Thesis

so for the degree of Ph. D.

Applications of Quaternions to Field Equations.

By Feza Gürsey.

from the Imperial College of Science and Technology

Mathematics Dept.

4, Lonsdale Sq.

N. 1

NORTH 3379.

# ACKNOWLEDGEMENTS.

The author would like to express his sincere gratitude to Prof. H. Jones for continuous advice and encouragement throughout the course of this work.

He wishes to thank Dr. P. Weiss and Dr. C. W. Kilmister for stimulating discussions and for bringing to his notice several papers on quaternions. His thanks are also due to his friends Ann Gould and Cavid Erginsoy for their invaluable assistance in the preparation of the final copies.

Lastly he is indebted to the Turkish Ministry of Education for the award of a scholarship.

# CONTENTS

# ABSTRACT


The object of the thesis is to show that the quaternion algebra can be applied successfully to the discussion and solution of Dirac's equation and that it can be extended to general relativity to give an invariant formulation of field equations.

In the first three chapters which deal with the quaternion formulation of the wave equation, a connection is established with the matrix formalism of Dirac on one hand and with different quaternion formulations on the other. Attention is drawn to two space-like vectors, which form an orthogonal frame of reference together with the current and spin density four-vectors, and new divergence equations and algebraic identities satisfied by tensorial densities involving the charge conjugate function are derived. A generalization of the wave equation by means of a space like four-vector is given.

In the two chapters that follow a new method for solving the wave equation is developed and exact solutions are obtained in several problems in a rapid and concise manner. The formalism is also shown to be useful in the non-relativistic approximation.

In the remaining chapters wave equations for particles of spin $\frac{1}{2}$ and 1 are put in a quaternion form invariant for general coordinate transformations and valid in the affine space-time of distant parallelism. By means of the new concept of covariant quaternions these invariant equations may be also expressed in covariant form. Application to Dirac's equation leads to new tensor forms for this equation and throws some light on its connection with general relativity. A new form for Dirac's matrix equation in a general coordinate system is found which only involves the torsion tensor associated with the basis vectors that have been chosen.

# APPLICATIONS OF QUATERNIONS TO FIELD EQUATIONS

## General Introduction and Summary

The advantages of the quaternion calculus over the tensor calculus in dealing with special relativity and Maxwell's electrodynamics have first been recognized by Conway (1911), Silberstein (1912, 1913), Klein (1927) and others. The chief reasons which bring about a mathematical simplification are the avoidance of tensor indices and the fact that quaternions form an algebra. It is true that complex quaternions which possess null-divisors have to be used in this connection, but even this mathematical defect turns out to be physically significant as it is associated with the existence of null-vectors in special relativity. An elegant application of the use of these null-vectors in classical electrodynamics has been given by Weiss (1941).

After the relativistic wave equation was discovered by Dirac in 1928, its apparent lack of symmetry and its non-tensorial transformation properties led to the extension of tensor calculus known as the spinor calculus with its heavy apparel of dotted and undotted indices. It was then natural to see whether the quaternion algebra which had proved such an excellent tool in dealing with relativistic invariance would provide a simpler formulation of the wave equation. Unfortunately the answer given to this question by Lanczos (1929) was not very encouraging. The wave equation could be expressed in a very symmetrical quaternion form only by doubling the number of the components of the wave function. Another attempt was made in 1937 by Conway, who, on noting the equivalence between the algebra of linear quaternion functions and the Dirac-Eddington algebra succeeded in translating Dirac's equation into quaternion language. The difference with Lanczos was that in Conway's method the wave function was represented by a single complex quaternion instead of two, so that a correspondence between Conway's quaternion and Dirac's column matrix became possible. But the apparent lack of symmetry in Dirac's formalism was even worse in Conway's case to the extent that the relativistic invariance of the equation was not obvious at all and no simplified methods of solution were suggested by the quaternion formalism. On the other hand Blaton (1935), Scherrer (1935) and Sommerfeld (1936) had considered the connection between spinors and quaternions, and found that quaternions corresponding to Dirac's ~~wa~~ spinor wave function (Blaton called such quaternions binions) obeyed a very simple law under Lorentz transformation. And yet, in the only reasonable quaternion equivalent of Dirac's equation, namely in Conway's equation, the wave function did not obey this simple transformation law. This state of affairs was remedied by Watson (1947) who devised a quaternion equation more symmetrical in form than Conway's, in which the wave function transformed like a binion. But Watson did not apply his formalism to the solution of Dirac's equation and the connection of all these different quaternion formulations and the matrix formulation of quantum mechanics was not very clear.

Our aim in this investigation is to show that the quaternion algebra is an ideal mathematical tool for discussing and solving Dirac's equation, leading to many new results and elegant methods of solution. Moreover its applications are shown not to be limited to special relativity as it is generally believed but to be capable of extension to general relativity where it achieves a unified treatment of relativistic wave equations for particles of spin $\frac{1}{2}$ and 1. Hence the present work may be regarded as consisting of three parts:

The first part covered by the first three chapters is devoted to the discussion of Dirac's equation in quaternion form. In order to translate all the results obtained into the usual formalism of quantum mechanics a new matrix form for complex quaternions is given and the connection between the two methods is worked out in detail. In Chapter II Dirac's equation is expressed in quaternion form and the connection with other quaternion treatments is found. Two-component wave equations are shown to be degenerate cases of the general equation and applicable to neutral particles only. The very simple transformation law of the wave function leads to the introduction of an orthogonal frame of reference composed of a time-like vector and three space-like vectors. The time-like vector is shown to represent the current density 4-vector and one of the space-like 4-vectors corresponds to the spin density 4-vector whereas the other two are associated with transitions between positive and negative energy levels and are identical with the real and imaginary parts of a complex vector first introduced by de Broglie (1940). Chapter III deals with the properties of these 4-vectors and other tensorial quantities. A new method for finding the identities they satisfy leads to new relations for the tensors constructed by means of the charge conjugate function. Finally, to remove the last cause of lack of symmetry in Watson's equation a formal generalization of it is suggested. Unlike the original equation it leads to a symmetrical vector form of Dirac's equation involving a constant space-like vector. When the length of this vector does not vanish the new equation can always be transformed back into the standard Dirac equation. But when the constant space-like vector is a null-vector we obtain a new equation for a particle of spin $\frac{1}{2}$ and rest mass 0. It is shown how its plane wave solutions may be used to find plane wave solutions of Maxwell's equations in vacuum.

Chapters IV and V form the second part which deals with the quaternion solutions of Dirac's equation. The quaternion formalism is seen to be useful in the non-relativistic approximation since the real and imaginary parts of the wave quaternion correspond to the large and small components in Darwin's (1928) method. The essence of our method for solving Dirac's equation exactly may be traced to Sauter's (1931) suggestion to take the wave function not as a column matrix but a square matrix which is a hypercomplex number in the Dirac - Eddington algebra. Hence Sauter gave solutions of the wave equation in simple cases without specifying the representation of the Dirac operator. In this investigation we show that the quaternion formalism is equivalent to the assumption that the wave function depends on three of Dirac's operators only and the redundant variables in Sauter's method are automatically eliminated. In the case the potential depends on one coordinate only the quaternion method suggests a new canonical transformation which reduces the problem to the solution of a single complex linear equation. Taub's recent treatment of an electron in the external field of a plane wave is further simplified and a compact solution obtained. The application to the angular momentum operators proves to be fruitful. It is found that the generalized relativistic angular momentum operator satisfies a very simple commutation relation which has the same structure as the relation satisfied by the ordinary angular momentum operator. The rather complicated commutation relations satisfied by its different components were worked out by Born & Fuchs (1939) and are here combined in a single formula. This greatly simplifies the task of finding its eigenfunctions and the results are applied to the solution of Dirac's equation in angular coordinates and to the case an invariant scalar Coulomb field is present.

The third part is devoted to the quaternion formulation of field equations in general relativity. The extension of Dirac's equation to general relativity was first discussed by Tetrode (1928). Other generalizations were given by Weyl (1929), Fock (1929), Taub (1937), Frenkel (1934) and others. The most important and useful of these generalizations is due to Schrödinger (1932) whose results were carried further by Bargmann (1932) and very recently by Symonds (1950). The connection with the ~~In Chapter~~ affine space of distant parallelism was discussed by Flint (1928), Zaycoff (1929), Podolsky (1931), Haskey (1940) and more recently by Kilmister (1949, 1950) who first, and independently from the author connected the quaternion algebra with an affine space. KILMISTER'S Kilmister's use of quaternions in this connection ~~is~~, however, is different from ours, and his treatment suggests that unlike other relativistic equations Dirac's equation does not depend on the metric tensor. In our opinion the advantage of the quaternion method is due to the fact that it leads to a unified of Dirac's equation and other wave equations.

In order to achieve this unified treatment, the basis vectors which characterize an ordinary coordinate transformation or an affine space-time of distant parallelism are expressed as covariant hermitian quaternions by means of which other tensorial quaternions of the second, third and fourth rank can be constructed. Combination of these with ordinary tensors gives invariant quaternions. The procedure is very similar to the method of constructing invariant tensors from basis vectors and ordinary tensors a full account of which may be found in Brand's Vector and Tensor Analysis (1947). The chief difference is that our basis vectors are quaternions so that our invariant quaternions correspond to the contracted forms of Brand's invariant tensors. It is then shown that equations for particles of spin $\frac{1}{2}$ and 1 can be written in terms of invariant quaternions only. Equations for spin 1, including an equation recently discovered by Bhabha (1949) are all shown to be degenerate cases of Lanczos' equation for spin $\frac{1}{2}$. The application of the theory leads to two ~~tensor~~ distinct tensor forms for the generalized DIRAC Dirac equation given at the end of the first part. This may sound paradoxical since Dirac's wave function is known not to transform like a tensor.

The explanation is as follows: it has already been remarked by Uhlenbeck and Laporte (1931), Mott and Sneddon (1948, p.302) that there are two possible ways in which one may discuss the transformation properties of the Dirac equation. The wave function may be regarded as an invariant, then the matrix differential operator is also an invariant and the matrix coefficients of the four differential operators transform like the components of a 4-vector. Alternatively, if Dirac's matrices are regarded simply as determining the coefficient in the four wave equations equivalent to Dirac's equation and if they are kept constant it is found that the wave function is no longer an invariant but transforms like a spinor. This latter point of view is generally adopted in the literature. If, however a tensor formulation of the wave equation is needed then we must adopt the former point of view. It will be shown that in that case, although the wave function is an invariant quaternion it may be written as a linear combination of covariant scalar quantities and covariant basis vectors. Hence it may be regarded as possessing covariant components in the same way as the invariant vectorial quaternion which represents the electromagnetic field has for its six components the components of an antisymmetrical tensor. The second point of view is useful when we are considering "form invariance" under conformal transformations and the former when we deal with general coordinate transformations. Thus there is no a priori reasons against putting Dirac's equation in tensor form. Our theory shows that not only Dirac's equation ~~may be dis~~ but all relativistic field equations may be discussed from these two different points of view.

In the final chapter the Frenkel - Madelung equation is shown to have a simple geometrical interpretation. It describes an affine space in which a 4-vector and a pseudo 4-vector derived from the torsion tensor vanish identically. Equations of Dirac's type may also be derived if suitable restrictions are applied on the torsion tensor.

A new formula depending on the torsion tensor only is found for Dirac's matrix equation in general coordinate systems. This turns out to be a generalization of a formula known to hold only for orthogonal coordinates and may be regarded as an advance on the Schrödinger - Bargmann theory. Finally, conformal invariance of Dirac's equation is discussed and our conclusions compared with those of Infeld & Schild (1946) and Hill (1947).

It is the writer's hope that the present work may prove to be one more step towards unification of general relativity and Quantum Mechanics. However a severe limitation of the quaternion formalism is employed here ~~must~~ should be pointed out: all our investigation is only concerned with the one electron problem, the wave equation being regarded as a field equation. One yet needs to show that the quaternion algebra can be applied successfully to the quantization of field equations and to the formulation of a relativistic quantum electrodynamics.

~~A letter to the Phys.~~ Reprints of a letter to the Physical Review and of a paper in the Proceeding of the Cambridge Philosophical Society are given as appendices as they were written during the period in which the thesis was prepared. The letter entitled "On two-component wave equations" gives an application of quaternions to these special equations and the paper ~~on~~ the "Statistical mechanics of a Rectilinear Assembly" deals with the application of the Laplace transformation to the case of a linear assembly.

# CHAPTER I.
# Quaternion Introduction

## §1. Complex quaternions: notations and matrix representations.

By a quaternion, unless otherwise specified, we mean a complex quaternion, namely the linear form
$$Q = q_0 + e_1 q_1 + e_2 q_2 + e_3 q_3$$
where the numbers
$$q_\nu = a_\nu + i b_\nu \qquad (\nu = 0,1,2,3)$$
are complex and the operators $e_n$ $(n = 1,2,3)$ satisfy the usual relations
$$e_1^2 = e_2^2 = e_3^2 = -1 \quad , \; e_1 e_2 = e_3 \; , \; e_2 e_3 = e_1 \; \text{and} \; e_3 e_1 = e_2 . \qquad (1.1)$$
The imaginary unit $i$ is an additional operator which commutes with $e_n$ and obeys the equation $i^2 = -1$.
The quaternion conjugate of $Q$ is defined by
$$\bar{Q} = q_0 - e_1 q_1 - e_2 q_2 - e_3 q_3 \qquad (1.2)$$
and its complex conjugate by
$$Q^* = q_0^* + e_1 q_1^* + e_2 q_2^* + e_3 q_3^* . \qquad (1.3)$$
Since the operations $Q \to \bar{Q}$ and $Q \to Q^*$ commute we can define the hermitian conjugate of $Q$ by
$$Q^\dagger = \bar{Q}^* = q_0^* - e_1 q_1^* - e_2 q_2^* - e_3 q_3^* . \qquad (1.4)$$
It follows that if $Q$ and $R$ are two arbitrary quaternions and if $QR$ denotes their product calculated by means of the multiplication rules (1.1) we have
$$\overline{QR} = \bar{R}\,\bar{Q} \quad , \; (QR)^* = Q^* R^* \; \text{and} \; (QR)^\dagger = R^\dagger Q^\dagger . \qquad (1.5)$$
Our notation is different from that used by Weiss (1941) but in our opinion is justified by the notations used for the regular matrix representations of quaternions.

Any quaternion $Q$ can always be written as the sum of two special quaternions, such as
$$Q = SQ + VQ = \mathcal{R}Q + \mathcal{I}Q$$
where the scalar, vectorial, real and imaginary parts of $Q$

are given by

$$\mathcal{S}Q = \tfrac{1}{2}(Q+\bar{Q}) = a_0 + ib_0 \,, \quad \mathcal{V}Q = \tfrac{1}{2}(Q-\bar{Q}) = \Sigma e_n(a_n+ib_n) \quad (1.6)$$

$$\mathcal{R}Q = \tfrac{1}{2}(Q+Q^*) = a_0 + \Sigma e_n a_n \,, \quad \mathcal{J}Q = \tfrac{1}{2}(Q-Q^*) = ib_0 + i\Sigma e_n b_n . \quad (1.7)$$

Now we introduce two new definitions which will be found useful. A quaternion with real scalar part and imaginary vectorial part will be called "hermitian". It is identical with its hermitian conjugate. Similarly a quaternion which is equal to minus its hermitian conjugate will be called "antihermitian". It has an imaginary scalar part and a real vectorial part. Hence any quaternion $Q$ can also be written as the sum of its hermitian and antihermitian parts. Thus

$$Q = \mathcal{H}Q + \mathcal{A}Q$$

where

$$\mathcal{H}Q = \tfrac{1}{2}(Q+Q^{\dagger}) = a_0 + i\Sigma e_n b_n \,, \quad \mathcal{A}Q = \tfrac{1}{2}(Q-Q^{\dagger}) = ib_0 + \Sigma e_n a_n \quad (1.8)$$

Beside the quaternion multiplication of two quaternions $Q$ and $R$ we also use in this work their scalar product defined by

$$Q \cdot R = \mathcal{S}(QR) = \tfrac{1}{2}(QR + \bar{R}\bar{Q}) \quad (1.9)$$

with a dot inserted between the two quaternions to distinguish the scalar from the quaternion multiplication.

Three arbitrary quaternions $Q$, $R$ and $S$ satisfy the identity

$$Q\cdot(RS) = R\cdot(SQ) = S\cdot(QR) . \quad (1.10)$$

The components of the product $P = QR$ of two quaternions can be easily expressed with the help of the scalar product. Thus if

$$P = p_0 + e_1p_1 + e_2p_2 + e_3p_3$$

$p_0$ and $p_n$ are given by

$$p_0 = Q\cdot R \quad \text{and} \quad p_n = -\mathcal{S}(e_nQR) = -R\cdot e_nQ . \quad (1.11)$$

In the physical applications we shall also use the three dimensional vector notation

$$\underline{q} = \Sigma e_n q_n$$

and the multiplication rule in its vectorial form

$$P = QR = (q_0 + \underline{q})(r_0 + \underline{r}) = (q_0r_0 - \underline{q}\cdot\underline{r}) + (q_0\underline{r} + \underline{q}r_0 + \underline{q}\wedge\underline{r}) \quad (1.12)$$

where $\underline{q}\cdot\underline{r}$ and $\underline{q}\wedge\underline{r}$ denote respectively the scalar and the vectorial products of the complex vectors $\underline{q}$ and $\underline{r}$.

We now proceed to give various matrix representations for quaternions. First of all there are the two well-known regular representations of the quaternion $Q$, namely the complex matrices

$$\Psi = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \text{ and } \Psi' = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \quad (1.13)$$

belonging respectively to the equations

$$QR = P \qquad \text{and} \qquad RQ = P'. \qquad (1.14)$$

In the case of a real ~~number~~ quaternion $A$, these representations are irreducible in the field of real numbers but reducible in the field of complex numbers. We can choose, among other possibilities the following $2\times2$ complex matrices to represent the quaternion units

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \qquad (1.15)$$

Thus the real quaternion $A$ can also be represented by the matrix

$$A = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}. \qquad (1.16)$$

Our purpose is to derive new matrix representations for the complex quaternion $Q = A + iB$, using irreducible complex representations of the type (1.16) for $A$ and $B$. To this end we rewrite the first of the equations (1.14) as two separate equations between the real quaternions only. If we replace $Q$ by $A+iB$, $R$ by $C+iD$ and $P$ by $F+iG$ in that equation we obtain the equivalent real equations

$$\begin{cases} AC - BD = F \\ BC + AD = G \end{cases} \qquad (1.17)$$

Using $2\times2$ matrices the elements of which are quaternions we can combine the equations (1.17) into either of the matrix equations

We now proceed to give various matrix representations for quaternions. First of all there are the two well-known regular representations of the quaternion $Q$, namely the complex matrices

$$\Psi = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \text{ and } \Psi' = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \qquad (1.13)$$

belonging respectively to the equations

$$QR = P \qquad \text{and} \qquad RQ = P'. \qquad (1.14)$$

In the case of a real ~~number~~ quaternion $A$, these representations are irreducible in the field of real numbers but reducible in the field of complex numbers. We can choose, among other possibilities the following 2×2 complex matrices to represent the quaternion units

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \qquad (1.15)$$

Thus the real quaternion $A$ can also be represented by the matrix

$$A = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix}. \qquad (1.16)$$

Our purpose is to derive new matrix representations for the complex quaternion $Q = A + iB$, using irreducible complex representations of the type (1.16) for $A$ and $B$. To this end we rewrite the first of the equations (1.14) as two separate equations between the real quaternions only. If we replace $Q$ by $A + iB$, $R$ by $C + iD$ and $P$ by $F + iG$ in that equation we obtain the equivalent real equations

$$\begin{cases} AC - BD = F \\ BC + AD = G \end{cases} \qquad (1.17)$$

Using 2×2 matrices the elements of which are quaternions we can combine the equations (1.17) into either of the matrix equations

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \tag{1.18}$$

or
$$\begin{pmatrix} A & iB \\ iB & A \end{pmatrix}\begin{pmatrix} C \\ iD \end{pmatrix} = \begin{pmatrix} F \\ iG \end{pmatrix} . \tag{1.19}$$

Hence to the complex quaternion $A+iB$ corresponds one of the operators

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & iB \\ iB & A \end{pmatrix} . \tag{1.20}$$

As both types of operators possess the group property their algebras are isomorphic to the algebra of complex quaternions.

We can now replace the real quaternions $A$ and $B$ in (1.20) by their irreducible representations of the type (1.16) and thus obtain new matrices of the fourth rank to represent $A+iB$ in the field of complex numbers. The second of the matrices (1.20) occurs in Dirac's Theory and for that reason may be called Dirac's representation of the quaternion $A+iB$. In order to write the full matrix of the fourth rank we denote the elements of its first column by $\psi_1, \psi_2, \psi_3$ and $\psi_4$ so that we have

$$\psi_1 = a_0 - ia_3 \quad , \quad \psi_3 = ib_0 + b_3 \tag{1.21}$$
$$\psi_2 = a_2 - ia_1 \quad , \quad \psi_4 = ib_2 + b_1 .$$

The new representation of the quaternion $Q$ now reads

$$\Psi_D = \begin{pmatrix} \psi_1 & -\psi_2^{\times} & \psi_3 & \psi_4^{\times} \\ \psi_2 & \psi_1^{\times} & \psi_4 & -\psi_3^{\times} \\ \psi_3 & \psi_4^{\times} & \psi_1 & -\psi_2^{\times} \\ \psi_4 & -\psi_3^{\times} & \psi_2 & \psi_1^{\times} \end{pmatrix} . \tag{1.22}$$

It can be verified directly that complex matrices of this type form an algebra. The elements of the corresponding quaternion are then given by the formulae (1.21) which we may call Darwin's transformation formulae since similar equations were first introduced by this author soon after the discovery of the relativistic wave equation (Darwin 1928). The important difference is that in our case $a_\nu$ and $b_\nu$ are real whereas the corresponding quantities in Darwin's formulae are complex.

In any of the matrix forms given above the hermitian conjugate of the matrix corresponds to the hermitian conjugate of the associated quaternion in the sense of the definition (1.4). It follows that a hermitian quaternion is always represented by a hermitian matrix. This incidentally justifies our definition of a hermitian quaternion.

The correspondence between a quaternion and its regular representation also extends to the operations of taking the complex conjugate and the quaternion conjugate. In fact the transposed and the complex conjugate of either the matrices (1.13) correspond respectively to the quaternion conjugate and the complex conjugate of the associated quaternion. However, if we use Dirac's representation, its complex conjugate no longer represents the complex conjugate of the quaternion. To find the Dirac representation of complex conjugation we must consider the transformation which transforms the matrix

$$^2\Psi_D = \begin{pmatrix} A & iB \\ iB & A \end{pmatrix}$$

into

$$(^2\Psi_D)^c = \begin{pmatrix} A & -iB \\ -iB & A \end{pmatrix}.$$

This can be expressed by a canonical transformation

$$(^2\Psi_D)^c = \beta\, {}^2\Psi_D \beta^{-1}$$

where

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \qquad (1.23)$$

Here $I$ is the unit matrix of the second rank. The matrix $\beta$ has the property

$$\beta^2 = 1$$

so that the correspondence between $Q^\times$ and $(^2\Psi_D)^c$ may be written

$$Q^\times \longleftrightarrow \beta\, {}^2\Psi_D \beta .$$

## §2. Connection with Dirac's matrices.

Let $\alpha_n$ $(n=1,2,3)$ and $\beta$ be four hermitian matrices which satisfy the relations

$$\alpha_n\alpha_m + \alpha_m\alpha_n = 2\,\delta_{mn}\ ,\ \beta\alpha_m + \alpha_m\beta = 0 \text{ and } \beta^2 = 1 \qquad (2.1).$$

It follows that the three operators

$$e_1' = \alpha_3\alpha_2\ ,\quad e_2' = \alpha_1\alpha_3 \quad \text{and } e_3' = \alpha_2\alpha_1 \qquad (2.2)$$

satisfy the same commutation rules (1.1) as the quaternion units $e_n$. Furthermore the operator

$$i' = \alpha_1\alpha_2\alpha_3 \qquad (2.3)$$

commutes with $e_n'$ and obeys the equation

$$i'^2 = -1 .$$

With the help of this operator we can write

$$\alpha_n = i'e_n' \qquad (2.4)$$

so that the algebra of the linear forms

$$Q' = a_0 + \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_3\alpha_2 a_1 + \alpha_1\alpha_3 a_2 + \alpha_2\alpha_1 a_3 + \alpha_1\alpha_2\alpha_3 b_0 \quad (2.5)$$

or, with the new notations introduced in this section

$$Q' = (a_0 + i'b_0) + \sum (a_n + i'b_n)e_n \qquad (2.6)$$

is seen to be isomorphic to the algebra of the quaternions

$$Q = (a_0 + ib_0) + \sum (a_n + ib_n)e_n . \qquad (2.7)$$

Hence we have proved that the largest sub~~group~~algebra of the Dirac-Eddington algebra generated by $\alpha_1, \alpha_2, \alpha_3$ and $\beta$ which is the algebra generated by the three operators $\alpha_1, \alpha_2$ and $\alpha_3$ is isomorphic to the algebra of complex quaternions. As the numbers $a_\nu$, $b_\nu$ are real the hypercomplex number (2.5) may be represented by any of the matrices (1.13) or (1.20).

As a consequence of the hermitian property of $\alpha_n$ we have

$$\alpha_n^\dagger = \alpha_n\ ,\ (\alpha_m\alpha_n)^\dagger = -\alpha_m\alpha_n \text{ and } (\alpha_1\alpha_2\alpha_3)^\dagger = -(\alpha_1\alpha_2\alpha_3).$$

Hence in the isomorphism

$$Q' \longleftrightarrow Q$$

the image of $(Q')^\dagger$ is $Q^\dagger$. Again, denoting the linear form $\beta Q'\beta$ by $(Q')^c$ we find that the image of $(Q')^c$ is $Q^*$ since the total effect of this operation is to change the sign of $i'$ in (2,6).

Now supposing that $Q'$ is represented by the matrix (1.22) where $\psi_r$ is determined by (1.21), we know that $\beta$ can be represented by (1.23). Hence if $\beta$ has the diagonal form (1.23) the matrices $\alpha_n$ are the coefficient of $b_n$ in (1.22) and may be written as

$$\alpha_n = \begin{pmatrix} 0 & ie_n \\ ie_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_n \\ \sigma_n & 0 \end{pmatrix} \qquad (2.8)$$

where the complex matrices $\sigma_n$ stand for $i$ times the matrices (1.15) and are identical with Pauli's spin matrices.

To sum up: any complex quaternion (2.6) can be represented by (2.5) where the operators $\alpha_n$ denote Dirac's matrices (2.8). The resulting matrix is identical with (1.22). The matrix corresponding to the complex conjugate of this quaternion can be derived from the original matrix by the canonical transformation (1.23) where $\beta$ is Dirac's remaining matrix operator. The matrix which corresponds to the hermitian conjugate of the quaternion is obtained by taking the hermitian conjugate of the original matrix.

In quantum mechanics the wave equation is usually written as a matrix operator operating on the column matrix $\psi$ which represents the wave function. We shall ~~ca~~ see later that Dirac's wave equation may be regarded as the first column of a matrix equation in which the wave function itself is a square matrix of the form $\Psi_D$ (1.22). ~~so~~ The first column of this matrix wave function is identical with $\psi$. Let $\Psi$ the complex quaternion associated with the matrix wave function. $\Psi$ can always be put in the form

$$\Psi = A_1 + e_1 A_2 + i B_1 + ie_2 B_2 \qquad (2.9)$$

where $A_1, A_2, B_1$ and $B_2$ are real quaternions which commute with $e_3$. If we denote by $A_1', A_2', B_1'$ and $B_2'$ the same quantities where $e_3$ has been replaced by $-i$, Darwin's formulae can be written

$$\psi_1 = A_1', \quad \psi_2 = A_2', \quad \psi_3 = iB_1', \quad \psi_4 = iB_2'. \qquad (2.10)$$

Thus from the quaternion $\Psi$ we can derive the first column $\psi$ of its Dirac matrix representation through (2.10). We shall call $\psi$ the column matrix corresponding to the quaternion $\Psi$. Conversely if ~~any~~ the complex column matrix $\psi$ is given, from its elements we can derive a quaternion $\Psi$ which we call the quaternion corresponding to (or associated with) the column matrix $\psi$. Now to an operation applied on the quaternion $\Psi$ will correspond an operation applied to the column matrix $\psi$ associated with it and vice-versa. What we need is a list of equivalent operations applied to $\psi$ and to $\Psi$ to enable us to translate quaternion equations involving $\Psi$ into matrix equations involving the column matrix $\psi$. Such a code is easily found if we remember that $\psi$ is the first column of the Dirac matrix representation of $\Psi$. Thus we find the correspondence relations which will be used throughout this investigation

$$\begin{cases} ie_n\Psi \longleftrightarrow \alpha_n\psi \quad , \quad i\Psi \longleftrightarrow \alpha_1\alpha_2\alpha_3\psi \quad , \quad \Psi^\times \longleftrightarrow \beta\psi \\ \Psi ie_1 \longleftrightarrow -i\alpha_2\beta\psi^\times \quad , \quad \Psi ie_2 \longleftrightarrow \alpha_2\beta\psi^\times \quad , \quad \Psi ie_3 \longleftrightarrow -i\alpha_1\alpha_2\alpha_3\psi \end{cases} \qquad (2,11)$$

Other useful correspondences which may be deduced from (2.11) are

$$\begin{cases} e_1\Psi e_3 \longleftrightarrow i\alpha_2\alpha_3\psi \quad , \quad e_2\Psi e_3 \longleftrightarrow i\alpha_3\alpha_1\psi \quad , \quad e_3\Psi e_3 \longleftrightarrow i\alpha_1\alpha_2\psi \\ \Psi e_3 \longleftrightarrow -i\psi \quad , \quad ie_2\Psi^\times ie_2 \longleftrightarrow \psi^\times \text{ , etc.} \end{cases} \qquad (2.12)$$

As an immediate application of these formulae we note that the four columns of the matrix $\Psi_D$ (1.22) are given by

$$\psi \quad , \quad \psi' = \alpha_1\alpha_3\beta\psi^\times \quad , \quad \psi'' = -i\alpha_1\alpha_2\alpha_3\psi \quad \text{and} \quad \psi''' = -i\alpha_2\beta\psi^\times \qquad (2.13)$$

and correspond respectively to the quaternions

$\Psi$ , $\Psi e_2$ , $\Psi ie_3$ and $\Psi ie_1$.

In order to complete this code of correspondences we still need to give the quaternion representation of the product $\psi^{\times}\cdot\varphi$ where $\psi$ is the row matrix and $\varphi$ the column matrix corresponding respectively to the quaternions $\Psi$ and $\Phi$. The reason is that all expectation values in wave mechanics are of the form $\psi^{\times}\cdot\varphi$. Now $\psi^{\times}$ is the first row of the hermitian conjugate $\Psi_D^{\dagger}$ of $\Psi_D$ and $\varphi$ is the first column of $\Phi_D$. Hence the complex number $\psi^{\times}\cdot\varphi$ is the first element of the matrix $(\Psi^{\dagger}\Phi)_D$ and we have

$$\mathcal{R}(\psi^{\times}\cdot\varphi) = \mathcal{R}(\Psi^{\dagger}\cdot\Phi)$$

$$-i\,\mathcal{I}(\psi^{\times}\cdot\varphi) = \mathcal{R}[\psi^{\times}\cdot(-i\varphi)] = \mathcal{R}(\Psi^{\dagger}\cdot\Phi e_3)$$

so that

$$\psi^{\times}\cdot\varphi = \mathcal{R}(\Psi^{\dagger}\cdot\Phi) + i\,\mathcal{R}(\Psi^{\dagger}\cdot\Phi e_3). \qquad (2.14)$$

Another useful relation is

$$-i\,\mathcal{I}(\Psi^{\dagger}\cdot\Phi) = \mathcal{R}[\Psi^{\dagger}\cdot(-i\Phi)] = -\mathcal{R}(\psi^{\times}\cdot\alpha_1\alpha_2\alpha_3\varphi).$$

Hence

$$\Psi^{\dagger}\cdot\Phi = \mathcal{R}(\psi^{\times}\cdot\varphi) - i\,\mathcal{R}(\psi^{\times}\cdot\alpha_1\alpha_2\alpha_3\varphi) \qquad (2.15)$$

In particular if $\omega$ is a hermitian matrix and if

$$\varphi = \omega\psi$$

then the scalar $\psi^{\times}\cdot\omega\psi$ is real. In quaternion form the quaternion $\Psi$ associated with $\psi$ is related to $\Phi$ associated with $\varphi$ by

$$\Phi = \Omega(\Psi)$$

where $\Omega$ is an operator which corresponds to $\omega$. For instance if $\omega = \beta$ where $\beta$ is the fourth Dirac matrix then $\Omega(\Psi) = \Psi^{\times}$. If $\omega = \alpha_1$ then $\Omega(\Psi) = ie_1\Psi$. From (2.14) we obtain the important formula

$$\psi^{\times}\cdot\omega\psi = \mathcal{R}[\Psi^{\dagger}\cdot\Omega(\Psi)] \qquad (2.16).$$

---

## §3 – Canonical forms for complex quaternions.

In this section we show the different ways in which quaternions can be expressed in terms of their angular coordinates. The polar form

$$Q = q_0 + \underline{q} = s\, e^{\underline{\kappa}\gamma} \qquad (3.1)$$

is well known. Here

$$s^2 = \mathcal{N}(Q) = Q\bar{Q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \qquad (3.2)$$

is the norm of the quaternion, and the complex number

$$s = |Q| = (Q\bar{Q})^{\frac{1}{2}}$$

is called its modulus. The exponential is defined as

$$\exp(\underline{\kappa}\gamma) = \cos\gamma + \underline{\kappa}\sin\gamma$$

where

$$\underline{\kappa} = (q_1^2 + q_2^2 + q_3^2)^{-\frac{1}{2}}\, \underline{q}$$

is a unit vector with complex components. The square of $\underline{\kappa}$ is minus one and its norm is unity. The angle $\gamma$ is complex and is defined by

$$\tan\gamma = q_0^{-1}(q_1^2 + q_2^2 + q_3^2)^{\frac{1}{2}}.$$

In physical applications it is more advantageous to employ two new canonical forms where real angles and real unit vectors occur only. We first note that if $H$ is a hermitian quaternion with norm unity (3.1) gives

$$H = \exp(i\underline{\varepsilon}\alpha) = \cosh\alpha + i\underline{\varepsilon}\sinh\alpha \qquad (3.3).$$

Similarly a real quaternion $R$ with norm unity can be written

$$R = \exp(\underline{\omega}\beta) = \cos\beta + \underline{\omega}\sin\beta. \qquad (3.4)$$

In (3.3) and (3.4) $\underline{\varepsilon}$, $\underline{\omega}$ are unit vectors with real components and $\alpha, \beta$ are real numbers. ~~It~~

We now try to put the quaternion

$$V = s^{-1}.Q \qquad (V\bar{V} = 1)$$

in one of the forms

$$V = HR \quad \text{or} \quad V = R'H' \qquad (3.5)$$

where $R'$ is also real and $H'$ hermitian. This can be achieved ~~by means of the square root of V~~ in the following way.

$$\cancel{V^{\frac{1}{2}} = W = \exp(\tfrac{1}{2}\underline{\kappa}\gamma)}$$

~~which satisfies the relations~~

Let

$$V = A + iB$$

where $A$ and $B$ denote respectively the real and the imaginary parts of $V$. Since the norm of $V$ is one we have

$$A\bar{A} - B\bar{B} = 1 \qquad (3.6)$$

$$B\bar{A} + A\bar{B} = 2\,B.\bar{A} = 2\,\bar{B}.A = 0 \qquad (3.7)$$

Now we can write

$$V = |A|(1 + iB\bar{A})\,A\,|A|^{-1}.$$

From (3.7) it follows that $B\bar{A}$ is real and purely vectorial. Hence, if we put

$$H = |A|(1 + iB\bar{A})$$

$$R = A|A|^{-1}$$

we see that $H$ is hermitian and $R$ real. Moreover, using (3.6) we have

$$|H| = 1.$$

In the same way we can write

$$V = R'H'$$

with

$$R' = R = A|A|^{-1}$$

and

$$H' = |A|(1 + i\bar{A}B).$$

Using the polar forms (3.3) and (3.4) we ~~go~~ ~~obtain~~ conclude that any complex quaternion $Q$ with non-vanishing norm can be put in one of the forms

$$Q = s\, e^{i\underline{\varepsilon}\alpha}\, e^{\underline{\omega}\beta} \qquad (3.8)$$

or $$Q = s\, e^{\underline{\omega}\beta}\, e^{i\underline{\varepsilon}'\alpha'} \qquad (3.9).$$

where $s$ is complex and $\underline{\varepsilon}, \alpha, \underline{\varepsilon}', \underline{\alpha}', \underline{\omega}, \beta$ are all real.

## § 4 – Representation of ~~four-vectors~~ time-like and space-like four-vectors. ~~and~~ Orthogonal transformations.

Following P. Weiss (1941) we shall represent the position vector $(x^0, x^1, x^2, x^3)$ in the space-time of special relativity by the hermitian quaternion

$$X = I_\nu x^\nu$$

where $x^0 = ct$ and

$$I_0 = 1 \quad , I_n = i e_n \qquad (n = 1,2,3) \qquad (4.1)$$

A summation over $\nu$ ($\nu = 0,1,2,3$) is implied. The infinitesimal displacement vector $(dx^0, dx^1, dx^2, dx^3)$ is likewise represented by

$$dX = I_\nu dx^\nu .$$

If $a^{\mu\nu}$ is the metric tensor of special relativity ~~we~~ ~~find~~ by which indices are raised and lowered, we find

$$I^\lambda = a^{\lambda\nu} I_\nu = \bar{I}_\lambda .$$

so that the norm of $dX$

$$\mathcal{N}(dX) = dX\, d\bar{X} = I_\lambda . \bar{I}_\nu\, dx^\lambda dx^\nu = dx_\nu\, dx^\nu$$

is positive if $dX$ is time like and negative if it is space-like.

Now consider the coordinate transformation

$$(x')^\nu = f^\nu(x^\mu)$$

and let

$$dX' = I_\nu (dx')^\nu .$$

The transformation is conformal if

$$dX' = Q\, dX\, Q^\dagger \qquad (4.2)$$

where $Q$ is a complex quaternion, because taking the norm of both sides of (4.2) we get

$$(dx')^\nu (dx')_\nu = l^2\, dx^\nu dx_\nu$$

where the function

$$l^2 = (Q\bar{Q})(Q\bar{Q})^* .$$

is positive definite so that a time-like vector is transformed into another time-like vector. The physical ~~transfor~~ meaning of the transformation becomes clear if we replace $Q$ by its canonical expression (3.8). Thus (4.2) can be rewritten in the form

$$dX' = ss^* e^{i\varepsilon\alpha} e^{\omega\beta} dX e^{-\omega\beta} e^{i\varepsilon\alpha} .$$

It is clearly equivalent to a rotation round the axis $\underline{\omega}$ through the angle $2\beta$, followed by a Lorentz transformation in the $\underline{\varepsilon}$ direction with a velocity of translation $c \tanh 2\alpha$ and combined with an isotropic dilatation of the space and time coordinates by the factor $ss^*$.

It may be noted that (4.2) is invariant for the transformation

$$Q \to Q e^{i\lambda} . \qquad (4.3)$$

Hence we can always choose $\lambda$ so that $|Q|$ is real. For reasons which will become clear later (§6) (4.3) will be called a pseudo-gauge transformation.

Now we give canonical forms for time-like and space-like hermitian vectors with non-zero modulus. If $a = |A|$ denotes the modulus of a hermitian quaternion $A$ which represents a time-like four-vector, it is always possible to find a frame of reference where this vector is represented by the positive scalar $a$ which becomes its time component. It follows that $A$ can be derived from $a$ through an orthogonal transformation which is specified by a certain quaternion $C$ with unit norm. Hence any time-like hermitian quaternion may be written

$$A = |A| \, CC^{\dagger} \quad , \quad (|C| = 1) . \qquad (4.4)$$

Similarly a space-like vector will, in a suitable frame, possess only one non-vanishing component along a specified spatial direction which can always be taken as the $O_z$ direction. In this "rest system" the hermitian quaternion $B_3$ which corresponds to this vector becomes

$$(-B_3 \bar{B}_3)^{\frac{1}{2}} i e_3 ,$$

and therefore can always be written in the form

$$B_3 = (-B_3 \bar{B}_3)^{\frac{1}{2}} C' i e_3 C'^{\dagger} = |B_3| C' e_3 C'^{\dagger}$$

where again $C'$ determines a certain orthogonal transformation. In particular if $B_3$ is along the $O_z$ axis in the same frame of reference in which $A$ is scalar we can take $C' = C$.

Finally ~~if~~ let $B_1$ and $B_2$ two other space-like vectors which are Lorentz orthogonal to each other and to $B_3$ and $A$. In the rest system where $A = |A|$ and $B_3 = |B_3| e_3$, $B_1$ will reduce to its component along the $Ox$ axis and $B_2$ to its component along the $Oy$ axis. Hence they must be of the form

$$B_1 = |B_1| C e_1 C^\dagger \quad \text{and} \quad B_2 = |B_2| C e_2 C^\dagger$$

where $|B_1|$ and $|B_2|$ are imaginary.

Now ~~the~~ if $R$ denotes a real quaternion with unit norm the transformation

$$C \to CR \qquad (4.5)$$

which defines a spatial rotation in the rest system leaves $A$ invariant but affects the space-like vectors $B_n$. If $R$ commutes with one of the quaternion units, say $e_3$, then it is of the form

$$R = \exp(e_3 \gamma)$$

and the transformation

$$C \to C e^{e_3 \gamma} \qquad (4.6)$$

which defines a rotation in the rest system round the $Oz$ axis leaves both $A$ and $B_3$ invariant but transforms a linear combination of $B_1$ and $B_2$ into another linear combination of the same vectors. It is a special case of the general spatial rotation in the rest system (4.5) which has been called "an inner rotation" by Weiss (1941). (4.6) will be called a "gauge transformation" and this ~~de~~finition of the gauge will be justified in §6.

We shall now find the transformation law for the quaternion $C$ when the four-vectors $A$ and $B_n$ undergo a constant orthogonal transformation specified by the quaternion $\Gamma$. Thus, when the coordinates transform according to

$$dX' = \Gamma \, dX \, \Gamma^\dagger \qquad (|\Gamma| = 1) \qquad (4.7)$$

we have

$$A' = |A| \Gamma C C^\dagger \Gamma^\dagger \quad \text{and} \quad B'_n = |B_n| \Gamma C e_n C^\dagger \Gamma^\dagger$$

so that we can write

$$A' = |A'| C' C'^{\dagger} \quad \text{and} \quad B'_n = |B'_n| C' e_n C'^{\dagger}$$

assuming for $C$ the transformation law

$$C' = \Gamma C \qquad \text{or} \qquad C \to \Gamma C. \tag{4.8}$$

This is the celebrated spinor transformation law. In the next section it will be shown that in the quaternion form of Dirac's equation the wave function transforms according to (4.8).

The behaviour of six-vectors in an orthogonal transformation is also found easily. If $P_1$ and $P_2$ are two four-vectors, the components of the purely vectorial quaternion

$$G = V(P_1 \bar{P}_2) = \tfrac{1}{2}(P_1 \bar{P}_2 - P_2 \bar{P}_1)$$

form a six vector. Since in an orthogonal transformation

$$P_1' = \Gamma P_1 \Gamma^{\dagger} \quad \text{and} \quad P_2' = \Gamma P_2 \Gamma^{\dagger}$$

we find

$$G' = \tfrac{1}{2} \Gamma (P_1 \Gamma^{\dagger} \Gamma^{\times} \bar{P}_2 - P_2 \Gamma^{\dagger} \Gamma^{\times} . \bar{P}_1) \bar{\Gamma} = \Gamma G \bar{\Gamma}. \tag{4.9}$$

# CHAPTER II

## Quaternion Formulation of Dirac's Equation.

### § 5. Dirac's equation in Watson's quaternion form.

Dirac's equation in its algebraic form and in field free conditions reads

$$(p_0 - \Sigma \alpha_n p_n)\, \psi = m_0 c \beta \psi \qquad (5.1)$$

where $\alpha_n$, $\beta$ are defined by (2.1) and $\psi$ is a column matrix with complex elements. We have seen that $\psi$ can be regarded as the first column of the matrix form $\Psi_D$ (1,22) of a complex quaternion $\Psi$. With the help of the code (2.11) the corresponding quaternion equation can be written immediately as

$$\bar{P}\, \Psi = m_0 c\, \Psi^{\times} \qquad (5.2)$$

where

$$P = p_0 + i \Sigma e_n p_n$$

is the hermitian quaternion which represents the energy-momentum four-vector.

The Lorentz invariance of (5.2) is obvious. Under the constant orthogonal transformation (4.7), $\bar{P}$ becomes

$$\bar{P}' = \Gamma^{\times} \bar{P}\, \Gamma . \qquad (5.3)$$

Hence (5.2) is invariant if $\Psi$ obeys the same law (4.8) as $C$, i.e.

$$\Psi' = \Gamma \Psi . \qquad (5.4)$$

We now consider the improper Lorentz transformation which consists in changing the sign of the space coordinate (space reflection). It is defined by the equation

$$P' = P^{\times}$$

and (5.2) is restored to it original form by taking

$$\Psi' = \Psi^{\times} \qquad (5.5)$$

which is the transformation law for a space reflection.

By choosing Dirac's matrix form (1.22) for the quaternion $\Psi$ we get the following matrix equation equivalent to (5.2)

$$(p_0 - \Sigma \alpha_n p_n)\, \Psi_D = m_0 c \beta \Psi_D \beta \qquad (5.6)$$

since $\Psi^*$ is represented by $\beta \Psi_D \beta$ in this case. Equating the first columns of both sides in (5.4) we obtain (5.1). The three remaining columns give equations which correspond to the quaternion equations derived from (5.2) by multiplying both sides to the right by $e_2$, $ie_3$ and $ie_1$. They are

$$(p_0 - \Sigma \alpha_n p_n)\, \psi' = m_0 c \beta \psi'$$

$$(p_0 - \Sigma \alpha_n p_n)\, \psi'' = -m_0 c \beta \psi''$$

and $(p_0 - \Sigma \alpha_n p_n)\, \psi''' = -m_0 c \beta \psi'''$

where $\psi'$, $\psi''$ and $\psi'''$ are given by (2.13).

The quaternion formulation (5.2) leads immediately to the solution of the algebraic Dirac equations for $P$. Taking the complex conjugate of (5.2), using (1.5) and multiplying to the right by $(\Psi^*)^{-1}$ we obtain

$$P = m_0 c\, \Psi (\Psi^*)^{-1} \qquad (5.7).$$

We have assumed that the norm of $\Psi$ does not vanish. Hence $\Psi$ has the form

$$\Psi = r e^{i\lambda/2} C$$

where $r$ and $\lambda$ are real numbers and $C$ is a quaternion with unit norm. Hence (5.7) becomes

$$P = m_0 c\, e^{i\lambda} CC^\dagger.$$

Now, since in this equation $P$ and $CC^\dagger$ are hermitian we must have $\lambda = n\pi$. On the other hand, taking the modulus of both sides we find that $|P|$ must be equal to $\pm m_0 c$, so that (5.7) reduces to

$$P = \pm m_0 c\, CC^\dagger \qquad (C\bar{C} = 1). \qquad (5.8)$$

Comparing (5.8) with the canonical form (4.4) for a time-like hermitian quaternion we can state that the algebraic form of Dirac's equation merely expresses in quaternion language that the energy-momentum four-vector is time-like, the rest mass, hence the energy being either positive or negative.

