PROSPECTIVE MATHEMATICS TEACHERS' QUANTITATIVE REASONING ON THE DEVELOPMENT OF DECIMAL REPRESENTATION OF REAL NUMBERS AND ITS EFFECT ON THEIR COMPREHENSION OF A RELATED PROOF

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ABSTRACT

PROSPECTIVE MATHEMATICS TEACHERS' QUANTITATIVE REASONING ON THE DEVELOPMENT OF DECIMAL REPRESENTATION OF REAL NUMBERS AND ITS EFFECT ON THEIR COMPREHENSION OF A RELATED PROOF

This study investigated prospective mathematics teachers' quantitative reasoning on the development of decimal representation of real numbers and its effect on their comprehension of a related proof. The proof comprehension can be assessed according to seven dimensions developed by Mejia-Ramos and his colleagues (2012). Embedded experimental design was adopted in the study. Considering proof comprehension dimensions, a proof comprehension test about the proof of a statement on decimal representation of real numbers was formed. The proof comprehension test was used as both the pre-test prior to and the post-test upon completion of an instruction given to 19 prospective mathematics teachers to determine the effect of their quantitative reasoning on their performance of proof comprehension. Also, one hour long post-clinical interviews were conducted with six of them. Results from Wilcoxon Signed Rank test showed that there was a significant difference between pre and post proof comprehension test performances of prospective teachers. The results were also supported with the data from the post-interviews.Prospective teachers' thinking processes through quantitative reasoning during instruction and during the interviews are shared. Results from both the proof comprehension test and the interviews suggest that once prospective teachers engage in quantitative reasoning during instruction, their proof comprehension might develop.

ÖZET

REEL SAYILARININ ONDALIK AÇILIMININ GELİŞTİRİLMESİNDE MATEMATİK ÖĞRETMENİ ADAYLARININ NİCEL MUHAKEMESİ VE NİCEL MUHAKEMENİN KONUYLA BAĞLANTILI BİR İSPATI ANLAMALARINA ETKİSİ

Bu araştırmanın amacı matematik öğretmeni adaylarının reel sayıların ondalık açılımının geliştirilmesi sırasında gösterdikleri nicel muhakemeyi ve nicel muhakemenin matematik öğretmeni adaylarının konuyla ilgili verilen bir ispatı anlamalarına etkisini araştırmaktır. İspat anlama ile ilgili literatüre dayanarak, ispat anlama Mejia-Ramos ve arkadaşlarının geliştirmiş olduğu 7 boyut altında ölçülebilir (2012). Bundan dolayı, bu araştırmada öğretmen adaylarının reel sayıların ondalık açılımı ile ilgili verilen bir ispata ilişkin anlamalarının ölçülmesi için 7 boyut göz önünde bulundurularak açık uçlu ispat anlama testi geliştirilmiştir. Matematik öğretmen adaylarının reel sayıların ondalık açılımının geliştirilmesi sırasındaki nicel muhakemesinin onların ispat anlama tesindeki performansına etkisini ölçmek amacıyla, 19 matematik öğretmeni adayına 2 oturumluk öğretim uygulanmıştır. İspat anlama testi hem ön-test hem de son-test olarak kullanılmıştır. Elde edilen veriler Wilcoxon Sıralı işaretler testiyle değerlendirildiğinde, öğrencilerin ön-test ve son-test performansları arasında anlamlı bir fark olduğu görülmüştür. Sonuçlar, son-test sonrasında 6 öğretmen adayıyla yapılan yarı-yapılandırılmış mülakatlarla desteklenmiştir. Ayrıca öğretmen adaylarının reel sayıların ondalık açılımının gelişimi sırasındaki nicel muhakemeleri verilmiştir.

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LIST OF ACRONYMS/ABBREVIATIONS

APOS	Action Process Object Schema
D	Proof Comprehension Dimension
M	Mean
Min	Minimum
Max	Maximum
MoNE	Ministry of National Education
N	The number of students
NCTM	National Council of Teachers of Mathematics
Q	Question
R	Researcher
S	Student
SD	Standard Deviation

1. INTRODUCTION

Learning mathematics is more than applying formulas and using the same techniques to solve similar problems. Therefore, it requires to reason and do mathematics which is more than procedural understanding. A student may know how to use a formula or procedure to solve a problem but s/he may not explain why to use and how to determine that the chosen method is suitable for the problem. Therefore, students need to think critically, make connections between mathematical concepts and give justifications to problems through reasoning. These steps can be assumed as doing and knowing mathematics (Wood, 1998). In this regard, proof can be described as a necessary part for mathematics because it may help students to develop reasoning skills and answer ‘why’ questions and also develop critical thinking ability (Ball *et al.*, 2003). Similarly, Hanna (2000) asserts that “the key role of proof is the promotion of mathematical understanding” (Hanna, 2000, p.5). These contributions of proof makes the teaching of proof in all grades necessary to mathematics education in order to make students engage in reasoning and help them gather justification skills (NCTM, 1998; Ball *et al.*, 2003).

In this regard, in NCTM’s Principles and Standards for School Mathematics it is ascertained that reasoning and proof should “be a consistent part of students’ mathematical experience in prekindergarten through grade 12”(NCTM, 2000, p.56).

In Turkish High School Mathematics Curriculum it is also emphasized that one of the necessary considerations in mathematics teaching is to help students acquire proving skills to learn mathematics (MoNE, 2013, p.I). In order to focus on developing students’ proving skills teachers should know how to develop students’ reasoning and proof skills and also they should be expert in reasoning and proof (Stylianides *et al.*, 2005; Steele and Rogers, 2012).

There are many research studies related to proof and proving and many of them has shown that students have difficulties about proof and proving (Edwards, 1998; Heally and Hoyles, 2000; Chin and Lin, 2009; Al-Murani and Stylianides, 2010; Martinez *et al.*, 2011; Reyhani *et al.*, 2012; Moore, 1994; Almeida, 2000; Stylianides *et al.*, 2004; Stylianides *et al.*, 2005; Baştürk, 2010; İmamoğlu, 2010; Güler and Dikici, 2014). Just as students' conception of proof and proving is important (Heally and Hoyles, 2000), the researchers also emphasize the need for students' proof comprehension (Mejia-Ramos *et al.*, 2012). Though there is abundant research on students' conception of proof and proving, there are few research studies on proof comprehension (Zazkis and Zazkis, 2016).

In order to develop students' proving skills and proof comprehension, quantitative reasoning is seen as a tool of teaching (Weber *et al.*, 2014). Therefore, in this research study, the effect of prospective mathematics teachers' quantitative reasoning on their proof comprehension will be examined. The topic chosen for this investigation is decimal representation of real numbers. The number concept, specifically real numbers is seen as one of the core concepts in high school mathematics (NCTM, 1989). Yet, research has shown that students have misconceptions (Tall and Schwarzenberger, 1978; Ely, 2010; Voskoglou and Kosyvas, 2012) and difficulties (Voskoglou and Kosyvas, 2012) regarding real numbers which can be taught through quantitative reasoning (Karagöz-Akar, 2016). Therefore, in this study focusing on real numbers, the effect of prospective mathematics teachers' quantitative reasoning on their proof comprehension will be examined.

In the following section, research on proof, proof comprehension, quantitative reasoning and real numbers will be shared.

2. LITERATURE REVIEW

The aim of this section is to review research studies related to proof, real numbers and quantitative reasoning. Firstly, proof and proving will be defined. Then, research findings on students' proof conceptions according to different grade levels will be examined. Following that, studies on proof comprehension will be introduced and how to assess one's proof comprehension will be explained. After that, conceptions on real numbers will be discussed in detail. As a theoretical framework of the study, proof comprehension assessment model and quantitative reasoning with its relation to proof and proving will be emphasized. Lastly, how proof comprehension, quantitative reasoning and real numbers come together for this study will be shared.

2.1. Proof and Proving

In this section several definitions of proof and proving are given and the role of proving activity in learning mathematics is discussed.

Griffiths (2000) defined proof as "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion." (p.2). On the other hand, Rav (1999) defined the proof as "mathematicians' way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p.13) and additionaly made an analogy with the roads and bus stops referring to proof and statements to be proved respectively. Since the definition emphasizes mathematical machinery for solving problems, it puts emphasis on the role of proof in mathematics education such that it could be considered as problem solving and doing mathematics (Polya, 1981; Stylianides and Stylianides, 2009b). Similarly, Stylianides and Stylianides (2009b) explained proof as "general, valid, and accessible to the members of the community" (p. 239). Based on these definitions, it can be inferred that proof has a formal language of mathematics with the use of axioms, definitions and serves general and valid argument, accepted by mathematicians' community.

Since proofs are obtained via set of activities which is the action of proving, it is also necessary to define what proving is. Stylianides defined proving as "the activity associated with the search for a proof" (2007). Morris emphasized in detail that for proving one needs to form deductive arguments by inferring to definitions, axioms and already proven statements (2007). Moreover, Weber considered proving as "a complex mathematical activity with logical, conceptual, social, and problem-solving dimensions" (Weber, 2005, p.351).

According to Hersh (1993) the role of proof changes with respect to the field of use. It is different if it functions in a mathematical research and if it is used in a classroom environment. For the research area, the role of proof is to convince (Hersh, 1993) and also proof influences mathematical growth and development (Searle, 1977) but in a classroom setting proof helps conceptual understanding (Polya, 1981) and the role it has, depends on two different views: absolutist view and humanistic view. Absolutist view holds that "Without complete, correct proof, there is no mathematics" (Hersh, 1993, p.396) and humanistic view advocates that proof is an explanation. A teacher with absolutist view prefers to present more general and shorter proofs to the class however, a teacher with humanistic view maintains the idea of presenting proofs with explanatory power and argues that students need to use, play with enjoy mathematical tools.

Considering the role of proof in a classroom setting, Stylianides (2007) gave another definition of proof. He views proofs as mathematical arguments and asserts that it has certain properties. Firstly, proof contains statements which are familiar to all students in a classroom. Secondly, the reasoning behind the steps of proof should be valid for and known to all students in class. Lastly, the representation of proof should be accepted by the classroom community.

Since mathematical proof is seen as a problem solving activity and classified as doing mathematics, the function of proof in a learning environment is improvement of understanding. Also it is emphasized that the meaning of the statements and how they are connected in a proof is more important than procedural aspects of the proof (Hanna, 1995).

For the current study, since a teaching session took place, the definition of proof given by Stylianides (2007) considering a classroom environment was important and also, the other definition of proof that "general, valid, and accessible to the members of the community" (Stylianides and Stylianides, 2009b, p.239) was crucial.

2.1.1. Students' Conceptions of Proof and Proving

While teaching the process of proof, it is important to consider how students conceptualize and use proof in mathematics. The construct of proof conception was defined by Conner (2007) as "a person's ability to prove and analyze proofs and his or her beliefs about proof and justification." (p.11). Till then, some researchers examined students' way of reasoning and proof (Maher and Martino, 1996; Edwards, 1998; Healy and Hoyles, 2000; Stylianides, 2007; Chin and Lin, 2009; Al-Murani and Stylianides, 2010; Palla *et al.*, 2012; Reyhani *et al.*, 2012; Moore, 1994; Almeida, 2000; Stylianides *et al.*, 2004; Stylianides *et al.* 2005; Baştürk, 2010; İmamoğlu, 2010; Güler and Dikici, 2014). These research studies included different grade levels (elementary, middle, high school levels and college level). Findings from these studies showed that students' proof conceptions change with respect to different grade levels. In this regard, in the following sections students proof conceptions in different grade levels will be discussed separately.

2.1.1.1. Students' Proof Conceptions in K-12. Several research studies about proof were conducted in elementary (Maher and Martino, 1996; Stylianides, 2007) middle (Bieda, 2010) and secondary school levels (Edwards, 1998; Healy and Hoyles, 2000;

Chin and Lin, 2009; Al-Murani and Stylianides, 2010; Martinez *et al.*, 2011; Palla *et al.*, 2012; Reyhani *et al.*, 2012). The focus of these studies was students' thinking processes while generating proofs for several mathematical problems. In general, the findings of these studies pointed to the fact that students might develop different proof methods with the increase of grade levels. However, especially in high school level, some students have weaknesses in constructing proofs and understanding the structural characteristics of a proof construction.

In particular, Maher and Martino (1996) study focused on a student's proof actions within the first 5 years of schooling. They found that although this student firstly rejected a solution without a justification to the mathematics problems, within the continuum, she asked for justifications and developed different proof methods such as indirect proof, proof by cases. This suggests that once focused on reasoning about the solutions to the mathematical problems, even the students at the very early stages of education, might develop proof methods.

Similarly, Stylianides, (2007) also focused on third grade students' proof development, aim of which was to explain the role of teacher in the learning process of proving. It was found out that students firstly tried to engage in providing justification empirically (i.e. justification with examples) but the teacher did not accept empirical justifications as proof therefore, students were stimulated for generating formal proofs. Additionally, at the stages where students did not have enough knowledge to proceed for proving, teacher taught new concepts. For instance, students did not know about algebraic structures so teacher tried to explain algebraic structures to encourage students to construct more valid proofs. Therefore, once provided with algebraic structures such as using boxes and thinking with variables, they were able to explain the algebraic arguments which is the more general way of representing empirical arguments. This also indicates that once students were given opportunities to engage in reasoning while solving problems even in elementary school, their proof conceptions might develop to some extent.

Bieda (2010) focused on middle students' proof developments while dealing with proof-related activities as well as teachers' actions promoting such development. She found that students not only were able to produce conjectures to the tasks but also almost half of the conjectures they provided were proofs. However, as considering teachers' actions, it was found that in some situations teachers cannot provide sufficient feedback for students' conjectures to proceed to proving and also they complained about time restrictions to facilitate proof-related tasks in class.

As already mentioned, when the grade levels increase to high school, students start to use abstractions and algebraic forms. In this respect, students are expected to use algebraic proofs in proof-related tasks and move from empirical arguments to algebraic arguments because algebraic proofs help students both to understand and reason about algebra (Martinez *et al.*, 2011). However, some research results indicated that students were not able to construct algebraic proofs and they preferred using empirical arguments, although they knew that empirical arguments are not generalizable and they valued algebraic arguments. Even, some students stated that they did not understand algebraic proofs. Therefore, when they are expected to choose an algebraic proof as their approach, they either chose invalid algebraic proofs as true unconsciously or selected an invalid algebraic argument as giving the best mark. (Healy and Hoyles, 2000). Similarly, Reyhani *et al.* (2012), Edwards (1998), and Chin and Lin's (2009) study findings showed that majority of the students could not use algebraic expressions and also understand the steps of algebraic proofs. For instance, Edward (1998) showed that for problems asking the "truthness of the statements" such as "Even × odd makes even" (p.493), none of the students gave algebraic justifications; they even chose to give empirical proofs. This suggested that students had weaknesses in terms of constructing algebraic proofs. By the same token, Martinez *et al.*'s, (2011), study focusing on students' integration of algebra and proof showed that students had difficulties:

> "Shifting from using numerical examples towards using algebra to prove, defining the independent and dependent variables in the algebraic expression that they found, and shifting from using algebra in an equation solving mode

> towards using algebra to deductively generate equivalent expressions to ultimately prove the conjecture." (p.38).

Some other studies focusing on specific methods of proofs further pointed to students' difficulties in constructing and understanding proof statements. For instance, Al-Murani and Stylianides' (2010) study focusing on mathematical induction showed that majority of the students applied mathematical induction procedurally without understanding the relationship between the steps of construction. Also, some students could not explain the reason for the applicability of the method using natural numbers (Al-Murani and Stylianides, 2010). Furthermore, Palla *et al.*'s (2012) study about proof and refutation revealed that some students believed that a statement may have both a valid proof and a counter example. Participants with this idea were interviewed. During the process they had a chance to rethink if a problem might have both a proof and counter-example at the same time. Some students changed their mind during the thinking but still rejected to accept the validity of the proof for the given problem (Palla *et al.*, 2012).

All these studies point that high school students are expected to generate formal proofs rather than stating empirical arguments, understand proof steps conceptually and apply them for justification, and differentiate counter-example and proof. However, they have difficulties in both constructing and understanding algebraic proofs, differentiating counter-example and formal proof and understanding the structure of a proof conceptually.

In sum, aforementioned research on K-12 students' proof conceptions indicate the following issues: First, regardless of the grade level, students might engage in proving and develop different proof methods. Secondly, their development in proving and proofs gets sophisticated within years once they are provided with proof-related tasks. Third, teachers might stimulate such development by providing activities such as providing new concepts to facilitate the students' development of formal proof (Stylianides, 2007). However, although they are expected to construct and understand proofs, they have difficulties in doing both.

Taking into account the fact that someone might progress in doing and understanding proofs and teachers might foster such development, in this research study, prospective teachers were provided with problems starting from examples and ending with general structural statements. Also, they were asked to think about the problems in a specific way, focusing on quantities given in diagrams.

2.1.1.2. Students' Proof Conceptions in College Level. As already mentioned under the section 2.1.1.1. that while the grade level increases, students are expected to perform formal proving activities that is to use formal language of mathematics and reduce the use of informal proofs.

Harel and Sowder (1998) stated that it is essential to think about the ideas that stimulate students' thinking about mathematical proof. In this sense, they classified college mathematics students proof schemes cognitively. By 'proof scheme' they considered students' justification types. Using this idea, they found out the following schemes: External conviction proof scheme, empirical proof scheme and analytic proof scheme. External conviction proof scheme includes ritual, authoritarian and symbolic proof schemes. A student having this proof scheme can rely on appearance of a proof, symbolic language used in the proof and an authority's way of representing proof to accept that the proof is valid. Empirical proof scheme consists of inductive and perceptual proof schemes in which a student can trust to the evidence obtained by examples. Lastly, a student with analytic proof scheme can transfer the mental images that s/he create to a formal mathematical proof. Although, these different proof schemes seem to be separate, a student can have more than one proof scheme which can affect producing formal mathematical proof.

Although college students are expected to give more sophisticated proofs, some research studies showed that students have difficulties as producing mathematical proofs. Specifically, Moore (1994) conducted a research study to determine prospective mathematics teachers and mathematics majors difficulties about proving. The data, collected through lesson observations and interviews with participants and the instructor, indicated the following sources of difficulties considering proof practices: lack of knowledge about definitions of terms and statements and how to use them in the proof, lack of understanding the concepts and also mathematical language, students' inadequate concept images, students' failure to generate or to use their own examples about the proof statement, and lack of knowledge about how to start proving. The studies that are given below also point to the same difficulties about proof and proving.

In particular, Almeida (2000) conducted research on undergraduate freshmen and sophomore mathematics students' perceptions and preferences about proof. The results indicated that freshmen and sophomore students' perception of proof were similar to each other. Additionally, most of the students thought that proving is an external activity. In terms of proof preferences, they appreciated informal proofs and some students found formal proofs difficult to construct.

Some studies were about students' understandings or performances considering a particular proof strategy. Specifically, Stylianides *et al.*, (2004) examined mathematics and mathematics education majors understanding of contrapositive rule. The participants were given 3 questions about contrapositive rule. One of them was in symbolic expression and two of them were in verbal expression (one is invalid). Results showed that understanding contrapositive rule was affected by the department of the participants. Most of the mathematics education majors performed better on questions given with verbal expression than on question given with symbolic expressions. Among mathematics majors, there was no significant difference between the performances on three questions. The study indicated that some participants had the following incorrect belief: if p, then q is equivalent to if not p, then not q.

In a similar study, Stylianides *et al.* (2005) focused on mathematics and mathematics education students' understanding of mathematical induction. they were given the following statement 'For every natural number n $\geq$5 the following is true:1 * 2 * ... * (n – 1) * n >2^n' (p.242) and the proof of it by mathematical induction and also the truth of the statement for the cases n= 3,4,6,10 was asked. Most of the mathematics students responded correctly that the proof is valid for the statement whereas almost half of the mathematics education students responded correctly. Although the statement is true for n=4, some students argued that the statement is not valid for n=4 and based on this idea some of the students responded the the statement is true for some cases. The reason behind this idea can be the following: 4 is not an element of the set given in the statement. Researchers contended that some students have the idea that the truth set and the domain set should be the same for a statement to be true.

In an another study about teaching mathematics students' conceptions of proof and proving, freshman teaching mathematics students emphasized that they have difficulties in constructing proof however, they valued the role of proof in mathematics lesson. While preparing for mathematics exams, they mostly prefer to memorize the proofs and focus on the procedure of the proof rather than the ideas and connections between them in the proof (Baştürk, 2010).

İmamoğlu (2010) searched about freshman and senior teaching mathematics and mathematics majors' conceptions and belief concerning proof. The results demonstrated that there is no significant difference between the beliefs and the proof performances of freshman teaching mathematics and mathematics students. However, among senior students, mathematics majors performed better in proof exam and proof evaluation exam than teaching mathematics majors.

Güler and Dikici's (2014) study of prospective mathematics teachers proof performances on algebraic concepts indicated that prospective teachers found difficult to determine how to start proving that is deciding the proof strategy to use and some prospective teachers did not know proof methods. Additionally, some prospective teachers could not use prior knowledge required for proving in some algebraic proofs since they could not connect the ideas to each other. In consideration of the findings, researchers stated that starting to prove in a proper proof strategy aids to complete the proof in a correct way.

In conclusion, most of the students in college level have some obstacles related to proof such as determining the proof strategy, connecting the ideas in a proof, use of prior knowledge, generating and using examples. Under the guidance of aforementioned studies, the current study focused on generating and use of examples, and use of prior knowledge to comprehend a proof.

2.2. Proof Comprehension

In this section, first, proof comprehension is defined. Then a proof comprehension assessment model is shared (See also Section 2.5.1) and research studies on proof comprehension are summarized.

Yang and Lin (2008) defined proof comprehension as follows:

> "....reading comprehension of proofs means understanding proofs from the essential elements of knowing how a proof operates and why a proof is right, in addition to knowing what a proof can prove." (Yang and Lin, 2008, p. 60)

As stated by Yang and Lin (2008), although proof comprehension involves reading proofs, Samkoff and Weber (2015) pointed to the distinction between proof reading and proof comprehension: They stated that proof reading activity gives students a general idea about the proof read; however, proof comprehension requires students having the dimensions developed by Mejia-Ramos *et al.* (2012). These dimensions are (i) meaning of terms and statements, (ii) logical status of statements and proof framework,(iii) justification of claims, (iv) summarizing via high-level ideas, (v) identifying the modular structure, (vi) transferring the general ideas or methods to another context and (vii) illustrating with examples (Mejia-Ramos *et al.*, 2012).

Although these dimensions were specifically identified for the proof comprehension on undergraduate level (Mejia-Ramos *et al.*, 2012), their roots stood upon some previous research (Conradie and Frith, 2000; Yang and Lin, 2008). Particularly, Conradie and Frith (2000) focused on the idea of comprehension test. They emphasized that testing students' understanding of a proof via "state and prove" format can have some drawbacks. For instance, students may have a tendency to memorize the proof word by word and write the proof as appeared in the textbook or in their notebooks. Thus, it might be hard to determine students' level of understanding. Also, students with low proving ability can do their best if they memorize the proof(s) correctly. This might hinder what they understand about the proof(s), too. In contrast, comprehension tests might lead students to work on proofs to understand rather than to memorize. In addition, multiple dimensions can be assessed through these tests, such as "understanding of the specific steps in a proof", "understanding of the structure of proof", "understanding of concepts used in the proof", "understanding of assumptions and conclusions" and "understanding of some of the more subtle aspects of a proof" (Conradie and Frith, 2000).

2.2.1. Research Studies on Proof Comprehension

There are several research studies about proof comprehension but they are scarce. Therefore, there can be a need for this research area (Mejia-Ramos and Inglis, 2009). The current studies on proof comprehension have different foci. For instance, some of them examines the effect of proof representation on proof comprehension, some of them are about the effects of teaching method on proof comprehension and also there are studies focusing on students' approaches to comprehend proofs. Apart from these studies, researchers investigated the prospective teachers ways to assess their students' proof comprehension and students' beliefs on proof comprehension. In the following sections, studies on proof comprehension are discussed.

2.2.1.1. Proof Representation and Proof Comprehension. Due to the idea that proof representation can affect proof comprehension, there are some research studies examining this issue (Fuller *et al.*, 2014). Therefore, some alternative presentations for proof comprehension were suggested by researchers: Structured proofs in which proof is given as divided to levels including summary of the proof focusing on methods and general idea, logical connections between steps of the proof (Leron, 1983), generic proofs (also called transparent proof), which focus on a specific example (Rowland, 2002), e-proofs that is presented through the slides on the internet (Alcock, 2009).

In particular, Movshovitz-Hadar (1997), presented a transparent proof of a summation formula (proof of a particular case) to the prospective teachers:

$$1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2 \tag{2.1}$$

The proof was in pictorial form and a prospective teacher wrote the proof in a formal way by examining the visual and expressed the case for any number n using the particular case (transforming geometrical expression to algebraic expression). Via interviews one of the participants emphasized that just one example leads to reach the general structure and the other participant concluded that by pictorial representation s/he has an understanding of the concept so s/he expressed it in a formal way (Movshovitz-Hadar, 1997).

Similarly, the effect of transparent proofs on students' ability to comprehend and construct proofs was the aim of Malek and Movshovitz-Hadar's Study (2011). Participants who were taking Linear Algebra at the time of study, were presented transparent proofs about linear algebra concepts. The data collected both quantitatively and qualitatively showed that transparent proofs had a positive effect on students' proof comprehension of formal proofs and proof construction. Since, via transparent proofs one understands the meanings of the statements and connection between them and transparent proofs can be considered as a concrete object which helps to develop the formal structure.

Different than presenting transparent proofs, Fuller's *et al.*, (2014) focus was the effect of presenting structured proofs on proof comprehension of mathematics majors. The results indicated that there is no significant difference between proof comprehension test performances of participants who were presented structured proofs and who were presented linear proofs. However, participants of structured group emphasized that they valued the summary part of this kind of representation and also they did better on questions about the summary of the proof since they comprehend the main idea of the proof. On the other hand, some participants found structured proofs as complex to comprehend (Fuller *et al.*, 2014).

Roy *et al.* (2010) determined the effect of presenting a proof in three different ways on undergraduate mathematics students' proof comprehension. The three forms are presenting proof from a textbook, by a lecture and as a computer based e-proof. All the mathematics majors in their second or third year and took the real analysis course participated in the study. Participants were randomly divided into three groups as reading group, lecture group and and e-proof group. At the end of the intervention all took a comprehension test which was prepared according to Yang and Lin's model and two weeks later they took the delayed post comprehension test. Results indicated that lecture group did better in both post and delayed post test and there was no significant difference between textbook and e-proof groups.

The research studies explained previously showed that representation type affects proof comprehension. Generic proofs develops comprehension and transition to the general case that is to the proof. Also, lecture is superior to presenting proofs in a textbook or as an e-proof. This findings contributes to the current research study. Although the focus was the effect of prospective mathematics teachers' quantitative reasoning on proof comprehension, in the lecture which was based on quantitative reasoning, at least two specific examples were used to reach generalization. Therefore, use of generic examples focusing on quantitative reasoning might have an effect on prospective teachers' proof comprehension.

2.2.1.2. Teachers' and Students' Beliefs on Proof Comprehension. The ways preferred by teachers to teach proofs affect their instruction since their role affect students' learning processes (Malek and Movshovitz-Hadar, 2011). Also, students' ways of working on proofs and their belief about what to do to comprehend proofs are essential to consider.

Weber and Mejia-Ramos (2014) examined students' and their instructors' beliefs related to proof reading. their findings showed that students and instructors had contradictory beliefs about proof reading. For instance, most of the students thought that they do not need to give justifications to the statements given in the proof, not need to spend a long time to understand a proof and to draw diagrams or shapes. Additionally, their opinion is that in order to comprehend a proof, it is essential to understand all the statements and their justifications. on the other hand, most of the instructors gave importance to working on the proof while reading it and drawing diagrams for comprehension. For them the time alloted for studying proof is more than 15 minutes. One reason about the appearance of these contradictory beliefs was the less emphasis of proof reading in class and assessment of proof comprehension (Weber and Mejia-Ramos, 2014).

In another study, Zazkis and Zazkis (2016) evaluated how prospective teachers can determine their students' comprehension of a given proof. For the study proof of Pythagorean Theorem was chosen. Researchers prepared a proof comprehension test based on the model suggested by Mejia-Ramos and his colleagues (2012) but they focused on three dimensions as follows: Meanings of terms and statements, justification of claims and summarizing main ideas. The comprehension test was used while analyzing the data. Researchers expected twenty four prospective teachers to write teacher-student dialogue about the proof of the Pythagorean Theorem which has the aim of assessing students' comprehension of the given proof. In the analysis, based on the comprehension test question prepared previously, they determined whether dialogues meet the requirements of the three dimensions. The results figured out that teachers less focused on holistic comprehension which refers in this study to summarizing main ideas. Only six dialogues included questions about holistic comprehension. Also, it was found that for justification of claims dimension, in some cases prospective teachers wrote in dialogues incomplete or incorrect justifications. Prospective teachers ignored some necessary points for the comprehension of proof which are included in the comprehension test. This research study showed that teachers' view of proof comprehension affects their assessment process on students' proof comprehension.

The studies showed that teachers' perspectives on proof comprehension affect their instructional decisions therefore, students' beliefs and comprehension levels can be influenced by their instructional implementations. Its relationship to the current study was due to the effect of instruction on students' thinking.

2.2.1.3. Teaching Strategy and Proof Comprehension. In terms of the effect of teaching strategy or instruction on proof comprehension, different research studies were conducted focusing on different methods.

One of the studies was about the effect of self-explanation training on proof comprehension (Hodds *et al.*, 2014). Researchers conducted three different experiments on proof comprehension and self-explanation training and gathered data via proof comprehension test prepared according to the dimensions of proof comprehension (Mejia-Ramos *et al.*, 2012). Two of them were computer-based studies and the last one was instruction based on self-explanation. Analysis of three experiments indicated that self-explanation method improves students' performances on proof comprehension test.

In a recent study of Samkoff and Weber (2015), the effect of instruction about the proof comprehension strategies on students' proof comprehension was examined. The study was conducted in two iterations and for the first iteration participants were six students from real analysis course and in the second iteration two students from teaching mathematics department who completed real analysis course were participated. During the iterations, participants focused on proofs using the following strategies by the guidance of the researcher and they were asked about the contribution of each strategy:

(i) "Make sure you understand the definitions of all the terms,
(ii) Illustrate the theorem with and example,
(iii) Describe how you would try to prove the theorem,
(iv) See if the proof can be broken into independent parts,
(v) Identify what proof methods are being used,
(vi) See why confusing statements are true with a particular example,
(vii) Compare the approach you chose to the proof you read" (Samkoff and Weber, 2015, p.30).

The results showed that apart from the strategy "See if the proof can be broken into independent parts", all other strategies contributed students' proof comprehension. For instance, illustration of the theorem with an example helped them see the general picture and comprehend the proof. Also, emphasis on definitions of terms and concepts enriched the comprehension.

These studies contributed to the current study in terms of the effects of example usage, definitions of terms and concepts on proof comprehension since the study included a sequence on examples and emphasis on definitions and terms in the training of participants.

2.3. How Proof and Proving can be taught

Proofs should promote mathematical understanding in a mathematics classroom (Weber and Mejia-Ramos, 2014). That is why the presentation method of a proof and teaching of proof is essential to consider. The studies about students' proof conceptions showed that students' difficulties on proof can be due to their misinterpretations of definitions, statements or concepts (Moore, 1994). Therefore, teachers have a great role on how to improve students' proving abilities and comprehension abilities. In this section, the suggested ways for teaching proof will be explained.

According to Dormolen (1977), for teachers it is important to consider that students cannot be expected to perform in a higher level, for instance giving a formal, deductive proof, without performing in lower levels satisfactorily. Therefore, in order to make students to prove conjectures and comprehend proofs and eliminate their misconceptions and difficulties on proving, they should be taught extensively on reasoning and principles and methods of proving. Therefore, teachers' knowledge for teaching proof plays an important role in this process (Stylianides *et al.*, 2005; Stylianides, 2007; Steele and Rogers, 2012; Güler and Dikici, 2014).

