complexity $O(n)$.

(iv) Set union: The union set of k sorted sets can be computed by linear scan with time complexity $O(kn)$.

(v) Set initialization: A set consisting of n elements can be initialized in time $O(n)$.

In Figure 5.2, as a first step, the transitive closure of $\mathcal{G}$ is computed in time $O(n^3)$ and the set L_p is initialized in time $O(n)$. The sets S_i, $i = 1, \ldots, n$, can be sorted in time $O(n)O(nlog(n)) = O(n^2log(n))$. Then, for each vertex pair (v_i, v_j), the sorted reachability sets are compared (set difference) which requires time $O(n^2)O(n) = O(n^3)$. Finally, the computation of the sets $\bar{S}_i$ requires set union and set difference operations which yield the time complexity $O(n)O(n^2)O(n) = O(n^4)$. Therefore, we conclude that the total time complexity of primary layer subgraph detection algorithm is $O(n^3) + O(n) + O(n^2log(n)) + O(n^3) + O(n^4) = O(n^4)$.

In secondary layer subgraph detection algorithm, the variable l is initialized in time $O(1)$. For the computation of set L_s, both set union and set difference operations need to be applied and all (v_j, v_k) pairs need to be checked whether there exist links between them. Therefore, the computation of L_s requires time $O(n^2)O(n^2)O(n) =$

$O(n^5)$. The computation of $\tilde{\mathcal{G}}_i$ requires time $O(n^2)$ since in the worst case $2n$ links have to be removed. The sets $\tilde{S}_i$ are the sets of reachable vertices (transitive closure) and their computation requires $O(n^3)$ time. For the computation of $\bar{S}_l$, set union and set difference operations must be applied at most n times and therefore it requires time $O(n)O(n)O(n^2) = O(n^4)$. Thus, in each while loop execution, the time complexity is $O(n^5) + O(n^2) + O(n^3) + O(n^4) = O(n^5)$. Since at least one secondary layer subgraph is determined in each iteration, the while loop will be executed at most $n-2$ times. Consequently, the time complexity of secondary layer subgraph detection algorithm is $O(n-2)O(n^5) = O(n^6)$.

Note that the time complexities of primary and secondary layer subgraph detection algorithms are both polynomials of n and hence for any given graph, the primary and secondary layer subgraphs can be determined in polynomial time.

5.3. Convergence Analysis of Multi-Equilibria Consensus

Armed with the powerful concepts of primary and secondary layer subgraphs, we can proceed to determine the number of consensus equilibria of a given network in the sense of Definition 2.9. To this end, note that any network matrix W can be transformed into the following form

$$W=\begin{bmatrix} W_{1,1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & W_{l_p,l_p} & 0 & \cdots & 0 \\ W_{l_p+1,1} & \cdots & \cdots & W_{l_p+1,l_p+1} & \cdots & W_{l_p+1,l_p+l_s} \\ \vdots & & & & \ddots & \vdots \\ W_{l_p+l_s,1} & \cdots & \cdots & \cdots & \cdots & W_{l_p+l_s,l_p+l_s} \end{bmatrix}, \tag{5.1}$$

where the first l_p blocks correspond to the primary layer subgraphs and the rest correspond to the secondary layer subgraphs. In this setting, vertices in the primary layer subgraphs have smaller indices (as compared to those of in the secondary layer subgraphs) and the vertices in the same subgraph are labeled successively.

Let $\mathcal{I}_p = \{1, \dots l_p\}$ and $\mathcal{I}_s = \{1, \dots l_s\}$ be index sets and let $n_{p,i}$, $i \in \mathcal{I}_p$, and $n_{s,j}$, $j \in \mathcal{I}_s$, be the number of vertices in the i-th primary layer and j-th secondary layer subgraphs, respectively. Let the total number of vertices in the primary and secondary layer subgraphs be denoted by $\bar{n}_p = \sum_{i=1}^{l_p} n_{p,i}$ and $\bar{n}_s = \sum_{j=1}^{l_s} n_{s,j}$, respectively. Let $x_p(k) \in R^{\bar{n}_p}$ and $x_s(k) \in R^{\bar{n}_s}$ denote the state vectors of the agents in primary and secondary layer subgraphs, respectively.

Then, we can express (2.4) as

$$\begin{bmatrix} x_p(k+1) \\ x_s(k+1) \end{bmatrix} = \begin{bmatrix} W_p & 0 \\ B & W_s \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_s(k) \end{bmatrix} \tag{5.2}$$

where W_p =blkdiag$(W_{i,i}) \in R^{\bar{n}_p \times \bar{n}_p}$, $i \in \mathcal{I}_p$,

$$W_s = \begin{bmatrix} W_{l_p+1,l_p+1} & \cdots & W_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ W_{l_p+l_s,l_p+1} & \cdots & W_{l_p+l_s,l_p+l_s} \end{bmatrix}_{\bar{n}_p \times \bar{n}_p}$$

and

$$B = \begin{bmatrix} W_{l_p+1,1} & \cdots & W_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ W_{l_p+l_s,1} & \cdots & W_{l_p+l_s,l_p} \end{bmatrix}_{\bar{n}_s \times \bar{n}_p} .$$

We now present Lemmas 5.1, 5.2 and 5.3 which will be instrumental in proving the main contributions of the chapter.

Lemma 5.1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a graph of n vertices which includes a spanning tree and let the i–th vertex be the root of the spanning tree. Let $\bar{A}$ be a non–negative matrix with the same zero pattern as $A^T + I$ where I is the $n \times n$ identity matrix and let the $i-th$ row of $\bar{A}$ sum up to a value that is less than one while all other rows sum up to one. Then all eigenvalues of $\bar{A}$ are strictly less than one in magnitude.

Proof. In at most $n-1$ steps, all vertices in $\mathcal{G}$ are reachable by vertex i. Therefore $\bar{A}^n$ is a matrix with all row sums less than one. From Lemma 5.1, we conclude that all eigenvalues of $\bar{A}^n$ lie inside the unit circle, i.e., $|\lambda_i(\bar{A}^n)| < 1$, $i = 1, \ldots, n$. Hence each eigenvalue of $\bar{A}^n$ has magnitude less than one. ☐

Lemma 5.2. Let $\bar{A}$ be an $n \times n$ non-negative matrix. Let the $i - th$ row of $\bar{A}$ sum up to $1 - \gamma$ where $\gamma > 0$ and all other rows sum up to one. Then the $i - th$ column of $(I - \bar{A})^{-1}$ is equal to $\frac{1}{\gamma}\mathbf{e}$.

Proof. Since $\bar{A}$ is an $n \times n$ non-negative matrix, given that the i-th row of $\bar{A}$ adds up to $1 - \gamma$, we have

$$\bar{A}\mathbf{e} = \mathbf{e} - \gamma q_i \tag{5.3}$$

where q_i is the basis vector which has 1 as its i-th component and zeros elsewhere. It follows from Lemma 5.1 that $(I - \bar{A})$ is invertible. Hence, we can re-arrange (5.3) as

$$\mathbf{e} = (I - \bar{A})^{-1}\gamma q_i \tag{5.4}$$

where I is the $n \times n$ identity matrix. Therefore, the i-th column of $(I - \bar{A})^{-1}$ is computed as $\frac{1}{\gamma}\mathbf{e}$. ☐

Lemma 5.3. Let W be a row-stochastic matrix that satisfies Assumption 2.1. Then $\lim_{k\to\infty} W^k$ exists.

Proof. From Lemma 2.2, the largest eigenvalue (in magnitude) of W is 1 (with multiplicity not necessarily equal to 1) and there is no other eigenvalue with magnitude 1. Since $||W^k||_\infty$ are uniformly bounded for all k, the algebraic and geometric multiplicity of the eigenvalue 1 are equal [65]. Therefore, we conclude that $\lim_{k\to\infty} W^k$ exists. ☐

One of the fundamental contributions of this chapter is given in Theorem 5.1 below.

Theorem 5.1. The number of equilibrium states for a network with graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is given by

$$K = l_p + l_s \tag{5.5}$$

where l_p and l_s are the number of primary and secondary layer subgraphs, respectively.

Proof. From Lemma 5.3, we conclude that $\lim_{k\to\infty} W^k$ exists. As $k \to \infty$, the state vector can be computed as

$$x_\infty = [x_{1,\infty}^T, \dots, x_{n,\infty}^T]^T = \lim_{k\to\infty} x(k) = \lim_{k\to\infty} W^k x(0).$$

Since we are interested in the number of equilibrium states, we will investigate the number of distinct $x_{i,\infty}$. Due to the lower triangular structure of W in (5.1), for any $k \geq 1$, the first l_p diagonal blocks of W^k are simply the k-th powers of the corresponding diagonal blocks of W. Therefore, the first l_p blocks of W^k can be represented as row stochastic matrices $(W_{i,i})^k$, $i = 1, \dots, l_p$. As $k \to \infty$, $(W_{i,i})^k$ converge to rank one matrices [3] which results in l_p equilibria states in l_p primary layer subgraphs.