The quaternion $C$ can always be written in the canonical form

$$C = e^{\frac{1}{2} i \underline{k} \alpha} R$$

with the help of (3,6). Here $R$ is a real quaternion with norm unity. Then (5.8) determines $\underline{k}$ and $\alpha$ in terms of $p_\nu$ and we obtain

$$p_0 = m_0 c \cosh\alpha \quad \text{and} \quad \underline{p} = \underline{k}\, m_0 c \sinh\alpha \tag{5.9}$$

$$\text{or } p_0 = -m_0 c \cosh\alpha \quad \text{and} \quad \underline{p} = -\underline{k}\, m_0 c \sinh\alpha .$$

We now pass to the quaternion formulation of the matrix wave equation which is derived from (5.1) by replacing $p_0$ by $\hbar i \partial_0$ and $p_n$ by $-\hbar i \partial_n$. When the operand is no longer the column matrix $\psi$ but the quaternion $\Psi$, the relevant formula of the code (2.12) provides the equivalent quaternion operators

$$p_0 \Psi = -\hbar \partial_0 \Psi e_3 \quad , p_n \Psi = \hbar \partial_n \Psi e_3 . \quad \cancel{(5.10)}$$

If we put

$$D = \bar{I}^\nu \partial_\nu = \partial_0 + \Sigma i e_n \partial_n$$

we see that $\bar{P}$ is replaced by the operator

$$-\hbar D(\ )e_3 .$$

Thus an explicit reference to the $Oz$ direction in the rest system is introduced in the wave equation through the unit quaternion $e_3$. More symmetrical quaternion representations of the operators $p_\nu$ will be discussed in §§ 7 and 14. ~~If (5.10) is adopted the~~ With the present choice for $p_\nu$ the wave equation becomes

$$D \Psi = \mu \Psi^\times e_3 . \qquad (\mu = m_0 c / \hbar) \tag{5.10}$$

In the presence of an electromagnetic field, $\bar{P}$ gets replaced by

$$\bar{P} + e\bar{A}$$

where $A$ is the hermitian quaternion which corresponds to the electromagnetic potential with components $A_\nu$, i.e.

$$A = I^\nu A_\nu = A_0 \mp \Sigma i e_n A_n$$

so that we obtain

$$\hbar D \Psi = (m_0 c \Psi^\times + e \bar{A} \Psi) e_3 \tag{5.11}$$

This equation is essentially equivalent to the quaternion form of Dirac's equation proposed by Watson (1947).

If Dirac's matrix form is used to represent the quaternions in (5.11) we get

$$\hbar(\partial_0 + \sum \alpha_n \partial_n)\,{}^{2}\Psi_D = \{ m_0 c \beta\,{}^{2}\Psi_D \beta + (eA_0 + e\sum \alpha_n A_n)\,{}^{2}\Psi_D \}\alpha_2\alpha_1 . \quad (5.12)$$

Writing that the first columns of the matrices on both sides of this equation are equal we find

$$i\hbar(\partial_0 + \sum \alpha_n \partial_n)\psi = m_0 c \beta\psi + (eA_0 + e\sum \alpha_n A_n)\psi \quad (5.13)$$

that is, the usual form of Dirac's equation where the wave function is represented by the column matrix $\psi$.

Writing that the remaining three columns on both sides are equal we find three more matrix equations of the type (5.13). They correspond to the equations obtained by multiplying the quaternion equation (5.11) to the right by $e_2$, $ie_3$ and $ie_1$. Translated into Dirac's formalism which involve the column matrix $\psi$, they read

$$i\hbar(\partial_0 + \sum \alpha_n \partial_n)\psi' = -(m_0 c\beta + eA_0 + e\sum \alpha_n A_n)\psi'$$

$$i\hbar(\partial_0 + \sum \alpha_n \partial_n)\psi'' = (-m_0 c\beta + eA_0 + e\sum \alpha_n A_n)\psi''$$

$$i\hbar(\partial_0 + \sum \alpha_n \partial_n)\psi''' = (m_0 c\beta - eA_0 - e\sum \alpha_n A_n)\psi''';$$

where $\psi'$, $\psi''$ and $\psi'''$ are the second, third and fourth columns of the matrix ${}^{2}\Psi_D$ and are given by (2.13). All these equations may of course be derived directly from the usual wave equation (5.13). Thus the matrix equation (5.12) groups in one equation four different forms of (5.13). It will be noted that $\psi'''$ is the charge conjugate of $\psi$ since it satisfies Dirac's equation with $e$ changed into $-e$. The column matrix $\psi''$ satisfies the wave equation with a negative rest-mass and in the equation satisfied by $\psi'$ ~~bo~~ the sign of both the rest mass and the charge is changed.

## § 6. Gauge and pseudo-gauge invariance – Charge conjugation.

Consider a Dirac particle in a pseudo-vector field. If $B_0$, $B_n$ are the components of the pseudo-vector, the additional terms in Dirac's matrix equation have the form

$$\kappa\,(i\alpha_1\alpha_2\alpha_3 B_0 + i\alpha_2\alpha_3 B_1 + i\alpha_3\alpha_1 B_2 + i\alpha_1\alpha_2 B_3)\,\psi$$

where $\kappa$ is a coupling constant. With the help of (2,11) and (2,12) we are led to add the term

$$\kappa\,(iB_0 + \Sigma e_n B_n)\,\Psi$$

on the right hand side of the quaternion equation (5,11) if we suppose that an electromagnetic field is also present. Putting

$$B = B_0 + \Sigma i e_n B_n$$

we write the wave equation with vector and pseudo-vector interactions

$$\hbar D\Psi = (m_0 c\,\Psi^* + e\bar{A}\,\Psi)\,e_3 + i\kappa\bar{B}\,\Psi \qquad (6,1)$$

Now let $\Psi$ be subject to the transformation

$$\Psi \to \Psi \exp(e_3\gamma/\hbar) \qquad (6,2)$$

where $\gamma$ is a real scalar function. (6,1) can be restored to its original form if $eA$ is transformed according to

$$e\bar{A} \to e\bar{A} - D\gamma\,.$$

As $\Psi$ has the same transformation properties as $C$ in § 4 this justifies the name of gauge transformation given to Weiss' special inner rotation (4,5). The transformation of the column matrix $\psi$ which corresponds to (6,2) is obviously the familiar gauge transformation

$$\psi \to \psi \exp(-i\gamma/\hbar)\,.$$

This gauge transformation (6,2) should not be confused with the rotation of the coordinate system round the $Oz$ axis through the angle $\beta$ and which is expressed by

$$\Psi \to \exp(\tfrac{1}{2}e_3\beta)\,\Psi\,.$$

We now consider the transformation which replaces $\kappa\bar{B}$ by $(\kappa\bar{B} - D\lambda)$ in (6,1). It is given by

$$\Psi \to \Psi \exp(i\lambda/\hbar) \qquad (6,3)$$

and leads to a gauge transformation of the pseudo-vector potential. This is why we call (6.2) and (6.3) pseudo-~~vector~~ gauge transformations. The matrix equivalent of (6.3) involving $\psi$

is found to be

$$\psi \to \exp(\alpha_1\alpha_2\alpha_3\lambda/\hbar)\,\psi\,. \qquad (6.4)$$

The main difference between the pseudo-gauge transformation (6.3) and the gauge transformation (6.2) is that the rest mass, although invariant under the latter, is changed into

$$m_0 \to m_0 \exp(-2i\lambda/\hbar)$$

by the former. In the matrix form of the wave equation the additional term resulting from the transformation (6.4) reads

$$(c_1 + c_2\alpha_1\alpha_2\alpha_3)\beta\psi$$

where

$$c_1 = m_0c\,[\cos(2\lambda/\hbar) - 1]\,, \quad c_2 = -m_0c\sin(2\lambda/\hbar)\,. \qquad (6.5)$$

Hence as a result of (6.4) the pseudo-vector potential suffers a gauge transformation while new scalar and pseudoscalar ~~terms~~ interaction terms $c_1\beta$ and $c_2\alpha_1\alpha_2\alpha_3\beta$ appear in the wave equation.

Now ~~let~~ suppose $B = 0$ and let

$$\Psi_+ = \Psi(E) = \Phi_+ e^{-e_3Et/\hbar}$$

where $E$ is a positive value of the energy and $\Phi$ independent of time. (6.1) can be written as

$$E\,\Phi_+ - i\underline{\nabla}\,\Phi_+e_3 = m_0c\,\Phi_+^* + e\bar{A}\,\Phi_+ \cancel{- ie\bar{B}\Phi_+}\,. \qquad (6.6)$$

If we consider a solution corresponding to a negative value of the energy

$$\Psi_- = \Psi(-E) = \Phi_-\, e^{e_3Et/\hbar}$$

we find that $\Phi_-$ satisfies

$$-E\,\Phi_- - i\underline{\nabla}\,\Phi_-e_3 = m_0c\,\Phi_-^* + e\bar{A}\,\Phi_- \cancel{- ie\bar{B}\Phi_-}\,. \qquad (6.7)$$

We are looking for a relation between $\Psi_-$ and $\Psi_+$. Multiplication of (6,6) to the right by $ie_1$ gives, after changing the sign of both members of the equation

$$-E(\Phi_+ie_1) - i\underline{\nabla}(\Phi_+ie_1)e_3 = m_0c(\Phi_+ie_1)^* - e\bar{A}(\Phi_+ie_1)\,. \qquad (6.8)$$

We see that $\Phi_+ie_1$ satisfies the same equation as $\Phi_-$ with the opposite sign of the charge $e$. Hence

$$\Psi_- = \Psi(-E,e) = \Phi_-\,e^{e_3Et/\hbar} = \underset{e\to-e}{\Phi_+}\,ie_1\,e^{e_3Et/\hbar} = \Phi_+(E,-e)\,e^{-e_3Et/\hbar}\,ie_1 \cancel{= \Psi(E,-e)}$$

so that

$$\Psi(-E,e) = \Psi(E,-e)\,ie_1\,. \qquad (6,9)$$

In other words if we know a ~~positive~~ energy solution of the wave equation, by changing the sign of the charge in this ~~equation~~ solution and by multiplying it to the right by $ie$, we obtain a negative energy solution of the original equation.

The operation of multiplication of $\Psi$ to the right by $ie$ is called "charge conjugation" since the charge conjugate function

$$\cancel{\Psi} \quad \Psi ie,$$

satisfies the wave equation with an opposite value of the charge. We have seen[2,13] that to the transformation

$$\Psi \to \Psi ie,$$

corresponds the matrix transformation

$$\psi \to -i\alpha_2 \beta \psi^*$$

which is the standard form (Hill & Landshoff Rev. Mod. Phys. 1938) for the charge conjugation.

## §7. Comparison with other quaternion forms of Dirac's equation.

Dirac's equation in the absence of field may be written
$$\beta^\nu \partial_\nu \psi = \mu \psi \qquad (7,1)$$
where the operators $\beta^\nu$ obey the commutation relations
$$\beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda = -2 g^{\lambda\nu}. \qquad (7,2)$$
$g^{\lambda\nu}$ is the metric tensor of special relativity.

It is possible to find various realizations of the operators $\beta$ in terms of linear quaternion functions. The first in date was given by Lanczos (1929) and can be expressed by the matrix formula
$$\beta^\nu = \begin{pmatrix} 0 & \bar{I}^\nu \\ -I^\nu & 0 \end{pmatrix} \qquad (7,3)$$
where the hermitian quaternions are given by (1,8). The operator (7,3) is assumed to operate on a column matrix composed of two complex quaternions $\chi$ and $\eta$. Remembering that
$$D = \bar{I}^\nu \partial_\nu = \sum I_\nu \partial_\nu$$
and using the representation (7,3) in (7,1) we obtain Lanczos' equations
$$\begin{cases} D\eta = \mu \chi \\ -\bar{D}\chi = \mu \eta \end{cases} \qquad (7,4)$$
which are relativistically symmetrical in form and represent a particle of spin $\frac{1}{2}$. However, as the wave function has sixteen real components it cannot be put into a one to one correspondence with the usual Dirac column matrix which depends on eight real functions.

Another quaternion representation of $\beta^\nu$ was given by Conway (1937). It can be written explicitly as
$$\beta^0 = (\ )\underline{a} \quad , \quad \beta^n = e_n(\ )\underline{b} \qquad (7,5)$$
where $\underline{a}$ and $\underline{b}$ are two real vectorial quaternions such that
$$S(\underline{a}\,\underline{b}) = 0 \quad , \quad N(\underline{a}) = N(\underline{b}) = 1.$$
It is understood that the wave function is represented by a complex quaternion $\Psi$ which has to be inserted in the brackets in (7,5). In particular, taking

$\underline{a} = e_1 \quad , \quad \underline{b} = e_2$

one obtains one form of Conway's equation, namely

$$\partial_0 \Psi e_1 + \underline{\nabla} \Psi e_2 = \mu \Psi \qquad (7,6)$$

where

$$\underline{\nabla} = \sum e_n \partial_n \qquad (7,7)$$

As the imaginary unit does not appear in (7,6) $\Psi$ might be taken as a real quaternion. The wave equation in this case would only involve 4 real functions. In the next section it will be shown that it leads to a two-component wave function. In order to introduce the electro-magnetic field through the substitution

$$\hbar \partial_\nu \to \hbar \partial_\nu \pm ie A_\nu$$

we must take $\Psi$ complex. Conway has shown that the equation obtained in this way is equivalent to Dirac's matrix equation when a real representation is ~~not~~ chosen for $\alpha_1$, $\alpha_2$ and $\alpha_3$. In Conway's equation the charge conjugate function is simply $\Psi^*$ and the function which corresponds to a time reflection is $\Psi e_2$. The chief defect of Conway's representation is the non symmetric form of (7,5) which makes impossible to bring the gradient operator D into the wave equation. Even in the very simple case of the free electron (see Conway, ~~Conw~~ 1948) the solution of (7,6) is difficult and the equation does not allow a simple treatment of the problem as in our § 13.

Watson's form (5,9) of the wave equation corresponds to the following choice for the operators $\beta$

$$\beta^\nu = -\bar{I}^\nu (\ )^\times e_3 \qquad (7,8)$$

and will be used throughout this investigation.

An immediate generalization of this representation is

$$\beta^\nu = -\bar{I}^\nu (\ )^\times \underline{a}$$

where $\underline{a}$ is again a purely vectorial ~~qu~~ real quaternion with unit norm. For instance if we choose $\underline{a} = e_1$, we can write Dirac's equation as

$$D \Psi = \mu \Psi^* e_1$$

in the absence of field and as

If $\Psi = A + iB$ $\quad A, B$ real

$= U\tfrac{1}{2}(1+ie_1) + V\tfrac{1}{2}(1-ie_1)$

$U = A - Be_1$

$V = A + Be_1$

$g_0 + ie_1h_1 = (g_0+h_1)\tfrac{1}{2}(1+ie_1) + (g_0-h_1)\tfrac{1}{2}(1-ie_1)$

$e_3g_3 + ie_2h_2 = e_3(g_3 + ie_1h_2) = e_3(g_3+h_2)\tfrac{1}{2}(1+ie_1) + e_3(g_3-h_2)\tfrac{1}{2}(1-ie_1)$

$(A - Be_1)\tfrac{1}{2}(1+ie_1) + (A+Be_1)\tfrac{1}{2}(1-ie_1)$

$\tfrac{1}{2}A + \tfrac{1}{2}Bi + \tfrac{1}{2}Aie_1 - \tfrac{1}{2}Be_1 - \tfrac{1}{2}Aie_1 + \tfrac{1}{2}Be_1$

$= A + iB.$

$$\boxed{A + iB \equiv (A - Be_1)\tfrac{1}{2}(1+ie_1) + (A+Be_1)\tfrac{1}{2}(1-ie_1)}$$

$A - Be_1 = U \qquad 2A = U + V$

$A + Be_1 = V \qquad 2B = (U - V)e_1$

$\varepsilon^2 = \varepsilon$

Cf. Bleton. $a = a_1 + a_2$ where $a_1 = a\varepsilon$, $a\bar{\varepsilon} = a_2$.

Here $a_1$ is of the form

$a_1 = \tfrac{1}{2}U(1+i\varepsilon)$ where $U$ is a real quaternion



$$D\Psi = (\mu\Psi^* + e\bar{A}\Psi)e_1 \qquad (7,9)$$

when the electromagnetic field is present. We can restore (7,9) to Watson's form by making the transformation

$$\Psi \to \Psi \exp(-e_2\pi/4).$$

Finally we can take

$$\beta^\nu = \bar{I}^\nu(\ )^x T \qquad (7,10)$$

where $T$ is an antihermitian quaternion with unit norm. This is formally a relativistic generalisation of (7,8) the consequences of which will be studied in §14.

Our purpose now is to discuss the relationship between Watson's and Conway's equation in order that a solution of Conway's equation may be derived from a corresponding solution of Watson's equation. First we note that any complex quaternion

$$\Psi = G + iH$$

with real part $G$ and imaginary part $H$ can be written in the form

$$\Psi = \tfrac{1}{2}U(1+ie_1) + \tfrac{1}{2}Ve_3(1-ie_1) \qquad (7,11)$$

where $U$ and $V$ are again real quaternions. The relations between these two sets of real quaternions are easily found to be

$$2G = U + Ve_3$$
$$2H = (U - Ve_3)e_1$$

and, when solved for $U$ and $V$ give

$$\begin{cases} U = G - He_1 \\ V = He_2 - Ge_3 . \end{cases} \qquad (7,12)$$

Inserting (7,11) into Dirac's equation

$$\partial_0\Psi + i\underline{\nabla}\Psi = \mu\Psi^*e_3 \qquad (7,13)$$

where $\underline{\nabla}$ is given by (7,7), and using the identity

$$i(1+ie_1) = -e_1(1+ie_1)$$

we can write (7,13) in the form

$$\tfrac{1}{2}(\partial_0U - \underline{\nabla}Ue_1)(1+ie_1) + \tfrac{1}{2}(\partial_0V - \underline{\nabla}Ve_1)e_3(1-ie_1) = \tfrac{1}{2}\mu Ue_3(1+ie_1) + \tfrac{1}{2}\mu Ve_1e_3(1-ie_1).$$

From the transformation equations (7,12) it is clear that equating the real and the imaginary part of both sides in this equation is equivalent to equating the real coefficients of $(1+ie_1)$ and $(1-ie_1)$. Thus we obtain

$$\partial_0 U + \underline{\nabla}\, U e_1 = \mu\, U e_3 \qquad (7,14)$$

and the same equation holds for $V$. multiplying the latter by $i$ and adding it to (7,14) we get one form of Conway's equation which reads

$$\partial_0 (U+iV) + \underline{\nabla}(U+iV)e_1 = \mu\,(U+iV)e_3 \;.$$

In the same way the general equation

$$\partial_0 \Psi + i\underline{\nabla}\,\Psi = \mu\,\Psi^{x} e_3 + \lambda A \Psi e_3 \qquad (\lambda = e/\hbar c)$$

leads to the following equations for $U$ and $V$:

$$\partial_0 U - \underline{\nabla} U e_1 - \mu U e_3 = -\lambda A_0 V + \lambda \underline{A} V e_1 \qquad (7,15)$$

$$\partial_0 V - \underline{\nabla} V e_1 - \mu V e_3 = \lambda A_0 U - \lambda \underline{A}\, U e_1 \;.$$

Hence the complex quaternion

$$\Phi = U + iV$$

satisfies the equation

$$(\partial_0 - i\lambda A_0)\,\Phi - (\underline{\nabla} - i\lambda \underline{A})\,\Phi e_1 = \mu\,\Phi e_3 \;.$$

Thus in Conway's quaternion formulation the electromagnetic potential is introduced in exactly the same way as in the matrix equation through the substitution

$$\hbar\,\partial_\nu \rightarrow \hbar\,\partial_\nu + ie A_\nu \;.$$

We conclude that a solution of Conway's equation $\Phi = U + iV$ may be deduced from a solution of Watson's equation $\Psi = G + iH$ with the help of the relations (7,12).

## §8. Two-Component Wave equations as degenerate forms of the general equation.

Let $\Psi_s$ be a solution of Dirac's equation which is equal to its charge conjugate. We shall call it a self charge-conjugate solution. We have

$$\Psi_s = \Psi_s i e_1$$

since $\Psi i e_1$ is the charge-conjugate of $\Psi$. From (7,10) it is clear that $\Psi_s$ must be of the form

$$\Psi_s = \tfrac{1}{2} U_s (1 + i e_1). \qquad (8,1)$$

Hence the corresponding Conway function is

$$\Phi_s = U_s$$

since $V_s = 0$. From (7,14) it also follows that $U_s$ satisfies the real quaternion equation

$$\partial_0 U_s - \nabla U_s e_1 - \mu U_s e_3 = 0 \qquad (8,2)$$

which is the same as the equation (7,13) for a free particle because it does not involve the electromagnetic potential. Now (8,2) being a real quaternion equation may be put in a $2 \times 2$ matrix form. Let us take

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad (8,3).$$

The first column of the matrix

$$U_s = u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3$$

is represented by

$$u = \begin{pmatrix} u_0 + i u_1 \\ u_2 - i u_3 \end{pmatrix}.$$

Another form of (8,2) is

$$\beta^\nu \partial_\nu U_s + \mu U_s = 0 \qquad (8,4)$$

where the operators $\beta^\mu$ are defined by

$$\beta^0 U_s = U_s e_3$$

$$\beta^n U_s = e_n U_s e_2 \qquad (n = 1, 2, 3).$$

and satisfy the commutation relations

$$\beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda = -2 g^{\lambda\nu}.$$

$g^{\lambda\nu}$ is the metric tensor of special relativity. It will now be shown that (8,4) can also be written in the form

$$\gamma^\nu \partial_\nu U_s + \mu U_s^\times = 0 \qquad (8,5).$$

where $U_s^{\times}$ is the complex conjugate of the matrix which represents the real quaternion $U_s$. We note, that, as $e_2$ is real and $e_1, e_3$ are imaginary in the representation (8,3) we have

$$-e_2 U_s e_2 = u_0 - e_1 u_1 + e_2 u_2 - e_3 u_3 = U_s^{\times}. \qquad (8,6).$$

Hence, multiplying (8,4) to the left by $-e_2$ and to the right by $e_2$ we find (8,5) where the operators $\gamma^\nu$ are defined by

$$\gamma^\nu U_s = -e_2(\beta^\nu U_s)e_2 = (\beta^\nu U_s)^{\times}.$$

It follows that

$$\gamma^{\times\lambda}\gamma^\nu U_s + \gamma^{\times\nu}\gamma^\lambda U_s = (\beta^\lambda\beta^\nu + \beta^\nu\beta^\lambda) U_s$$

Thus the operators in (7,5) satisfy the commutation relations

$$\gamma^{\times\lambda}\gamma^\nu + \gamma^{\times\nu}\gamma^\lambda = -2g^{\lambda\nu} \qquad (8,7).$$

The first column of the matrix equation (7,5) gives

$$\gamma^\nu \partial_\nu u + \mu u^{\times} = 0 \qquad (8,8)$$

where $\gamma^\nu$ now represents a matrix operating on the two components of $u$ and corresponding to the operator $-e_2(\beta^\nu U_s)e_2$. But (8,8) is just the two-component wave equation postulated by Jehle (1949) and (8,7) is identical with his commutation relations for the matrices $\gamma^\nu$.

It is now clear that since the two-component equation is satisfied by the self charge-conjugate function, the electromagnetic field cannot be introduced in (8,8) through the familiar ~~relation~~ substitution

$$\partial_\nu \to \partial_\nu + i\lambda\phi_\nu \qquad (\lambda = e/\hbar c).$$

If this is done, then $\phi_\nu$ no longer represents an electromagnetic ~~interaction~~ potential, but as Serpe (1949) has shown directly, corresponds to a pseudo-vector interaction.

C. Kikuchi (1949, Phys. Rev.) has proposed another two component equation which, in quaternion notation reads

$$(i\partial_0 + \sum e_n \partial_n)\phi = \mu e_2 \phi^{\times} \qquad (8,9)$$

and where $2\times 2$ matrix representations are employed for $e_n$ and $\phi$. However, it can be easily shown that (8,9) is in fact equivalent to two different equations of the type (8,8). Using (8,6) we can replace $\phi^{\times}$ in (8,9) by $-e_2\phi e_2$. Hence we obtain the equation

$$(i\partial_0 + \sum e_n \partial_n)\phi = \mu \phi e_2 \qquad (8,10)$$

which, operated on by $-e_2(\ )e_2$ gives

$$i\partial_0\phi^\times - e_1\partial_1\phi^\times + e_2\partial_2\phi^\times - e_3\partial_3\phi^\times = \mu\phi^\times e_2 .$$

Remembering that $e_2$ is real and that $e_1$ and $e_3$ are imaginary, by taking the complex conjugate of this last equation we find

$$(-i\partial_0 + \Sigma e_n\partial_n)\phi = \mu\phi e_2 \qquad (8,11).$$

Thus both (8,10) and (8,11) follow from (8,9). Addition and subtraction of (7,10) and (7,11) also gives

$$\partial_0\phi = 0 \qquad (8,12)$$

$$\underline{\nabla}\phi = \mu\phi e_2 \qquad (8,13).$$

Hence Kilmister's equation (8,9) cannot be physically significant since the time variable is automatically eliminated from it. The two columns of (8,10) ~~give~~ written separately give rise to two equations of the type (8,8) which differ by the sign of their first terms only.

The equation (8,10) is related to Watson's equation

$$(\partial_0 + i\underline{\nabla})\Psi = \mu\Psi^\times e_3 .$$

Which is transformed into

$$(i\partial_0 + \underline{\nabla})\Psi = \mu\Psi^\times e_2 \qquad (8,14)$$

by the substitution

$$\Psi \rightarrow \Psi^\times e^{i\frac{\pi}{4}} e^{e_1\frac{\pi}{4}} .$$

Now, under the time reflection $x_0 \rightarrow -x_0$, $\Psi$ goes over into $\Psi^\times$. Hence the real part of $\Psi$ which we shall denote by $\phi$ is the self time-conjugate solution of (8,14) and satisfies the equations (8,12) and (8,13). This leads us to another remark made by Jehle. He states incorrectly that the two-component wave equation cannot be made covariant with respect to reflections. But we have shown that Jehle's equation is equivalent to (8,7). Under the time reflection $x_0 \rightarrow -x_0$, $U_0$ is changed into $U_0 e_2$. Hence (8,8) is covariant with respect to this transformation, but not with respect to charge conjugation. On the other hand Kilmisters equation (8,11) is covariant with respect to charge conjugation but ~~invari~~ not with respect to a time reflection. Thus both are degenerate cases of the general equation which is covariant with respect to both time reflection and charge conjugation.

We conclude that 2-component wave equations can only have physical significance if they represent neutral particles of spin $\frac{1}{2}$.

## §9. Introduction of an orthogonal frame of reference. The de Broglie vectors.

In §4 we have seen that if $C$ is a quaternion which transforms according to the law

$$C' = \Gamma C$$

when the coordinates are subject to the orthogonal transformation

$$dX' = \Gamma \, dX \, \Gamma^{\dagger}$$

then the four hermitian quaternions $C I_{\mu} C^{\dagger}$ represent four four-vectors mutually orthogonal in Lorentz' sense. But from (5,4) $\Psi$ has the same transformation properties as $C$ so that an orthogonal frame of reference is defined by the four hermitian quaternions

$$J = \Psi \Psi^{\dagger} \qquad (9,1)$$

$$-S = \Psi i e_3 \Psi^{\dagger} \qquad (9,2)$$

$$-U = \Psi i e_1 \Psi^{\dagger} \qquad (9,3)$$

$$-V = \Psi i e_2 \Psi^{\dagger} \qquad (9,4).$$

$J$ is a time-like four-vector the time component of which is positive definite. In a frame of reference where $J$ is reduced to its scalar component, $\Psi$ may be taken as a scalar quaternion. In such a frame of reference $S$, $U$ and $V$ would then reduce to their components in the $e_3$, $e_1$ and $e_2$ directions respectively. Thus they are seen to be space-like four-vectors. We now want to show that $J$ represents the current (or probability) density four-vector while $S$ corresponds to the spin density four-vector. The vectors $U$ and $V$ which are orthogonal to each other and orthogonal to the current and spin vectors are related to a complex four-vector introduced by de Broglie (1940; p. 136) and for that reason will be called the de Broglie four-vectors. The space-like vectors $S, U, V$ are also pseudo-vectors since under ~~the~~ a space reflection the transformation law (5,5) implies that

$$-S' = \Psi^{\times} i e_3 \overline{\Psi} = S^{*}$$

that is, the scalar part of $S$ changes its sign under this transformation. The same proof holds for the de Broglie vectors.

We write

$$J = j^0 + i e_1 j^1 + i e_2 j^2 + i e_3 j^3 = I_\nu j^\nu \qquad (9.4).$$

From (1,11) it follows that

$$j^\nu = \Psi^{\dagger} . \overline{I}^{\nu} \Psi = \Psi^{\dagger} . I_\nu \Psi$$

and the code (2,11) in conjunction with (2,16) gives

$$j^0 = \Psi^{\dagger} . \Psi = \psi^{\times} . \psi \quad , \quad j^n = \Psi^{\dagger} . i e_n \Psi = \psi^{\times} . \alpha_n \psi \qquad (9.5)$$

so that the functions $j_\nu$ represent the components of the current four-vector.

Likewise we can express the components $s^\nu$ of the space-like vector $S$ by means of the column matrix $\psi$. From (2,11), (2,12) and (2,16) we obtain

$$s^0 = -\Psi^{\dagger} . \Psi i e_3 = \psi^{\times} . i \alpha_1 \alpha_2 \alpha_3 \psi$$

$$s^1 = -\Psi^{\dagger} . i e_1 \Psi i e_3 = \psi^{\times} . i \alpha_2 \alpha_3 \psi$$

$$s^2 = -\Psi^{\dagger} . i e_2 \Psi i e_3 = \psi^{\times} . i \alpha_3 \alpha_1 \psi$$

$$s^3 = -\Psi^{\dagger} . i e_3 \Psi i e_3 = \psi^{\times} . i \alpha_1 \alpha_2 \psi$$

or

$$s^\nu = \psi^{\times} . \sigma_\nu \psi \qquad (9.6)$$

where $\sigma_1, \sigma_2, \sigma_3$ are the spin operators and $\sigma_0 = i \alpha_1 \alpha_2 \alpha_3$. Thus $s^\nu$ are the components of the spin density pseudovector.

Now the particular gauge transformation

$$\Psi \to \Psi \exp(e_3 \tfrac{\pi}{4})$$

transforms $U$ into $V$, so that we need only consider the physical meaning of $U$. From (6,9) we infer that in the field free case $e = 0$, the charge conjugate function $\Psi_+ i e_1$ of the positive energy solution $\Psi_+$ represents a negative energy solution. Hence we can write

$$-U = \Psi i e_1 \Psi^{\dagger} = \Psi_- \Psi_+^{\dagger} = \Psi_+ \Psi_-^{\dagger}$$

Thus $U$ is connected with the transition probabilities from positive to negative energy states. De Broglie in his "theory of light" considered the complex densities

$$W_0 = \varphi^{\times} . \psi \qquad W_n = -\varphi^{\times} . \alpha_n \psi \qquad (9.7)$$

where $\varphi = -i \alpha_2 \rho \psi^{\times}$ is the charge conjugate function. Now it is easily ~~shown~~ seen that the real and imaginary parts of $w_\mu$ give the components of the vectors $U$ and $V$. To show this we first notice that the quaternion

$$\Psi i e_2 = -(\Psi i e_1) e_3$$

corresponds to the column matrix $i\varphi$. Hence

$$u^0 = -\Psi^\dagger . \Psi i e_1 = -\mathcal{R}(\varphi^\times . \varphi)$$
$$u^n = -\Psi^\dagger . i e_n \Psi i e_1 = -\mathcal{R}(\varphi^\times . \alpha_n \varphi)$$
$$v^0 = -\Psi^\dagger . \Psi i e_2 = -\mathcal{R}(i\varphi^\times . \varphi) = -i\mathcal{J}(\varphi^\times . \varphi)$$
$$v^n = -\Psi^\dagger . i e_n \Psi i e_2 = -\mathcal{R}(i\varphi^\times . \alpha_n \varphi) = -i\mathcal{J}(\varphi^\times . \alpha_n \varphi)$$

so that the complex combinations

$$\begin{cases} -(u^0 + i v^0) = (\varphi^\times . \varphi)^\times = w^0 \\ -(u^n + i v^n) = (\varphi^\times . \alpha_n \varphi)^\times = (\varphi . \alpha_n^\times \varphi^\times)^\times = -w_n = w^n \end{cases} \tag{9.8}$$

are seen to be identical with the contravariant components of de Broglie's complex vector.

When the field is present, however, the transition probabilities from ~~the~~ positive to negative energy states are determined by the four-vector

$$\Psi_- \Psi^\dagger = \Psi(-e)\, i e_1 \Psi^\dagger(e)$$

which is different from the vector

$$U = \Psi(e)\, i e_1 \Psi^\dagger(e).$$

Thus our physical interpretation of de Broglie's vectors ~~in~~ as giving transition probabilities between positive and negative energy states breaks down in the general case. From the geometric point of view $U$ and $V$ are two space-like vectors which together with the current density and spin density four-vectors form an orthogonal frame of reference. The considerations of § 4 show that the vectors $J, S, U, V$ are all invariant under the pseudo-gauge transformation (6,3). But only the current density vector $J$ and the spin density are gauge invariant. The gauge transformation (6,2) transforms a linear combination of the de Broglie vectors $U$ and $V$ into another linear combination of the same vectors.

# CHAPTER III.

## New divergence equations and algebraic identities between tensors attached to a particle of spin $\frac{1}{2}$.

—

### § 10. Divergence equations satisfied by the basis vectors of the orthogonal frame.

In this section we propose to show that the quaternion method of deriving the equation for the conservation of probability gives simultaneously three other divergence equations of which ~~the fourth~~ one is Uhlenbeck & Laporte's equation and the other two concern the de Broglie vectors introduced in § 9.

Multiplying the wave equation

$$\hbar\, D\Psi = (m_0 c\, \Psi^{*} + e\, \bar{A}\Psi)\, e_3$$

to the left by $\Psi^{\dagger}$, and the hermitian conjugate equation

$$\hbar\, \Psi^{\dagger}\overleftarrow{D} = -e_3\, (m_0 c\, \bar{\Psi} + e\, \Psi^{\dagger}\bar{A})$$

to the right by $\Psi$ and adding them together we obtain

$$\hbar(\Psi^{\dagger}D\Psi + \Psi^{\dagger}\overleftarrow{D}\Psi) = m_0 c\, (\Psi^{\dagger}\Psi^{*} - \bar{\Psi}\Psi)\, e_3 + e\, (\Psi^{\dagger}\bar{A}\Psi e_3 - e_3\Psi^{\dagger}\bar{A}\Psi) \qquad (10.1)$$

Here we have employed the notation

$$\Psi^{\dagger}\overleftarrow{D} = \partial_0\Psi^{\dagger} + \Sigma\, \partial_n \Psi^{\dagger} i e_n = (\partial_\nu \Psi^{\dagger})\, \bar{I}^{\nu} \qquad (10.2).$$

If we multiply the quaternion equation (10.1) scalarly by $I_\nu$ we obtain four scalar equations. By (1,10) we find

$$D.\, \Psi\Psi^{\dagger} = 0 \qquad (10.3)$$

$$\hbar\, D.\, \Psi i e_1 \Psi^{\dagger} = 2e\, (\Psi i e_2 \Psi^{\dagger}).\, \bar{A} \qquad (10.4)$$

$$\hbar\, D.\, \Psi i e_2 \Psi^{\dagger} = -2e\, (\Psi i e_1 \Psi^{\dagger}).\, \bar{A} \qquad (10.5)$$

$$\hbar\, D.\, \Psi i e_3 \Psi^{\dagger} = m_0 c\; i\, (\Psi\bar{\Psi} - \Psi^{*}\Psi^{\dagger}) \qquad (10.6)$$

From (9.1) and (9.5) it is seen that (10.3) may be written as

$$\partial_\nu j^{\nu} = 0$$

and expresses the conservation of the probability density.

Let

$$\Psi\bar{\Psi} = \bar{\Psi}\Psi = \omega_1 + i\omega_2 \qquad (10.7)$$

where $\omega_1$ and $\omega_2$ are two real scalars. They are seen to be invariant under the orthogonal transformation (5.4). The space reflection (5.5) leaves $\omega_1$ invariant but changes the sign of $\omega_2$. We conclude that $\omega_2$ is a pseudo-invariant scalar (also called pseudoscalar). We now give the expressions of $\omega_1$ and $\omega_2$ in Dirac's matrix theory. Taking the complex conjugate of (10.7) and using (2.15) together with (2.16) we obtain

$$\Psi^{\dagger}\Psi^{*} = \psi^{*}\cdot\beta\psi - i\psi^{*}\cdot\alpha_1\alpha_2\alpha_3\beta\psi .$$

Thus

$$\omega_1 = \psi^{*}\cdot\beta\psi \qquad (10.8)$$

and $$\omega_2 = \psi^{*}\cdot\alpha_1\alpha_2\alpha_3\beta\psi \qquad (10.9)$$

are identical with the scalar and pseudo-scalar densities attached to a Dirac particle. The equation (10.6) now takes the form

$$-\hbar\, D\cdot S = 2m_0 c\, \omega_2$$

or

$$\hbar\, \partial_\nu s^\nu = 2m_0 c\, \omega_2 \qquad (10.10)$$

and may be recognized as the divergence equation for the spin density vector first given by Uhlenbeck & Laporte (1931).

Finally the equations (10.4) and (10.5) give the divergencies of the space-like de Broglie vectors U and V defined by (9.3) and (9.4) ~~and~~. They can be written in the form

$$\hbar\, \partial_\nu u^\nu = 2e A_\nu v^\nu \qquad (10.11)$$

$$\hbar\, \partial_\nu v^\nu = -2e A_\nu u^\nu . \qquad (10.12)$$

We see that unlike (10.3) or (10.10) these equations depend on the electromagnetic potential. Using (9.8) we find the divergence equation satisfied by De Broglie's complex vector $w^\nu$

$$\cancel{\hbar\,\partial_\nu w} \quad (\hbar\,\partial_\nu + 2ieA_\nu)\, w^\nu = 0 \qquad (10.13)$$

In the absence of field the divergence of de Broglie's vector vanishes, a result already found by de Broglie. The equation (10.13) generalizes this result to the case a field is present. Moreover for the free electron we shall see that $\omega_2$ vanishes. Hence in this case the four basis vectors of the orthogonal frame introduced in §9 have vanishing divergences. If this orthogonal frame is used to define an affine space-time of distant parallelism we shall see in Chapter VIII that ~~it must~~ the divergences of its basis vectors vanish if the contracted torsion tensor ~~of~~ in this space-time vanishes. This gives a clue to a possible geometric interpretation of both the wave function and the electromagnetic potential. We leave the discussion of this question to the final chapter.

## § 11. The same divergence equations in a general field

Besides the electromagnetic field there are other types of fields (scalar, pseudoscalar, pseudovectorial) with which a general particle of spin $\frac{1}{2}$ may interact. These more general interactions occur, for instance in the case of nucleons. It is also possible that a Pauli interaction term due to an anomalous magnetic moment may ~~exist~~ be present. We have already seen how to express vector, pseudovector, scalar and pseudoscalar interactions in quaternion form. The effect of an anomalous magnetic moment on the quaternion wave equation will now be examined.

The relation between the field quantities and the electromagnetic potential, namely

$$\underline{E} = -\partial_0 \underline{A} - \underline{\nabla} A_0 \qquad \underline{H} = \text{curl}\, \underline{A} = \underline{\nabla} \wedge \underline{A}$$

assume the quaternion form

$$V(DA) = \underline{F}^{\times}$$

where

$$\underline{F} = i\underline{E} + \underline{H}$$

When the coordinates are subject to the constant orthogonal transformation (4.7) we have

$$\bar{D}' = \Gamma \bar{D} \Gamma^{+} \; , \quad \bar{A}' = \Gamma^{\times} \bar{A} \bar{\Gamma}$$

so that the transformation law for $\underline{F}$

$$\underline{F}' = \Gamma \underline{F} \bar{\Gamma}$$

is the same as (4,8) for a general six-vector. Hence the product $F^{\times} \Psi^{\times}$ transforms like $\Psi^{\times}$ and a term of the form $le\, F^{\times} \Psi^{\times}$ may be added to the right-hand side of (6.1) without altering the relativistic invariance of Dirac's equation. In matrix form this additional term turns out to be equivalent to a Pauli interaction. ~~If we~~

If we also assume a scalar invariant potential $f_1 \Omega_1$, and a pseudo-scalar potential $f_2 \Omega_2$, Dirac's equation with the most general interaction terms may be written as

$$\hbar D \Psi = m_0 c \Psi^{\times} e_3 + (f_1 \Omega_1 + i f_2 \Omega_2) \Psi^{\times} e_3 + el\, \underline{F}^{\times} \Psi^{\times} + e\bar{A} \Psi e_3 + i\kappa \bar{B} \Psi \tag{11.1}$$

We note that the charge conjugate function $\Psi i e_1$ satisfies the same equation with $e$ changed into $-e$.

$$\hbar D\Psi = (m_0 c + l\,\underline{F}^\times)\Psi^\times e_3$$

$$\hbar\,\Psi^\dagger \square = -e_3\bar\Psi(m_0 c + l\,\underline{\bar F})$$

$$\hbar(\Psi^\dagger D\Psi + \Psi^\dagger\square\Psi) = (\Psi^\dagger\Psi^\times - \bar\Psi\Psi)m_0 c\, e_3 + l(\Psi^\dagger\underline{F}^\times\Psi^\times e_3 - e_3\bar\Psi\underline{\bar F}\Psi)$$



The extension of the divergence equation (10.1) to this general case is immediate. Starting from (11.1) we proceed as in §10, and instead of (10.1) we obtain

$$\hbar(\Psi^\dagger D\Psi + \Psi^\dagger\square\Psi) = -2m_0c\,\omega_2\, ie_3 - 2(f_1\Omega_1\omega_2 - f_2\Omega_2\omega_1)ie_3 + el(\Psi^\dagger F^\times\Psi^\times - \bar\Psi F\Psi) + e(\Psi^\dagger\bar A\Psi e_3 - e_3\Psi^\dagger\bar A\Psi)$$

where $\omega_1$ and $\omega_2$ are given by (10.7) or (10.8) and (10.9). This equation is equivalent to the four divergence equations

$$\hbar D.\Psi\Psi^\dagger = 0 \qquad (11.2)$$

$$\hbar D.\Psi ie_1\Psi^\dagger = 2e\Psi ie_2\Psi^\dagger.\bar A + eli(\underline{F}^\times.\Psi^\times e_1\Psi^\dagger - \underline{F}.\Psi e_1\bar\Psi) \qquad (11.3)$$

$$\hbar D.\Psi ie_2\Psi^\dagger = -2e\Psi ie_1\Psi^\dagger.\bar A + eli(\underline{F}^\times.\Psi^\times e_2\Psi^\dagger - \underline{F}.\Psi e_2\bar\Psi) \qquad (11.4)$$

$$\hbar D.\Psi ie_3\Psi^\dagger = -2(m_0c\,\omega_2 - f_1\omega_1\Omega_2 + f_2\Omega_2\omega_1) + eli(\underline{F}^\times.\Psi^\times e_3\Psi^\dagger - \underline{F}.\Psi e_3\bar\Psi) \qquad (11.5)$$

Here (11.2) again ~~represents~~ expresses the conservation of the probability density. In order to put the remaining equations in tensor form we must first give matrix representations of the three purely vectorial quaternions

$$\underline{M} = \Psi e_3\bar\Psi, \quad \underline{M}' = \Psi e_1\bar\Psi \quad \text{and} \quad \underline{M}'' = \Psi e_2\bar\Psi$$

which clearly represent six-vectors, since they obey the law (4.8) under an orthogonal transformation. Let $F_{\mu\nu}$ denote the components of the antisymmetrical tensor which corresponds to the field six-vector. We can write

$$\underline{F} = \tfrac{1}{2}\,\bar I^\mu I^\nu F_{\mu\nu}$$

with the notation (4.12). Similarly writing

$$\underline{M} = i\underline{P} - \underline{N} = \Psi e_3\bar\Psi = \tfrac{1}{2}\,\bar I^\rho I^\sigma M_{\rho\sigma}$$

we find that

$$P_n = M_{0n} = \mathcal{R}(ie_n.\underline{M}) = \psi^\times i\alpha_n\beta\psi \qquad (11.6)$$

since

$$\mathcal{R}(ie_n.\underline{M}) = \mathcal{R}(ie_n.\underline{M})^\times = -\mathcal{R}(\Psi^\dagger ie_n\Psi^\times e_3)$$

and also that

$$N_p = -M_{mn} = \mathcal{R}(e_p.\underline{M}) = \psi^\times\sigma_p\beta\psi \qquad (11.7)$$

since

$$\mathcal{R}(e_p.\underline{M}) = \mathcal{R}(e_p.\underline{M})^\times = -\mathcal{R}(\Psi^\dagger ie_p\Psi^\times ie_3).$$

The index $n$ runs from 1 to 3, $(m,n,p)$ is a circular permutation of (1,2,3) and the matrices

$$\sigma_p = i\alpha_m\alpha_n$$

are the spin operators. As before $\psi$ is the column matrix which corresponds to the quaternion $\Psi$.

The formulae (11.6) and (11.7) show that $\underline{P}$ and $\underline{N}$ are respectively the electric and the magnetic moment densities associated with the Dirac particle.

With the help of the completely antisymmetrical tensor of rank four $\varepsilon^{\lambda\mu\nu\varrho}$ which is equal to $+1$ or $-1$ according to whether $(\lambda\mu\nu\varrho)$ is an even or an odd permutation of $(0,1,2,3)$ and equal to zero otherwise, we obtain

$$-i\,(\underline{F}\cdot\Psi e_3\overline{\Psi} - \underline{F}^{\times}\cdot\Psi^{\times}e_3\Psi^{\dagger}) = -2(\underline{H}\cdot\underline{P} + \underline{E}\cdot\underline{N}) = -\tfrac{1}{2}\varepsilon^{\lambda\mu\nu\varrho}F_{\lambda\mu}M_{\nu\varrho} \quad (11.8)$$

so that (11.5) is equivalent to the tensor equation

$$\hbar\,\partial_\nu s^\nu = 2(m_0 c\,\omega_2 - f_1\Omega_1\omega_2 + f_2\Omega_2\omega_1) + \tfrac{1}{2}\varepsilon^{\lambda\mu\nu\varrho}F_{\lambda\mu}M_{\nu\varrho} \quad (11.9)$$

with the notations of §§ 9 and 10. This equation is a generalization of Uhlenbeck and Laporte's equation and contains the pseudo invariant formed by means of the electromagnetic field tensor $F_{\lambda\mu}$ and the electromagnetic moment tensor $M_{\nu\varrho}$.