In the research study of Steele and Rogers, one prospective mathematics teacher and one experienced mathematics teacher were observed in their teachings of a particular proof of a topic and data collected through observation, a written test on proving statements and semi-structured interviews to determine their mathematical knowledge for teaching on proof revealed that the achievement on teaching proof to students is strongly affected by mathematical knowledge of teachers (Steele and Rogers, 2012)

Ball and her colleagues see proving as a branch of problem-solving activity and suggest that lessons on proof can facilitate students through the following activities: "to mix deduction and experiment, tinker with ideas, shift between representations, conduct thought experiments, sketch and transform diagrams." (Ball, *et al.*, 2003, p.912). In this sense, Almeida (2000) pointed out the mathematicians' thinking process while providing a proof as 'intuition, trial, error, speculation, conjecture, proof' but this is inconsistent with in class proof teaching since the process continues as definition, theorem and proof which can be the reason for students' difficulties in proof and proving (Atiyah *et al.*, 1994). In order to facilitate the learning of formal proof, Blum and Kirsh (1991) they suggested to use informal proofs (use of visuals, diagrams, generic examples). Ball *et al.* (2003) also argue that transparent proofs (called also generic proofs which focus on a specific example leading to generalization) can be beneficial for teaching proof, because transparent proofs presents a particular but a generalizable case of a conjecture and students may shift to the formal structure and write proofs. Also, they defend that use of transparent proofs can enhance students' comprehension of a formal proof of a conjecture (Ball *et al.*, 2003).

In a study (Weber, 2012) about mathematicians' pedagogical preferences while teaching proof, it was found that some mathematicians suggested to use diagrams and examples to make students to comprehend proofs. This was also evident from the study of Gibson (1998) that students who used diagrams succeeded in completing the tasks that they could not complete before as only thinking through verbal-symbolic representations. In terms of use of examples it was stated that "Most mathematicians spend a lot of time thinking about and analyzing particular examples." (Epstein and Levy, 1995, p.670). The reason can be due to the fact that they are taken as strategies in problem-solving and proving is a kind of problem-solving. Also Ellis and her colleagues (2013) argued that use of examples can enhance students understanding and constructing conjectures and proofs but students should be taught how to analyze examples since students should also be aware of the limitations of the empirical arguments and use them as transition to formal proof. Also, "understanding what processes can be used to reap the benefits from diagrams and examples and designing instruction so that students can employ diagrams and examples more effectively" is important to consider (Stylianides, Stylianides and Weber, 2016).

As conclusion, studies showed that empirical justifications and diagrams or visuals can enhance students' proof constructions and proof comprehension since, they provide a transition to formal structure. In this respect, the current study contributed to the literature by providing the effect of using generic examples along with quantitative reasoning on the proof comprehension.

2.4. Real Numbers

Numbers are in the center of mathematics learning and curriculum (NCTM, 1989; MoNE, 2013). Starting from national numbers students try to construct integers, rational numbers, irrational numbers, real numbers and complex numbers. The conceptual understanding of students about the properties of numbers system, components and transition (for instance, construction of rational numbers from integers, real numbers from rational numbers) are essential to consider. Misconception in one level can hinder conceptual understandings in higher levels (Voskoglou and Kosyvas, 2012).

In this study, decimal representation of real numbers was considered as a case on the effect of prospective mathematics teachers' quantitative reasoning on their proof comprehension. The reason for chosing real numbers was due to the fact that students have misconceptions about real numbers (Tall and Schwarzenberger, 1978; Ely, 2010; Voskoglou and Kosyvas, 2012, Voskoglou, 2013) (can be considered separately as rational numbers and irrational numbers) and also the construction of real numbers could be taught by quantitative reasoning (Karagöz-Akar, 2016).

In this section, students' misconceptions about real numbers and some suggested teaching techniques will be explained.

2.4.1. Students' Conceptions of Real Numbers

Due to the importance of the numbers in mathematics education, some research studies conducted to investigate students, pre-service and in-service teachers conceptions of numbers. In this section, specifically research studies focusing on conceptions about real numbers will be explained.

Tall and Schwarzenberger (1978) emphasized that students have misconceptions about the relationships between decimal and limit concept and decimal and fractions. Difficulty aroused about real numbers was that students had a confusion when two different decimals represented the same real number (Tall and Schwarzenberger, 1978). They also stated that when the series and sequences were taught without students' complete understanding of real numbers, decimals and fractions, students might not undestand these concepts.

Similarly, in an another study (Ely, 2010) about students' conceptions on infinitesimals, an undergraduate calculus student was interviewed. In addition to misconceptions on infinitesimals, the researcher revealed the following misconception: the student did not think that 3.99999.... and 4 represents the same number. Also, when asked whether 1/3=0.33333...., the student answered as no however, when asked whether she believed that 1/2=0.5, she answered as yes (Ely, 2010). This may show that student could not explain the decimal-limit relationship as in the previous research.

Voskoglou and Kosyvas (2012) examined students' difficulties in understanding real numbers via quantitative and qualitative studies. The results of the study investigated the followings: students had an incomplete understanding of rational numbers. It is observed that they prefer to deal with decimal representation more than fractions. It may give the intuition that they may not combine fraction and its decimal representation. The researchers found that students having difficulties in rational numbers could not answer the questions about irrational numbers. Therefore, they claimed that for conceptual understanding of irrational numbers, it is essential to understand rational numbers conceptually since the former is built on the latter.

2.4.2. How Real Numbers and Their Decimal Expansions can be taught

For teaching of real numbers, there is a need to consider students' prior knowledge on numbers (natural numbers, integers, rational numbers) and how they are constructed based on the previous set of numbers (Voskoglou and Kosyvas, 2012). Based on the findings of Voskoglou and Kosyvas (2012) about students' difficulties on fractions and decimal representations, it can be suggested that in order to teach fractions and decimals it is necessary to combine fractions and division algorithm because division algorithm leads students to relate decimal representation and fractions (Schwanke, 2008). After the complete understanding of rationals, it can be easy to construct irrational numbers and form the set of real numbers. The suggested way of teaching real numbers is the use of "multiple representations of real numbers (rational numbers written as fractions and periodic decimals, irrational numbers considered as non rational ones and as incommensurable decimals which are limits of sequences of rational numbers, geometric representations, etc) and on flexible transformations among them." (Voskoglou, 2013, p.41).

In another study, pre-service teachers understandings of the relation between the fraction or integer and its decimal representation was examined (Weller *et al.*, 2009). As teaching strategy, APOS Theory (Asiala *et al.*, 1997; Dubinsky and McDonald, 2001), which is developed based on the ideas of Piaget, was used in the experimental group. APOS refers to the initials of action, process, object, schema. According to this theory, "a mathematical concept develops as one tries to transform existing physical or mental objects" (Weller *et al.*, 2009, p.8). In the study, an instruction focusing on fraction and decimal representation based on APOS Theory and supported by activities on the computer, classroom discussion, and exercises done outside of class was given to the experimental group when the control group took a traditional instruction. The results of the study showed that pre-service teachers in the experimental group construct the relationship between fraction or integer and its decimal representation conceptually.

Since students have misunderstandings about the relationship between limit and decimal representation, decimal representation and its fraction, the existence of two different decimals representing the same number, the current study focused on these misconceptions and the suggestions for the teaching of real numbers given here were used. For instance, researcher emphasized the relation between division algorithm, fraction and its decimal representation and also the concept of limit, sequences was used to show how to reach an approximate value of a real number. Also, through the quantitative reasoning, participants were encouraged to construct decimal representations via mental operations.

2.5. Theorethical Framework

In this research study, since the effect of prospective mathematics teachers' quantitative reasoning on their proof comprehension was examined, this section was dedicated to the proof comprehension assessment model (Mejia-Ramos *et al.*, 2012) which was used in this study and quantitative reasoning which was considered during the instruction. First proof comprehension assessment model will be explained and then quantitative reasoning will be defined under the light of literature.

2.5.1. Proof Comprehension Assessment Model

In this section, how proof comprehension assessment model (Mejia-Ramos *et al.*, 2012) evolved will be explained in detail.

Following Conradie and Frith (2000), Yang and Lin (2008) made an attempt about proof comprehension of geometric arguments. They proposed a model of proof comprehension with four levels (See Figure 2.1): surface, recognizing elements, chaining elements, and encapsulation. The surface level consists of students' defining and explaining mathematical terms, concepts and meanings of symbols. Identifying the proof method is also included in the surface level. The level recognizing elements involve students' identifying the components of the proof such as recognizing premises and conclusions and the properties of the concepts. The chaining elements level includes students' connecting the premises to each other and to the conclusion such that they are expected to identify logical relations between them. Finally, the encapsulation level embraces comprehending the proof holistically and interiorizing the proof with the ability to apply it to other contexts (Yang and Lin, 2008).

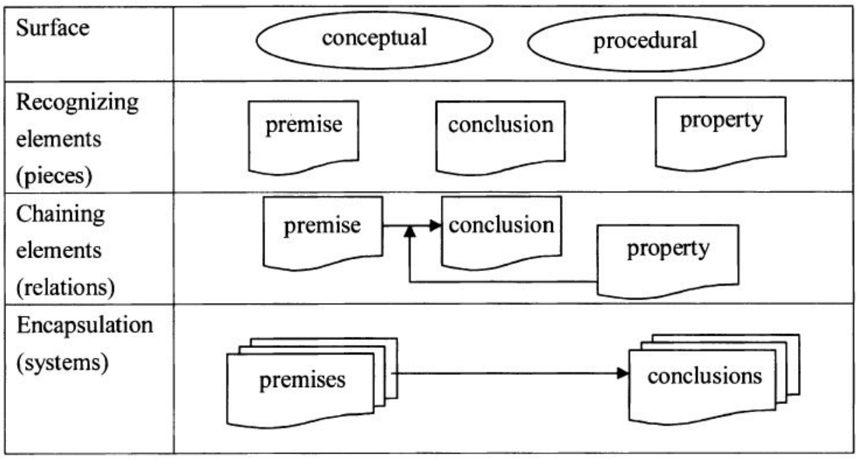


Figure 2.1. Yang and Lin's Hypothetical Level of Geometric Proof Comprehension (Yang and Lin, 2008, p.63)

Yang and Lin (2008) also defined five facets connecting the levels of understanding to each other as shown in Figure 2.2: basic knowledge, logical status and summarization, generality and application.

Based on Yang and Lin's (2008) model of levels of reading comprehension, Mejia-Ramos *et al.* (2012) developed of an assessment model of proof comprehension at undergraduate level. They directly adopted first three levels (Yang and Lin, 2008) and classified them as local comprehension of a proof. They then named the fourth dimension as the holistic comprehension of a proof. This model might enable instructors to develop questions for the comprehension of different proofs. Since this study will be based on Mejia-Ramos *et al.* (2012) study, the dimensions, their explanations and how they could be assessed are given in detail below.

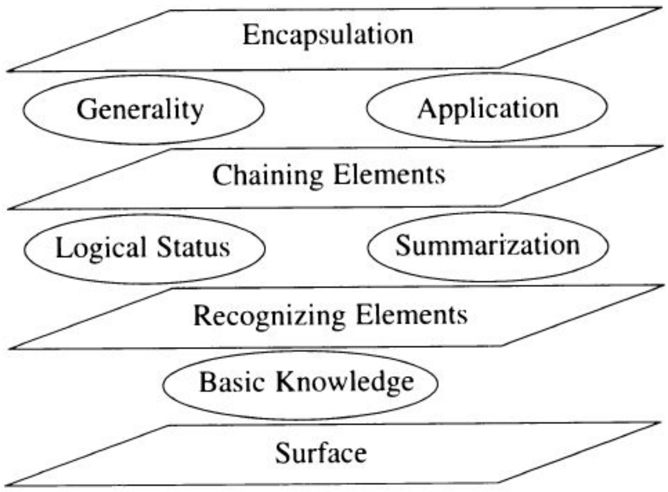


Figure 2.2. The Relationship between Levels of Comprehension and Facets (Yang and Lin, 2008, p. 71)

The dimensions of proof comprehension can be categorized in the following way: Local comprehension of a proof and holistic comprehension of a proof.

Dimensions of local comprehension of a proof are the following:

(i) Meaning of terms and statements: involves students' knowing the meanings of terms and statements in the proof. Once asked, Students are expected to define the terms or statements and provide examples about the use of them.

(ii) Logical status of statements and proof framework: involves knowing several ways of proving such as direct proof, proof by contradiction, proof by contrapositive, mathematical induction since each proof strategy has a different logic. Students are expected that they realize the logic of a given particular proof For assessing this dimension, it is suggested to ask the type of proof technique and purposes of using a particular technique or use of assumptions.

(iii) Justification of claims: involves knowing that mathematical sentences in a proof are logically connected to each other. Examining this connection one reaches the conclusion. Therefore, in order to comprehend a proof, students are expected to make connections among the statements in the proof. For assessing this dimension, it is suggested to ask questions focusing on the purpose of using particular statements and their relations to each other.

Dimensions of holistic comprehension of a proof are the following:

(iv) Summarizing via high-level ideas: includes realizing the aim of the proof (i.e. big idea of proof). Therefore, to assess this dimension, it is suggested to ask the summary of the proof or the summary of a piece of proof crucial for comprehension.

(v) Identifying the modular structure: involves how to use "lemmas, corollaries and sub-theorems". (Mejia-Ramos *et al.*, 2012, p.12). Therefore, students are expected to understand them and why they are used in the proof. In this way, students can focus on modular structure of the proof. For assessing this dimension, it is suggested that students can be asked to divide a given proof in to modules, determine the aim of using any particular module and the connection between the modules.

(vi) Transferring the general ideas or methods to another context: involves understanding the proof holistically such as the general ideas and the methods in the proof.In order to assess this understanding it is suggested to ask transferring the method to another context.

(vii) Illustrating with examples includes the idea of using the proof in a problem or in an example. Therefore, it is suggested that students can be asked to apply the proof to an example or explain the proof with a diagram.

The proof comprehension dimensions explained above have been developed recently. For this reason, the researchers suggested that the proof comprehension assessment model can be used as a methodological tool by educational researchers. They proposed that in order to evaluate "the effectiveness of instructional interventions designed to increase the comprehensibility of mathematical presentations" (Mejia-Ramos *et al.*, 2012, p.16), the proof comprehension assessment model can be used. By the same token, in this research study the model helped to form the instruction to evaluate the development of students' performances in the comprehension of a specific proof.

2.5.2. Quantitative Reasoning

In this section, quantitative reasoning which was used as a tool in the instruction of this research study will be explained.

Quantitative reasoning was first developed by Patrick W. Thompson and defined as the analysis of a situation into structure -a network of quantities and quantitative relationships" (Thompson, 1990, p.13). Another definition was given by Weber *et al.* (2014) as "a way to describe the mental actions of a mathematical situation, construct quantities in that situation, and then relates, manipulates and uses those quantities to make a problem situation" (p.25). In order to conceptualize these two similar definitions, it is important to understand the meaning of quantity and quantification. Also, which operations lead to quantitative reasoning is essential to consider. In this section, quantity, quantification and operations as reasoning quantitatively will be explained by the support of examples.

Quantity is defined as "a conceived attribute of something that admits a measurement process, where this 'something' could be image of a situation interpreted from a problem statement or a mathematical object" (Moore *et al.*, 2009, p.3) From this definition it can be inferred that quantity is formed in the mind (Thompson, 1993; Thompson, 2011). For instance, if the fraction 1/3 is considered by a student, s/he can recognize it as a selecting one piece from three equal pieces of a whole. This process occurs in the mind since student makes partitioning on an object and gives a numerical value to it. Referring to the definition of quantity, image of a situation in the example is thinking a whole and making partitioning and mathematical object is the fraction 1/3. This process of doing a mental construction leads students to think through quantities and supported by mathematics educators as an essential tool for mathematics (Thompson, 2011).

Quantification defined as “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure and the attribute’s measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit.” (Thomson, 2011, p.37). The example for quantification is given by Thompson and Thompson (1994) as thinking through a speed of a car. In order to determine the speed of a car which is called as object and here attribute refers to having a velocity which can be measurable by determining the distance traveled and times passed. If students think over this situation, they make a quantification for the speed of the car. They measure the speed via traveled distance per time.

The quantitative operations which leads to quantitative reasoning are asserted by referencing to the studies of Thompson (1994) and Thompson (2011) and Steffe (1991) as unitizing, segmenting, measuring, grouping, equal partitioning, counting and matching (Karagöz-Akar, 2016).

Quantitative operations enhance students to think more on the situations and the meaning of the numerical values obtained through quantification. Therefore, quantitative reasoning is more than numerical operations (Thomson, 2011). Based on this argument Ellis (2007) argued that students who focus on generalizations by quantitative reasoning can perform better in constructing proofs than students who focus on empirical examples as obtaining generalizations. The relationship between proof and quantitative reasoning is due to the development of generalization. It is emphasized that proof is a kind of generalization (Otte, 1994) and generalization can be supported by quantitative reasoning. Additionally, Weber *et al.*, (2014) emphasized based on the research studies that quantitative reasoning is essential for developing students' thinking process on proof.

By the support of aforementioned studies, the current study focused on the effect of quantitative reasoning, which includes basically the operations, the equal partitioning and unitizing operations, on proof comprehension ability. Our hypothesis was that quantitative reasoning can eliminate possible difficulties related to proof such as lack of knowledge about definitions of terms and statements and how to use them in the proof, lack of understanding the concepts and also mathematical language, students' inadequate concept images, students' failure to generate or to use their own examples about the proof statement, and lack of knowledge about how to start proving which are expressed by Moore (1994) as students' difficulties in learning proof and proving.

2.6. Proof Comprehension, Quantitative Reasoning and Real Numbers

In this section, how three disconnected parts, proof comprehension, quantitative reasoning and real numbers, come together will be discussed. Research studies suggested that generalization and proof can be supported by quantitative reasoning (Ellis, 2007; Weber *et al.*, 2014) that is students who deal with concepts via quantitative reasoning were able to reach generalizations compared to the students who dealt with only empirical examples (Ellis, 2007). In this sense, quantitative reasoning helps students to understand the topics conceptually and obtain generalizations. Since proof is a kind of generalization (Otte, 1994), understanding topics is also an issue for proof comprehension. For instance, in the proof comprehension assessment model (Mejia-Ramos *et al.*, 2012), dimension 1 (Meaning of terms and statements) was included. This dimension aims to reveal whether students have backround knowledge about the concepts and statements or they understand them in the proof. Therefore, this alligns with quantitative reasoning. Other than dimension 1, dimension 6 (Application to examples) is also related to quantitative reasoning. For this dimension, students are expected to apply the theorem in examples. In quantitative reasoning, students deal with not only the numbers but also the quantities. Therefore, they need to think through examples. When the real numbers and proof are considered. It can be said that they match in terms of the suggestions about teaching of them. For the teaching of real numbers, researchers suggested to use multiple representations of real numbers and also transitions between representations (Voskoglou, 2013). By the same token, for teaching of proof, students are expected "to mix deduction and experiment, tinker with ideas, shift between representations, conduct thought experiments, sketch and transform diagrams." (Ball, *et al.*, 2003, p.912). In terms of the use of multiple representations, real numbers and teaching proof align. Another important point is that for the teaching of proof, use of diagrams is given as suggestion which might also contribute to the students' comprehension of a proof. This suggestion matches with quantitative reasoning because in order to think through quantities, students deal with some objects which could be represented by diagrams. In sum, quantitative reasoning, proof comprehension, and real numbers could be seen as inter-related.

3. SIGNIFICANCE OF THE STUDY

The main significance of this research study is due to the importance and place of proof and proving in both mathematics and in the curricula (e.g. NCTM standards and MoNE high school mathematics curriculum). By many researchers proof is seen as an activity classified as doing and understanding mathematics (Polya, 1981; Kitcher, 1984; Hanna, 2000; Stylianides, 2007). The most of the research studies showed that students, prospective teachers have difficulties in proving and comprehending proofs (Moore, 1994; Stylianides *et al.*, 2004; Stylianides *et al.*, 2005; Baştürk, 2010; İmamoğlu, 2010, Güler and Dikici, 2014). Therefore, there is a need to find a solution to decrease difficulties related to proving and comprehending proofs. One solution for narrowing down problems on proving and proof comprehension is to focus on how to teach proof. Some researchers pointed out that definition, theorem, proof sequence is insufficient to expect a development on students' and prospective teachers' ability to prove (Almeida, 2000). This kind of sequence may lead to memorization and misunderstanding of proofs. It is suggested that use of generic proofs (starting with a particular example which can lead to generalization) can be effective (Moshovitz-Hadar, 1996; Ball *et al.*, 2003). Also, there is a need for research studies about teaching strategies about proof (Ball, *et al.*, 2003; Stylianides and Stylianides, 2009a). In this sense, current research study contributes to teaching strategies.

Proof comprehension was the focus of the study. There are many research studies conducted on proof conceptions and difficulties however, there are a few research studies focusing on proof comprehension (Selden, Selden, 2003; Mejia-Ramos *et al.*, 2010; Roy *et al.*, 2010; Zazkis and Zazkis, 2016). Also, it is emphasized that it is essential to conduct research studies on proof comprehension (Mejia-Ramos, *et al.*, 2012). There is a need for empirical research studies focusing on teaching about proof and analysis of the processes in the development of students' proving abilities. (Ball, *et al.*, 2003)

The instruction based on quantitative reasoning about decimal representation of real numbers was given to the prospective mathematics teachers and in this instruction a particular case was used with the support of diagrams and visuals. One important aspect of the lesson is to focus on quantitative reasoning since it is emphasized that quantitative reasoning improves students' thinking about proof (Weber *et al.*, 2014). Also, Amy Ellis who is assistant professor in the field of middle-school and secondary mathematics education started to a research named "Supporting Students' Proof Practices Through Quantitative Reasoning in Algebra" emphasized that the research study will connect proof and quantitative reasoning although initially they seem to be disconnected (Ellis, 2010). Therefore, this current research study also examined the effect of quantitative reasoning on proof comprehension and contributes to the literature by connecting quantitative reasoning and proof. The second aspect of the lesson is to focus on a particular example to the general structure and the use of diagrams and visuals. Researchers emphasized that diagrams and visuals can contribute to proof understanding and proof construction (Hersh, 1993; Movshovitz-Hadar, 1997). In addition, it is emphasized that there is a need to conduct research studies related to the use of visuals and diagrams while teaching proof (Gibson, 1998; Fuller *et al.*, 2014).

4. STATEMENT OF THE PROBLEM

The purpose of this research study is to explore prospective mathematics teachers' quantitative reasoning on developing the decimal representation of real numbers and its effect on their comprehension of a proof of statement related to decimal representation of real numbers.

4.1. Variables of the Study

The independent variable of the study is the type of instruction and the dependent variable of the study is participants' performance on proof comprehension.

4.2. Research Questions

During the research study the following research questions were investigated:

(i) What are the current levels of prospective mathematics teachers in the pre-proof comprehension test according to the proof comprehension dimensions?

(ii) What are the current levels of prospective mathematics teachers in the post-proof comprehension test according to the proof comprehension dimensions?

(iii) Is there any significant difference between pre and post proof comprehension test results of prospective mathematics teachers who took an instruction about construction of real numbers as decimal expansion via quantitative reasoning?

(iv) How do prospective mathematics teachers reason while developing the decimal representation of real numbers?

(v) How do prospective mathematics teachers reason on the post-proof comprehension test upon completion of an instruction on the construction of real numbers as decimal expansion through quantitative reasoning?

4.2.1. Hypothesis of the Research Question

H0: There is no significant difference between pre and post proof comprehension test results of the group of prospective mathematics teachers who took an instruction about construction of real numbers as decimal expansion via quantitative reasoning.

5. METHOD

The aim of this study was to investigate prospective mathematics teachers' thinking on the development of decimal representation of real numbers based on quantitative reasoning and the effect of instruction on the comprehension of a related proof about decimal representation of real numbers. In this chapter, participants, sampling technique, research design, instruments, procedure, data analysis will be explained. Lastly, information on pilot study and contributions of the pilot study to the actual study will be shared.

5.1. Participants

Target population of this study was prospective primary and secondary mathematics teachers (N=76) in an English-Medium state university in İstanbul. The sample of the study (N=19) was the prospective primary and secondary mathematics teachers in the same university. The sampling technique is convenience sampling because the participants were available and willing to participate in this research study (Creswell, 2012). The common characteristic of the participants is that they have already taken Math 111-Introduction to Mathematical Structures course from the mathematics department which covers logic and proof techniques.

5.2. Design of the Study

The design of the study was a mixed methods study. Specifically, it can be considered as embedded experimental study. It is stated that embedded experimental design can be fruitful in the case that the researcher needs to support quantitative data as embedding qualitative data (Creswell and Clark, 2007). However, it is important to determine at what point qualitative data are collected. The qualitative data can be collected before, during or after the instruction (Creswell and Clark, 2007). In the current research study, the qualitative data were collected during and after the instruction. The aim of the data collection during the instruction was to determine students' thinking process while engaging the activities and discussion on decimal representation of real numbers. On the other hand, qualitative data collection after instruction aimed to support qualitative data obtained from pre and post proof comprehension tests.

The quantitative part of the study was one group pre-test post-test design (Gay *et al.*, 2011). The qualitative part of the study consisted of the instruction and semi-structured interviews conducted after the post-test. The instruction part can be considered as a classroom teaching experiment (Cobb, 2000): In classroom teaching experiment "the researchers engage in promoting the development as part of cycle of interaction and reflection" (Simon and Tzur, 1999, p. 253). During the interaction phase, in this study, the researcher attempted to promote the further development of students' ideas based on their current knowing in the pre proof comprehension test. On the basis of their ideas, the researcher together with the supervisor developed a lesson plan based on their hypotheses about how development of decimal representation of real numbers might proceed. Simultaneously, during the teaching sessions, the researcher elicited evidence of the students' evolving understandings. During the reflection phase, both the researcher and the supervisor analyzed the researcher's interactions with the students such that students' reasoning, their difficulties, and the limitations of their current state of knowing were revealed. This analysis led to modified hypotheses about students' development of decimal representation of real numbers; these hypotheses in turn guided the next interaction phase.

The second part of the qualitative data came from semi-structured interviews after the post-test. a semi-structured interview was used to focus on individual student's reasoning on the proof comprehension test. Therefore, although the main questions were about how they thought on the answers, the researcher asked detailed questions depending on the individual student's answers to the proof comprehension test (Drever, 1995). Semi-structured interview enabled students' freely talk about the questions and decide the extent of explanation and how to express their reasoning.

(Drever, 1995)

5.3. Instruments

During the study in order to collect data, a proof of a statement about real numbers and a proof comprehension test were used. Explanations about proof and its comprehension test is given in the following sections. In addition, development process and content validity of the test are discussed.

In this section, assessment tools for the research study will be introduced and development process of the tools will be explained. Also, the reason for choosing the specific statement for assessing the effect of prospective mathematics teachers' quantitative reasoning on their proof comprehension will be emphasized. The details of the instruments are given in Appendix A (Statement and its proof), Appendix B (Proof Comprehension Test) and Appendix C (Rubric for the Proof Comprehension Test).

5.3.1. Proof of the Statement (Real numbers as decimal expansion)

For the current study, a statement and its proof were used and along with the proof, a proof comprehension test was developed based on the proof comprehension assessment model of Mejia-Ramos *et al.* (2012). Since the study focuses on as a specific case about the decimal representation of real numbers, the following statement was determined to be used.

> "Prove that if x is a positive real number, then for each natural number k there is an integer d_k, $0 \leq d_k \leq 9$, such that $D+d_1/10+d_2/10^2+.....+d_k/10^k \leq x < D+d_1/10+d_2/10^2+.....+d_k/10^k +1/10^k$ where D is a natural number." (Usiskin *et al.*, 2003, p.34)

In addition to the above statement and its proof, a definition which can be inferred from them was used. The definition is the following:

> "Every increasing and bounded rational number sequence represents a real number. Two sequences which have a difference approaching to zero are called sequences representing the same real number."(Taner, 1991, p.3)

After determining the statement, the proof of the statement was written by the researcher. While writing the proof, Matematik Dünyası (Mathematics World, 1991) which was about real numbers was examined. The statement was proved via using the method of direct proof since in the process, construction of a real number was required. After completing the proof, the researcher and thesis advisor checked the steps together and improved the use of proper mathematical language in the proof. Then, the statement and the proof were sent to two mathematicians and one mathematics educator in order to check the correctness of the proof written by the researcher. Based on the feedback given by mathematicians and mathematics educators, proof was revised in terms of the mathematical language and each step of the proof was numbered in order to ask questions about the steps. Other than small corrections, the mathematicians and mathematics educator did not mention any serious mistake. They agreed that the method of proving and mathematical expressions were correct.

The reason for using this particular statement, its proof and the definition as a conclusion is that due to the nature of the statement, proof of it requires construction and also this aligns with the instruction which will be explained in the following sections. Briefly in the instruction students were expected to construct the decimal representations of real numbers using the earlier concepts such as rational numbers, fractions, long division, sequences and the concept of limit. They were able to connect different concepts to reach the decimal representation of real numbers. Therefore, the instruction and the proof matched due to the construction of the concepts.

5.3.2. Proof Comprehension Test

Proof comprehension test was developed based on the model created by Mejia-Ramos *et al.* (2012). They classified proof comprehension questions into seven dimensions which are given as the following:

(i) Meaning of terms and statements,
(ii) Justification of claims
(iii) Logical status of statements and proof framework
(iv) Summarizing via high-level ideas
(v) Transferring the general ideas or methods to another context
(vi) Illustrating with examples
(vii) Identifying the modular structure (Mejia-Ramos, *et al.*, 2012)

In the process of forming the proof comprehension test questions, the researcher and her supervisor prepared at least 2 questions for each dimension separately focusing on the proof of the statement. They took the explanations given for the dimensions into account in order to match the dimensions with the questions. After the individual work of the researcher and her supervisor, they eliminated some questions and selected some of them for the comprehension test via the analysis of proof comprehension test dimensions. Totally, the comprehension test consisted of 13 open-ended questions with the total score of 27 points which was determined via the rubric formed previously.

<u>5.3.2.1. Validity and Inter-Rater Reliability of the Proof Comprehension Test.</u> The validity analysis of the test was done in the following way: Two mathematicians and one mathematics teacher educator checked the content validity of the test. All the reviewers found questions suitable for the dimensions and thought that the test assesses the comprehension of the given proof. Based on their feedback, structure of some questions were revised. For instance, initially a question was written as the following: 'How can the statement in line 14 be written following the earlier steps?

Could you explain?' Reviewers suggested to ask the question in the following way: 'How does the statement in line 14 follow starting from line 1? Could you explain?'

In addition to the validity of proof comprehension test questions, the reviewers also checked the suitability of the rubric for the test. Both the two mathematicians and the mathematics educator found scoring efficient but they suggested to check the rubric after piloting study. At the end of the pilot study, the scoring was found suitable for the evaluation of the test performances.

The inter-rater reliability analysis of the test was conducted after the data collection process. The inter-rater reliability was used to check the consistency in the scoring of the items. A mathematics educator scored the items using the rubric developed by the researcher and the thesis advisor. After that Cohen's Kappa were calculated for both pre and post proof comprehension tests. Cohen's Kappa value for the pre-test was calculated as 0.818 and Cohen's Kappa value for the post-test was calculated as 0.942 which are considered as high for inter-rater reliability.

<u>5.3.2.2. Proof Comprehension Test Dimensions and Corresponding Questions.</u> In this section, proof comprehension test questions are given and how each question is related to a different dimension is explained.

For the first dimension named as "Meanings of the Terms and Statements", it is required to ask about understandings of participants about terms and statements used in the proof. Therefore, the following questions were asked in the proof comprehension test (Figure 5.1).

The Q1 (Question 1) requires participants to give an example for a situation. Therefore, this question assess their understanding of an assumption used in the proof. The Q2 directly asks some definitions of the terms and expects participants to give examples to see whether they internalize the definitions or not. These questions are helpful and pre-requisite for answering further questions in the comprehension test.

D1: Meaning of the terms and statements	1. Let x be a positive real number. a. Give an example for the case where x is a rational number. i. Find r_1. ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm? a) Dividend-Divisor b) Quotient-Divisor c) Remainder-Divisor d)Remainder- Dividend b. Give an example for the case where x is an irrational number. Find r_1.
	2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used. a. What does sequence mean? b. What does bounded sequence mean? c. Can you give an example of 'bounded sequence'? d. What does "rational number sequence" mean? e. Can you give an example of a rational number sequence?