Let n_i, $i = 1, \dots, l_p + l_s$, be the number of vertices in the i–th subgraph and let $\bar{x}_i \in R^{n_i \times 1}$, $i = 1, \dots, l_p + l_s$, be the state vectors for the corresponding subgraphs. Then the update equation (2.3) can be re-written as

$$\bar{x}_i(k+1) = W_{i,i}\bar{x}_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{l_p+l_s} W_{i,j}\bar{x}_j(k), \tag{5.6}$$

where $W_{i,j}$ are sub-matrix blocks of W defined in (5.1). In the rest of the proof, we consider the secondary layer subgraphs, i.e., $l_p < i \leq l_p + l_s$. Applying z-transform to both sides of (5.6), we obtain

$$z\bar{X}_i(z) - z\bar{X}_i(0) = W_{i,i}\bar{X}_i(z) + \sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\bar{X}_j(z) \tag{5.7}$$

$$(zI - W_{i,i})\bar{X}_i(z) = z\bar{X}_i(0) + \sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\bar{X}_j(z) \tag{5.8}$$

$$\bar{X}_i(z) = (zI - W_{i,i})^{-1}z\bar{X}_i(0) + (zI - W_{i,i})^{-1}\sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\bar{X}_j(z). \tag{5.9}$$

Since $\bar{x}_{i,\infty} = \lim_{k\to\infty} \bar{x}_i(k)$ exists, we apply the final value theorem to obtain

$$\bar{x}_{i,\infty} = \lim_{z\to 1}(1 - z^{-1})\bar{X}_i(z) \tag{5.10}$$

$$= \lim_{z\to 1}(1 - z^{-1})(zI - W_{i,i})^{-1}z\bar{X}_i(0) \tag{5.11}$$

$$+ \lim_{z\to 1}(1 - z^{-1})(zI - W_{i,i})^{-1}\sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\bar{X}_j(z). \tag{5.12}$$

For secondary layer subgraphs, we have $\lim_{z\to 1}(1 - z^{-1})(zI - W_{i,i})^{-1}z\bar{X}_i(0) = 0$ and therefore

$$\bar{x}_{i,\infty} = \lim_{z\to 1}(1 - z^{-1})(zI - W_{i,i})^{-1}\sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\bar{X}_j(z) \tag{5.13}$$

$$= (I - W_{i,i})^{-1}\sum_{\substack{j=1\\j\neq i}}^{l_p+l_s} W_{i,j}\lim_{z\to 1}(1 - z^{-1})\bar{X}_j(z) \tag{5.14}$$

$$= \sum_{\substack{j=1\\j\neq i}}^{l_p+l_s}(I - W_{i,i})^{-1}W_{i,j}\bar{x}_{j,\infty}. \tag{5.15}$$

Let r_i be the index of the vertex that is the root of the spanning tree for the subgraph i. From Lemma 5.2, the r_i-th column of $(I - W_{i,i})^{-1}$ has identical components and all rows of $W_{i,j}$ except for the r_i-th row are composed of zeros which results in $(I - W_{i,i})^{-1}W_{i,j}$ to have identical rows. Hence we conclude that $\bar{x}_{i,\infty} = \mathbf{e}c_i$ where c_i is calculated in terms of the primary and remaining secondary layer subgraphs. This implies that vertices in a specific secondary layer subgraph agree on a specific value that is not necessarily equal to a consensus value of any other layer. Consequently, the total number of equilibria can be represented as the sum of the number of equilibria in primary and secondary layer subgraphs, i.e., $K = l_p + l_s$. ☐

Theorem 5.1 delineates the number of consensus equilibria for a given digraph in terms of primary and secondary layer subgraphs. Theorem 5.1 is novel in the sense that it extends the previous consensus results in terms of expressing the equilibria. For instance, the traditional consensus problem of achieving a single common state ($K = 1$) corresponds to the case with a single primary layer subgraph ($l_p = 1$), i.e., the graph has a spanning tree (this is the condition given in [3]). This can be easily seen by Theorem 5.1 for the case $K = 1$, since from Definition 5.2, if there exists only one primary layer subgraph, then the network cannot have any secondary layer subgraphs.

As another direct consequence of Theorem 5.1 is that a graph with n vertices will achieve $K = n$ consensus equilibria if and only if each primary and secondary layer subgraph is composed of a single vertex. One shall see in the next example that the network reaches $K = l_p + l_s$ consensus equilibria as implied by Theorem 5.1.

Example 5.4. Revisit the network with 18 agents given in Figure 5.4. For a random choice of initial state values $x(0)$ and weighting coefficients w_{ij} which satisfy Assumption 2.1, the evolution of the states is depicted in Figure 5.6. The network converges to 9 equilibrium consensus states with $l_p = 3$ primary and $l_s = 6$ secondary layer subgraphs which verifies our theoretical results.

Theorem 5.1 can be further manipulated to yield the following important results regarding two and three equilibria, respectively.

Corollary 5.1. A directed network of agents utilizing protocol (2.3) under Assumption 2.1 converges to two equilibrium consensus states ($K = l_p + l_s = 2$) if and only if there exist two primary layer subgraphs.

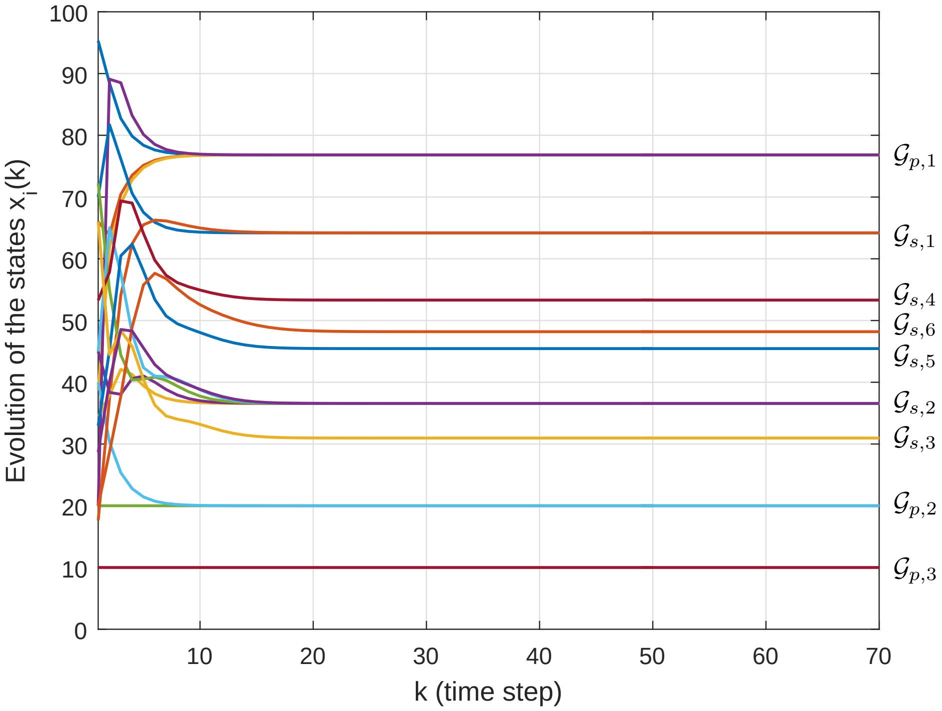

Figure 5.6. The simulation results for the network given in Figure 5.4.

Proof. From Theorem 5.1, a directed network of agents utilizing protocol (2.3) under Assumption 2.1 can have $K = l_p + l_s = 2$ equilibria if and only if one of the three cases $(l_p, l_s) \in \{(1,1),(0,2),(2,0)\}$ holds. In the following we will show that the cases $(l_p, l_s) \in \{(1,1),(0,2)\}$ are not feasible whereas $(l_p, l_s) \in \{(2,0)\}$ is the only feasible case that is stated in the Corollary.

Case (i): $(l_p, l_s) = (1,1)$ Suppose that the network has only one primary layer subgraph (l_p=1). By Definition 5.3, a secondary layer subgraph has to receive information from at least two other subgraphs. In this case, this is not feasible as there is only one other subgraph (which is primary).

Case (ii): $(l_p, l_s) = (0,2)$ From Definition 5.2, every graph must have at least one primary layer subgraph, which contradicts $l_p = 0$.

Case (iii): $(l_p, l_s) = (2,0)$ This case is feasible by Definitions 5.2 and 5.3. ☐

Remark 5.2. Note that the condition stated in the Corollary 5.1 being sufficient is somewhat straightforward based on the spanning tree condition in [3], i.e., when there exist two primary layer subgraphs, each will converge to two separate final values. On the other hand, what makes Corollary 5.1 significant is that it states that this condition is also necessary, i.e., there cannot exist a graph which does not obey the condition but still has 2 equilibria.

Corollary 5.2. A directed network of agents utilizing protocol (2.3) under Assumption 2.1 converges to three equilibrium consensus states ($K = l_p + l_s = 3$) if and only if

(i) there exist three primary layer subgraphs, or
(ii) there exist two primary and one secondary layer subgraphs.

Proof. From Theorem 5.1, a directed network of agents utilizing protocol (2.3) under Assumption 2.1 can have $K = l_p + l_s = 3$ equilibria if and only if one of the four cases $(l_p, l_s) \in \{(1,2), (0,3), (3,0), (2,1)\}$ holds. We now show that the cases $(l_p, l_s) \in \{(1,2), (0,3)\}$ are not feasible whereas the feasible cases $(l_p, l_s) \in \{(3,0), (2,1)\}$ are the conditions stated in the Corollary.

Case (i): $(l_p, l_s) = (1,2)$ Suppose that there exist one primary layer and two secondary layer subgraphs in the network. From Definition 5.3, the roots of the secondary layer subgraphs have to receive information from at least two vertices that are in different subgraphs. For $(l_p, l_s) = (1,2)$, both roots of the secondary layer subgraphs must receive information from a vertex (not necessarily the same) in the primary layer subgraph. This condition implies that there exist directed paths to the roots of the secondary layer subgraphs from the root of the primary layer subgraph, i.e., the whole graph itself contains a spanning tree. This contradicts $l_s = 2$.

Case (ii): $(l_p, l_s) = (0,3)$ Similar to the Case (ii) of Corollary 5.1, from Definitions 5.2 and 5.3 there must exist at least one primary layer in any graph which contradicts $l_p = 0$.

Case (iii): $(l_p, l_s) = (3,0)$ This case which is identified as Condition (i) in the Corollary is feasible based on Definition 5.2.

Case (iv): $(l_p, l_s) = (2,1)$ This case which is identified as Condition (ii) in the Corollary is feasible based on Definitions 5.2 and 5.3. □

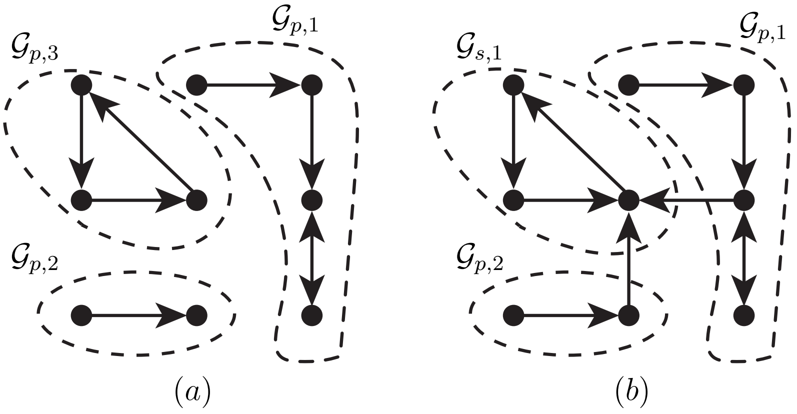

Figure 5.7. Illustration of Corollary 5.2: (a) Condition (i), (b) Condition (ii).