Similarly we find that the tensor components of the six-vectors $\underline{M}'$ and $\underline{M}''$ are given by the complex equations

$$M'_{0n} + iM''_{0n} = \varphi^{\times}\cdot i\alpha_n\beta\,\psi \quad (11.10)$$

$$-(M'_{mn} + iM''_{mn}) = \varphi^{\times}\cdot\sigma_p\beta\,\psi \quad (11.11)$$

where $\varphi$ is the charge conjugate of $\psi$. In other words the six-vectors $\underline{M}'$ and $\underline{M}''$ are connected with the expectation values of the electromagnetic moment operators for the transitions between positive and negative energy states in the case of a free ~~D~~ Dirac particle. The equations (11.3) and (11.4) can be put in tensor notation in the same way as the equation (11.5). Hence the divergences of de Broglie's vectors in the case of a general field are given by

$$\hbar\,\partial_\nu u^\nu = 2e v^\nu A_\nu + \tfrac{1}{2}\,el\,\varepsilon^{\lambda\mu\nu\varrho}F_{\lambda\mu}M'_{\nu\varrho} \quad (11.12)$$

$$\hbar\,\partial_\nu v^\nu = -2e u^\nu A_\nu + \tfrac{1}{2}\,el\,\varepsilon^{\lambda\mu\nu\varrho}F_{\lambda\mu}M''_{\nu\varrho}. \quad (11.13)$$

It may be noted that the pseudovector $B$ does not appear in any of the ~~equa~~ divergence equations. The scalar and the pseudoscalar potentials are only present in (11.9) and the electromagnetic potential enters only the equations (11.12) and (11.13). The complex six vector given by (11.10) and (11.11) has been first considered by de Broglie in an attempt to interpret it as describing the field of a photon. A photon for de Broglie is composed of a Dirac particle and an antiparticle.

## §12. Algebraic Identities.

We now proceed to establish some algebraic identities which hold between various tensorial quantities that have arisen in the preceding sections, namely the scalar $\omega_1$ and the pseudoscalar $\omega_2$ defined by

$$\Psi\bar{\Psi} = \omega_1 + i\omega_2 \quad ,$$

the four four-vectors

$$J = \Psi\Psi^{\dagger} \;,\; S = -\Psi i e_3 \Psi^{\dagger} ,\; U = -\Psi i e_1 \Psi^{\dagger} ,\; V = -\Psi i e_2 \Psi^{\dagger}$$

and the three six-vectors

$$\underline{M} = \Psi e_3 \bar{\Psi} \;,\; \underline{M}' = \Psi e_1 \bar{\Psi} \quad \text{and} \quad \underline{M}'' = \Psi e_2 \bar{\Psi} .$$

All these tensorial quantities are expressed by means of Dirac's column matrix $\psi$ in the formulae (10.8), (10.9) for $\omega_1$ and $\omega_2$, (9.5) for $J$, (9.6) for $S$, (9.8) for $U$ and $V$, (11.6) and (11.7) for $\underline{M}$ and finally (11.9) and (11.10) for $\underline{M}'$ and $\underline{M}''$. It is obvious in quaternion notation that the vectors $J$, S, U, V have all the same norm. This property ~~for J~~ was known for J and S and was pointed out for U and V in the case of a free electron by de Broglie. Thus we write

$$\mathcal{N}(J) = -\mathcal{N}(S) = -\mathcal{N}(U) = -\mathcal{N}(V) = (\Psi\bar{\Psi})(\Psi\bar{\Psi})^{\times} = \omega_1^2 + \omega_2^2 \qquad (12.1)$$

The norms of the six vectors $\underline{M}$, $\underline{M}'$ and $\underline{M}''$ are also obviously equal.

$$\mathcal{N}(\underline{M}) = \mathcal{N}(\underline{M}') = \mathcal{N}(\underline{M}'') = (\Psi\bar{\Psi})^2 = \omega_1^2 - \omega_2^2 + 2i\,\omega_1\omega_2 \qquad (12.2)$$

~~The basis vectors of~~ The orthogonality of the basis vectors is expressed by

$$J.\bar{S} = J.\bar{U} = J.\bar{V} = S.\bar{U} = S.\bar{V} = U.\bar{V} = 0 \qquad (12.3)$$

The six-vectors are also orthogonal since

$$\underline{M}.\underline{M}' = \underline{M}.\underline{M}'' = \underline{M}'.\underline{M}'' = 0 . \qquad (12.4)$$

~~Finally the same six vectors obey the ~~the same quaternion~~ quaternion multiplication rules~~

$$\underline{M}\,\underline{M}' = (\omega_1 + i\omega_2)\,\underline{M}''$$
$$\underline{M}'\underline{M}'' = (\omega_1 + i\omega_2)\,\underline{M}$$
$$\underline{M}''\underline{M} = (\omega_1 + i\omega_2)\,\underline{M}' .$$

~~Apart from the factor $(\omega_1 + i\omega_2)$ these have the same form as the multiplication rules for $e_1$, $e_2$ and $e_3$.~~

The quaternion identities we have derived can also be expressed in tensor form. The ~~orthogonality~~ relations (12,1) become

$$j^\nu j_\nu = -s^\nu s_\nu = -u^\nu u_\nu = -v^\nu v_\nu = \omega_1^2 + \omega_2^2 .$$

Now we have

$$\mathcal{N}(\underline{M}) = \underline{M}\,\overline{\underline{M}} = -(i\underline{P}-\underline{N})(i\underline{P}-\underline{N}) = \underline{N}.\underline{N} - \underline{P}.\underline{P} - 2i(\underline{P}.\underline{N})$$

From (11.8) it follows that

$$-2\,\underline{P}.\underline{N} = -\frac{1}{4}\,\varepsilon^{\lambda\mu\nu\rho} M_{\lambda\mu} M_{\nu\rho} .$$

We also find that

$$\underline{N}.\underline{N} - \underline{P}.\underline{P} = \frac{1}{2} M^{\alpha\beta} M_{\alpha\beta} .$$

Hence the tensor equations which correspond to (12.2) are

$$\frac{1}{2} M^{\alpha\beta} M_{\alpha\beta} = \frac{1}{2} M'^{\alpha\beta} M'_{\alpha\beta} = \frac{1}{2} M''^{\alpha\beta} M''_{\alpha\beta} = \omega_1^2 - \omega_2^2$$

$$\frac{1}{2} M^{\alpha\beta} \tilde{M}_{\alpha\beta} = \frac{1}{2} M'^{\alpha\beta} \tilde{M}'_{\alpha\beta} = \frac{1}{2} M''^{\alpha\beta} \tilde{M}''_{\alpha\beta} = 2\,\omega_1 \omega_2$$

where we have introduced the dual tensors $\tilde{M}_{\alpha\beta}, \tilde{M}'_{\alpha\beta}, \tilde{M}''_{\alpha\beta}$.
For example $\tilde{M}_{\alpha\beta}$ is defined by

$$\tilde{M}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} M^{\rho\sigma} .$$

The orthogonality relations (12,3) take the form

$$j^\nu s_\nu = j^\nu u_\nu = j^\nu v_\nu = s^\nu u_\nu = s^\nu v_\nu = u^\nu v_\nu = 0$$

and to (12,4) correspond the two sets of relations

$$M^{\alpha\beta} M'_{\alpha\beta} = M^{\alpha\beta} M''_{\alpha\beta} = M'^{\alpha\beta} M''_{\alpha\beta} = 0$$

and

$$M^{\alpha\beta} \tilde{M}'_{\alpha\beta} = M^{\alpha\beta} \tilde{M}''_{\alpha\beta} = M'^{\alpha\beta} \tilde{M}''_{\alpha\beta} = 0 .$$

Now we pass to another category of identities obtained by multiplying the various quaternions under consideration ~~between~~ among themselves. Thus if we multiply the four-vectors we find

$$-iJ\overline{S} = U\overline{V} = (\omega_1 - i\omega_2)\,\underline{M} \qquad (12,5)$$

$$-iJ\overline{U} = V\overline{S} = (\omega_1 - i\omega_2)\,\underline{M}'$$

$$-iJ\overline{V} = S\overline{U} = (\omega_1 - i\omega_2)\,\underline{M}''$$

and multiplying the six vectors among themselves we have

$$\underline{M}\,\underline{M}' = (\omega_1 + i\omega_2)\,\underline{M}''$$

$$\underline{M}'\underline{M}'' = (\omega_1 + i\omega_2)\,\underline{M}$$

$$\underline{M}''\underline{M} = (\omega_1 + i\omega_2)\,\underline{M}'$$

It may be noticed that apart from the factor $(\omega_1 + i\omega_2)$ these last equations have the same form as the multiplication rules for $e_1$, $e_2$ and $e_3$.

Finally the multiplication of six-vectors by four-vectors gives the identities

$$\underline{M}'V = -i\underline{M}J = -\underline{M}''U = (\omega_1 + i\omega_2)S$$
$$\underline{M}''S = -i\underline{M}'J = -\underline{M}V = (\omega_1 + i\omega_2)U$$
$$\underline{M}U = -i\underline{M}''J = -\underline{M}'S = (\omega_1 + i\omega_2)V$$
$$\underline{M}S = \underline{M}'U = \underline{M}''V = i(\omega_1 + i\omega_2)J. \qquad (12,6)$$

The equations of this category are somewhat more difficult to translate into tensor equations than the identities considered in the beginning of this section. A general technique for transforming any invariant quaternion equation into an equivalent tensor equation in a general system of coordinates will be developped in Chapter 6. Pending this investigation we shall give two examples to show how the quaternion identities can be put in tensor form. We have

$$i\underline{M} = \tfrac{1}{2} i \bar{I}^{\lambda} I^{\mu} M_{\lambda\mu} = i(i\underline{P} - \underline{N}) = \tfrac{1}{2} \bar{I}^{\lambda} I^{\mu} \tilde{M}_{\lambda\mu}.$$

Hence one of the equations (12,5) namely

$$-iJ\bar{S} = (\omega_1 - i\omega_2)\underline{M}$$

can be written, ~~after separation of~~

$$j_\lambda s_\mu - j_\mu s_\lambda = \omega_2 M_{\lambda\mu} + \omega_1 \tilde{M}_{\lambda\mu}$$

and the other equation

$$U\bar{V} = (\omega_1 - i\omega_2)\underline{M}$$

takes the form

$$u_\lambda v_\mu - u_\mu v_\lambda = \omega_1 M_{\lambda\mu} - \omega_2 \tilde{M}_{\lambda\mu}.$$

Similarly one of the identities (12,6) which reads

$$\underline{M}S = -(\omega_2 - i\omega_1)J$$

split into two tensor equations, namely

$$M^{\alpha\beta} s_\beta = -\omega_2 j^\alpha$$
$$\text{and } \tilde{M}^{\alpha\beta} s_\beta = -\omega_1 j^\alpha.$$

Among the identities derived in this section, those involving only the gauge invariant quantities $J$, $S$ and $\underline{M}$, the pseudoscalar $\omega_2$ and the scalar $\omega_1$ are well known and have been first derived by Pauli and by Kofink (1940). They were expressed in tensor form by Petiau (1946). All the other identities are new and they illustrate the power of the quaternion methods.

## §13. Equations satisfied by the de Broglie vectors in the case of a free particle.

In this section we start by solving the wave equation for a free electron and evaluating the various tensorial quantities considered in the preceding sections. Then we shall show that the de Broglie vectors corresponding to plane wave solutions satisfy equations of Proca's type.

We seek a solution of Dirac's equation

$$\hbar D \Psi = m_0 c \Psi^{*} e_3$$

of the form

$$\Psi = X \exp(e_3 \lambda / \hbar)$$

where $X$ is a constant quaternion and

$$\lambda = Et - p_1 x_1 - p_2 x_2 - p_3 x_3 = p^{\nu} x_{\nu} = P . \bar{X} .$$

The equation satisfied by $X$ is identical with the algebraic form of the wave equation (5,2) and reads

$$(D \lambda) X = \bar{P} X = m_0 c X^{*} .$$

Hence $X$ must satisfy the two conditions

$$P = m_0 c X (X^{*})^{-1} \quad \text{and} \quad X \bar{X} = \text{real}.$$

In particular if we put

$$X = e^{\frac{1}{2} i \underline{k} \alpha} R$$

where $R$ denotes any real quaternion, from (5,9) we obtain

$$E = m_0 c^2 \cosh \alpha \quad \text{and} \quad \underline{p} = \underline{k}\, m_0 c \sinh \alpha \qquad (13,1)$$

The plane wave solution of (5.10) which corresponds to the values (13,1) of the energy and the momentum is therefore given by

$$\Psi = e^{\frac{1}{2} i \underline{k} \alpha} R \exp\{ e_3 m_0 c (ct \cosh \alpha - \underline{r} . \underline{k} \sinh \alpha) / \hbar \} \qquad (13.2)$$

We can normalize $\Psi$ in a cube of volume $l^3$ so that

$$(\Psi^{\dagger} . \Psi) l^3 = 1 .$$

Since the probability vector $J$ is

$$J = \Psi \Psi^{\dagger} = R \bar{R} \exp(i \underline{k} \alpha)$$

the normalization condition becomes

$$R \bar{R} \cosh \alpha = l^{-3} . \qquad (13.3)$$

The invariant $\omega_1$ and the pseudo-invariant $\omega_2$ are given by

$$\Psi \bar{\Psi} = \omega_1 + i \omega_2 = R \overline{R}$$

Hence $\omega_2$ vanishes identically. The normalization condition (13.3) determines $\omega_1$, and we find

$$\omega_1 = m_0 c^2 / (E l^3) .$$

The spin density vector

$$S = - \Psi i e_3 \Psi^\dagger = - \chi i e_3 \chi^\dagger$$

is seen to be constant. But the de Broglie vectors $U$ and $V$ are functions of position in space-time. They are given by

$$- U = \Psi i e_1 \Psi^\dagger = \chi \, e^{e_3 \lambda/\hbar} \, i e_1 \, e^{-e_3 \lambda/\hbar} \chi^\dagger$$

$$- V = \Psi i e_2 \Psi^\dagger = \chi \, e^{e_3 \lambda/\hbar} \, i e_2 \, e^{-e_3 \lambda/\hbar} \chi^\dagger .$$

The electromagnetic moment density six-vector

$$\underline{M} = \Psi e_3 \bar{\Psi} = \chi e_3 \bar{\chi}$$

is also constant unlike the six vectors

$$\underline{M}' = \Psi e_1 \bar{\Psi} = \chi e^{e_3 \lambda/\hbar} e_1 e^{-e_3 \lambda/\hbar} \bar{\chi}$$

and $\underline{M}'' = \Psi e_2 \bar{\Psi} = \chi e^{e_3 \lambda/\hbar} e_2 e^{-e_3 \lambda/\hbar} \bar{\chi}$

which depend on the position coordinates.

We now wish to show that the four-vectors $U$, $V$ and the six-vectors $\underline{M}'$, $\underline{M}''$ constructed from plane wave solutions satisfy equations of Proca's type. Remembering that

$$D \lambda = m_0 c \, \chi^\times \chi^{-1}$$

we get

$$\hbar \, D U = - (D\lambda) \chi e^{2 e_3 \lambda/\hbar} e_3 \, i e_1 \chi^\dagger = - 2 m_0 c \, i \underline{M}''^\times$$

$$\hbar \, D \underline{M}'' = 2 (D\lambda) \chi e^{2 e_3 \lambda/\hbar} e_3 \, e_2 \bar{\chi} = 2 i m_0 c \, U^\times .$$

Hence

$$\hbar^2 D \bar{D} U = - 4 m_0^2 c^2 U . \qquad (13.4)$$

In the same way we find

$$\hbar \, D V = 2 i m_0 c \, \underline{M}'^\times$$

and $\hbar^2 D \bar{D} V = - 4 m_0^2 c^2 V .$ (13.5)

The equations (13.4) and (13.5) show that if the pseudo-vectors $U$ and $V$ correspond to plane wave solutions they obey second order wave equations associated with a particle of mass $2 m_0$ and spin 1. Complex equations of a similar form were first derived by de Broglie. The question whether the Maxwell equations can be solved by means of plane wave solutions of an equation associated with a particle of spin $\frac{1}{2}$ will be answered in the next section.

## §14. A form of Dirac's equation involving a constant space-like vector. A new equation for a particle of rest mass 0 and spin $\frac{1}{2}$. Application to the plane wave solutions of Maxwell's equations.

In the last section we have seen that the pseudo-scalar $\omega_2$ vanishes for plane wave solutions of the field-free Dirac equation since in this case the norm of the wave function is real. Although for a general solution this may not be true we can always transform the wave function by means of the pseudo gauge transformation (6.3) as to make its norm real. Such a transformation will not affect the basis vectors of the orthogonal frame of reference introduced in §9. If we assume that only this orthogonal system has a physical significance (and some arguments supporting this view will be put forward in §31) then we can always assume that the pseudo-invariant $\omega_2$ can be made to vanish and the spin-density vector to satisfy the conservation equation. Thus we come to the conclusion that for a free electron the wave function $\Psi$ satisfies the divergence equation

$$\Psi^\dagger D\Psi + \Psi^\dagger \overleftarrow{D}\Psi = 0 \qquad (14.1)$$

with the help of the supplementary condition that the norm of $\Psi$ is real. We now look for a linear equation of the form

$$D\Psi = \Psi^* L \qquad (14.2)$$

which leads to (14.1) assuming that $N(\Psi)$ is real. From (14.2) we obtain

$$\Psi^\dagger D\Psi + \Psi^\dagger \overleftarrow{D}\Psi = \Psi^\dagger\Psi^* L + \bar{\Psi}\Psi L^\dagger .$$

Hence if $L$ is an antihermitian quaternion (14.1) is satisfied since $\omega_2$ vanishes. Moreover if $L$ is constant the second order equation derived from (14.2) reads

$$\bar{D}D\Psi = \Psi L^* L = -\Psi \bar{L} L . \qquad (14.3)$$

This last equation shows that if the norm of $L$ is positive we can put

$$\bar{L}\,L = m_0^2c^2/\hbar^2 = \mu^2 \qquad (14.4)$$

and (14.3) becomes the second order wave equation for a particle of mass $m_0$. We can also put

$$L = \mu T$$

where

$$T = -T^{\dagger} \quad \text{and} \quad T\bar{T} = 1 \qquad (14.5)$$

and (14.2) becomes identical with the equation obtained if the representation (7.10) is chosen for the operators $\beta^{\nu}$ so that it is a possible quaternion form for Dirac's equation. We shall show presently that by a suitable transformation this equation may be transformed into the standard form (5.10) adopted in this investigation. A quaternion $T$ satisfying (14.5) can be always put in the form (see § 4)

$$T = C e_3 C^{\dagger}.$$

Thus (14.2) can be written as

$$D\Psi = \mu \Psi^{\times} C e_3 C^{\dagger}. \qquad (14.6)$$

As $C$ is constant the transformation

$$\Psi' = \Psi C^{\times}$$

reduces (14.6) to ~~the~~ Watson's quaternion form

$$D\Psi' = \mu \Psi'^{\times} e_3.$$

The advantage of Dirac's equation in the form (14.2) is that a general space-like constant vector $L$ appears in the equation instead of the spatial direction specified by $e_3$ in Watson's form. In this more symmetrical form it is found that Dirac's equation can be written in a symmetrical vector form. Thus, if we put

$$L = i l_0 + e_1 l_1 + e_2 l_2 + e_3 l_3$$

where

$$l_1^2 + l_2^2 + l_3^2 - l_0^2 = \mu^2 \qquad (l_\mu = \text{constant})$$

and

$$\Psi = a_0 + i b_0 + i\underline{a} + \underline{b}.$$

With the help of (1.12) we find the vector form of (14.2)

$$\partial_0 a_0 + \underline{\nabla} . \underline{a} = l_0 b_0 + \underline{l} . \underline{b}$$
$$\partial_0 b_0 + \underline{\nabla} . \underline{b} = l_0 a_0 + \underline{l} . \underline{a}$$
$$\partial_0 \underline{a} + \underline{\nabla} a_0 - \underline{\nabla} \wedge \underline{b} = -l_0 \underline{b} - \underline{l} b_0 + \underline{l} \wedge \underline{a}$$
$$\partial_0 \underline{b} + \underline{\nabla} b_0 + \underline{\nabla} \wedge \underline{a} = -l_0 \underline{a} - \underline{l} a_0 - \underline{l} \wedge \underline{b}$$

The general tensor form of these equations will be given in § 30. ~~Take~~ The choice of Watson's form is equivalent to the choice of a coordinate system for the vector equations for which $l_3 = \mu$ and $l_1 = l_2 = l_0 = 0$. Hence it is interesting to know the Dirac form of (14.2). If again $\psi$ denotes the column matrix associated with $\Psi$ the formulae (2.11) and (2.12) lead to the equation

$$(\partial_0 + \Sigma \alpha_m \partial_m)\psi = (l_0 \alpha_1\alpha_2\alpha_3 - i l_3)\beta\psi - (l_1 + i l_2)\alpha_1\alpha_2\alpha_3\beta\varphi \quad (14.7)$$

where $\varphi = -i\alpha_2\beta\psi^{*}$ is the charge conjugate function. It may be verified directly that (14.7) leads to the second order equation for $\psi$ and that it reduces to the familiar Dirac equation if we take $l_3 = \mu$, $l_1 = l_2 = l_0 = 0$. (14.7) can be transformed into this standard form through a transformation which corresponds to the quaternion transformation

$$\Psi \rightarrow \Psi C$$

and which involves the charge conjugate function.

We now turn to the important case where the norm of $L$ vanishes, i.e.

$$l_1^2 + l_2^2 + l_3^2 - l_0^2 = 0 \quad (14.8)$$

A real quaternion $R$ with norm unity can always be found such that

$$l_1 e_1 + l_2 e_2 + l_3 e_3 = \sqrt{(l_1^2 + l_2^2 + l_3^2)}\; R e_3 \bar{R}.$$

and $L$ takes the form

$$L = l_0 R(i + e_3)\bar{R}.$$

Inserting this value for $L$ in (14.2) and making the transformation

$$\Phi = \Psi R$$

we obtain the canonical form of (14.2) in the case of zero rest-mass, i.e.

$$D\Phi = l_0 \Phi^{*}(i + e_3). \quad (14.9)$$

Consequently, when the norm of $L$ vanishes the equation (14.7) can always be transformed into

$$(\partial_0 + \Sigma \alpha_m \partial_m)\xi = l_0(\alpha_1\alpha_2\alpha_3 - i)\beta\xi \quad (14.10)$$

where $\xi$ is the column matrix which corresponds to $\Phi$.

The equation (14.7) with the condition (14.8) or the canonical equation (14.10) leads to the second order equation

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\,\xi = 0 .$$

Hence (14.10) represents a particle of rest mass zero and spin $\frac{1}{2}$. The wave equation of such a particle is usually taken as

$$(\partial_0 + \Sigma\, \alpha_n \partial_n)\,\xi = 0 .$$

The equation (14.10) suggested by the quaternion solution of the divergence equation (14.1) provides an alternative solution of the problem.

The plane wave solutions of (14.9) can be found easily by means of quaternions. Putting

$$\Phi = \xi \exp(e_3 \lambda / \hbar)$$

where

$$\lambda = p^\nu x_\nu = \bar{P} . X$$

we can transform (14.9) into the algebraic equation

$$\bar{P}\,\xi = \hbar\, l_0\, \xi^\times (1 - i e_3) .$$

Hence the energy-momentum vector is given by

$$P = \hbar\, l_0\, \xi (1 + i e_3)(\xi^\times)^{-1}$$

This relation shows that the norm of $P$ must vanish, so that a solution exists only if

$$p^\nu p_\nu = 0 .$$

The general plane wave solution of (14.9) reads

$$\Phi = \xi \exp\{ e_3\, l_0\, \xi^\times (1 - i e_3)\, \xi^{-1} . X \} . \qquad (14.11)$$

With the help of any solution $\Phi$ of (14.9) we can construct four mutually orthogonal four-vectors, namely

$$\Phi \Phi^\dagger,\quad \Phi\, i e_1 \Phi^\dagger,\quad \Phi\, i e_2 \Phi^\dagger \text{ and } \Phi\, i e_3 \Phi^\dagger .$$

In particular if $\Phi_n$ represents a plane wave solution (14.11) associated with the engy-momentum vector $P_n$, the vectors $\Phi_n \Phi_n^\dagger$ and $\Phi_n\, i e_3 \Phi_n^\dagger$ are seen to be constant quaternions. The vectors $\Phi_n\, i e_1 \Phi_n^\dagger$ and $\Phi_n\, i e_2 \Phi_n^\dagger$, however, depend on the coordinates and we have

$$A'_{nn} = \Phi_n\, i e_1 \Phi_n^\dagger = \xi_n\, e^{2 e_3 \lambda_n / \hbar}\, i e_1\, \xi_n^\dagger$$

$$A''_{nn} = \Phi_n\, i e_2 \Phi_n^\dagger = \xi_n\, e^{2 e_3 \lambda_n / \hbar}\, i e_2\, \xi_n^\dagger$$

where

$$\lambda_n = \bar{P}_n . X$$

It can be easily verified that $A'_{nn}$ and $A''_{nn}$ are plane wave solutions of the Maxwell equations in vacuum

$$\bar{D} D A = 0$$

$$D . A = 0$$

and correspond to circularly polarized plane waves. Conversely such plane waves solutions of Maxwell's equations can be expressed in terms of the plane wave solutions of the equation (14.10). We also note that the ~~con~~ hermitian quaternions $A'_{nn}$ and $A''_{nn}$ correspond to the de Broglie vectors in the case of Dirac's equation.

# CHAPTER IV – Some Simple Quaternion Solutions.

## § 15. Second order Equations.

The second order equation satisfied by the wave function can be easily deduced from the first order equation

$$\hbar D \Psi e_3 = m_0 c \Psi^{\times} + e \bar{A} \Psi \qquad (15,1)$$

and its complex conjugate

$$\hbar \bar{D} \Psi^{\times} e_3 = m_0 c \Psi + e A \Psi^{\times}. \qquad (15,2)$$

On multiplying (15,1) to the left by $\hbar \bar{D}$ and to the right by $e_3$, and using (15,2) we get

$$-\hbar^2 D\bar{D} \Psi = m_0^2 c^2 \Psi + e\hbar \bar{D}\bar{A} \Psi e_3 - m_0 c e A \Psi^{\times}.$$

In this equation we insert the value of $m_0 c \Psi^{\times}$ as given by (15,1) and find

$$-\hbar^2 \bar{D} D \Psi = (m_0^2 c^2 - e^2 A\bar{A}) \Psi + e\hbar (\bar{D}\bar{A} + AD) \Psi e_3.$$

On the other hand (see Weiss 1941 or § 28 of this thesis) Maxwell's equations can be written

$$\begin{cases} DA = -F \\ -\bar{D} F = J \end{cases} \qquad (15,3)$$

where
$$A = A_0 + ie_1 A_1 + ie_2 A_2 + ie_3 A_3$$
$$F = e_1(iE_1 + H_1) + e_2(iE_2 + H_2) + e_3(iE_3 + H_3) = i\underline{E} + \underline{H}$$
$$J = \rho + ie_1 j_1 + ie_2 j_2 + ie_3 j_3.$$

Now

$$(\bar{D}\bar{A} + AD)\Psi = (\bar{D}\bar{A})\Psi + (\bar{D}\underset{\circ}{\bar{A}} + AD)\Psi$$
$$= (\bar{D}\bar{A})\Psi + 2(A.D)\Psi.$$

Thus using (15,3) we obtain the following quaternion form of the second order equation

$$-\hbar^2 \bar{D} D \Psi = (m_0^2 c^2 - e^2 A\bar{A})\Psi + 2e\hbar (A.D)\Psi e_3 + e\hbar(i\underline{E} + \underline{H})\Psi e_3. \qquad (15,4)$$

The last term is the spin term as it is seen from the quaternion representation of the spin operators in § 8. A different quaternion form for the second order equation was proposed by Fischer. But in his equation the multiplication of $\Psi$ to the right by $e_3$ in (15,4) is replaced by ordinary multiplication by $i$. Fischer's equation is clearly incorrect since it cannot be deduced from any quaternion equation of the first order. As we have pointed out in § 9 a similar confusion between the operators $i$ and $(\,)e_3$ occurs in Watson's paper, leading this author to a wrong interpretation of the rotation of the coordinate system round the $Oz$ axis.

Another difficulty which arises in connection with the second order equation is the following: in Dirac's formalism, the term in $\underline{E}$ occurs as an energy due to ~~an~~ imaginary electric moment of the electron (cf. Dirac 1947 p. 264). We shall now show that for a very slow electron the imaginary part of the wave quaternion $\Psi$ may be neglected in the first approximation. Thus the equation (15,4) reduces to its real part which is of the same form, but with the term in $\underline{E}$ missing. We write

$$\Psi = (\chi + i\eta)\, e^{-e_3 Et/\hbar} \qquad (15,5)$$

where $\chi$ ~~and~~ $\eta$ are ~~respectively the~~ real. ~~and the imaginary parts of $\Psi$~~. The Separation of the real and the imaginary parts of (15,4) gives the real quaternion equations

$$\cancel{-\hbar^2 \bar{D} D \chi = (m_0^2 c^2 - e^2 A\bar{A})\chi + 2e\hbar(\underline{A}.D)\chi e_3 + e\hbar \underline{H}\chi e_3 - e\hbar \underline{E}\eta e_3}$$

$$\cancel{-\hbar^2 \bar{D} D \eta = (m_0^2 c^2 - e^2 A\bar{A})\eta + 2e\hbar(\underline{A}.D)\eta e_3 + e\hbar \underline{H}\eta e_3 + e\hbar \underline{E}\chi e_3}$$

~~Now the current density is~~

$$\cancel{\Psi\Psi^\dagger = \chi\bar{\chi}^\dagger + \eta\bar{\eta}^\dagger + i(\eta\bar{\chi}^\dagger - \chi\bar{\eta}^\dagger)}$$

$$\frac{1}{c}E\chi - \hbar\underline{\nabla}\eta e_3 = m_0 c\chi + \frac{e}{c}A_0\chi + \frac{e}{c}\underline{A}\eta \qquad (15,6)$$

$$\frac{1}{c}E\eta + \hbar\underline{\nabla}\chi e_3 = -m_0 c\eta + \frac{e}{c}A_0\eta - \frac{e}{c}\underline{A}\chi. \qquad (15,7)$$

For positive energy solutions we can put

$$E = m_0 c^2 + W$$

In the non relativistic case $W$ is small compared with $m_0c^2$. Hence if we keep ~~keeping~~ only the terms in $\frac{1}{c}$ and $\frac{1}{c^2}$, (15,7) becomes

$$\eta = -\frac{\hbar}{2m_0c}\underline{\nabla}\chi e_3 - \frac{e}{2m_0c^2}\underline{A}\chi \qquad (15,8)$$

This shows that $\eta$ is small for positive energy solutions. Hence $\Psi$ reduces to its real part $\chi\,\exp.(-e_3Et/\hbar)$.

The Pauli equation may be obtained by inserting $\eta$ as given by (15,8) in the equation (15,6). The resulting equation for the real quaternion $\chi$ is:

$$W\chi = -\frac{\hbar^2}{2m_0}(\underline{\nabla}.\underline{\nabla})\chi + \frac{e\hbar}{m_0c}(\underline{A}.\underline{\nabla})\chi e_3 - \frac{e\hbar}{2m_0c}\underline{H}\chi e_3 \qquad (15,9).$$

As $\chi$ is a real quaternion we can use the irreducible representations

$$\chi = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

to obtain Pauli's form

$$W\psi = \left(-\frac{\hbar^2}{2m_0}\nabla^2 + i\frac{e\hbar}{2m_0 c}\underline{A}.\underline{\nabla} - \frac{e\hbar}{2m_0 c}\underline{\sigma}.\underline{H}\right)\psi$$

where $\underline{\sigma}$ is ~~the~~ Pauli's spin operator.

In nuclear theory more general types of interaction besides the electromagnetic interaction have to be considered. As the mass of a nucleon is large the non-relativistic approximation is of special interest. ~~This problem has been considered by ____ and ____. Here the quaternion method gives the result with the minimum of labour~~. The most general equation with the Pauli interaction term (anomalous magnetic moment) a scalar field $\omega$ and a pseudo-vector field $B$, ~~reads~~ is given by

$$\hbar(\partial_0 + i\underline{\nabla})\Psi e_3 = m_0 c\,\Psi^{\times} + \frac{\omega}{c}\psi^{\times} + \frac{ek}{m_0 c^2}(i\underline{E} + \underline{H})\psi^{\times} e_3 + \frac{1}{c}(B_0 + i\underline{B})\psi e_3 + \frac{e}{c}(A_0 + i\underline{A})\psi$$

Assuming the solution $(15, 5^1)$ we obtain, in the first approximation

(15,10) $$\frac{1}{c}(W - \omega - eA_0)\chi - \frac{ek}{m_0 c^2}\underline{H}\chi e_3 - \frac{1}{c}\underline{B}\chi e_3 = \hbar\underline{\nabla}\eta e_3 + \frac{ek}{m_0 c^2}\underline{E}\eta e_3 + \frac{e}{c}\underline{A}\eta + \frac{1}{c}B_0\eta e_3$$

(15,11) $$2m_0 c\,\eta = -\hbar\underline{\nabla}\chi e_3 - \frac{ek}{m_0 c^2}\underline{E}\chi e_3 - \frac{e}{c}\underline{A}\chi + \frac{1}{c}B_0\chi e_3$$

The equation in $\chi$ is obtained by inserting the approximate value of $\eta$ as given by (15,11) in (15,10). Retaining only terms in $c^{-1}$ and $c^{-2}$ we find

$$\begin{aligned} W\chi = {} & (\omega + eA_0)\chi - \frac{\hbar^2}{2m_0}\nabla^2\chi + \frac{e\hbar}{m_0 c}(\underline{A}.\underline{\nabla})\chi e_3 + \underline{B}\chi e_3 + \frac{e^2A^2}{2m_0c^2}\chi \\ & + \frac{ek}{m_0 c}\underline{H}\chi e_3 - \frac{e\hbar}{2m_0 c}\underline{H}\chi e_3 - \frac{1}{2m_0 c}(\underline{\nabla}B_0)\chi \\ & + \frac{ek\hbar}{2m_0^2c^2}(\underline{\nabla}\underline{E})\chi - \frac{1}{2m_0c^2}B_0^2\chi + \frac{e^2k}{m_0^2c^2}(\underline{A}.\underline{E})\chi e_3 \end{aligned}$$

Using the irreducible representation for $\chi$ and Pauli's matrices we finally have

$$\begin{aligned} W\psi = \Big\{ & \omega + eA_0 - \cancel{\frac{\hbar^2}{2m_0}\nabla^2} + \frac{1}{2m_0}\left(i\hbar\underline{\nabla} + \frac{e}{c}\underline{A}\right).\left(i\hbar\underline{\nabla} + \frac{e}{c}\underline{A}\right) \\ & + \frac{e\hbar}{2m_0 c}(1 - 2k)\,\underline{H}.\underline{\sigma} - \underline{B}.\underline{\sigma} + \frac{\hbar}{2m_0 c}(\underline{\sigma}.\underline{\nabla})B_0 - \frac{1}{2m_0c^2}B_0^2 \\ & - i\frac{e^2k}{m_0^2c^2}(\underline{A}.\underline{E}) + \frac{ek\hbar}{m_0^2c^2}\underline{E}.\underline{\nabla} + i\frac{ek\hbar}{m_0^2c^2}(\underline{\sigma}.\underline{\nabla})(\underline{\sigma}.\underline{E})\Big\}\psi. \end{aligned}$$

The problem of finding the non relativistic approximation of the wave equation in the case of a general field has been studied in detail by L. A. Radicati (1948) and G. Petiau (J. de Phys. 1949). The method used by this author is essentially the same as that proposed by Darwin (1928, p. 654). We have shown in this section that the algebraic properties of the quaternions provide a simple and rapid method of solution.

## §16. Free electron Solutions.

§10. In §13 we have already given the plane wave solution of Dirac's equation for the free electron

$$\hbar D \Psi = m_0 c \Psi^* e_3$$

in the form

$$\Psi = e^{\frac{1}{2} i \underline{k} \alpha} R \exp. (e_3 \lambda / \hbar) \qquad (16,1)$$

where

$$\lambda = m_0 c (ct \cosh\alpha - \underline{r}.\underline{k} \sinh\alpha)$$

the energy-momentum 4-vector being given by

$$E/c + i\underline{p} = m_0 c (\cosh\alpha + i\underline{k} \sinh\alpha)$$

and $R$ is an arbitrary real quaternion.

The wave function is normalized in such a way that

$$R\bar{R} \cosh\alpha = l^{-3}$$

where $l^3$ is the volume of the cube enclosing the electron.

Putting

$$K = R \cosh\alpha = A + e_2 B \qquad (16,2)$$

where

$$A = a_1 - e_3 a_2 \quad , \quad B = b_1 - e_3 b_2$$

we get

$$\Psi = (1 + i\underline{k} \tanh\tfrac{\alpha}{2}) K \exp(e_3 \lambda/\hbar)$$

or

$$\Psi = [1 + ic\underline{p}/(E + m_0 c^2)] K \exp (e_3 \lambda/\hbar). \qquad (16,3)$$

since

$$\underline{k} \tanh\frac{\alpha}{2} = (\underline{p} \tanh\frac{\alpha}{2}) / (m_0 c \sinh\frac{\alpha}{2}) = \underline{p} / [m_0 c (1 + \cosh\alpha)].$$

The standard solution for the free electron in Dirac's formalism can be derived easily from (16,3) as follows: First we replace $K$ in (16,3) by its value (16,2). We find

$$\underline{p} K = [e_3 p_3 + e_2 (p_2 + e_3 p_1)] (A + e_2 B)$$
$$= [e_3 p_3 A - (p_2 - e_3 p_1) B] + e_2 [(p_2 + e_3 p_1) A - e_3 p_3 B]$$

going over to Dirac's representation and applying the rules expressed by (2,9) and (2,10) we obtain for the elements of the column matrix $\psi$ the familiar expressions

$$\psi_1 = A' e^{-iS} \qquad \psi_2 = B' e^{-iS}$$
$$\psi_3 = - \frac{p_3 A' + (p_1 - i p_2) B'}{E/c + m_0 c} e^{-iS} \quad , \quad \psi_4 = \frac{(p_1 + i p_2) A' - p_3 B'}{E/c + m_0 c} e^{-iS}$$

where

$$S = \lambda/\hbar \quad , \quad A' = a_1 + i a_2 \quad \text{and} \quad B' = b_1 + i b_2 \,.$$

The plane wave solution corresponds to an energy $\pm m_0 c^2 \operatorname{ch} \alpha$ with an absolute value larger than $m_0c^2$. We now seek a solution such that $|E| < m_0 c^2$. Accordingly one is led to put

$$E = m_0 c^2 \cos\beta \qquad \text{where } -\tfrac{\pi}{2} < \beta < \tfrac{\pi}{2}$$

and $\Psi = \eta \exp(e_3 E t/\hbar)$.

Dirac's equation takes the form

$$\cos\beta\, \eta - l_0\, i \underline{\nabla} \eta e_3 = \eta^\times . \qquad (l_0 = \hbar / m_0 c)$$

. Further, let

$$\eta = K \exp(-l_0^{-1} \sin\beta\, \underline{a}.\underline{r})$$

where $K$ satisfies the algebraic equation

$$\cos\beta\, K + i\underline{a} \sin\beta\, K e_3 = K^\times .$$

The solution of this algebraic equation is

$$K = (\underline{a}F + Fe_3) e^{i\beta/2} = L e^{i\beta/2}$$

where $F$, ~~hence~~ L is ~~are~~ ~~real~~ arbitrary real quaternion~~s~~.

and $L = \underline{a}F + Fe_3$

is a real quaternion such that

$$\underline{a} L e_3 = -L .$$

Hence the solution of Dirac's equation for $E = m_0 c^2 \cos\beta$ is

$$\Psi = (\underline{a}F + Fe_3) \exp\left\{ i\tfrac{\beta}{2} - \tfrac{m_0 c}{\hbar}\left( \sin\beta\, \underline{a}.\underline{r} - e_3 c \cos\beta\, t \right)\right\} \qquad (16,4)$$

~~This is a~~ and is seen to represent a wave decaying exponentially in the direction specified by $+\underline{a}$.

This solution is not bounded since the probality density

$$\Psi \Psi^\dagger = L\bar{L}\, e^{-\underline{b}.\underline{r}}.$$

where $\underline{b} = \frac{m_0 c}{2\hbar} \sin\beta\, \underline{a}$

increases exponentially in the negative $\underline{a}$ direction. However, corresponding to the same value of $E$ another solution can be constructed by changing the sign of $\beta$ in (16,4). Any linear combination and in particular half the sum of these two solutions is also a solution of Dirac's equation. Thus we obtain the bounded $\Psi$ solution

$$\Psi = L \cos\tfrac{1}{2}(i\beta - \underline{b}.\underline{r}) \exp.\left(\tfrac{m_0 c^2}{\hbar} \cos\beta\, t\right). \qquad (16,5)$$

The probality density reduces to its scalar part and it has the bounded value

$$\Psi\Psi^\dagger = \cos\tfrac{1}{2}(i\beta - \underline{b}.\underline{r}) \cos\tfrac{1}{2}(i\beta + \underline{b}.\underline{r}) = L\bar{L}\left(\cos^2\tfrac{1}{2}\underline{b}.\underline{r} + \operatorname{sh}^2\tfrac{1}{2}\beta\right).$$

as the velocity vanishes in both cases the solutions (16,4) and (16,5) correspond to stationary waves.

## §17. One dimensional motion of the electron.

The particular case where the potential depends on one variable only has been studied by ~~F. Sauter~~ (1° K. Nikolsky (1930), F. Sauter (1931) and M. S. Plesset (1932). In this ~~para~~ section we give a quaternion solution of the problem. Sauter's method is nearest our own since he takes the wave function as a linear function of the matrices generated by Dirac's operators $\alpha_1 \alpha_2 \alpha_3$ and $\beta$. Hence his wave function depends on sixteen complex functions and the elimination of the arbitrary constants thus introduced is effected in a rather arbitrary way. Plesset introduces a canonical transformation which simplifies the problem of splitting Dirac's equation into two identical two-component equations. A similar method is proposed by Mott & Sneddon (1947?). The

All these methods have one defect in common: unlike the normal Schrödinger equation, the simplified two-component equation obtained after the elimination of the variables on which the potential does not depend bears very little resemblance to the equation one would obtain by omitting those variables from the general Dirac equation. In short one would expect a relation between the solution of the equation

$$hi(\partial_t + iV(z) + \alpha_1 \partial_x + \alpha_2 \partial_y + \alpha_3 \partial_z)\psi = \mu\beta\psi$$

and the solution of the reduced equation

$$hi(\partial_t + iV(z) + \alpha_3 \partial_z)\eta = \mu'\beta\eta$$

(We are considering a motion in the $Oz$ direction).
In this last equation as the only operators are $\alpha_3$ and $\beta$ such that $\alpha_3^2 = \beta^2 = 1 \quad \alpha_3\beta + \beta\alpha_3 = 0$ they may be represented by $2 \times 2$ matrices. This is exactly what the quaternion method achieves to do. ~~Di~~

Dirac's equation in this case is

$$D\,{}^{-}\Psi e_3 = \mu\,{}^{2}\Psi^{x} - V(z)\,{}^{2}\Psi$$

~~First by making the transformation~~

$$\cancel{{}^{2}\Psi \to {}^{2}\Psi e^{\frac{\pi}{4}e_2}}$$

~~we obtain the equivalent equation~~

$$\cancel{D\,{}^{2}\Psi e_1 = \mu\,{}^{2}\Psi^{x} + V(z)\,{}^{2}\Psi}$$

~~(we have taken $\hbar = c = 1$)~~

Plemets' transf.

$$\psi = \frac{1}{\sqrt{2}}(\rho\chi + \alpha_1\chi)$$



By putting

$$^{2}\Psi = \psi\, e^{e_3(Et - p_1 x - p_2 y)}$$

we obtain

$$[E \pm V(z)]\psi + i e_3 \frac{d\psi}{dz} e_3 = \mu(\psi^\times + i\underline{a}k\psi) \qquad (17.1)$$

where

$$k = (p_1^2 + p_2^2)^{\frac{1}{2}} \quad \text{and} \quad \underline{a} = k^{-1}(p_1 e_1 + p_2 e_2).$$

Now consider the transformation

$$\psi = \frac{1}{\sqrt{2}}(\chi + \underline{a}\chi^\times). \qquad (17.2)$$

Its inverse is given by

$$\chi = \frac{1}{\sqrt{2}}(\psi - \underline{a}\psi^\times)$$

Substituting in (17.1), ~~and noting~~ that $\underline{a}$ anticommutes with $e_3$ we obtain

$$A + \underline{a}A^\times = 0 \qquad (17.3)$$

where

$$A = (E \pm V)\chi + i e_3 \frac{d\chi}{dz} e_3 - \mu(1 - ik)\chi^\times$$

But, since the transformation (17.2) has an inverse (17.3) implies

$$A = 0.$$

Further, let

$$\cancel{l = (1+k^2)^{\frac{1}{2}} \text{ and }} \quad k = \tan\gamma, \quad l = (\cos\gamma)^{-1}$$

we have $1 - ik = l e^{-i\gamma}$.

We make the transformation

$$\chi = e^{-i\frac{\gamma}{2}}\eta$$

and get the equation

$$(E \pm V)\eta + i e_3 \frac{d\eta}{dz} e_3 = \mu l \eta^\times \qquad (17.4)\ \cancel{(17.3)}$$

which has exactly the same form as the general equation if the wave function is assumed to be independent of $x$ and $y$.

Now the quaternion $\eta$ can always be put in the form

$$\eta = \eta_1 + \underline{a}\eta_2^\times$$

where $\eta_1$ and $\eta_2$ commute with $e_3$. We find that both $\eta_1$ and $\eta_2$ satisfy the equation

$$(E \pm V)\zeta - i\frac{d\zeta}{dz} = \mu l \zeta^\times \qquad (17.5)$$

$\zeta$ is of the form

$$\zeta = \zeta_1 + e_3\zeta_2$$

where $\zeta_1$ and $\zeta_2$ are complex scalar functions which both satisfy (17.5). Hence if $\zeta_1$ is a scalar solution of (17.5) we can obtain ~~its general~~ a solution which satisfies the same boundary conditions by taking

$$\cancel{\zeta = u_1\zeta_1 +} \qquad \zeta = (u_1 + e_3 u_2)\zeta_1$$

where $u_1$ and $u_2$ are ~~complex~~ real numbers.

Now $\eta$ can be taken as

$$\cancel{\eta = \{v_1\zeta + \underline{a} v_2 \zeta^x\}} \qquad \eta = v_1\zeta + \underline{a}\, v_2 \zeta^x$$

where $v_1$ and $v_2$ are real numbers.