Figure 5.1. Comprehension test questions for dimension 1

The second dimension “Justification of claims” requires participants to explain why the claims are true and how they are connected to previous claims. Therefore, for this dimension the following questions are asked (Figure 5.2):

D2: Justification of claims	3. Could you justify why a. the rigth side of the inequality expressed in line 8 holds? b. the rigth side of the inequality expressed in line 16 holds?
	4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.

Figure 5.2. Comprehension test questions for dimension 2

The third dimension “Logical relationship” requires participants to understand mathematical language and methods of proving. Therefore, the following questions are asked to assess this dimension (Figure 5.3):

D3: Logical Structure	5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?
	6. How does the statement in line 14 follow starting from line 1? Could you explain?

Figure 5.3. Comprehension test questions for dimension 3

The fourth dimension “Higher Level Ideas” expects participants to explain proof in a general way summarizing the ideas in the proof. Therefore, the below two questions asked for this dimension (Figure 5.4):

D4: Higher Level Ideas	**7.** Could you explain the main idea (big idea) of the proof?
	8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

Figure 5.4. Comprehension test questions for dimension 4

The fifth dimension "General methods" requires participants to apply the proof to another context. For this dimension participants are asked to explain a particular case and the explanation of the definition given after the proof as follows (Figure 5.5):

D5: General Method	**9.** Please explain the following using the method of the proof you have been working with. **'Some real numbers have a terminating expansion.'**
	13. Using the proof, can you explain why the definition given at the end of the proof holds? (**Hint:** Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + \ldots\ldots\ldots + \frac{d_k}{10^k}$ Let $\{y_k\}$ be a sequence such that any term $y_k = D + \frac{d_1}{10} + \ldots\ldots + \frac{d_k}{10^k} + \frac{1}{10^k}$)

Figure 5.5. Comprehension test questions for dimension 5

The sixth dimension "Application to examples " requires participants to use the proof and the statement in the examples. Therefore, for this aim the following questions are asked (Figure 5.6):

D6: Application to examples	**10.** Find the decimal representation of 13/7 by refering to the proof.
	11. The decimal representation of $\sqrt{2}$= 1.4142135623730950488016887242096980785 6… is given. Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which $\sqrt{2}$ fall in?

Figure 5.6. Comprehension test questions for dimension 6

The seventh dimension "Identifying the modular structure" requires participants to comprehend the modular structure and components of the proof and how they are related to each other. Therefore, the following question is asked for this aim (Figure 5.7):

D7: Identifying modular structure	**12.** Continuing from line 13 could you construct the first 5 steps of decimal representation for any positive real number x (**until k=5**) and explain the steps you followed?

Figure 5.7. Comprehension test questions for dimension 7

5.4. Data Collection

In this section, data collection process is explained in detail. Additionally, the lesson plan is shared. From now on prospective teachers will be called as ‘students’ since they are the people taking the instruction.

Firstly, a mathematical statement about decimal representation of real numbers and its proof were given to the students and they were expected to read the statement and the proof for 5 minutes. Then, proof comprehension test related to the given proof was applied to the prospective secondary and primary mathematics teachers along with given proof. 42 primary and 16 secondary mathematics education students took the pre test. However, 26 of primary and 14 of secondary mathematics education students volunteered to participate in the whole research process. After that, the researcher tried to arrange time for the instruction by considering students’ exam dates or other works but at this stage some drop-outs occurred again. Due to the high rate of drop-outs from the study, totally 14 secondary and 5 primary mathematics education students participated in the study. With 19 students, the schedule for the instruction was determined. The instruction was given by the researcher after 5 pm on weekdays and took two sessions. Each session was completed in 2 hours so in total instruction lasted in 4 hours. Throughout the instruction, a master student in mathematics education video taped and recorded students’ speech and students’ class work was gathered.

After two weeks from the instruction, students took the same proof comprehension test as a post-test. According to the scores (low score, high score and average score) of the students, 2 students for each category of score, that is in total 6 students, were selected for semi-structured interviews. The aim of the semi-structured interviews was to make students elaborate on their answers' in the post proof comprehension test and to have an idea about their thought processes. Therefore, although the main questions were about how they thought on the answers, the researcher asked detailed questions depending on the individual student's answers to the proof comprehension test (Drever, 1995). Semi-structured interview enabled students to freely talk about the questions and decide the extent of explanation and how to express their reasoning. (Drever, 1995).

5.4.1. Instruction Based on Quantitative Reasoning

A lesson plan about decimal representation of real numbers was prepared. The lesson was based on the emphasis of quantitative reasoning which was the treatment in the research study. The lesson plan originated and was extended from Karagöz-Akar's (2016) study. The details of the lesson plan is given in Appendix D.

<u>5.4.1.1. Phase I of the Instruction.</u> In the first session of implementation, the focus was the definition of rational numbers, fractions, equivalent fractions and similarities and differences between them. Additionally, the relationship between the division algorithm and decimal representation of real numbers was given with the focus of quantitative reasoning. The quantitative skills which are considered to contribute to quantitative reasoning as equal partitioning and grouping was emphasized. The reason for including these concepts was to stimulate students to connect them while representing decimals of real numbers. As mentioned under the section 2.5.1 , it was found that students having difficulties in rational numbers, could not answer questions about irrational numbers and their claim was that for conceptual understanding of irrational numbers, it is necessary to understand rational numbers conceptually (Voskoglou and Kosyvas, 2012). Therefore, for the current research study, focus on previous concepts fostered students to comprehend the decimal representation of real numbers.

At the beginning, students were asked to define rational number and fraction. Then, they were asked to explain the difference between them and gave examples to both. In the lesson, fraction was thought as a quantity not a symbolic notation although fraction can also be considered as ratio. After that students were asked to define equivalent fractions and show how to obtain equivalent fractions using diagrams. The focus was the equal partitioning. For instance, students gave examples that 1/2 and 6/12 are equivalent fractions. They also showed it via diagrams with the focus of equal partitioning as follows:

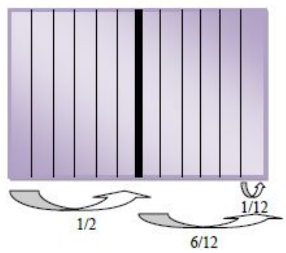


Figure 5.8. The equivalence of 1/2 and 6/12 in one diagram

After that, prospective teachers were asked to explain decimal representation of fractions and the relationship between decimal representation and equivalent fractions. The expected explanation is that decimal representation is obtained via the use of equivalent fractions. After giving examples for different decimal representations using equivalent fractions, the researcher asked another question using the case of mixed (improper) fractions that one can explain the relationship between improper fractions and division algorithm. For instance, as an example prospective teachers may explain the following using the fraction 7/2 and obtain:

$$\begin{array}{r|l} 7 & 2 \\ \cline{2-2} -6 & 3 \\ \cline{1-1} 1 & \end{array} \qquad 7=2\times3+1 \rightarrow 7/2=3+1/2$$

Figure 5.9. Division algorithm for 7/2

After this example, the researcher asked to express 11/3 using division algorithm and continue with long division using diagrams. The expression can be given as the following:11= 3*3+2 and 11/3 = 3 + 2/3. And using the long division we can continue with division adding an ',' and dividing 2 by 3. At this stage the researcher asked how 2 can be divided by 3. What can be done? The researcher wanted prospective teachers to draw diagram for 11/3 as in Figure 5.10 and explain from where the long division continues. Then, the question is 'How can this happen?' At this stage, the researcher wanted students think through equivalent fractions and expressing 2/3 as 20/30=20/3*1/10.

Figure 5.10. Diagram for 11/3

At this stage, the researcher wanted students think through equivalent fractions and expressing 2/3 as 20/30=20/3*1/10.

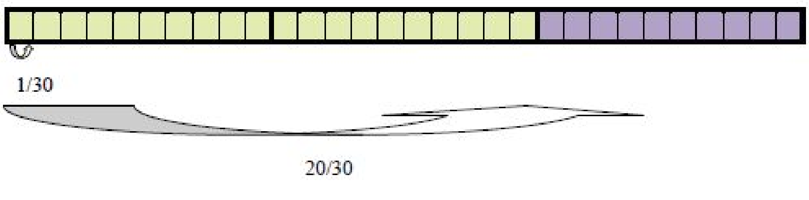


Figure 5.11. The equivalence of 2/3 and 20/30

Then, they were asked 'How many 3 pieces are there in 20 pieces and what will be the amount of them compared to the whole shape? The answer is that 6 groups of 3 pieces and 2/3 is remaining. The amount of pieces is 1/10 and it can be written as follows: 2/3 = 20/30 = 20/3 * 1/10 = (6 + 2/3)*1/10=6/10+2/30.

The researcher asked what to do to continue division. The expected answer is to divide 2 by 30. Then, they were asked to draw diagrams of the portion 2/30 as in Figure 5.12 to explain the division.

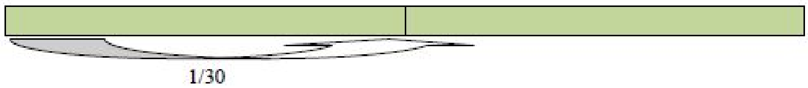


Figure 5.12. Zoom in 2/30

They were expected to write the expression for the division 2/30 as 2/30 = 20/300 = 20/3 * 1/100 and draw the diagram.

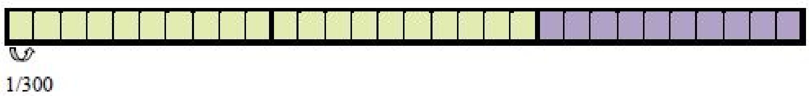


Figure 5.13. The equivalence of 2/30 and 20/300 in one diagram

The researcher again asks 'How many 3 pieces are there in 20 pieces?' the expected answer is 6 many 3 pieces and the remaining is 2/30. The amount of each 3 piece is 1/100. Then, they may write as 2/30=20/300=20/3*1/100=(6+2/3)*1/100= 6/10+2/300. Then, the researcher asked to compose all the findings about the division 11 by 3 expressed as 11/3. The students will be expected to write an expression as follows:

$11/3 = 3+2/3=3+6/10+2/30=3+6/10+6/100+2/300$

$= 3+2/3=3+6/10+(2/3)/10=3+6/10+6/10^2+(2/3)/10^2.$

As a last exercise students worked on the decimal representation of 8/7 combining the 3 representations that is long division, diagram and decimal representation. The reason for working on 8/7 was that decimal representation of it is also periodic but all the digits are not the same. The first session of the lesson focusing on quantitative reasoning finished at this stage. The aim of the first session was to combine the three different representations via quantitative reasoning and make them meaningful to students.

<u>5.4.1.2. Phase II of the Instruction.</u> In the second session, the researcher started with the expressions of 11/3 as a reminder and asked to squeeze 11/3 between two numbers and the interval was narrowed down using the decimal representation of 11/3 as the following:

$$3+6/10 \leq 3+6/10+(2/3)/10 < 3+6/10+1/10$$
$$3+6/10+6/10^2 \leq 3+6/10+6/10^2+(2/3)/10^2 < 3+6/10+6/10^2+1/10^2$$

$$3+6/10+6/10^2+6/10^3+.....+6/10^n$$
$$\leq 3+6/10+6/10^2+6/10^3+.....+6/10^n+(2/3)/10^n < 3+6/10+6/10^2+6/10^3+.....+6/10^n+1/10^n.$$

Figure 5.14. Intervals for decimal representation of 11/3

The aim of representing the decimal representations of 11/3 in an interval is to pass on to the concept of sequences and reach a different definition of real numbers using sequences. Therefore, students were asked to analyze first the lower bounds and the researcher gave the expression $a_n=f(n)=3+6/10+......+6/10^n$ for n being a positive natural number and asked whether it is a mapping or not? The reason for asking this question was to go through the concept of sequences because sequence is a special kind of mapping having the domain as natural numbers. Depending on students' answers, researcher asked probing questions to promote students through the sequence concept. After defining that the expression as sequence, they investigated the properties of the sequence formed by lower bounds as increasing, bounded and rational number sequence. The other question was about the properties of upper bound and similarly, the researcher gave the expression $b_n=g(n)=3+6/10+.....+6/10^n+1/10^n$ and asked whether the expression is a mapping or not? The reason of asking this question was the same with the case of lower bounds. In this case, students again investigated the properties of the sequence and found that it is a decreasing bounded rational number sequence. The next step was to check the differences between the terms of the sequences that is the difference between b_1-a_1, b_2-a_2 and b_n-a_n. The inference from the differences between the terms of the sequence was that as n increases the difference approaches to zero and thus both sequences represent the rational number 11/3. Therefore, an increasing bounded rational number sequence and a decreasing bounded rational number sequence can represent a rational number.

Up to this point students dealt with positive rational number examples. Therefore, in order to generalize findings to positive rational numbers, the researcher introduce x as any positive rational number and with the class the decimal representation of x was considered. After that the number x was squeezed with the help of its decimal representation and then the class again discussed about whether the lower bounds and upper bounds of x constitute a mapping or not. Again as in the case of 11/3, the students later focused on the issue of sequences about the bounds of positive rational number x. Students analyzed the sequences formed by upper and lower bounds and deduced the properties of them as follows: the lower bounds form increasing bounded rational number sequence and the upper bounds form decreasing bounded rational number sequence. After the analysis of the two sequences, students focused on the differences of the terms of the sequences. They found the differences as $1/10$, $1/10^2$,...., $1/10^n$ and tried to answer the question that what happens if becomes larger?. The aim of the question was to generalize the students' finding in the case of 11/3 to a general case for positive rational numbers. While thinking through the question students' obtained that every positive rational number can be represented by an increasing bounded and a decreasing bounded rational number sequences. The next was to investigate whether their findings can work for the irrational number case. For this aim, they worked on the example $\sqrt{3}$. The researcher asked how to squeeze the irrational number $\sqrt{3}$. The procedure to follow was to divide the intervals on the number line by ten at each step as follows (Figure 5.15):

By expressing the irrational number between two intervals as seen above and obtaining an approximate value of $\sqrt{3}$ as the intervals narrowed down. After that, the

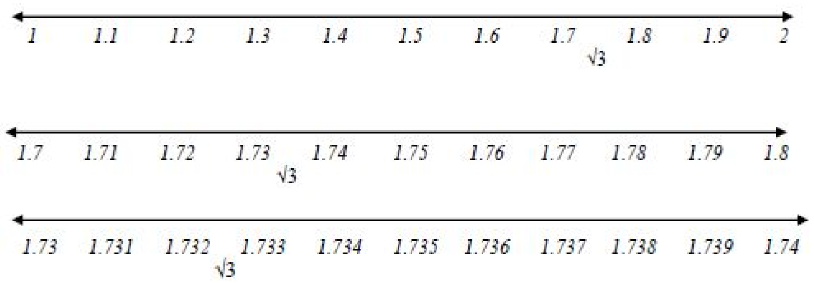


Figure 5.15. Placing $\sqrt{3}$ in number line

students asked to generalize the findings as stated previously. They found that irrational numbers can also be represented by increasing and decreasing rational number bounded sequences. That is, we can converge to a real number from above and below with rational number sequences.

5.4.1.3. The Relationship of Instruction and Proof Comprehension Dimensions. For the comprehension of a proof Mejia-Ramos et. al (2012) proposed a model which includes seven dimensions as Dimension 1 (Meaning of terms and statements), Dimension 2 (Justification of claims), Dimension 3 (Logical structure), Dimension 4 (Higher level ideas), Dimension 5 (General methods) and Dimension 6 (Application to examples), Dimension 7 (Identifying the modular structure). These dimensions also supported by the instruction used for this research study. As Mejia-Ramos *et al.* suggested the proof comprehension model was used as evaluating the effectiveness of the instruction on students' comprehension of a specific proof. Therefore, in this section, how the instruction and proof comprehension dimensions are related will be shared.

Dimension 1 is related to the understanding of terms and statements. In the instruction, students used different concepts for the construction of decimal representation of real numbers. They started with the definitions of rational numbers, fractions, equivalent fractions as considering them quantitatively and explaining via diagrams and then in the second phase of the instructions they used the concept of sequence and the concept of limit. For each concept students were asked to give definitions because by the help of these concepts they were able to obtain an alternative definition of real numbers. In addition to the definitions of concepts, during the instruction students were expected to generalize their findings from an example. Therefore, they used some mathematical notations and explained the meaning of them. Since definitions of terms and statements were given a special importance, dimension 1 in proof comprehension assessment model was also a focus in the instruction.

Dimension 2 requires the students to explain the truthness of the claims or expressions. How this dimension was supported by the instruction was that students were asked to explain their reasonings about their inferences and their construction of decimal expansions. Since students' reasoning was an important component of the instruction, explaining the truthness of claims and expressions were supported.

Dimension 5 expects students to apply the proof to another context. For instance, in the proof comprehension test, students were asked to explain the condition for obtaining terminating decimal expansion referring to the proof. Also, throughout the lesson students dealt with some specific cases such as decimals with terminating expansion, decimals with repeating non-terminating expansion and non-repeating non-terminating expansions. While expressing decimal using 3 different ways, they focused on the remaining parts of the diagrams to find the decimals. They were able to explain the relationship between long division and re-partitioning and grouping operations on the diagram. Therefore in the proof comprehension test, they were able to reason in which condition they will get terminating or non-terminating decimal expansions. In the post-proof comprehension test, some of the students explained the question about terminating expansion referring to the proof and also support their answers by the help of instruction. An another important point was that students dealt with sequences and the concept of limit in the second part of the instruction to define real numbers in a different way. Along with this, they were asked to explain the definition referring to the proof in the proof comprehension test.

Dimension 6 requires students to apply the proof to an example. Generic proofs are associated with this dimension and in the phase I of the instruction students focused on examples to find the decimal representation via triangulation of long division, division algorithm and diagrams along with a focus of quantitative reasoning and in the phase II of the instruction they continued with examples. Using examples they obtained a general case. In this way students applied the idea of the proved statement to an example and saw how proof operates. In this sense, the instruction might support Dimension 6.

Dimensions 3, 4 and 7 require abstract mathematical understanding which were promoted by Phase II of the instruction since in the second part students tried to generalize their observations on the example 11/3 and defined the decimal representation of real numbers using the concept of sequences. Students were able to summarize their findings and produce conclusions for the construction of decimal representation of real numbers along with rational number sequences.

5.5. Data Analysis

Due to the nature of the design of the study, both quantitative and qualitative data analysis were used. For the research questions 1, 2 and 3 quantitative analysis methods were implemented. For the research questions 4 and 5 qualitative analysis methods were conducted.

5.5.1. Analysis of the Pre-test and the Post-test Results (RQ1, RQ2 and RQ3)

For the analysis of the data in terms of research questions 1 and 2 descriptive statistics were used. Therefore, means, standard deviations, minimum and maximum values of the scores for both the pre and the post proof comprehension tests were obtained. Also, percentages of correct, incorrect and partially correct responses for each dimension and each question were obtained.

For the analysis of the data in terms of research questions 3, that is the comparison of students' performances in pre and post proof comprehension test, Wilcoxon-Signed Rank test was used. This statistics is useful when the number of participants are low or the data are not normally distributed and one group comparison is needed. It is the non-parametric equivalent of one sample t-test (Baştürk, 2010). The reason for using a non-parametric statistics in the current research study was that the number of students participated in the study was low (N=19) although the pre-test scores and the post-test scores were normally distributed which was checked via Kolmogorov-Smirnov Test of Normality and $p > 0.05$ was obtained.

5.5.2. Analysis of Qualitative Data

For the analysis of the data in terms of research questions 4 and 5, qualitative analysis methods were used.

5.5.2.1. Analysis of Instruction (RQ4). Firstly, since some part of the data was based on the teaching sessions, an ongoing analysis (Simon, 2000) was conducted during the teaching sessions. The ongoning analysis was considered as "the basis for spontaneous and planned interventions with the students, interactions that gather additional information, test hypotheses, and promote further development."(Simon, 2000, p.341). In this sense, throughout the ongoing analysis, the researcher and her supervisor met after each session and evaluate students' learning by considering how the plan fulfilled the expectations, the difficulties' that students met and how to overcome the difficulties in the following sessions. Therefore, in some stages of the plan, some changes were made due to students' performances on the tasks. For instance, at the end of the first phase of the instruction, the researcher examined that some students had a difficulty in expressing the decimal representation using diagram by doing re-partitioning and grouping. Therefore, at the beginning of the second phase, the researcher returned to the example in the first phase and expected students to find the decimal representation of the number by explaining via diagrams and connecting it to long division.

After the completion of teaching sessions, the data were transcribed by the researcher. To document the analysis of instruction, retrospective analysis (Steffe and Thompson, 2000) was followed because "one of the primary aims of this analysis is to place classroom events in a theoretical context" (Cobb, 2000, pp.325, 326). That meant framing students' thinking in the instruction. Before the retrospective analysis, first, the researcher conducted a line by line analysis. During the retrospective analysis, the researcher came back to the instruction and reflected on the activities occurred during the instruction by focusing on cognitive and social perspectives to understand students' mathematical thinking. It meant that the researcher thought of both the individuals' mathematical thinking and class' mathematical thinking as a group (Heinz, 2000). The reason of focusing on individual students' mathematical reasoning was that "the individual students' mathematical activity and the activity of the classroom was related"(Heinz, 2000, pp.59). Since the focus of analysis was students' quantitative reasoning throughout the teaching sessions, while doing that individual students' and also classroom community's making sense of the relationship between fractions and positive rational numbers; the meaning of equivalent fractions, the relationship between equivalent fractions, long division and decimal representation of fractions were illustrated and the inferences about their performances were revealed.

5.5.2.2. Analysis of Interview Data along with the Pre-test and the Post-test (RQ5). The interview excerpts used as a support in order to see how answers changed from the pre-test to the post-test. While analyzing the data, in the dimensions 1, 2, 5 and 6 the interpretive analysis (Clement, 2000) was maintained. During the interpretive analysis, researcher shared interview excerpts which helped her infer students' thinking process during the interview (Clement, 2000). The main focus for the dimensions 1, 2, 5 and 6 was students' thinking process through quantities while explaining their reasoning on the questions of proof comprehension test. For the dimensions 3, 4 and 7, descriptive analysis (Patton, 2002). That is, the question considered for the analysis of the data was 'What does the data show?'. In the dimensions 3, 4 and 7 students were asked about the general structure and framework of the proof, so the data related to these dimensions were analyzed descriptively.

Prior to the analysis of the interview data, as in the descriptive analysis, to be able to determine the extent of change in six participants' proof comprehension, the researcher classified their answers as no answer/incorrect answer, partially correct answer and correct answer based on the results of pre and post proof comprehension test scores. Particularly, if the student answered all the questions in the dimensions correctly, then his/her performance was classified as correct answer. If the student could not answer or incorrectly answer all the questions in the dimensions, then his/her performance was classified as no/incorrect answer. Since each dimension included more than one question except Dimension 7 (Identifying Modular Structure), if a student gave correct answer to one of the questions in the dimension and no answer or incorrect answer to the other question, then the student's performance in the dimension was classified as partially correct. This classification was maintained both for the pre-test and the post-test performances. After determining student's status in each dimension for the pre test and the post test, the transitions between the answers from the pre-test and the post-test were illustrated. For instance, for proof comprehension dimension 1, if a student was classified as partially correct in the pre-test and correct in the post-test, for the analysis the case was taken as from partially correct to correct answer for the student in dimension 1. In some cases, although students could not answer questions in some dimensions in both the pre and the post test, they tried to correct themselves during the interview. Additionally, if a student's transition between the pre and the post tests were considered as 'no change' and if the student showed an improvement in the interview, his/her performance was also shared.

5.6. Pilot Study

The pilot study was conducted in Spring 2015 on April 15 and April 16. Two prospective secondary mathematics teachers from the department of teaching secondary mathematics participated in the pilot study. Prior to the treatment, on April 15, two students took pre-proof comprehension test and in the following day, on April 16, they participated in the lesson based-on quantitative reasoning. The lesson and students' work were videotaped with two cameras and their classwork were collected at the end of the lesson. 2 weeks later the students took post-proof comprehension test.

Pilot study contributed to the revision of lesson plan, rubric and determining the time alloted for both pre-and post-tests and the lesson. The observations about the lesson in the pilot study and revisions based on them are the following:

- The students completed the tests in approximately 2 hours and the lesson lasted in 2 hours 45 minutes. While revising the lesson plan, it is decided to divide the lesson to two parts because just one session is too long to concentrate.
- At the beginning of the lesson, students defined the fraction. One of them defined it as the ratio between two numbers. Throughout the lesson fraction was going to be considered as a quantity which is one of the meanings of the fraction. Therefore, the researcher and the thesis supervisor decided to emphasize this meaning and show a diagram to the experimental group explaining the similarities and differences between ratio and fraction in case that prospective teachers in experimental group may define fraction as ratio in the main study.
- While analyzing the data from the pilot lesson and the data from the pre- and post-proof comprehension tests, the researcher and the thesis supervisor realized that students have a confusion about the meaning of sequences and series. Therefore, it was decided that in the main study the researcher would try to reveal whether students in experimental group have the same misunderstanding and then try to eliminate it via classroom discussion.

- The first part of the lesson focuses on rational numbers as decimal expansion and the last part focuses on irrational numbers as decimal expansion. Through the teaching investigation, the researcher and thesis advisor decided to include more examples for irrational numbers as decimal expansion part because one example can be inadequate to comprehend the procedure of irrational number construction.

6. RESULTS

In this section, results of the analysis of both quantitative and qualitative data will be shared. For the analysis of the data, both quantitative and qualitative data analysis methods were used. For research questions 1 and 2, the descriptive information about pre and post test scores was provided. Then, for research question 3, the performances of the participants on the proof comprehension test before and after the instruction were compared. Finally, the data of the instruction and semi-structured interviews were analyzed qualitatively in order to investigate research questions 4 and 5.

In the following sub-sections, results of the analysis of both quantitative and qualitative data will be shared.

6.1. Analysis of the Pre-test and the Post-test Results

6.1.1. Descriptive Analysis of the Results

Descriptive analysis was used to find an answer to research questions 1 and 2. The aim was to investigate current levels of prospective teachers' understanding of a proof assessed via proof comprehension test prior to and after the instruction. For this aim, means, standard deviation, maximum and minimum scores were obtained for the pre-test and the post-test separately. Also, correct, incorrect and partially correct percentages both for each question and for each dimension were obtained.

As seen from Table 6.1, for the pre-test, the mean was found as 6,37 and in the post-test mean value increased to 13,26. The maximum score increased from 13 in the pre-test to 23 in the post test and also an increase occurred in the minimum scores from the pre-test to the post-test.

Table 6.1. Means-Standard Deviations- Maximum and Minimum Scores for the Pre and the Post Proof Comprehension Tests.

	N	M	SD	Min	Max.
Pre-proof comprehension test	19	6.37	3.435	1	13
Post-proof comprehension test	19	13.26	4.241	7	23

When correct, partially correct, the incorrect, percentages for each of the question both in the pre-test and post-test were examined, the results were as follows: Along with the Tables 6.2 and 6.3 , Figure 6.1, Figure 6.2 and Figure 6.3 represented correct answer, partially correct answer and incorrect answer percentages for both the pre-test and the post-test respectively.

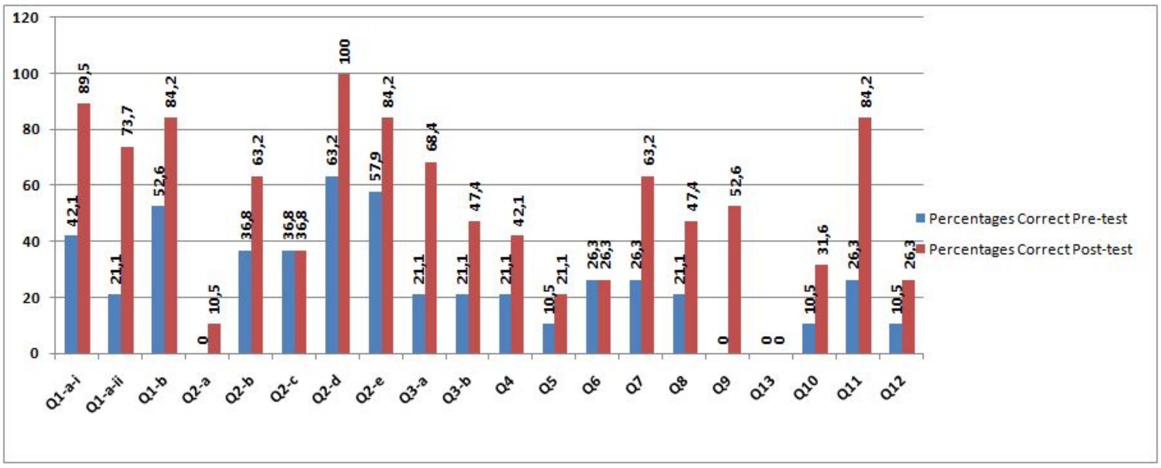


Figure 6.1. Correct Answer Percentages for Proof Comprehension Test Questions

Results showed that in majority of the questions the percentages of correct answers increased in the post-test compared to the pre-test. In particular, for all the questions except Q2-c and Q6, students' correct answer percentages increase from the pre-test to the post-test.

However, in questions(Q2c and Q6) percentages for both the pre-test and the post-test stayed the same. For instance, for question 2c percentages in the pre-test and in the post-test were the same. The reason for this stability is that in the post test although some students gave an example of bounded sequence, they did not indicate the interval that the sequence lies in. Therefore, they did not get point for this question in the post-test. The highest increase occurred in question 9 which belonged to the dimension 5: General Method. Initially no one could answer the question correctly, but in the post test more than half of the students gave a correct answer to this question. The reason could be that they were able to generalize the proof to a special case because the proof was for decimal representation of positive real numbers and the question asked in Q9 was about the terminating expansion of numbers. It seems that students were able to reason of a special case from a general case.

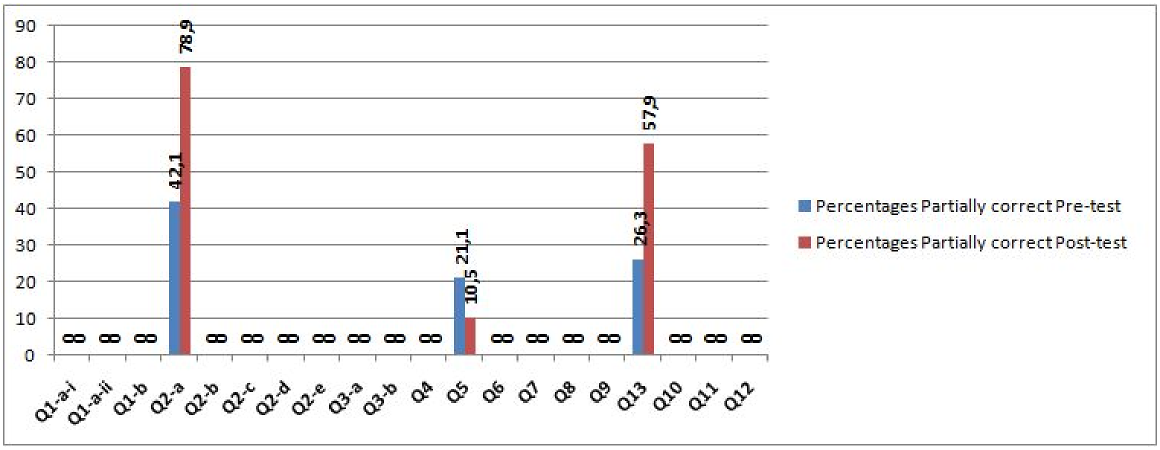


Figure 6.2. Partially Correct Answer Percentages for Proof Comprehension Test Questions

When the percentages of partially correct answers in the pre-test and the post-test were examined, as seen in Figure 6.2 for 3 questions students could give partially correct answers. If their pre-test answer percentages and post-test answer percentages were compared, it could be said that for Q2a and for Q13 the percentages increased from the pre-test to the post-test. For instance, the question 2a asked the definition of sequence and if students defined it without referring to function property of it, students got partial point. As seen, 42,1% of students could give partial answers in the pre-test and 78,9% of them give partial answers in the post-test. The increase in the percentages of partially correct answers could be that in the second phase of the instruction they thought of sequences and they defined what a sequence is. Therefore, the instruction might contribute their answers. For Q13 which asked about the validation of definition about real numbers via the given proof, students showed a progress. The reason again could be due to the instruction because students focused on the generic cases (Example of rational number and irrational number) and reached the definition asked to be validated for a general case. However, for Q5 asking the method of proof, students' partially correct answer percentages decreased from the pre-test to the post-test. It might seem that it was a negative outcome but the correct percentages for Q5 was increased.