Remark 5.3. Note that Case (iii) stated in the Corollary 2 is somewhat straightforward from Corollary 5.1 and the spanning tree condition of [3]. On the other hand, a network with two primary and one secondary layer subgraphs $(l_p, l_s) = (2, 1)$ has also three consensus equilibria. Corollary 5.2 presents not only sufficient but also necessary conditions on the network structure for three consensus equilibria.

In the following example, we illustrate Corollary 5.2.

Example 5.5. Consider the two 9 agent networks shown in Figure 5.7. Note that the network depicted in Figure 5.7a consists of $l_p = 3$ primary layer subgraphs whereas the network shown in Figure 5.7b has $l_p = 2$ primary and $l_s = 1$ secondary layer subgraphs. Both networks achieve 3 equilibrium consensus states which is consistent with the simulation results shown in Figures 5.8 and 5.9.

Corollaries 5.1 and 5.2 give the necessary and sufficient conditions for two and three consensus equilibria, respectively. For $K \geq 3$, the conditions on the network structure are generalized in the following corollary.

Corollary 5.3. If a directed network using protocol (2.3) achieves $K \geq 3$ consensus equilibria, there exist at least two primary layer subgraphs ($l_p \geq 2$).

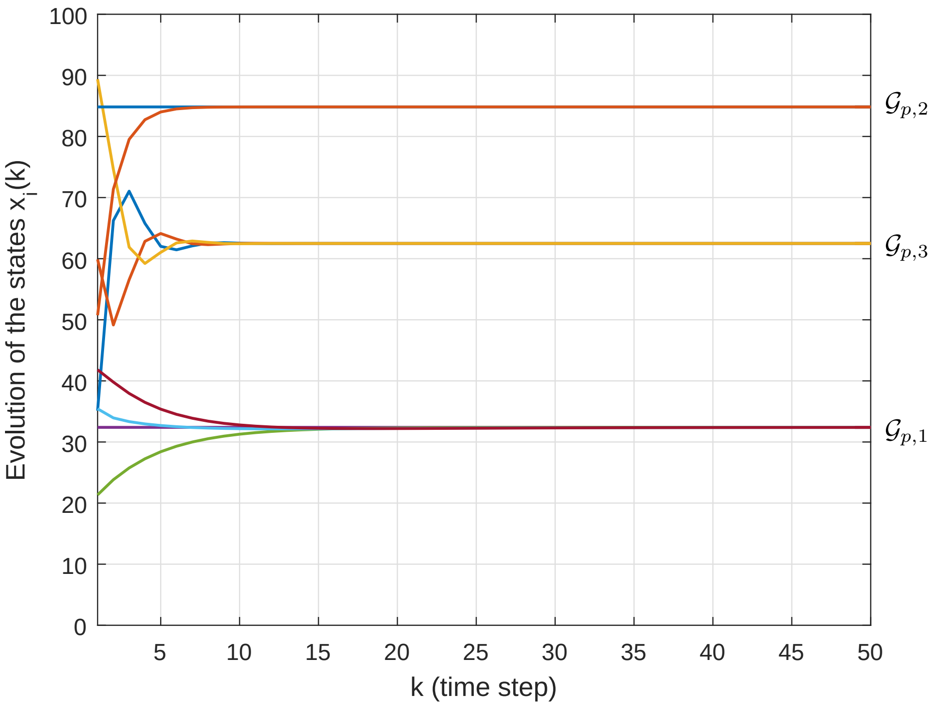

Figure 5.8. The simulation results for the network given in Figure 5.7 (a).

Proof. The proof follows by eliminating the cases $l_p = 0$ and $l_p = 1$ by Definitions 5.2 and 5.3. Therefore, for $K = l_p + l_s \geq 3$, the set of possible (l_p, l_s) combinations are given by $\{(2, K-2), \ldots, (K-1, 1), (K, 0)\}$. ☐

The following example shows that Theorem 5.1 can also be applied to the networks of agents with continuous-time integrator dynamics (2.5).

Example 5.6. Reconsider the network given in Figure 5.4 where the agents update their states by utilizing protocol (2.5) and the weighting coefficients are chosen arbitrarily according to the Assumption 2.2. Although the evolution of the states are different from its discrete-time counterpart, the network achieves $K = l_p + l_s = 9$ equilibria as depicted in Figure 5.10 which verifies that the results obtained for networks with discrete-time dynamics also hold for networks with continuous-time dynamics.

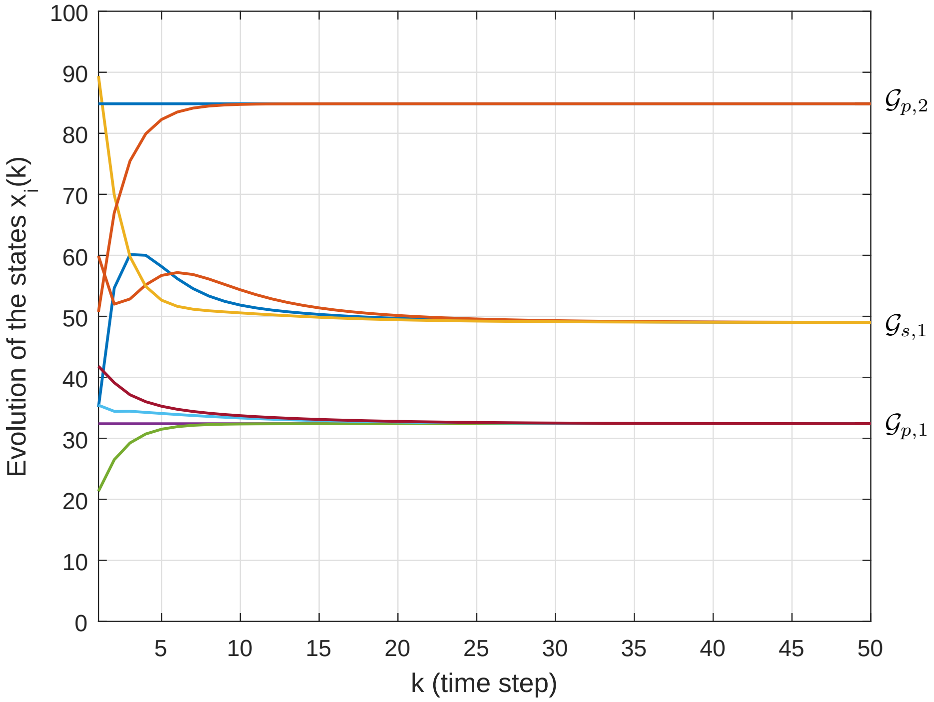

Figure 5.9. The simulation results for the network given in Figure 5.7b.

5.4. Convergence Rate of the Protocol

The convergence rate of a consensus protocol is a very important metric for the evaluation of its performance. It has been studied by many authors for the traditional consensus problem and there exist numerous results in the literature (see e.g., [66]). It is well-known that the convergence rate of traditional consensus protocols is related to the magnitude of the second largest eigenvalue (SLEM) of the system matrix. In the problem of multi-equilibria consensus, the multiplicity of the largest eigenvalue 1 is greater than one. In fact, it can be concluded from the block diagonal structure of W that the multiplicity of the eigenvalue 1 is equal to the number of primary layer subgraphs, i.e., l_p. Therefore, the eigenvalue of interest is the largest eigenvalue of W that is less than one, which can be expressed as

$$\max \left\{ \lambda_2(W_{1,1}), \ldots, \lambda_2(W_{l_p,l_p}), \lambda_1 \left(\begin{bmatrix} W_{l_p+1,l_p+1} & \cdots & W_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ W_{l_p+l_s,l_p+1} & \cdots & W_{l_p+l_s,l_p+l_s} \end{bmatrix} \right) \right\}$$

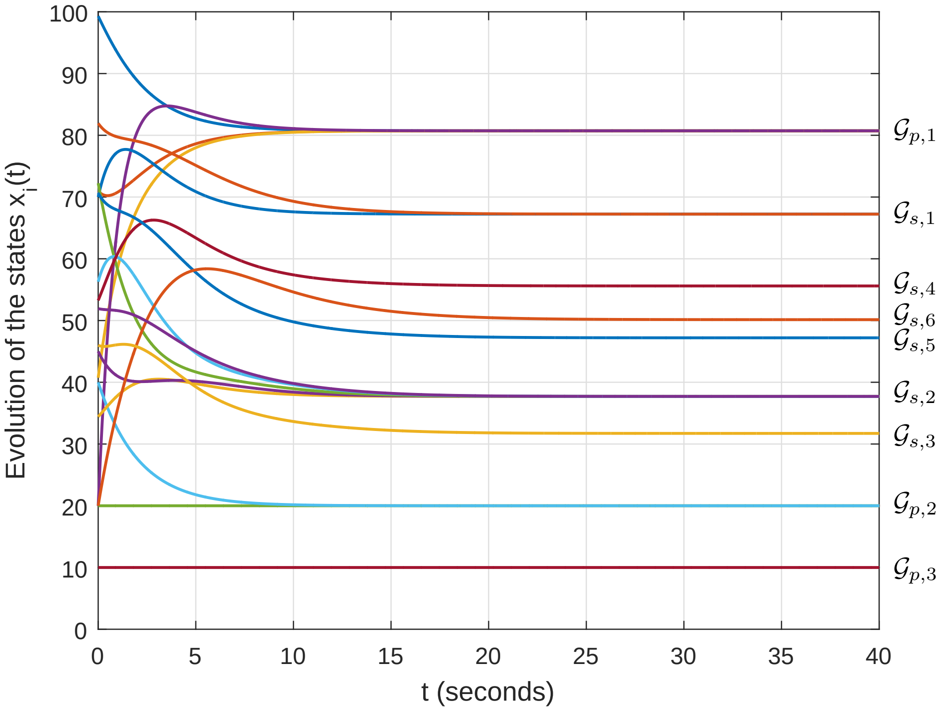

Figure 5.10. The simulation results for the network given in Figure 5.4 with continuous-time dynamics.

where $\lambda_i(\cdot)$ denotes the i-th largest eigenvalue in magnitude. In other words, once the primary and secondary layer subgraphs are determined, the convergence rate of the protocol can be easily analyzed.

5.5. Numerical Examples

The theoretical results presented in this chapter are demonstrated by two further examples. In the first case, we illustrate how the selection of weighting coefficients of communication links affects convergence rate even if the network topology remains the same. The latter case provides a useful insight into the network design with the desired number of equilibria, which also paves the way for practical implementations.