Hence $\eta$ has the form

$$\eta = (\lambda_1 + e_3\lambda_2)\zeta_1 + \underline{a}(\lambda_1' + e_3\lambda_2')\zeta_1^x = \lambda\zeta_1 + \underline{a}\lambda'\zeta_1^x,$$

and the solution of (17.1) is

$$\zeta = \frac{1}{\sqrt{2}}\left\{e^{-i\frac{\gamma}{2}}\eta + \underline{a}\, e^{i\frac{\gamma}{2}}\eta^x\right\}$$

which can also be written as

$$\zeta = \frac{1}{\sqrt{2}}\left\{\left(e^{-i\frac{\gamma}{2}}\lambda - e^{i\frac{\gamma}{2}}\lambda'\right)\zeta_1 + \underline{a}\left(e^{-i\frac{\gamma}{2}}\lambda' + e^{i\frac{\gamma}{2}}\lambda\right)\zeta_1^x\right\}$$

~~Hence~~ In short, the equation (17.1) can be solved if ~~the~~ a solution of the one-component equation (17.5) is known. This brings about a very substantial simplification of the problem.

This method can be transposed directly into Dirac's formalism as follows:

The equation to be solved is

$$(E - V)\psi + i\alpha_3\frac{d\psi}{dz} = \mu\beta\psi + \alpha_1\alpha_2\alpha_3\tau k\psi \qquad (17.6)$$

where $k$ is defined as before and the operator $\tau$ is defined by

$$\tau = k^{-1}(p_1\alpha_3\alpha_2 + p_2\alpha_1\alpha_3)$$

We have

$$\tau^2 = -1, \quad \tau\beta = \beta\tau, \quad \tau\alpha_3 = -\alpha_3\tau \quad \text{and } \tau(\alpha_1\alpha_2\alpha_3) = (\alpha_1\alpha_2\alpha_3)\tau$$

~~The equation (5) becomes~~, On putting

$$\psi = \frac{1 - \tau\beta}{\sqrt{2}}\chi.$$

The equation (17.6) becomes

$$\frac{1 - \tau\beta}{\sqrt{2}}\left\{(E - V)\chi + i\alpha_3\frac{d\chi}{dz}\right\} - \mu(\beta + \alpha_1\alpha_2\alpha_3\tau k)\frac{1 - \tau\beta}{\sqrt{2}}\chi = 0$$

and as ~~$\tau\beta$~~

$$\alpha_1\alpha_2\alpha_3\tau(1 - \tau\beta) = (1 - \tau\beta)\alpha_1\alpha_2\alpha_3\beta$$

we obtain

$$(E - V)\chi + i\alpha_3\frac{d\chi}{dz} - \mu(1 + \alpha_1\alpha_2\alpha_3 k)\beta\chi = 0$$

We next make the transformation

$$\chi = e^{\alpha_1\alpha_2\alpha_3\frac{\gamma}{2}}\varphi \quad \text{where } k = \operatorname{tg}\gamma.$$

The new function $\varphi$ satisfies the equation

$$(E - V)\varphi + i\alpha_3\frac{d\varphi}{dz} = \mu l\beta\varphi \quad \text{where } l = \sqrt{1 + k^2}$$

Using Dirac's matrices we get the matrix equation

~~This may be compared with Pleroti's transformation~~

~~$$\psi = \frac{\beta + \alpha_1}{\sqrt{2}}\chi$$~~

$$(E-V)\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} + i\frac{d}{dz}\begin{pmatrix}\varphi_3\\ -\varphi_4\\ \varphi_1\\ -\varphi_2\end{pmatrix} = \mu l\begin{pmatrix}\varphi_1\\ \varphi_2\\ -\varphi_3\\ -\varphi_4\end{pmatrix}$$

We note that the pairs $(\varphi_1, \varphi_3)$ and $(-\varphi_4, \varphi_2)$ satisfy the same 2-component equation

$$(E-V)\varphi' + i\alpha_3'\frac{d}{dz}\varphi' = \mu l \beta'\varphi' \qquad (17.7)$$

where

$$\varphi' = \begin{pmatrix} f \\ g \end{pmatrix} \qquad \alpha_3' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } \beta' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From a solution of (17.7) we derive

$$\varphi = \begin{pmatrix} Af \\ Bf \\ Ag \\ -Bg \end{pmatrix}$$

where $A$ and $B$ are complex numbers and the solution of (17.6) is given by

$$\psi = \frac{1+\beta\tau}{\sqrt{2}}\, e^{\alpha_1\alpha_2\alpha_3\frac{\gamma}{2}}\varphi.$$

Further if we put

$$f = u_1 + iv_1, \qquad g = iu_2 - v_2$$

$$U = u_1 \pm iu_2 \quad , \quad V = v_1 \pm iv_2$$

we find that $U$ and $V$ satisfy the same one-component equation

$$(E-V)U \pm i\frac{dU}{dz} = \mu l U^*$$

which is the same as (17.5) derived by the quaternion method.

~~Now if we look for solutions of (4) belonging to an energy value less than $\mu l$, we can put~~

~~$E = \mu l \cos\alpha$~~

## §18 - Electron between two potential barriers.

In this section we shall employ the two types of solution free electron waves (progressive and stationary) in connection with the problem of an electron moving between two potential barriers of magnitude $\Phi$ and separated by a distance $2L$. We shall suppose that $\Phi$ is the time component of an electromagnetic type of interaction in the frame of reference where the energy is $E$. If we take the potential barriers at the planes $z=-L$ and $z=L$ we have seen in the preceding section that we need only consider the one-dimensional Dirac equation (17,4). Putting $\frac{\mu l}{\hbar}=\mu'$ in (17,4) we have

$$E\eta - \hbar i e_3 \frac{d\eta}{dz} e_3 = \mu' \eta^* \qquad \text{for } -L<z<L \qquad (18,1)$$

$$(E-\phi)\eta - \hbar i e_3 \frac{d\eta}{dz} e_3 = \mu' \eta^* \qquad \text{for } |z|>L. \qquad (18,2)$$

Antisymmetrical in $z$ ~~can now be~~ and vanishing for $z=\pm\infty$ can now be obtained ~~by taking~~ for all values of $E$ such that

$$|E-\Phi| < m_0c^2.$$

We distinguish two cases

I. $E > m_0c^2$.

In this case we define $\alpha$ by

$$E = \hbar\mu \operatorname{ch}\alpha.$$

The solution of (18,1), (18,2) reads

$$\eta = \exp\{-i\tfrac{\beta}{2} + (\mu' \sin\beta\, z + e_3 E t)/\hbar\} \qquad \text{for } z<-L$$

$$\eta = A\left(e^{ie_3\frac{\alpha}{2}} e^{-e_3\frac{\mu'}{\hbar}\operatorname{sh}\alpha\, z} + e^{-ie_3\frac{\alpha}{2}} e^{e_3\frac{\mu'}{\hbar}\operatorname{sh}\alpha\, z}\right) e^{e_3 E t/\hbar} \qquad \text{for } -L<z<L$$

$$\eta = B\exp\{i\tfrac{\beta}{2} - (\mu' \sin\beta\, z - e_3 E t)/\hbar\} \qquad \text{for } z>L.$$

where

$$E-\Phi = \mu'\cos\beta \quad \text{and} \quad E = \mu'\cosh\alpha. \qquad (18,3)$$

The elimination of $E$ between the two equations (18,3) gives

$$\mu'\cos\beta = \mu'\cosh\alpha - \Phi.$$

The condition of continuity at the planes $z=L$ and $z=-L$ requires that

$$\cos\tfrac{\beta}{2}\, e^{-\mu' L \sin\beta/\hbar}\, B = 2\cosh\tfrac{\alpha}{2}\, \cos\tfrac{1}{\hbar}(\mu' L \sinh\alpha)\, A$$

$$\sin\tfrac{\beta}{2}\, e^{-\mu' L \sin\beta/\hbar}\, B = 2\sinh\tfrac{\alpha}{2}\, \sin\tfrac{1}{\hbar}(\mu' L \sinh\alpha)\, A$$

Therefore a solution exists only if $\alpha$ satisfies the equation

$$\operatorname{tg}\frac{\beta}{2} = \operatorname{th}\frac{\alpha}{2}\operatorname{tg}(\mu' L \operatorname{sh}\alpha)/\hbar . \qquad (18,4)$$

In the non relativistic approximation if we put

$$E = m_0c^2 + W , \quad k^2 = 2m_0(\phi - W) \quad \text{and} \quad a^2 = 2m_0 W$$

(18,4) reduces to

$$a \operatorname{tg} a L = k .$$

which is the eigenvalue equation for $W$ in Schrödinger's theory.

2.) $|E| < m_0c^2$

In this case we put

$$E = \mu' \cos\gamma$$

and obtain the solution

$$\Psi = B\, e^{-i\beta/2}\, e^{\frac{\mu'}{\hbar} z \sin\beta}\, e^{e_3 E t/\hbar} \qquad \text{for } z < -L$$

$$\Psi = A \cos\left(i\frac{\gamma}{2} - \frac{\mu'}{\hbar} z \sin\gamma\right) e^{e_3 E t/\hbar} \qquad \text{for } -L < z < L$$

$$\Psi = B\, e^{i\beta/2}\, e^{-\frac{\mu'}{\hbar} z \sin\beta}\, e^{e_3 E t/\hbar} \qquad \text{for } z > L.$$

The continuity conditions give

$$B\, e^{i\beta/2}\, e^{-\frac{\mu'}{\hbar}\sin\beta L} = A \cos\left(i\frac{\gamma}{2} - \frac{\mu'}{\hbar}\sin\gamma L\right)$$

and the complex conjugate equation derived from it.
Separating the real and imaginary parts we get

$$B \cos\frac{\beta}{2} = A \cosh\frac{\gamma}{2}\cos\left(\frac{\mu'}{\hbar}\sin\gamma L\right)$$

$$\text{and } B \sin\frac{\beta}{2} = A \sinh\frac{\gamma}{2}\sin\left(\frac{\mu'}{\hbar}\sin\gamma L\right).$$

These equations lead to the eigenvalue equation

$$\operatorname{tg}\frac{\beta}{2} = \operatorname{th}\frac{\gamma}{2}\operatorname{tg}\left(\frac{\mu'}{\hbar}\sin\gamma L\right)$$

where $\beta$ and $\gamma$ are functions of $E$ defined by

$$\mu' \cos\beta = E - \phi$$

$$\mu' \cos\gamma = E .$$

Hence, provided the potential barrier is finite for any values of $E$ such that $|E - \phi| < m_0c^2$ we can find eigensolutions which vanish at infinity.

## § 19. The Electron in the field of a plane wave.

The problem of the relativistic electron in the field of a monochromatic linearly polarized plane electro-magnetic wave has been solved rigorously by Volkow (1935) and Sengupta (1947). ~~Recently~~ Taub (1949) has given a more elegant solution with the help of the spinor calculus. In this section a very simple quaternion solution will be outlined.

The Dirac equation in this case can be written

$$\hbar D \Psi e_3 = m_0 c \Psi^* - \frac{e}{c} A \Psi . \qquad (19,1).$$

where

$$A = U \varphi(\lambda) .$$

$U$ is a constant hermitian quaternion and $\varphi$ is a scalar function of the phase

$$\lambda = \bar{N} . X$$

where

$$X = x_0 + ie_1 x^1 + ie_2 x^2 + ie_3 x^3$$

and $N = \frac{1}{2}(1 + i\underline{n})$

is a null-vector, $\underline{n}$ being the unit vector in the direction of propagation of the electromagnetic wave.

The Maxwell equations are

$$D\bar{D} A = 0 \qquad \bar{D} . A = 0$$

Hence $N . U = N . A = 0 .$ (19,2)

Let

$$\Psi = \Gamma \exp(-e_3 S/\hbar) .$$

(19,1) gets replaced by

$$\hbar D \Gamma e_3 + (DS)\Gamma + \frac{e}{c} A \Gamma = m_0 c \Gamma^* .$$

An exact solution ~~may be~~ is obtained if a quaternion $\Gamma$ and a real scalar function $S$ can be found such that

$$D\Gamma = 0 \quad \text{and} \quad DS + \frac{e}{c} A = m_0 c \, \Gamma^* \Gamma^{-1}$$

Clearly $\Gamma\bar{\Gamma}$ must be real. We can always normalize the wave function such that $\Gamma\bar{\Gamma} = 1$. Hence we must have

$$\bar{\Gamma}\Gamma = 1 \qquad (19,3)$$

$$D\Gamma = 0 \qquad (19,4)$$

$$m_0 c \, \Gamma^* \bar{\Gamma} = DS + \frac{e}{c} A \qquad (19,5).$$

A quaternion which satisfies (19,3) and (19,4) is clearly

$$\Gamma = (1 + kNA)\xi \qquad (19,6)$$

where $\xi$ is a constant quaternion such that

$$\xi\bar{\xi} = 1$$

and $k$ is a constant scalar to be determined later.

In the absence of field

$$\bar{V} = \xi\xi^{\dagger} = \Psi\Psi^{\dagger} \text{ ~~would~~}$$

would be the velocity vector of the free electron. With the value (19,6) for $\Gamma$ the velocity vector is

$$\Psi\Psi^{\dagger} = (1 + kNA)\xi\xi^{\dagger}(1 + kAN)$$

$$= \bar{V} + 2k\,\mathcal{H}(NA\bar{V}) + k^2 NA\bar{V}AN.$$

Here $\mathcal{H}(NA\bar{V})$ denotes the hermitian part of $NA\bar{V}$.

Now, if ~~A~~ $C_1, C_2, C_3$ are hermitian quaternions we have the identity

$$\mathcal{H}(C_1\bar{C}_2C_3) = (C_1.\bar{C}_2)C_3 + (C_2.\bar{C}_3)C_1 - (C_3.\bar{C}_1)C_2$$

Hence, using (19,2) we get

$$\mathcal{H}(NA\bar{V}) = (A.\bar{V})N - (N.V)\bar{A}$$

and

$$NA\bar{V}AN = \mathcal{H}\{N(A\bar{V}A)N\} = 2\{N.(A\bar{V}A)\}N.$$

But

$$A\bar{V}A = 2(A.\bar{V})A - (A\bar{A})V,$$

$$N.(A\bar{V}A) = \cancel{2(A.\bar{V})(N.A)} - (\bar{A}A)(N.V)$$

~~since~~ in virtue of (19,2), so that

$$NA\bar{V}AN = -2(\bar{A}A)(N.V)N.$$

Thus we find

$$\Psi\Psi^{\dagger} = \Gamma\Gamma^{\dagger} = \bar{V} - 2k(N.V)\bar{A} + \{2k(A.\bar{V}) - 2k^2(A\bar{A})(N.V)\}N.$$

In order that in the equation (19,5) the coefficient of $\bar{A}$ be zero we must choose

$$k = -\frac{e}{m_0c^2}\frac{1}{2(N.V)}.$$

The resulting equation is

$$DS = m_0cV - \frac{\bar{N}}{(N.V)}\left\{\frac{e}{c}(A.\bar{V}) + \frac{1}{2}\frac{e^2}{m_0c^2}\bar{A}A\right\}$$

and the phase $S$ is found to be

$$S = m_0c\,V.X - \frac{1}{N.V}\int_{\lambda_0}^{\lambda}\left\{\frac{e}{c}(U.\bar{V})\varphi(\lambda) + \frac{1}{2}\frac{e^2}{m_0c^2}U\bar{U}\varphi^2(\lambda)\right\}d\lambda.$$

since $\bar{N}.dX = d\lambda$.

$\underline{\Lambda} = e_1 m^2{}_3 + e_2 m^3{}_1 + e_3 m^1{}_2$



# CHAPTER V. The Wave Equation in angular coordinates.

## § 20. Commutation relations and eigenfunctions of the ~~general~~ angular momentum operator.

Before proceeding to the solution of Dirac's equation in angular coordinates we give a quaternion treatment of the ~~are~~ properties of the angular momentum operator. The components of this operator are in Dirac formalism

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \qquad (\mu, \nu = 1, 2, 3). \qquad (20,1)$$
$$= -i\hbar (x_\mu \partial_\nu - x_\nu \partial_\mu)$$

according to the quaternion representation (5,10) of the momentum operators, if $\Psi$ is the quaternion wave function we have

$$m_{\mu\nu} \Psi = \hbar (x_\mu \partial_\nu - x_\nu \partial_\mu) \Psi e_3 .$$

Hence we need only study the properties of the quaternion

$$\underline{\Lambda} = \underline{r} \wedge \underline{\nabla} = e_1 (x^2 \partial_3 - x^3 \partial_2) + e_2 (x^3 \partial_1 - x^1 \partial_3) + e_3 (x^1 \partial_2 - x^2 \partial_1) \qquad (20,2)$$

where

$$\underline{r} = e_1 x^1 + e_2 x^2 + e_3 x^3 \quad \text{and} \quad \underline{\nabla} = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3 . \qquad (7,7)$$

In polar coordinates ~~where~~ we have

$$\underline{r} = r B e_3 \bar{B} \qquad (20,3)$$

where the real quaternion $B$ depends on the angular coordinates only and is given by

$$B = e^{e_3 \frac{\varphi}{2}} e^{e_2 \frac{\theta}{2}} . \qquad (20,4)$$

since (20,4) is equivalent to

$$x^1 = r \sin\theta \cos\varphi , \quad x^2 = r \sin\theta \sin\varphi , \quad x^3 = r \cos\theta .$$

We shall also use the notation

$$\underline{\varepsilon} = r^{-1} \underline{r} = B e_3 \bar{B} \qquad (20,5)$$

for the unit radial vector.

We find that the polar form of the operator $\underline{\Lambda}$ is

$$\underline{\Lambda} = B e_2 \bar{B} \partial_\theta - B e_1 \bar{B} \frac{1}{\sin\theta} \partial_\varphi \qquad (20,6)$$

The square of this operator is

$$\underline{\Lambda}\underline{\Lambda} = -\underline{\Lambda}.\underline{\Lambda} + \underline{\Lambda} \wedge \underline{\Lambda} = \left(B e_2 \bar{B} \partial_\theta - B e_1 \bar{B} \frac{1}{\sin\theta} \partial_\varphi\right)^2$$
$$= -\partial_\theta^2 - \frac{1}{\sin^2\theta} \partial_\varphi^2 - \cot\theta\, \partial_\theta - B e_2 \bar{B} \partial_\theta + B e_1 \bar{B} \frac{1}{\sin\theta} \partial_\varphi$$

Hence, on separating the scalar and vectorial parts we obtain

$$\underline{\Lambda}\cdot\underline{\Lambda} = \frac{1}{\sin\theta}\,\partial_\theta \sin\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\,\partial_\varphi^2 \qquad (20,7)$$

and

$$\underline{\Lambda}\wedge\underline{\Lambda} = -\,\underline{\Lambda} \qquad (20,8).$$

The operator (20,7) appears in the polar form of Laplace's equation and (20,8) gives the well-known commutation relations for the angular momentum operator.

In order to find the eigen functions of the operator $\underline{\Lambda}$ we first transform the expression (20,6), in the following way: since

$$\bar{B}\,\partial_\theta\Psi = \partial_\theta(\bar{B}\Psi) - (\partial_\theta\bar{B})\Psi$$

or, symbolically

$$\bar{B}\,\partial_\theta = \partial_\theta\bar{B} - (\partial_\theta\bar{B})$$

where $(\partial_\theta\bar{B})$ is a quaternion which depends on $\theta$ and $\varphi$, we can write

$$\underline{\Lambda} = B\left(e_2\,\partial_\theta - e_1\frac{1}{\sin\theta}\,\partial_\varphi\right)\bar{B} - \left\{B\left(e_2\partial_\theta - e_1\frac{1}{\sin\theta}\,\partial_\varphi\right)\bar{B}\right\}.$$

Now the term between curly brackets is easily evaluated. One finds

$$\left(e_2\,\partial_\theta - e_1\frac{1}{\sin\theta}\,\partial_\varphi\right)\bar{B} = \left(1 - \tfrac{1}{2}e_2\cot\theta\right)\bar{B}. \qquad (20,9).$$

Hence

$$1 + \underline{\Lambda} = B\,\underline{L}\,\bar{B} \qquad (20,10)$$

where

$$\underline{L} = e_2\left(\partial_\theta + \tfrac{1}{2}\cot\theta\right) - e_1\frac{1}{\sin\theta}\,\partial_\varphi. \qquad (20,11)$$

Let $X'_\ell$ be a real quaternion which is an eigen-function of the operator $\underline{L}$, belonging to the eigenvalue $-\ell$. We have

$$\underline{L}\,X'_\ell = -\ell\,X'_\ell \qquad (20,12)$$

Writing

$$X_\ell = \bar{B}\,X'_\ell \qquad (20,13).$$

in (20,12) we obtain

$$B\,\underline{L}\,\bar{B}\,X_\ell = -\ell\,X_\ell$$

or, by (20,10)

$$\underline{\Lambda}\,X_\ell = -(\ell+1)\,X_\ell. \qquad (20,14).$$

Thus, if $X'_\ell$ ~~of~~ is an eigenfunction of $\underline{L}$, then $X_\ell$ (20,13) is an eigenfunction of $\underline{\Lambda}$. We now prove that $\varepsilon\,X_\ell$ is also an eigenfunction of $\underline{\Lambda}$ and we have

$$\underline{\Lambda}\,(\underline{\varepsilon}\,X_\ell) = (\ell-1)\,\underline{\varepsilon}\,X_\ell \qquad (20,15).$$

To see this we multiply (20,12) to the left and to the right by $e_3$. as $e_3$ anticommutes with $\underline{L}$ we have

$$e_3\,X'_\ell\,e_3 = X'_{-\ell} \qquad (20,16)$$

and from (20,13)

$$\underline{\varepsilon}\,X_\ell\,e_3 = X_{-\ell} \qquad (20,17).$$

$X_{-\ell}$ and $\underline{\varepsilon}\,X_\ell = -X_{-\ell}\,e_3$ both satisfy (20,15).

The eigenvalues of the operator $-\underline{\Lambda}\cdot\underline{\Lambda}$ which is given by (20,17) in polar coordinates can be deduced simply from (20,14). ~~We have~~, Using the identity

$$\cancel{-(\underline{\Lambda}\cdot\underline{\Lambda})\,X_\ell =}\ \underline{\Lambda}\,\underline{\Lambda} = -\underline{\Lambda}\cdot\underline{\Lambda} - \underline{\Lambda}$$

which is a consequence of (20,8), we have

$$-(\underline{\Lambda}\cdot\underline{\Lambda})\,X_\ell = (1+\underline{\Lambda})\,\underline{\Lambda}\,X_\ell = \ell(\ell+1)\,X_\ell \qquad (20,18)$$

Hence $X_\ell$ is an eigenfunction of this scalar operator ~~and~~ corresponds to the eigenvalue $\ell(\ell+1)$. The eigenfunction which corresponds to the eigenvalue $\ell(\ell-1)$ is $X_{-\ell}$ given by (20,17).

Hence the solution of the equation

$$\{\underline{\Lambda}\cdot\underline{\Lambda} + \ell(\ell+1)\}\,X_\ell = 0$$

can be derived from the very simple linear equation (20,12).

We now proceed to show that $X'_\ell$ can be simply expressed in terms of Legendre functions. Let

$$X'_\ell = q(\theta)\,e^{-e_3(m+\frac{1}{2})\varphi} \qquad (20,19)$$

where $q(\theta)$ is a quaternion which depends on $\theta$ only.

The equation satisfied by $q$ is

$$\left(\frac{d}{d\theta} + \frac{1}{2}\cot\theta\right) q + \frac{m+\frac{1}{2}}{\sin\theta}\,e_3\,q\,e_3 = \ell\,e_2\,q$$

The further transformation

$$q = (u + e_2 v)\,e^{-e_2\frac{\theta}{2}} = q'\,e^{-e_2\frac{\theta}{2}} \qquad (20,20)$$

where

$$q' = u + e_2 v$$

and $u$ and $v$ are scalar functions of $\theta$, leads to the equation

$$\cancel{\left(\frac{d}{d\theta} + \frac{1}{2}\cot\theta\right) q' = \ell\,q'\,e_2 + \frac{m+\frac{1}{2}}{\sin\theta}\,\bar{q}'}$$

~~since $e_2$ commute with $q'$ and~~

$$\cancel{e_3\,q'\,e_3 = -\bar{q}'}.$$

$$\left(\frac{d}{d\theta} + \frac{1}{2}\cot\theta\right) q' - (m+\tfrac{1}{2})\cot\theta\,\bar{q}' = (m+\tfrac{1}{2})\,e_2\,\bar{q}' + (\ell+\tfrac{1}{2})\,e_2\,q' \qquad (20,21)$$

since $e_2$ commutes with $q'$ and

$$e_3\,q'\,e_3 = -\bar{q}'$$

The quaternion equation (20, ~~16~~ 21) is equivalent to the scalar equations

$$\left(\frac{d}{d\theta} - m\cot\theta\right) u = -(l-m) v$$

and

$$\left[\frac{d}{d\theta} + (m+1)\cot\theta\right] v = (l+m+1) u$$

But these equations ~~beco~~ are identical with the recurrence relations between associated Legendre functions so that we can take

$$u = (l-m)\, P_{l,m}(\cos\theta) \quad \text{and} \quad v = P_{l,m+1}(\cos\theta)$$

Using (20, ~~15~~ 20) (20,19) and (20, ~~1~~ 4) we obtain

$$X'_{l,m} = \{(l-m)\, P_{l,m} + e_2 P_{l,m+1}\}\, \bar{B}\, e^{-e_3 m\varphi} \qquad (20, \cancel{17}\, 22)$$

and finally, from (20, ~~9~~ 13)

$$X_{l,m} = (l-m)\, P_{l,m}\, e^{-e_3 m\varphi} + e_2 P_{l,m+1}\, e^{-e_3 (m+1)\varphi} \qquad (20, \cancel{18}\, 23)$$

If we are looking for single valued solutions of (20, ~~10~~ 14) the numbers $l$ and $m$ must be integers. Denoting the spherical harmonics by ~~$Y_{l,m}$~~

$$Y_{l,m} = P_{l,m}(\cos\theta)\, e^{-e_3 m\varphi}$$

we can also write

$$X_{l,m} = (l-m)\, Y_{l,m} + e_2 Y_{l,m+1}.$$

In section (9) we have seen that the quaternion equivalent of the spin operator in the $Oz$ direction is ~~$e_3(\ )e_3$~~.

$$\sigma_3 = \sigma_{12} = -e_3(\ \ )e_3.$$

where the function on which $\sigma_{12}$ operates has to be inserted between the brackets. The total angular momentum is

$$\mathcal{M}_{12} = m_{12} + \tfrac{1}{2}\hbar\,\sigma_{12}.$$

We have

$$\mathcal{M}_{12} X_{l,m} = \hbar\, \partial_\varphi X_{l,m}\, e_3 \cancel{+} - \tfrac{1}{2}\hbar\, e_3 X_{l,m} e_3$$

But, from (20, ~~7~~ 13) and (20, ~~17~~ 22)

$$X_l = B\,\{u(\theta) + e_2 v(\theta)\}\,\bar{B}\, e^{-e_3 m\varphi}.$$

Since

$$\partial_\varphi B = \tfrac{1}{2} e_3 B$$

we obtain

$$\mathcal{M}_{12} X_{l,m} = (m + \tfrac{1}{2})\, X_{l,m}$$

Hence the eigenvalues of $\mathcal{M}_{12}$ are of the form $m+\frac{1}{2}$, where $m$ is integer. Thus the spin term arises from the fact that $X_{l,m}$ cannot be expressed as the product of two quaternions: one depending on $\theta$ and one on $\varphi$ only.

## §21. Commutation relations and eigenfunctions of the generalized angular momentum operator.

Born & Fuchs (1939) p.100 have given the relativistic generalization of (20,1) as

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = i\hbar (x_\mu \partial_\nu - x_\nu \partial_\mu). \qquad (\mu,\nu = 0,1,2,3)$$

and deduced some of the properties of this generalized angular momentum operator. The eigenfunctions and the eigenvalues of this operator become important when the problem has a four-dimensional spherical symmetry (e.g. if the interaction terms depend on the four-dimensional interval only, or if the boundary conditions possess pseudo-spherical symmetry).

In this section ~~its~~ properties ~~of~~ will be studied by means of ~~the~~ quaternions. As before we need only consider the vectorial quaternion

$$\underline{K} = I^\mu \bar{I}^\nu (x_\mu \partial_\nu - x_\nu \partial_\mu) = I_\mu \bar{I}^\nu (x^\mu \partial_\nu - x_\nu \partial^\mu)$$

where $I^\mu$ is given by ( ). ~~We have~~ and $\bar{I}^\nu = I_\nu$ from ( ). We have since $x^0 = x_0$

$$\underline{K} = V(I_\mu x^\mu \bar{I}^\nu \partial_\nu) = V(x^0 + i\underline{r})(\partial_0 + i\underline{\nabla})$$

$$= i(x_0 \underline{\nabla} + \underline{r}\,\partial_0) - \underline{r} \wedge \underline{\nabla}.$$

Or, on putting

$$\underline{N} = x_0 \underline{\nabla} + \underline{r}\,\partial_0 \qquad (21.1)$$

$$\underline{K} = i\underline{N} - \underline{\Lambda}.$$

~~The~~ four-dimensional angular coordinates $\tau, \alpha, \theta, \varphi$ are defined ~~by~~ the equations:

$$x^0 = \tau \operatorname{ch}\alpha, \quad x^1 = \tau \operatorname{sh}\alpha \sin\theta \cos\varphi$$

$$x^2 = \tau \operatorname{sh}\alpha \sin\theta \sin\varphi, \quad x^3 = \tau \operatorname{sh}\alpha \cos\theta$$

which are equivalent to the quaternion relation

$$X = I_\mu x^\mu = \tau Q^\dagger Q \qquad (21.2)$$

where

$$Q = e^{ie_3 \frac{\alpha}{2}} e^{-e_2 \frac{\theta}{2}} e^{-e_3 \frac{\varphi}{2}}. \qquad (Q\bar{Q} = 1) \quad (21.3)$$

We find that

$$D = \sum_\nu I_\nu \partial_\nu = \bar{I}^\nu \partial_\nu = \bar{Q} Q^* \partial_\tau + \bar{Q}\, ie_1 Q^* \frac{1}{\tau \operatorname{sh}\alpha} \partial_\theta + \bar{Q}\, ie_2 Q^* \frac{1}{\tau \operatorname{sh}\alpha \sin\theta} \partial_\varphi + \bar{Q}\, ie_3 Q^* \frac{1}{\tau} \partial_\alpha.$$

Hence

$$\underline{K} = V(XD) = Q^\dagger ie_1 Q^* \frac{1}{\operatorname{sh}\alpha} \partial_\theta + Q^\dagger ie_2 Q^* \frac{1}{\operatorname{sh}\alpha \sin\theta} \partial_\varphi + Q^\dagger ie_3 Q^* \partial_\alpha \qquad (21.4)$$

depends only on the angular coordinates $\alpha$, $\theta$ and $\varphi$.

From (20,6) we find after some simple transformations

$$\underline{K} = i\varepsilon(\partial_\alpha - \coth\alpha\,\underline{\Lambda}) - \underline{\Lambda}$$

~~so that~~ where $\varepsilon$ is given by (20,5) ~~so that from (21,7)~~.

$$\underline{N} = \varepsilon(\partial_\alpha - \coth\alpha\,\underline{\Lambda}) \qquad (21,5)$$

~~Then~~ We shall now prove the following commutation relation

$$\underline{N}\,\underline{\Lambda} + \underline{\Lambda}\,\underline{N} = -2\underline{N}. \qquad (21,6)$$

as $\underline{\Lambda}$ commutes with the ~~term~~ bracketed term in (21,5) we have

$$\underline{N}\,\underline{\Lambda} + \underline{\Lambda}\,\underline{N} = (\varepsilon\underline{\Lambda} + \underline{\Lambda}\varepsilon)(\partial_\alpha - \coth\alpha\,\underline{\Lambda}).$$

From (20,5) and (20,10):

$$\varepsilon\underline{\Lambda} + \underline{\Lambda}\varepsilon = B(e_3\underline{L} + \underline{L}e_3)\bar{B} - 2Be_3\bar{B} = -2\varepsilon$$

since $\underline{L}$ anticommutes with $e_3$. Hence, with the help of (21,5) this proves (21,6).

The next commutation relation we want to prove is

$$\underline{N}\wedge\underline{N} = \underline{\Lambda}. \qquad (21,7)$$

It is convenient to use the cartesian form (21,1) for $\underline{N}$.

We have

$$\underline{N}\wedge\underline{N} = V(\underline{N}\,\underline{N}) = V\{x_0\partial_0(\underline{\nabla}\,\underline{r} + \underline{r}\,\underline{\nabla}) + \underline{r}\,\underline{\nabla}\}$$

since

$$\partial_0 x_0 = x_0\partial_0 + 1.$$

On the other hand

$$\underline{\nabla}\,\underline{r} = (\underline{\nabla}\,\underline{r}) + e_1\underline{r}\partial_1 + e_2\underline{r}\partial_2 + e_3\underline{r}\partial_3,$$

and since $(\underline{\nabla}\,\underline{r}) = -3$, the operator

$$(\underline{\nabla}\,\underline{r} + \underline{r}\,\underline{\nabla}) = -3 - 2\underline{r}\cdot\underline{\nabla}$$

is a scalar operator. Therefore

$$\underline{N}\wedge\underline{N} = V(\underline{N}\,\underline{N}) = V(\underline{r}\,\underline{\nabla}) = \underline{r}\wedge\underline{\nabla}$$

and this completes the proof of (21,7).

We can now show easily that

$$\underline{K}\wedge\underline{K} = 2\underline{K}. \qquad (21,8)$$

This relation is a straightforward generalization of (20,8).

We have

$$\underline{K}\wedge\underline{K} = V(\underline{K}\,\underline{K}) = V(i\underline{N} - \underline{\Lambda})^2 = V\{\underline{\Lambda}\,\underline{\Lambda} - \underline{N}\,\underline{N} - i(\underline{N}\,\underline{\Lambda} + \underline{\Lambda}\,\underline{N})\}$$

From (21,6), (21,7) and (20,8) we find

$$\underline{K}\wedge\underline{K} = 2i\underline{N} - 2\underline{\Lambda}$$

which proves (21,8).

$K' = \frac{1}{2}K$

$K' \times K' = K'$

We also find

$$-\underline{K}\cdot\underline{K} = S(\underline{K}\,\underline{K}) = S(\underline{\Lambda}\,\underline{\Lambda} - \underline{N}\,\underline{N} + 2i\underline{N}) = -\underline{\Lambda}\cdot\underline{\Lambda} + \underline{N}\cdot\underline{N}\,,$$

and since

$$\underline{K}\,\underline{K} = -\underline{K}\cdot\underline{K} + \underline{K}\wedge\underline{K}\,,$$

from (21, 8) we deduce that

$$-\underline{K}\cdot\underline{K} = \underline{N}\cdot\underline{N} - \underline{\Lambda}\cdot\underline{\Lambda} = \underline{K}(\underline{K}-2) \qquad (21, 9).$$

The scalar operator $-\underline{K}\cdot\underline{K}$ occurs in ~~the~~ D'Alembert's equation, so that if an eigenfunction $U_k$ of the operator $\underline{K}$ ~~and~~ corresponds to the eigenvalue $-k$, i.e.

$$\underline{K}\,U_k = -k\,U_k \qquad (21, 10).$$

then $U_k$ is also an eigenfunction of the scalar operator (21, 9) and corresponds to the eigenvalue $k(k+2)$ because

$$-(\underline{K}\cdot\underline{K})\,U_k = \underline{K}(\underline{K}-2)\,U_k = k(k+2)\,U_k$$

again we notice that (21, 9) is a relativistic generalization of the 3-dimensional equation

$$-\underline{\Lambda}\cdot\underline{\Lambda} = \underline{\Lambda}(\underline{\Lambda}+1)\,.$$

We now seek the quaternion solutions of (21, 10). Let

$$U_k = Q^{\dagger}\,U'_k \qquad (21, 11)$$

where $Q^{\dagger}$ is defined by $(21, 3^{\dagger})$.

Now, going back to the expression (21, 4) we can write

$$\underline{K} = Q^{\dagger}\Big\{ie_3\Big[\partial_\alpha - \frac{1}{sh\,\alpha}\Big(e_2\partial_\theta - e_1\frac{1}{\sin\theta}\partial_\varphi\Big)\Big]Q^{\times}$$
$$- Q^{\dagger}ie_3\Big\{\Big[\partial_\alpha - \frac{1}{sh\,\alpha}\Big(e_2\partial_\theta - e_1\frac{1}{\sin\theta}\partial_\varphi\Big)\Big]Q^{\times}\Big\} \qquad (21, 12)$$

where the second term is a function of $\alpha$, $\theta$ and $\varphi$ and is not a differential operator. Remembering that

$$Q^{\times} = e^{-ie_3\frac{\alpha}{2}}\,\overline{B} = \Big(\cosh\frac{\alpha}{2} - ie_3\,sh\frac{\alpha}{2}\Big)\overline{B}$$

and using (20, 9) we obtain

$$\Big(e_2\partial_\theta - e_1\frac{1}{\sin\theta}\partial_\varphi\Big)e^{-ie_3\frac{\alpha}{2}}\,\overline{B} = e^{ie_3\frac{\alpha}{2}}\Big(1 - \frac{1}{2}e_2\cot\theta\Big)\overline{B}$$
$$= \Big(e^{ie_3\alpha} - \frac{1}{2}e_2\cot\theta\Big)Q^{\times}.$$

The contribution of the term in $\partial_\alpha$ to the curly bracket is

$$\partial_\alpha e^{-ie_3\frac{\alpha}{2}}\overline{B} = -\frac{ie_3}{2}\,Q^{\times}$$

Hence the second term of (21, 12) reads

$$Q^{\dagger}ie_3\Big\{\frac{ie_3}{2} + \frac{1}{sh\,\alpha}\Big(e^{ie_3\alpha} - \frac{1}{2}e_2\cot\theta\Big)\Big\}Q^{\times}$$
$$= \frac{3}{2} + Q^{\dagger}\Big(ie_3\coth\alpha - \frac{1}{2}ie_1\frac{\cot\theta}{sh\,\alpha}\Big)Q^{\times}$$

so that we have finally

$$\left[(1-\zeta^2)\frac{d}{d\zeta}+l\zeta\right] f = -(l+m)\, g$$

$$\left[(1-\zeta^2)\frac{d}{d\zeta}-l\zeta\right] g = (l-m) f.$$

$$\left\{(1-\zeta^2)\frac{d}{d\zeta}-l\zeta\right\}\left\{(1-\zeta^2)\frac{d}{d\zeta}+l\zeta\right\} = (1-\zeta^2)^2\frac{d^2}{d\zeta^2} - l^2\zeta^2 + (1-\zeta^2)x - 2\zeta\frac{d}{d\zeta} + l(1-\zeta^2)$$

$$= m^2 - l^2$$

$$(1-\zeta^2)\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} - l^2\zeta^2 + l^2 - m^2 + l(1-\zeta^2).$$

$$= (1-\zeta^2)\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} + l^2(1-\zeta^2) - m^2 + l(1-\zeta^2)$$

$$(1-\zeta^2)\left\{\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} + l(l+1) - m^2\right\}$$

Hobson p. 289, 290.

$$\left\{(\zeta^2-1)\frac{d}{d\zeta}+\zeta(n+m+1)\right\}P_n^m(\zeta) = (n-m+1)P_{n+1}^m(\zeta)$$

$$\left\{(\zeta^2-1)\frac{d}{d\zeta}+\zeta(n-m)\right\}P_n^m = -(n+m)P_{n-1}^m$$

or:

$$\left\{(\zeta^2-1)\frac{d}{d\zeta}+\zeta(n-m+1)\right\}P_{n+1}^m = -(n+m+1)P_n^m.$$

2.



$$\underline{K} = \frac{3}{2} + Q^{\dagger} K' Q^{\times} \qquad (21,13)$$

where

$$K' = i e_3\left(\partial_\alpha + \coth\alpha - \frac{1}{sh\,\alpha}\underline{L}\right). \qquad (21,14)$$

$\underline{L}$ is given by (20,11).

Inserting (21,13) and (21,11) in (21,10) we obtain the equation

$$K' U'_k = -\left(k+\frac{3}{2}\right) U'_k. \qquad (21,15)$$

Hence $U'_k$ is an eigenfunction of the operator $K'$ and corresponds to the eigenvalue $-\left(k+\frac{3}{2}\right)$. The equation (21,15) written explicitly as

$$\left(\partial_\alpha + \coth\alpha - \frac{1}{sh\,\alpha}\underline{L}\right) U'_k = -\left(k+\frac{3}{2}\right) i e_3\, U'_k \qquad (21,16)$$

becomes separable if we put

$$U'_k = V_k(\alpha)\, X'_l(\theta,\varphi) \qquad (21,17)$$

or ~~$U'_k = V$~~ where $X'_l$ is the real quaternion given by (20,22) and where $V(\alpha)$ is a complex quaternion which depends on $\alpha$ only and commutes with $e_3$. We define the real scalar functions $F(\alpha)$ and $G(\alpha)$ by the equation

$$V = (F + i e_3 G)\, e^{i e_3 \frac{\alpha}{2}}. \qquad (21,18)$$

Substituting ~~this~~ (21,17) in (21,16) and remembering that $\underline{L}$ anticommutes with $e_3$, with the help of (20,12) we obtain

$$\left(\frac{d}{d\alpha} + \coth\alpha\right) V = -\left(k+\frac{3}{2}\right) i e_3 V - \frac{l}{sh\,\alpha} V^{\times} \qquad (21,19)$$

which, by (21,18) leads to the real scalar equations

$$\left[\frac{d}{d\alpha} + (l+1)\coth\alpha\right] F = -(k+l+2)\, G$$

$$\left[\frac{d}{d\alpha} - (l-1)\coth\alpha\right] G = -(k-l+2)\, F.$$

These equations are further simplified on putting

$$F + i e_3 G = (f + i e_3 g)/sh\,\alpha$$

and $\zeta = \coth\alpha$.

We have

$$\left[(1-\zeta^2)\frac{d}{d\zeta} + l\zeta\right] f = -(k+2+l)\, g \qquad (21,20)$$

$$\left[(1-\zeta^2)\frac{d}{d\zeta} - l\zeta\right] g = -(k+2-l)\, f. \qquad (21,21)$$

Comparing this system of equations with the recurrence relations between the Legendre functions $P_l^{k+2}(\zeta)$ and $P_{l-1}^{k+2}(\zeta)$ (cf. Hobson p. 290), namely

$$\left[(1-\zeta^2)\frac{d}{d\zeta} + l\zeta\right] P_l^{k+2} = (k+2+l)\, P_{l-1}^{k+2}$$

$$\left[(1-\zeta^2)\frac{d}{d\zeta} - l\zeta\right] P_{l-1}^{k+2} = (k+2-l)\, P_l^{k+2}$$

we see that (21,20) and (21,21) are satisfied if we take

$$f = P_l^{k+2}(\zeta) \quad \text{and} \quad g = -P_{l-1}^{k+2}(\zeta).$$

In physical applications we are interested in those solutions which vanish at infinity on the space-like surface defined by

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = \tau^2 = \text{cons}.$$

Hence, we must have $f = g = 0$ for $\alpha \to \pm\infty$ or $\zeta = \pm 1$. But by Hobson (p.94) such solutions of (21,20) and (21,21) correspond to integral values of $k<l$. Hence $f$ and $g$ are associated Legendre polynomials.

To ~~calculate~~ evaluate $V$ (21,18) we need the recurrence relations (Hobson p.290)

$$\zeta P_l^{k+2} - P_{l-1}^{k+2} = (l-k-1)\, P_l^{k+1}\sqrt{\zeta^2-1}$$

$$P_l^{k+2} - \zeta P_{l-1}^{k+2} = (l+k+1)\, P_{l-1}^{k+1}\sqrt{\zeta^2-1}$$

which are valid since $\zeta > 1$. We obtain

$$e^{ie_3\alpha}(f + ie_3 g) = \left(\frac{\zeta}{\sqrt{\zeta^2-1}} + ie_3\frac{1}{\sqrt{\zeta^2-1}}\right)\left(P_l^{k+2} - ie_3 P_{l-1}^{k+2}\right)$$

$$= (l-k-1)\, P_l^{k+1} + ie_3(l+k+1)\, P_{l-1}^{k+1}.$$

But by (21,11) ~~and~~, (21,17) and (21,18)

$$U_k = B e^{ie_3\alpha}(f + ie_3 g)\, X_l' \,(\mathrm{sh}\,\alpha)^{-1}.$$

Hence, from (20,~~7~~ 13)

$$U_k = (\mathrm{sh}\,\alpha)^{-1} B\left\{(l-k-1)P_l^{k+1}(\zeta) + ie_3(l+k+1)P_{l-1}^{k+1}(\zeta)\right\}\bar{B}\, X_l(\theta,\varphi).$$

and the full expression of $U$ ~~[illegible]~~ is

$$U_l^{k,m} = (\mathrm{sh}\,\alpha)^{-1}\left\{(l-k-1)P_l^{k+1}(\coth\alpha) + i\underline{\varepsilon}\,(l+k+1)P_{l-1}^{k+1}(\coth\alpha)\right\}\left\{(l-m)P_l^{m}(\cos\theta)e^{-e_3 m\varphi} + e_2 P_l^{m+1}(\cos\theta)e^{-e_3(m+1)\varphi}\right\} C \qquad (21,22)$$

where $C$ is a constant real quaternion.