The percentages of incorrect answers in the pre-test and the post-test were represented in the following figure:

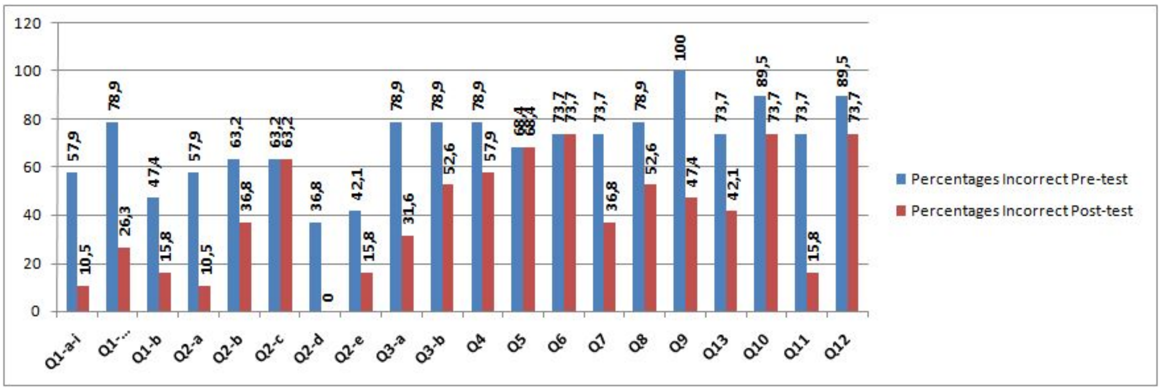


Figure 6.3. Incorrect Answer Percentages for Proof Comprehension Test Questions

The percentages of incorrect answers for the question except Q2-c, Q5 and Q6, decreased from the pre-test to the post-test. This again showed students' progression. However, for questions Q2-c, Q5 and Q6 percentages stayed the same because students gave similar answers in the pre-test and the post-test.

Table 6.2. Response Percentages of Pre-proof Comprehension Test Items.

Dimensions	Questions	Incorrect	Partial	Correct
Meaning of the terms and statements	1a(i)	57.9	0	42.1
	1a(ii)	78.9	0	21.1
	1b	47.4	0	52.6
	2a	57.9	42.1	0
	2b	63.2	0	36.8
	2c	63.2	0	36.8
	2d	36.8	0	63.2
	2e	42.1	0	57.9
Justification of claims	3a	78.9	0	21.1
	3b	78.9	0	21.1
	4	78.9	0	21.1
Logical structure	5	68.4	21.1	10.5
	6	73.7	0	26.3
Higher level ideas	7	73.7	0	26.3
	8	78.9	0	21.1
General Method	9	100	0	0
	13	73.7	26.3	0
Application to examples	10	89.5	0	10.5
	11	73.7	0	26.3
Identifying modular structure	12	89.5	0	10.5

Table 6.3. Response Percentages of Post-proof Comprehension Test Items.

Dimensions	**Questions**	**Incorrect**	**Partial**	**Correct**
Meaning of the terms and statements	**1a(i)**	10.5	0	89.5
	1a(ii)	26.3	0	73.7
	1b	15.8	0	84.2
	2a	10.5	78.9	10.5
	2b	36.8	0	63.2
	2c	63.2	0	36.8
	2d	0	0	100
	2e	15.8	0	84.2
Justification of claims	**3a**	31.6	0	68.4
	3b	52.6	0	47.4
	4	57.9	0	42.1
Logical structure	**5**	68.4	10.5	21.1
	6	73.7	0	26.3
Higher level ideas	**7**	36.8	0	63.2
	8	52.6	0	47.4
General Method	**9**	47.4	0	52.6
	13	42.1	57.9	0
Application to examples	**10**	68.4	0	31.6
	11	15.8	0	84.2
Identifying modular structure	**12**	73.7	0	26.3

When the data were considered in terms of dimensions, as shown in Figure 6.4 there is a decrease in the percentage of incorrect answer from the pre-test to the post-test. Therefore, percentages of partially correct and correct answers increased.

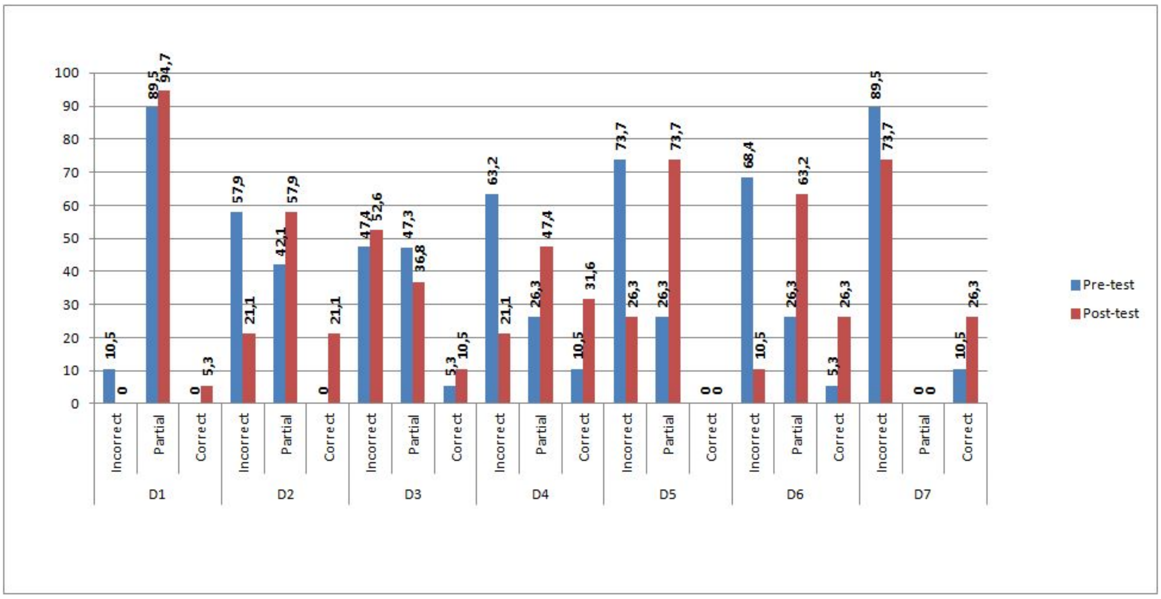


Figure 6.4. Response Percentages for Proof Comprehension Dimensions

In D3: Logical status of statements and proof framework the percentage of incorrect answers in the post-test is higher than the pre-test and the percentage of partially correct answers in the post-test is lower than the pre-test. It seems that some students' performance turned to incorrect answers from partially correct answers. The questions in this dimension was about the method and framework of the proof. One question asked the method of proof and the other one was about the framework. The reason for the decrease in students' performances in this dimension was their wrong answers about the method of the proof. The number of students stating that the proof was mathematical induction was higher than the number of students stating that the proof was direct proof.

6.1.2. Comparison of Pre and Post Test Results

In order to answer research question 3, the scores of the students in the pre proof comprehension test and the post proof comprehension test were compared. For this aim, Wilcoxon signed rank test was used.

Table 6.4. Wilcoxon Signed Rank Test Results-Ranks.

		N	Mean Rank	Sum of Ranks
Post Total-Pre Total	**Negative Ranks**	0	,00	,00
	Positive Ranks	18	9,50	171,00
	Ties	1		
	Total	19		

Table 6.5. Wilcoxon Signed Rank Test Results-Test Statistics.

	Post Total-Pre Total
Z	-3,731
Asymp.Sig.(2-tailed)	,000

The Wilcoxon Signed rank test results showed that there was a significant difference between the pre and the post proof comprehension test results. There was a significant increase in students' scores from the pre-test to the post-test in terms of proof comprehension $z=-3.731$ and $p<0.05$. 18 students' post proof comprehension test scores were higher than the pre proof comprehension test scores. One of the students' post proof comprehension test score were the same as the pre proof comprehension test score. There were no students with a lower score in the post test than the pre-test.

6.2. Analysis of Qualitative Data

The focus of the analysis of the instruction was to reveal students' thinking pathways to reach the decimal representation of real numbers connecting mathematical concepts such as fractions, equivalent fractions, division algorithm, sequences, bounded sequences, rational number sequences and limit. The focus of the analysis of interview data was to reveal students' improvement on thinking about decimal representation of real numbers through the analysis of their answers to the proof comprehension test.

6.2.1. Analysis of Instruction

One purpose of this study was to gain insight into the reasoning while developing the decimal representation of real numbers. For that, the researcher focused on the reasoning of individuals as well as the reasoning that seemed to be underlying taken-as-shared meanings and practices that emerged within the group. At times there were aspects of the mental activity of an individual that seemed to shed light on the key aspects of what it means to meaningfully develop the decimal representation of real numbers and at times there were aspects of the activity of the class as a whole that provided insight into that issue. Therefore, in the following paragraphs, the researcher depicted the sample common thoughts and how students reached these ideas during the instructional sequence. There are two types of data sources: i. class discussions and ii. written artifacts. To provide breath and depth to the analysis, in both of these data types both from students interviewed and other students will be shared in the following sections.

The phase I of the instruction consists of two parts: Part I includes the review of the concepts of rational numbers, fractions, equivalent fractions. The second part (Part II) involves obtaining decimal representation of fractions through equivalent fractions and its relation to long division. In phase II of the instruction, the main focus is the relationship between rational number sequences and decimal representation.

Phase II has three parts. At the beginning of Phase II, students tried to squeeze a rational number to nested intervals. After that in Part I, they investigated again the decimal representation of 11/3 considering through sequences. In Part II, they generalized their observations to a decimal representation of a positive rational number considering through sequences as in the case of 11/3. In Part III, they analyzed the decimal representation of an irrational number through considering sequences.

6.2.1.1. Phase I of the Instruction.

- Part 1: Review of the concepts of rational numbers, fractions, equivalent fractions and relationship between them:

The purpose of the first part of the instruction was to have a common language and meaning for rational numbers, fractions, equivalence of fractions and their relationships to each other. This was important because this study dwelled on the meaning of fractions as a quantity, representing an amount, and developing the meaning of decimal representation of real numbers from such understanding. Therefore, it was important for the whole class to have a taken-as-shared understanding for the meaning of fractions as a quantity. A second purpose was to engage students in the partitioning activity. This was important for two reasons: First, they were expected to go through such quantitative operation to be able to explain the meaning of equivalent fractions. Second, they were expected to re-use that activity while developing the meaning of the decimal representation of fractions.

The instruction started with the following question: 'What is rational number?'. The following excerpts show common answers of the students.

> R: .....What is a rational number? How did you define it? Anyone who wants to answer?
> S19: Any number written as the division of any number to another is named as a rational number.
> R: As you say, division of one number to another, how is this number?
> Class: That can be types as a/b.

R: What is important about a and b?
S8: a and b are integers.
S19: It has to be Integer.
R: a and b are integers. Anything else? Are there any conditions?
S14, S19, S2: b is not equal to zero.

As the excerpt showed, initially students defined rational number in a very broad way without considering the properties of it. However, by questioning, the researcher got what they knew about the properties of rational numbers. At the end, the students collectively defined rational numbers that emphasized the properties: a and b are integers and b is different from zero.

The next question of the sequence was 'What is fraction?' and students had different approaches to the fractions. As shown in the following excerpts, some students stated that a fraction is a ratio and one of them stated that it is just a name for rational numbers. Then, the following discussion occurred:

S10: I have just said quantity
R: What else?....
S9: I have said that it is the image that appears in mind while thinking of rational numbers.
R: What do you mean?
S9: When I think of 2/3, it is the display that I divide a whole to 3 pieces and taking 2 of them.
R: OK. Any objections or agreements with this idea? What do you think?
S1: Actually, when I said a name, I meant what S9 just said. While saying a name... R: If you explain a little bit more..
S1: I mean, I say 1/2 and a representation of 1/2 comes to my mind. But there is no such thing in the nature. I am giving this to it. This becomes its name. I think what S9 wanted to say is something like this.
S9: Yes.
R: What do you name? You define 1/2 with something. What is that?
S1: I revive a number in my mind as 3/4.
R: OK. How does it revive on your mind? It does not revive as 3/4, right?
A Student: Fraction is the ratio of 1 piece or pieces to the whole.
R: OK. What do you think about this definition?
S6: Yes, I think the same too...
R: What would you say if you compare it with the definition of quantity? There have been some people calling it quantity.
S7: I think it indicates a quantity.

In the dialogue, most of the students defined fraction considering quantity although they used different ways to explain fraction as quantity. For instance, S10 directly emphasized that the fraction is a quantity. However, S9's explanation also lead to the quantity meaning of fraction as she said 'when considering 2/3, in my mind I can show by dividing a whole to 3 pieces and taking 2 pieces from the 3 pieces'. S9's explanation can be taken as a shared explanation since most of the students accepted this explanation and explained in a way that the same ideas appeared. For instance, S1 initially defined fraction as a name for rational numbers. After S9's explanation, S1 stated that he meant what S9 said. Although some of the students defined fraction as ratio or representation, since the flow of the instructions were based on the meaning of the fraction as quantity, the researcher summed up the students' answers and emphasized the quantity meaning:

> R: We have said the ratio of two quantities. We can also define fraction also as ratio. But another meaning is quantity. We can call it as a representation of rational number, but today in our lecture we will move through quantity. So what will we say. When I divide the whole into equal number of pieces....the piece I take is...fraction...

After defining fraction as a quantity, the discussion continued with the difference between fractions and rational numbers:

> R: What is the difference between the rational numbers and fraction?
> S17: Rational numbers can also be negative.
> R: Fraction?
> S17: It cannot be.
> R: Why ?
> S17: When you divide a whole into its pieces, those pieces cannot be negative.
> A Student: Because there is a quantity.
> R: What does negativity in rational numbers indicate? We said that fraction cannot be negative.
> S1, S4: It indicates direction.
> R: It indicates direction. What can we say if we talk the quantity meaning of a fraction? I can define the positive rational numbers as fraction. Because I am talking about the quantity.Right?
> Class: Yes.

In the presented dialogue one student explained that since we defined the fractions considering a whole and pieces of a whole considering the quantity (amount) meaning of fraction, it cannot be negative. When asked as the meaning of minus sign, some of the students answered that it represents direction. After the researcher summarized the explanations, the class agreed with the idea of difference between fractions and rational numbers. From this dialogue it can be inferred that students have already known the difference.

Next step of the sequence was the definition of equivalent fractions. First, students tried to define equivalent fractions and draw diagrams of equivalent fractions individually. Then, some of them shared their explanation with the class. The taken-as-shared meaning within group for the equivalent fractions was shown in some examples of below (See Figure 6.5, Figure 6.6 and Figure 6.7):

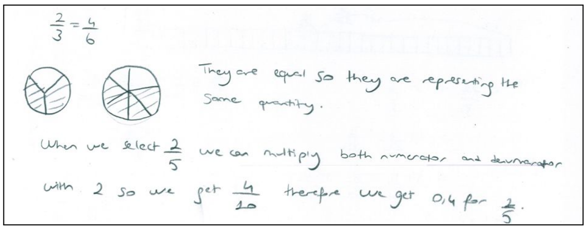


Figure 6.5. S5's explanation 1

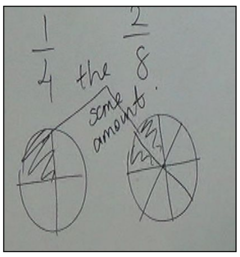


Figure 6.6. S19's explanation 1

Although the representations (Figure 6.5, Figure 6.6 and Figure 6.7) differed from each other, all three explanations focused on the fact that equivalent fractions represent the same quantity.

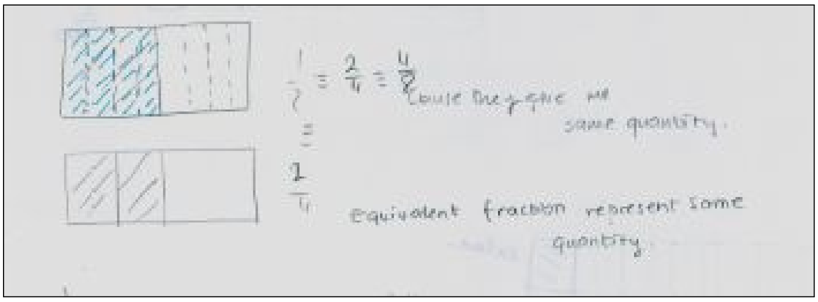


Figure 6.7. S7's explanation 1

After students worked individually S19 volunteered to explain her thought:

> S19: For instance, when I eat half of an apple and when I divide it to four and eat two parts of it; it is the same thing. You eat the same amount. I have given 1/4 and 2/8 as example. (Draws two circles on the board). This can be a melon, an apple, or a pizza; depending on how you interpret it. I have divided the previous one into 4 pieces and took 1 piece of it. I did the same thing with other one and I have also divided the intervals. So I divided it into 8 pieces and shaded 2 pieces of it. When I look at these two sides, I see that both are "same quantity". It is completely about this. The more you divide, the more you need to eat for the same amount of eating. In other words, I eat the same amount when I divide it into 4 and eat 1 piece, an when I divide it into 8 and eat 2 pieces.

As the excerpt shows, S19 first partitioned a whole with a circle representation into 4 equal pieces and took one of them. Then, she re-partitioned each of the equal pieces into two. Her activity of re-partitioning the equal pieces resulted in the re-partitioning of the whole. This way she was able to think that the amounts did not change. Her statement "whatever the number of partitions you have, you eat at the same rate. That is, the amount that you eat when you divide into 4 equal pieces and eat 1 piece will be the same as when you divide into 8 pieces and eat 2 pieces" shows that she knows that although the number of partitions change the amount stays the same.

A similar explanation came from S17 as follows:

S17: I have moved from 2/3 and 4/6 fractions. When we multiply the numerator and denominator with the same number or when we divide them, I have said that we have equivalent fractions as result. And I have indicated with a figure like this. (First he draws two figures in the same size. First, he divided both figures into 3 pieces. Then he divides each piece of first figure into 2 pieces. In total, he has 6 pieces.) This is 2/3 fraction (shows the figure below). This is 4/6 fraction. They indicate the same quantity.

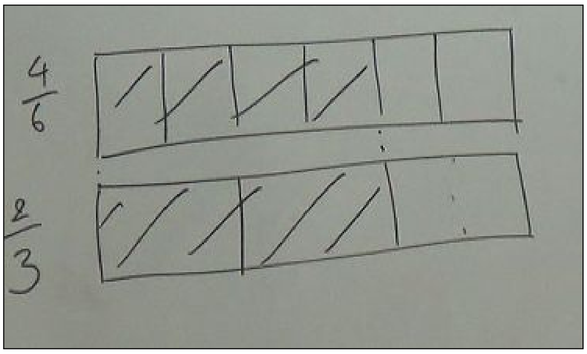


Figure 6.8. S17's explanation 1

Different from S19, S17 (See Figure 6.8) considered two rectangular shapes. Then she partitioned both shapes into three equal pieces and crossed. Two pieces to show 2/3. Yet, similar to S19, in one of the shapes, she re-partitioned the equal pieces into two more parts. Her statement " now that is the fraction 2/3 (pointing to the shape below) and that is the fraction 4/6 (pointing to the shape above). They represent the same amount". Since most of the students used two figures to explain the equivalence of fractions, and agreed with S17's and S19's ideas, researcher asked them to show the equivalence in one figure. They were all able to show it and explain their reasoning in the same way: That re-partitioning did not change the amount. Just the number of partitions changed.

Based on the discussions above, researcher decided that the whole group had the following ideas as taken as shared: First, although some might have held different meanings of fractions in their knowledge repertoire, they also acknowledged that fractions represent amounts too. And, that rational numbers could have negative signs and such negative signs represent direction. Also, through the quantitative operation of repartitioning of the given whole, they were able to reason that equivalent fractions represent the same amount. At this point of the instruction, the second part started to take place.

- Part II: Obtaining Decimal Representation of Fractions through Equivalent Fractions and Its Relation to Long Division:

The aim of part II was to connect different mathematical ideas to each other, that is connecting decimal representation with equivalent fractions and long division through quantities. Therefore, the researcher asked whether we can use equivalent fractions to obtain decimal representation or not. Students first worked individually and shared their thoughts with the class. Some sample students' explanations were given below:

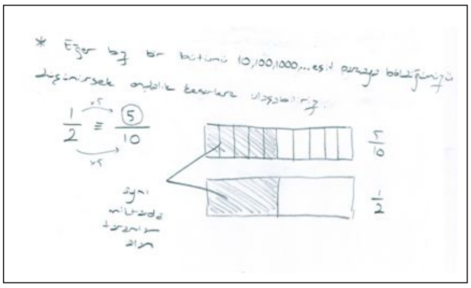


Figure 6.9. S8's explanation 1

> "If we think of dividing a whole into 10,100,1000 equal parts. . . .we can reach decimal fractions.Both diagrams represent the same amount of shaded area." (S8's explanation 1-Figure 6.9)

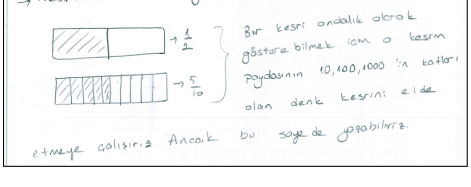


Figure 6.10. S12's explanation 1

"In order to show a fraction by decimal representation, we need to obtain equivalent fraction being the denominator of the fraction, 10, 100 and 1000 multiples. Thus we can express fraction with decimal representation." (S12's explanation 1-Figure 6.10)

The excerpts and the figures from both students explain the same idea. S8's statement (See Figure 6.9) "if we divide the whole into 10, 100, 1000,... pieces, we get the decimal fractions" shows that he knows that for the decimal representation the whole has to be partitioned into the powers of 10. Similarly, his statement on both figures "the shaded area showing the amount" shows that he thinks of equivalent fractions. Though he does not provide any further explanation, S12's statement (See Figure 6.10) "to be able to show a fraction in decimal forms we construct fractions with the denominator of 10, 100, 1000 ... equivalent to that fraction. Only this way we can get decimal forms". This shows that both S12 and S18 reasoned that they could use equivalent fractions to get decimal expansion, since equivalent fractions represent the same amount and since they need fractions with denominators being powers of 10.

Then, the researcher asked 'what if we have improper fractions, what can we do to find the decimal representation'. Therefore, the next question was about the decimal representation of 7/2. The reason was to develop the relationship between the decimal representations the long division algorithm.

R: Who wants to explain now? How do we represent 7/2 with decimal? Can someone explain on the board please? (S4 comes to the board).

S4: I have named 7/2 as 3 whole 1/2. We have written 1/2 as decimals before. And it has been 3.5.

R: If you mention it with a figure..

S4: (After drawing the figures, he writes the fractions below. Then he says that he has to mention as 5/10 for the last figure)

I have to show this as 5/10. We will move to decimal..

R: Yes. What was that last piece for you? You have 1 whole, 1 whole and 1 more whole. And what was your final piece at 7/2?

S4: It was 1/2.

R: OK. Draw it again. (S4 redraws 1/2 piece.)

R: How will you express 5/10 on that figure? (S4 makes a partitioning on the last figure)

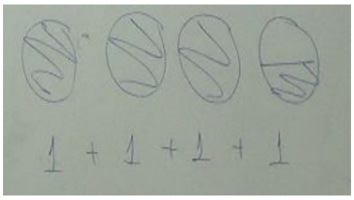


Figure 6.11. S4's explanation 1

R: So what have you done? In the last figure....Each piece

S4: I have divided into 5 equal pieces.

R: You have divided into 5 pieces. What would happen if you did the same thing on these 3 pieces? (I mean the 3 full) How many pieces did you divide this figure in total? (I mean the last figure)

S4: 10.

R: What would happen if you did the same thing for these, would anything change? What do you think? (Turns to class)

S4 and class: It would not change.

So far the excerpt shows that first S4 drew 3 wholes and one more whole divided into two equal pieces. He thought of 7/2 as 3 wholes and $\frac{1}{2}$ and stated "this is equal to 3.5". Then, he was able reason that he could re-partition each of the two equal pieces into 5 more partitions and therefore the whole had 10 partitions. That is, he was able reason through repartitioning of the part that was not a whole. Then the researcher asked them the following intentionally to take their attention to the whole number division.

R: Alright, we have written 7/2 in two different ways. If I said....If I wrote according to the rules of division. I have divided 7 to 2. I found 3, right? The remainder is 1. Is it possible for me to continue dividing?

Students make comments with low voices...

R: What will we do?

S7: We put a zero and a comma. R: We put zero next to 1..

S7: And we put a comma next to 3.

R: OK. Why ?

S7: We never thought about it...

R: Why do we put a zero and a comma?

S1: We take decimals. Like in the subtraction algorithm we take 10 from the neighbor and this time we take 10 from . . . the right base.

R: From where?

S1: From the right digit.

S6: From the 1/10 th s base.

As the data indicated, once asked to think about the whole number division and how to continue the division with the left over part, students could not provide a reasonable rationale. S7's statement "we never thought about it" , S1's comment "like in the subtraction algorithm we take 10 from the neighbor and this time we take 10 from . . . the right base" and S6's uttering "from the 1/10th s base" all indicate that any of these students did not know how to proceed the division with the left over part.

> R: Yes. OK. I put a zero next to 1, but what was the quantity I had? Where do I continue dividing from? Are there 3 wholes in my figure now?
> Class: Yes.
> R: Where will I continue to division?
> Class: From 1/2, other side
> R: Will I continue from here? (I show the shaded part)
> Class: Yes.
> R: Do I have 1/2 in my hand?
> Class: Yes

The excerpt above show that the whole class as a chorus could state that the division has to continue with the the shaded area ($\frac{1}{2}$) which corresponds to the division of the left over part with the divisor. The excerpt is important in the following way: As the data showed once asked about the reasoning behind putting "0" right to the left over in the long division algorithm, students could not provide any explanation. On the contrary, even one of them stated that they have never thought of it before. However, once the researcher provided them with the diagram S4 has drawn earlier and asked them to think through the algorithm with the diagram, they realized that they could continue the division with the shaded area. This was evident in the whole class' agreement. That is, the researcher asked them to think about the meaning of putting "0" to the right of the left over part "1" in the long division using the diagram. The discussion continued in the following way:

> R: We have said that a whole was divided into 2 and one piece is taken. If I continue dividing from here. I put a zero next to 1, and it is 10. Will there be any change about 2?
> A Student: Now you have divided each piece to 10 equal pieces, right?
> A Student: Become complicated.

R: Can you think with equivalent fractions?
S7: (inaudible)
R: Can you say a bit louder please? Again please?
S7: If that is 10, then the other should be 20..
R: What happens if so? Did we say 1/2=10/20?
Class: Yes.
R: OK. Can we represent this in any other way? For instance 10/2*1/10? What do you think? 10/2; what is the number of pieces with 2 in a 10 piece? What is the answer?
S17: It is 5.
R: How many 1/10 do I have now?
Class: 5

In the excerpt above, once the researcher asked in 1/2, 'the numerator "1" becomes "10" then will there be any change about 2?', a student's explanation "Now you have divided each piece to 10 equal pieces, right" seems to indicate that he might have repartitioned each of the equal pieces in 1/2 into 10 more equal pieces. Unfortunately, the researcher did not probe him further about his thinking. Yet, later, the class' as a chorus agreeing with 1/2=10/20 together with their earlier reasoning on the equivalence of fractions seems to suggest that they might have realized how it was possible to continue with the long division through getting equivalent fractions. Still, we need to be cautious about the extent of their making sense of the relationship between the equivalence of fractions and the long division with whole numbers while constructing the decimal representation of positive rational numbers. Further data is needed to conclude whether they acknowledged such relationship or not.

In sum, in the excerpts above, students tried to explain why in the long division, it is required to add a zero to the remainder and add a comma in order to continue the division. Firstly, S4 showed the decimal representation of 7/2 using diagrams. He did partitioning in the left over piece $\frac{1}{2}$ to write it in decimal form. After that the class focused on reasoning behind putting zero to right of the left over parts and comma in the quotient in long division. Initially some students stated that they did not think about the reason before. After a while, they gave general answers such as 'we need to find the 1/10th decimal place so we add zero and comma'. In order to narrow down the scope, the researcher tried to make them think through the diagrams and asked 'from where can we continue with division?' While thinking through diagrams, they realized that the division could be continued with getting equivalent fractions. Therefore, students matched the representations and long division via the concept of equivalent fractions.

Decimal Representation of 11/3:

After the class had a consensus on this idea, the researcher asked "if we do not have powers of 10 or we don't have any chance to obtaining powers of 10 by equivalent fractions what can we do for decimal representation?". Therefore, the next question was about the decimal representation of 11/3. As shown earlier, the first question was asked to focus students on reasoning about a terminating positive rational number, 7/2, and the next question was used to take students' attention to non-terminating positive rational number.

Firstly, students worked individually to represent 11/3 in decimal form with the help of long division and diagrams. However, some students had obstacles while reasoning on the diagrams. For example, S8 tried to answer the question by combining diagram, long division and the representation of the improper fraction as a mixed fraction (See Figure 6.12). Although, while working individually he could not obtain the decimal representation by explaining via diagrams, he was able to do so after reasoning through the questions. The following dialogue based on his individual work represents this issue:

> R: Well I will ask you something. How did you write that down?
> S8: (Explains with the figure above) This is whole now. (He refers to the last whole)
> R: How did you equalize 2/3? (asks for the last drawn figure) (she refers to 2/3=6,666.../10)
> S8: How did I equalize? These lines (he refers to the lines he drew further on the last shape) do not fit at the same place; but when I divide it into 10 equal pieces, it passes line 6 and does not reach line 7. It is in the middle according to 2/3.
> R: How do you now that the one in the middle comes to 6?
> S8: I know how it equalizes 6,6666. I know it from here (showing the

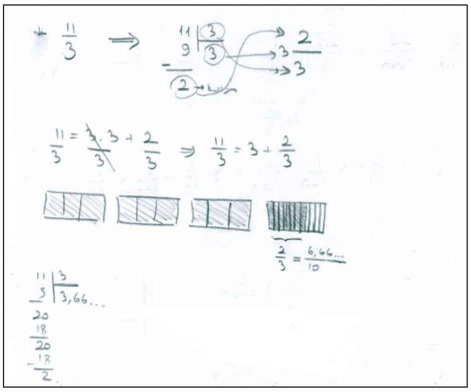


Figure 6.12. S8's explanation 2

division). I know it from the division.

Although S8 used three different forms for obtaining decimal representation, he could not directly explained why he subdivided the last diagram and obtained the portion for the 6/10 in the portion of 2/3. However, he referred to the long division as a reason and he could not explain actually what he did in the diagram. He was only able to explain his reasoning on the division algorithm procedurally. That is, he was not able to call on the re-partitioning activity to explain the reasoning behind how to continue with the long division: The fact that he was able to re-partition each of the equal pieces into 10 more equal parts, he would be able to come up with equivalent fractions. This would enable him to continue with the long division.

Like S8, another student S1 also had some difficulties. S1 also drew the diagram of 11/3. However, he continued from the part that were not shaded that is the portion 1/3 (See Figure 6.13). Then, he enlarged the fraction by 10 and showed it in the diagram by just zooming in the portion 1/3. After that he found the number of 1/10's in that portion by equal-partitioning and grouping every 3 equal pieces. This was evident in the following excerpt. The data also show how S1 corrected his reasoning.

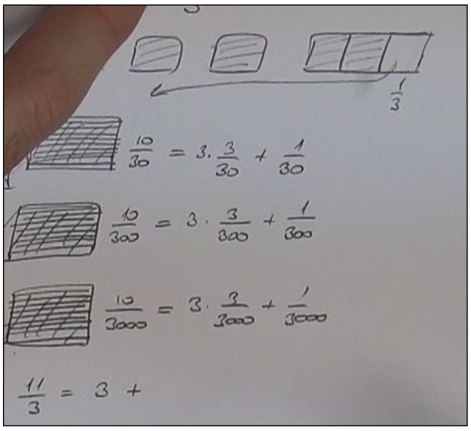


Figure 6.13. S1's explanation 1

(S1 takes 1/3 as the remaining piece and continues with equal partitioning and grouping!)