Example 5.7. Reconsider the network given in Example 5.4. For a different choice of weighting coefficients, evolution of the states $x_i(k)$ is depicted in Figure 5.11. Although evolution of the states and the final values are different from those of in Example 5.4, the network also achieves $K = l_p + l_s = 9$ equilibria. In Example 5.4, the

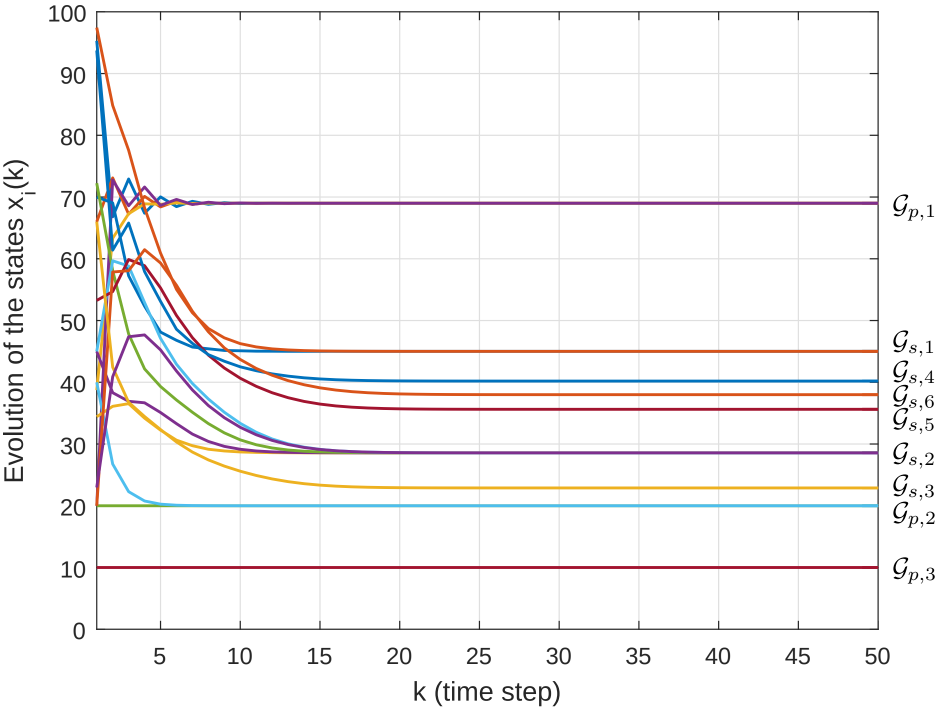

Figure 5.11. The simulation results for Example 5.7.

largest eigenvalue of W that is less than one is 0.8931 which is greater than the largest eigenvalue of W in this example, which is computed as 0.8809. Note that the choice of weighting coefficients directly affects the convergence rate of the algorithm, where in this case the network in Example 5.7 converges faster than the network in Example 5.4. Optimization of the weighting coefficients to achieve optimal convergence speed is an open problem which we leave as a future work.

Example 5.8. The results presented in Section 5.3 can be utilized in the design of networks in a distributed context. For instance, reconsider the network with 18 agents depicted in Figure 5.4. The number of subgraphs can be reduced significantly if the link between $\mathcal{G}_{p,1}$ and $\mathcal{G}_{s,1}$ is removed. In this scenario, the network converges to three equilibria as illustrated in Figure 5.12 since there will be only three primary layer subgraphs ($K = l_p + l_s = 3$). On the other hand, if the link between the vertices in $\mathcal{G}_{p,2}$ is removed, the network converges to $K = l_p + l_s = 4 + 6 = 10$ equilibria as shown in Figure 5.13. Similar logic can be worked out for adding new links.

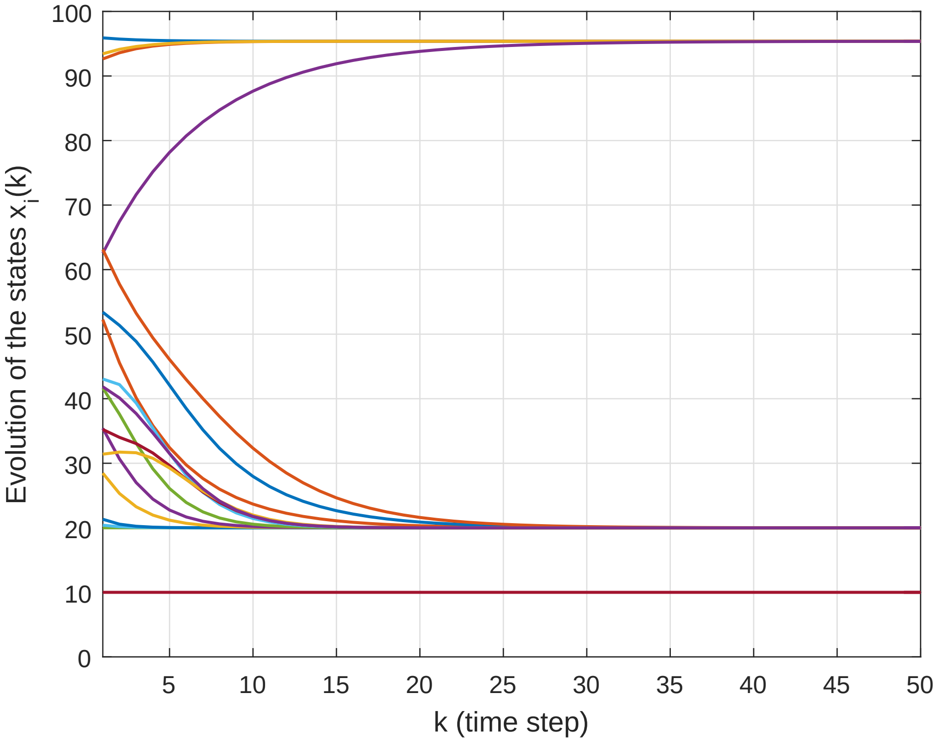

Figure 5.12. The simulation results for the network given in Figure 5.4 when the link between $\mathcal{G}_{p,1}$ and $\mathcal{G}_{s,1}$ is removed.

5.6. Chapter Summary & Concluding Remarks

This chapter has investigated the problem of multi-equilibria consensus for directed graphs under fixed interaction. A novel method has been developed to analyze the convergence properties such networks which rely on two new notions of primary and secondary layer subgraphs. Furthermore, we have proposed algorithms that detect primary and secondary layer subgraphs in any given network. We have theoretically shown that total number of primary and secondary layer subgraphs is equal to the number of equilibria of the network. Finally, some examples are presented to illustrate the validity of our theoretical results.

We believe that the concepts of primary and secondary layer subgraphs not only provide fundamental structures for designing networks with desired number of equilibria but also enable one to better understand the behavior of consensus protocols under different topologies, and network sizes. One potential example may be determination

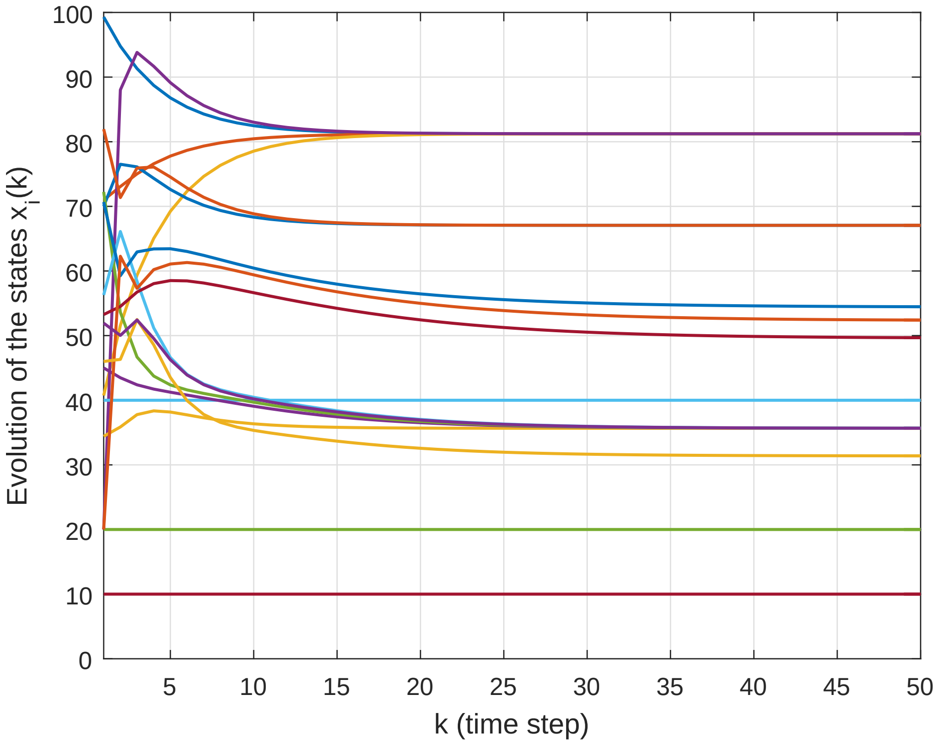

Figure 5.13. The simulation results for the network given in Figure 5.4 when the link between the vertices in $\mathcal{G}_{p,2}$ is removed.

of the number of opinions in a social network in which there exists local information exchange between individuals. Therefore, one can gain more insight about the behavior of a society by modelling its interactions. Another interesting example may be developing a communication network for unmanned air vehicle groups where each group achieves a different task.

6. MULTI-EQUILIBRIA CONSENSUS IN DIRECTED NETWORKS UNDER TIME-DELAYS

In Chapters 3, 4 and 5, we have studied the multi-equilibria consensus problem in the absence of delays. However, in most of the practical systems, delay is encountered due to packet losses or sampling [8,67]. Delays in networked systems can roughly be classified into three types, namely, computation delay, waiting delay and transmission delay. The first two types can be negligible to some extent when compared with transmission delays [67]. As transmission delays are one of the potential causes of instability and degraded performance in networked systems, the possible effects of delay in networks with multi-equilibria have to be examined properly. Therefore, we first introduce the mathematical model of consensus protocol with time-delays. Afterwards, we provide stability analysis for directed networks under uniform and nonuniform transmission time-delays. Additionally, we discuss the performance of time-delayed protocol in terms of its convergence rate.

6.1. Distributed Consensus Protocol with Time-Delays

In this section, we formulate the consensus problem in the presence of communication (transmission) delays between agents, and present the assumption imposed on the protocol.

Consider a network of n agents where each agent updates its state according to the following dynamics:

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j=1, j\neq i}^{n} w_{ij}x_j(k-\tau_{ij}(k)) \tag{6.1}$$

where $w_{ij} \geq 0$ is the averaging coefficient; and $\tau_{ij}(k) \leq \tau_{max}$ is the amount of bounded delay in communication between agents j and i at time step k. We will require the following assumption throughout this chapter.