Noticing that $P_{-l} = P_{l-1}$, we obtain another independent solution which reads

$$U_{-l}^{k,m} = (\mathrm{sh}\,\alpha)^{-1}\left\{+(l+k+1)P_{l-1}^{k+1}(\coth\alpha) + i\underline{\varepsilon}\,(l-k-1)P_l^{k+1}(\coth\alpha)\right\}$$

$$\times\left\{+(l+m)P_{l-1}^{m}(\cos\theta)e^{-e_3 m\varphi} - e_2 P_l^{m+1}(\cos\theta)e^{-e_3(m+1)\varphi}\right\} C$$

It may be noted that, as defined by (21,14) $K'$ is purely imaginary. Hence taking the complex conjugate of (21,15) we have

$$K'(U_k')^{\times} = (k+\tfrac{3}{2})(U_k')^{\times} = -\left\{-(k+3)+\tfrac{3}{2}\right\}(U_k')^{\times}.$$

Hence the relation:

$$(U_k')^{\times} = U'_{-(k+3)} \qquad (21,23)$$

Now, replacing $k$ by $-k-3$ in (22,10) we have

$$\underline{K}\,U_{-(k+3)} = (k+3)\,U_{-(k+3)}. \qquad (21,24)$$

But, from (21, 11)

$$(U_k')^{\times} = Q(U_k)^{\times}, \qquad (21,25)$$

Hence, using (21,23),

$$U'_{-(k+3)} = Q(U_k)^{\times}.$$

and multiplying this last relation to the left by $Q^{\dagger}$.

$$U_{-(k+3)} = Q^{\dagger}Q(U_k)^{\times} = e^{i\varepsilon\alpha}(U_k)^{\times}. \qquad (21,26)$$

Thus we see from (21,24) that if $U_k$ is an eigenfunction of $\underline{K}$ with the eigenvalue $-k$, then the expression (21,25) is an eigenfunction with the eigenvalue $k+3$. Here is an immediate application of this theorem: unity is an eigenfunction of $\underline{K}$ and corresponds to ~~the~~ $k=0$. Hence

$$U_0 = 1, \quad U_{-3} = e^{i\varepsilon\alpha}$$

and we have

$$\underline{K}\,e^{i\varepsilon\alpha} = 3e^{i\varepsilon\alpha}.$$

~~The other eigenfunctions which correspond to the value $k=3$ with different values for $l$ and $m$ can be deduced from the general formula (22,22).~~

## §22 – Solution of Dirac's equation in four-dimensional angular coordinates.

First consider the quaternion equation

$$\underline{\nabla}\,\chi = 0 \qquad (22,1)$$

which leads to the second order equation

$$(\partial_1^2 + \partial_2^2 + \partial_3^2)\,\chi = 0 .$$

We write

$$\chi = X_l^m(\theta,\varphi)\, R(r) \qquad (22,2)$$

where $X_l^m$ is given by (20, 23) and is an eigenfunction of $\underline{\Lambda}$. R depends on r only. Now, multiplication of (22, 1) to the left by $\underline{r}$ gives

$$(-r\,\partial_r + \underline{\Lambda})\,\chi = 0$$

since

$$\underline{r}\,\underline{\nabla} = -\underline{r}.\underline{\nabla} + \underline{r}\wedge\underline{\nabla} = -r\,\partial_r + \underline{\Lambda} .$$

Substituting the expression (22,2) for $\chi$ we obtain an equation for R, namely

$$r\frac{dR}{dr} + (l+1)R = 0$$

which is satisfied by

$$R = a\, r^{-(l+1)} . \qquad (a = \text{cons. quaternion})$$

Hence a particular solution of (22,1) is

$$\chi = r^{-(l+1)} X_l^m(\theta,\varphi)\, a . \qquad (22,3)$$

From (20, 17) we obtain another solution

$$\chi' = r^{l-1}\, \underline{\varepsilon}\, X_l^m\, e_3\, a' . \qquad (22,4)$$

Now consider the Fränkel-Madelung equation to which Dirac's equation reduces if the rest mass of the particle vanishes, namely

$$D\,\eta = 0 \qquad (22,5)$$

which leads to the second order equation $\bar{D}D\eta = 0$.

From (21,4) we have

$$X D = X.D + \underline{K} .$$

and since

$$X.D = \tau\,\partial_\tau$$

on multiplying (22,5) to the left by X we obtain

$$(\tau\,\partial_\tau + \underline{K})\,\eta = 0 . \qquad (22,6)$$

One is immediately led to put

$$\eta = U_l^{k,m}\, T(\tau) \qquad (22,7)$$

where $U_l^{k,m}$ given by (21, 22) is an eigenfunction of $\underline{K}$ and $T$ is a quaternion which depends on $\tau$ only. Inserting (22,7) in (22,6) we find that $T$ satisfies the equation

$$\tau \frac{dT}{d\tau} - kT = 0 \qquad (22,8)$$

since

$$\underline{K}\,\eta = -k\,\eta .$$

The solution of (22,8) is of the form

$$T = \tau^k C$$

where $C$ is a constant quaternion. Hence a particular solution of the Fränkel - Madelung equation is

$$\eta = \tau^k U_l^{k\,m} C . \qquad (22,9)$$

Another particular solution can be ~~found~~ deduced immediately from the relation (21,25), namely

$$\eta' = \tau^{-(k+3)} e^{i\varepsilon\alpha} (U_l^{k\,m})^{\times} C' .$$

In particular if we take $k = 0$ we have the constant solution

$$\eta = C$$

and another solution

$$\eta' = \tau^{-3} e^{i\varepsilon\alpha} C'$$

which can also be written as

$$\eta' = -\tfrac{1}{2}(\bar{D}\tau^{-2})\, C' .$$

As $\tau^{-2}$ is harmonic it can be verified directly that

$$D\eta' = 0 .$$

We now turn to Dirac's equation with an invariant scalar interaction term depending on $\tau$ only, namely

$$D\,\Psi = [\mu + \omega(\tau)]\,\Psi^{\times} e_3 . \qquad (22,10)$$

For the sake of simplicity we put

$$\mu' = \mu + \omega(\tau) .$$

Now, on account of (21,2$^3$) and (21,13) we have

$$X D = \tau Q^{\dagger} Q D = \tau \partial_\tau + \underline{K} = Q^{\dagger}(\tau\partial_\tau + \tfrac{3}{2} + K')Q^{\times} .$$

Hence on multiplying both sides to the left by $\tau^{-1}\bar{Q}Q^{\times}$ we find

$$D = \bar{Q}\,(\partial_\tau + \frac{3}{2\tau} + \frac{1}{\tau}K')\,Q^{\times} \qquad (22,11)$$

and on making the substitution

$$\Psi = Q^{\dagger}\,\Phi$$

$$\frac{d}{d\tau} J_0(\tau) = - J_1(\tau) \qquad \frac{\partial}{\partial r} J_0(\tau) = \frac{\partial \tau}{\partial r} \frac{d}{d\tau} J_0(\tau)$$

$$\tau = \sqrt{t^2 - r^2} = (t^2 - r^2)^{\frac{1}{2}} \qquad \frac{\partial \tau}{\partial r} = -\frac{1}{2} (t^2 - r^2)^{-\frac{1}{2}} 2r = -\frac{r}{\tau}$$

$$\frac{1}{r}\frac{\partial}{\partial r} J_0(\tau) = -\frac{1}{\tau}\frac{d}{d\tau} J_0(\tau) = \frac{1}{\tau} J_1(\tau).$$



the equation (29, 10) is replaced by ~~(23, 12)~~

$$\left(\partial_\tau + \frac{3}{2\tau} + \frac{1}{\tau} K'\right) \Phi = \mu' \Phi^{\times} e_3 , \qquad (29, 12)$$

where $K'$ is given by (28, 14).

We try to satisfy this equation by a function of the form

$$\Phi = U'^{km}_{l} A(\tau)\, \tau^{-1} + (U'^{km}_{l})^{\times} B(\tau)\, \tau^{-1} e_3 .$$

Here $U'^{k}_{l,m}$ is an eigenfunction of the operator $K'$ and is given by (28, 17). From (28, 15) and (28, 23) it follows that

$$K' \Phi = -\left(k + \tfrac{3}{2}\right) \left[ U'^{km}_{l} A \tau^{-1} - (U'^{km}_{l})^{\times} B \tau^{-1} e_3 \right].$$

Inserting this expression in (29, 12) we obtain the equation

$$0 = U'^{km}_{l}\left\{\left(\partial_\tau + \frac{3}{2\tau} - \frac{k+3/2}{\tau}\right) A \tau^{-1} + \mu' B \tau^{-1}\right\} + (U'^{km}_{l})^{\times}\left\{\left(\partial_\tau + \frac{3}{2\tau} + \frac{k+3/2}{\tau}\right) B \tau^{-1} - \mu' A \tau^{-1}\right\} e_3$$

which can be satisfied by taking $A$ and $B$ scalar functions of $\tau$ such that the curly brackets vanish, i.e.

$$\left\{ \begin{aligned} \left(\frac{d}{d\tau} - \frac{k+1}{\tau}\right) A &= -\mu'(\tau) B = -[\mu + \omega(\tau)] B \\ \left(\frac{d}{d\tau} + \frac{k+2}{\tau}\right) B &= \mu'(\tau) A = [\mu + \omega(\tau)] A \end{aligned} \right. \qquad (29, 13)$$

It would be very interesting to see if these equations lead to an eigenvalue equation for the rest mass $\mu$ when the invariant potential $\omega(\tau)$ takes special forms.

In the simple case of a free electron $\omega$ vanishes. Comparing the equations (29, 13) with the recurrence relations for Bessel functions (cf. Whittaker & Watson p. 360.)

$$\left(\frac{d}{d\tau} + \frac{n}{\tau}\right) J_n(\tau) = J_{n-1}(\tau)$$

$$\left(\frac{d}{d\tau} - \frac{n-1}{\tau}\right) J_{n-1}(\tau) = - J_n(\tau)$$

we find that (29, 13) are satisfied if we take

$$A = J_{k+1}(\mu\tau) \quad \text{and} \quad B = J_{k+2}(\mu\tau).$$

Hence a particular solution of (29, 10) for the free electron, such that $\Psi$ vanishes at infinity on the space-like surface $\tau = \text{const.}$ is

$$\Psi = \tau^{-1} Q^{\dagger}\left\{ U'^{km}_{l} J_{k+1}(\mu\tau) + (U'^{km}_{l})^{\times} J_{k+2}(\mu\tau)\, e_3 \right\}.$$

From (28, 11) we have

$$U^{km}_{l} = Q^{\dagger} U'^{km}_{l}$$

and from (28, 25)

$$Q^{\dagger}(U'^{km}_{l})^{\times} = Q^{\dagger} Q (U^{km}_{l})^{\times} = e^{i\varepsilon\alpha}(U^{km}_{l})^{\times} = U^{-(k+3),m}_{l}.$$

Hence

$$\Psi = \tau^{-1}\left\{ U^{k,m}_{l} J_{k+1}(\mu\tau) + U^{-(k+3),m}_{l} J_{k+2}(\mu\tau)\, e_3 \right\}$$

This solution ~~when~~ expressed as a column matrix is

of the same form as the solution given by ~~Born and~~ Fuchs (1940, p. 147). ~~This~~ solution however is quadratically integrable on the time like surface $r^2 - c^2 t^2 = a^2$, ~~whereas the solution instead~~ and not on the space-like surface $\tau = \text{cons.}$ as in this section.

§23. Solution of Dirac's equation in a conformal space-time with zero scalar curvature.

In §35 it will be shown that the formulation of Dirac's equation in an affine conformal space-time with zero scalar curvature leads to the equation

$$\hbar D \Phi = m_0 c \left(1 - \frac{a}{r}\right) \Phi^{\times} e_3 \qquad (23,1)$$

where $a$ is a certain constant. This equation has been solved ~~by~~ independently by Caldirola (1948). In this section we shall give a shorter quaternion treatment. The problem is simplified in three-dimensional polar coordinates. The polar form of the differential operator $\underline{\nabla}$ (~~20,1~~ 7,7) is easily obtained from the relation

$$\underline{r}\,\underline{\nabla} = -\underline{r}\cdot\underline{\nabla} + \underline{r}\wedge\underline{\nabla} = -r\partial_r + \underline{\Lambda}$$

where $\underline{\Lambda}$ is given by (20,~~2~~2). From (20,~~5~~6) and (20,~~8~~10) it follows that

$$r B e_3 \bar{B} \underline{\nabla} = B\{-(1 + r\partial_r) + \underline{L}\}\bar{B}.$$

Hence

$$\underline{\nabla} = r^{-1} B \{e_3(\partial_r r - \underline{L})\}\bar{B}.$$

Substitution of

$$\Phi = r^{-1} B \chi e^{e_3 E t/\hbar} \qquad (23,2)$$

in (23,1) gives

$$E\chi/\hbar + i e_3(\partial_r - r^{-1}\underline{L})\chi e_3 = (m_0 c/\hbar - \gamma r^{-1})\chi^{\times} \qquad (23,3)$$

since

$$D = \partial_0 + i\underline{\nabla}.$$

Here $\gamma$ is a new constant ~~such that~~ equal to $a/\hbar$.

Now let

$$\chi = f(r) X'_{l,m}(\theta,\varphi) + i e_3 g(r) X'_{l,m}(\theta,\varphi) e_3$$

where $X'_{l,m}$ is an eigenfunction of $\underline{L}$ given by (20,~~14~~22). With the help of (20,~~8~~12) and (20,~~12~~16) and assuming that $f$ and $g$ are real scalar functions we find that (23,3) is transformed into

$$E(f X'_l + i e_3 g X'_l e_3)/\hbar + i e_3\left(\frac{df}{dr} X'_l + i e_3 \frac{dg}{dr} X'_l e_3 + \frac{l}{r} f X'_l - i e_3 \frac{l}{r} g X'_l e_3\right) e_3$$
$$= (m_0 c/\hbar - \gamma/r)(f X'_l - i e_3 g X'_l e_3).$$

The separation of real and imaginary parts yields

$$\left(\frac{d}{dr} + \frac{l}{r}\right) f + \frac{1}{\hbar c}\left(E + m_0 c^2 - \frac{\gamma \hbar c}{r}\right) g = 0$$
$$\left(\frac{d}{dr} - \frac{l}{r}\right) g - \frac{1}{\hbar c}\left(E - m_0 c^2 + \frac{\gamma \hbar c}{r}\right) f = 0$$

It is advantageous to use the complex combination

$$v = f + ig$$

and to set

$$E = m_0 c^2 \cos\beta$$

as we are interested in the stationary values of the energy which are less in absolute value than $m_0 c^2$. Thus we obtain the single complex equation

$$\frac{dv}{dr} - i\left(\frac{m_0 c}{\hbar}\cos\beta\right)v + \frac{l}{r}v^* + i\left(\frac{m_0 c}{\hbar} - \frac{\gamma}{r}\right)v^* = 0 \qquad (23,3)$$

The further transformation

$$v = e^{i\beta/2}\, e^{-m_0 c r \sin\beta/\hbar}\, u \qquad (23,4)$$

yields the equation

$$e^{i\beta}\frac{du}{dr} + \frac{l - i\gamma}{r}u^* = i\frac{m_0 c}{\hbar}(u - u^*). \qquad (23,5)$$

Assume a polynomial solution of the form

$$u = \sum_{n=0}^{n=s} a_n r^{\nu+n}$$

where the coefficients $a_n$ are complex numbers and $s$ is an integer. If such a solution exists, then

$$\lim_{r\to\infty} v = 0 \qquad \text{provided } \sin\beta > 0$$

and $\Phi$ becomes bounded.

From (23,5) the recurrence relations for the coefficients $a_n$ ~~are~~ is found to be

$$e^{i\beta}(\nu + n)a_n + (l - i\gamma)a_n^* = i\frac{m_0 c}{\hbar}(a_{n-1} - a_{n-1}^*). \qquad (23,6)$$

As the polynomial starts with $a_0$ and ends with $a_s$ we must have

$$a_{-1} = a_{s+1} = 0.$$

~~Putting~~ Making $n = 0$ in (23,6) we get

$$e^{i\beta}\nu\, a_0 = (i\gamma - l)\, a_0^*.$$

The absolute values of both sides are equal, hence

$$\nu^2 = \gamma^2 + l^2. \qquad (23,7)$$

Now (23,6), ~~becomes~~ for $n = s+1$, reduces to

$$a_s - a_s^* = 0.$$

~~Hence~~ Therefore $a_s$ must be real. For $n = s$ we have

$$[e^{i\beta}(\nu + s) + (l - i\gamma)]\, a_s = i\frac{m_0 c}{\hbar}(a_{s-1} - a_{s-1}^*).$$

Now, as the right hand side is real the imaginary part of the left hand side must vanish, so that

$$(\nu + s)\sin\beta - \gamma = 0 \qquad (23,8)$$

In order that the condition $\sin\beta > 0$ be satisfied we must take

$$0 < \frac{\gamma}{\nu + s} < 1 .$$

The elimination of $\nu$ between (23,8) and (23,7) leads to the eigenvalue equation

$$\sin\beta = \frac{\gamma}{s + \sqrt{l^2 + \gamma^2}}$$

for $\beta$ ~~and~~ hence to

$$E/m_0 c^2 = \left[1 - \frac{\gamma^2}{(s + \sqrt{l^2 + \gamma^2})^2}\right]^{\frac{1}{2}}$$

for the energy.

Now, if the constant $a = \hbar\gamma$ is interpreted as the gravitational constant (see § 35), then $\gamma$ is very small and putting

$$E = m_0 c^2 + W$$

we have approximately

$$W = -\frac{1}{2}\frac{\gamma^2}{(s+l)^2}$$

which is of the same form as the non-relativistic energy levels of an electron in a Coulomb field with a much smaller separation constant for the energy levels.

Thus the electron ~~has~~ possesses discrete but quasi-continuous energy levels in a space-time with zero scalar curvature.

---

# CHAPTER VI.

# Properties of the Affine Space-time of Distant Parallelism.

## §24. General coordinate transformations in flat space-time. The covariant basis vectors.

In §4 we have ~~given the~~ represented a 4-vector as a hermitian quaternion in terms of its cartesian components. In particular, if $d\xi^\mu$ denote the real cartesian components of the infinitesimal displacement vector $dX$, we have

$$dX = I_\mu \, d\xi^\mu \qquad (24.1)$$

where $I_\mu$ are the hermitian units given by (4.1). The indices associated with cartesian coordinates are raised and lowered by means of the constant metric tensor of special relativity

$$\gamma_{\mu\nu} = I_\mu \cdot \bar{I}_\nu \,, \quad (\gamma_{00} = -\gamma_{11} = -\gamma_{22} = -\gamma_{33} = 1) \,. \qquad (24.2)$$

We note immediately that the displacement vector may be written in the form (24.1) without interpreting $I_\mu$ as quaternions. In the ordinary notation of vector calculus ~~would~~ if we specify $dX$ by the complex coordinates

$$\eta^0 = d\xi^0 \,, \quad \eta^n = i \, d\xi^n \qquad (n = 1, 2, 3)$$

$dX$ would be written as

$$dX = (\eta^0, \eta^1, \eta^2, \eta^3)$$

or, ~~putting~~ denoting the unit vectors by $e_\mu$, namely

$$e_0 = (1, 0, 0, 0) \,, \quad e_1 = (0, 1, 0, 0)$$
$$e_2 = (0, 0, 1, 0) \,, \quad e_3 = (0, 0, 0, 1)$$

$dX$ would take the form

$$dX = e_\mu \eta^\mu$$

which is the same as (24,1) if we take

$$e_0 = I_0 \,, \quad i e_n = I_n \,.$$

Now, if a multiplication rule for the units vectors $e_\mu$ is defined by means of (1.1) then $e_0$ becomes unity and the $e_n$ become quaternion units so that (24.1) represent a hermitian quaternion and if a matrix undors (Belinfante), expansors (Comtois 1946).



representation is adopted for the unit quaternions we can express the displacement vector $dX$ as a matrix. This is no more mysterious than the representation of complex numbers by matrices ~~and~~ yet several authors insist on calling a matrix representation of (24.1) by such strange names as matrix length (Flint, 1928), quantum geometric operator (Fock & Iwanenko (1929) or vectrix (Duffin 1950).

In this section we give a generalization of (24.1) and (24.2) for a general coordinate system in flat space.

Under the coordinate transformation

$$\xi^\alpha = f^\alpha(x^\beta) \tag{24.3}$$

the contravariant components of $dX$ transform according to the equation

$$d\xi^\alpha = {}^\alpha t_\beta \, dx^\beta \tag{24.4}$$

where

$${}^\alpha t_\beta = \frac{\partial f^\alpha}{\partial x^\beta}. \tag{24.5}$$

Hence, putting

$$E_\beta = I_\alpha \, {}^\alpha t_\beta \tag{24.6}$$

we can write

$$dX = I_\beta \, d\xi^\beta = E_\alpha \, dx^\alpha \tag{24.7}$$

It is clear that the quaternion $dX$ may be ~~considered~~ regarded as invariant under the transformation (24.3) since it has exactly the same form in the two coordinate systems, the hermitian quaternions $E_\alpha$ replacing the unit hermitian quaternions $I_\alpha$ and the generalized (or curvilinear) coordinates $dx^\alpha$ replacing the cartesian coordinates $d\xi^\alpha$. A remark here is necessary before we proceed any further. With the help of the new coordinates $dx^\alpha$ of the displacement vector it is also possible to construct the hermitian quaternion

$$dX' = I_\alpha \, dx^\alpha \tag{24.8}$$

which is obviously different from $dX$ given by (24.7) and hence is not invariant under coordinate transformations. ~~It is~~ The quaternion (24.8) is useful when one is dealing with conformal transformations. ~~We have for this~~ In this case

$$E_\alpha = f\,\zeta^\dagger I_\alpha \zeta \qquad (|\zeta| = 1) \qquad (24.8)$$

where $f$ is scalar and $\zeta$ a quaternion with unit norm, and (24.7) takes the form

$$dX = f\,\zeta^\dagger I_\alpha \zeta\, dx^\alpha = f\,\zeta^\dagger dX' \zeta\,. \qquad (24.9)$$

Lorentz transformations which come under this group have been discussed in §4 and the general conformal transformations will be studied in detail in §35.

The interval associated with the invariant displacement (24.7) is the invariant scalar given by the norm of $dX$

$$ds^2 = \mathcal{N}(dX) = \gamma_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = g_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (24.10)$$

where the constant tensor ~~which cha~~ $\gamma_{\alpha\beta}$ which belongs to the cartesian coordinates $\xi^\alpha$ is given by (24.2) and the metric tensor $g_{\alpha\beta}(x^0, x^1, x^2, x^3)$ which corresponds to the transformation (24.3) to the curvilinear coordinates $x^\alpha$ is given by

$$g_{\alpha\beta} = \mathcal{S}(E_\alpha \bar{E}_\beta) = E_\alpha . \bar{E}_\beta\,. \qquad (24.11)$$

Using (24.6) we can also write

$$g_{\alpha\beta} = \gamma_{\mu\nu}\, {}^\mu t_\alpha\, {}^\nu t_\beta = {}^0t_\alpha\, {}^0t_\beta - {}^1t_\alpha\, {}^1t_\beta - {}^2t_\alpha\, {}^2t_\beta - {}^3t_\alpha\, {}^3t_\beta\,.$$

To the doubly covariant tensor $g_{\alpha\beta}$ corresponds the doubly contravariant reciprocal tensor $g^{\alpha\beta}$ defined by

$$g^{\alpha\beta} g_{\beta\epsilon} = \delta^\alpha_\epsilon \qquad (24.12)$$

where $\delta^\alpha_\epsilon$ is the Kronecker symbol. In the cartesian system we have

$$\gamma^{\mu\nu} = \gamma_{\mu\nu}\,.$$

We now ~~make~~ adopt the convention that indices ~~to~~ on the left of ${}^\alpha t_\beta$ will be raised and lowered by $\gamma_{\alpha\beta}$ and those on the right by $g_{\alpha\beta}$ and $g^{\alpha\beta}$. Thus we can write

$$
\begin{aligned}
g_{\alpha\beta} &= {}^\mu t_\alpha\, {}_\mu t_\beta = {}_\mu t_\alpha\, {}^\mu t_\beta\,. \\
g^\alpha{}_\beta &= \delta^\alpha_\beta = {}_\mu t^\alpha\, {}^\mu t_\beta && (24.13) \\
\gamma_{\mu\nu} &= {}_\mu t^\alpha\, {}_\alpha t_\nu = {}_\mu t_\alpha\, {}^\alpha t_\nu \\
\gamma^\mu{}_\nu &= \delta^\mu_\nu = {}^\mu t_\alpha\, {}^\alpha t_\nu\,. && (24.14)
\end{aligned}
$$

The contravariant basis quaternions

$$E^\alpha = g^{\alpha\beta} E_\beta = I^\mu\, {}_\mu t^\alpha = I_\mu\, {}^\mu t^\alpha\,. \qquad (24.15)$$

are seen, from (24.13), to satisfy the relations

$$E^\alpha . \bar{E}_\beta = \delta^\alpha_\beta \qquad (24.16)$$

Multiplying both sides of the first equation (24.15) scalarly by $\bar{E}^\beta$, or raising the suffix $\beta$ in (24.16) we also obtain

$$g^{\alpha\beta} = E^\alpha . \bar{E}^\beta . \qquad (24.17)$$

Now let $a^\alpha$ the contravariant components of a 4-vector. ~~By de~~ It follows that the hermitian quaternion

$$A = E_\alpha a^\alpha = E^\alpha a_\alpha \qquad (24.18)$$

is an invariant since $a^\alpha$ transform like $dx^\alpha$. The covariant components of $A$ are here denoted by $a_\alpha$. Multiplying both sides scalarly by $\bar{E}^\beta$ or $\bar{E}_\beta$ and using (24.16) we find

$$a^\alpha = \bar{E}^\alpha . A$$

$$a_\alpha = \bar{E}_\alpha . A .$$

Since $A$ is invariant, it can also be written in the cartesian form

$$A = I_\alpha \, {}^\alpha a$$

where the cartesian coordinates ${}^\alpha a$ of $A$ are given by

$${}^\alpha a = \bar{I}^\alpha . A . \qquad (~~24.1~~$$

Replacing $A$ by its expression in curvilinear coordinates (24.18) and using (24.6) and (24.2) we find

$${}^\alpha a = {}^\alpha t_\beta \, a^\beta . \qquad (24.19)$$

This important formula will be used to connect covariant and contravariant coordinates of a vector in a general curvilinear coordinate system ~~with d~~ specified by right indices with its cartesian coordinates specified by left indices. When this rule is applied to the components of the vector $dX$ we ~~fi~~ have

$${}^\alpha(dx) = \bar{I}^\alpha . dX = d\xi^\alpha$$

and $${}^\alpha E = {}^\alpha t_\beta E^\beta = I^\alpha$$

so that we can write

$$dX = E_\alpha dx^\alpha = {}_\alpha E \; {}^\alpha(dx)$$

and $$A = E_\alpha a^\alpha = {}_\alpha E \; {}^\alpha a .$$

where the right indices refer to the curvilinear coordinate system and the left indices to the cartesian frame of reference. It will be noticed that if the vector $A$ is constant its cartesian components will also be constant ~~whereas~~ while its curvilinear components will change from point to point.

We shall now derive two identities between covariant and contravariant basis quaternions which will be employed frequently in the remainder of this work. Consider the hermitian quaternion

$$G_{\lambda\mu\nu} = \mathcal{H}(E_\lambda \bar{E}_\mu E_\nu)$$

where the symbol $\mathcal{H}$ defined by (1.8) means the hermitian part of the complex quaternion inside the brackets. To evaluate $\mathcal{H}$ we need the following lemma: if $A$, $B$ and $C$ are three hermitian quaternions then we have the identity

$$\mathcal{H}(A\bar{B}C) = (A.\bar{B})C + (B.\bar{C})A - (C.\bar{A})B \qquad (24.20)$$

which is easily proved by writing out the components of ~~the three terms~~ $A$, $B$ and $C$. If now we take

$$A = E_\lambda \quad , \quad B = E_\mu \quad , \quad C = E_\nu$$

(24.20) gives by (24.11)

$$G_{\lambda\mu\nu} = \mathcal{H}(E_\lambda \bar{E}_\mu E_\nu) = g_{\lambda\mu} E_\nu + g_{\mu\nu} E_\lambda - g_{\nu\lambda} E_\mu . \qquad (24.21)$$

Now again making use of a notation given in (1.8) let $E_{\lambda\mu\nu}$ denote the antihermitian part of $E_\lambda \bar{E}_\mu E_\nu$

$$E_{\lambda\mu\nu} = \mathcal{A}(E_\lambda \bar{E}_\mu E_\nu) \doteq \frac{1}{2}(E_\lambda \bar{E}_\mu E_\nu - E_\nu \bar{E}_\mu E_\lambda) . \qquad (24.22)$$

To derive a new expression for the antihermitian quaternion (24.22) we need the lemma: if $A$, $B$, $C$ are hermitian and their components defined by

$$A = I_\mu a^\mu \quad , \quad B = I_\mu b^\mu \quad , \quad C = I_\nu c^\nu$$

then we have

$$\mathcal{A}(A\bar{B}C) = -i \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + e_1 \begin{vmatrix} a^0 & b^0 & c^0 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + e_2 \begin{vmatrix} a^0 & b^0 & c^0 \\ a^3 & b^3 & c^3 \\ a^1 & b^1 & c^1 \end{vmatrix} + e_3 \begin{vmatrix} a^0 & b^0 & c^0 \\ a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \end{vmatrix} . \qquad (24.23)$$

The lemma is proved by the application of (1.12) which gives

$$\mathcal{A}(A\bar{B}C) = \frac{1}{2}(A\bar{B}C - C\bar{B}A) = -i\,\underline{a}.(\underline{b}\wedge\underline{c}) + a_0(\underline{b}\wedge\underline{c}) + b_0(\underline{c}\wedge\underline{a}) + c_0(\underline{a}\wedge\underline{b})$$

thus proving (24.23). From (24.23) it follows that the components of $E_{\lambda\mu\nu}$ are all determinants, hence $E_{\lambda\mu\nu}$ is antisymmetrical in all its three indices. It is then easily seen that the real number

$$k_{\kappa\lambda\mu\nu} = \frac{1}{i} \bar{E}_\kappa . E_{\lambda\mu\nu} \qquad (24.24)$$

is also antisymmetrical in all its four indices. Let $k$ denote the determinant of the matrix ${}^{\alpha}t_{\beta}$, that is

$$k = \mathrm{Det}\,({}^{\alpha}t_{\beta}) = \delta^{\kappa\lambda\mu\nu}_{0123}\;{}^{0}t_{\kappa}\;{}^{1}t_{\lambda}\;{}^{2}t_{\mu}\;{}^{3}t_{\nu} \qquad (24.25)$$

where $\delta^{\kappa\lambda\mu\nu}_{0123}$ is equal to one or minus one according to whether $(\kappa\lambda\mu\nu)$ is an even or odd permutation of $(0123)$. With the help of (24.23) we find that

$$k = -k_{0123} = -\frac{1}{i}\,\bar{E}_0 \cdot E_{123}\,. \qquad (24.26)$$

Multiplication of both sides by $E^{\kappa}$ and summation over the index $\kappa$ gives

$$E_{\lambda\mu\nu} = i\,k_{\kappa\lambda\mu\nu}\,E^{\kappa}\,. \qquad (24.27)$$

which is the second identity we were looking for. Combining (24.21) and (24.27) we obtain

$$E_{\lambda}\,\bar{E}_{\mu}\,E_{\nu} = \theta_{\lambda\mu\nu} + E_{\lambda\mu\nu} = g_{\lambda\mu}E_{\nu} + g_{\mu\nu}E_{\lambda} - g_{\nu\lambda}E_{\mu} + i k_{\kappa\lambda\mu\nu}E^{\kappa} \qquad (24.28)$$

The determinant $k$ depends only on the metric tensor $g_{\mu\nu}$ since

$$g = \mathrm{Det}\,(g_{\mu\nu}) = \mathrm{Det}\,(\gamma_{\alpha\beta}\;{}^{\alpha}t_{\mu}\;{}^{\beta}t_{\nu}) = (\mathrm{Det}\,\gamma_{\alpha\beta})\cdot(\mathrm{Det}\,{}^{\alpha}t_{\mu})^2$$

and hence, from (24.2) and (24.25)

$$g = -k^2 \quad \text{or} \quad k = (-g)^{\frac{1}{2}}\,. \qquad (24.29)$$

Since $k_{\kappa\lambda\mu\nu}$ is a tensor, its suffixes can be raised by means of the tensor $g^{\alpha\beta}$. Thus

$$k^{0123} = k_{\kappa\lambda\mu\nu}\,g^{0\kappa}g^{1\lambda}g^{2\mu}g^{3\nu}.$$

On the other hand

$$k_{\kappa\lambda\mu\nu} = k_{0123}\,\delta^{0123}_{\kappa\lambda\mu\nu} = -k\,\delta^{0123}_{\kappa\lambda\mu\nu}\,,$$

Therefore

$$k^{0123} = -k\,\delta^{0123}_{\kappa\lambda\mu\nu}\,g^{0\kappa}g^{1\lambda}g^{2\mu}g^{3\nu} = -k\,\mathrm{Det}\,(g^{\mu\nu})$$

But, from (24.12) and (24.29)

$$\mathrm{Det}\,(g^{\mu\nu}) = g^{-1} = -k^{-2}$$

hence

$$k^{0123} = k^{-1}.$$

In the same manner it can be proved that

$$k^{\kappa\lambda\mu\nu} = -(k_{\kappa\lambda\mu\nu})^{-1}$$

since

$$k^{\kappa\lambda\mu\nu} = k^{0123}\,\delta^{\kappa\lambda\mu\nu}_{0123} = k^{-1}\,\delta^{\kappa\lambda\mu\nu}_{0123}\,.$$

It can also be shown, ~~that~~ from the properties of the generalized Kronecker symbols that

$$k^{\alpha\lambda\mu\nu}\,k_{\beta\lambda\mu\nu} = -3!\,\delta^{\alpha}_{\beta}\,. \qquad (24.30)$$

Hence, from (24.27) we derive the relation

$$3!\,E^{\kappa} = -\frac{1}{i}\,E_{\lambda\mu\nu}\,k^{\kappa\lambda\mu\nu} = \frac{3!}{i}\,E_{\lambda\mu\nu}\,(k_{\kappa\lambda\mu\nu})^{-1} \qquad (24.31).$$

To conclude this section two more identities related to basis quaternions will be given. The first of these is

$$E_{\alpha}\,\bar{A}\,E^{\alpha} = -2A \qquad (24.32)$$

where $A$ is any hermitian quaternion. The proof is as follows: we can always write

$$A = E^{\mu} a_{\mu}.$$

As a special case of (24.21) we have

$$E_{\alpha}\,\bar{E}^{\mu} E^{\alpha} = 2\mathcal{H}(E_{\alpha}\,\bar{E}^{\mu} E^{\alpha}) = 2E^{\mu} - \delta^{\alpha}_{\alpha} E^{\mu} = -2E^{\mu},$$

hence (24.31) follows.

The other identity is that, if $\underline{\omega}$ is an arbitrary purely vectorial complex quaternion, then

$$\bar{E}^{\alpha}\,\underline{\omega}\,E_{\alpha} = \bar{E}_{\alpha}\,\underline{\omega}\,E^{\alpha} = 0. \qquad (24.33)$$

This follows from the obvious identity

$$\bar{I}_{\alpha}\,\underline{\omega}\,I^{\alpha} = 0$$

if we use (24.15) and (24.13).

## § 25. The affine space-time of distant parallelism. The torsion tensor.

In flat space the sixteen functions ${}^{\alpha}t_{\beta}$ which define a coordinate transformation are not arbitrary but derive from four real functions $f^{\alpha}$ (24.5). If we put

$$X = I_{\alpha} \xi^{\alpha} = I_{\alpha} f^{\alpha}(x^0, x^1, x^2, x^3)$$

the equations (24.5) may be written as

$$E_{\alpha} = \partial_{\alpha} X \qquad (25.1)$$

where $\partial_{\alpha} = \partial / \partial x^{\alpha}$ and the differential quaternion (24.7) is seen to be integrable, since

$$dX = dx^{\alpha}\, \partial_{\alpha} X .$$

From (25.1) it follows that the basis vectors in flat space satisfy the condition

$$\partial_{\alpha} E_{\beta} - \partial_{\beta} E_{\alpha} = 0 \qquad (25.2)$$

We now define a ~~vario~~ frame of reference which varies from point to point by means of sixteen arbitrary real functions, or by the four arbitrary hermitian basis vectors $E_{\alpha}$ given by (24.6). This generalized coordinate system will not correspond to a coordinate transformation (24.3) since (25.2) is not satisfied and (24.7) is not integrable. A ~~quate~~ hermitian quaternion $A$, referred to this frame of reference will have coordinates $a^{\alpha}$ so that the functions ${}^{\alpha}t_{\beta}$ may be regarded as characterizing the geometry of a space based on such coordinates. Now a reciprocal frame of reference may be defined at each point by means of the functions ${}_{\alpha}t^{\beta}$ given by (24.13) associated with the ~~basis~~ contravariant basis vectors $E^{\beta}$ which are given by (24.16). We have shown that (24.16) ~~may~~ can be solved directly and the contravariant basis quaternions expressed in terms of the covariant quaternions $E_{\alpha}$ by means of the formula (24.314). Having defined the reciprocal frames by $E^{\alpha}$ and $E_{\alpha}$, by scalar multiplication of both sides of (24.18) by $E_{\beta}$ we derive the relation between covariant and contravariant components of $A$, namely

$$a_\alpha = (E_\alpha \cdot \bar{E}_\beta)\, a^\beta .$$

Since this is the rule for lowering the indices it is natural to define ~~it~~ the metric in this space by

$$g_{\alpha\beta} = E_\alpha \cdot \bar{E}_\beta = \tfrac{1}{2}(E_\alpha \bar{E}_\beta + E_\beta \bar{E}_\alpha) . \qquad (25.3)$$

The relation (24.15) follows from this definition, hence the indices of the basis vectors are also raised and lowered by means of the metric tensor (25.3) and the reciprocal metric tensor is given by

$$g^{\alpha\beta} = E^\alpha \cdot \bar{E}^\beta .$$

It must be pointed out that although the metric (25.3) is the most convenient other symmetrical tensors could also be introduced to define the metric.

If the origin of the vector A is moved to a neighbouring point by the amount and direction specified by $dX$, the basis vectors at this point will differ from the basis vectors at the original point by

$$d E_\beta = \Gamma^\gamma_{\alpha\beta}\, E_\gamma\, dx^\alpha \qquad (25.4)$$

where the scalar quantities $\Gamma^\gamma_{\alpha\beta}$ are the coefficients of the affine connection. As $E_\beta$ are given functions of $x^\alpha$, (25.4) is integrable and we can write

$$d E_\beta = (\partial_\alpha E_\beta)\, dx^\alpha .$$

Hence the 64 coefficients $\Gamma^\gamma_{\alpha\beta}$ are not arbitrary but satisfy the condition

$$\partial_\alpha E_\beta = \Gamma^\gamma_{\alpha\beta}\, E_\gamma . \qquad (25.5)$$

Using (24.16) we get

$$\Gamma^\gamma_{\alpha\beta} = \bar{E}^\gamma \cdot \partial_\alpha E_\beta = \mu t^\gamma\, \partial_\alpha\, {}^\mu t_\beta \qquad (25.6)$$

In flat space where ~~we have~~ (25.1) is satisfied we have the relation

$$\Gamma^\gamma_{\alpha\beta} = (g^{\gamma\epsilon}\, \partial_\epsilon \bar{X}) \cdot \partial_\alpha \partial_\beta X$$

which shows that in this case $\Gamma^\gamma_{\alpha\beta}$ are symmetrical in their lower indices.

We now define the parallel displacement of the vector A as in the case of flat space-time by assuming that in the displacement $dX$ the vector A has suffered no change. Thus for a parallel displacement

$$+\quad \partial_\alpha g_{\beta\gamma} = \Gamma^{\rho}_{\alpha\beta}\, g_{\gamma\rho} + \Gamma^{\rho}_{\alpha\gamma}\, g_{\beta\rho}$$

$$+\quad \partial_\beta g_{\gamma\alpha} = +\Gamma^{\rho}_{\beta\gamma}\, g_{\alpha\rho} + \Gamma^{\rho}_{\gamma\beta}\, g_{\alpha\rho} \qquad + - \Gamma^{\rho}_{\gamma\beta}\, g_{\alpha\rho} + \Gamma^{\rho}_{\gamma\beta}\, g_{\alpha\rho}$$

$$-\quad \partial_\gamma g_{\alpha\beta} = -\Gamma^{\rho}_{\gamma\alpha}\, g_{\beta\rho} - \Gamma^{\rho}_{\beta\alpha}\, g_{\gamma\rho}$$

$$2\,[\alpha\beta,\gamma] = 2\,\Lambda^{\rho}{}_{\alpha\beta}\, g_{\gamma\rho} + 2\,\Lambda^{\rho}{}_{\alpha\gamma}\, g_{\beta\rho} + 2\,\Lambda^{\rho}_{\beta\gamma}\, g_{\alpha\rho} + 2\,\Gamma^{\rho}_{\gamma\beta}\, g_{\alpha\rho}$$



we must have

$$d\,(E_\alpha a^\alpha) = 0$$

or, using (25.4)

$$E_\gamma\,(\Gamma^\gamma_{\alpha\beta}\, a^\beta\, dx^\alpha + da^\gamma) = 0$$

and since the basis vectors are linearly independent,

$$da^\gamma = -\Gamma^\gamma_{\alpha\beta}\, a^\beta\, dx^\alpha \qquad (25.7).$$

This is the parallel displacement formula for the contravariant components of a vector in an affine space. From

$$d\,(E^\gamma a_\gamma) = 0$$

it also follows that

$$dE^\gamma = -\Gamma^\gamma_{\alpha\beta}\, E^\beta\, dx^\alpha$$

is a parallel displacement. That is, the contravariant basis vectors are displaced like the contravariant components of a vector. We also find

$$da_\beta = \Gamma^\gamma_{\alpha\beta}\, a_\gamma\, dx^\alpha.$$

It is easily shown that the curvature tensor

$$R^\gamma_{\alpha\beta,\varepsilon} = \partial_\beta \Gamma^\gamma_{\alpha\varepsilon} - \partial_\alpha \Gamma^\gamma_{\beta\varepsilon} + \Gamma^\mu_{\alpha\varepsilon}\Gamma^\gamma_{\beta\mu} - \Gamma^\mu_{\beta\varepsilon}\Gamma^\gamma_{\alpha\mu} \qquad (25.8)$$

corresponding to the affine connection (25.7), vanishes identically. This can be ~~shown~~ proved by writing

$$\partial_\alpha \partial_\beta E_\varepsilon = \partial_\beta \partial_\alpha E_\varepsilon$$

and using (25.5). $R^\gamma_{\alpha\beta,\varepsilon}$ has the same form as the Riemann-Christoffel tensor, but in the Riemannian case the coefficients $\Gamma^\gamma_{\alpha\beta}$ are identical with the Christoffel symbols

$$\left\{{\gamma \atop \alpha\beta}\right\} = \tfrac{1}{2}\, g^{\gamma\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}) \qquad (25.9)$$

and are symmetrical in $\alpha$ and $\beta$. It is because the curvature tensor vanishes that we were able to interpret the basis vectors as defining a frame of reference varying from point to point in flat space.

The connection between the metric tensor and the coefficients $\Gamma^\gamma_{\alpha\beta}$ is found by differentiating (25.3) and using (25.5). It is given by

$$\partial_\varepsilon g_{\alpha\beta} = \Gamma^\gamma_{\varepsilon\beta}\, g_{\alpha\gamma} + \Gamma^\gamma_{\varepsilon\alpha}\, g_{\beta\gamma}. \qquad (25.10)$$

We have seen that the condition for $dX$ to be a perfect differential is expressed by (25.2) which can also be written

$$(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha})\, E_\gamma = 0 \qquad (25.11)$$

Identity:

$$B_{\alpha,\beta\gamma} B^{\gamma\beta\alpha} = 2\Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\gamma,\beta} + \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\beta,\gamma}$$

$$\Lambda_{\alpha\beta,\gamma}\Lambda^{\gamma\beta,\alpha} = \Lambda_{\beta\alpha,\gamma}\Lambda^{\beta\gamma,\alpha} = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\gamma,\beta}$$

Einstein:

$$J_1 = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\gamma,\beta}$$

$$J_2 = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\beta,\gamma}$$

so that

$$B_{\alpha,\beta\gamma} B^{\gamma,\beta\alpha} = 2J_1 + J_2 = 4\left(\tfrac{1}{2}J_1 + \tfrac{1}{4}J_2\right)$$

$$R = 2\Delta_\alpha B_\nu{}^{\nu\alpha} - g_{\alpha\beta} B_\mu{}^{\mu\alpha} B_\nu{}^{\nu\beta} + B_{\alpha,\beta\gamma}B^{\gamma,\beta\alpha}$$

$$B_\nu{}^{\nu\alpha} = 2\Lambda^\alpha{}_\nu{}^\nu = 2\Lambda^\alpha$$

$$J_3 = \Lambda^\alpha{}_\nu{}^\nu \Lambda_\alpha{}_\mu{}^\mu = \Lambda^\alpha\Lambda_\alpha$$

$$R = 4(\Delta_\alpha\Lambda^\alpha - J_3) + 4\left(\tfrac{1}{2}J_1 + \tfrac{1}{4}J_2\right)$$

$$\frac{1}{4}R = \Delta_\alpha\Lambda^\alpha + \underbrace{\tfrac{1}{2}J_1 + \tfrac{1}{4}J_2 - J_3}_{\text{Einstein's Lagrangian}}$$



It is proved easily (cf. Eisenhart p. ) that

$$\Lambda_{\alpha\beta}{}^{\cdot\gamma}_{\cdot} = \tfrac{1}{2}\left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right) \qquad (25.12)$$

is a tensor of the third rank. It is antisymmetrical in its lower indices and is called the torsion tensor. Its indices are here written on ~~separate~~ different vertical lines to enable us to raise and lower them separately. The condition (25.11) for the space to be Euclidian can now be written

$$\Lambda_{\alpha\beta}{}^{\cdot\gamma}_{\cdot} = 0. \qquad (25.13)$$

From (25.10) we can now derive the relation:

$$\Gamma^\gamma_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} + B_{\alpha\beta}{}^{\cdot\gamma}_{\cdot} \qquad (25.14)$$

where the tensor

$$B_{\alpha\beta}{}^{\cdot\gamma}_{\cdot} = \Lambda_{\alpha\beta}{}^{\cdot\gamma}_{\cdot} - \Lambda_{\alpha}{}^{\cdot\gamma}_{\cdot\beta} - \Lambda_{\beta}{}^{\cdot\gamma}_{\cdot\alpha} \qquad (25.15)$$

is the same as Ricci's rotation coefficients. Ricci's tensor has the property

$$B_{\alpha,\beta\gamma} = -B_{\alpha,\gamma\beta} \qquad (25.16)$$

since

$$B_{\alpha,\beta\gamma} = \Lambda_{\alpha\beta,\gamma} - \Lambda_{\beta\gamma,\alpha} - \Lambda_{\alpha\gamma,\beta}.$$

Other obvious properties of the torsion tensor and the Ricci tensor are expressed by

$$\Lambda^{\alpha}_{\cdot\alpha\gamma} = g^{\alpha\beta}\Lambda_{\alpha\beta,\gamma} = 0$$

$$B_{\alpha,\beta}{}^{\beta} = g^{\beta\gamma}B_{\alpha,\beta\gamma} = 0.$$

Contracting the tensors $\Lambda_{\alpha\beta}{}^{\cdot\gamma}_{\cdot}$ and $B^{\alpha}_{\cdot\beta\gamma}$ we find the 4-vectors

$$\Lambda_\beta = \Lambda_{\beta\alpha}{}^{\cdot\alpha}_{\cdot} = \Lambda_{\beta}{}^{\alpha}{}_{\alpha} \qquad (25.17)$$

and

$$B_\beta = B^{\alpha}_{\cdot\beta\alpha} = B_{\alpha\beta}{}^{\cdot\alpha}_{\cdot} \qquad (25.18)$$

which are related by the relation

$$B_\beta = -2\Lambda_\beta. \qquad (25.19)$$

Two more 4-vectors may be derived from the torsion tensor and the Ricci tensor by means of the antisymmetrical tensor of the fourth rank $k^{\epsilon\alpha\beta\gamma}$ of the preceding section. They are (the factor $\frac{1}{4}$ will prove useful)

$$L^\epsilon = \tfrac{1}{4}k^{\epsilon\alpha\beta\gamma}\Lambda_{\alpha\beta,\gamma} = \tfrac{1}{8}k^{\epsilon\alpha\beta\gamma}\left(\Lambda_{\alpha\beta,\gamma} - \Lambda_{\alpha\gamma,\beta}\right) \qquad (25.20)$$

and

$$R^\epsilon = k^{\epsilon\alpha\beta\gamma}B_{\alpha,\beta\gamma} = \tfrac{1}{2}k^{\epsilon\alpha\beta\gamma}\left(B_{\alpha,\beta\gamma} - B_{\beta,\alpha\gamma}\right) \qquad (25.21)$$

$$B = E^\alpha \bar{E}^\beta \Delta_\alpha E_\beta = 4E^\alpha(\Lambda_\alpha + iL_\alpha)$$

$$16 L^\alpha L_\alpha = 2 J_2 - 4 J_1$$

$$\mathcal{R} B\bar{B} = 16\Lambda^\alpha\Lambda_\alpha - 16 L^\alpha L_\alpha$$

$$\mathcal{R} B\bar{B} = 16 J_3 - 2J_2 + 4J_1$$

$$\frac{1}{8}\mathcal{R} B\bar{B} = 2J_3 - \frac{1}{4}J_2 + \frac{1}{2}J_1$$

$$\frac{1}{8}\mathcal{R} B\bar{B} = \frac{1}{2}J_1 - \frac{1}{4}J_2 + 2J_3$$

$$\frac{1}{4}R = \Delta_\alpha\Lambda^\alpha + \frac{1}{2}J_1 + \frac{1}{4}J_2 - J_3 .$$

If $\Lambda_\alpha = 0$

$$R = 2J_1 + J_2$$



We find, from (25.15)

$$R^\varepsilon = k^{\varepsilon\alpha\beta\gamma}(\Lambda_{\alpha\beta,\gamma} - \Lambda_{\beta\gamma,\alpha} - \Lambda_{\alpha\gamma,\beta}) = 4L^\varepsilon \qquad (25.22)$$

since the contributions from the second and the third terms in the bracketted expression cancel one another. To sum up, whether we start from the torsion tensor or the Ricci tensor we find the two important 4-vectors $\Lambda_\alpha$ and $L^\alpha$.