S1: What do we do now?

R: Tell us what you do.

S1: It was 3 whole 2/3... This 1/3 remains... I am searching for 1/10's in these.

R: Can you also write division next to it? If you move from division... Write, 11/3.

S1: This is.

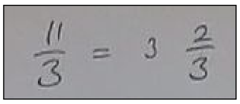


Figure 6.14. S1's representation 1

R: OK. how would you proceed to long division?

S1: Will I continue from here (Showing Figure 6.15)? No.

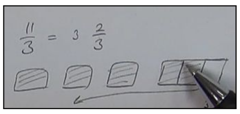


Figure 6.15. S1's representation 2

S1: (Writing the normal division as in Figure 6.16)

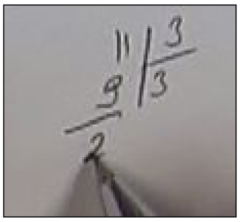


Figure 6.16. S1's representation 3

R: Where will you continue from?
S1: From that shaded field...
R: What is the quantity of that shaded field?
S1: 2/3.

From the researcher's perspective, it seems that S1 thought that although he used the word remainder in the division, he could not match what it meant in terms of the diagrams. That is, he considered that remaining part meant the unshaded part when taking the diagrams into account. However, when the researcher asked "how would you proceed with the long division?" This took S1's attention to the shaded region in the last whole. That is, after S1 focused on what the remainder in the long division represents as a quantity; in his own words; "the shaded area" he was able to realize from where to continue in the diagram.

After students worked individually, realizing that S8 had difficulties, the researcher asked him to come to the board to explain his reasoning: S8 wrote 3 types of representations on the board: the long division, the diagram and 11/3 as a mixed fraction. Via questioning, he tried to explain how he can use diagram to show the decimal representation 11/3.

R: Can you please tell where you continue division from in each representation?

S8: (By showing the normal division) That is how I do (Adds zero next to the remaining number, and mentions adding comma)

R: OK; where do I continue from in the figure?

S8: I will continue from here in the figure (Shows the last whole he had drawn, 2/3 shaded field)

R: Where will you continue from to division at 3+2/3?

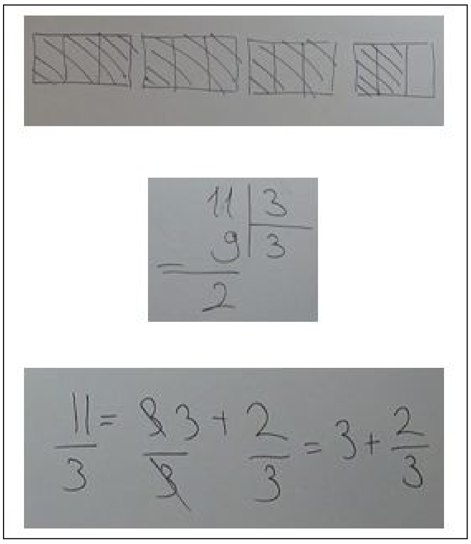


Figure 6.17. S8’s representation 1

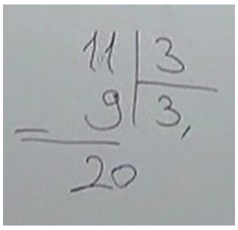


Figure 6.18. S8’s representation 2

S8: I will continue from here (shows 2/3).

In the above excerpt, S8 showed where he could continue with division and he was aware of the remainder in each representation, while continuing with the division. Later, he explains his thought process in the following way:

S8: If I divide 2/3 into 10 pieces.
R: Say again, why did you divide it into 10 pieces.
S8: Why did we divide in 10 pieces?
R: Show again the part you have divided into 10 pieces.
S8: (Showing above the figures) This is 2/3. What do I do now. If I multiply these by 10, this means that I divide these pieces into 10. I divide one piece into 10 equal pieces
R: You divide them all, right?
S8: Of course, I divide them all. I divide this separately, I divide that (showing each 1/3 pieces).

S18: Now we have divided the whole into 30. But now this is not a decimal representation.

R: OK.

S8: Now it means this (shows 20/30).

R: When you divided each part in the shaded field to 10 equal pieces, what is the shaded area equivalent with? Which fraction does it represent?

S8: It equivalent to 20/30.

R: OK.

S8: Now what does this mean?(He refers to 20/30) This means that what subdivisions did I do in these parts. 1/10 more pieces. How many of them do I need now?

S14: 20/3.

The excerpt above is important because first it shows that S8 overcame his difficulty in the first dialog when he worked individually. From the researcher's perspective, this was possible on the part of S8 because he paid attention to the meaning of the remainder: That is, firstly, he added "zero" right to the remainder in the long division. Then, he was asked to explain how he would proceed with the diagram. This enabled him to focus on what the remainder represented as a quantity: the shaded area in the diagram. Then, he thought of repartitioning each of the equal parts in to 10 more pieces. This way, he was able to reason that he would get an equivalent fraction, 20/30, to 2/3. Then, his uttering "what does this mean?(He refers to 20/30) This means that what more subdivisions did I do in these parts. 1/10 more pieces. How many of them do I need now?" shows that he thought that he re-partitioned each equal part (1/3) into 10 more equal pieces which resulted in the 20 of 1/30 parts. Then, his reasoning to find out how many of 1/10th s are there in the 20/30 shows that he was ready to engage in measuring the 20/30 in terms of 1/10th s. Interestingly, S14, another student reasoned with him simultaneously and answered him. Data also shows that S8 was able to flexibly reasoned in between different representations. That is, he was able to explain how each representation was correspondent with each other.

Then, he continued with the following figure given to him. This was because, his drawing would be getting confusing for him the researcher provided him a more proper drawing of what he drew already (See Figure 6.19):

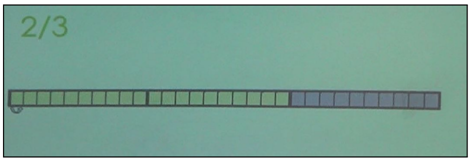


Figure 6.19. Representation1

S8: Can I make an explanation?
R: Sure.
S8: I have 20 many 1/3 pieces. How many do I have from whole 1/10 piece? 20/3.

Although the other student, S14, provided an answer, it is interesting that S8 wanted to continue with his reasoning: The data show that S8 knew that each of the re-partitioned pieces represented 1/30. This shows that he knew that when he partitioned each of the three equal parts into ten more pieces, the whole was also re-partitioned into equal parts which resulted in each equal piece representing 1/30 as an amount. Then, asking how many of 1/10 are in 20/30, he was able to answer his question. After explaining that there are 20/3 many 1/10 in 2/3, as shown in the following excerpt, the researcher asked how to find the exact number of 1/10's from the figure. Asking such a question was important since it guided S8 to re-focus on what 1/10 represented as a quantity.

R: OK. Can you now indicate it on the figure, how many 1/10 will you have?
S8: How many 1/10 will I have?
R: Now the littlest piece is 1/30, right?
S8: 1/30 , OK.
R: OK. What do I need to do in order to find the quantity of 1/10?
S8: This is 1/10, right? (showing the first three pieces)
S19: This is the answer; I have 20/3 pieces of 1/10, right?
R: Yes, but.
S19: OK, the answer is 20/3.
S8: 1 piece of 1/10, 2 pieces of 1/10, 3, 4 , 5, 6.
S12: Are we not going to do the same thing always?
S8: OK, now it is fine.
R: What do our 3 pieces mean?
S8: 1/10. 2/10. 3/10.
S15: There are 6, and 2 remains.
S8: 4/10. 5/10. 6/10. 2 pieces are left in here.

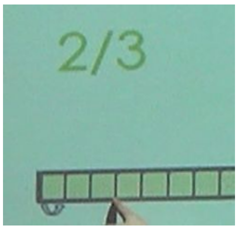


Figure 6.20. S8's representation 3

S15: We will divide that 2, too.

R: OK. Can you write down your findings? How many 1/10 did you have?

S15, S8: 6 pieces of 1/10.

R: You had 6 pieces of 1/10, and how many pieces are left that you cannot count?

S8: 2.

R: Then what is left, if you look at the whole.

S8: Hmm. 1 second. 2/30

R: OK.

S8: We had 2 of 1/30s.

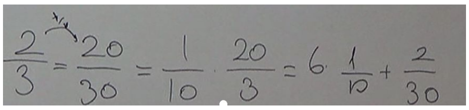


Figure 6.21. S8's representation 4

Engaging in counting, he was able to determine how many of 1/10's are there in the shaded amount of 20/30 and engaging in grouping activity, he realized how much 1/10 in the diagram represented: 3 groups of 1/30 being equal to 1/10. That is, he thought each piece as quantity of 1/30 and concluded that 3 of them makes 1/10. Then he counted 3 groups of 1/30 in 20/30 which resulted in 6 whole groups and a remaining part equal to 2/30 th. That is, with the goal of measuring 20/30 with 1/10, he counted the number of groups of 1/10. At the end, he wrote 11/3 = 3+6/10+2/30.

After finding 1/10 th decimal place of 11/3, the class continued with the 1/100th decimal place. Again students first worked individually and then S15 shared his work on the board. S15 continues from the part that is not grouped in the case of 1/10 th digit and the quantity was 2/30.

S15: Now only these two are left and it is 2/30. I have written this as 20/300. Like this. This means 20 over 3 times 1 over 100. I have written this part as decimal.

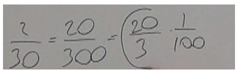


Figure 6.22. S15's representation 1

6 comes out when I divide 20 into 3. This becomes 2 divide 300.

R: If we show with a diagram? Can you draw a diagram?

S15: This one. (Draws 2 pieces on the board). Now as we have divided this into tens, we will search for hundreds at 2/30

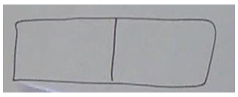


Figure 6.23. S15's drawing for 2/30

R: Write down the quantity as well. Write down the quantity of that figure you just drawn.

S15: 2/30 is the whole. I have divided to 10 after this. (Meaning each equal piece) We have divided each of them into pieces (mentions dividing 2 pieces into 10). What is the result? 20/300. We have extended it. We will search for 1/100's. This means 20/3 times 1/100.

Initially, S15 explained operationally (numerically) that is just saying that 'the remaining portion is this one. It is 2/30. I wrote it as 20/300. It means 20/3*1/100.' In this case, S15 did not show any reasoning about his operations. However, the researcher forced him to show what he did on the diagram and S15 drew two pieces representing 2/30. He was aware of that he was continuing from the piece that is not grouped while finding the number of 1/10's. At this stage, S15 started to dealt with quantities rather than just numbers and continued from the diagram to find exact number of 1/100 s in 11/3. The following excerpt illustrates this process:

S15: (Explains the process 20/3*1/100). I have separated this as 1/100. I mean (continues with figure) I am grouping them by 3 pieces . Now we have 6 wholes.

R: We have 6 many which groups with?

S19: 1/100.

R: Do we have 3 groups of pieces?

Class: Yes.

R: What do the 3 groups of pieces represent for me?

S2, S5, S13:1/100.

S15: 2 remains. One of them is 1/300, and the 2 of them are 2/300. In 300 pieces, we have 2 pieces left.

R: 6? You have 6 of what?

S15: 6 pieces of 1/100.

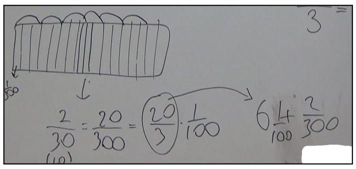


Figure 6.24. S15's representation 2

S15: We will continue to write.

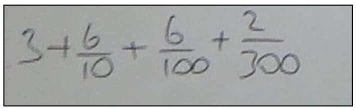


Figure 6.25. S15's representation 3

S15: Now we are here (showing 2/300). I have to zoom in 2 pieces out of 300

R: Can we also say it like this? 3 plus 6/10 plus /10 square plus 2 dividend 3 dividend 10 square. Is that fine?

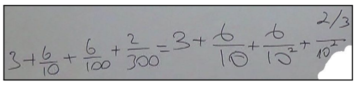


Figure 6.26. S15's representation 4

Together with the first excerpt, data shows that S15 knew that he was working with the quantity 2/30. This was evident in his statement that the figure represented 2/30 upon drawing the figure. Then, his uttering from the first excerpt, "This one. (Draws 2 pieces on the board). Now as we have divided this into tens, we will search for hundreds at 2/30" shows that he knew that he would be measuring 2/30 with 1/100 pieces. That is why later on he grouped each three pieces. As shown in the second excerpt, while drawing the diagram, first S15 (Figure 6.24) partitioned each piece by 10. Then, he grouped each three pieces that represented 1/100. That is, he measured the remaining portion 2/30 by 1/100 th s. Therefore, he engaged in the partitioning and grouping activity. In addition, although he just used two portion by zooming in 2/30 while doing partitioning he expressed the quantity he dealt with thinking through the whole and said that it became 20/300. Again, for finding the exact number of 1/100's he grouped each 3 pieces of 1/300 to make 1/100 and counted the groups of 3 in the portion 20/300. That is, his drawing each three pieces representing 1/100 th s and stating that he is looking after 1/100 th s in 2/30 also shows that he knew that each equal piece represented 1/300. As he emphasized in the dialogue he could not grouped 2 pieces which have the quantity 2/300 and his statement that each piece represented 1/300 also shows that he knew that all the equal pieces in 20/300 was representing 1/300. At the end, he came back to the fraction 11/3 and expressed it 11/3=3+6/10+6/100+2/300. Again, long division, diagram and decimal form were matched because when asked the step for 1/100 th digit students both explained the situation using diagrams and long division. He shifted through the diagram easily and used the concept of equivalent fractions and quantitative reasoning operations as equal partitioning, grouping and counting. After S15's work the researcher suggested another way of representing 11/3 decimally mainly changing the last added part: $3+6/10+6/10^2+2/3/10^2$. The reason behind this representation was based on the flow of the instruction. Later, they considered $2/3/10^2$ and showed some properties of the expression and generalize it using symbols.

The class continued one more step and lastly found the 1/1000 digit of 11/3. Each student worked individually. They proceeded in the same way using equivalent fractions and equal partitioning and grouping.The reason for expecting students work as individually is to observe every students' work during the process. Since the first two steps were discussed collaboratively, the researcher wanted to focus on students' works and their thinking process individually.

Although it is possible to show all the students' written work showing their thinking process one by one, based on the written work from all of them and their agreement during the whole class discussions when individually students mentioned their reasoning, the individual works given below could be taken as depicting such process. In this respect, three students' written work from their continuing engagement in the left over part is shown below. Particularly, S17 's written work depicts students' reasoning who showed the whole process from the very beginning holistically (i.e. starting from the first left over 2/3) on one diagram. On the other hand, S9's written work depicts some other students' reasoning who especially showed both the long division and the diagram drawing simultaneously. Similarly, S7's written work is shown to represent some students' reasoning who did not much used the long division but worked with diagrams and the decimal representation.

S17's (See Figure 6.27) started her work by writing the long division along with diagram. However, later she continued with only the diagram. In the diagram, she firstly took the piece of 2/3 of the whole and the divided the each piece by 10 to get 20/30. Although she did not depict the process of grouping in 20/30 to find the number of 1/10's in the portion 2/3, she showed the process for finding the number of 1/10's and 1/100 th s in a single shape emphasizing re-partitioning. Also, she showed how she counted 1/100 th s as grouping 3 pieces in the portion of 2/30 after re-partitioning. Her representation of a single diagram was an explanatory one because it showed that S17 was aware of that she was dealing with the portion which became smaller as calculating the further decimals of 11/3.

On the other hand, although S9 (See Figure 6.28) proceeded similarly for finding the decimals, she used 3 representations simultaneously. For instance, as finding 1/10's through diagrams using equal-partitioning and grouping, she showed the correspondent step in long division and she used the same logic in further steps. It seems that she made sense of the relationship between quantitative operations (equal-partitioning and grouping) and long division (adding zero to right side of remainder and comma to the right side of quotient).

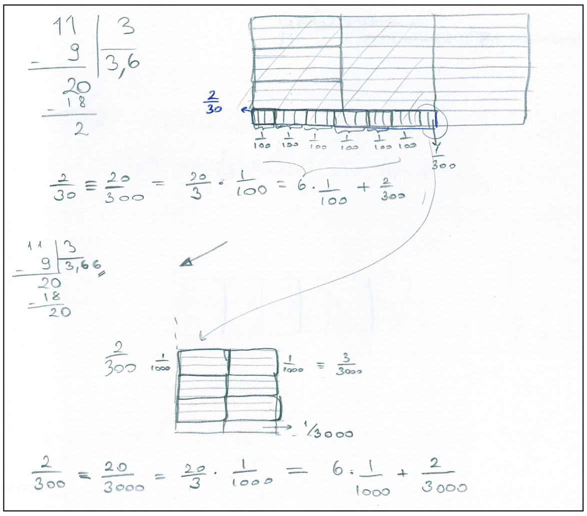


Figure 6.27. S17's explanation 2

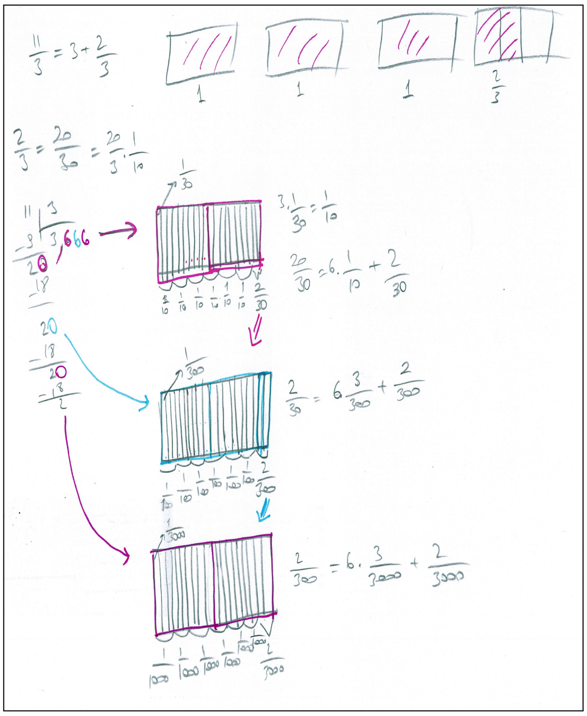


Figure 6.28. S9's explanation 1

S7 (See Figure 6.29) just used diagrams and decimal representations. She proceeded similarly but she also explained how to find the decimals as writing "I am looking for 1/100 th s so I will make 3 groups of them". From her explanation it can be understood that she did not apply the procedure just memorizing the steps and she also made sense of it.

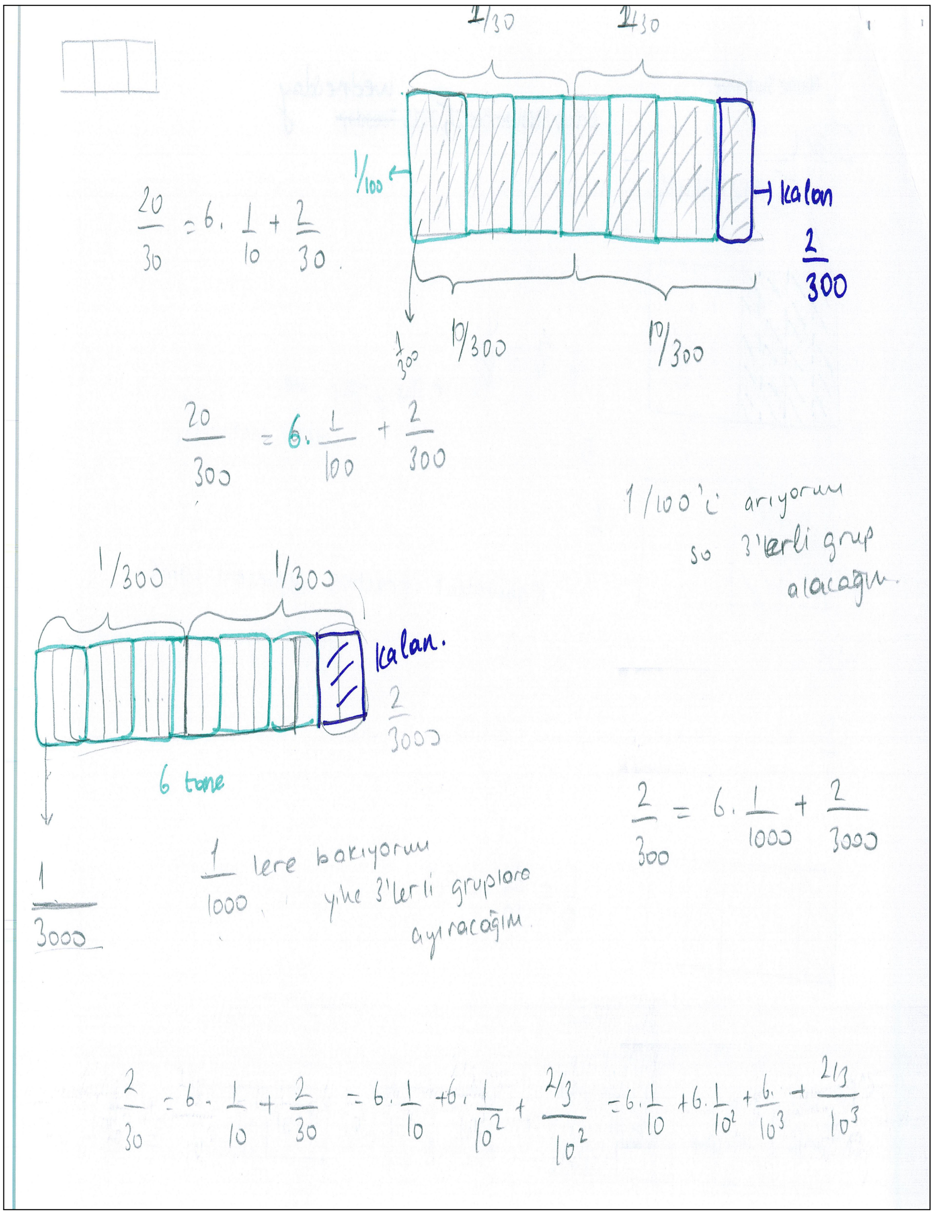


Figure 6.29. S7's explanation 2

Although the students S17, S9 and S7 differed slightly in terms of their focus of representations that is in one side the student continued with 3 representations simultaneously, in other side one focused on 2 representations, in another side the student showed the big picture focusing on one big diagram, their conclusion about their learning coincided with each other. S17 (See Figure 6.31), S9 and S7 (See Figure 6.30) explained that they dealt with equivalent fractions to find the decimal representation along with equal-partitioning and grouping. Also, they were aware of that the grouping operation was related to the divisor they dealt with.

> S9: We have made equal partitioning to the remaining parts and then we have grouped them and tried to find the decimals.
> R: Which fraction logic did we follow?
> S9: Which fraction? Well we followed equivalent fractions

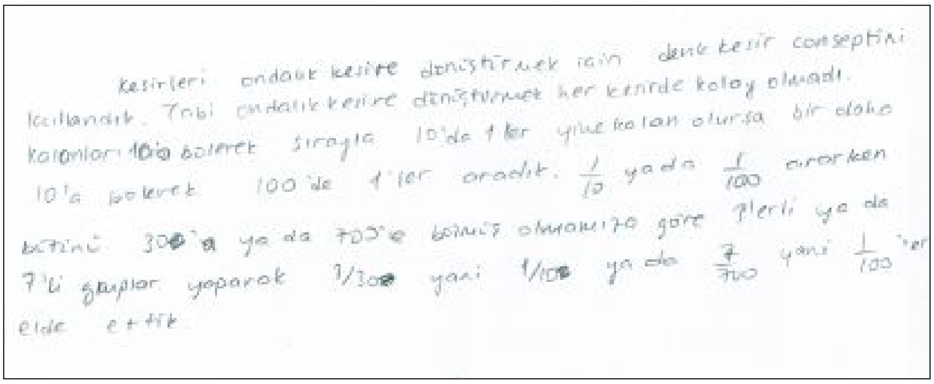
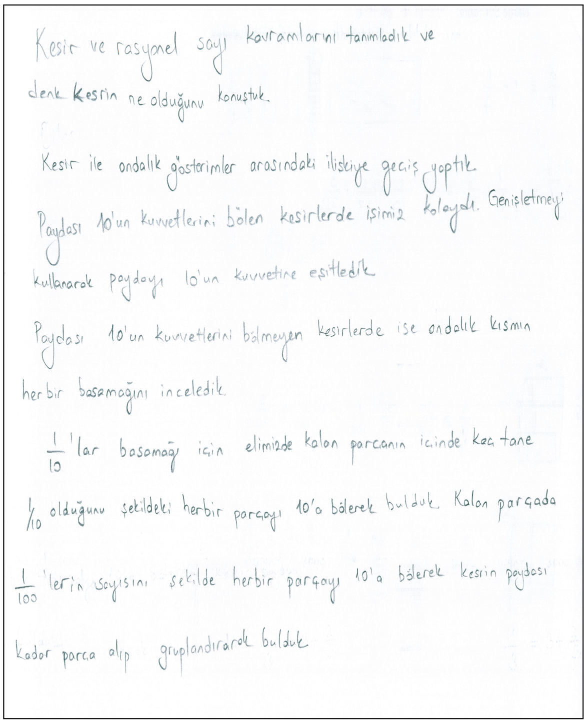


Figure 6.30. S7's explanation 3

> "To convert fractions to decimal representation, we used the concept of equivalent fractions. Converting each fraction to decimal representation wasn't easy. As partitioning the remaining parts by 10, we searched for 1/10's. If there is also remaining parts after finding 1/10's, we again partitioned the remaining parts by 10 and searched 1/100 th s. As searching for 1/10 th s or 1/100 th s according to the partitioning with 300 or 700 pieces in total, we grouped the pieces by 3 or 7 and obtained 3/300 that is 1/100 th s or 7/700 that is 1/100 th s." (S7 's explanation 3-Figure 6.30)

> "We have defined fraction and rational number concepts and talked about what equivalent fraction is. We have moved to the relation between fraction and decimal representation. For fractions of which the denominator divisible by powers of 10, we had an easy job. We have used expansion and equalized it with powers of 10. For fractions of which the denominator not divisible by powers of 10, we examined each step of the decimal part. For finding the 1/10 th digit, we examined the remaining pieces as partitioning each piece by 10 and finding the number of 1/10 th s in the remaining pieces. Then again in the remaining pieces, in order to find the number of 1/100 th s, we partitioned each piece into 10 again and grouped the pieces according to the denominator." (S17's explanation 3-Figure 6.31)



Kesir ve rasyonel sayı kavramlarını tanımladık ve denk kesrin ne olduğunu konuştuk.

Kesir ile ondalık gösterimler arasındaki ilişkiye geçiş yaptık

Paydası 10'un kuvvetlerini bölen kesirlerde işimiz kolaydı. Genişletmeyi kullanarak paydayı 10'un kuvvetine eşitledik

Paydası 10'un kuvvetlerini bölmeyen kesirlerde ise ondalık kısmın herbir basamağını inceledik

$\frac{1}{10}$'lar basamağı için elimizde kalan parcanın içinde kaç tane $1/10$ olduğunu şekildeki herbir parçayı 10'a bölerek bulduk. Kalan parçada $\frac{1}{100}$'lerin sayısını şekilde herbir parçayı 10'a bölerek kesrin paydası kadar parca alıp gruplandırarak bulduk

Figure 6.31. S17's explanation 3

Decimal representation of 8/7:

In the last stage of the first phase, students found the decimal representation of 8/7. The reason for giving this fraction is that the remainder is changing. Students worked individually for this exercise and after that the phase I of the instruction ended.

6.2.1.2. Phase II of the Instruction. In the phase 2 of the instruction, researcher continued with the example 11/3 and asked students about the previous instruction to assess whether they comprehend the topic and connect the mathematical ideas such as equivalent fraction, long division and decimal representation. After that she wanted students to remind the decimal representation of 11/3 as given below:

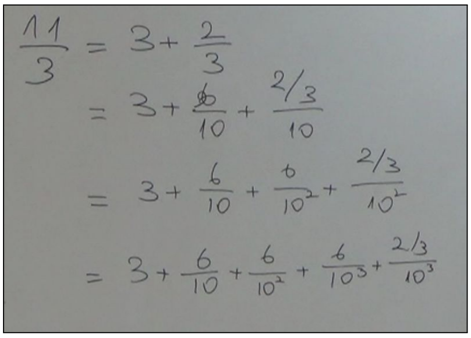


Figure 6.32. Decimal representation of 11/3

The reason for representing 11/3 in this form is to continue thinking through the boundaries of 11/3 that is to determine the lower and upper bounds as the number of decimal places of 11/3 increase. Therefore, the researcher firstly asked the interval that 11/3 lies in and as students gave answers researcher asked to narrow down the interval:

R: Now, we will try to squeeze 11/3 into an interval. First of all, between which two numbers that 11/3 lies in?

S4: 3 and 4.

R: Can I narrow down the interval by looking at the second expression?

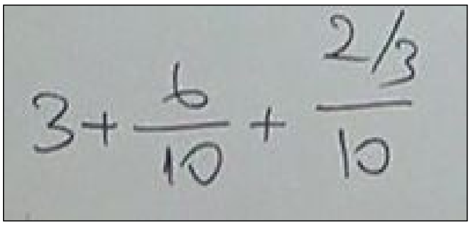


Figure 6.33. Decimal Representation of 11/3 up to 1/10 th digit

You said that it is between 3 and 4. It is somewhere between 3 and 4... Somewhere in here..

I will narrow my interval and approach to number. Do you think that I can narrow my interval by using this expression?

S1: Yes

R: We have expressed 11/3 like this; 3 plus 6/10 plus (2/3)/10. Now I have said it is between 3 and 4, but I want to narrow my interval. Can I narrow my interval by thinking from this expression? I want to change both my upper and lower bounds. Not only one of them. I want to change both.

S1: Lower bound changes, but we could not change the upper bound.

Figure 6.34. Interval for 11/3

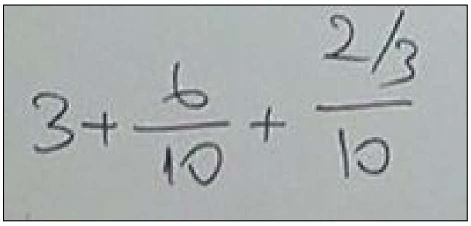


Figure 6.35. Decimal Representation of 11/3 up to 1/10 th digit

In the above excerpt, the researcher expressed what she expects from the students about the narrowing down the interval. However, students had a difficulty in changing both upper and lower bound as expressed above excerpt. S1 could change the lower bound but could not change the upper bound. A similar case occurred while students tried to find the intervals individually. As seen S19 just changed the lower bound as 3.6 and the upper bound remained the same. Then, the researcher leaded her to think about changing the upper bound.

R: Write it down and let's see.
S19: This is how I reached.

Figure 6.36. Decimal Representation of 11/3 up to 1/10 th digit and interval

(Shows 3+6/10)
R: Now you have increased lower bound. I want you to decrease the upper bound. I want you to change both the upper and lower bound.

Although some students thought just one boundary, there were also students narrowing down the interval as possible and explaining their reasoning. For instance S9 could change the two boundaries and explained as follows:

R: (I am next to S9) I said use this.

S9: Yes. I have used it. This is how it will be. 3 plus 6/10 plus (2/3)/10. This (2/3)/10 is smaller than 1/10. R: Can you write down those? Your thoughts.

S9: How should we write?

R: Did you say 2/3 is smaller than 1, or smaller than 1/10?

S9: Smaller than 1/10. I thought that if this is 3/3, then it will be 1/10.

R: Let's have a look.

S9: Because if it is 3/3 then this (shows 2/3) will be 1. 1/10.

S16: 2/3/10 is smaller than 1/10.

S9: Yes. Yes, exactly.

R: Yes. OK. Write down that statement.

S9: (2/3)/10 is smaller than 1/10. Because when we look at numerator, if it was 3/3 then it would be equal to 1/10.So, relying on that, what should I do.

R: What did you say 2/3/10 is smaller than 1/10?

S9: I said 11/3 is between 3.6 to 3.7.