Assumption 6.1. (i) $w_{ii} \geq \delta,$ for all $i \in \mathcal{I}$ and some positive constant δ.

(ii) $w_{ij} = \begin{cases} \in [\delta, 1), & \text{if} \quad (v_j, v_i) \in \mathcal{E} \\ 0, & \text{if} \quad (v_j, v_i) \notin \mathcal{E} \end{cases}$, for all $\quad i, j \in \mathcal{I}, \quad i \neq j.$

(iii) $\sum\limits_{j=1}^{n} w_{ij} = 1,$ for all $i \in \mathcal{I}$.

(iv) $\tau_{ij}(k) \leq \tau_{max},$ for all $i, j \in \mathcal{I}$.

Assumption 6.1(i) states that each agent uses its own information in its update. Assumption 6.1(ii) ensures that the information received from a neighbor should be used with strictly positive weighting. Assumption 6.1(iii) indicates that the sum of weighting coefficients for each agent adds up to one. The first three conditions of Assumption 6.1 guarantee that the resulting network matrix W is a stochastic matrix. Finally, Assumption 6.1(iv) states that the delays are upper bounded by τ_{max} for all agents. Note that agents use their local values without delay, since delay only affects the state information coming from the neighbors.

In this chapter, we revisit the multi-equilibria consensus problem given in Definition 2.9 and analyze it for the networks with (i) uniform fixed delay and (ii) non-uniform fixed delay.

6.2. Analysis of the Protocol under Uniform Fixed Delay

In this section, we examine the convergence properties of the protocol (6.1) where the delayed network achieves multi-equilibria consensus.

Let $\hat{x}_p$ and $\hat{x}_s$ denote the augmented state vectors of the primary and secondary layer subgraphs, respectively, and given by

$$\begin{aligned} \hat{x}_p(k) &= [x_p(k)^T, x_p(k-1)^T, \ldots, x_p(k-\tau_{max})^T]^T \\ \hat{x}_s(k) &= [x_s(k)^T, x_s(k-1)^T, \ldots, x_s(k-\tau_{max})^T]^T. \end{aligned} \tag{6.2}$$

Consider the case in which there exists a fixed uniform communication delay between vertices, i.e., $\tau_{ij}(k) = \tau < \infty \ \forall \ i \neq j \in \mathcal{I}$ and $k \in \mathbb{N}$. With this assumption, (6.1) reduces to

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j=1, j\neq i}^{n} w_{ij}x_j(k-\tau). \tag{6.3}$$

For the primary layer subgraphs, the delayed consensus protocol (6.3) becomes

$$x_p(k+1) = W_{p,D}x_p(k) + (W_p - W_{p,D})x_p(k-\tau) \tag{6.4}$$

where $W_{p,D} \in R^{\bar{n}_p \times \bar{n}_p}$ is a diagonal matrix which consists of the diagonal components of W_p. For the vertices in the secondary layer subgraphs, distributed consensus protocol (6.3) can be rewritten as

$$x_s(k+1) = W_{s,D}x_s(k) + (W_s - W_{s,D})x_s(k-\tau) + Bx_p(k-\tau) \tag{6.5}$$

where $W_{s,D}$ is a diagonal matrix which consists of the diagonal components of W_s.

Then, we can express (6.4) and (6.5) in the matrix form as

$$\begin{bmatrix} \hat{x}_p(k+1) \\ \hat{x}_s(k+1) \end{bmatrix} = \begin{bmatrix} \hat{W}_p & 0 \\ \hat{B} & \hat{W}_s \end{bmatrix} \begin{bmatrix} \hat{x}_p(k) \\ \hat{x}_s(k) \end{bmatrix} \tag{6.6}$$

where

$$\hat{W}_p = \begin{bmatrix} W_{p,D} & 0 & \cdots & 0 & W_p - W_{p,D} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{(\tau+1)\bar{n}_p \times (\tau+1)\bar{n}_p} \tag{6.7}$$

and

$$\hat{W}_s = \begin{bmatrix} W_{s,D} & 0 & \cdots & 0 & W_s - W_{s,D} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{(\tau+1)\bar{n}_s \times (\tau+1)\bar{n}_s} \tag{6.8}$$

are the augmented matrices for the primary and secondary layer subgraphs, respectively, and

$$\hat{B} = \begin{bmatrix} 0 & \cdots & 0 & B \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(\tau+1)\bar{n}_s \times (\tau+1)\bar{n}_p} \tag{6.9}$$

is the augmented input matrix for the secondary layer subgraphs and $\hat{x}_p$ and $\hat{x}_s$ are as defined in (6.2). The following theorem allows us to determine the number of consensus equilibria of a directed network under delayed information.

Theorem 6.1. Consider a network of agents with underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which has l_p primary and l_s secondary layer subgraphs. When the agents utilize protocol (6.3), under Assumption 6.1, the number of consensus equilibria is given by

$$K = l_p + l_s. \tag{6.10}$$

Proof. Let T_p and T_s be transformation matrices given by $T_p = [T_{p,1}, \ \ldots, \ T_{p,\bar{n}_p}]^T$ and $T_s = [T_{s,1}, \ \ldots, \ T_{s,\bar{n}_s}]^T$ where $T_{p,i} = [q_i, \ q_{\bar{n}_p+i}, \ \ldots, \ q_{\tau_{max}\bar{n}_p+i}]$ and $T_{s,i} = [q_i, \ q_{\bar{n}_s+i}, \ \ldots, \ q_{\tau_{max}\bar{n}_s+i}]$. Let the block diagonal transformation matrix T be equal to

$$T = \begin{bmatrix} T_p & 0 \\ 0 & T_s \end{bmatrix}. \tag{6.11}$$

Multiplying (6.6) from left by T and from right by T^{-1}, we obtain

$$\begin{bmatrix} \tilde{x}_p(k+1) \\ \tilde{x}_s(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{W}_p & 0 \\ \tilde{B} & \tilde{W}_s \end{bmatrix} \begin{bmatrix} \tilde{x}_p(k) \\ \tilde{x}_s(k) \end{bmatrix} \tag{6.12}$$

where $\tilde{x}_p(k) = T_p\hat{x}_p(k)$, $\tilde{x}_s(k) = T_s\hat{x}_s(k)$, $\tilde{W}_p = T_p\hat{W}_pT_p^{-1}$, $\tilde{B} = T_s\hat{B}T_p^{-1}$ and $\tilde{W}_s = T_s\hat{W}_sT_s^{-1}$. Note that $\tilde{x}_p$ and $\tilde{x}_s$ are reordered versions of the augmented state vectors $\hat{x}_p$ and $\hat{x}_s$ where a vertex and its delayed versions are ordered successively. Therefore, (6.1) and (6.12) are equivalent consensus protocols for the networks with uniform fixed communication delay. Since the system matrix of (6.12) satisfies the conditions of Theorem 5.1, the number of consensus equilibria for a network with uniform fixed delay is equal to $K = l_p + l_s$. □

Remark 6.1. Throughout this chapter, we consider that each agent uses its own information without delay. Although it is not meaningful in terms of practical applications, if each agent uses its own information with the same amount of delay τ, the consensus protocol can be represented by $x(k+1) = Wx(k-\tau)$ which shares the same convergence properties as the consensus protocol in networks without delay.

6.3. Analysis of the Protocol under Nonuniform Fixed Delay

Since the amount of delay between agents is not identical in practical systems, we further investigate the multi-equilibria consensus problem in networks with nonuniform communication delays. We now consider the case where the delay is nonuniform and fixed, i.e., $\tau_{ij}(k) = \tau_{ij}$ for all $i \neq j \in \mathcal{I}$ and $k \in \mathbb{N}$. Then, (6.1) boils down to

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j=1, j\neq i}^{n} w_{ij}x_j(k-\tau_{ij}). \tag{6.13}$$

Similarly, for the primary and secondary layer subgraphs, the delayed consensus protocol (6.13) becomes

$$x_p(k+1) = \sum_{i=0}^{\tau_{max}} W_{p,i}x_p(k-i) \tag{6.14}$$

and

$$x_s(k+1) = \sum_{i=0}^{\tau_{max}} W_{s,i} x_s(k-i) + \sum_{i=0}^{\tau_{max}} B_i x_p(k-i) \tag{6.15}$$

where $W_{p,i}$, $W_{s,i}$ and B_i are the matrices that consist of the weighting coefficients related to the amount of time-delay i and $\sum_{i=0}^{\tau_{max}} W_{p,i} = W_p$, $\sum_{i=0}^{\tau_{max}} W_{s,i} = W_s$ and $\sum_{i=0}^{\tau_{max}} B_i = B$.

The consensus protocols (6.14) and (6.15) can be represented in the matrix form as

$$\begin{bmatrix} \hat{x}_p(k+1) \\ \hat{x}_s(k+1) \end{bmatrix} = \begin{bmatrix} \hat{W}_p & 0 \\ \hat{B} & \hat{W}_s \end{bmatrix} \begin{bmatrix} \hat{x}_p(k) \\ \hat{x}_s(k) \end{bmatrix} \tag{6.16}$$

where

$$\hat{W}_p = \begin{bmatrix} W_{p,0} & W_{p,1} & \cdots & W_{p,\tau_{max}} \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & I & 0 \end{bmatrix}_{(\tau_{max}+1)\bar{n}_p \times (\tau_{max}+1)\bar{n}_p} \tag{6.17}$$

and

$$\hat{W}_s = \begin{bmatrix} W_{s,0} & W_{s,1} & \cdots & W_{s,\tau_{max}} \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & I & 0 \end{bmatrix}_{(\tau_{max}+1)\bar{n}_s \times (\tau_{max}+1)\bar{n}_s} \tag{6.18}$$

are the augmented matrices for the primary and secondary layer subgraphs, respectively, and

$$\hat{B} = \begin{bmatrix} B_0 & \cdots & B_{\tau_{max}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{(\tau_{max}+1)\bar{n}_s \times (\tau_{max}+1)\bar{n}_p} \tag{6.19}$$

is the augmented input matrix for the secondary layer subgraphs.

Another fundamental contribution of this chapter states that if there exist nonuniform time-delays on the information exchange in a network, the number of consensus equilibria is equal to the sum of primary and secondary layer subgraphs.