The space-time of distant parallelism was first introduced into physics by Einstein (1928). In ~~the same~~ flat space ~~way as we can~~ we were able to find the cartesian coordinates ${}^\alpha a$ of a vector with curvilinear coordinates $a^\alpha_\xi$ ~~in flat space~~ by means of the formula (24.19). In the same way, from the generalized coordinates $a^\alpha$ we can derive the invariants

$${}^\alpha a = {}^\alpha t_\beta \, a^\beta$$

which are called "Bein-components" ~~by~~ (Einstein) or "ennuple components" (Kilmister 1949). We shall call them the invariant components of the vector A. In flat space two vectors at distant points are regarded as parallel and congruent if their cartesian components are the same. Similarly in the space of distant parallelism two vectors are regarded as congruent if their invariant components are equal. In particular the invariant components are kept constant in a parallel displacement. In the unitary theory proposed by Einstein in 1928 the 4-vector $\Lambda_\alpha$ was interpreted as the electromagnetic potential 4-vector. We shall discuss this interpretation in § 32 in connection with the transformation properties of Dirac's equation.

## § 26. The two covariant differentiations $w_\alpha$ and $\Delta_\alpha$. Differential equations satisfied by the fundamental quaternions.

In the space-time of distant parallelism introduced in the preceding section we define the affine covariant derivative of a tensor $T_{\alpha\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot}$ with respect to the generalized coordinate $x^\varepsilon$ by

$$w_\varepsilon T_{\alpha\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} = \partial_\varepsilon T_{\alpha\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} - \Gamma_{\varepsilon\alpha}^{\gamma} T_{\gamma\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} - \Gamma_{\varepsilon\beta}^{\gamma} T_{\alpha\gamma}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} + \Gamma_{\varepsilon\gamma}^{\lambda} T_{\alpha\beta}{}^{\cdot\cdot\gamma\mu}{}_{\cdot\cdot} + \Gamma_{\varepsilon\gamma}^{\mu} T_{\alpha\beta}{}^{\cdot\cdot\lambda\gamma}{}_{\cdot\cdot} \qquad (26.1)$$

and the metric covariant derivative of the same tensor by

$$\Delta_\varepsilon T_{\alpha\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} = \partial_\varepsilon T_{\alpha\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} - \left\{ {\gamma \atop \varepsilon\alpha} \right\} T_{\gamma\beta}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} - \left\{ {\gamma \atop \varepsilon\beta} \right\} T_{\alpha\gamma}{}^{\cdot\cdot\lambda\mu}{}_{\cdot\cdot} + \left\{ {\lambda \atop \varepsilon\gamma} \right\} T_{\alpha\beta}{}^{\cdot\cdot\gamma\mu}{}_{\cdot\cdot} + \left\{ {\mu \atop \varepsilon\gamma} \right\} T_{\alpha\beta}{}^{\cdot\cdot\lambda\gamma}{}_{\cdot\cdot} \qquad (26.2)$$

where $\left\{ {\gamma \atop \alpha\beta} \right\}$ is given by (25.9) and $\Gamma_{\alpha\beta}^{\gamma}$ by (25.6). ~~They~~ The two coefficients are related by (25.14) where $B_{\alpha\beta}{}^{\cdot\gamma}$ is given by (25.15) in terms of the torsion tensor. For integrable coordinate systems in flat space the torsion tensor and the Ricci tensor vanish and we have 1

$$\Delta_\varepsilon = w_\varepsilon .$$

But in the space of distant parallelism, from a given tensor we can derive two different tensors by means of the two types of differentiation. ~~From~~ With these definitions it follows from (25.5) that

$$w_\alpha E_\beta = 0 \qquad (26.3)$$

that, is the covariant affine derivatives of the basis vectors vanish identically. We also find

$$w_\alpha E^\beta = 0 . \qquad (26.4)$$

If the vector A is ~~dis~~ subject to a parallel displacement, we have, from (25.7) and from the similar relation for $a_\beta$

$$w_\alpha a^\beta = w_\alpha a_\beta = 0 . \qquad (26.5)$$

Thus the affine derivatives of the components of a vector vanish in a parallel displacement. (25.10) can also be written

$$w_\varepsilon g_{\alpha\beta} = 0 \qquad (26.6).$$

The corresponding results for the metric derivation are

$$\Delta_\alpha E_\beta = B_{\alpha\beta}{}^{\cdot\gamma} E_\gamma \qquad (26.7)$$

$$\Delta_\alpha E^\gamma = - B_{\alpha\beta}{}^{\cdot\gamma} E^\beta \qquad (26.8)$$

for the basis vectors the metric derivatives of which are seen

$\Lambda^\gamma_{\alpha\beta}$ $\quad V^\gamma_\alpha(\partial_\alpha V_\beta - \partial_\beta V_\alpha)$

$\gamma = \alpha$

$$\Lambda_\beta = V^\alpha \partial_\alpha V_\beta - V^\alpha \partial_\beta V_\alpha$$
$$= \frac{dV_\beta}{ds} - \frac{1}{2}\partial_\beta V^\alpha \bar V_\alpha \qquad V^\alpha \bar V_\alpha = 1$$

$$\Lambda_\beta = \frac{dV_\beta}{ds}$$

$$\Delta_\varepsilon g_{\alpha\beta} \equiv 0$$



not to vanish. The rotation tensor was defined by Ricci by means of (26.7). In a parallel displacement, we have

$$\Delta_\alpha a_\beta = B_{\alpha\beta}{}^{\gamma} a_\gamma \qquad (26.9)$$
$$\Delta_\alpha a^\gamma = -B_{\alpha\beta}{}^{\gamma} a^\beta$$

and the metric tensor satisfies the identity

$$\Delta_\varepsilon g_{\alpha\beta} = B_{\varepsilon\beta}{}^{\gamma} g_{\alpha\gamma} + B_{\varepsilon\alpha}{}^{\gamma} g_{\beta\gamma} = B_{\varepsilon\beta\alpha} + B_{\varepsilon\alpha\beta} = 0 \qquad (26.10)$$

It is well known that both types of covariant derivatives are distributive with respect to the ordinary multiplication of scalar tensors. We now proceed to show that covariant derivation is also distributive with respect to quaternion multiplication of tensorial quaternions.

Let ${}^\mu p_\alpha$ be a set of four ($\mu = 0,1,2,3$) 4-vectors. From these sixteen quantities we can construct the four quaternions

$$P_\alpha = I_\mu {}^\mu p_\alpha$$

which are seen to transform like the basis vectors $E_\alpha$. We call them covariant ~~cov~~ quaternions of the first rank. Similarly, from a tensor

$${}^{\lambda\mu}T^{\alpha\beta\gamma\cdots}$$

we can construct the contravariant quaternions of the third rank

$$Q^{\alpha\beta\gamma} = I_\lambda \bar I_\mu {}^{\lambda\mu}T^{\alpha\beta\gamma} \text{ or } R^{\alpha\beta\gamma} = I_\lambda I_\mu {}^{\lambda\mu}T^{\alpha\beta\gamma}.$$

Now let $P_\alpha$ ~~covariant qu~~ denote a set of covariant quaternions of the first ~~order~~ rank and $Q^\alpha$ contravariant quaternions of the first rank. We define covariant differentiation of a tensorial quaternion in the same way as in (26.1) and (26.2). Hence

$$w_\varepsilon(P_\alpha Q^\beta) = \partial_\varepsilon(P_\alpha Q^\beta) - \Gamma^\gamma_{\varepsilon\alpha} P_\gamma Q^\beta + \Gamma^\beta_{\varepsilon\gamma} P_\alpha Q^\gamma \qquad \cancel{(26.11)}$$

and since

$$\partial_\varepsilon(P_\alpha Q^\beta) = (\partial_\varepsilon P_\alpha) Q^\beta + P_\alpha \partial_\varepsilon Q^\beta$$

we obtain

$$w_\varepsilon(P_\alpha Q^\beta) = (w_\varepsilon P_\alpha) Q^\beta + P_\alpha (w_\varepsilon Q^\beta). \qquad (26.11)$$

A similar proof holds for the differentiation $\Delta_\varepsilon$ and for the multiplication of two general tensorial quaternions of arbitrary rank.

With the help of this simple theorem we shall now derive some differential quaternion identities that hold in the space-time of distant parallelism. First let us call fundamental quaternions the tensorial quaternions derived from the basis vectors by means of quaternion multiplication.

Thus the fundamental quaternions of the first rank are the basis quaternions $E_\alpha$.

From the product $E_\alpha \bar{E}_\beta$ we can derive two fundamental quaternions of the second rank. The first is the symmetrical part of this product and is identical with the scalar metric tensor $g_{\alpha\beta}$. The second is the antisymmetrical part of the product. It is a purely vectorial quaternion and will be denoted by

$$\underline{E}_{\alpha\beta} = \frac{1}{2}\left(E_\alpha \bar{E}_\beta - E_\beta \bar{E}_\alpha\right). \qquad (26.12)$$

Hence we can write

$$E_\alpha \bar{E}_\beta = g_{\alpha\beta} + \underline{E}_{\alpha\beta} \qquad (26.13)$$

From the product $E_\alpha \bar{E}_\beta E_\gamma$ which can be written

$$E_\alpha \bar{E}_\beta E_\gamma = G_{\alpha\beta\gamma} + E_{\alpha\beta\gamma}$$

we have already derived the hermitian quaternion $G_{\alpha\beta\gamma}$ symmetrical in $\alpha$ and $\gamma$ and given by (24.21) and the antihermitian quaternion $E_{\alpha\beta\gamma}$ antisymmetrical in all its indices and given by (24.27).

Similarly we can derive other fundemental quaternions of the fourth rank from the product $E_\alpha \bar{E}_\beta E_\gamma \bar{E}_\varepsilon$. The most important of these is the scalar quaternion $k_{\alpha\beta\gamma\varepsilon}$ antisymmetrical in all its indices and whose absolute value is given by (24.25) or (24.29). It is the imaginary part of the scalar part of $E_\alpha \bar{E}_\beta E_\gamma \bar{E}_\varepsilon$.

Now from (26.11) we obtain

$$w_\varepsilon E_\alpha \bar{E}_\beta = w_\varepsilon E^\alpha \bar{E}^\beta = 0.$$

The scalar part of one of these equations gives the identity (26.6) and the vectorial part gives

$$w_\varepsilon \underline{E}^{\alpha\beta} = 0 \qquad (26.14)$$

We also find

$$w_\varepsilon k_{\alpha\beta\gamma\varphi} = 4\,\Gamma^{\mu}_{\varepsilon\alpha}\, k_{\mu\beta\gamma\varphi} = 0$$

Multiplying both sides by $k^{\alpha\beta\gamma\eta}$ and using (24.30) we have

$$k^{\alpha\beta\gamma\eta}\, \partial_\varepsilon k_{\alpha\beta\gamma\eta} = -4!\, \Gamma_{\varepsilon\mu}^{\mu}$$

and from (24.26)

$$k^{-1} \partial_\varepsilon k = \Gamma_{\varepsilon\mu}^{\mu} \qquad \cancel{(26.15)}$$

But, since

$$B_{\varepsilon\mu}{}^{\mu} = 0$$

we have, from (25.14)

$$k^{-1} \partial_\varepsilon k = \left\{ {\mu \atop \varepsilon\mu} \right\} \qquad (26.15)$$

We now have

$$\Delta_\varepsilon E^\varepsilon = \partial_\varepsilon E^\varepsilon + \left\{ {\mu \atop \varepsilon\mu} \right\} E^\varepsilon = k^{-1} \partial_\varepsilon k E^\varepsilon$$

But from (26.8) and (25, 19)

$$\Delta_\varepsilon E^\varepsilon = - B_{\alpha\beta}{}^{\cdot\cdot\alpha} E^\beta = 2 \Lambda_{\beta\alpha}{}^{\cdot\cdot\alpha} E^\beta = \cancel{\pm} 2 \Lambda_\beta E^\beta \qquad (26.16)$$

Similarly we find

$$\Delta_\alpha \underline{E}^{\alpha\beta} = k^{-1} \partial_\alpha k\, \underline{E}^{\alpha\beta} = \cancel{\pm} 2 \Lambda_\alpha \underline{E}^{\alpha\beta} - \Lambda_{\mu\nu}{}^{\cdot\cdot\beta} \underline{E}^{\mu\nu} \qquad (26.17)$$

and

$$\Delta_\alpha \underline{E}_{\beta\gamma} + \Delta_\beta \underline{E}_{\gamma\alpha} + \Delta_\gamma \underline{E}_{\alpha\beta} = \partial_\alpha \underline{E}_{\beta\gamma} + \partial_\beta \underline{E}_{\gamma\alpha} + \Delta_\gamma \underline{E}_{\alpha\beta}$$
$$= 2 (\Lambda_{\alpha\beta}{}^{\varepsilon} \underline{E}_{\varepsilon\gamma} + \Lambda_{\beta\gamma}{}^{\varepsilon} \underline{E}_{\varepsilon\alpha} + \Lambda_{\gamma\alpha}{}^{\varepsilon} \underline{E}_{\varepsilon\beta}) \qquad (26.18)$$

It is seen that if the torsion tensor vanishes we have

$$\partial_\varepsilon k E^\varepsilon = 0 \qquad (26.19)$$

and each component of the quaternion $\underline{E}_{\alpha\beta}$ obeys Maxwell's equations ~~for~~ in vacuum. ~~If~~ $\underline{E}_{\alpha\beta}$ is given by

$$\underline{E}_{\alpha\beta} = \mathcal{V}(E_\alpha \bar{E}_\beta) = \frac{1}{2} I_\mu \bar{I}_\nu ({}^\mu t_\alpha\, {}^\nu t_\beta - {}^\mu t_\beta\, {}^\nu t_\alpha)$$

so that for instance

$$f_{\alpha\beta} = \frac{1}{2}({}^0 t_\alpha\, {}^3 t_\beta - {}^0 t_\beta \cdot {}^3 t_\alpha)$$

is a solution of Maxwell's equations in vacuum if the torsion tensor corresponding to ${}^\mu t_\alpha$ vanishes.

Besides tensorial quaternions we can also consider invariant quaternions. Let, for example $T_{\alpha\beta}{}^{\cdot\cdot\gamma}$ denote a tensor. Then the quaternion

$$Q = E^\alpha \bar{E}^\beta E_\gamma T_{\alpha\beta}{}^{\cdot\cdot\gamma} \qquad (26.20)$$

will be an invariant. Of course other invariant quaternions may be contructed by means of such a tensor. Each complex component of $Q$, namely

$${}_\lambda Q = \bar{I}_\lambda \cdot Q$$

will represent some invariant derived from $T_{\alpha\beta}{}^{\cdot\cdot\gamma}$ by means of the functions ${}^{\mu}t_{\alpha}$. The components of each quaternion will be eight invariants derived from the ~~generaliti~~ original tensor. For instance from the symmetrical tensor $S_{\alpha\beta}$ we can derive the invariant scalar

$$E^{\alpha}\bar{E}^{\beta}S_{\alpha\beta} = g^{\alpha\beta}S_{\alpha\beta} = S_{\alpha}^{\cdot\alpha}$$

which is the contracted form of $S_{\alpha}{}^{\beta}$ or the ~~eight~~ four invariant components of the hermitian quaternion

$$S = E^{\alpha}E^{\beta}S_{\alpha\beta} = \tfrac{1}{2}(E^{\alpha}E^{\beta} + E^{\beta}E^{\alpha})S_{\alpha\beta}.$$

We have

$$S = \tfrac{1}{2}(I^{\mu}I^{\nu} + I^{\nu}I^{\mu})\,{}_{\mu}t^{\alpha}\,{}_{\nu}t^{\beta}S_{\alpha\beta}$$
$$= \tfrac{1}{2}(I^{\mu}I^{\nu} + I^{\nu}I^{\mu})\,{}_{\mu\nu}S.$$

If we put

$$\Gamma^{\mu\nu}_{\lambda}\eta = \tfrac{1}{2}\bar{I}_{\lambda}\cdot(I^{\mu}I^{\nu} + I^{\nu}I^{\mu}) \qquad (26.21)$$

then the four invariants that we can derive from $S_{\alpha\beta}$ are

$${}_{\lambda}S = (\Gamma^{\mu\nu}_{\lambda}\eta)({}_{\mu\nu}S) = (\Gamma^{\mu\nu}_{\lambda}\eta)\,{}_{\mu}t^{\alpha}\,{}_{\nu}t^{\beta}S_{\alpha\beta}.$$

Differentiation of the invariant quaternion (26.20) gives

$$\partial_{\varepsilon}Q = w_{\varepsilon}Q = (w_{\varepsilon}E^{\alpha}\bar{E}^{\beta}E_{\gamma})T_{\alpha\beta}{}^{\gamma} + E^{\alpha}\bar{E}^{\beta}E_{\gamma}\,w_{\varepsilon}T_{\alpha\beta}{}^{\gamma}$$

and since the first term vanish we have

$$\partial_{\varepsilon}Q = w_{\varepsilon}(E^{\alpha}\bar{E}^{\beta}E_{\gamma})T_{\alpha\beta}{}^{\gamma} = E^{\alpha}\bar{E}^{\beta}E_{\gamma}\,w_{\varepsilon}T_{\alpha\beta}{}^{\gamma}, \qquad (26.22)$$

that is the affine differentiation operator commutes with fundamental quaternions. We shall make constant use of this property.

## § 27. Further identities satisfied by the fundamental quaternions. Tensor components of invariant quaternions.

Before deriving certain algebraic identities we define duality. Consider the vector components $a_\varepsilon$, the antisymmetrical tensor $b_{\varrho\sigma}$ and the antisymmetrical tensor of the third rank $c_{\varrho\sigma\tau}$. Duality will be defined by means of the antisymmetrical tensors of the fourth rank $k_{\alpha\beta\gamma\varepsilon}$ and $k^{\alpha\beta\gamma\varepsilon}$ that we have introduced in § 24. Thus:

$$\text{dual } a_\varepsilon = \tilde{a}^{\alpha\beta\gamma} = k^{\alpha\beta\gamma\varepsilon} a_\varepsilon \tag{27.1}$$

$$\text{dual } b_{\varrho\sigma} = \tilde{b}^{\alpha\beta} = \frac{1}{2!} k^{\alpha\beta\varrho\sigma} b_{\varrho\sigma} \tag{27.2}$$

$$\text{dual } c_{\varrho\sigma\tau} = \tilde{c}^{\alpha} = \frac{1}{3!} k^{\alpha\varrho\sigma\tau} c_{\varrho\sigma\tau} \tag{27.3}$$

or, on lowering and raising the indices

$$\text{dual } a^\varepsilon = \tilde{a}_{\alpha\beta\gamma} = k_{\alpha\beta\gamma\varepsilon} a^\varepsilon \tag{27.4}$$

$$\text{dual } b^{\varrho\sigma} = \tilde{b}_{\alpha\beta} = \frac{1}{2!} k_{\alpha\beta\varrho\sigma} b^{\varrho\sigma} \tag{27.5}$$

$$\text{dual } c^{\varrho\sigma\tau} = \tilde{c}_{\alpha} = \frac{1}{3!} k_{\alpha\varrho\sigma\tau} c^{\varrho\sigma\tau} \tag{27.6}$$

With these definitions, from (24.30) we derive

$$\text{dual dual } a_\varepsilon = \text{dual } \tilde{a}^{\alpha\beta\gamma} = \frac{1}{3!} k_{\varepsilon\alpha\beta\gamma} k^{\alpha\beta\gamma\sigma} a_\sigma = a_\varepsilon .$$

$$\text{dual dual } b_{\varrho\sigma} = \frac{1}{2!} \text{dual } k^{\alpha\beta\lambda\mu} b_{\lambda\mu} = \frac{1}{2!} k_{\varrho\sigma\alpha\beta} k^{\alpha\beta\lambda\mu} b_{\lambda\mu}$$

But we have

$$k_{\varrho\sigma\alpha\beta} = -k\, \delta^{0123}_{\varrho\sigma\alpha\beta} \tag{27.7}$$

and

$$k^{\alpha\beta\lambda\mu} = k^{-1} \delta^{\alpha\beta\lambda\mu}_{0123} . \tag{27.7}$$

Hence, using the properties of the generalized Kronecker symbols (cf. Brand pp. 353, 369) we find

$$k_{\varrho\sigma\alpha\beta} k^{\lambda\mu\alpha\beta} = -\delta^{\lambda\mu\alpha\beta}_{\varrho\sigma\alpha\beta} = -2!\, \delta^{\lambda\mu}_{\varrho\sigma}$$

where

$$\delta^{\lambda\mu}_{\varrho\sigma} = (\delta^\lambda_\varrho \delta^\mu_\sigma - \delta^\lambda_\sigma \delta^\mu_\varrho)$$

~~so that~~ Hence

$$\text{dual dual } b_{\varrho\sigma} = -b_{\varrho\sigma} .$$

When applied to fundamental quaternions the same definitions give

$$\text{dual } E_\varepsilon = k^{\alpha\beta\gamma\varepsilon} E_\varepsilon = \frac{1}{i} E^{\alpha\beta\gamma}$$

so that we have

$$\tilde{E}^{\alpha\beta\gamma} = -i\, E^{\alpha\beta\gamma} \quad \text{or} \quad E_{\alpha\beta\gamma} = i\, \tilde{E}_{\alpha\beta\gamma} \,. \qquad (27.8)$$

We know wish to prove the following important result:

$$\underline{E}_{\alpha\beta} = i\, \underline{\tilde{E}}_{\alpha\beta} \qquad (27.9)$$

which is of the same form as (27.8). $\underline{\tilde{E}}_{\alpha\beta}$ is defined by

$$\underline{\tilde{E}}_{\alpha\beta} = \text{dual } E^{\rho\sigma} = \frac{1}{2!}\, k_{\alpha\beta\rho\sigma}\, E^{\rho\sigma} \,. \qquad (27.10)$$

To prove (27.9) we need the lemma

$$\underline{E}_{\alpha\beta} \cdot \underline{E}_{\rho\sigma} = (g_{\alpha\sigma} g_{\beta\rho} - g_{\alpha\rho} g_{\beta\sigma}) - i\, k_{\alpha\beta\rho\sigma} \qquad (27.11)$$

which is proved by replacing $E_{\alpha\beta}$ by its expression (26.12) and by using the identity (24.20). Now let $f^{\alpha\beta}$ denote an arbitrary real antisymmetrical tensor and consider the ~~qu~~ vectorial invariant quaternion

$$\underline{F} = \frac{1}{2}\, \underline{E}_{\alpha\beta}\, f^{\alpha\beta} \,.$$

We have, using (27.11) and the definition (27.5)

$$\underline{E}_{\rho\sigma} \cdot \underline{F} = \frac{1}{2} (\underline{E}_{\alpha\beta} \cdot \underline{E}_{\rho\sigma})\, f^{\alpha\beta} = (f_{\sigma\rho} - i\, \tilde{f}_{\rho\sigma}) \qquad (27.12)$$

Hence we can write

$$\underline{F} = \frac{1}{2}\, \underline{E}_{\alpha\beta}\, f^{\alpha\beta} = \frac{1}{4}\, \underline{E}^{\rho\sigma}\, \mathcal{R}\{ \underline{E}_{\rho\sigma} \cdot \underline{E}_{\alpha\beta}\, f^{\alpha\beta} \} \,.$$

~~Hence~~ Similarly the quaternion $i\underline{F}$ is represented by

$$\frac{1}{2}\, i\, \underline{E}_{\alpha\beta}\, f^{\alpha\beta} = \frac{1}{4}\, \underline{E}^{\rho\sigma}\, \mathcal{R}\{ \underline{E}_{\rho\sigma} \cdot i\underline{E}_{\alpha\beta}\, f^{\alpha\beta} \} = \frac{1}{4}\, \underline{E}^{\rho\sigma}\, k_{\rho\sigma\alpha\beta}\, f^{\alpha\beta} \,.$$

or, on using (27.10)

$$i\,\underline{F} = \frac{1}{2}\, i\, E_{\alpha\beta}\, f^{\alpha\beta} = \frac{1}{2}\, \tilde{E}_{\alpha\beta}\, f^{\alpha\beta} \,.$$

so that for an arbitrary tensor $f^{\alpha\beta}$ we must have

$$(E_{\alpha\beta} - i\,\tilde{E}_{\alpha\beta})\, f^{\alpha\beta} = 0$$

and this proves (27.9).

We now come to our main result:

Any invariant quaternion $Q$ may be expressed in one of the forms

$$Q = E_\alpha\, a^\alpha + \frac{1}{3!}\, E^{\lambda\mu\nu}\, b_{\lambda\mu\nu} \qquad (27.13)$$

or

$$Q = u + i\, \frac{1}{4!}\, k^{\lambda\mu\nu\varepsilon}\, v_{\lambda\mu\nu\varepsilon} + \frac{1}{2}\, \underline{E}^{\rho\sigma}\, f_{\rho\sigma} \qquad (27.14)$$

where $a^\alpha$ represents a real 4-vector, $b_{\lambda\mu\nu}$ a real antisymmetrical tensor of rank 3, $u$ a real invariant scalar, $v_{\lambda\mu\nu\varepsilon}$ a real scalar antisymmetrical tensor of rank 4 and $f_{\rho\sigma}$ a real antisymmetrical tensor of rank 2.

To prove (27.13) we multiply both sides scalarly by $\bar{E}^{\varrho}$:

$$\bar{E}^{\varrho}.\,Q = a^{\varrho} - i\frac{1}{3!}k^{\varrho\lambda\mu\nu}b_{\lambda\mu\nu} = a^{\varrho} - i\tilde{b}^{\varrho}.$$

Hence $a^{\varrho}$ and $\tilde{b}^{\varrho}$ (hence $b_{\lambda\mu\nu}$) are determined uniquely by the relations

$$a^{\varrho} = \mathcal{R}(\bar{E}^{\varrho}.\,Q)$$
$$\tilde{b}^{\varrho} = \mathcal{R}(i\bar{E}^{\varrho}.\,Q).$$

The dual vector of an antisymmetrical tensor of rank 3 is a pseudo-vector. Therefore any complex quaternion may be regarded as having vector and pseudo-vector components. Its hermitian part corresponds to the vector and its antihermitian part corresponds to the pseudovector.

~~The~~ In the relation (27.14), the dual quantity

$$\tilde{v} = \frac{1}{4!}k^{\lambda\mu\nu\varepsilon}v_{\lambda\mu\nu\varepsilon} \qquad (27.15)$$

of the antisymmetrical tensor of the fourth rank is a pseudo-scalar. $u$ and $\tilde{v}$ are determined uniquely by

$$u + i\tilde{v} = \mathcal{S}(Q) \qquad (27.16)$$

and from (27.12) $f_{\alpha\beta}$ is determined by

$$f_{\alpha\beta} = -\mathcal{R}(\underline{E}_{\alpha\beta}.\,\underline{F})$$

where

$$\underline{F} = \mathcal{V}(Q) \qquad (27.17)$$

Hence any ~~real~~ invariant quaternion may be also regarded as having an invariant scalar component, a pseudo-invariant scalar component and ~~an~~ real antisymmetrical tensor components. The first two are associated with its scalar part and the six-vector with its vectorial part.

There is also an alternative way of expressing (27.17) by means of an antisymmetrical tensor. Let

$$\underline{F} = \frac{1}{2i}E^{\varrho\sigma}f'_{\varrho\sigma}$$

where $f'_{\varrho\sigma}$ is again real. ~~From~~ By (27.9), this gives

$$\underline{F} = \frac{1}{2}\tilde{E}^{\varrho\sigma}f'_{\varrho\sigma} = \frac{1}{2}E^{\varrho\sigma}\tilde{f}'_{\varrho\sigma}$$

Hence we find

$$f'_{\varrho\sigma} = \tilde{f}_{\varrho\sigma}$$

so that (27.14) may be written as

$$Q = u + i\tilde{v} + \frac{1}{2}E^{\varrho\sigma}f_{\varrho\sigma} = u + i\tilde{v} + \frac{1}{2i}E^{\varrho\sigma}\tilde{f}_{\varrho\sigma}.$$

We note that $f_{\varrho\sigma}$ and $\tilde{f}_{\varrho\sigma}$ may be regarded as different representations of the same six-vector $\underline{F}$.

# CHAPTER VII
# Invariant and Covariant Formulation of Field Equations.

## § 28. Invariant and covariant forms of the equations of Maxwell and Lanczos.

Let again $x^\alpha$ denote the generalized coordinates in the affine space-time of distant parallelism or the coordinates in curvilinear coordinate system in flat space-time. We introduce the invariant differential operator

$$D = \bar{E}^\alpha \partial/\partial x^\alpha = \bar{E}^\alpha \partial_\alpha \qquad (28.1)$$

In the cartesian coordinate system defined by (24.1) we have

$$D = \bar{I}^\alpha \partial/\partial \xi^\alpha = \Sigma I_\alpha \partial/\partial \xi^\alpha = \frac{\partial}{\partial \xi^0} + ie_1 \frac{\partial}{\partial \xi^1} + ie_2 \frac{\partial}{\partial \xi^2} + ie_3 \frac{\partial}{\partial \xi^3} \qquad (28.2)$$

It is well known (cf. Conway 1911, Klein 1927) that Maxwell's equations in flat space and in cartesian coordinates ~~may~~ can be written as

$$\underline{E}(DA) = \underline{F}^\times \qquad (28.3)$$

$$D\underline{F} = J^\times \qquad (28.4)$$

where $D$ is given by (28.2). $\underline{F}$ corresponds to the electromagnetic field, ~~and~~ $A$ to the potential vector and $J$ to the current density 4-vector. Now we can regard these quaternion equations as invariant. In the case of generalized coordinates $x^\alpha$, the invariant operator $D$ takes the form (28.1). ~~as~~ The components of $A$, $J$ and $\underline{F}$ are determined by

$$A = E^\beta A_\beta \qquad (28.5)$$

$$J = E_\beta J^\beta \qquad (28.6)$$

$$\underline{F} = \frac{1}{2} \underline{E}^{\alpha\beta} f_{\alpha\beta} = \frac{1}{2} E^\alpha \bar{E}^\beta f_{\alpha\beta} \qquad (28.7)$$

where $f_{\alpha\beta}$ are the antisymmetrical tensor components of the invariant vectorial quaternion $\underline{F}$. Inserting these expressions in (28.3) ~~we obtain~~ and using (26.22) we obtain

$$\bar{E}^\alpha E^\beta w_\alpha A_\beta = \frac{1}{2} \bar{E}^\alpha E^\beta f_{\alpha\beta} \qquad (28.8)$$

Taking the scalar parts of both sides we find, by (25.3)

$$g^{\alpha\beta} w_\alpha A_\beta = w_\alpha A^\alpha = 0 \qquad (28.9)$$

which is the Lorentz condition. Taking the vectorial parts of both sides in (28.8) we find

$$V^{\times}(\bar{E}^\alpha E^\beta w_\alpha A_\beta) = \tfrac{1}{2}(\underline{E}^{\alpha\beta})^\times (w_\alpha A_\beta - w_\beta A_\alpha) = \tfrac{1}{2}(\underline{E}^{\alpha\beta})^\times f_{\alpha\beta}$$

Hence, since $f_{\alpha\beta}$ is real

$$w_\alpha A_\beta - w_\beta A_\alpha = f_{\alpha\beta} \qquad (28.10)$$

which is a covariant relation between the covariant components of the potential vector and those of the electromagnetic field. (28.10) and (28.9) are equivalent to (28.3) and they depend on the torsion tensor. For instance (28.9) can be written, from (26.2) and (25.19)

$$\Delta_\alpha A^\alpha = k^{-1}\partial_\alpha k A^\alpha = -2\Lambda_\alpha A^\alpha$$

where $\Delta_\alpha$ denotes metric differentiation. (28.10) gives

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - 2\Lambda_{\alpha\beta}{}^{\cdot\gamma} A_\gamma = f_{\alpha\beta} \qquad (28.11)$$

again, for integrable coordinate systems the torsion tensor vanishes and (28.11) reduces to the familiar relation between field and potential.

Now substitution of (28.6) and (28.7) in (28.4) gives

$$\tfrac{1}{2}\bar{E}^\alpha E_\beta \bar{E}_\gamma w_\alpha f^{\beta\gamma} = \bar{E}_\gamma J^\gamma$$

Taking the complex conjugate of both sides and using the identity (24.28) we have

$$\tfrac{1}{2} G^{\alpha}{}_{\cdot\beta\gamma}^{\;\cdot\cdot} w_\alpha f^{\beta\gamma} + \tfrac{1}{2} E^{\alpha\beta\gamma} w_\alpha f_{\beta\gamma} = E_\gamma J^\gamma \qquad (28.12)$$

where the hermitian quaternion $G_{\alpha\beta\gamma}$ is given by (24.21) and the antihermitian quaternion $E_{\alpha\beta\gamma}$ by (24.27). Taking the antihermitian parts of both sides of (28.12) we obtain

$$\tfrac{1}{2} E^{\alpha\beta\gamma} w_\alpha f_{\beta\gamma} = \tfrac{i}{2} k^{\varepsilon\alpha\beta\gamma} E_\varepsilon w_\alpha f_{\beta\gamma}$$
$$= \tfrac{i}{2} E_\varepsilon w_\alpha k^{\varepsilon\alpha\beta\gamma} f_{\beta\gamma} = 0$$

Hence

$$w_\alpha \tilde{f}^{\varepsilon\alpha} = 0 \qquad (28.13)$$

Taking the hermitian part of (28.12) we find

$$\tfrac{1}{2} G^{\alpha}{}_{\cdot\beta\gamma}^{\;\cdot\cdot} w_\alpha f^{\beta\gamma} = E_\gamma J^\gamma$$

But, by (24.21)

$$\tfrac{1}{2} G^{\alpha}{}_{\cdot\beta\gamma} w_\alpha f^{\beta\gamma} = E_\gamma w_\alpha f^{\alpha\gamma},$$

Hence

$$w_\alpha f^{\alpha\gamma} = J^\gamma \qquad (28.14)$$

This last relation can also be written

$$k^{-1}\partial_\alpha k f^{\alpha\gamma} + 2\Lambda_\alpha f^{\alpha\gamma} + \Lambda_{\mu\nu}{}^{\gamma} f^{\mu\nu} = J^\gamma \qquad (28.15)$$

and it takes the familiar form when the torsion tensor vanishes.

In § 7 we have given the quaternion form of Lanczos' equations for a particle of spin $\frac{1}{2}$ in cartesian coordinates. The complex quaternions $\psi$ and $\chi$ which appear in these quaternion may be regarded as invariant so that in the space-time of distant parallelism or in general coordinate systems we can still write

$$D\psi = \mu \chi^\times \qquad (28.15)$$

$$D\chi = -\mu \psi^\times . \qquad (28.16)$$

Now let us choose for $\psi$ the tensor representation (27.13) and for $\chi$ the representation (27.14).

$$\psi = E^\beta a_\beta + \frac{1}{3!} E_{\lambda\mu\nu} b^{\lambda\mu\nu} = E^\beta(a_\beta + i\tilde{b}_\beta) \qquad (28.17)$$

$$\chi = u + \frac{i}{4!} k^{\lambda\mu\nu\epsilon} v_{\lambda\mu\nu\epsilon} + \frac{1}{2} E^{\alpha\beta} f_{\alpha\beta} = u + i\tilde{v} + \frac{1}{2} E^{\alpha\beta} f_{\alpha\beta} \qquad (28.18)$$

We have, using (28.1)

$$\mathcal{S}(D\psi) = w_\alpha a^\alpha + i w_\alpha \tilde{b}^\alpha = u - i\tilde{v} \qquad (28.19)\ (28.20)$$

$$\mathcal{V}(D\psi) = \frac{1}{2}(E^{\alpha\beta})^\times[(w_\alpha a_\beta - w_\beta a_\alpha) + i(w_\alpha \tilde{b}_\beta - w_\beta \tilde{b}_\alpha)] = \frac{1}{2}(E^{\alpha\beta})^\times f_{\alpha\beta}$$

Now from (27.11) it follows that, for any two real antisymmetrical tensors $R_{\alpha\beta}$ and $S_{\alpha\beta}$ we have

$$R_{\alpha\beta} = \tilde{S}_{\alpha\beta} = \frac{1}{2} k_{\alpha\beta\rho\sigma} S^{\rho\sigma}$$

if

$$E^{\alpha\beta}(R_{\alpha\beta} + i S_{\alpha\beta}) = 0 . \qquad (28.21)$$

The equation (28.20) has the form (28.21) if we put

$$-S_{\alpha\beta} = w_\alpha \tilde{b}_\beta - w_\beta \tilde{b}_\alpha$$

$$\text{and } R_{\alpha\beta} = w_\alpha a_\beta - w_\beta a_\alpha - f_{\alpha\beta}$$

Now we evaluate $\tilde{S}_{\alpha\beta}$. We have

$$\tilde{S}_{\alpha\beta} = -\frac{1}{2} k_{\alpha\beta\rho\sigma}(w^\rho \tilde{b}^\sigma - w^\sigma \tilde{b}^\rho) = -\frac{1}{3!} k_{\alpha\beta\rho\sigma} k^{\sigma\lambda\mu\nu} w^\rho b_{\lambda\mu\nu}$$

and since

$$k_{\alpha\beta\rho\sigma} k^{\sigma\lambda\mu\nu} = \delta^{\sigma\lambda\mu\nu}_{\sigma\alpha\beta\rho} = \delta^{\lambda\mu\nu}_{\alpha\beta\rho} = (\delta^{\lambda\mu}_{\alpha\beta}\delta^\nu_\rho + \delta^{\lambda\mu}_{\beta\rho}\delta^\nu_\alpha + \delta^{\lambda\mu}_{\rho\alpha}\delta^\nu_\beta)$$

we obtain

$$\tilde{S}_{\alpha\beta} = -\frac{1}{2}\delta^{\lambda\mu}_{\alpha\beta} w^\rho b_{\lambda\mu\rho} = w^\rho b_{\rho\alpha\beta}$$

Hence (28.20) leads to the covariant equation

$$\mu f_{\alpha\beta} = w_\alpha a_\beta - w_\beta a_\alpha - w^\rho b_{\rho\alpha\beta} \qquad (28.22)$$

We now turn to the equation (28.16) which can be written in the form

$$E^{\alpha} w_{\alpha}(u - i\tilde{v}) + \tfrac{1}{2}(G^{\alpha\beta\gamma} + E^{\alpha\beta\gamma}) w_{\alpha} f_{\beta\gamma} = -\mu E^{\alpha} a_{\alpha} - \mu \tfrac{1}{3!} E^{\lambda\mu\nu} b_{\lambda\mu\nu}$$

where $G^{\alpha\beta\gamma}$ is given by (24.21) and $E^{\alpha\beta\gamma}$ by (24.27).

Taking the hermitian parts of both sides we find

$$w_{\alpha} u + w^{\beta} f_{\beta\alpha} = -\mu a_{\alpha} \qquad (28.23)$$

By (24.31), the antihermitian parts of both sides give

$$w^{\lambda} v_{\lambda\alpha\beta\gamma} + (w_{\alpha} f_{\beta\gamma} + w_{\beta} f_{\gamma\alpha} + w_{\gamma} f_{\alpha\beta}) = -\mu b_{\alpha\beta\gamma} \qquad (28.24)$$

We also write (28.19) as two separate real equations:

$$w_{\alpha} a^{\alpha} = \mu u \qquad (28.25)$$

$$w_{\alpha} \tilde{b}^{\alpha} = -\mu \tilde{v}$$

This last equation can also be written as

$$w_{\alpha} b_{\lambda\mu\nu} - w_{\lambda} b_{\alpha\mu\nu} - w_{\mu} b_{\lambda\alpha\nu} - w_{\nu} b_{\lambda\mu\alpha} = -\mu v_{\alpha\lambda\mu\nu} \qquad (28.26)$$

Now (28.25), (28.26), (28.22), (28.23) and (28.24) are the covariant equivalents of the invariant equations (28.15) and (28.16).

That Lanczos' equations describe a particle of spin $\frac{1}{2}$ is seen from the fact that if we put

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

the equations (28.15) and (28.16) may be written as

$$\gamma^{\mu} \partial_{\mu} \Psi = \mu \Psi$$

where $\gamma^{\mu}$ represent the operators

$$\gamma^{\mu} = \begin{pmatrix} 0 & -\bar{E}^{\mu}(\;)^{\times} \\ \bar{E}^{\mu}(\;)^{\times} & 0 \end{pmatrix}.$$

Here $(\;)^{\times}$ stands for the operation of taking the complex conjugate of $\Psi$. We have

$$\tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}) = \begin{pmatrix} -\frac{1}{2}(\bar{E}^{\mu}E^{\nu} + \bar{E}^{\nu}E^{\mu}) & 0 \\ 0 & -\frac{1}{2}(\bar{E}^{\mu}E^{\nu} + \bar{E}^{\nu}E^{\mu}) \end{pmatrix} = -g^{\mu\nu}$$

These are Tetrode's commutation relations for the operators $\gamma^{\mu}$ in a general coordinate system in the case of a particle of spin $\frac{1}{2}$.