R: OK. Can you write it like this statement? 3.6 and 3.7 (3 plus 6/10.......this is how I want him to write.)

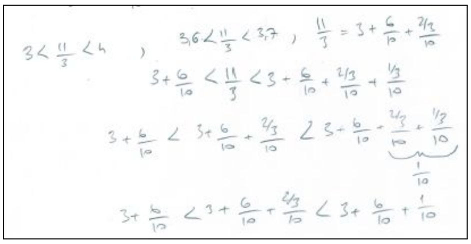


Figure 6.37. S9's Representation

(R asked how I will write 11/33 and S9 stated like this)

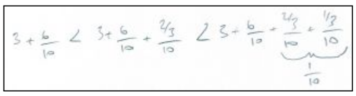


Figure 6.38. Interval of 11/3 up to 1/10 th digit

R: This is how you could write this. Because...you said 2/3/10 is smaller than what?

S9: Because 2/3/10 is smaller than 2/3/10+1/3/10.

R: So you can say 1/10 there.

S9: Yes and this is 1/10. Let me write it again from here. (Writes the last line)

In this excerpt, S9 compares the numerator of the last portion that is 2/3 with 3/3 and 0/3. Then, she concludes that 2/3/10 is less than 3/3/10 and bigger than 0/3/10. With this thought she changed both the lower and upper bound.

Different than S9, S8 stated that instead of focusing on 1/3, 2/3 or 3/3 as separate parts, he focused on the left over parts.

> S8: I did not do 1/3, 2/3, 3/3. As we work directly with decimals. What was our statement? This.

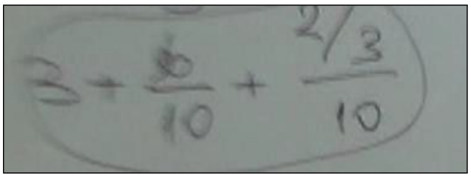


Figure 6.39. Decimal Representation of 11/3 up to 1/10 th digit

> R: We were moving upon this.
> S8: I have moved upon this. This was the remaining piece (showing the last added part)

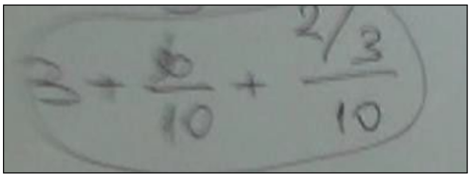


Figure 6.40. Decimal Representation of 11/3 up to 1/10 th digit

> S8: It was endless. That is why I have taken as 3 plus 6/10. 11/3 is bigger than this one (means 3+6/10), because it has this excess. (means (2/3)/10) What does this complete itself to? It will complete to 1/10.This is how it goes:

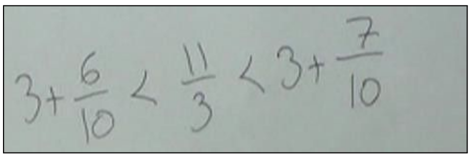


Figure 6.41. Interval of 11/3 up to 1/10 th digit

He focused on 2/3/10 as a whole. Therefore, from 2/3/10 he tried to obtain portions of 1/10 and he added that 2/3/10 completes itself to 1/10. From his expression it seems that he tried to think considering quantities. He knew that he obtained 6 many 1/10's and the remaining part 2/30 cannot give some amount of 1/10. Therefore, in order to find the upper bound he completed 2/3/10 to 1/10. Although some other students just focused on the numerator and compared 2/3 and 3/3 without considering the denominator, at the end the students could understand each other and agreed that they thought similarly. The above excerpt illustrates this case:

> S19: The same logic is not it?
> S15: Same logic.
> S19: It seems that you ignore the remaining part.
> S8: Yes, the remaining part. Probably, after that we obtain this one don't we? 6/100 and here 7/100.

Later S19 wrote the intervals for second, third and n-th steps as:

1.Step:

$3 + 6/10 < 3 + 6/10 + 2/3/10 < 3 + 6/10 + 1/10$

2.Step:

$3 + 6/10 + 6/10^2 < 3 + 6/10 + 6/10^2 + 2/3/10^2 < 3 + 6/10 + 6/10^2 + 1/10^2$

3.Step:

$3 + 6/10 + 6/10^2 + 6/10^3 < 3 + 6/10 + 6/10^2 + 6/10^3 + 2/3/10^3 < 3 + 6/10 + 6/10^2 + 6/10^3 + 1/10^3$

n.Step:

$3 + 6/10 + 6/10^2 + \ldots\ldots. + 6/10^n < 3 + 6/10 + 6/10^2 + \ldots.. + 6/10^n + 2/3/10^n < 3 + 6/10 + 6/10^2 + \ldots.. + 6/10^n + 1/10^n$

- Part I: Students' Re-Writing And Examining The Properties of The Decimal Expression of 11/3 as Sequences

Since the instruction continued with the analysis of the lower and upper bounds it was important to comprehend how the upper and lower bounds obtained. The students had no problems in determining the intervals. Later, first focusing on the lower bounds, and then focusing on the upper bounds, the researcher wrote the following expressions on the board

f(n) = 3 + 6/10 + 6/10^2 + ...+ 6/10^n for n in N^+ g(n) = 3 + 6/10 + 6/10^2 + ...+ 6/10^n+1/10^n for n in N^+

and asked them to think about whether these could be functions or not. Firstly, since students could not give answers to the question, researcher shifted the discussion through the definition of function. Students collaboratively defined the function by giving two necessary conditions for being a function. Therefore, students did not have any misconception or misinterpretation about the definition of function. After that, again the researcher went back to the question about f(n) and g(n). First, students had a consensus on the fact that the first expression would yield a function because for all the natural numbers there is a corresponding decimal expression and one natural number did not match with two different decimal expressions. S8 was one of the students showing how f(n) yielded a function as follows:

Here, S8 was able to write corresponding decimal expressions to natural numbers (Figure 6.42). The rest of the class also agreed with the explanation of S8 and confirmed that f(n) formed a function. Upon their agreement, the researcher asked them for the special name of this particular function. Some answered:

S14: We call these special functions from natural numbers to real numbers as sequence.
S7: Sequence is this mapping isn't it?

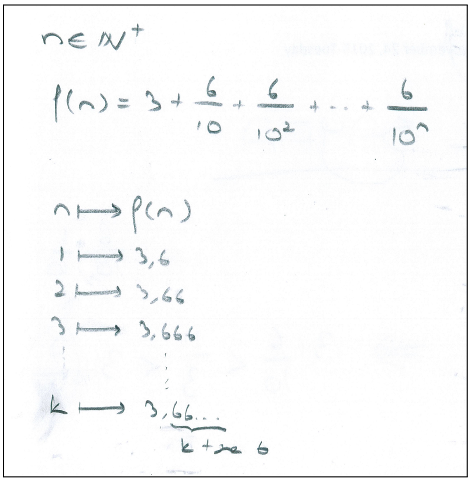


Figure 6.42. S8's interpretion of f(n) as a function

At that point, the researcher confirmed what these students said. After that the researcher focused on the upper bound and asked whether it was a sequence or not. They concluded that it was also a sequence since the only difference was the $1/10^n$ part. After classifying the functions, f(n) and g(n), as sequences, the class started to examine the properties of the two sequences. The increasing and decreasing properties of the two sequences were discussed in the excerpt below:

S5: A sequence of which the terms come closer.
S15: Convergent sequence. There was such a sequence.
S14: Convergent sequence.
R: It was 3 plus 6/10, what did we add then?
S19, S5, S14: $6/10^2$
S15: It comes closer to 3,7.
R: OK. How are the terms changing......
S19: It approximates.
S5: The last part we have added approach to zero.
S15: It comes closer to 3,7.
S19: It is increasing. Increasing sequence.
R: Do you agree? What do you think?
S15: Monotone increasing.
S5: Yes. But it increases by decreasing. (Rate of increase reduces, but the terms increase)
R: It increases by decreasing. OK. Why does it increase by decreasing?
S5: Because we add smaller things.
S19: It increases in smaller amounts, yes.
R: So we look at the increase. We called the lower bound as increasing

Based on their explanations, the researcher wrote down the following on the board (Figure 6.43):

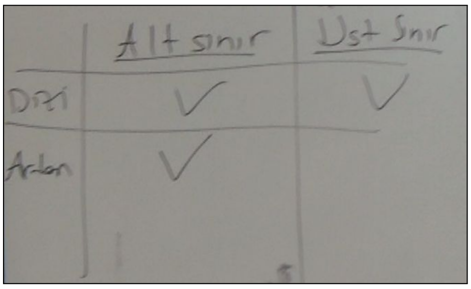


Figure 6.43. Table of the properties of sequences

Initially, students could not figure out the increasing property of the sequence and said that was a convergent sequence. Although their answers were not wrong, the researcher focused their attention on increasing property of the sequence by asking the question "how are the terms changing?". Students focused on the different terms of the sequence, numerical values, as well as how those terms were changing. They could realize that not only the sequence was increasing but also the nature of such increase was at a decreasing rate because they were adding amounts smaller than the previous ones each time. Thus, the students agreed that the lower bounds formed an increasing sequence at a decreasing rate since the following terms are bigger than the previous terms with lesser amounts added. The next discussion then occurred about the increasing/ decreasing property of the upper bound. As opposed to the property of the sequence formed by lower bound, the students had a difficulty in figuring out the property of the sequence formed by upper bounds.

Initially, students could not figure out the increasing property of the sequence and said that it is a convergent sequence. Although the answer was not wrong, the researcher asked more questions to focus on increasing property of the sequence. At the end, the students agreed that the lower bounds formed an increasing sequence since the following terms are bigger than the previous terms. The next discussion occurred about the increasing/ decreasing property of the upper bound. The students had a difficulty in figuring out the property of the sequence formed by upper bounds.

R: What do you think about the upper bound?
S14: Upper bound is also increasing.
S15: Upper is decreasing.
R: Why ?
S15: Because they will become close to each other. In order to find the exact value of the number in the middle.
R: Just look at upper bound.
S15: Ok, it is like this. It was 3,67. Other one will be 3,667. It has decreased to 667 from 670. For instance, we look for n=3, I look for n=4. 3rd term is bigger than 4th term. It is smaller, that is why it decreases.
S14: Yes, it decreases because from 3,67.
S15: It means 3,67.
R: How did you write the first term?
S15, S19: 3.7
S15: Other one is 3,67. It decreases more.
S19: Yes, it is very logical
S15: That is how I wanted to say.
R: What is my upper bound in this case?
S19: Decreasing.
S15: It decreases. That is why they approach more to each other. We define 11/3 better
S7: We squeeze it already and an forms lower bound. It increases already. When I squeeze the number in number line, I increase the terms of a_n and decrease the terms of b_n. This is correct.

For the upper bounds when they just looked at the general term $3 + 6/10 + 6/10^2 +.....+ 6/10^n + 1/10^n$, S14 said that it was also increasing seems to suggests that he thought that like in the previous sequence, in this sequence they were adding amounts too It seemed from S14's reaction that some of the other students might have had a similar difficulty in deciding whether the sequence was increasing or decreasing. Then, this difficulty was compromised by S15's comments because focusing on the numerical values of the terms such as 3,67 and 3,667, S15 realized that the second value comes down to 3,67 from 3,7 so he concluded that it was decreasing and he was aware of that as changing both the lower and upper bounds (increasing the lower bound and decreasing the upper bound) the decimal representation of 11/3 could be more precise. That is, during the discussion, focusing on and checking the value of each term in the sequence, S15 could convince his friends to understand that adding something may not necessarily increase the amount.

The other investigated property was boundedness. First they examined the sequence formed by upper bounds and later the sequence formed by upper bounds. They collectively agreed upon that fact that the sequences were bounded since the value of the terms were always between 3 and 4. The discussion given below illustrates the students' thought processes about boundedness of both sequences:

R: Now if we think... Between which two numbers can I mention?
S14: 3 and 4.
S19: 3 and 4.
R: 3 and 4, right? 3 plus 6/10, between 3 and 4?
S19, S5, S14: Yes.
R: 3 plus 6/10 plus $6/10^2$.
S15: That is also 3 and 4.
R: Which interval will it be in?
S15: That is also 3 and 4.
R: Will it reach to 4?
S15, S19: No.
R: Why ?
S15: Because upper part is also decreasing.
S19: None of the additions make a whole.
R: When we look at the general terms of our sequence, we have named the lower bound as bounded. What do you think about the upper bound? Can you please write down what you think about the bound of upper bounds?
S15: It cannot be bigger than 3,7.
S19: Yes.
R: If you squeeze that between two natural numbers?
S15: That is also 3 and 4.
S19: 3 and 4.
S15: This is also bounded.

Although students focused on the numerical values of the terms of the sequences, S19's reasoning that the sequence formed by the lower bound could not become 4 because the added amounts after comma could not make a whole whatever amount/quantity could added, suggested that she focused on these numbers as amounts/quantity rather than examining them just as numerical values. The last property investigated was about being a rational number sequence. Students first defined what a rational number sequence is and then concluded that these two sequences were rational number sequences.

After investigating the properties of the sequences that a_n is increasing and bounded rational number sequence and b_n is decreasing bounded rational number sequence, the researcher probed students to look for the differences of the terms of these two sequences. That is, students were asked to check the differences b_1-a_1, b_2-a_2, and b_n-a_n. The researcher focused their attention of these consecutive differences because students were expected to conclude that the two sequences can represent the rational number 11/3, since the differences between the terms of the sequences become smaller as n increases. The below excerpt illustrates their inferences from the differences of the terms of the sequences:

> R: Let us have a look at the difference between the terms of these two sequences. What do you think b_1-a_1 is?
> S4, S1: 1/10.
> R: What is b_2-a_2?
> S4, S1, S7: $1/10^2$.
> R: What does the difference in the n th step?
> S4: $1/10^n$.
> R: $1/10^n$ is now n th step. What will the difference be if I increase n?
> Class: It decreases.
> R: It decreases. OK. Does it go to a certain value?
> S4: It goes to zero.
> R: What happens if I send n to infinity? Where will b_n-a_n value go?
> Class: To zero.
> R: It will go to zero. What will you say in this case?
> S1: They might be equal.
> R: Which number did I mention between a_n and b_n sequences, 11/3, right? What did we say about the difference when I increase the terms of sequence?
> S7: We said it goes to zero.
> S2: a_n and b_n are equal.
> S19: Both will be like equal to 11/3, right?
> S4: All 3 of them equal at infinity.

S4 and S1 told the differences and after that the researcher asked what happens if n becomes larger S4 said that $1/10^n$ approaches to zero. The researcher again probed the students asking 'What could you say then?' S1 and S2 emphasized that the sequences could become equal. As S4 emphasized, students went further and concluded that the sequences and 11/3 would become equal at infinity. Their consensus on the fact that both the 11/3 and the two bounded rational number sequences one of which is increasing and the other is decreasing might have been taken as an evidence that for these students those three representations referred to the same quantity.

Furthermore, some students expressed their inferences referring to the properties of the sequences. For instance S7 (See Figure 6.44) and S8(See Figure 6.45) gave explanations below:

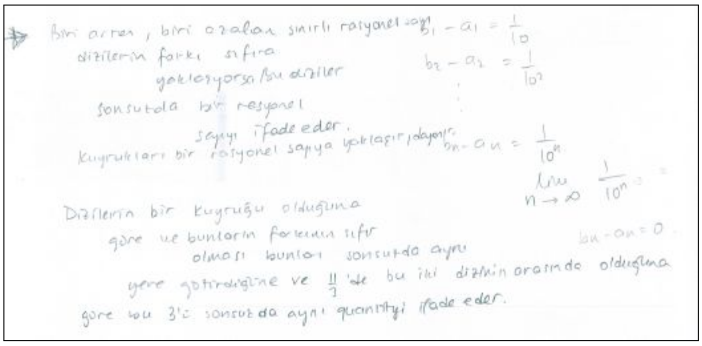


Figure 6.44. S7's Explanation 4

"If the difference of rational number sequences one is increasing and the other decreasing approaches to zero, then at infinity they represent the same number. They approach to the same rational number. Since sequences have tail and difference between them zero means that they go to the same point at infinity and 11/3 is between these two sequences. Therefore, all 3 will represent the same quantity at infinity." (S7's Explanation 4- Figure 6.44)

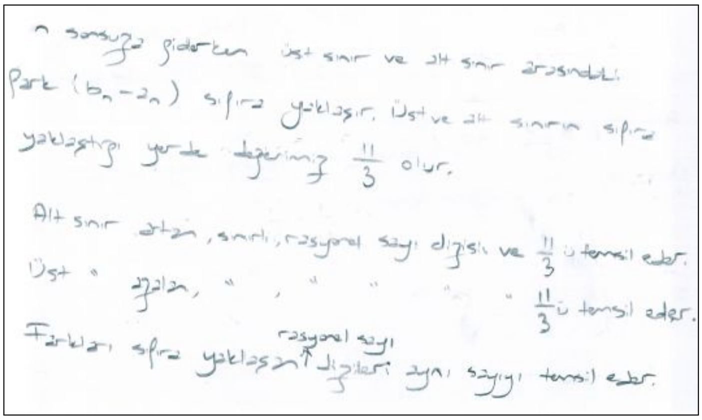


Figure 6.45. S8's Explanation 4

"As n goes to infinity the difference between the upper bound and lower bound approaches to zero. At the place where the difference approaches to zero, the value would be 11/3. The increasing bounded rational number sequence will represent 11/3. The decreasing bounded rational number sequence will represent 11/3. The rational number sequence whose differences will approach to zero will represent the same number." (S8' Explanation 4-Figure 6.45)

They both emphasized that they obtained increasing and decreasing bounded rational number sequences and stated the meaning of the differences of the terms as the number of the terms increases. Therefore, they were able to reach the conclusion that 11/3 can be represented by two bounded rational number sequences one of them increasing and the other one is decreasing.

- Part II: Students' Re-Writing and Examining the Properties of the General Decimal Expressions as Sequences

Until this point students had dealt with specific examples especially with positive rational numbers. They also observed some properties and relationships as expressed previously. In order to be able to generalize their observations to all positive rational numbers they needed to use notations such as symbols and letters. For this reason, the researcher introduced the following notations for the decimal representation of any positive rational number X making inferences using 11/3.

$X=D+d_1/10+r_1/10$

After that students were asked to express the decimal representation of X up to 1/100 th decimal and go on further steps. The class helped the researcher to write the expression as $D + d_1/10 + \ldots.. + d_n/10^n + r_n/10^n$.

Since using notations is not enough to convey the findings in the previous example to the general case, students were asked to explain what do D, d_n's, and r_n's refer to. They also correctly identified what kind of numbers D, d_n's, and r_n's should be referring to. They collectively agreed upon the idea that D is a natural number and d_n are the natural numbers between 0 and 9. Also, for the current case they stated that r_n's are rational numbers between 0 and 1 since they were dealing with positive rational number X. Although the researcher did not investigate further whether the students were able to correctly determine which kind of numbers were D, d_n's, yet the following discussion evidenced that students were referring to the examples they have engaged in previously while thinking about the interval the r_n's belonged to:

R: And r_1?
S1: Rational number.
R: You said rational number. In which interval?
S1: Between zero and one.
R: Do we agree to this?
Class: Yes.
R: S17?
S17: Yes.
R: For instance, we think r_n?
S1: That is also between zero and one.
R: What will happen if all r's become zero? Or can it be zero?
S1: Yes.
S18: Yes.
R: What happens if so?
S7: Then it will have a rational expression..
R: So?
S7: What was the first example we did? I do not remember now.
R: 7/2.
S1: For instance, 3.5 is there. Second digit was zero.
S8: It is not non-terminating..

The important point is that upon one student's statement that when r was zero, then the decimal representation was non-repeating (i.e. terminating), other students agreed that the possibility of r_n's being zero was meaningful. Some of them even explained the idea referring to the example 7/2 examined previously. This might have been taken as evidence that since they investigated the decimal representation of 11/3 through quantitative reasoning and focused on the un-grouped parts of the pieces, they were able to represent r_n's between 0 and 1.

As done in the example of 11/3, students were asked to squeeze the positive rational number X in the intervals which can be considered as nested. For this aim, first students worked individually and then one of them shared the solution with the class. Although other students reasoned similar to S1 whose explanation was given below, only S4's work (See Figure 6.46) and S9's work (See Figure 6.47) were similar to each other. They both thought the case where r could be zero so they added an equality sign to the relationship between the lower bound and the decimal representation of the rational number X.

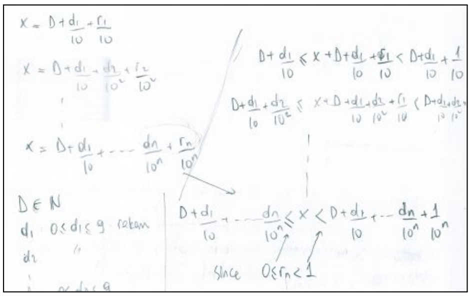


Figure 6.46. S4's explanation 2

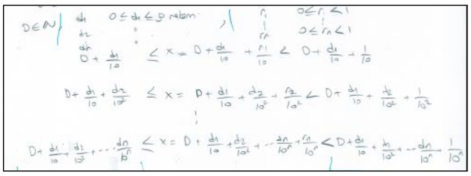


Figure 6.47. S9's explanation 2

After individual work, S1 explained what he did to the class. In contrast to S4 and S9 and similar to the other students, he did not consider the case of r being zero. Therefore, the researcher asked what would happen if r was zero. After that he thought about the equality sign. The discussion followed:

> S1: r_1 can be zero. That is why I give zero to r_1 while finding lower bound. So this goes.(shows $r_1/10$ piece)

Figure 6.48. Intervals for the general case up to 1/10 th digit

R: What will happen if it is zero? We said it will go when it is zero. We can have such a case. Then, what will come?

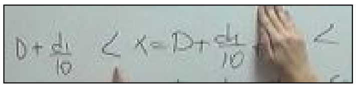


Figure 6.49. Intervals for the general case up to 1/10 th digit

S1: Equality. This is what happens to our lower bounds. Because r's can be zero.Let us look at upper bounds now. Now, r_1 will be smaller than 1 at upper bound.So if I give 1 to r_1, this (upper bound) will be bigger than this (decimal expansion of x). Because r_1 can never be 1. So this will stay like this (case of not adding any equal sign to inequality)

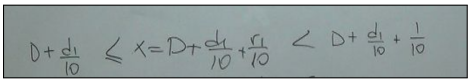


Figure 6.50. Intervals for the general case up to 1/10 th digit

S1: Then I will do the same in second step. So when I reach to n step, it will be something like this.

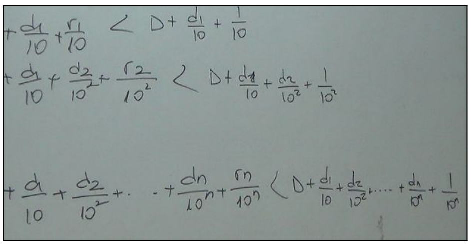


Figure 6.51. Intervals for the general case

The next thing that the students thought is whether the lower and upper bounds formed sequences or not. S5 uttered that it is a kind of function as previously (referring to the case of 11/3) so it is a sequence and all the students agreed upon S5's explanation. The reason behind their answer in their written arguments was similar to their previous explanations in the case of 11/3. Even, one student, S19, came to the board and stated the following along with how she considered the properties of the sequences:

R: Exactly. How about you? Can someone come on board and explain upper and lower bound.

S19: I will.

R: OK.

S19: Now will this lower bound generate a sequence? Yes it will. Because $D+d_1/10$ comes to 1st term. This one is the 2 nd term. This is 3 rd term; this is n th term.

R: But now the first term is $D+d_1/10$. Why do you say that the sequence is increasing? (I am asking due to what is written on paper)

S19: Why do I say that it increased? Because when it passes to term 2, it has $1/10^2$ extra and it increases as such. So the statement 2 is more than statement 1. Statement 3 is more than statement 2 and statement 1. n. statement is more than previous ones. Therefore, it is increasing. Bounded. It is bigger than D, but smaller than D+1. And it is a rational sequence as I can write it with rational expressions. I pass to the upper bounds now. Is it increasing or decreasing, how will I decide? This is D,$\underline{d_1+1}$. This is the second digit (indicates the underlined statement). 1/10's. How will I write? d_1 this is 1/10 th digit, and I wrote d_2+1 to 1/100 th s. D,$d_1(d_2+1)$. Now this expression (Second one) is smaller than this (First one). Why small? Because there is d_1 in 1/10 th digit. There is d_1+1. The 1/10 th digit here is bigger than other one's. So this statement decreases. Decreasing. Is it bounded? Bounded. The same reason again. (between D and D+1) and a rational number sequence.

While explaining her reasoning about whether the expressions were sequences or not she reasoned that the correspondent decimal expansions changed while n changes. That is, she was able to relate each term of the decimal expansion with the natural numbers by constructing a one-to-one correspondence between the natural numbers and the each term of the decimal expansion. For the increasing property of the sequence formed by lower bounds, she reasoned that for example for the second term there was an increase of $1/10^2$ and, compared the first and the second terms of the sequence and the second and the third terms of the sequence. That is, mentally, she was simultaneously co-varying the quantities corresponding to n=1, 2, 3...and the added amounts in the consecutive terms of the sequence. For boundedness, she concluded that the sequence was bounded since its values would always between D and D+1. Thus, also being aware of the fact that the terms were rational numbers, she concluded that the sequence formed by lower bounds was increasing and bounded rational number sequence. For the sequence formed by upper bounds she reasoned in the same way except for decreasing property of the sequence. For this case, she compared the decimal digits. For instance, she wrote the first and the second terms of the sequence formed by the upper bounds as D,(d_1+1) and D,d_1(d_2+1) and compared 1/10 th digits and concluded that the sequence was decreasing. Since they were dealing with notations it could have been more difficult for her reason about the increasing or decreasing cases. For this reason, she used different approach for explaining the decreasing property of the sequence.

After determining the properties of sequences, the researcher asked to find the differences of the terms of decreasing bounded rational number sequence and increasing bounded rational number sequence. The aim was to foster students examine the meaning of the differences as the number of terms increases. They were expected to conclude that both sequences represented the positive rational number x.

As in the case of 11/3, students found the differences between the first terms as 1/10, between second terms as $1/10^2$ and between n th terms as $1/10^n$ and thinking that the differences were getting smaller and smaller, they concluded that the difference was approaching to zero. Once asked to explain further what this would mean, some students stated the following:

> S19: What will happen. y_n acts like equal with x_n.
> S8: We come to same point again. Lower bound is equal with upper bound.
> R: OK.
> S8: They represent the same number.
> R: Which number will it represent in our case?
> S19: X

- Part III: Students' Way of Examining Irrational Numbers:

Until this stage, the examples were positive rational numbers and the generalizations were based on the rational number examples, therefore the researcher later shifted to the irrational numbers and asked 'if the number was an irrational number, can we make the same generalization?' and S15 answered as 'no, we cannot'. Then the researcher challenged the students as asking what if we try for $\sqrt{3}$.

R: We have dealt with positive rational number so far. Would this still be valid if we had an irrational number? $\sqrt{3}$ for instance. Let us move from this..

S1: $\sqrt{3}$ when we took the left side there will be a number that increases each time.

R: Now $\sqrt{3}$ number maybe we can find geometrically the correct place, but will we be able to express the number as the value exactly

S1: Somewhere between 1 and 2.

R: OK. $\sqrt{3}$ u got to say.

S1: That's roughly 1.7 like

R: Which interval does $\sqrt{3}$ lie ?

Class: 1 and 2.

S1: We will use triangle.

R: It is the geometric interpretation. now we will not use that triangle. 11/3 narrowing down we went, right? What could we do for $\sqrt{3}$? Now, it is somewhere here (Showing Figure 6.52), but I say. Exactly where could it be? I could not find the place in a way I approach the value. How close can I get? Now when we goes back through our first lesson, what did we do? Our figures are always divided into how many parts?

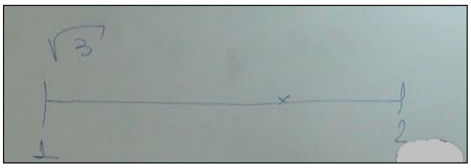


Figure 6.52. Interval of $\sqrt{3}$

S1: 10

S18: We divided 10.

R: Here, too, what happens if I use the same logic? 1.1 next count after 1?

S1: 1.1

R: 1.2 ..

S1: Last one will be 1.9

R: Okay, here I write 1.9.

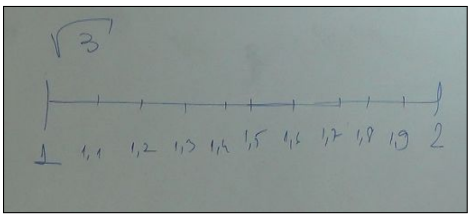


Figure 6.53. Interval of $\sqrt{3}$ up to 1/10 th digit

How do you think I can find now $\sqrt{3}$ where is it? So, we know that it will fall into one of the ranges, ultimately it's not? Is a number between 1 and 2. How do you think I can find it?

S1: So do we get square?

A Student: Take the square..

S1 first thought that they try to use triangles for the position of the number on the number line. Then, researcher cleared the situation as saying that they were not expected to use triangles but to converge to the number. As a clue she came back to the previous lesson and asked what they did while they were dealing with diagrams and S1 said that they divided each portion of the diagram to 10. Then the researcher triggered the students to try the same idea for the case of irrational numbers. After dividing the interval (1, 2) into 10 pieces, the researcher asked how to determine to which interval $\sqrt{3}$ falls and S1 and a student recommended to take the squares of the values.

After that students work on the example until the 1/100 th decimal and shared what they found. Initially some of them tried to guess the interval that $\sqrt{3}$ might be in. However, they also took the squares of the bounds and they found the 1/10 th digit. S12 found that $\sqrt{3}$ is between 1,7 and 1,8. Later, the researcher asked about how to continue to find the decimals of $\sqrt{3}$ and S1 emphasized that they again do partitioning by 10 on the interval (1.7, 1.8) to find the 1/100th digit. They calculated the squares of the bounds and determined the interval that $\sqrt{3}$ falls in as shown in Figure 6.54.

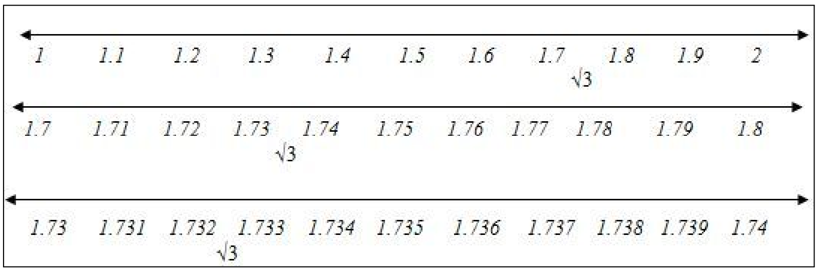


Figure 6.54. Interval of $\sqrt{3}$

S12: Square of 1.1 is 1.21
S6: But it sounds like it is between 1.4 and 1.5.
R: Let us calculate. So, what do you need to do to see if it was between 1.4 and 1.5?
A Student: I'll have to get their square.
R: OK.
A student: 1.8's square?

S12: Between 1.7 and 1.8.

R: Between 1.7 and 1.8. So, can I put it this way? 1+with 7/10 1+7/10+1/10 between, can I call?

Class: Yes, we did.

S1: We said that.

R: Now, how do I continue?

S1: We will still divide a 10 interval.

S4: It's the same process.

R: Where?

S12: That range.

S1: Between 1.7 and 1.8 we will divide 10. Thus one's place in we'll be on the ones in hundreds.

R: Which range do you think?

S1: We will take the square.

R: OK.

A Student: Between 1.73 and 1.74

R: Between 1.73 and 1.74. 1+7/10+3/10^2 can I say?

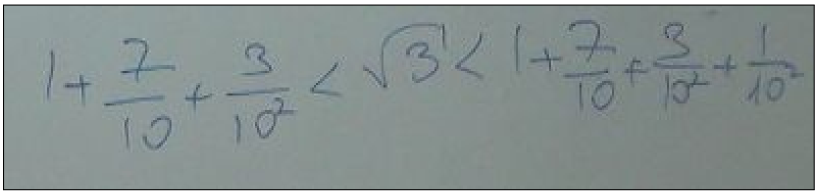


Figure 6.55. Interval of $\sqrt{3}$ up to 1/100 th digit

Well, do you see a similarity between this process and that we'd done before?