Theorem 6.2. Consider a network of agents with underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which has l_p primary and l_s secondary layer subgraphs. When the agents utilize protocol (6.13), under Assumption 6.1, the number of consensus equilibria is given by

$$K = l_p + l_s. \tag{6.20}$$

Proof. By multiplying (6.16) from left by T given in (6.11) and from right by its inverse, we obtain an equation in the same form as (6.12). Since the system matrix of (6.12) satisfies the conditions of Theorem 5.1, the number of equilibria consensus states for a network with nonuniform fixed delay is equal to $K = l_p + l_s$. $\square$

Remark 6.2. From Definitions 5.2 and 5.3, the vertices in a primary (or equivalently secondary) layer subgraph of $\mathcal{G}$ together with their delayed versions form a primary (or equivalently secondary) layer subgraph in the delayed network graph. This is the underlying reason why bounded time-delay does not affect the number of consensus equilibria.

6.4. Effect of Delay on the Convergence Rate

As pointed out in Section 5.4, convergence rate is an important performance measure for distributed consensus protocols. In Sections 6.2 and 6.3, it is proved that delay does not cause instability in the multi-equilibria consensus problem under Assumption 6.1. In the rest of this section, we investigate the effect of delay on the rate of convergence in networks with multi-equilibria consensus. In particular, we will consider three cases: (i) delayed system converges slower, (ii) delayed system converges faster, (iii) delay does not have a detrimental effect on the rate of convergence. In following example, we examine case (i).

Example 6.1. (Delayed system converges slower) Consider the multi-equilibria consensus problem for the a network of 4 vertices forming 2 primary and 1 secondary layer subgraphs, where the system matrix is chosen as

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

In the absence of delay, the largest eigenvalue of W that is less than 1 is calculated as $\lambda_2(W) = 1/2$. However, the one-step uniform delayed network converges slower since we have $\lambda_2(\hat{W}) = \frac{1}{8}(1 + \sqrt{17}) > 1/2$.

In order to demonstrate case (ii), we study the following example.

Example 6.2. (Delayed system converges faster) Consider the multi-equilibria consensus problem for the a network of 4 vertices forming 2 primary and 1 secondary layer subgraphs, where the system matrix is chosen as

$$W = \begin{bmatrix} 19/20 & 1/20 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 7/15 & 0 & 1/3 & 1/5 \end{bmatrix}.$$

In the absence of delay, the largest eigenvalue of W that is less than 1 is calculated as $\lambda_2(W) = 9/20$. However, the one-step uniform delayed network converges faster since we have $\lambda_2(\hat{W}) \approx 0.3724 < 9/20$.

The above examples illustrate that delay may have different effects on the convergence rate for different choices of weighting coefficients. In order to investigate case (iii), we present the following lemma which is a crucial ingredient in the proof of Theorem 6.3.

Lemma 6.1. [8] Let the system matrix $W(k)$ be a lower triangular matrix which satisfies Assumption 2.1. Then the delayed and nominal system matrices have the same non-zero spectra at each time step under bounded non-uniform varying delay.

Theorem 6.3. Let the network graph $\mathcal{G}$ be composed of l_p primary and l_s secondary layer subgraphs. Let $\tilde{\mathcal{G}}$ be the graph where a vertex in $\tilde{\mathcal{G}}$ corresponds to a subgraph in $\mathcal{G}$, and the edges of $\tilde{\mathcal{G}}$ correspond to the information flow between the subgraphs of $\mathcal{G}$. Regardless of the choice of weighting coefficients, delay is not detrimental on the convergence speed of the multi-equilibria consensus protocol (6.13) if

(i) all primary and secondary layer subgraphs in $\mathcal{G}$ are acyclic[1] graphs, and
(ii) the graph $\tilde{\mathcal{G}}$ is an acyclic graph.

Proof. When each subgraph in $\mathcal{G}$ and the graph $\tilde{\mathcal{G}}$ are acyclic, the system matrix $W(k)$ can be transformed into a lower (or equivalently upper) triangular matrix by using

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[1] A graph is called acyclic if its adjacency matrix is permutation-similar to lower (or equivalently upper) triangular matrix).

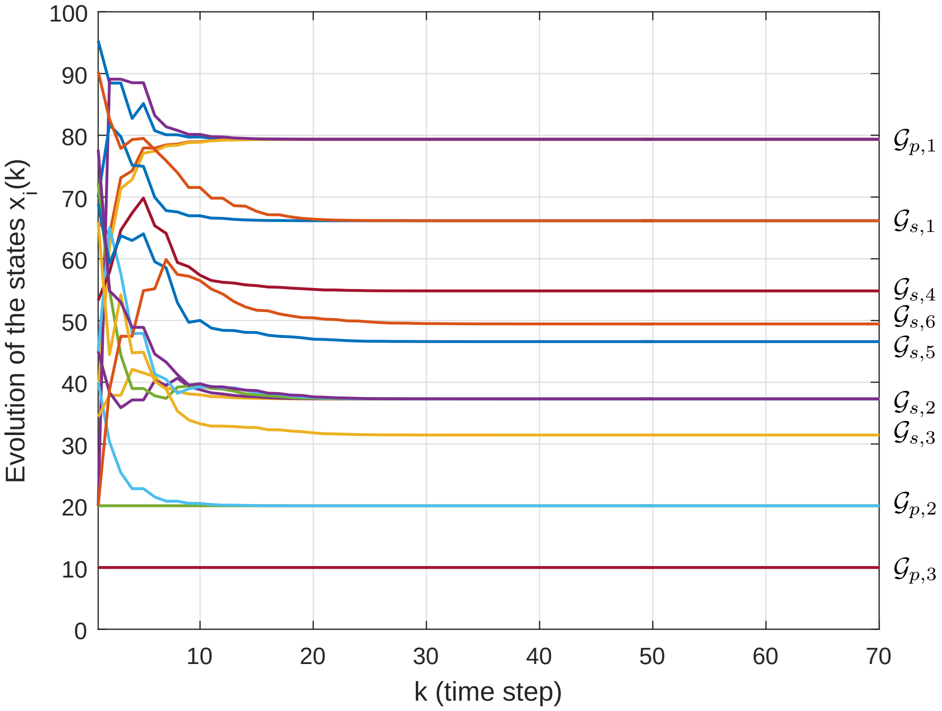


Figure 6.1. The simulation results for the delayed network given in Figure 5.4.

a similarity transformation matrix T. From Lemma 6.1, we conclude that at each iteration, $W(k)$ and its delayed version have the same non-zero spectra and hence delay is not detrimental on the rate of convergence of the multi-equilibria consensus. $\square$

6.5. Numerical Examples

In this section, we demonstrate the effectiveness of our results with a set of simulations. These simulations cover both examples of directed networks which has uniform and nonuniform bounded communication delays. Throughout this section, the weighting coefficients among the agents in the networks are assumed to satisfy Assumption 6.1.

Example 6.3. (Illustration of Theorem 6.1) In order to investigate the effect of delay on the number of consensus equilibria, reconsider the network given in Figure 5.4. Recall from Example 5.4 that the network reaches $K = l_p + l_s = 9$ consensus equilibria states in the absence of time-delays. From Theorem 6.1, we expect the delayed network to converge to the same number of consensus equilibria as the non-delayed network.

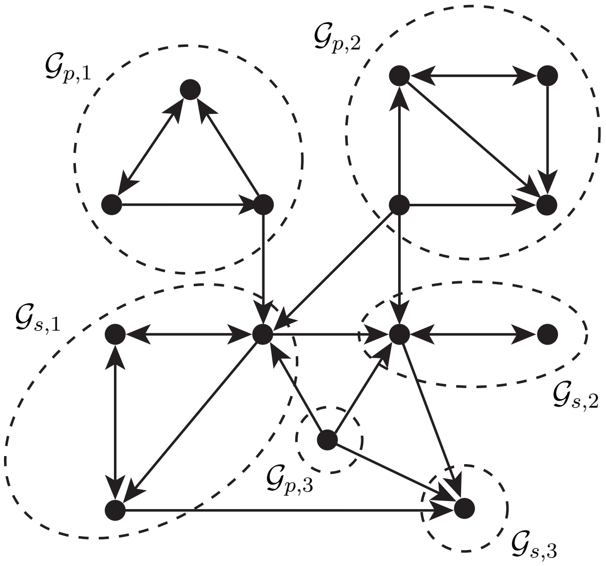


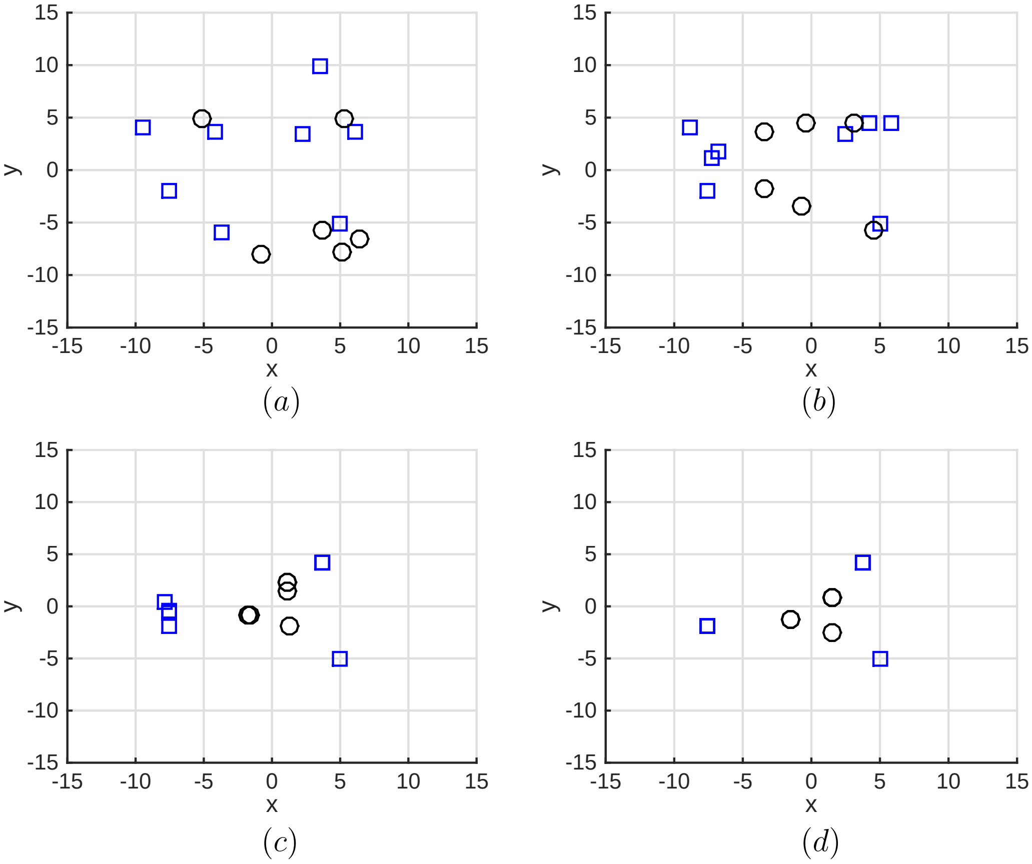
Figure 6.2. A directed graph with 14 vertices and 26 edges.