## §29. Wave equations for particles of spin 1 as degenerate cases of Lanczos' equations.

Lanczos' equations involve a wave function with 16 real or 8 complex components. We propose to show in this section that all equations for a particle of spin 1, including an equation recently discovered by Bhabbha (1949) may be regarded as degenerate cases of Lanczos' equations for a particle of spin $\frac{1}{2}$. We distinguish the following cases:

a – $\psi$ is hermitian and given by

$$\psi = E^{\beta} a_{\beta}$$

$\chi$ is a real scalar. We can represent $\chi$ by the invariant $u$.

$$\chi = u .$$

The covariant Lanczos equations reduce to the two equations

$$\begin{cases} w_{\alpha} a^{\alpha} = \mu u \\ w_{\alpha} u = -\mu a^{\alpha} \end{cases} \qquad (29.1)$$

b – $\psi$ is antihermitian and $\chi$ is an imaginary scalar. We put

$$\psi = \frac{1}{3!} E^{\rho\sigma\tau} b_{\rho\sigma\tau} \quad , \quad \chi = \frac{i}{4!} k^{\rho\sigma\tau\varepsilon} v_{\rho\sigma\tau\varepsilon} .$$

The surviving covariant equations are

$$\begin{cases} w_{\rho} b_{\sigma\tau\varepsilon} - w_{\sigma} b_{\rho\tau\varepsilon} - w_{\tau} b_{\sigma\rho\varepsilon} - w_{\varepsilon} b_{\sigma\tau\rho} = -\mu v_{\rho\sigma\tau\varepsilon} \\ w^{\lambda} v_{\lambda\rho\sigma\tau} = -\mu b_{\rho\sigma\tau} \end{cases} \qquad (29.2)$$

(29.1) and (29.2) were shown to be tensor forms of equations for particles of spin 1 when the wave function has five components by Kemmer (1938).

c – $\psi$ is hermitian and $\chi$ purely vectorial. We put

$$\psi = E^{\alpha} a_{\alpha} \quad , \quad \chi = \frac{1}{2} E^{\alpha\beta} f_{\alpha\beta}$$

so that the ~~particle~~ field is described by a 4-vector and by an antisymmetrical tensor. Lanczos' equations take the form

$$\begin{cases} w_{\alpha} a_{\beta} - w_{\beta} a_{\alpha} = \mu f_{\alpha\beta} . \\ w^{\beta} f_{\beta\alpha} = -\mu a_{\alpha} , \end{cases} \qquad (29.3)$$

that is we obtain Proca's equations describing a vector particle of spin 1.

d – $\psi$ is antihermitian and $\chi$ purely vectorial. Let

$$\psi = \frac{1}{3!} E^{\varepsilon\alpha\beta} b_{\varepsilon\alpha\beta} \quad , \quad \chi = \frac{1}{2} E^{\alpha\beta} f_{\alpha\beta}$$

The field is described by a pseudo-vector and an antisymmetrical tensor. We obtain

$$\begin{cases} w^\varepsilon b_{\varepsilon\alpha\beta} = -\mu f_{\alpha\beta} \\ w_\varepsilon f_{\alpha\beta} + w_\alpha f_{\beta\varepsilon} + w_\beta f_{\varepsilon\alpha} = -\mu b_{\varepsilon\alpha\beta} \end{cases} \tag{29.4}$$

that is the equations for a pseudo-vector particle of spin 1. Kemmer (1939) has shown that (29.3) and (29.4) are the only two equations which describe a particle of spin 1 when its wave function has 10 components. All the four sets of equations (29; 1, 2, 3, 4) can be put in the form

$$\beta^\alpha \partial_\alpha \Psi = i\mu \Psi \tag{29.5}$$

where the operators $\beta^\alpha$ satisfy the Duffin-Kemmer commutation relations

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = g^{\lambda\mu} \beta^\nu + g^{\nu\mu} \beta^\lambda \tag{29.6}$$

These relations are more general than the Dirac-Tetrode commutation relations for particles of spin $\frac{1}{2}$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 g^{\mu\nu} \tag{29.7}$$

since the operators $i\gamma^\mu$ satisfy (29.6), but $\beta^\mu$ do not satisfy (29.7) in general. Now Bhabha has shown recently that there exists a commutation relation more general than (29.7) and (29.6), namely

$$(\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda) + (\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu) + (\beta^\nu \beta^\lambda \beta^\mu + \beta^\mu \beta^\lambda \beta^\nu) = 2 g^{\lambda\mu} \beta^\nu + 2 g^{\nu\mu} \beta^\lambda + 2 g^{\lambda\nu} \beta^\mu \tag{29.8}$$

which is obviously satisfied by a solution of (29.6) or $i$ times a solution of (29.7), but whose general solution need not satisfy either (29.6) or (29.7). He has also shown that (29.5) where $\beta^\alpha$ satisfy (29.8) may still be regarded as describing a particle of spin 1 and has given equivalent tensor equations when $\Psi$ in (29.5) has 14 components. Then $\beta^\alpha$ are 14×14 matrices. Now we shall see presently that his equations are identical with another special case of Lanczos' equations.

e – $\psi$ is a complete complex quaternion

$$\psi = E^\alpha a_\alpha + \frac{1}{3!} E^{\alpha\beta\delta} b_{\alpha\beta\delta}$$

but $\chi$ is purely vectorial

$$\chi = \frac{1}{2} E^{\alpha\beta} f_{\alpha\beta}$$

so that we must put $u = \tilde{v} = 0$ in the covariant Lanczos equations. Thus we obtain Bhabbha's equations

$$\begin{cases} w_\alpha a_\beta - w_\beta a_\alpha - w^\varepsilon b_{\varepsilon\alpha\beta} = \mu f_{\alpha\beta} \\ w^\varepsilon f_{\varepsilon\alpha} = -\mu a_\alpha \\ w_\alpha f_{\beta\gamma} + w_\beta f_{\gamma\alpha} + w_\gamma f_{\alpha\beta} = -\mu b_{\alpha\beta\gamma} \end{cases} \quad (29.9)$$

which again lead to the relativistic second order wave equations for the vector $a_\alpha$, the pseudo-vector $b_{\alpha\beta\gamma}$, the antisymmetrical tensor $f_{\alpha\beta}$ and which describe a particle of spin 1.

f – Another incomplete form of Lanczos' equations is obtained by taking $f_{\alpha\beta} = 0$. Then the wave function has 10 components but consists of (29.1) and (29.2) written together. In the form (29.5) the $\beta^\alpha$ still satisfy (29.6).

g – $\psi$ is hermitian and the pseudo-scalar component $\tilde{v}$ of $\chi$ is zero. In this case we have the 11-component wave equation:

$$\begin{cases} w^\alpha a_\alpha = \mu u \\ w_\alpha a_\beta - w_\beta a_\alpha = \mu f_{\alpha\beta} \\ w_\alpha u + w^\beta f_{\beta\alpha} = -\mu a_\alpha \end{cases} \quad (29.10)$$

h – $\psi$ is antihermitian and the scalar invariant component $u$ of $\chi$ is zero. We get another 11-component equation

$$\begin{cases} w_\alpha b_{\beta\gamma\varepsilon} - w_\beta b_{\alpha\gamma\varepsilon} - w_\gamma b_{\beta\alpha\varepsilon} - w_\varepsilon b_{\beta\gamma\alpha} = -\mu v_{\alpha\beta\gamma\varepsilon} \\ w^\varepsilon b_{\varepsilon\alpha\beta} = -\mu f_{\alpha\beta} \\ w^\varepsilon v_{\varepsilon\alpha\beta\gamma} + (w_\alpha f_{\beta\gamma} + w_\beta f_{\gamma\alpha} + w_\gamma f_{\alpha\beta}) = -\mu b_{\alpha\beta\gamma} \end{cases} \quad (29.11)$$

i – $\psi$ is complete, but $u$ is missing from $X$. We get a 15-component equation.

$$\begin{cases} w_\alpha b_{\beta\gamma\varepsilon} - w_\beta b_{\alpha\gamma\varepsilon} - w_\gamma b_{\beta\alpha\varepsilon} - w_\varepsilon b_{\beta\gamma\alpha} = -\mu v_{\alpha\beta\gamma\varepsilon} \\ w_\alpha a_\beta - w_\beta a_\alpha - w^\varepsilon b_{\varepsilon\alpha\beta} = \mu f_{\alpha\beta} \\ w^\varepsilon f_{\varepsilon\alpha} = -\mu a_\alpha \\ w^\varepsilon v_{\varepsilon\alpha\beta\gamma} + (w_\alpha f_{\beta\gamma} + w_\beta f_{\gamma\alpha} + w_\gamma f_{\alpha\beta}) = -\mu b_{\alpha\beta\gamma} \end{cases} \quad (29.12)$$

j.– We have another 15-component equation when $\tilde{v}$ is missing

$$\begin{cases} w^\alpha a_\alpha = \mu u \\ w_\alpha a_\beta - w_\beta a_\alpha - w^\varepsilon b_{\varepsilon\alpha\beta} = \mu f_{\alpha\beta} \\ w_\alpha u + w^\beta f_{\beta\alpha} = -\mu a_\alpha \\ w_\alpha f_{\beta\gamma} + w_\beta f_{\gamma\alpha} + w_\gamma f_{\alpha\beta} = -\mu b_{\alpha\beta\gamma}. \end{cases} \quad (29.13)$$

It can be seen that if (29; 10, 11, 12, 13) are put in the form (29.5) $\beta^\alpha$ do not satisfy any of the relations (29; 7, 6 or 8).

## §30. New tensor forms of Dirac's equation.

Lanczos' equations, though describing a particle of spin $\frac{1}{2}$ are not equivalent to Dirac's equation as they involve a wave function with 8 complex components. We now want to give tensor equations equivalent to Dirac's equation. We have seen that this can be done if it can be put in an invariant quaternion form. Now Watson's form of Dirac's equation for the free electron is, in cartesian coordinates $\xi^\alpha$

$$(\bar{I}^\alpha \partial/\partial \xi^\alpha)\, {}^2\Psi = \mu\, {}^2\Psi^\times e_3 . \quad (\bar{I}^0 = 1,\ \bar{I}^n = ie_n).$$

~~Now~~ We have also seen that if Dirac's equation is ~~asked~~ wanted to retain its cartesian form with the same constant operators $\bar{I}^\alpha$ under a coordinate transformation $\xi^\alpha \to x^\alpha$, then ${}^2\Psi$ cannot be regarded as an invariant quaternion but obeys the spinor transformation law (cf. §5). On the other hand we have seen that in their covariant formulation other field equations do not retain their form in cartesian coordinates. Hence if we want to find a covariant formulation for Dirac's equation we must assume that under $\xi^\alpha \to x^\alpha$, the Dirac operators are also transformed into variable matrices which obey the Tetrode commutation relations ~~in a~~ (29.7) in a general coordinate system. In this case the differential operator $\gamma^\alpha \partial_\alpha$ in Dirac's equation

$$\gamma^\alpha \partial_\alpha \psi = \mu \psi \qquad (30.1)$$

is an invariant, hence $\psi$ is also an invariant. Now this point of view in discussing the transformation properties of ~~Dirac's equation~~ (30.1) is generally adopted for writing it in a general coordinate system, but it is the other point of view that is supposed to have physical significance. For writing (30.1) in quaternion form we take the following representation of the operators $\gamma^\alpha$ when the basis vectors $E^\alpha$ which correspond to the generalized coordinates $x^\alpha$ are known:

$$\gamma^\mu = E^\mu(\ )^\times e_3$$

so that, if $\Psi$ is a complex quaternion

$$\gamma^\mu \Psi = E^\mu \Psi^\times e_3 .$$

These operators satisfy (29.7) and Dirac's equation becomes

$$D\Psi = -\mu\, \Psi^\times e_3 \qquad (30.2)$$

where $D$ is given by (28.1) and is invariant. $\Psi$ is also an invariant quaternion. To obtain a covariant ~~comp~~ representation we ~~must~~ write $\Psi$ and $e_3$ in terms of their covariant components. Now for the invariant quaternion $\Psi$ we can always choose a ~~the~~ representation of the form (27.13)

$$\Psi = E_\alpha c^\alpha + \frac{1}{3!} E^{\alpha\beta\gamma} b_{\alpha\beta\gamma} = E_\beta(c^\beta - i\tilde{b}^\beta). \qquad (30.3)$$

Now we write

$$-e_3 = i\, I^\alpha{}_\alpha l = i E^\alpha l_\alpha \qquad (30.4)$$

so that the cartesian coordinates of $e_3$ are ${}_\alpha l$ and its general coordinate $l_\alpha$. We have

$${}_0 l = {}_1 l = {}_2 l = 0 \quad , \quad {}_3 l = -1$$

in cartesian coordinates. But $l_\alpha \neq 0$ in general. Since $e_3$ is constant and has unit norm, $l_\alpha$ must satisfy the conditions

$$l_\mu l^\nu = -1 , \qquad w_\alpha l_\nu = 0 \qquad (30.5)$$

where $w_\alpha$ denotes the affine covariant differentiation. Thus (30.2) may be written as

$$\bar{E}^\alpha E^\beta w_\alpha c_\beta + \frac{1}{3!} \bar{E}^\varepsilon E^{\alpha\beta\gamma} w_\varepsilon b_{\alpha\beta\gamma} = \mu \bar{E}^\alpha E^\beta i a_\alpha l_\beta$$
$$+ \frac{1}{3!} (E^{\alpha\beta\gamma})^\times E^\varepsilon i\, b_{\alpha\beta\gamma} l_\varepsilon \qquad (30.6).$$

Proceeding in exactly the same way as for the first Lanczos equation (28.15) we find

$$w_\alpha c^\alpha + i \frac{1}{4!} k^{\varepsilon\alpha\beta\gamma}(w_\varepsilon b_{\alpha\beta\gamma} - w_\alpha b_{\varepsilon\beta\gamma} - w_\beta b_{\alpha\varepsilon\gamma} - w_\gamma b_{\alpha\beta\varepsilon})$$
$$+ \frac{1}{2} E^{\alpha\beta}\left[(w_\alpha c_\beta - w_\beta c_\alpha) + i(w_\alpha \tilde{b}_\beta - w_\beta \tilde{b}_\alpha)\right]$$
$$= b_\alpha l^\alpha + i c_\alpha l^\alpha + \frac{1}{2} E^{\alpha\beta}\left[(l_\beta \tilde{b}_\alpha - l_\alpha \tilde{b}_\beta) - i(l_\beta c_\alpha - l_\alpha c_\beta)\right] \qquad (30.7)$$

From this invariant equation, using the dual notations of § 27, we have

$$\left\{\begin{array}{l} w^\alpha c_\alpha = \mu\, l^\alpha \tilde{b}_\alpha \\ w^\alpha \tilde{b}_\alpha = \mu\, l^\alpha c_\alpha \\ w_\alpha c_\beta - w_\beta c_\alpha + k_{\alpha\beta\gamma\varepsilon} w^\gamma \tilde{b}^\varepsilon = \mu(l_\beta \tilde{b}_\alpha - l_\alpha \tilde{b}_\beta + k_{\beta\alpha\gamma\varepsilon} l^\gamma c^\varepsilon) \\ w_\alpha \tilde{b}_\beta - w_\beta \tilde{b}_\alpha - k_{\alpha\beta\gamma\varepsilon} w^\gamma c^\varepsilon = \mu(l_\beta c_\alpha - l_\alpha c_\beta - k_{\beta\alpha\gamma\varepsilon} l^\gamma \tilde{b}^\varepsilon). \end{array}\right. \qquad (30.8)$$

These covariant relations exhibit a curious symmetry between dual quantities. From §1 and §5 it follows that if ~~b~~ ${}_{\alpha}c$, ${}_{\alpha}\tilde{b}$ are the cartesian components of the vector and pseudo vector whose components in the general coordinate system are $c_\alpha$ and $\tilde{b}_\alpha$, then the four invariant ~~compo~~ elements of the column matrix $\psi$ in Dirac's representation (30.1) are given by

$$\psi_1 = {}_0c - i\,{}_3\tilde{b} \qquad \psi_3 = i\,{}_0\tilde{b} + {}_3c$$
$$\psi_2 = {}_2\tilde{b} - i\,{}_1\tilde{b} \qquad \psi_4 = i\,{}_2c + {}_1c$$

In this case we obtain the ordinary vector equations (14. ).

It may be noticed that we could obtain another tensor form of Dirac's equation by keeping (30.4) but adopting the form (27.14) for $\Psi$. Then the electron would be described by an antisymmetrical tensor, a scalar invariant and a scalar pseudo-invariant, all these quantities being real.

When an electromagnetic field is present the additional term on the right hand side of (30.2) is the invariant quaternion

$$\Omega = \bar{A}\,\Psi\,e_3. \tag{30.9}$$

Let

$$A = E^\alpha A_\alpha.$$

The additional term on the right of (30.8) is

$$\Omega = \bar{E}^\alpha E^\beta E^\gamma\, i\, l_\gamma A_\alpha (c_\beta - i\tilde{b}_\beta)$$

which can always be put in the form

$$\Omega = \omega_1 + i\omega_2 + \tfrac{1}{2}E^{\alpha\beta}\omega_{\alpha\beta}$$

where $\omega_{\alpha\beta}$ is an antisymmetrical tensor, $\omega_1$ scalar and $\omega_2$ pseudo scalar. The calculation of these quantities which will appear in the equations (30.8) in terms of $A_\alpha$, $c_\beta$, $\tilde{b}_\beta$, $l_\gamma$ is rather tedious and the resulting covariant equations too complicated. This is why the tensor form of Dirac's equation need not be used for solving it. We have seen in chapters <u>IV</u> and <u>V</u> how the invariant quaternion form can be solved directly.

# CHAPTER VIII

## Dirac's Equation and General Relativity

### §31. Spaces characterized by the vanishing of two 4-vectors derived from the torsion tensor.

All the attempts to set up unitary field theories show at least that some field equations for particles of spin 1 (like Maxwell's and Proca's equations) may be given some geometrical interpretation if a suitable space-time model is set up. However, in all the investigations connected with general relativity no equations to describe the structure of a space-time model have been encountered which are, even in the first approximation, similar to wave equations for particles of spin $\frac{1}{2}$. A satisfactory unitary theory must, on the other hand, take account of the matter field described by Dirac's equation. In this section we propose to show that the simplest of such equations for particles of spin $\frac{1}{2}$, namely the Fränkel-Madelung equation (that is Dirac's equation when the rest mass zero) may be given a very simple geometrical interpretation.

We consider the affine space-time of distant parallelism described by the basis vectors $E_\alpha$. If the space is euclidian, that is if a coordinate transformations can be found such that it transforms $E_\alpha$ into the constant cartesians basis vectors $I_\alpha$, then the torsion tensor vanishes and we have the 24 equations

$$\Lambda_{\alpha\beta,\gamma} = 0 \tag{31.1}$$

In general the torsion tensor does not vanish. Now we have seen that in such a space we can derive from the torsion tensor two important 4-vectors, namely $\Lambda_\alpha$ given by (25.17) and $L_\varepsilon$ given by (25.20). They vanish if (31.1) holds but they may also vanish without (31.1) being true. In this case they define a special type of affine space

$$\bar{E}^{\alpha} = f \psi I^{\alpha} \psi^{\dagger}$$

$$E^{\alpha} \bar{E}^{\beta} E^{\gamma}_{\alpha,\beta\gamma} = f^3 \psi I^{\alpha} \bar{I}^{\beta} I^{\gamma} \psi^{\dagger} B_{\alpha,\beta\gamma}$$

$$= f^3 \psi \left( I^{\alpha} \bar{I}^{\beta} I^{\gamma} B_{\alpha,\beta\gamma} \right) \psi^{\dagger}$$

$$\mathrm{Re}\, I^{\alpha} \bar{I}^{\beta} I^{\gamma} = \gamma^{\alpha\beta} I^{\gamma} + \gamma^{\beta\gamma} I^{\alpha} - \gamma^{\alpha\gamma} I^{\beta}$$



just as Einstein's gravitational equations define a special type of Riemannian space. Thus we ~~write~~ assume the 8 equations

$$\Lambda_{\varepsilon} = 0 \tag{31.2}$$

$$L_{\varepsilon} = 0 . \tag{31.3}$$

We now want to prove that from (31.2) and (31.3) we can derive an equivalent quaternion equation which is identical with Dirac's equation with rest mass zero. Consider the ~~can~~ invariant quaternion constructed by means of Ricci's rotation tensor

$$B = E^{\alpha} \bar{E}^{\beta} E^{\gamma} B_{\alpha,\beta\gamma} \tag{31.4}$$

where $B_{\alpha,\beta\gamma}$ is given by (25.15). Using (24.28) we find

$$B = (G^{\alpha\beta\gamma} + E^{\alpha\beta\gamma}) B_{\alpha,\beta\gamma} = E^{\varepsilon}(-2 g^{\alpha\gamma} B_{\alpha,\varepsilon\gamma} + i k_{\varepsilon\alpha\beta\gamma} B^{\alpha,\beta\gamma}).$$

or, By (25.18) and (25.21)

$$B = E^{\varepsilon}(-2 B_{\varepsilon} + i R_{\varepsilon}) .$$

It now remains to express the vectors $B_{\varepsilon}$ and $R_{\varepsilon}$ in terms of the torsion tensor ~~#~~ with the help of (25.19) and (25.22) Thus

$$B = 4 E^{\varepsilon}(\Lambda_{\varepsilon} + i L_{\varepsilon}) . \tag{31.5}$$

We see that (31.2) and (31.3) are equivalent to the quaternion equation

$$B = 0 \tag{31.6}$$

where $B$ is given by (31.4). Another expression for $B$ derived from (26.7) by multiplying both sides to the left by $E^{\alpha} \bar{E}^{\beta}$ is

$$B = E^{\alpha} \bar{E}^{\beta} \Delta_{\alpha} E_{\beta} . \tag{31.7}$$

(31.4), (31.7) and (31.5) are equivalent.

At this point let us introduce another set of basis vectors $E'_{\alpha}$ which lead to the same metric tensor $g_{\alpha\beta}$ as $E_{\alpha}$. Hence

$$E'_{\alpha} . \bar{E}'_{\beta} = E_{\alpha} . \bar{E}_{\beta} = g_{\alpha\beta} . \tag{31.8}$$

It follows that at each point $E'_{\alpha}$ are related to $E_{\alpha}$ by an orthogonal transformation. Thus we can write

$$E_{\alpha} = \xi^{\dagger} E'_{\alpha} \xi \tag{31.9}$$

where $\xi$ is an invariant quaternion such that

$$\xi \bar{\xi} = 1 \tag{31.10}$$

On account of (31.8) the metric covariant differentiation $\Delta_{\alpha}$

is the same, whether associated with the frame $E_\alpha$ or the frame $E'_\alpha$. The Ricci rotation tensor which corresponds to $E'_\alpha$ will be denoted by $B'_{\alpha,\beta\gamma}$ and the torsion tensor by $\Lambda'_{\alpha\beta,\gamma}$. They are defined in the same way as for $E_\alpha$ and satisfy the relations

$$\Delta_\alpha E'_\beta = B'_{\alpha,\beta\gamma} E'^\gamma \qquad (31.11)$$

$$B' = E'^\alpha \bar{E}'^\beta \Delta_\alpha E'_\beta = E'^\alpha \bar{E}'^\beta E'^\gamma B'_{\alpha,\beta\gamma} = 4E'^\epsilon(\Lambda'_\epsilon + iL'_\epsilon) \qquad (31.12)$$

where

$$\Lambda'_\epsilon = \Lambda'^{\cdot\,\cdot\,\alpha}_{\epsilon\alpha\cdot} \qquad (31.13)$$

$$L'_\epsilon = \frac{1}{4} k_{\epsilon\alpha\beta\gamma} \Lambda'^{\alpha\beta,\gamma} \qquad (31.14)$$

which are similar to (25.20) and (25.17). The tensor $k_{\epsilon\alpha\beta\gamma}$ is unchanged because it depends only on $g_{\alpha\beta}$. The dashed torsion tensor is defined by

$$\partial_\alpha E'_\beta - \partial_\beta E'_\alpha = \Lambda'^{\cdot\,\cdot\,\gamma}_{\alpha\beta\cdot} E'_\gamma . \qquad (31.15)$$

~~Now~~ We can evaluate $B$ in terms of the new basis vectors $E'_\alpha$ as follows:

$$\Delta_\alpha E_\beta = (\Delta_\alpha \zeta^\dagger) E'_\beta \zeta + \zeta^\dagger E'_\beta (\Delta_\alpha \zeta) + \zeta^\dagger (\Delta_\alpha E'_\beta)\zeta . \qquad (31.16)$$

as $\zeta$ is invariant we have

$$\Delta_\alpha \zeta = \partial_\alpha \zeta .$$

A consequence of (31.10) is that

$$\underline{\omega}_\alpha = (\partial_\alpha \zeta)\bar{\zeta} \qquad (31.17)$$

is purely vectorial. With this definition (31.16) takes the form

$$\Delta_\alpha E_\beta = \zeta^\dagger(\underline{\omega}^\dagger_\alpha E'_\beta + E'_\beta \underline{\omega}_\alpha + \Delta_\alpha E'_\beta)\zeta \qquad (31.18)$$

which, after multiplication to the left by $E^\alpha \bar{E}^\beta$ gives, with the help of (31.7), (31.9) and (31.12)

$$B = \zeta^\dagger(E'^\alpha \bar{E}'^\beta \underline{\omega}^\dagger_\alpha E'_\beta + E'^\alpha \bar{E}'^\beta E'_\beta \underline{\omega}_\alpha + B')\zeta$$

Now using the identity (24.33) (which is also true for the dashed basic vectors) we see that the first term ~~between~~ in the bracketted expression vanishes identically. We also have

$$\bar{E}'^\beta E'_\beta = 4 .$$

Hence we derive the important transformation formula between the invariant quaternions $B$ and $B'$:

$$B = \zeta^\dagger(4\, E'^\alpha \underline{\omega}_\alpha + B')\zeta \qquad (31.19).$$

Now we have seen that the equations (31.2) and (31.3) are equivalent to (31.6) which now takes the form

$$E'^{\alpha}\partial_{\alpha}\xi + \frac{1}{4}B'\xi = 0 \qquad (31.20)$$

On the other hand, since $B'_{\alpha,\rho\sigma}$ is antisymmetrical in $\rho$ and $\sigma$ we can write

$$B' = E^{\alpha'}\bar{E}'^{\rho}E'^{\sigma}B'_{\alpha,\rho\sigma} = E'^{\alpha}(E'^{\rho\sigma})^{\times}B'_{\alpha,\rho\sigma}$$

so that (31.20) is equivalent to

$$\bar{E}'^{\alpha}\left(\partial_{\alpha} + \frac{1}{4}B'_{\alpha,\rho\sigma}E'^{\rho\sigma}\right)\xi^{\times} = 0 \qquad (31.21)$$

Now Dirac's equation for zero rest mass ~~reads, in Bargmann's~~ can be derived ~~…~~ from the generalized equations given by Bargmann (1932) and Symonds (1950) if we assume that the rest mass vanishes. It reads

$$\gamma^{\alpha}\left(\partial_{\alpha} - \frac{1}{4}B'_{\alpha,\rho\sigma}I^{\rho\sigma}\right)\psi = 0 \qquad (31.22)$$

where

$$I^{\rho\sigma} = \frac{1}{2}(\gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho}) \qquad (31.23)$$

and $\gamma^{\alpha}$ obey (29.7). The quaternion operators

$$\gamma^{\alpha} = E'^{\alpha}(\ )^{\times}e_3$$

provide a representation of $\gamma^{\alpha}$ and we find

$$I^{\rho\sigma} = -\frac{1}{2}(E'^{\rho}\bar{E}'^{\sigma} - E'^{\sigma}\bar{E}'^{\rho}) = -E'^{\rho\sigma}.$$

Hence the quaternion form of (31.22) is (31.21).

In the case of a conformal space we have

$$E_{\alpha} = f\,\xi^{+}I_{\alpha}\xi \qquad (31.24)$$

$$E^{\alpha} = f^{-1}\xi^{+}I^{\alpha}\xi$$

where $f$ is scalar and $I_{\alpha}$ given by (4.1). Hence in this case we can take

$$E'_{\alpha} = f\,I_{\alpha} \qquad (31.25$$

From (31.15) we find

$$\Lambda'^{\cdot\cdot\cdot\gamma}_{\alpha\beta\cdot} = \frac{1}{2}(\tau_{\alpha}\delta^{\gamma}_{\beta} - \tau_{\beta}\delta^{\gamma}_{\alpha}) \qquad (31.26)$$

where

$$\tau_{\alpha} = \partial_{\alpha}\log f. \qquad (31.27)$$

We find

$$B' = E'^{\mu}\bar{E}'_{\mu}E'^{\lambda}\tau_{\lambda} - E'^{\mu}\bar{E}'^{\lambda}E'_{\mu}\tau_{\lambda}$$

and ~~using~~ by (24,32)

$$B' = 6E'^{\lambda}\tau_{\lambda}. \qquad (31.28)$$

Thus (31.20) becomes

$$I^{\nu}\partial_{\nu}\Phi = 0 \qquad (31.29)$$

where

$$\Phi = f^{3/2}\xi. \qquad (31.30)$$

## § 32. Possible relations between the rest mass, the electromagnetic potential and the torsion tensor.

We now consider a conformal space-time of affine parallelism where the basis vectors are given by (31.24). The fundamental relation (31.19) can be written ~~as~~ in the form

$$4\, E'^{\alpha} \partial_\alpha \xi + B' \xi = \xi^{\times} B \tag{32.1}$$

In conformal space, replacing $B'$ by (31.28) and making the substitution (31.30) we find

$$4\, I^{\alpha} \partial_\alpha \Phi = f\, \Phi^{\times} B \tag{32.2}$$

Now let us consider a conformal space-time such that the contracted curvature tensor still vanishes, that is (31.2) holds, but that (31.3) is no longer true. Then from (31.5) it follows that $B$ is antihermitian. Let $l_\varepsilon = {}_\varepsilon L$ be the cartesian coordinates of the vector $L_\varepsilon$. We have

$$B = 4i\, E^{\varepsilon} L_\varepsilon = 4i\, I^{\varepsilon}\, {}_\varepsilon L = 4i\, I^{\varepsilon} l_\varepsilon$$

If the vector $I^{\varepsilon} l_\varepsilon$ is space-like and is constant we have seen in §14 that (32.2) is identical with Dirac's equation. Choosing a cartesian frame of reference such that ~~$l_1 = l_2 = l_0 = 0$~~

$$l_1 = l_2 = l_0 \quad \text{and} \quad l_3 = \mu$$

we find

$$I^{\alpha} \partial_\alpha \Phi = \mu f\, \Phi^{\times} e_3 . \tag{32.3}$$

The only difference is that the rest mass term has the coefficient $f$, that is, (32.3) ~~is Dirac's~~ represents Dirac's equation with a scalar invariant potential $\mu(f-1)$. The metric of the space is

$$g_{\alpha\beta} = f^2 \gamma_{\alpha\beta}$$

where $\gamma_{\alpha\beta}$ is the constant metric of special relativity. It has been shown by Infeld & Schild (1946) that Dirac's equation must be taken as (32.3) if we are to ensure conformal invariance. This point will be discussed in § 35.

To sum up, the transformation equation (32.1) is equivalent to Dirac's equation for a free particle if $B$ satisfies the following conditions

$$B = -B^{\dagger}$$

$$\partial_{\varepsilon} B = 0$$

$$B\bar{B} = \mu^2 = m_0^2 c^2/\hbar^2$$

where $m_0$ is ~~of~~ the rest mass.

Now let

$$E^{\alpha}\Lambda_{\alpha} = \Lambda \qquad (32.4)$$

$$E^{\alpha}L_{\alpha} = L. \qquad (32.5)$$

Both $\Lambda$ and $L$ are invariant hermitian quaternions. Using (31.5) we can write

$$B = 4(\Lambda + iL)$$

and the equation (32.2) becomes

$$I^{\alpha}\partial_{\alpha}\Phi = f\,\Phi^{\times}(\Lambda + iL) \qquad (32.6)$$

which we compare with Dirac's equation

$$I^{\alpha}\partial_{\alpha}\Phi = (\mu\,\Phi^{\times} + \lambda A\,\Phi)e_3 \quad (\lambda = e/\hbar c) \qquad (32.7)$$

satisfied by the complex conjugate of $\Psi$ in §5. We see that (32.6) can be written in the form (32.7) if between the geometrical quantities $\Lambda$, $L$ and the physical quantities $\mu$, $A$ we have the relation

$$f\,\Phi^{\times}(\Lambda + iL) = (\mu\,\Phi^{\times} + \lambda A\,\Phi)e_3 \qquad (32.8)$$

where $\Phi$, the wave function, is given by (31.30) and connected with the basis vectors by (31.24). Multiplying (32.8) to the left by $\xi^{\dagger}$ and using (31.30) we find

$$f(\Lambda + iL) = (\mu + \lambda\,\xi^{\dagger}A\xi)\,e_3. \qquad (32.9)$$

Hence $B$ obeys the condition that $Be_3$ is hermitian. That is

$$Be_3 + e_3B^{\dagger} = 0 \qquad (32.10)$$

From (32.9) we derive

$$f\Lambda = \frac{\lambda}{2}(\xi^{\dagger}A\xi\,e_3 - e_3\,\xi^{\dagger}A\xi) \qquad (32.11)$$

$$\text{and } fL = \mu e_3 + \frac{\lambda}{2}(\xi^{\dagger}A\xi\,e_3 + e_3\,\xi^{\dagger}A\xi). \qquad (32.12)$$

In Einstein's unitary field theory in the space-time of distant parallelism (~~E~~ 1928) $\Lambda_{\varepsilon}$ were regarded as the components of the electromagnetic potential. That is he took

$$\Lambda = A.$$

This, however would lead to a wrong interaction term in Dirac's equation. If a unitary theory is possible on these lines it seems that $A$ should be related to both vectors $\Lambda$ and $L$ through a relation of the form (32.9).

## §33. Other quaternion solutions of the divergence equation. Lanczos' equation.

In the last section it has been found that in conformal space where the basis vectors have the form

$$k\epsilon^{\mu}_{\cancel{\mu}} = \Psi^{\dagger} I^{\mu} \Psi \qquad (k = \sqrt{-g}) \qquad (33,1)$$

if the contracted torsion tensor vanishes we have

$$\cancel{\partial}\,\Delta_{\mu}\,\epsilon^{\mu} = \partial_{\mu}\, k\epsilon^{\mu} = 0. \qquad (33,2)$$

The resulting equation for $\Psi$ is

$$\Psi^{\dagger} D \Psi + \Psi^{\dagger} \overleftarrow{D} \Psi = 0 \qquad (33,3)$$

where $D = I^{\mu}\partial_{\mu}$. We have seen that Dirac's equation for free space

$$D\Psi = \mu \Psi^{\times} e_3$$

satisfies (33,3). It will now be shown that a similar connection between the basis vectors and the quaternions in Lanczos' equation can also be formulated. Let the basis quaternions have the form

$$k\epsilon^{\mu} = \lambda_1 (\Phi^{\dagger} I^{\mu} \Phi)^{\times} + \lambda_2 \Psi^{\dagger} I^{\mu} \Psi. \qquad (\lambda_1, \lambda_2 = \text{constant scalars}).$$

Hence we are not dealing with a conformal space. Let the contracted ~~tor~~sion tensor vanish so that the divergence equation (33,2) still holds. Instead of (33,3) we obtain

$$\lambda_1 \bar{\Phi}\, \bar{D}\, \Phi^{\times} + \lambda_1 \bar{\Phi}\, \bar{\overleftarrow{D}}\, \Phi^{\times} + \lambda_2 \Psi^{\dagger} D \Psi + \lambda_2 \Psi^{\dagger} \overleftarrow{D} \Psi = 0 \qquad (33,4)$$

This quaternion equation is satisfied if we take

$$D\Psi = \lambda_1 \Phi^{\times} \qquad (33,5)$$

$$D\Phi = -\lambda_2 \Psi^{\times}. \qquad (33,6).$$

These equations become identical with Lanczos' equations if $\lambda_1 = \lambda_2$. As we have seen in §29 all the equations for spin 1 are degenerate cases of the equations (33; 5, 6). For instance if $\Psi = A$ is hermitian and $\Phi = \underline{F}$ is purely vectorial we have Proca's equations when $\lambda_1 = 1, \lambda_2 = \lambda^2$ and Maxwell's equations when $\lambda_1 = 1$, $\lambda_2 = 0$. Thus, in all these ~~of~~ cases (33,4) is valid. The quaternion equation (33,4) is equivalent to the 4 equations

$$D\cdot(\lambda_2 \Psi\Psi^{\dagger} + \lambda_1 \Phi\Phi^{\dagger}) = 0$$

$$D\cdot(\lambda_2 \Psi i e_n \Psi^{\dagger} - \lambda_1 \Phi i e_n \Phi^{\dagger}) = 0 \qquad (n = 1,2,3)$$

These are the generalisations of the equations ~~as~~ §10; 1,2) in

Dirac's case. Thus for Lanczos' equations ($\lambda_1 = \lambda_2$) the vector

$$J = \underline{\Psi}\,\underline{\Psi}^{+} + \Phi\,\Phi^{+}$$

with vanishing divergence and with positive definite scalar part represents the current density. But in the case of particles with spin 1 the divergence equations (33,4) represent the vanishing of the divergence of the energy-momentum tensor. Thus in Maxwell's case ($\lambda_1 = 1$, $\lambda_2 = 0$) they read

$$\underline{F}^{+} D \underline{F} + \underline{F}^{+} \bar{D} \underline{F} = 0$$

or $\partial_\mu \, \underline{F}\, I^\mu \underline{F}^{+} = 0$.

Multiplying scalarly by $\bar{I}^\nu$ we have

$$\partial_\mu T^{\mu\nu} = 0$$

where

$$T^{\mu\nu} = \bar{I}^\nu . \underline{F}\, I^\mu \underline{F}^{+} = I^\mu F^{+} . \bar{I}^\nu F \qquad (33,7)$$

Now we shall prove that (33,7) is the quaternion expression for Maxwell's energy momentum tensor. Let

$$\underline{F} = \tfrac{1}{2} I^\rho \bar{I}^\sigma f_{\rho\sigma}$$

we have

$$\underline{F}^{+} = \tfrac{1}{2} \bar{I}_\alpha I_\beta f^{\alpha\beta}.$$

Hence we can write

$$T^\mu{}_\nu = \tfrac{1}{4} I^\mu \bar{I}_\alpha I_\beta . \bar{I}_\nu I^\rho \bar{I}^\sigma f^{\alpha\beta} f_{\rho\sigma}.$$

This tensor equation is obviously valid if we replace $I^\mu$ by other basis vectors $E^\mu$ and $f^{\alpha\beta}$ by the components of the quaternion $\underline{F}$ with respect to the new basis vectors.

~~We have~~ By (24,28) we have

$$I^\mu \bar{I}_\alpha I_\beta . I_\nu I^\rho \bar{I}^\sigma f^{\alpha\beta} f_{\rho\sigma} = (2\,\delta^\mu_\alpha I_\beta + i g^{\mu\lambda} k_{\lambda\alpha\beta\varepsilon} I^\varepsilon)$$
$$\times (2\,\delta^\rho_\sigma \bar{I}^\sigma + i g_{\nu\kappa} k^{\kappa\rho\sigma\chi} \bar{I}_\chi) f^{\alpha\beta} f_{\rho\sigma}$$

since $f^{\alpha\beta}$ is antisymmetrical. Hence

$$T^\mu{}_\nu = f^{\mu\sigma} f_{\nu\sigma} - \tfrac{1}{4} g_{\nu\kappa} g^{\mu\lambda} \delta^{\kappa\rho\sigma}_{\lambda\alpha\beta} f^{\alpha\beta} f_{\rho\sigma}.$$

since

$$k^{\kappa\rho\sigma\chi} k_{\lambda\alpha\beta\chi} = \delta^{\kappa\rho\sigma}_{\lambda\alpha\beta} \qquad ?$$

by (27,7). But

$$\delta^{\kappa\rho\sigma}_{\lambda\alpha\beta} = \delta^\kappa_\lambda \delta^{\rho\sigma}_{\alpha\beta} + \delta^\kappa_\alpha \delta^{\rho\sigma}_{\beta\lambda} + \delta^\kappa_\beta \delta^{\rho\sigma}_{\lambda\alpha}$$

so that

$$\tfrac{1}{4} g_{\nu\kappa} g^{\mu\lambda} \delta^{\kappa\rho\sigma}_{\lambda\alpha\beta} f^{\alpha\beta} f_{\rho\sigma} = \tfrac{1}{2} \delta^\mu_\nu f^{\alpha\beta} f_{\alpha\beta} + \tfrac{1}{2} g_{\nu\kappa} g^{\mu\lambda} \delta^{\rho\sigma}_{\beta\lambda} f^{\kappa\beta} f_{\rho\sigma}$$

The second term of this sum can also be written as

$$g_{\nu\kappa}\, g^{\mu\lambda}\, f^{\kappa\rho} f_{\rho\lambda} = f_{\nu\rho} f^{\rho\mu}$$

and we obtain finally

$$T^{\mu}{}_{\nu} = 2\left(f^{\mu\sigma} f_{\nu\sigma} - \tfrac{1}{4}\,\delta^{\mu}_{\nu}\, f^{\alpha\beta} f_{\alpha\beta}\right).$$

Thus (33,7) is equivalent to the energy-momentum tensor.

In the case of Proca's equations we find

$$T^{\mu\nu} = \bar{I}^{\nu}.\left(\underline{F}\, I^{\mu} F^{\dagger} + \lambda^2 \bar{A}\, \bar{I}^{\mu} \bar{A}\right).$$

Six more divergence equations can be deduced from Lanczos's equations (33;5,6). We have

$$D.\left(\Psi\, i e_n\, \Phi^{\dagger}\right) = S\left\{\Phi^{\dagger} D \Psi\, i e_n + i e_n\, \Phi^{\dagger} D \Psi\right\}$$

$$= S\left\{\lambda_1 \Phi^{\dagger} \Phi^{*}\, i e_n - \lambda_2\, i e_n\, \Psi^{-} \Psi\right\} = 0.$$

Hence the hermitian and antihermitian parts of $\Psi\, i e_n\, \Phi^{\dagger}$ satisfy the divergence equations

$$D.\left(\Psi\, i e_n\, \Phi^{\dagger} + \Phi\, i e_n\, \Psi^{-\dagger}\right) = 0$$

$$D.\tfrac{1}{i}\left(\Psi\, e_n\, \Phi^{\dagger} - \Phi\, e_n\, \Psi^{-\dagger}\right) = 0$$

In both Maxwell's and Proca's cases the vectors with vanishing divergence are

$$A\, i e_n\, \underline{F}^{\dagger} + \underline{F}\, i e_n\, A$$

and $\quad A\, e_n\, F^{\dagger} - \underline{F}\, e_n\, A.$

In Kemmer's theory for particles of spin ~~0~~ 1 the field variables $A_\alpha$ and $f_{\beta\gamma}$ are assumed to be complex quantities and a real current vector with vanishing divergence is derived from the first order equation

$$\beta^{\alpha} \partial_{\alpha} \psi = \mu \psi$$

and is given by

$$s_\alpha = \psi^{*}.\left(2\beta_0^2 - 1\right)\beta_\alpha \psi.$$

When expressed in terms of $A_\alpha$ and $f_{\beta\gamma}$ we find (de Broglie p. 135)

$$s_\alpha = \tfrac{1}{i}\left(f^{*}_{\alpha\beta} A^{\beta} - f_{\alpha\beta} A^{*\beta}\right).$$

Here we have shown that a four-vectors, bilinear in $A$ and $F$ and with a vanishing divergence can also be obtained in the case of real fields.

—

## § 34. Derivation of a new form of Dirac's equation in a general coordinate system in terms of the torsion tensor.

Leaving aside possible geometrical interpretations of the quantities which occur in Dirac's equation we turn to the more practical problem of writing Dirac's equation in a general coordinate system. We have seen that if $E_\alpha$ are the basis vectors of the transformation, we have

$$\Delta_\varepsilon E_\alpha = 0 \qquad (34.1)$$

where $\Delta_\varepsilon$ are the symbols for covariant metric differentiation and Dirac's equation in this coordinate system is

$$\bar{E}^\alpha \partial_\alpha \Psi = (\mu \Psi^\times + \lambda \bar{A} \Psi) e_3 . \qquad (34.2)$$

Let $g_{\alpha\beta}$ be the metric tensor associated with this transformation. If we choose other basis vectors $E'_\alpha$ such that (31.8) is satisfied they correspond to the same metric and are given by (31.9). Our purpose is to write (34.2) in terms of the vectors $E'_\alpha$. To this end we put

$$\Psi = \xi^{+} \Psi' \qquad (34.3)$$
$$A = \bar{\xi} A' \xi^\times .$$

With the definition (31.17) we find

$$\bar{E}'^\alpha (\partial_\alpha - \omega^\times_\alpha) \Psi' = (\mu \Psi'^\times + \lambda \bar{A}' \Psi') e_3 \qquad (34.4)$$

This equation may be written in Dirac's matrix form as

$$\gamma^\alpha (\partial_\alpha - \Gamma_\alpha + i\lambda A'_\alpha) \psi' = \mu \psi' \qquad (34.5)$$

where the operators $\Gamma_\alpha$ correspond to the quaternion operators $\omega^\times_\alpha (\ )^\times e_3$. The form (34.5) was first given by Schrödinger and the operators $\Gamma_\alpha$ were evaluated by himself and by Bargmann (1932) in terms of Ricci's rotation tensor $B'_{\alpha,\rho\sigma}$ defined by

$$\Delta_\alpha E'_\rho = B'_{\alpha,\rho\sigma} E'^\sigma . \qquad (34.6)$$

From (34.1) it follows that $B$ vanishes for the basis vectors $E_\alpha$. From (31.19) we derive immediately

$$\bar{E}'^\alpha \omega^\times_\alpha = -\frac{1}{4} B'^\times = -\frac{1}{4} \bar{E}'^\alpha E'^\rho E'^\sigma B'_{\alpha,\rho\sigma} \qquad (34.7)$$

Hence, another form for (34.4) is

$$\bar{E}'^{\alpha}\left(\partial_\alpha + \frac{1}{4} E'^{\rho\sigma} B'_{\alpha,\rho\sigma}\right){}^2\Psi' = \left(\mu\,{}^2\Psi'^{\times} + \lambda \bar{A}'\,{}^2\Psi'\right)e_3 \qquad (34.8)$$

In § 31 we have shown that in Dirac's formalism the operators $E'^{\rho\sigma}$ are represented by the spin operators $I^{\rho\sigma}$ defined by (31.23). Hence the equation which corresponds to (34.8) in matrix formalism is

$$\gamma^\alpha\left(\partial_\alpha - \frac{1}{4} B'_{\alpha,\rho\sigma} I^{\rho\sigma}\right)\psi' = \mu\psi' + \lambda\gamma^\alpha A'_\alpha \psi' \qquad (34.9)$$

and is seen to be identical with the equations given by Bargmann (1932) and Symonds (1950).

Now as the result of our investigations we can improve on the formula (34.9) which is not practical as the terms $B'_{\alpha,\rho\sigma} I^{\rho\sigma}$ are difficult to evaluate.

~~From (31.12)~~ Substitution of (31.12) in (34.7) gives

$$\bar{E}'^{\alpha}\underline{\omega}^{\times}_{\alpha} = -\bar{E}'^{\alpha}(\Lambda'_\alpha - iL'_\alpha) \qquad (34.10)$$

Hence a new form for (34.4) is

$$\bar{E}'^{\varepsilon}(\partial_\varepsilon + \Lambda'_\varepsilon - iL'_\varepsilon)\,{}^2\Psi' = \left(\mu\,{}^2\Psi'^{\times} + \lambda\bar{A}'\,{}^2\Psi'\right)e_3 \qquad (34.11)$$

where $\Lambda'_\varepsilon$ and $L'_\varepsilon$ are given respectively by (31.13) and (31.14) in terms of the torsion tensor ~~given~~ defined by (31.15). To translate (34.11) into the matrix formalism we need an ~~covariant~~ invariant representation of the operator $i$ which in cartesian coordinates is equal to $\alpha_1\alpha_2\alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are Dirac's constant matrices. Now (24.31) can be written as

$$i\,3!\,E^{\varepsilon} = -k^{-1}\delta^{\varepsilon\rho\sigma\tau}_{0123} E_\rho \bar{E}_\sigma E_\tau . \qquad (k = \sqrt{-g}) \qquad (34.12)$$

Multiplying both sides by $\bar{E}^{\varepsilon}$ we have

$$i = -\frac{1}{4!}(-g)^{-\frac{1}{2}}\delta^{\varepsilon\rho\sigma\tau}_{0123}\cdot\bar{E}_\varepsilon E_\rho \bar{E}_\sigma E_\tau \qquad (34.13)$$

which is the invariant representation we wanted. Now since (34.12) is also valid for $E'_\alpha$ we ~~also have~~ can also replace the basis vectors in (34.13) by the dashed vectors.