Class: Yes.

S18: The Same.

R: I've made a generalization over examples of rational numbers only rational numbers we go to the rational numbers. Is this for irrational numbers when I looked up, do you see a similarity?

After writing the intervals as $1 + 7/10 < \sqrt{3} < 1 + 7/10 + 1/10$ and $1 + 7/10 + 3/10^2 < \sqrt{3} < 1 + 7/10 + 3/10^2 + 1/10^2$, students again thought about sequence properties they previously found focusing on lower bounds and upper bounds of $\sqrt{3}$. They collectively concluded that lower bounds form increasing bounded rational number sequence and upper bounds form decreasing bounded rational number sequences by justifying their decisions. The excerpt below illustrates this discussion:

R: if I look at upper bound, what did I get again?

S19: Sequence.

S5: Increasing, rational, a series of numbers.

R: A sequence of increasing rational numbers ...
S5: Bounded.
R: Do you agree? What do you think?
S14: Yes.
S19:Yes.
R: What did we call for the lower bound? What did S5 say? Bounded.
S5: I said increasing, I said a sequence of rational numbers.
R: Why bounded?
S5: Again, because we are able to squeeze in between the two numbers..
R: Which two numbers?
S15: Between 1 and 2.
S5: Between 1 and 2.
R: OK. A sequence of rational numbers you said ... Why?
S5: It is the sum of the terms rational numbers, because rational numbers.
R: You said increasing?
S5: Yes. Increases. So, each term is more from the previous one.
R: OK. What can you say for upper bound?
S15: it decreases.
R: Decreasing?What kind of a sequence?
S5: A sequence of rational numbers.
R: A sequence of rational numbers. What else? Bounded or not?
S15: Bounded.
R: Between which values?
S15: Between 1 and 2.

As done in the previous cases, that is the case of 11/3 and the generalization of the findings to positive rational numbers, students followed the same pathways. They considered the differences of the terms of sequences and and found the difference for any n natural number as $1/10^n$.

R: Between 1 and 2. But again, let's make generalizations? Now what happens if you take the differences? (We're talking about the difference between the terms of the sequence.)
S5: $1/10^n$.
S15: $1/10^n$.
R: As n grows, where does the difference go?
S5: It goes to zero.
R: Goes to zero. those two boundaries, what does it represent?
S14: $\sqrt{3}$.
R: $\sqrt{3}$ OK. For the irrational case we did.
S5: Yes.
R: Well, we had a conclusion for the lower bound.
S5: Increasing, Bounded, a sequence of rational numbers.
R: What would it represent?
S5: $\sqrt{3}$.

R: It would represent $\sqrt{3}$. We said that. The case for a rational number we did. Also for the case of irrational numbers we did. In its most general form if we say. . . .

S5: By using two arrays for positive numbers, we can write all positive real numbers. There will be two sequences of rational numbers.

R: But how?

S19, S14: Increasing.

R: The lower bound.

S15: It will Increase. My upper bound will get decreased.

S5 generalized the case to positive real numbers stating that using two rational number sequences we can express positive real numbers. Also, students stated that the lower bounds are increasing and upper bounds are decreasing.

Throughout the lesson, students dealt with different concepts and combine them to form the decimal representation of real numbers. They started with fraction concept and analyzing three representations (diagram, division algorithm, decimal representation) of a fraction they engaged in quantitative reasoning operations and related three representations to each other. Lastly they considered sequences, to define decimal representation of real numbers from different perspective. From specific cases to general ones they constructed the definition via quantitative reasoning, questioning and discussion. Although in some cases individual student's work was shared, they also represented an overview of common learning occurred in the classroom environment.

6.2.2. Analysis of Selected Students' Performances on Proof Comprehension Test along with Interviews

In order to analyze selected students' performances about the proof comprehension test, firstly their answers thinking through proof comprehension dimensions were classified as no answer/incorrect answer, partially correct answer and complete answer. If the student answered all the questions in the dimensions correctly, then his/her performance was classified as correct answer. If the student could not answer or incorrectly answer all the questions in the dimensions, then his/her performance was classified as no/incorrect answer. Since each dimension included more than one question except Dimension 7 (Identifying Modular Structure), if a student gave correct answer to one of the questions in the dimension and no answer or incorrect answer to the other question, then the student's performance in the dimension was classified as partially correct. The classification were maintained both for the pre-test and the post-test performances. After determining student's status in each dimension for the pre test and the post test, the transitions between the answers from the pre-test and the post-test were illustrated. In some cases, although students could not answer questions in some dimensions in both the pre and the post test, they tried to correct themselves during the interview. Additionally, if a student's transition between the pre and the post tests were considered as 'no change' and if the student showed an improvement in the interview, his/her performance was also shared. Therefore, the interview excerpts used as a support for change in answers from the pre-test to the post-test. The interviewees were determined based on their performances on the test. The profile of the interviewees is given in Figure 6.56:

Interviewees	Interviewees Profile													
	Pre-test							Post-test						
	D1	D2	D3	D4	D5	D6	D7	D1	D2	D3	D4	D5	D6	D7
S1	Partial	Partial	Correct	Partial	No/incorrect	Partial	Correct	Correct	Partial	Correct	Partial	Partial	Partial	Correct
S5	Partial	Partial	No/incorrect	No/incorrect	Partial	Partial	Correct	Partial	No/incorrect	Partial	Correct	Partial	Partial	Correct
S7	Partial	Partial	No/incorrect	No/incorrect	No/incorrect	Partial	No/incorrect	Partial	Correct	No/incorrect	Correct	Partial	Partial	Correct
S8	Partial	Partial	No/incorrect	Partial	No/incorrect	No/incorrect	No/incorrect	Partial	Correct	Partial	Partial	Partial	Partial	No/incorrect
S9	Partial	Partial	Partial	Partial	No/incorrect	No/incorrect	No/incorrect	Correct	Correct	Correct	Partial	Partial	Correct	Correct
S15	No/incorrec	Partial	No/incorrect	No/incorrect	Partial	No/incorrect	No/incorrect	Partial	Partial	No/incorrect	Correct	Partial	Partial	No/incorrect

Meanings of the colors	
(yellow)	There is no change
(green)	There is a positive change
(pink)	There is a negative change

Figure 6.56. Interviewee's Profile

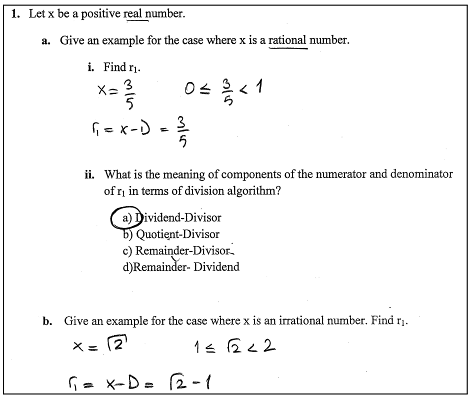
<u>6.2.2.1. Dimension 1: Meaning of terms and statements.</u> The questions Q1 and Q2 addressed the dimension 1 'meanings of terms and statements. Q1 was about some notations used at the beginning of the proof and students were expected to give example referring to the use of notations. Q2 was about definitions of sequence, bounded sequence and rational number sequence and examples of them. For this dimension, in the pre-test, 5 of the interviewees' performance were labeled as partially correct answer, 1 of them was classified as no/incorrect answer. In the post-test, 2 students' performances improved as correct answer. In addition, 3 performance remained the

same from the pre-test to the post-test as partially correct. The transitions between performances from pre to post test were shared as follows:

- Case I: From partially correct answer to correct answer'

The students S1 and S9 represented this case. They both gave partial answers to both questions (Q1 and Q2) in the pre-test and gave correct answers to both questions in the post-test. The following data from S1 is chosen to illustrate this change.

For Q1, in the pre-test he gave the following answer (Figure 6.57):


1. Let x be a positive real number.

a. Give an example for the case where x is a rational number.

i. Find r_1.

$x = \frac{3}{5}$ $\quad 0 \leq \frac{3}{5} < 1$

$r_1 = x - D = \frac{3}{5}$

ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm?

a) Dividend-Divisor
b) Quotient-Divisor
c) Remainder-Divisor
d) Remainder- Dividend

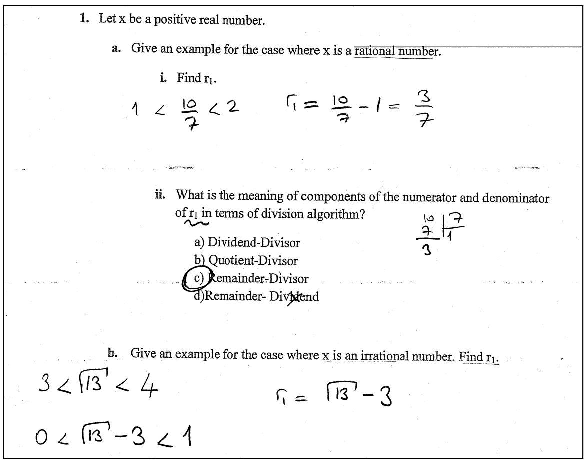
b. Give an example for the case where x is an irrational number. Find r_1.

$x = \sqrt{2}$ $\quad 1 \leq \sqrt{2} < 2$

$r_1 = x - D = \sqrt{2} - 1$

Figure 6.57. S1's answer to question 1 in the pre-test

For the pre-test, although he could answer parts a.i. and b. correctly, he could not determine the components of r_1 once x is a rational number. In the interview he stated that for the pre-test he could not define the meaning of r_1 and d_1 and uttered "what did d_1 mean in the pre-test? d_1 meant the number coming from r_1. What is r_1 There was an r_1 (in the proof)". As the data indicated he knew that r_1 and d_1 were numbers and used them since they were in the proof. However, he did not know what r_1 and d_1 represented. That was most likely the reason that, he could not answer part ii correctly. That is, data indicated that S1 had thought of x while answering ii because in the analysis of x, 3 refers to the dividend and 5 refers to the divisor. However, the question asked for the meaning of r_1 which was the same as numerically with his example x=3/5. It was evident in his statement "I am dividing 3 to 5 so 3 is dividend and 5 is divisor". He even emphasized in the interview that his thought changed in the post-test and answered Q1 correctly as follows (Figure 6.58):



1. Let x be a positive real number.

a. Give an example for the case where x is a rational number.

i. Find r_1.

$1 < \frac{10}{7} < 2$ $\quad r_1 = \frac{10}{7} - 1 = \frac{3}{7}$

ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm?

a) Dividend-Divisor
b) Quotient-Divisor
c) Remainder-Divisor
d) Remainder- Dividend

b. Give an example for the case where x is an irrational number. Find r_1.

$3 < \sqrt{13} < 4$ $\quad r_1 = \sqrt{13} - 3$

$0 < \sqrt{13} - 3 < 1$

Figure 6.58. S1's answer to question 1 in the post-test

In the interview, he reasoned about the meaning of r_1.

S1: The r_1 that I said was the remaining part. (The part that is obtained after determining the wholes) How can I think? Now, this is 3/7. 3 cannot be divisible by 7. We look for 1/10 th digit. Now I look to 30/7. I am searching 1/10 th s in 30/70.

R: How did you move to 30/70 from 3/7? Which concept did you use?

S1: Multiplying each part with 10 (numerator, denominator) What is Multiplying with 10? It means equal-partitioning in the diagram. Dividing each piece to 10 equal pieces. We partitioned each pieces into 10 and obtained 70 pieces. 30 of them were shaded. Now if I take 7/70's that is 1/10's in 30 pieces, there are 4 many 1/10's. Then, I found d_1 as 4.

Together with the post-test, the data from interview showed how S1 reasoned: As he stated for the case of x being a rational number, r_1, i.e. 3/7, represented the remaining portion for him. when he thought of diagrams, r_1 referred to the remaining piece that was shaded in Figure 6.59 after determining the whole, "1" in the long division as in figure from the post-test interview (Figure 6.60). Therefore, contrary to the pre-test, he was able to label correctly in (ii) the components of 3/7 as remainder and divisor respectively. Drawing the diagram in Figure 6.59 he first showed the

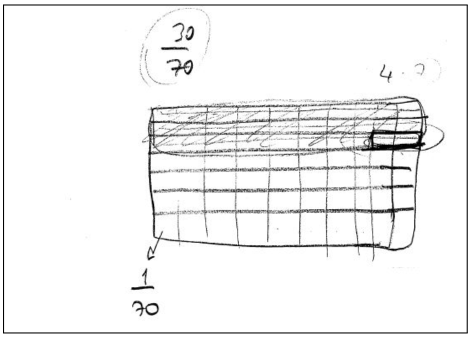


Figure 6.59. S1's representation 4

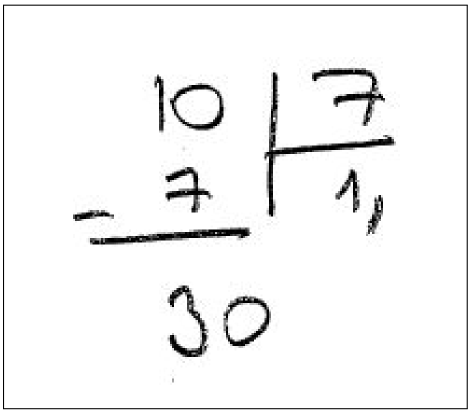


Figure 6.60. S1's representation 5

quantity 3/7. He then partitioned each portion into 10 because he knew that he could not continue diving 3 by 7. This was why he put a zero and comma in the long division in Figure 6.60. Partitioning the diagram into 10 more pieces, he was able to get 30/70 an equivalent fraction to 3/7. Data also showed that S1 knew that while finding the equivalent fraction to 3/7 multiplying both numerator and denominator with 10 meant equal partitioning for him. This also indicated he was able to think through different representations, long division and a diagram, simultaneously. Once partitioned, as the diagram showed, for him each piece was equal to 1/70. Since he was working with 1/10 th's to continue with the division, he engaged in finding out how many 1/10 th's there were in 30/70. Though knowing that each piece was equal to 1/70, he grouped the quantity of 1/70 th's by 7 to obtain 7/70 which was equal

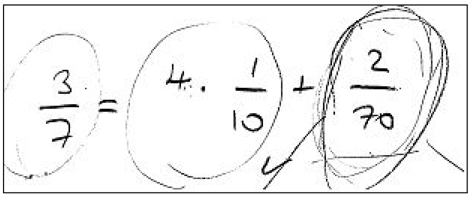


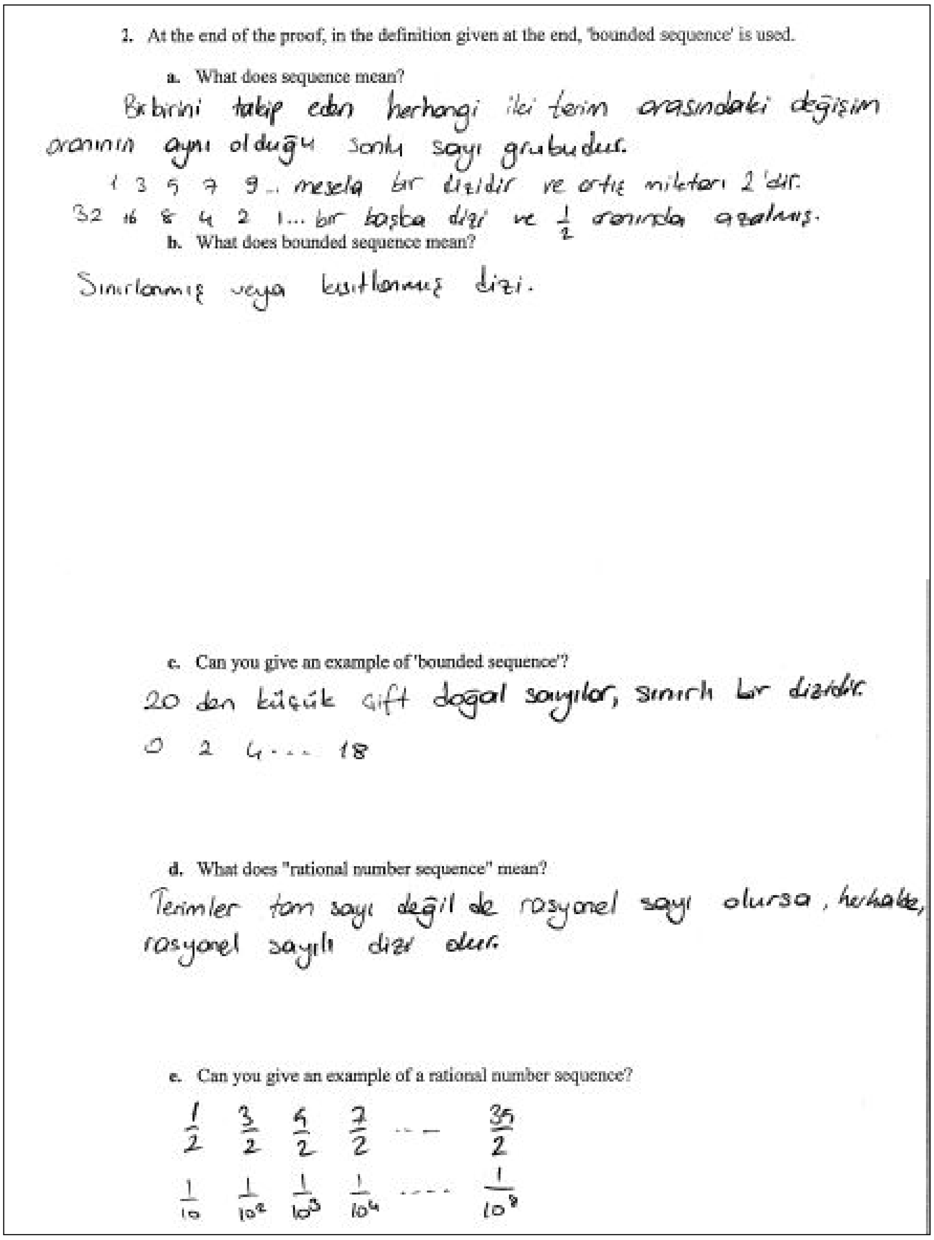
Figure 6.61. S1's representation 6

to 1/10's. Then, he found 4 groups 1/10 th's in the portion 30/70. As Figure 6.61 indicated in the decimal representation of 3/7 he came up with, he even stated that the remaining portion was 2/70. His referring to 2/70 in the diagram also indicated that he was able to flexibly relate decimal representation and diagram. The data therefore showed that, not only he was able to define r_1 for the case of x, but also he was able to explain how to use it and how it works in the division algorithm through diagrams. This indicated that while explaining the meaning of r_1 in the proof comprehension test, he did not consider just numerical operations but by focusing on the quantities he internalized the reasoning behind the numerical operations.

In question 2, for the pre-test S1 gave the following answer (Figure 6.62):

> "2.a.The finite number group which follow each other and the difference ratio between two terms is the same. For instance, 1 3 5 7 9... is a sequence and the increasing amount is 2. 32 16 8 4 2 1.... is another sequence and it decreased in the rate of 1/2.
>
> 2.b. Bounded or restricted sequence.
>
> 2.c. The even natural numbers less than 20 is a bounded sequence. 0 2 4 ....18.
>
> 2.d.If the terms are rational numbers instead of being integer than it is probably a rational number sequence." (Figure 6.62)

Data showed that his definition of sequence was limited because he defined sequence as finite collection of numbers. However, sequences need not to be finite. The other problem was that he could not define bounded sequence, he just translated the bounded sequence to Turkish. Although his use of the translation does not necessarily mean that he does not know the correct meaning of bounded sequences, his definition



2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used.

a. What does sequence mean?

Birbirini takip eden herhangi iki terim arasındaki değişim oranının aynı olduğu sonlu sayı grubudur.

1 3 5 7 9 .. mesela bir dizidir ve artış miktarı 2'dir.

32 16 8 4 2 1... bir başka dizi ve $\frac{1}{2}$ oranında azalmış.

b. What does bounded sequence mean?

Sınırlanmış veya kısıtlanmış dizi.

c. Can you give an example of 'bounded sequence'?

20 den küçük çift doğal sayılar, sınırlı bir dizidir.

0 2 4 . . . 18

d. What does "rational number sequence" mean?

Terimler tam sayı değil de rasyonel sayı olursa, herhalde, rasyonel sayılı dizi olur.

e. Can you give an example of a rational number sequence?

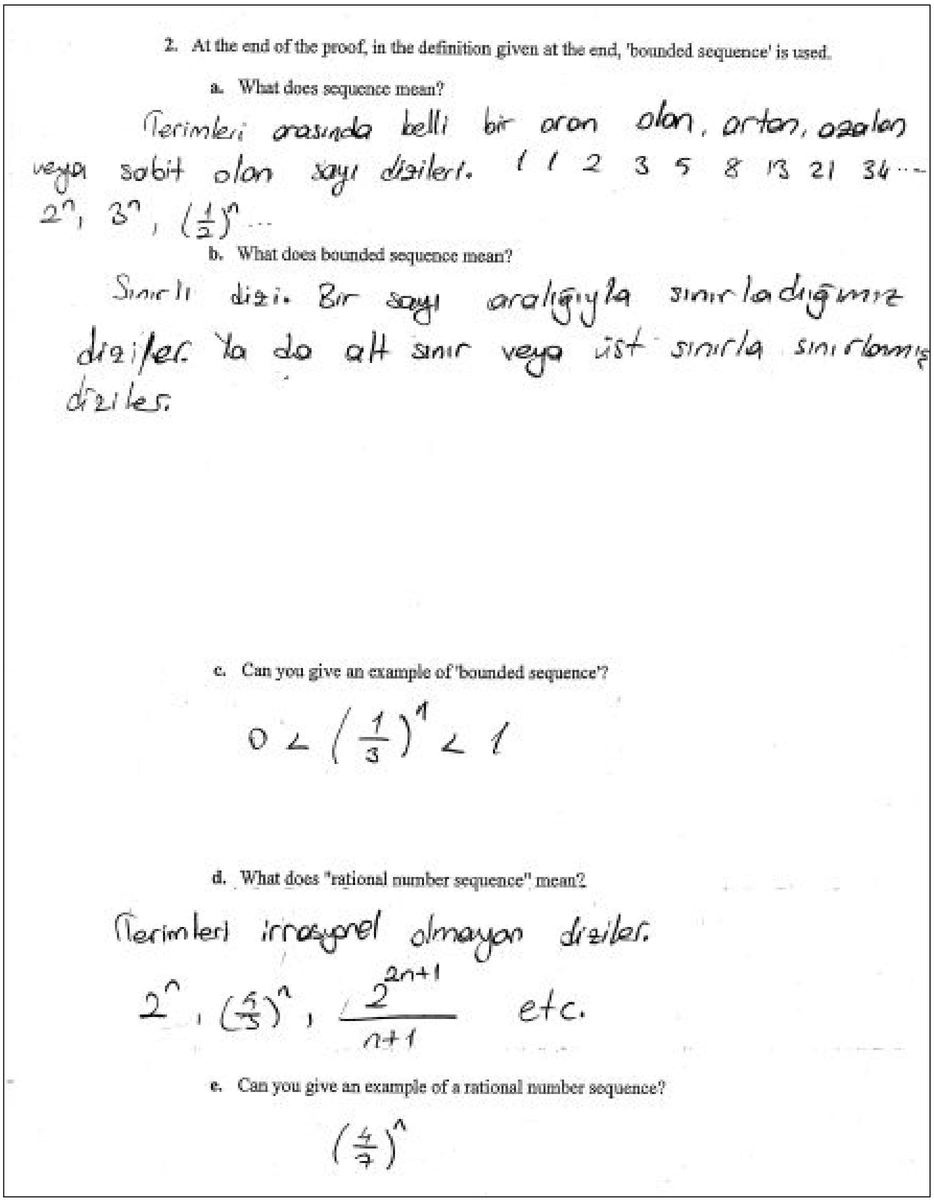
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ ... $\frac{35}{2}$

$\frac{1}{10}$ $\frac{1}{10^2}$ $\frac{1}{10^3}$ $\frac{1}{10^4}$ .... $\frac{1}{10^8}$

Figure 6.62. S1's answer to question 2 in the pre-test

of sequence together with his example for the bounded sequence as finite indicated that his knowledge of bounded sequences might have been limited.

Contrary to the pre-test his explanations in the post test were all correct (Figure 6.63):


2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used.

a. What does sequence mean?

Terimleri arasında belli bir oran olan, artan, azalan veya sabit olan sayı dizileri. 1 1 2 3 5 8 13 21 34 ... 2^n, 3^n, $(\frac{1}{2})^n$ ...

b. What does bounded sequence mean?

Sınırlı dizi. Bir sayı aralığıyla sınırladığımız diziler. Ya da alt sınır veya üst sınırla sınırlanmış diziler.

c. Can you give an example of 'bounded sequence'?

$0 < (\frac{1}{3})^n < 1$

d. What does "rational number sequence" mean?

Terimleri irrasyonel olmayan diziler.

2^n, $(\frac{4}{5})^n$, $\frac{2^{2n+1}}{n+1}$ etc.

e. Can you give an example of a rational number sequence?

$(\frac{4}{7})^n$

Figure 6.63. S1's answer to question 2 in the post-test

"2.a Being a rate between the terms as increasing, decreasing or constant sequences. 1 1 2 3 5 8 13 21 34....,2^n, 3^n, $(1/2)^n$.

2.b. Bounded sequence. The sequences which we squeeze them in a number interval. The sequences squeezed with lower bound or with upper bound.

2.d. The sequences whose terms are not irrational". (Figure 6.63)

For the post-test, data showed that he was able to express all the properties that a sequence might hold, such as increasing, decreasing, or constant sequences.

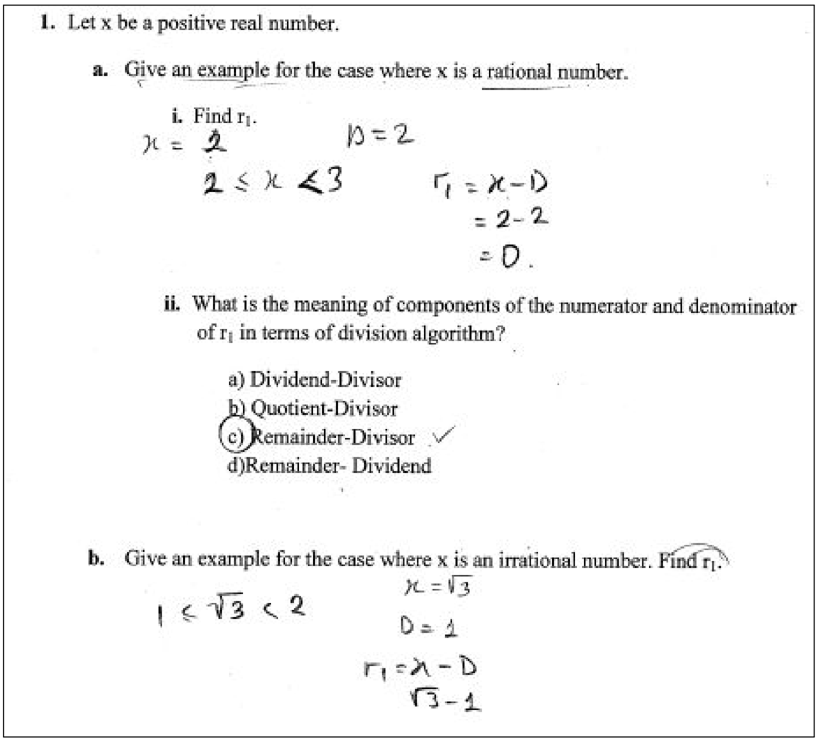
Moreover, in the interview he stated that the symbol “n” represented natural numbers and was able to correspond the first and the second terms of the sequence $(1/3)^n$ with n=1,n=2 etc. Data seemed to suggest that he was aware of the structure of sequences such that for each natural number n, he could show the correspondent term of the sequence formed as a_n. Again contrary to the pre-test, he was able to define bounded sequences correctly. In the interview, he further explained boundedness in terms of his example $(1/3)^n$. He stated that the terms of the sequence would be always between 0 and 1.

To conclude in the pre-test, S1 gave partial answers to both questions but in the post-test his answers turned to correct answers. Also, during the interview especially for question 1 he further explained his reasoning focusing on quantities. He switched between different representations and was able to explain the meaning of notations in the proof through diagrams focusing on quantities.

- Case II: No change (Partial answer)

S7, S8 and S9 gave partially correct answers to the questions in the dimensions in both the pre and the post test. S7 was chosen as a representative for this case. The reason for classifying her in the category was her answers in question 2. She gave correct answers to Q1 in both tests (Figure 6.64 and Figure 6.65) but her answer remained partial for Q2 in both tests.

Data showed that S7 gave correct answers to Q1 both in the pre-test and post-test. For the post test she found r=4/7 as the remainder and could determine the components of it as remainder and denominator. Shown in the post-test and her explanation in the interview ”Even here I did what was said (referring to the proof) I squeezed this (x=11/7) between two integers (D=1 and D+1=2). It was squeezed in 1 and 2. After that I changed the interval to the interval of 0 and 1. Then, I subtracted 1 from 11/7 and 4/7 remained for r_1” suggested that she was aware of the meanings of the notations used in the proof. She knew that D referred to the whole



1. Let x be a positive real number.

a. Give an example for the case where x is a rational number.

i. Find r_1.

$x = 2$ $D = 2$

$2 \leq x < 3$ $r_1 = x - D$
$= 2 - 2$
$= 0.$

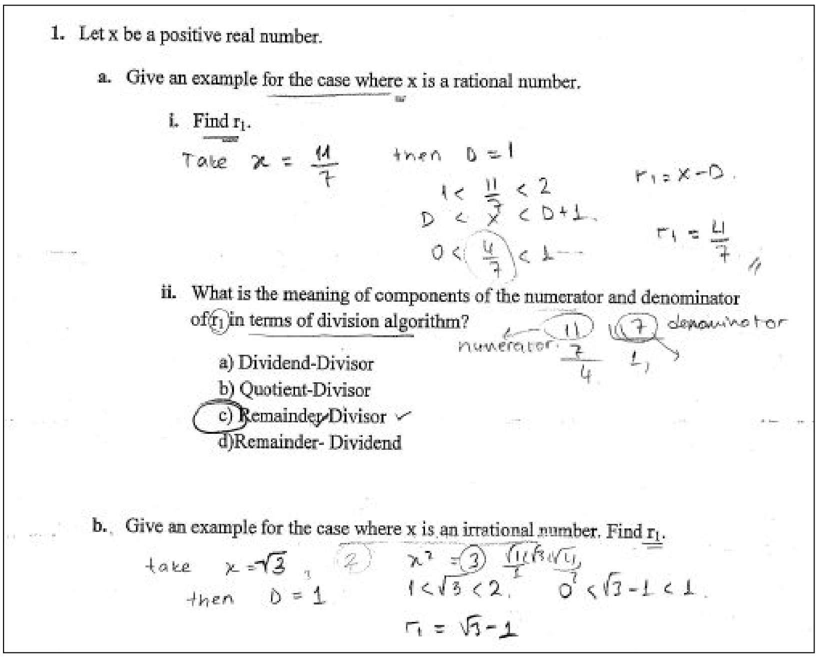
ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm?

a) Dividend-Divisor
b) Quotient-Divisor
c) Remainder-Divisor ✓
d)Remainder- Dividend

b. Give an example for the case where x is an irrational number. Find r_1.

$1 \leq \sqrt{3} < 2$ $x = \sqrt{3}$
$D = 1$
$r_1 = x - D$
$\sqrt{3} - 1$

Figure 6.64. S7's answer to question 1 in the pre-test



1. Let x be a positive real number.

a. Give an example for the case where x is a rational number.

i. Find r_1.