Figure 6.1 shows the state evolution of the agents for the network with time-delay $\tau_{ij} \leq \tau_{max} = 5$ and for an arbitrary choice of initial conditions and weighting coefficients. Note that although the final state values are different for the non-delayed and delayed networks, the number of consensus equilibria is still $K = l_p + l_s = 9$.

Example 6.4. (Illustration of Theorem 6.2) Consider a rendezvous problem in a network of 14 agents and 26 directed communication links as depicted in Figure 6.2. Let the agents be randomly placed inside the region $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$ of the 2D coordinate system. For a random choice of weighting coefficients w_{ij} and a random choice of bounded delay in the network $\tau_{ij} \leq \tau_{max} = 5$, let the agents update their positions with the protocols $x(k+1) = Wx(k)$ and $y(k+1) = Wy(k)$, i.e., they run the protocol in two dimensions. The positions of the agents are depicted for 4 different time steps in Figure 6.3. Since the network graph has $l_p = 3$ primary and $l_s = 3$ secondary layer subgraphs, the agents form 6 groups in the coordinate system. Note also that the final positions of the agents in secondary layer subgraphs are inside the convex hull of the final positions of the agents in primary layer subgraphs.

6.6. Chapter Summary & Concluding Remarks

In this chapter, we have analyzed the number of equilibria for a given network in the presence of communication delays, relying on the concepts of primary and sec-



(a) (b) (c) (d)

Figure 6.3. The positions of the primary layer agents (marked as squares) and secondary layer agents (marked as circles) in 2D coordinate system $(x_i(k), y_i(k))$ for (a) $k = 0$, (b) $k = 3$, (c) $k = 20$ and (d) $k = 80$.

ondary layer subgraphs given in Chapter 5. The main contribution of this chapter is to investigate convergence properties not only for the networks with uniform communication delay but also for the networks with nonuniform communication delay. Under the assumption that all delays are bounded, our first result shows that a network with uniform fixed delay achieves K equilibria consensus where K is the number of equilibria for the network in the absence of delay. We further investigate the convergence properties of multi-equilibria consensus problem with nonuniform communication delays. Both of the results show that bounded time-delay affects neither the number of consensus equilibria nor the convergence properties of the network. Our proof relies on the properties of nonnegative matrices and the results of system theory.

Furthermore, we provide insight on how delay affects the convergence rate of the consensus protocol. We also show that, delay does not adversely affect the convergence speed of the consensus protocol provided that the graphs considered are acyclic. Finally, we provide two numerical examples to illustrate the theoretical results. In the first example, we compare the number of consensus equilibria of a given network under non-delayed and delayed information exchange. In the second example, we consider the rendezvous problem of a delayed network of mobile agents equipped with sensors and communicating with others, where the ultimate goal is to arrive at a common position in 2D space. Both of the simulations show that the number of final states of the agents and the number of equilibria derived by the proposed protocol are equal which verifies the theoretical results of the chapter.

7. CONCLUSION AND OPEN PROBLEMS

Inspired by the variety of applications in networked controlled systems, this thesis has studied the multi-equilibria consensus problem of multi-agent networks with first order dynamics. The objective of this thesis is to provide a unified convergence analysis for networks of agents which utilize linear consensus protocols both in continuous-time and discrete-time. In particular, we have investigated how the number of consensus equilibria depends on the structural properties of the network. We have considered three cases in terms of communication structure of the network: (i) time-independent communication topologies with bidirectional links; (ii) time-dependent communication topologies with bidirectional links; (iii) time-independent communication topologies with unidirectional links and bounded delays.

The first part of the thesis is concerned with establishing convergence conditions for networks with bidirectional information exchange. The proposed condition for fixed topology networks relies on the results of classical spectral graph theory. In this context, we have established the result that connected components of the underlying graph determine the number of equilibria in the network. Although the result on fixed topology networks is predictable, it is not directly applicable to the networks where the communication topology varies over time. Two conditions which ensure convergence to $K, K \geq 2$ equilibria have been established for networks under dynamically changing topologies with bidirectional information flow. The first condition is stated in terms of the connectivity of the associated graph, which requires the graph to be K connected across specific time intervals. The second condition asserts that if the corresponding integral/sum graph is K connected, then the network achieves K consensus equilibria. In this case, the communication topology is assumed to be fully symmetric, that is, the weights of the communication between agents are exactly the same. With integral/sum graph equivalent of the original time-varying graph, the multi-equilibria consensus problem is converted into the convergence analysis of a static graph. This latter condition is much easier to check in the sense that it does not require finding specific intervals that satisfy connectivity condition of the original graph.

In the second part of the thesis, we focus on the multi-equilibria problem of a network of n agents where the interactions are static but unidirectional. A novel method has been developed to analyze the convergence properties such networks that rely on two new notions of primary and secondary layer subgraphs. These notions have been used subsequently in determining the number of equilibria for a given network with unidirectional information flow. In order to detect these subgraphs for a given graph, we have proposed two algorithms. Furthermore, we provide a polynomial-time bound on the convergence time of these algorithms. The striking result in this case is that the number of consensus equilibria is equal to the total number of primary and secondary layer subgraphs of the network. We then have extended these findings to networks with time-delays. When the influence of time-delays is taken into account, our analysis has revealed that the number of consensus equilibria of the network remains the same under the assumption that time-delays are bounded. We have also provided insight on how the delay affects the convergence rate of the consensus protocol. It is shown that delay does not adversely affect the convergence speed of the consensus protocol provided that the graphs considered are acyclic.

While this thesis provides novel theoretical contributions, many interesting and important problems are yet to be studied. The work presented in this thesis concentrates on directed graphs under fixed topologies. However, in most of the real-world applications the interactions among the agents in the network are represented by a dynamic graph. Therefore, the results on networks with static communication can be generalized to the case of time-dependent communication. An alternative topic could be analysis of multi-agent networks governed by second or higher order dynamics in which the analysis becomes more challenging. Another interesting problem is to analyze the behavior of the network in the presence of varying time-delays. We hope that the open problems presented herein stimulate further research on multi-equilibria consensus, both on the theoretical side as well as in the practical applications.

REFERENCES

1. Vicsek, T., A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, “Novel type of phase transition in a system of self-driven particles”, *Physical Review Letters*, Vol. 75, No. 6, p. 1226, 1995.

2. Jadbabaie, A., J. Lin and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules”, *IEEE Transactions on Automatic Control*, Vol. 48, No. 6, pp. 988–1001, June 2003.

3. Ren, W. and R. W. Beard, “Consensus seeking in multiagent systems under dynamically changing interaction topologies”, *IEEE Transactions on Automatic Control*, Vol. 50, No. 5, pp. 655–661, May 2005.

4. Moreau, L., “Stability of multiagent systems with time-dependent communication links”, *IEEE Transactions on Automatic Control*, Vol. 50, No. 2, pp. 169–182, Feb 2005.

5. Olfati-Saber, R. and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays”, *IEEE Transactions on Automatic Control*, Vol. 49, No. 9, pp. 1520–1533, 2004.

6. Akar, M. and R. Shorten, “Distributed probabilistic synchronization algorithms for communication networks”, *IEEE Transactions on Automatic Control*, Vol. 53, No. 1, pp. 389–393, 2008.

7. Lin, Z., B. Francis and M. Maggiore, “Necessary and sufficient graphical conditions for formation control of unicycles”, *IEEE Transactions on Automatic Control*, Vol. 50, No. 1, pp. 121–127, 2005.

8. Cihan, O. and M. Akar, “Effect of nonuniform varying delay on the rate of convergence in averaging-based consensus”, *Turkish Journal of Electrical Engineering &*

Computer Sciences, Vol. 23, No. 4, pp. 1069–1080, 2015.

9. Cihan, O. and M. Akar, “Fastest mixing reversible Markov chains on graphs with degree proportional stationary distributions”, *IEEE Transactions on Automatic Control*, Vol. 60, No. 1, pp. 227–232, 2015.

10. Wang, A. and Y. Zhao, “Event-triggered consensus control for leader-following multi-agent systems with time-varying delays”, *Journal of the Franklin Institute*, Vol. 353, No. 17, pp. 4754–4771, 2016.

11. Zhang, D., Z. Xu, D. Srinivasan and L. Yu, “Leader-Follower Consensus of Multi-agent Systems With Energy Constraints: A Markovian System Approach”, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, Vol. 47, No. 7, pp. 1727–1736, 2017.

12. DeGroot, M. H., “Reaching a consensus”, *Journal of the American Statistical Association*, Vol. 69, No. 345, pp. 118–121, 1974.

13. Zhang, D., P. Shi, W. A. Zhang and L. Yu, “Energy-Efficient Distributed Filtering in Sensor Networks: A Unified Switched System Approach”, *IEEE Transactions on Cybernetics*, Vol. 47, No. 7, pp. 1618–1629, 2017.

14. McLain, T. W. and R. W. Beard, “Trajectory planning for coordinated rendezvous of unmanned air vehicles”, *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Vol. 4369, pp. 1–8, AIAA Reston, VA, 2000.

15. Dörfler, F., M. Chertkov and F. Bullo, “Synchronization in complex oscillator networks and smart grids”, *Proceedings of the National Academy of Sciences*, Vol. 110, No. 6, pp. 2005–2010, 2013.

16. Amelina, N., A. Fradkov, Y. Jiang and D. J. Vergados, “Approximate consensus in stochastic networks with application to load balancing”, *IEEE Transactions on Information Theory*, Vol. 61, No. 4, pp. 1739–1752, 2015.

17. di Bernardo, M., A. Salvi and S. Santini, “Distributed consensus strategy for platooning of vehicles in the presence of time-varying heterogeneous communication delays”, *IEEE Transactions on Intelligent Transportation Systems*, Vol. 16, No. 1, pp. 102–112, 2015.