Hence if

$$\bar{E}'^{\alpha}\,{}^2\Psi'^{\times}e_3 = \gamma^\alpha\,{}^2\Psi'$$

(34.13) with dashed vectors may be written as

$$i = -\frac{1}{4!}(-g)^{-\frac{1}{2}}\delta^{\varepsilon\rho\sigma\tau}_{0123}\gamma_\varepsilon\gamma_\rho\gamma_\sigma\gamma_\tau$$

When $\gamma^\alpha$ are matrix representations of the same operators operating on the column matrix $\psi$ we put

$$\gamma_5 = \frac{1}{4!}(-g)^{-\frac{1}{2}}\delta^{\varepsilon\rho\sigma\tau}_{0123}\gamma_\varepsilon\gamma_\rho\gamma_\sigma\gamma_\tau \qquad (34.14)$$

Hence $\gamma_5\psi$ corresponds to $-i\,{}^2\Psi$.

$$\gamma^\varepsilon(\partial_\varepsilon + \Lambda'_\varepsilon + \gamma_5 L'_\varepsilon + i\lambda A'_\varepsilon)\psi = m\psi.$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$
$$\partial_\rho\gamma_\sigma - \partial_\sigma\gamma_\rho = \Lambda'_{\rho\sigma}{}^{\tau}\gamma_\tau$$

orthogonal system $\left\{ \begin{array}{l} \Lambda'_\beta = \frac{1}{2}\partial_\beta \log \dfrac{{}^0h_0\,{}^1h_1\,{}^2h_2\,{}^3h_3}{{}^\beta h_\beta} \\ L'_\beta = 0 \end{array} \right.$

$$\Lambda'_\beta = \tfrac{1}{2}\partial_\beta \log {}^0h_0\,{}^1h_1\,{}^2h_2\,{}^3h_3 - \tfrac{1}{2}\partial_\beta \log {}^\beta h_\beta$$

${}^0h_0 = 1$ $\quad \partial_0\,{}^0h_0 = 0$ $\quad \partial_1\,{}^1h_1 = \partial_\theta\, \tau\, sh\,\alpha = 0$



Thus we find the matrix form of (34.11) which reads
$$\gamma^\varepsilon(\partial_\varepsilon + \Lambda'_\varepsilon + \gamma_5 L'_\varepsilon + i\lambda A'_\varepsilon)\psi = \mu\psi \qquad (34.15).$$
This is the new formula which is valid in a general coordinate system and which replaces (34.9). The torsion tensor $\Lambda'_{\rho\sigma}{}^{\tau}$ which gives rise to $\Lambda'_\varepsilon$ and $L'_\varepsilon$ is determined by
$$\partial_\rho\gamma_\sigma - \partial_\sigma\gamma_\rho = \Lambda'_{\rho\sigma}{}^{\tau}\gamma_\tau. \qquad (34.16).$$
In an orthogonal coordinate system we ~~have~~ can write
$$E_\alpha = \zeta^\dagger E'_\alpha \zeta$$
where
$$E'_\alpha = {}^\alpha h_\alpha I_\alpha \quad (\text{no summation}). \qquad (34.17)$$
The only contributions to the vector $L_\varepsilon$ come from the components of $\Lambda'_{\rho\sigma,\tau}$ when the indices $\rho, \sigma, \tau$ are all different. ~~Hence~~ in this case $\Lambda'_{\rho\sigma,\tau}$ is zero for $\rho \neq \sigma \neq \tau$. The only non-vanishing components of $\Lambda'_{\rho\sigma,\tau}$ are
$$\Lambda'_{\beta\alpha,\alpha} = \tfrac{1}{2}\,{}_\alpha h_\alpha\, \partial_\beta\, {}^\alpha h_\alpha \quad (\text{no summation}).$$
Hence
$$\Lambda'_\beta = g^{\alpha\gamma}\Lambda'_{\beta\alpha,\gamma} = \tfrac{1}{2}\left({}_0h^0\,\partial_\beta\,{}^0h_0 + {}_1h^1\,\partial_\beta\,{}^1h_1 + \dots\right)$$
$$- \tfrac{1}{2}\left({}_0h^0\,\partial_0\,{}^0h_\beta + {}_1h^1\,\partial_1\,{}^1h_\beta + \dots\right)$$
$$= \tfrac{1}{2}\partial_\beta \log \frac{{}^0h_0\,{}^1h_1\,{}^2h_2\,{}^3h_3}{{}^\beta h_\beta}.$$
and
$$L'_\beta = 0$$
for orthogonal coordinates. Inserting these values for $\Lambda'_\beta$ and $L'_\beta$ in (34.15) we find the formula first given by Frenkel (1934, p. 366) with a wrong sign and corrected and generalized by Symonds (1950).
In four-dimensional spherical coordinates we have
$$\partial_\varepsilon\,{}^\varepsilon h_\varepsilon = 0 \quad \text{and} \quad {}^0h_0 = 1,$$
so that
$$\Lambda'_\beta = \tfrac{1}{2}\partial_\beta \log {}^1h_1\,{}^2h_2\,{}^3h_3.$$
The coordinates are (§ 22)
$$x^0 = \tau, \quad x^1 = \theta, \quad x^2 = \varphi, \quad x^3 = \alpha$$
and we find
$${}^1h_1 = \tau\, sh\,\alpha, \quad {}^2h_2 = \tau\, sh\,\alpha \sin\theta, \quad {}^3h_3 = \tau$$
so that $\Lambda'_\beta = \frac{1}{2}\partial_\beta \log \tau^3 sh^2\alpha \sin\theta$

conformal coordinates.

$$g_{\mu\nu} = f^2 \eta_{\mu\nu}.$$

$$f^{-1} f^{-\frac{3}{2}} \gamma_\mu \partial_\mu f^{\frac{3}{2}} \psi' = \mu \psi.$$

$\gamma_\mu$



or

$$\Lambda'_0 = \frac{3}{2\tau}, \quad \Lambda'_1 = \frac{1}{2}\cot\theta, \quad \Lambda'_2 = 0, \quad \Lambda'_3 = \coth\alpha$$

and we get (22.12), which can be further simplified by putting ~~$\eta = \tau^3 \operatorname{sh}\alpha$~~

$$\eta = \tau^{3/2} \operatorname{sh}\alpha \sqrt{\sin\theta}\, \Phi.$$

Thus if the original wave function in the quaternion equation is $\Psi$ we put

$$\eta = \tau^{3/2} \operatorname{sh}\alpha \sqrt{\sin\theta}\; e^{e_3\frac{\varphi}{2}} e^{e_2\frac{\theta}{2}} e^{-ie_3\frac{\alpha}{2}} \Psi$$

and the equation satisfied by $\eta$ is

$$\{\partial_\tau + (\tau \operatorname{sh}\alpha)^{-1} ie_1 \partial_\theta + (\tau \operatorname{sh}\alpha \sin\theta)^{-1} ie_2 \partial_\varphi + \tau^{-1} ie_3 \partial_\alpha\}\eta = (\mu\, \eta^\times + \lambda \bar{A}' \eta) e_3.$$

where

$$A' = Q^\times A \bar{Q}.$$

with $Q$ defined by (21.3)

In the case of conformal coordinates we have

$$E_\beta = f\, \zeta^\dagger I_\beta \zeta$$

so that

$$E'_\beta = f I_\beta \quad \text{and} \quad \Lambda'_\beta = f^{-\frac{3}{2}} \partial_\beta f^{3/2}.$$

Thus the quaternion equation becomes

$$f^{-1} \bar{I}^\varepsilon f^{-\frac{3}{2}} \partial_\varepsilon f^{3/2} \Psi' = \mu \Psi'^\times e_3 + f^{-1} \bar{I}^\varepsilon A_\varepsilon \Psi' e_3$$

We are led to put

$$\eta = f^{3/2} \Psi' = f^{3/2} \zeta \Psi \qquad (34.18)$$

and the equation satisfied by $\eta$ is

$$\bar{I}^\varepsilon \partial_\varepsilon \eta = \mu f \eta^\times e_3 + \bar{I}^\varepsilon A_\varepsilon \eta e_3. \qquad (34.19)$$

Thus for conformal transformations Dirac's equation can be transformed to its cartesian form by means of (34.18). In § 31 we have seen that for such conformal transformations we have

$$I^\varepsilon \partial_\varepsilon f^{3/2} \zeta = 0. \qquad (34.20)$$

## §35. Quaternion representation of conformal transformations. Conformal invariance of Dirac's equation.

Conformal transformations have been studied by Robertson, Hill, Infeld and Schild. Robertson has shown that the ~~acc~~ acceleration transformation of L. Page is a special conformal transformation. The general group depends on 15 parameters and is generated by

a – a constant uniform space-time expansion

b – a translation defined by a constant four-vector.

c – a constant Lorentz transformation

d – an inversion defined by

$$x_0' = x_0\tau^{-2}, \quad x_n' = -x_n\tau^{-2} \qquad (\tau^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2).$$

In §4 we have already given the quaternion equivalents of a, b and c. They are

a – $X' = fX$ ($f$: constant real scalar)

b – $X' = X + A$ ($A$: constant hermitian).

c – $X' = QXQ^\dagger$ ($Q$: constant quaternion with unit norm)

It is easily seen that the inversion is represented by

$$X' = X^{-1}.$$

Hence the general group may be written as

$$(X')^{-1} = f\,Q(X-A)^{-1}Q^\dagger + B \qquad (35.1)$$

where $B$ is another constant ~~opera~~ hermitian quaternion. Any one of the operations a, b, c, d leaves (35.1) formally unchanged with different values for the constants $A, B, f, Q$. For $f=1$, $A=B=0$ we obtain the Lorentz transformation.

If $f=1$, $Q=1$, $A=0$, $B=i\underline{\gamma}/c^2$ we have

$$(X')^{-1} = X^{-1} + i\underline{\gamma}/c^2. \qquad (35.2)$$

In the non relativistic limit this transformation reduces to the ordinary ~~accl~~ transformation formulae between frames of reference moving relatively with constant acceleration $\gamma$ in the direction $\underline{\gamma}$. If $\underline{\gamma} = e_1\gamma$, then

$$x_0' = x_0$$

$$x_1' = x_1 + \tfrac{1}{2}\gamma x_0^2.$$

$\psi\bar{\psi} = f^3$

$R = \frac{1}{\psi\bar{\psi}} \Box (\psi\bar{\psi})^{1/3}$



mines the Riemannian curvature ~~base~~, that is the gravitational effects of space-time. We may try to determine $f$ from the gravitational equations related to the metric (35.4). But as we have only one unknown function we need one equation which can be taken as the contracted Einstein equations. But the contracted Einstein equations

$$R^{\alpha}_{\ \alpha} = R = 0 \qquad (35.5)$$

simply express that the gaussian curvature of space-time is zero. Hence we determine $f$ by the condition (35.5).

The Riemann-Christoffel tensor for a conformal space such that

$$g_{\mu\nu} = e^{2\tau} a_{\mu\nu}$$

is given by Finzi's formula (Levi-Civita, p. )

$$-R_{\lambda\mu\nu\varepsilon} = g_{\lambda\varepsilon}(\tau_{\mu\nu} - \tau_\mu \tau_\nu) - g_{\lambda\nu}(\tau_{\mu\varepsilon} - \tau_\mu \tau_\varepsilon) - g_{\mu\varepsilon}(\tau_{\lambda\nu} - \tau_\lambda \tau_\nu)$$
$$+ g_{\mu\nu}(\tau_{\lambda\varepsilon} - \tau_\lambda \tau_\varepsilon) + (g_{\lambda\varepsilon} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\varepsilon}) \tau_\rho \tau_\sigma g^{\rho\sigma}$$

where

$$\tau_\mu = \partial_\mu \tau \quad \text{and} \quad \tau_{\mu\nu} = \partial_\mu \partial_\nu \tau .$$

Expressing that the scalar curvature vanishes we have

$$g^{\lambda\mu} g^{\nu\varepsilon} R_{\lambda\mu\nu\varepsilon} = 0 .$$

After obvious simplifications, this leads to the equation

$$a^{\mu\nu}(\tau_{\mu\nu} + \tau_\mu \tau_\nu) = 0$$

$R = 6 f^{-3} \Box f = 0$

or

$$a^{\mu\nu} \partial_\mu \partial_\nu e^{\tau} = \Box f = 0 . \qquad (35.6)$$

Hence in the case of conformal space the metric is determined by a linear equation. If we look for a solution of (35.6) which has a singularity on the world line $r = 0$ ~~we obtain~~ and which tends to unity for $r \to \infty$ we find

$$f = 1 + \frac{a}{r} .$$

Putting this in Dirac's equation (34.19) we see that it can be interpreted as an extra energy term due to a gravitational point mass at the origin. If $M$ is the mass of the ~~proton~~ nucleus in Kepler's problem, then we must take

$$a = GM/c^2$$

where $G$ is the gravitational constant. In § 23 we have already seen that such a gravitational field leads to ~~quasi~~-continuous energy levels for the ~~H~~ electron. The dimensionless coupling constant $m_0 MG/\hbar c = \gamma$ is of the order $n\,3 \times 10^{-42}$ if there are $n$ nucleons in the nucleus, and is entirely negligible compared with the electrostatic coupling constant $e^2/\hbar c = 1/137$.

With the help of the differentiation formula

$$dX^{-1} = -X^{-1}dX\,X^{-1}$$

(35.2) gives

$$(X')^{-1}dX'(X')^{-1} = X^{-1}dX\,X^{-1}$$

or

$$dX' = (X\,X'^{-1})^{-1}dX\,(XX'^{-1})^{\dagger -1}.$$

But from (35.2)

$$XX'^{-1} = 1 + iX\underline{\gamma}/kc^2.$$

Hence

$$dX' = (1+iX\underline{\gamma}/kc^2)^{-1}dX\,(1+i\underline{\gamma}X/kc^2)^{-1}.$$

Thus we can write

$$dX' = \Gamma^{\dagger}dX\,\Gamma \qquad (35.3)$$

where

$$\Gamma = (1+i\underline{\gamma}X/kc^2)^{-1}.$$

This shows that the transformation is conformal.

Let

$$f^{2} = \Gamma\bar{\Gamma} \qquad \zeta = f^{-1}\Gamma$$

Substitution in (35.3) gives

$$dX' = f\,\zeta^{\dagger}dX\,\zeta.$$

where

$$f^{-1} = (1+i\underline{\gamma}X/kc^2)(1-i\bar{X}\underline{\gamma}/kc^2)$$
$$= 1 + c^2\underline{\gamma}.\underline{r} - X\bar{X}\gamma^2/4c^4.$$

Hence in the first approximation

$$f = 1 - \underline{\gamma}.\underline{r}/c^2.$$

Putting in (34.19) we see that an invariant potential function $-m_0\underline{\gamma}.\underline{r}$ which is generated by the acceleration of the electron is added to Dirac's equation, in complete agreement with Page's interpretation of this special conformal transformation.

We now consider a conformal space time characterized by the function $f$ so that the metric is

$$g_{\mu\nu} = f^2 a_{\mu\nu} \qquad (35.4)$$

where $a_{\mu\nu}$ is the metric of the special relativity, and by the quaternion $\zeta$ ($\zeta\bar{\zeta}=1$) which defines a variable Lorentz transformation at each point of space-time. We assume that $\zeta$ is related to $f$ through (34.20). The function $f$ then deter-

Einstein & Mayer.

$$J^{\alpha} = E^{\alpha}{}_{\sigma\tau}\, \chi^{\sigma} \bar{\chi}^{\tau} + E^{\alpha\sigma\tau} \bar{\psi}_{\sigma} \psi_{\tau}.$$

Einstein

$$\mathfrak{J}_1 = 4 \Lambda_{\mu\beta}{}^{\alpha} \Lambda^{\beta}{}_{\mu\alpha} = 4 \Lambda_{\mu\beta}{}^{\alpha} \Lambda^{\mu}{}_{\alpha}{}^{\beta}$$

$$= 4 \Lambda_{\mu\beta,\alpha} \Lambda^{\mu\alpha,\beta} = 4 K_3.$$

$$\mathfrak{J}_2 = 4 \Lambda_{\mu\beta}{}^{\alpha} \Lambda^{\mu}{}_{\alpha\beta} = \Lambda_{\mu\beta,\alpha} \Lambda^{\mu\beta,\alpha} = \cancel{4}\, 4 K_2.$$

$$\mathfrak{J}_3 = 4 \Lambda_{\alpha} \Lambda^{\alpha} = 4 K_1.$$



## § 36. Concluding remarks

All the preceding chapters have made it clear that the quaternion treatment has put Dirac's equation, $^{\alpha}t_{\beta}$ on the same footing as other field equations, in particular abolishing the distinction between real and complex fields since both types of field may be expressed by complex quaternions. Thus, Dirac's equation may be derived from a Lagrangian which takes the quaternion form

$$\mathcal{L} = \tfrac{1}{2} \hbar c \left( \Psi^{\dagger} . D \Psi e_3 - e_3 \Psi^{\dagger} \square . \Psi \right) + m_0 c^2 \mathcal{R}(\Psi \bar{\Psi})$$
$$+ e \bar{A} . \Psi \Psi^{\dagger} \qquad (36.1)$$

with our usual notations. Now if $\Psi$ is expressed by means of a 4. vector $c^{\alpha}$ and a pseudo-vector $\tilde{b}^{\alpha}$ as in (30.3), then $\mathcal{L}$ depends only on these real quantities and the potential $A_{\alpha}$. On varying $c^{\alpha}$ and $\tilde{b}^{\alpha}$ separately we would again obtain 8 real equations equivalent to Dirac's equation.

Another point we want to make is this. If there is some truth in the relations between the basis vectors on the one hand, and ~~with~~ the physical quantities appearing in the wave equation on the other, we ~~shou~~ ought to be able to derive equations having a physical meaning from a suitable invariant Lagrangian connected with geometrical quantities. Einstein (1928) chose, in the affine space-time of distant parallelism, the Lagrangian

$$\mathcal{L} = K_3 = \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\gamma,\beta} \qquad (36.2)$$

and from this, derived approximate equations for the 16 components $^{\alpha}t_{\beta}$ of the basis vectors. Weitzenböck (1929) showed that there are two more invariants which can be derived from the torsion tensor, namely

$$K_2 = \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\beta,\gamma} \qquad (36.3)$$

and

$$K_1 = \Lambda_{\alpha} \Lambda^{\alpha} \qquad (36.4)$$

where $\Lambda_{\alpha}$ is the contracted torsion tensor. In Einstein's theory, $\Lambda_{\alpha}$ was interpreted as the electromagnetic potential

$$16 L_\varepsilon L^\varepsilon = (\delta^{\beta\gamma}_{\mu\nu} - \delta^{\beta\gamma}_{\nu\mu}) \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\mu,\nu} + \delta^{\beta\gamma}_{\lambda\mu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu\alpha}$$

$$= 2 \delta^{\beta\gamma}_{\mu\nu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\mu,\nu} + \delta^{\beta\gamma}_{\lambda\mu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu\alpha}$$

$$= 2 (\delta^\beta_\mu \delta^\gamma_\nu - \delta^\beta_\nu \delta^\gamma_\mu) \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\mu,\nu} + (\delta^\beta_\lambda \delta^\gamma_\mu - \delta^\beta_\mu \delta^\gamma_\lambda) \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\alpha}$$

$$= 2 \Lambda_{\alpha\beta,\gamma} (\Lambda^{\alpha\beta,\gamma} - \Lambda^{\alpha\gamma,\beta}) + \Lambda_{\alpha\beta,\gamma} (\Lambda^{\beta\gamma,\alpha} - \Lambda^{\gamma\beta,\alpha})$$

$$= 2 \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\beta,\gamma} - 2 \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\gamma,\beta} + 2 \Lambda_{\alpha\beta,\gamma} \Lambda^{\beta\gamma,\alpha}$$

$$2 \Lambda_{\beta\alpha,\gamma} \Lambda^{\alpha\gamma,\beta}$$

$$2 \Lambda_{\alpha\beta,\gamma} \Lambda^{\gamma\alpha,\beta}$$

$$+ 2 \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\gamma,\beta}$$

Einstein. $J_1 =$ .

$$8 L_\varepsilon L^\varepsilon = K_2 + \tfrac{1}{2} K_3 = \tfrac{1}{4}(J_2 + \tfrac{1}{2} J_1).$$

$$= \tfrac{1}{4} J_2 + \tfrac{1}{4}\tfrac{1}{2} J_1$$

$$16 L_\varepsilon L^\varepsilon =$$

Einstein finds.

$$\mathcal{L} = \tfrac{1}{2} J_1 + \tfrac{1}{4} J_2 - J_3 .$$

The field eq. are.

$$G^{\mu\alpha} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + \frac{\partial \mathcal{L}}{\partial \Lambda^\alpha_{\mu\nu}} = 0$$

$G^{\mu\alpha}$ = symmetrical.

$$16 L_\varepsilon L^\varepsilon = k^{\varepsilon\alpha\beta\gamma} \Lambda_{\alpha\beta,\gamma}\, k_{\varepsilon\lambda\mu\nu} \Lambda^{\lambda\mu,\nu}$$

$$= \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\nu} = \delta^\alpha_\lambda \delta^{\beta\gamma}_{\mu\nu} + \delta^\alpha_\mu \delta^{\beta\gamma}_{\nu\lambda} + \delta^\alpha_\nu \delta^{\beta\gamma}_{\lambda\mu}$$

$$16 L_\varepsilon L^\varepsilon = \delta^\alpha_\lambda \delta^{\beta\gamma}_{\mu\nu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\nu} + \delta^\alpha_\mu \delta^{\beta\gamma}_{\nu\lambda} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\nu} + \delta^\alpha_\nu \delta^{\beta\gamma}_{\lambda\mu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\nu}$$

$$= \delta^{\beta\gamma}_{\mu\nu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\alpha\mu,\nu} + \delta^{\beta\gamma}_{\nu\mu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\mu\alpha,\nu} + \delta^{\beta\gamma}_{\lambda\mu} \Lambda_{\alpha\beta,\gamma} \Lambda^{\lambda\mu,\alpha}$$

$$\overset{\lambda}{t}_\alpha\, {}_\lambda \overset{\times}{t}_\beta = g_{\alpha\beta}$$

$$g_{\alpha\beta} = a_{\alpha\beta} = \overset{\lambda}{t}_\alpha\, {}_\lambda \overset{\times}{t}_\beta$$

$$\overset{\times}{g}_{\beta\alpha} = (\overset{\lambda}{t}_\beta\, {}_\lambda \overset{\times}{t}_\alpha)^{\times} = {}_\lambda t_\alpha\, \overset{\lambda}{\overset{\times}{t}}_\beta = \overset{\lambda}{t}_\alpha\, {}_\lambda \overset{\times}{t}_\beta$$

$$\partial_\alpha \overset{\lambda}{t}{}^\beta + \Gamma^\beta_{\alpha\gamma} \overset{\lambda}{t}{}^\gamma = 0$$

$$\partial_\alpha \overset{\lambda}{t}{}^{\times\beta} + \Gamma^{\beta\,\times}_{\alpha\gamma} \overset{\lambda}{t}{}^{\gamma\times} = 0$$

$$\Gamma^{\beta\,\times}_{\alpha\gamma} = \Gamma^\beta_{\gamma\alpha}.$$

$$\Gamma^\beta_{\alpha\gamma} = \Gamma^\beta_{\underline{\alpha\gamma}} + i \Lambda^\beta_{\alpha\gamma}.$$

$$t_\alpha \bar{t}_\beta = \overset{\lambda}{t}_\alpha\, t_\beta$$

$$\overset{0}{t}_\alpha \overset{0}{t}_\beta - \overset{1}{t}_\alpha \overset{1}{t}_\beta + (\overset{0}{t}_\alpha \overset{1}{t}_\beta - \overset{1}{t}_\alpha \overset{0}{t}_\beta) e_3$$

$$\partial_\alpha \overset{\lambda}{A}{}^\beta + \Gamma^\beta_{\underline{\alpha\gamma}} \overset{\lambda}{A}{}^\gamma - \Lambda^\beta_{\alpha\gamma} \overset{\lambda}{B}{}^\gamma = 0$$

$$\partial_\alpha \overset{\lambda}{B}{}^\beta + \Gamma^\beta_{\underline{\alpha\gamma}} \overset{\lambda}{B}{}^\beta + \Lambda^\beta_{\alpha\gamma} \overset{\lambda}{A}{}^\gamma = 0$$

$$\overset{\lambda}{t}_\alpha = \overset{\lambda}{A}_\alpha + i \overset{\lambda}{B}_\alpha$$

$$E_\alpha \bar{E}_\beta = g_{\alpha\beta} = a_{\alpha\beta} + \underline{M}_{\alpha\beta}$$

$$w_\alpha E^\beta = K^\beta_{\alpha\gamma} E^\gamma$$

$$\partial_\varepsilon g_{\alpha\beta} = \Gamma^\gamma_{\varepsilon\beta}\, g_{\alpha\gamma} + \Gamma^\gamma_{\varepsilon\alpha}\, g_{\beta\gamma}$$

$$-\partial_\alpha g_{\beta\varepsilon} = -\Gamma^\gamma_{\beta\alpha}\, g_{\varepsilon\gamma} - \Gamma^\gamma_{\alpha\beta}\, g_{\varepsilon\gamma}$$

$$-\partial_\beta g_{\varepsilon\alpha} = -\Gamma^\gamma_{\alpha\varepsilon}\, g_{\beta\gamma} - \Gamma^\gamma_{\beta\varepsilon}\, g_{\alpha\gamma}$$

$$[\varepsilon, \alpha\beta] = \Lambda^\gamma{}_{\varepsilon\beta}\, g_{\alpha\gamma} + \Lambda^\gamma{}_{\varepsilon\alpha}\, g_{\beta\gamma} - \tfrac{1}{2}\Gamma^\gamma_{\alpha\beta} g_{\varepsilon\gamma} - \tfrac{1}{2}\Gamma^\gamma_{\beta\alpha} g_{\varepsilon\gamma}$$

$$+ \tfrac{1}{2}\cancel{\Gamma^\gamma_{\beta\alpha} g_{\varepsilon\gamma}} - \tfrac{1}{2}\Gamma^\gamma_{\beta\alpha} g_{\varepsilon\gamma}$$

$$-\cancel{\Lambda^\gamma_{\alpha\beta}\, g_{\varepsilon\gamma}} - \Gamma^\gamma_{\beta\alpha} g_{\varepsilon\gamma}$$

$B_{\alpha,\beta\gamma} - B_{\beta,\alpha\gamma}$

$B_{\alpha,\beta\gamma}$

$$\Lambda_{\alpha\beta,\gamma} - \Lambda_{\beta\gamma,\alpha} - \Lambda_{\alpha\gamma,\beta}$$
$$\underline{-\Lambda_{\beta\alpha,\gamma} + \Lambda_{\alpha\gamma,\beta} + \Lambda_{\beta\gamma,\alpha}}$$

$2\Lambda_{\alpha\beta,\gamma}$

$$R^\varepsilon = \tfrac{1}{2} k^{\varepsilon\alpha\beta\gamma} \cdot 2\Lambda_{\alpha\beta,\gamma} = k^{\varepsilon\alpha\beta\gamma}\Lambda_{\alpha\beta,\gamma}.$$

2a

$$L^\varepsilon = k^{\varepsilon\alpha\beta\gamma}\Lambda_{\alpha\beta,\gamma}.$$

$$L_\varepsilon = k_{\varepsilon\alpha\beta\gamma}\Lambda^{\alpha\beta,\gamma}.$$

$$L^\varepsilon L_\varepsilon = k^{\varepsilon\alpha\beta\gamma} k_{\varepsilon\lambda\mu\nu}\Lambda_{\alpha\beta,\gamma}\Lambda^{\lambda\mu,\nu}$$

$$= \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\Lambda_{\alpha\beta,\gamma}\Lambda^{\lambda\mu,\nu}$$

Weitzenbock -

if $R$ is the scalar curvature corresponding to $\left\{{\alpha \atop \beta\gamma}\right\}$, then.

$$R = 4(\Psi - \phi) - 2A - \cancel{4}B.$$

$$\boxed{\phi = \Lambda^\alpha_{\mu\alpha}\Lambda^\nu_{\nu\beta}} \qquad \Phi_\alpha = \Lambda^\mu_{\alpha\mu}$$

$$A = g^{\mu\nu}\Lambda^\alpha_{\mu\beta}\Lambda^\beta_{\nu\alpha} = \Lambda_{\mu\beta,\alpha}\Lambda^{\mu\alpha,\beta}$$

$$\boxed{A = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\gamma,\beta}}$$

$$B = g_{\beta\nu} g^{\alpha\sigma} g^{\beta\tau}\Lambda^\beta_{\alpha\rho}\Lambda^\nu_{\sigma\tau}$$

$$= \Lambda_{\alpha\beta}{}^\gamma \Lambda^{\alpha\beta}{}_\gamma = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\beta,\gamma}.$$

$$\boxed{B = \Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\beta,\gamma}}$$

$$\Psi = \Delta_\alpha \Phi^\alpha$$

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} = \delta^\alpha_\lambda \delta^{\beta\gamma}_{\mu\nu} + \delta^\alpha_\mu \delta^{\beta\gamma}_{\nu\lambda} + \delta^\alpha_\nu \delta^{\beta\gamma}_{\lambda\mu}.$$

$$= \delta^\alpha_\lambda(\delta^\beta_\mu\delta^\gamma_\nu - \delta^\beta_\nu\delta^\gamma_\mu) + \delta^\alpha_\mu(\delta^\beta_\nu\delta^\gamma_\lambda - \delta^\beta_\lambda\delta^\gamma_\nu) + \delta^\alpha_\nu(\delta^\beta_\lambda\delta^\gamma_\mu - \delta^\beta_\mu\delta^\gamma_\lambda)$$

$$= \delta^\alpha_\lambda\delta^\beta_\mu\delta^\gamma_\nu - \delta^\alpha_\lambda\delta^\beta_\nu\delta^\gamma_\mu + \delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\lambda - \delta^\alpha_\mu\delta^\beta_\lambda\delta^\gamma_\nu + \delta^\alpha_\nu\delta^\beta_\lambda\delta^\gamma_\mu - \delta^\alpha_\nu\delta^\beta_\mu\delta^\gamma_\lambda$$

$$\Lambda_{\alpha\beta,\gamma}\Lambda^{\lambda\mu,\nu}$$

$\Lambda^\varepsilon \Lambda_\varepsilon, \quad L^\varepsilon L_\varepsilon$

Third invariant

$$2m\, B\bar{B} = L_\varepsilon \Lambda^\varepsilon = L^\varepsilon \Lambda_\varepsilon = \Lambda_{\varepsilon\alpha,\beta}\, g^{\alpha\beta} g^{\varepsilon\gamma} k^{\gamma\lambda\mu\nu} \Lambda_{\lambda\mu,\nu}$$

$$= \Lambda^{\varepsilon\alpha,\beta}\, g_{\alpha\beta}\, g_{\varepsilon\gamma}\, k^{\gamma\lambda\mu\nu} \Lambda_{\lambda\mu,\nu}$$

$$= g_{\alpha\beta}\, g_{\varepsilon\gamma}\, k^{\gamma\lambda\mu\nu}\, \Lambda^{\varepsilon\alpha,\beta} \Lambda_{\lambda\mu,\nu}$$

$$k^{\gamma\lambda\mu\nu} = k\, \delta^{\gamma\lambda\mu\nu}_{0123}\, \Lambda_{\lambda\mu,\nu}$$

$$k\, g_{\varepsilon\gamma}\, \Lambda^\varepsilon \delta^{\gamma\lambda\mu\nu}_{0123} \Lambda_{\lambda\mu,\nu}$$

$$\Lambda_{\lambda\mu,\nu} + \Lambda_{\mu\nu,\lambda} + \Lambda_{\nu\lambda,\mu} -$$

$$\Lambda^0_\alpha(\Lambda^{12,3} - \Lambda^{21,3}) - A^1(\Lambda$$

$$a(1+ie_3) + a'(1 \pm ie_3) + e_2 b(1+ie_3) + e_2 b'(1-ie_3) \qquad F\bar{\Xi} = 0$$

$$(1+ie_3) \cdot (a + e_2 b)(1+ie_3) + (a' + e_2 b')(1-ie_3)$$

$$a(1+ie_3) + e_2(1 \pm ie_3)$$

$$(a' + e_2 b')(1 - ie_3) \cdot = a'(1-ie_3) + (1+ie_3) e_2 b' \qquad F \times F$$

$$X' = \Xi X \Xi^+ \qquad \Xi X' \Xi^+ = 0$$

$$(1+ie_3) e_2 b' \quad e_2$$

$$(e_2 + ie_1)(y_1 + iy_2)(1+ie_3) e_2$$

$$X' = \psi(ie_1 + e_2)\bar{\psi} X \psi^\times (ie_1 - e_2)\psi^\dagger = L(\bar{L} \cdot X)$$

$$\bar{\psi} X' \psi^\times = Y' \qquad \bar{\psi} X \psi^\times = Y$$

$$L = \psi(1+ie_3)\psi^\dagger$$

$$\bar{L} = \psi^\times (1 - ie_3)\psi$$

$$\bar{L} \cdot X = (1 - ie_3) \cdot (\bar{\psi} X \psi^\times)$$

$$= \tfrac{1}{2}(1 - ie_3)\bar{\psi} X \psi^\times + \tfrac{1}{2}\psi^\dagger \bar{X} \psi (1 + ie_3)$$

$$Y' = (1+ie_3) e_2\, Y e_2 (1 - ie_3)$$

$$= (e_2 + ie_1)(y_1 + iy_2)$$

$$Y = (y_0 + y_3)(1+ie_3) + (y_0 - y_3)(1-ie_3) + (y_1 + iy_2) e_2 (1+ie_3) \pm (y_1 - iy_2) e_2 (1 - ie_3)$$

$|X'| = 0 \qquad X'$ is weak

$$X' = \psi(1+ie_3)\psi^\dagger k = \psi(ie_1 + e_2)\psi^\dagger = \psi(1+ie_3)e_2 \bar{\psi} X \psi^\times (ie_1 - e_2)\psi^\dagger$$

$$X' = k\, \xi(1+ie_3)\xi^\dagger = k\, L = \psi(ie_1 + e_2)\bar{\psi} X \psi^\times (ie_1 - e_2)\psi^\dagger$$

determination of $k$ $\qquad \bar{\psi} X' \psi^\times = k(1+ie_3)$

$$k = \bar{\psi} \cdot X' \psi^\times$$

$X'$



We have seen that this interpretation cannot be correct since if we write Dirac's equation in such a space-time we obtain the equation (34.15) and the additional terms come from the substitution

$$\partial_\alpha \to \partial_\alpha + \Lambda_\varepsilon$$

and not from

$$\partial_\alpha \to \partial_\alpha + i\lambda A_\alpha$$

which alone gives the electromagnetic interaction. Other possible relationships between $A$ and the torsion tensor were discussed in §32. We have seen that, if a geometrical interpretation is possible then $A$ must be related to both the vectors $\Lambda_\varepsilon$ and $L_\varepsilon$. Now there is no a priori reason why the Lagrangian should involve only (36.2) and not (36.3) and (36.4). But as Weitzenböck has shown it is very difficult to make a choice. ~~Now~~ We have found on the other hand that the quaternion $B$ given by (31.5) plays a fundamental role in the transformation theory of Dirac's equation. It is then natural to try the Lagrangian

$$\mathcal{L} = \frac{1}{16}\mathcal{R}\, B\bar{B} = \Lambda_\varepsilon \Lambda^\varepsilon - L_\varepsilon L^\varepsilon \tag{36.5}$$

After some calculations we find

$$16 L_\varepsilon L^\varepsilon = 2\Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\beta,\gamma} - 4\Lambda_{\alpha\beta,\gamma}\Lambda^{\alpha\gamma,\beta} = 2K_2 - 4K_3$$

Hence

$$\mathcal{L} = K_1 - \frac{1}{8}K_2 + \frac{1}{4}K_3 \tag{36.6}$$

The direct variation of this Lagrangian would be very tedious. But the quaternion formula give, if we put

$$D' = E'^\alpha \partial_\alpha$$

$$B\bar{B} = |B|^2 = |4(D'\Xi)\bar{\Xi} + B'|^2 \tag{36.7}$$

In particular, for a conformal space defined by (31.24) we find, with the definition (31.30).

$$B = \cancel{4\,\Xi D'\Xi}\ 4(\Phi^{-1})^\times D'\Phi$$

Hence the Lagrangian (36.6) takes the very simple form

$$\mathcal{L} = \mathcal{R}\,|(\Phi^{-1})^\times D'\Phi|^2$$

The ~~stu~~ discussion of the field equations obtained in this way is outside the scope of this work and will be the object of a future investigation.

G. Beck. Rev. Mod. Phys. 17, 187-94 (April-July) 1945.
(Field concepts in quantum theory).
An. Acad. Brasil Ciênc. 19 (nº 4) 321-32. (1947)
(Solutions of Dirac's equation in hypercomplex form).

Belinfante. Physica.
(Undors).

J. Courtois. C.R. Acad Sci. Paris 222, pp. 377, 480 (1946)
(Expansors).

R. J. Duffin. Phys. Rev. 77, 683 (1950)
(On wave equation vector-matrices and their spurs)



# REFERENCES

(* Books)


V. BARGMANN. Sitz. Preuss. Akad., 346 (1932)
H.J. BHABHA. Rev. Mod. Phys. 21, 451 (1949)
J. BLATON. Zeits. f. Phys. 95, 337 (1935)
M. BORN & K. FUCHS. Proc. R.S. Edinburgh 60, 1, 100 (1939-40).
* L. BRAND. * Vector and Tensor Analysis. John Wiley, (1947).
* L. de BROGLIE. * Une Nouvelle Théorie de la Lumière. vol. 1, Paris (1940).

P. CALDIROLA. Nuovo Cim. nº 1, 5, 29 (1948).
A.W. CONWAY. Proc. R. Irish Ac. A29, 1 (1911)
Proc. Roy. Soc. A, 162, 145 (1937)
Acta Pont. Ac. Sci. 12, 259 (1948).
O. COSTA de BEAUREGARD. C.R. Ac. Sci. Paris, 214, 818 (1942).

C.G. DARWIN. Proc. Roy. Soc. A, 118, 654 (1928)
P.A.M. DIRAC. Proc. Roy. Soc. A, 117, 610 (1928)
* Quantum Mechanics (3rd ed.) Oxford (1947)
E. DURAND. Journ. de Math. 25, Fasc. 4 (1946)
id. 28, Fasc. 2 (1949).

A. EINSTEIN. Sitz. Preuss. Akad. pp. 217, 224 (1928)
id. pp. 2, 156, (1929)
A. EINSTEIN & W. MAYER id. p. 522, (1932)
L.P. EISENHART. * Non-Riemannian Geometry (Princeton), (1927).

O.F. FISCHER. Phil. Mag. 30, 135 (1940).
id., 39, 878 (1948).
H.T. FLINT. Phil. Mag. 39, 439 (1920)
Proc. Roy. Soc. A, 121, 676 (1928).
V. FOCK. Zeits. f. Phys. 56, 330 (1929)
id. 57, 261 (1929)

B. Kwall. C. R. Ac. Sci. Paris 226, 61-3 (Jan. 5, 1948)
(The approximation of geometrical optics in the theory of particles of spin).

T. Levi-Civita. Sitz. Preuss. Ak. p. 137 (1929)
id. p. 240 (1933).

T. Levi-Civita*. The absolute diff. Calculus (Blackie & Son Ltd) London, reprinted 1947.

G. Lemaître. Ann. Soc. Sci. de Bruxelles, 57, p 165-172, Dec. 31 (1937)
(Eddington's interpretation of Dirac's equation).

G. Mie. Ann. der Phys. 17, 465 (1933)
(Die Geometrie der Spinoren).

V. FOCK & D. IWANENKO. Zeits. f. Phys. 54, 798 (1929)

J. FRENKEL. * Wave Mechanics. Advanced general Theory., Oxford (1934).

K. FUCHS. Proc. R. S. Edinburgh 60, 2, 147 (1939-40).

H. W. HASKEY. Phil Mag. 30, 478 (1940)

E. L. HILL. Phys. Rev. 67, 358 (1945)
id. 72, 143 (1947)
id. 73, 910 (1948).

E. L. HILL & R. LANDSHOFF. Rev. Mod. Phys. 10, 120 (1938)

L. INFELD & A.E. SCHILD. Phys. Rev. 68, 250 (1945)
id. 70, 410 (1946)

L. INFELD & B.L. VAN der WAERDEN. Sitz. Preuss. Ak. 380 (1933)

H. JEHLE. Phys. Rev. 76, 568 (1949).

N. KEMMER. Proc. Roy. Soc. A166, 173 (1938)
id. 173, 91 (1939)

C.W. KILMISTER. Phys. Rev. 76, 568 (1949)
Proc. Roy. Soc. A, 199, 517 (1949)
Thesis, London (1950).

W. KOFINK. Ann. der Phys. 38, 421 (1940).

B. KWALL. C.R. Ac. Sci. Paris 203, pp. 1132, 983 (1936)
Journ. de Phys.: 7, p. 223 (1936).

F. KLEIN. * Vorlesungen über die Entwicklung der Math. im 19. Jahr. (Berlin) II (1927)

C. von LANCZOS. Zeits. f. Phys. 57 pp. 447, 474, 484 (1929)

O. LAPORTE & G.E. UHLENBECK. Phys. Rev. 37 pp. 1380, 1553 (1931)

W. H. MC CREA. Proc. Roy. Irish Ac. A 45, 66 (1939)
N.F. Mott & I. Sneddon * Wave Mechs & its applications Oxford (1948)
K. NIKOLSKY. Zeits. f. Phys. 62, 677 (1930)

G. Petiau C. R. Acad. Sci. Paris 226, p. 224-6 (Jan. 19, 1948)
(approximation of geometrical optics in Wave mechanics).

L. A. Radicati. Nuovo Cim. 5, 173-80 (1948)
(Second order equations for a Pauli-Dirac particle).

F. Sauter. Ann. Phys. Leipzig, (Folge 5) 43, Nos 6-7, ←
p. 404-16 (1943)
(The various representations of a homogeneous electric field in wave mechanics).

F. Van Bergen. Bull. Acad. Belge Sci. 33, 138-51 (Nov. 8 1947)
(an invariant form for Dirac's equation).

J. Van Isacker, C. R. Acad. Sci. Paris 222, 375-6 (Feb. 11, 1946)
(Formulae associated with plane waves of particles with spin $\frac{1}{2}$).

L. PAGE. Phys. Rev. 49, 254 (1936).

W. PAULI. Ann. Inst. H. Poincaré, V, 131.

G. PETIAU. Journ. de Math. vol. 25, fasc. 4 (1946)
id, VI, fasc 2, p. 109 (1936)
id. vol 26, fasc. 1 (1947)
Journ. de Phys. Rad. VIII, 10, 265 (1949).

M. S. PLESSET. Phys. Rev. 41, 278 (1932)

B. PODOLSKY. Phys. Rev. 37, 1398 (1931).

H. P. ROBERTSON. Phys. Rev. 49, 755 (1936)

H. S. RUSE. Trans. R. S. Edinburgh 57, 97 (1937).

F. SAUTER. Zeits. f. Phys. 69, ~~677~~ 742 (1931)
id. 73, 547 (1931)

W. SCHERRER. Comm. Math. Helv. 7, 141 (1935)

E. SCHRÖDINGER. Sitz. Preuss. Akad. 105 (1932)
Proc. Roy. Irish Ac. A46, 25 (1940)

J. SERPE. Phys. Rev. 76, 1538 (1949)

L. SILBERSTEIN. Phil. Mag. 23, 790 (1912)
id. 25, 135 (1913).

N. D. SINGUPTA. Bull. Calcutta Math. Soc. 39, 147 (1947).

A. SOMMERFELD. Sitz. Akad. Wiss. Wien II a. 145, 639 (1936)

N. SYMONDS. Proc. R. Irish Ac. 53A 7, 115 (1950)

A. H. TAUB. Phys. Rev. 51, 512 (1937)
Ann. of Math. 40, 937 (1939)
Rev. Mod. Phys. 21, 388 (1949)

H. TETRODE. Zeits. f. Phys. 49, 861 (1928)
id. 50, 336 (1928).

D. M. VOLKOW. Zeits. f. Phys. 94, 25 (1935)

A. G. D. WATSON. Proc. Cambridge Phil. Soc. 43 491, (1947)

2

Som. über die Klein'schen Parameters $\alpha, \beta, \gamma, \delta$ und ihre Bedeutung für die Dirac-Theorie.



P. WEISS. Proc. R. Irish Ac. 46A12, 129 (1941).

R. WEITZENBÖCK - Sitz. Preuss. Akad., 466 (1928).

H. WEYL - Zeits. f. Phys. 56, 330 (1929)
id. 75, 261 (1932)

E.T. WHITTAKER - Proc. Roy. Soc. 158A, 38 (1938)

E.T. WHITTAKER & G.N. WATSON. *Modern Analysis
4th. ed., Cambridge (1946).

ZAYCOFF - Zeit. f. Phys. 56, pp. 717, 862 (1929).

T. Levi-Civita - *The absolute diff. calculus
(Blackie & Son Ltd.) London (1947).

E.W. Hobson - *Spherical & Ellipsoidal Harmonics, Cambridge (1931)

Sequenz. 77, p. 844.

R. J. Duffin - On wave equation vector - Matrices and their spurs
Phys. Rev. 77, 683.

T. Levi-Civita - Sitz. Preuss. Ak. p. 137. (1929).
" " " p. 240 (1933)

G. Mie. Annalen der Phys. 17, 465 (1933).
(Die Geometrie der Spinoren)

$$\psi\psi^\dagger = P$$

$$p_\mu = E_\mu \cdot \psi\psi^\dagger = \psi^\dagger . E_\mu \psi$$

$$\psi = E^\lambda (a_\lambda + i\tilde{b}_\lambda)$$

$$p_\mu = S(E_\lambda E_\mu E_\nu)(a^\lambda - i\tilde{b}^\lambda)(a^\nu + i\tilde{b}^\nu) = S(E_\lambda E_\mu E_\nu)\{(a^\lambda a^\nu + \tilde{b}^\lambda \tilde{b}^\nu) + i(a^\lambda \tilde{b}^\nu - \tilde{b}^\lambda a^\nu)$$

E. Wigner. Zeit. f. Phys. 53 p. 592 (1929)
Einstein's new formulation of the general principle of Relativity.

G. Temple. Tensorial form of Dirac's wave eqs.
Proc. Roy. Soc. 122 p. 352 (1929).

J.A. Schouten & J. Haantjes: "Über die Konforminvariante Gestalt der Relativistischen Bewegungsgleichungen" Proc. Akad. Wetensch. Amsterdam 39, 1059 (1936)

$\Phi = \xi e^{\frac{1}{2}(\xi^+\xi) \cdot X}$ ie

then $V = \Phi i e_1 \Phi^+$ $\qquad M = \Phi i e_1 \bar{\Phi}$

$W = \Phi i e_2 \Phi^+$ $\qquad N = \Phi i e_2 \bar{\Phi}$.

$\Box V = - V$ $\qquad$ Similarly for W and M.

$$\begin{cases} DV = -2N^x \\ DN = 2V^x \end{cases} \qquad DV = 0.$$

$\Psi = \Phi e^{ie_2\frac{t}{2}}$ = constant - I

$\Psi i e_1 \Psi^+ = \Phi e^{ie_2\frac{t}{2}} i e_1 e^{ie_2\frac{t}{2}} \Phi^+ = \Phi i e_1 \Phi^+ = V$

$\Psi i e_2 \Psi^+ = \Phi e^{ie_2 t} i e_2 \Phi^+$.

$\Psi i e_2 \bar{\Psi} = \Phi e^{ie_2\frac{t}{2}} i e_2 e^{-ie_2\frac{t}{2}} \bar{\Phi} = \Phi i e_2 \bar{\Phi} = N$

$\Psi i e_1 \Psi^+$ and $\Psi i e_2 \bar{\Psi}$ are solutions of the Proca equation if $\Psi$ is a plane wave sol. of $D\Psi = \Psi^x e_3$.

$\Psi i e_1 \Psi^+ = U_{(1)}$.

$\Psi i e_2 \bar{\Psi}^{x} = i \Psi i e_1 \Psi^+ \Psi^x i e_3 \bar{\Psi} = -i\, U_{(1)} \overline{U_{(3)}} = - U \overline{U_{(2)}}$.

$\Psi\Psi^+ = \Phi e^{ie_2 t} \Phi^+ = \cosh \Phi\Phi^+ + \sinh \Phi i e_2 \Phi^+$.

$-\Psi^+ i e_1 \bar{\Psi}^{x} \Psi i e_2 \bar{\Psi} = \Psi^x e_3 \bar{\Psi}$.

$\mathcal{H}(\bar{V}N) = 0$. $\qquad \left| \overline{V_\alpha \tilde{N}_{\alpha\beta} = 0} \right.$ $\quad 4$

$4 + 6 = 10 \qquad 10 - 4 = 6$

ACKNOWLEDGEMENT

The author would like to express his sincere gratitude to Prof. H. Jones for continuous advice and encouragement throughout the preparation of this work.

He wishes to thank Dr. P. Weiss and Dr. C.W. Kilmister for stimulating discussions and for bringing to his notice several papers on quaternions.

Finally he is indebted to the Turkish Ministry of Education for the award of a scholarship.