18. Zhang, D., Q. G. Wang, D. Srinivasan, H. Li and L. Yu, “Asynchronous State Estimation for Discrete-Time Switched Complex Networks With Communication Constraints”, *IEEE Transactions on Neural Networks and Learning Systems*, Vol. PP, No. 99, pp. 1–15, 2017.

19. Chatterjee, S. and E. Seneta, “Towards consensus: Some convergence theorems on repeated averaging”, *Journal of Applied Probability*, Vol. 14, No. 1, pp. 89–97, 1977.

20. Wolfowitz, J., “Products of indecomposable, aperiodic, stochastic matrices”, *Proceedings of the American Mathematical Society*, Vol. 14, No. 5, pp. 733–737, 1963.

21. Tsitsiklis, J., D. Bertsekas and M. Athans, “Distributed asynchronous deterministic and stochastic gradient optimization algorithms”, *IEEE Transactions on Automatic Control*, Vol. 31, No. 9, pp. 803–812, 1986.

22. Blondel, V. D., J. M. Hendrickx, A. Olshevsky and J. N. Tsitsiklis, “Convergence in multiagent coordination, consensus, and flocking”, *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pp. 2996–3000, 2005.

23. Blondel, V. D., J. M. Hendrickx and J. N. Tsitsiklis, “On Krause’s multi-agent consensus model with state-dependent connectivity”, *IEEE Transactions on Automatic Control*, Vol. 54, No. 11, pp. 2586–2597, 2009.

24. Lorenz, J., “A stabilization theorem for dynamics of continuous opinions”, *Physica A: Statistical Mechanics and its Applications*, Vol. 355, No. 1, pp. 217–223, 2005.

25. Li, L., A. Scaglione, A. Swami and Q. Zhao, “Consensus, polarization and clustering of opinions in social networks”, *IEEE Journal on Selected Areas in Communications*, Vol. 31, No. 6, pp. 1072–1083, 2013.

26. Han, Y., W. Lu and T. Chen, “Cluster consensus in discrete-time networks of multiagents with inter-cluster nonidentical inputs”, *IEEE Transactions on Neural Networks and Learning Systems*, Vol. 24, No. 4, pp. 566–578, 2013.

27. Zheng, M., Y. Kashimori, O. Hoshino, K. Fujita and T. Kambara, “Behavior pattern (innate action) of individuals in fish schools generating efficient collective evasion from predation”, *Journal of Theoretical Biology*, Vol. 235, No. 2, pp. 153–167, 2005.

28. Krieger, M. J., J.-B. Billeter and L. Keller, “Ant-like task allocation and recruitment in cooperative robots”, *Nature*, Vol. 406, No. 6799, p. 992, 2000.

29. Yu, J. and L. Wang, “Group consensus of multi-agent systems with directed information exchange”, *International Journal of Systems Science*, Vol. 43, No. 2, pp. 334–348, 2012.

30. Chen, Y., J. Lü, F. Han and X. Yu, “On the cluster consensus of discrete-time multi-agent systems”, *Systems & Control Letters*, Vol. 60, No. 7, pp. 517–523, 2011.

31. Xia, W. and M. Cao, “Clustering in diffusively coupled networks”, *Automatica*, Vol. 47, No. 11, pp. 2395–2405, 2011.

32. Shang, Y., “Group consensus of multi-agent systems in directed networks with noises and time delays”, *International Journal of Systems Science*, Vol. 46, No. 14, pp. 2481–2492, 2015.

33. Qin, J. and C. Yu, “Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition”, *Automatica*, Vol. 49, No. 9, pp.

2898–2905, 2013.

34. Hou, B., F. Sun, H. Li, Y. Chen and G. Liu, “Observer-based cluster consensus control of high-order multi-agent systems”, *Neurocomputing*, Vol. 168, pp. 979–982, 2015.

35. Hu, H., L. Yu, W.-A. Zhang and H. Song, “Group consensus in multi-agent systems with hybrid protocol”, *Journal of the Franklin Institute*, Vol. 350, No. 3, pp. 575–597, 2013.

36. Qin, J., C. Yu and B. D. Anderson, “On leaderless and leader-following consensus for interacting clusters of second-order multi-agent systems”, *Automatica*, Vol. 74, pp. 214–221, 2016.

37. Han, G.-S., Z.-H. Guan, J. Li, R.-Q. Liao and X.-M. Cheng, “Multi-consensus of multi-agent networks via a rectangular impulsive approach”, *Systems & Control Letters*, Vol. 76, pp. 28–34, 2015.

38. Tan, C., G. Liu and G. Duan, “Group consensus of networked multi-agent systems with directed topology”, *Preprints of the 18th IFAC World Congress*, Vol. 44, pp. 8878–8883, 2011.

39. Angeli, D. and P.-A. Bliman, “Extension of a result by Moreau on stability of leaderless multi-agent systems.”, *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pp. 759–764, 2005.

40. Xiao, F. and L. Wang, “State consensus for multi-agent systems with switching topologies and time-varying delays”, *International Journal of Control*, Vol. 79, No. 10, pp. 1277–1284, 2006.

41. Lin, P., K. Qin, H. Zhao and M. Sun, “A new approach to average consensus problems with multiple time-delays and jointly-connected topologies”, *Journal of the Franklin Institute*, Vol. 349, No. 1, pp. 293–304, 2012.

42. Sun, Y., “Average consensus in networks of dynamic agents with uncertain topologies and time-varying delays”, *Journal of the Franklin Institute*, Vol. 349, No. 3, pp. 1061–1073, 2012.

43. Yu, J. and L. Wang, “Group consensus in multi-agent systems with switching topologies and communication delays”, *Systems & Control Letters*, Vol. 59, No. 6, pp. 340–348, 2010.

44. Xia, H., T.-Z. Huang, J.-L. Shao and J.-Y. Yu, “Group consensus of multi-agent systems with communication delays”, *Neurocomputing*, Vol. 171, pp. 1666–1673, 2016.

45. Erkan, Ö. F. and M. Akar, “Yönsüz Çizgelerde Tekli ve Çoklu Denge Noktası Üzerine Sonuçlar”, *Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı*, pp. 498–503, 2014.

46. Erkan, Ö. F. and M. Akar, “Çok Etmenli Sistemlerde Çoklu Denge Noktalarının Sürekli Zamanda Analizi”, *Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı*, pp. 457–461, 2015.

47. Erkan, Ö. F. and M. Akar, “Çoklu Denge Noktalı Onaylaşımda Bağlılık Koşulları”, *Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı*, pp. 496–499, 2016.

48. Erkan, Ö. and M. Akar, “Çok etmenli sistemlerde çoklu denge noktalarının sürekli zamanda analizi”, *Pamukkale University Journal of Engineering Sciences*, Vol. 22, No. 8, pp. 666–670, 2016.

49. Erkan, Ö. and M. Akar, “An Integral Connectivity Condition for Multi-Equilibria Consensus in Networks Evolving over Undirected Graphs”, *Anadolu University Journal of Science and Technology (AUJST-A)*, Vol. 18, No. 4, pp. 842–848, 2017.

50. Erkan, Ö. and M. Akar, “Cluster Consensus in Multi-Agent Networks with Mutual Information Exchange”, *to appear in AI&Society*, 2017.

51. Erkan, Ö. F., O. Cihan and M. Akar, "Distributed Consensus with Multi-Equilibria in Directed Networks", *Proceedings of the 2017 American Control Conference (ACC)*, pp. 4681–4685, 2017.

52. Erkan, Ö. F., O. Cihan and M. Akar, "Analysis of Distributed Consensus Protocols with Multi-Equilibria under Time-Delays", *to appear in Journal of the Franklin Institute*, 2017.

53. Horn, R. A. and C. R. Johnson, *Matrix analysis*, Cambridge University Press, 1987.

54. Olfati-Saber, R., J. A. Fax and R. M. Murray, "Consensus and cooperation in networked multi-agent systems", *Proceedings of the IEEE*, Vol. 95, No. 1, pp. 215–233, 2007.

55. Cao, L., Y. Zheng and Q. Zhou, "A necessary and sufficient condition for consensus of continuous-time agents over undirected time-varying networks", *IEEE Transactions on Automatic Control*, Vol. 56, No. 8, pp. 1915–1920, 2011.

56. Olshevsky, A. and J. N. Tsitsiklis, "Convergence Speed in Distributed Consensus and Averaging", *SIAM Review*, Vol. 53, No. 4, pp. 747–772, 2011.

57. Nedic, A., A. Ozdaglar and P. A. Parrilo, "Constrained Consensus and Optimization in Multi-Agent Networks", *IEEE Transactions on Automatic Control*, Vol. 55, No. 4, pp. 922–938, 2010.

58. Freedman, D., *Markov chains*, Springer Science & Business Media, 2012.

59. Bryan, K. and T. Leise, "The $25,000,000,000 eigenvector: The linear algebra behind Google", *SIAM Review*, Vol. 48, No. 3, pp. 569–581, 2006.

60. Erkan, Ö. and M. Akar, "Distributed Consensus with Multiple Equilibria in Continuous-Time Multi-Agent Networks under Undirected Topologies", *Proceed-*

ings of the 16th IFAC Conference on TECIS, Vol. 48, pp. 225–230, 2015.

61. Erkan, Ö. and M. Akar, “Discrete-Time Analysis of Multi-Agent Networks with Multiple Consensus Equilibria”, *Proceedings of 14th IFAC Symposium on Control in Transportation Systems*, Vol. 49, p. 389–394, 2016.

62. Nuutila, E., “Efficient Transitive Closure Computation in Large Digraphs”, *Acta Polytechnica Scandinavia: Mathematics and Computing in Engineering*, Vol. 74, pp. 1–124, Jul. 1995.

63. Bang-Jensen, J. and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer London, 2013.

64. Williams, J. W. J., “Algorithm 232: heapsort”, *Communications of the ACM*, Vol. 7, No. 6, p. 347–348, 1964.

65. Ding, J. and N. H. Rhee, “On the equality of algebraic and geometric multiplicities of matrix eigenvalues”, *Applied Mathematics Letters*, Vol. 24, No. 12, pp. 2211 – 2215, 2011.

66. Olshevsky, A. and J. N. Tsitsiklis, “Convergence speed in distributed consensus and averaging”, *SIAM Journal on Control and Optimization*, Vol. 48, No. 1, pp. 33–55, 2009.

67. Zhang, D., P. Shi, Q.-G. Wang and L. Yu, “Analysis and synthesis of networked control systems: A survey of recent advances and challenges”, *ISA Transactions*, Vol. 66, pp. 376–392, 2017.