Take $x = \frac{11}{7}$ then $D = 1$
$1 < \frac{11}{7} < 2$ $r_1 = x - D.$
$D < x < D+1.$
$0 < \frac{4}{7} < 1$ $r_1 = \frac{4}{7}$

ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm? numerator 11 7 denominator
$\frac{7}{4}$ 1,

a) Dividend-Divisor
b) Quotient-Divisor
c) Remainder-Divisor ✓
d)Remainder- Dividend

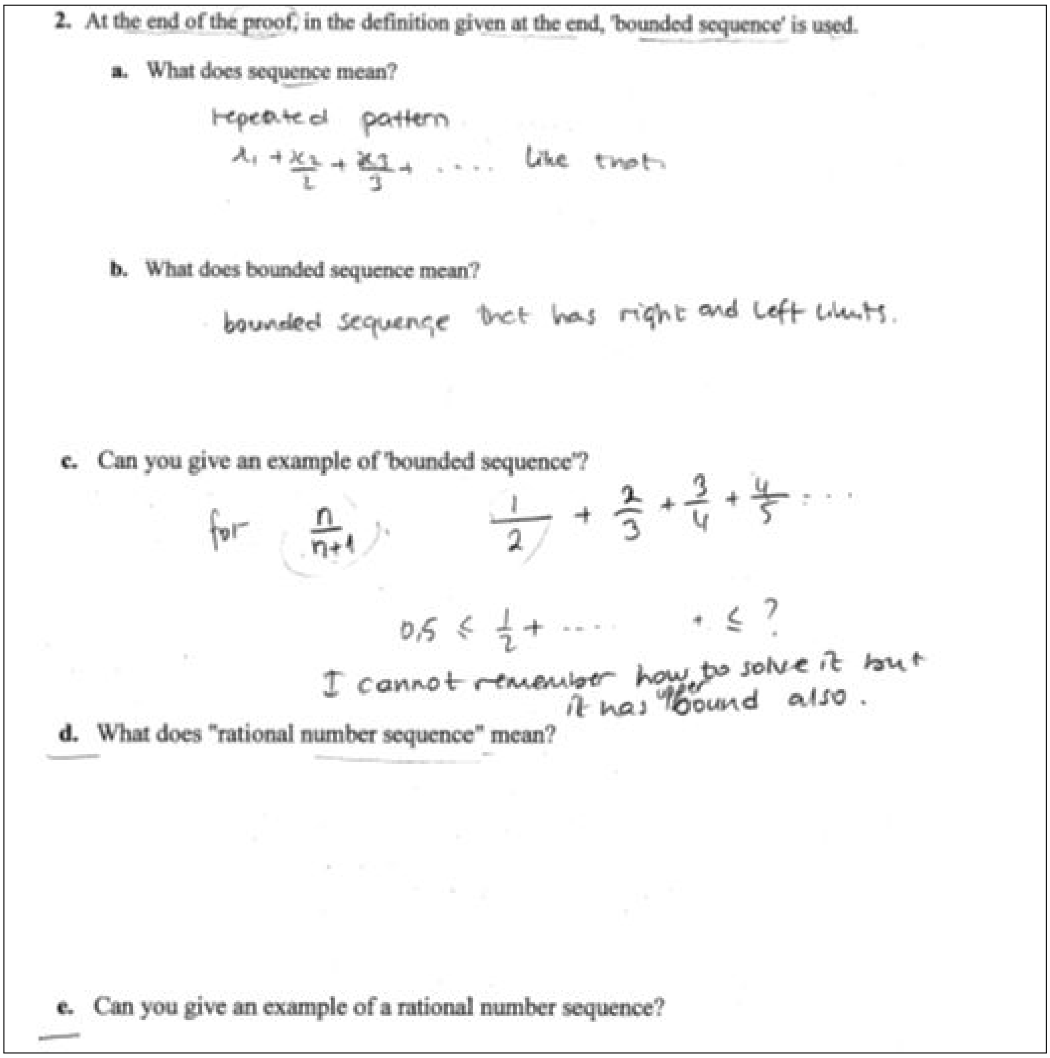
b. Give an example for the case where x is an irrational number. Find r_1.

take $x = \sqrt{3}$ 2 $x^2 = 3$ $\sqrt{1} < \sqrt{3} < \sqrt{4}$
then $D = 1$ $1 < \sqrt{3} < 2.$ $0 < \sqrt{3} - 1 < 1.$
$r_1 = \sqrt{3} - 1$

Figure 6.65. S7's answer to question 1 in the post-test

part of the number x and r referred to the remainder portion.Therefore, she was able to relate the example and the initial steps of the proof.

When S7's performance in question 2 was considered, she could not answer question 2 in the pre-test correctly (Figure 6.66).



2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used.

a. What does sequence mean?

repeated pattern

$x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots$ like that

b. What does bounded sequence mean?

bounded sequence that has right and left limits.

c. Can you give an example of 'bounded sequence'?

for $\frac{n}{n+1}$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \ldots$

$0{,}5 \leq \frac{1}{2} + \ldots + \leq ?$

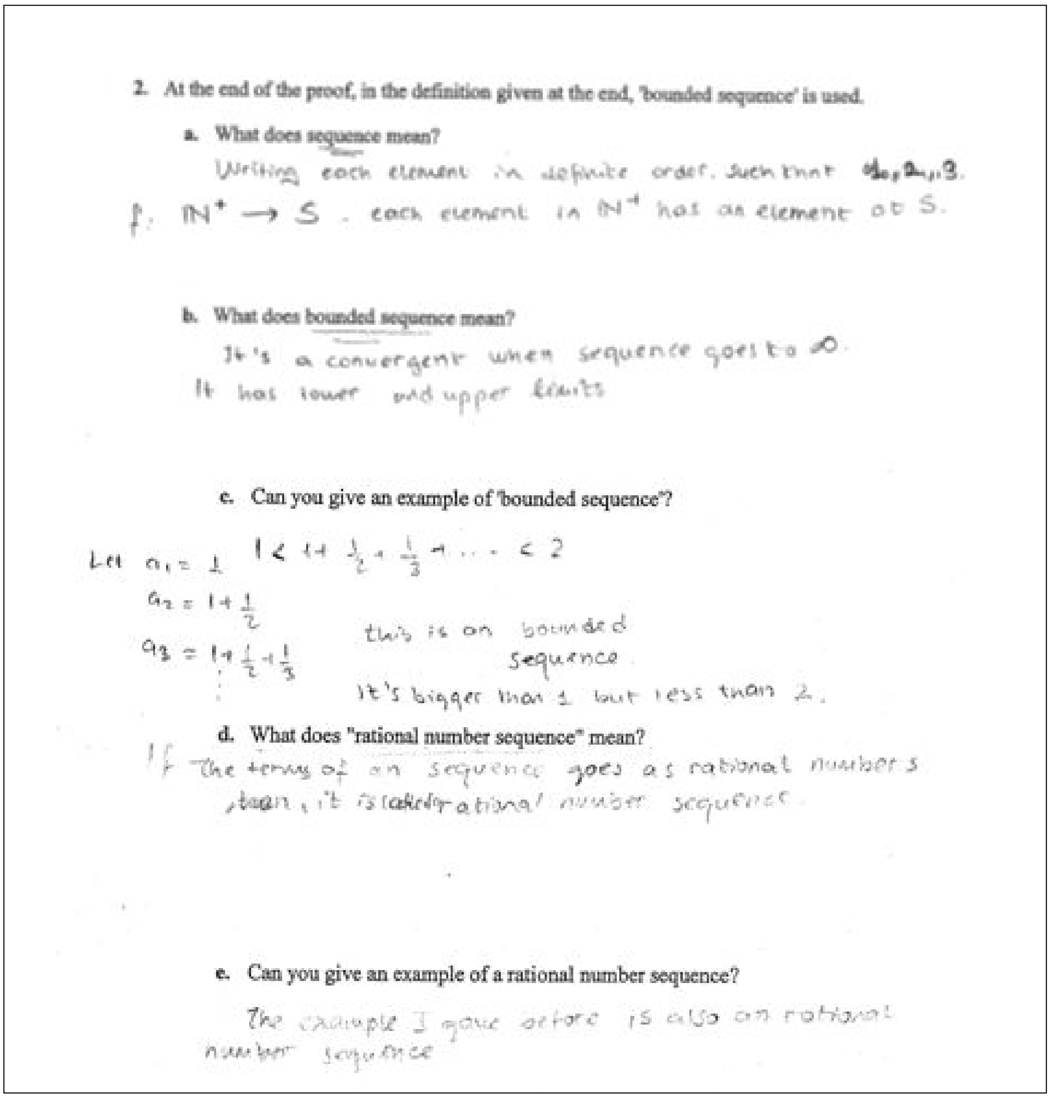
I cannot remember how to solve it but it has upper bound also.

d. What does "rational number sequence" mean?

e. Can you give an example of a rational number sequence?

Figure 6.66. S7's answer to question 2 in the pre-test

As the data showed, she even did not answer parts d and e. Her defining the sequence as a summative expression seemed to indicate that she might have confused the concept of sequences with series. Also, her answer for bounded sequence supported her confusion because although she wrote a general expression such as n/(n+1), she added the terms to each other. This showed that she thought of sequence as a summative expression. For the part 'b' she defined bounded sequences as the sequences which had the left and the right limits. In Calculus, while determining the limit of a function at a point, by the terminology the right limit and the left limit, mathematicians usually consider to approach a point from left and right. Therefore, her expression for bounded sequences became limited due to wording. In the post-test she answered in the following way (Figure 6.67):



2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used.

a. What does sequence mean?

Writing each element in definite order. Such that 1, 2, 3.
$f: \mathbb{N}^+ \to S$. each element in $\mathbb{N}^+$ has an element of S.

b. What does bounded sequence mean?

It's a convergent when sequence goes to ∞.
It has lower and upper limits

c. Can you give an example of 'bounded sequence'?

Let $a_1 = 1$ $\quad 1 < 1 + \frac{1}{2} + \frac{1}{3} + \dots < 2$
$a_2 = 1 + \frac{1}{2}$
$a_3 = 1 + \frac{1}{2} + \frac{1}{3}$
$\vdots$
this is an bounded sequence
it's bigger than 1 but less than 2.

d. What does "rational number sequence" mean?

If the terms of an sequence goes as rational numbers, then, it is called rational number sequence.

e. Can you give an example of a rational number sequence?

The example I gave before is also an rational number sequence

Figure 6.67. S7's answer to question 2 in the post-test

On the contrary, in the post test, she was able to answer all the parts about sequences correctly. There was a big difference in her definitions of sequence in the pre-test and the post-test. As the data from the post-test showed she defined the sequence as a function from positive natural numbers to a set. During the interview, she also stated the following:

> S7: If I can match positive integers with the terms,
> R: Which terms?
> S7: Terms of the sequence, I mean, a_1, a_2, a_3.
> R: Uh-Huh.
> S7: If I can write a function that matches this I said I'd call it a sequence.

This showed that for S7 a sequence was a special kind of function such that there is a correspondence (matching in her own words) between positive integers and the terms a_1 a_2 a_3....a_n. Also, for bounded sequence, she gave an acceptable answer. She thought convergence while defining boundedness which was meaningful. However, the example she gave for bounded sequence was not a bounded sequence. In the example, as the number of terms increases the terms of the sequence goes to infinity. Thus the sequence she gave was only bounded from below. Though since she used the wording upper and lower limits and also wrote (1+1/2+...). During the interview the researcher asked her to elaborate:

> R: What do you mean by upper and lower limit?
> S7: The tail of the sequence will approach to a value.
> R: Which value does the sequence approach to?
> S7: Here (pointing to the post test) for instance it approaches to 2 I said. Because I am adding values to 1 but the sum of these values does not make 1. (pointing to the values added after 1) Therefore, the sum may not pass 2.
> R: Then, can you say that the sequence is between 1 and 2? Or you are talking about just this expression.
> S7: I am talking about just this expression but it is a_n, the n th term of the sequence.

Data showed that the upper and lower limit would mean that the sequence was converging to a number. This suggested that she was reasoning in accordance with bounded sequence. Similarly, she used 1+1/2+1/3+... as the general term of the sequence. Although it was a summative expression, she was able to express the consecutive terms of the sequence.

To conclude, for the case 'No change', S7's work for Q1 and Q2 was shared. The reason to classify her work in this case was that she did not give complete answers to Q2 in the pre-test and the post-test but she showed an improvement in the Q2 for the post-test.

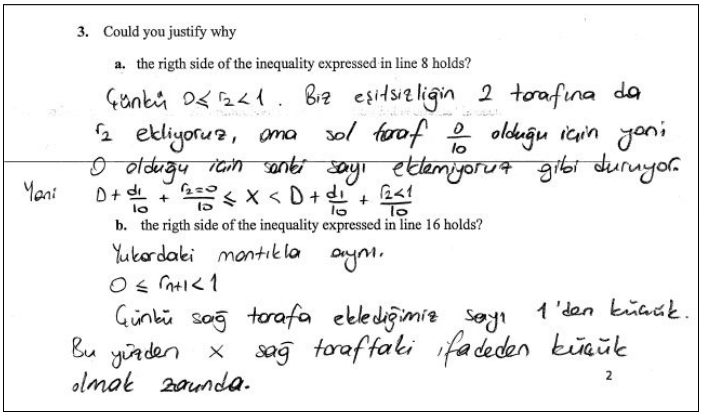
6.2.2.2. Dimension 2: Justification of claims. Q3 and Q4 addressed the dimension 'justification of claims'. In both questions students were expected to justify or explain how some inequalities were obtained. For this dimension, as the pre-test is considered, 6 of the interviewees' performance were labeled as partially correct answer. In the post-test, 3 students' performances were considered as correct answer, 2 of them were partially correct and 1 of them was no/incorrect answer. The transitions between performances from pre to post test were shared as follows:

- Case I: No change (Partial)

S1 and S15 was the representatives for this case. They both gave partially correct answers to the questions in this dimension. S1 was chosen for illustrating the case with the support of his performance in the tests and the interview.

For the Question 3 ($x<D+d_1/10+1/10$ and $x<D+d_1/10+......+d_k/10^k+1/10^k$) he provided similar answers in both the pre-test and the post-test (Figure 6.68 and Figure 6.69).

> "a.Because $0 \leq r_1 <1$. We add r_2 to both sides of the inequality but the left hand side becomes 0/10 that is 0, it seems that we did not add anything. That is, $d+d_1/10+ r_2=0/10 \leq x <D+d_1/10+r_2 <1/10$.
> b. It is the same with the above logic. $0 \leq r_n <1$. The number we added is less than 1. Therefore, x should be less than the right hand side." (S1's answer to Q3 in the pre-test)



3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

Çünkü $0 \le r_2 < 1$. Biz eşitsizliğin 2 tarafına da r_2 ekliyoruz, ama sol taraf $\frac{0}{10}$ olduğu için yani 0 olduğu için sanki sayı eklemiyoruz gibi duruyor.

Yani $D + \frac{d_1}{10} + \frac{r_2=0}{10} \le X < D + \frac{d_1}{10} + \frac{r_2<1}{10}$

b. the rigth side of the inequality expressed in line 16 holds?

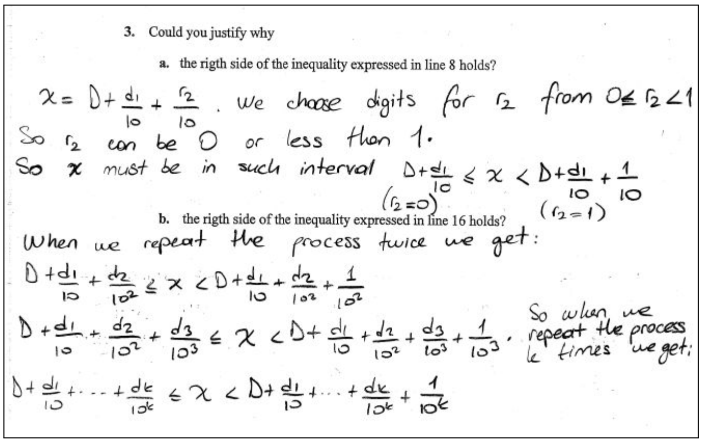
Yukardaki mantıkla aynı.

$0 \le r_{n+1} < 1$

Çünkü sağ tarafa eklediğimiz sayı 1'den küçük. Bu yüzden x sağ taraftaki ifadeden küçük olmak zorunda.

2

Figure 6.68. S1's answer to question 3 in the pre-test



3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

$x = D + \frac{d_1}{10} + \frac{r_2}{10}$. We choose digits for r_2 from $0 \le r_2 < 1$

So r_2 can be 0 or less than 1.

So x must be in such interval $\underset{(r_2=0)}{D+\frac{d_1}{10}} \le x < \underset{(r_2=1)}{D+\frac{d_1}{10}+\frac{1}{10}}$

b. the rigth side of the inequality expressed in line 16 holds?

When we repeat the process twice we get:

$D + \frac{d_1}{10} + \frac{d_2}{10^2} \le x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{1}{10^2}$

$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \le x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{1}{10^3}$, So when we repeat the process k times we get:

$D + \frac{d_1}{10} + \dots + \frac{d_k}{10^k} \le x < D + \frac{d_1}{10} + \dots + \frac{d_k}{10^k} + \frac{1}{10^k}$

Figure 6.69. S1's answer to question 3 in the post-test

S1 reasoned that since both $r_2$and r_{n+1} were between 0 and 1, the inequality held. However, for the post test how he obtained the inequality asked in part b was not clear from his writing. He just emphasized repeating the process to express the real number x in a more narrowed interval.

During the interview, interestingly, providing a rational number example, he was able to justify why the general inequality held.

> S1: Now, for example I have 10/7 and I divided 10 by 7. 3 is the remainder. Now, 10/7 is between 1 and 2. Now, the x here is this (10/7) D and D+1 are these (1 and 2).
> R: You said D to this (Pointing 1)
> S1: D+1 to this one also. (Pointing 2). Now I am trying to pass to the step 8 in th proof.I continue to division. I can put a zero and and a comma. Now, what does this mean? Now, I searching 1/10 in this (3/7). to look for 1/10th s, I am dividing each piece of 3/7 in to 10 pieces. The smallest piece is 1/70. counting 7 pieces in 30/70. Then, 7 times 4 is 28. That means there are 4 many 7 pieces in 3/7. Diagram is this (showing the diagram). I counted 7 pieces in this. Why? Because I looking for 1/10th s. The remaining part is 2/70. For instance, the 2 pieces here (pointing to the diagram). Now I am saying that actually 3/7 means that 4 times 1/10 and plus 2/70.
> R: Where are you in the proof?
> S1: I came to there. I found d_1.
> R: What does 2/70 mean to you?
> S1: 2/70. Now, I had 1 and I found 4/10. The number I found is bigger than this (1+4/10) because there is something extra in this number (pointing 2/70). But this number 2/70 is smaller than 7/70. If this were 7/70 what would happen? 5/10 (he added 1/10 to 4/10).But it is impossible because we have 2/70 not 7/70. Therefore, this number is smaller than this (1+4/10+1/10).
> R: If we turn to the question about proof...
> S1: If we turn to the question about proof, the number I called D is 1, d_1 is 4. Then The number 2/70 is this (pointing $r_2/10$) this cannot go to 1/10. I found something less than 1/10. Therefore, 8 th step holds.

He used quantities to explain the inequalities. He considered 10/7 as an example and used diagrams to find r_2 and later he made the comparison between the expression given as inequality in Q3 part a.

First he started with long division relating it with the general terms in the proof i.e x=10/7 D=1 and D+1=2 but then he continued reasoning with the diagram

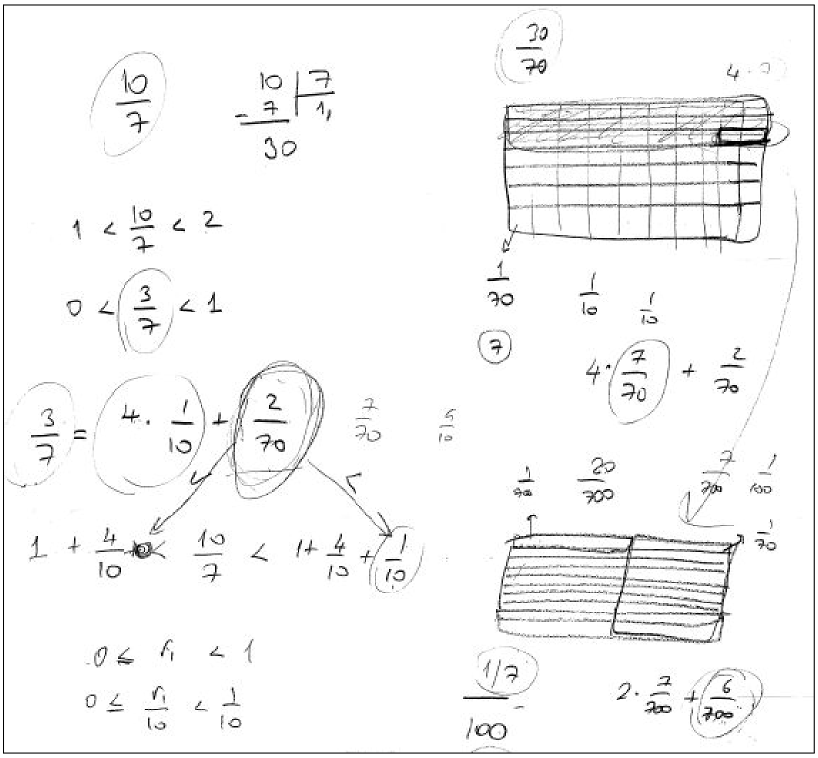


Figure 6.70. S1's representation 7

showing the remaining part 3/7. For him, in the long division the meaning of putting a zero right to the remainder and a comma in the quotient meant that in his drawing he was going to find out the number of 1/10th s in 3/7. He then partitioned the each piece of 3/7 by 10 and obtained 30/70 with the smallest piece 1/70. In order to find the number of 1/10th s in 3/7 he measured 30/70 with 7 pieces of 1/70 because he knew that 7 many 1/70 makes 1/10th s. At the end he obtained 4 many 1/10th s in 3/7 and there was a remaining part 2/70. Then, he returned to the Q3 part 'a' and explained again why the inequality at the right side held. He reasoned that he had no more 7/70 in the diagram because he was left with 2/70 after 4 groups of 7/70. Therefore, 2/70 was smaller than 7/70. This allowed him to reason that in the general expression d_1 was equal to 4 and $r_1/10$ was 2/70 so that he was able to deduce that $x=D+d_1/10+r_1/10<D+d_1/10+1/10$. This was evident in his writing comparing $r_1/10<1/10$. This again implied that thinking through quantities depicted in diagrams helped him to explain his reasoning. This also implied that he was able to shift flexibly between abstract expressions and concrete examples.

After he was asked to comment more on part b, he continued:

> R: What can you say for part b?
> S1: I explained here as follows: The 16 th step is the last step. That is the k th step. It is a generalized form. For instance, If I did this one 20 times, I will again obtain the inequality.
> R: How do we obtain the inequality? From which condition we obtain the inequality?
> S1: This is because..I had 2/70 here. Now, in 2/70.. The small piece here (showing 1 of them in the diagram) is 1/70. Now, I am dividing this to 10. the each piece becomes 1/700.Then, I count 7 of them to find 1/100 th s. There are 2 many 1/100. 6 pieces remained. quantity is 6/700. Now, I get 1+4/10+2/100+6/700. Now, my number is greater than this (1+4/10+2/100) because it has some extra (6/700). But smaller than this (1+4/10+2/100+1/100) because this (showing 2/700) is not 7/700.

For part b, he emphasized that part b was a generalized form of part a. He continued to explain using the diagram and suggesting that the same inequality would again be obtained. In particular, thinking the case k=2, he repartitioned 2/70 into 10 more equal pieces to find out how many 1/100 th s there were in 2/70 th. Knowing that the smallest equal piece was 1/700, he grouped them in piles of 7 to get 1/100. Counting the number of groups of 7/700, he found 2 many 1/100 th s. This way the remaining part was 6/700. Comparing 6/700 with 7/700 he concluded that it was smaller than 7/700.

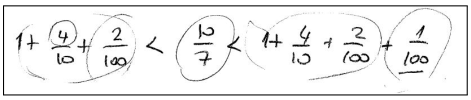


Figure 6.71. S1's representation 8

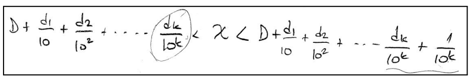


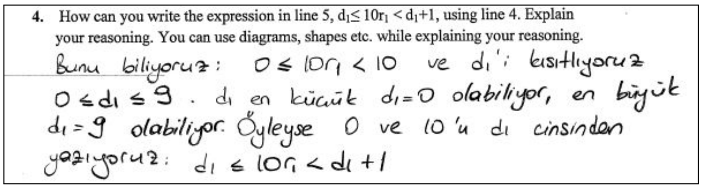
Figure 6.72. S1's representation 9

Then, he wrote the general expression up to k as given above and continued further.

> R: If you generalize.. What can you say if you generalize from this expression(showing proof)
> S1: I am writing until k. Now x is bigger than this ($D+d_1/10+...d_k/10^k$) because there is a remaining part ($r_k/10^k$).
> R: What is the difference between x and the right side?
> S1: x is smaller than the right side.
> R:Why?
> S1: The $1/10^k$ th digit of this (x) that is $d_k+r_k/10^k$ is less than this ($d_k/10^k+1/10^k$)

He compared the parts $d_k/10^k+r_k/10^k$ and $d_k/10^k+1/10^k$. From his previous explanations he was able to conclude that the right hand side of the inequality holds.

For Q4, S1 gave incorrect answers in both the pre-test and the post-test. His answer in the pre-test was given below (Figure 6.73):


4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.

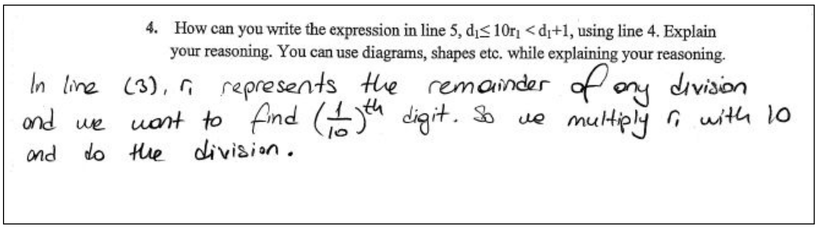
Bunu biliyoruz: $0 \leq 10r_1 < 10$ ve d_1'i kısıtlıyoruz $0 \leq d_1 \leq 9$. d_1 en küçük $d_1=0$ olabiliyor, en büyük $d_1=9$ olabiliyor. Öyleyse 0 ve 10'u d_1 cinsinden yazıyoruz: $d_1 \leq 10r_1 < d_1+1$

Figure 6.73. S1's answer to question 4 in the pre-test

> "We know this: $0 \leq 10r_1 < 10$ and we limit d_1, $0 \leq d_1 \leq 9$. d_1 the smallest value can be $d_1=0$, the greatest value can be $d_1=9$. Then, we wrote 0 and 10 in terms of d_1: $d_1 \leq 10r_1 < d_1+1$."

Thinking that he provided a better explanation for this question, he explained his reasoning from the pre-test. Still together with the interview, he thought the boundaries of d_1 and by using them he showed that the inequality asked in the question held. However, there was a misinterpretation in his answer. In order to find the boundaries of $10r_1$, he first thought that $d_1=0$, the lower bound, and then thought $d_1=9$, the upper bound. While finding the bounds, he should have assigned d_1 the same value. However, he considered different values of d_1.

In the post-test, he wrote the following (Figure 6.74):



4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.

In line (3), r_1 represents the remainder of any division and we want to find $(\frac{1}{10})^{th}$ digit. So we multiply r_1 with 10 and do the division.

Figure 6.74. S1's answer to question 4 in the post-test

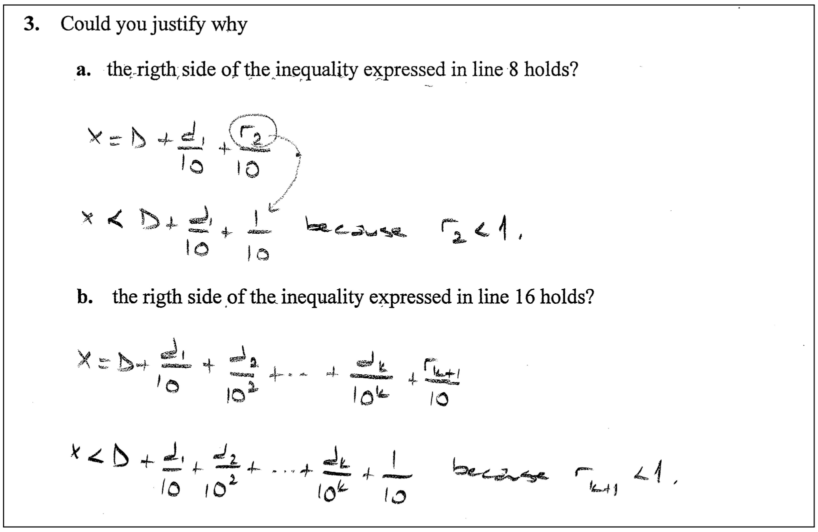
Data indicated that, he referred to multiplying the left over part with 10 to find pt the number of 1/10 th s. Although his reasoning was valid, he did not justify why $10r_1$ had to be between two consecutive numbers.

To conclude, S1's performance for Q3 and Q4 was shared. For this dimension her performance was considered as partial. In the pre-test, he gave correct answers to Q3 and incorrect answer to Q4. In the post-test, he answered only Q3 part (a) correctly. During the interview he elaborated on his answer for Q3 considering through quantities and hence he was able to reason why the asked inequalities held.

- Case II: From partially correct answer to correct answer

S7, S8 and S9's performance were considered under this case. S8 was chosen as a representative. He answered the Q3 in both the pre and the post test correctly but his answer was incorrect for Q4 in the pre-test. Also his answers in Q3, were very similar to S1's answers in Q3 shared in Case I. In the pre-test and the post-test, S8 answered Q3 (See Figure 6.75 and Figure 6.76) as follows:

During the interview, he stated the same reasoning for the part a and b as follows:


3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

$x = D + \frac{d_1}{10} + \frac{r_2}{10}$

$x < D + \frac{d_1}{10} + \frac{1}{10}$ because $r_2 < 1$.

b. the rigth side of the inequality expressed in line 16 holds?

$x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_k}{10^k} + \frac{r_{k+1}}{10}$

$x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_k}{10^k} + \frac{1}{10}$ because $r_{k+1} < 1$.

Figure 6.75. S8's answer to question 3 in the pre-test

S8: My number x is between in which interval it says here ($D+d_1/10 \leq x < D+d_1/10+1/10$) $r_2/10$ is always smaller than 1/10.

R: Why?

S8: The reason is this: r_2 is less than 1. Since r_2 is less than 1 we obtain this(showing that $r_2/10$ was less than 1/10)

R: Ok. Then, what can you say for 16 th step?

S8: I can say the same thing. The difference is this: What we did here? X was $D+d_1/10+r_2/10$. I can expand this. How can I expand? (Shows $r_2/10$). The expanding form is this (shows Figure 6.77)

That (referring to $d_2/100$) would never be bigger than 1/10. that is in fact it could not be bigger than 10/100 because this is 1/100 th s digit. The maximum value that it would take could only be 9/100. Therefore that (referring to 9/100) could not pass this(1/10), and x has to be less than this (referring to $D+d_1/10+d_2/10^2+.......+d_n/10^n+1/10^n$)

R: Then how can you say that the inequality holds?

S8: If I use this expression 16 th step, what happens now? We find the interval of x, don't we. I can say that while finding the interval: X now is from this number ($D+d_1/10+d_2/100+d_3/1000+\ldots+d_n/10^n$). X is greater than this because we do not include this.

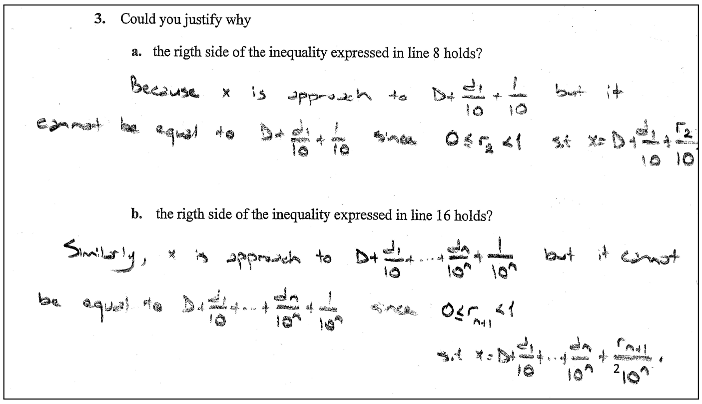
R: Yes.

S8: To find the upper bound of x.. I know that this number ($r_{n+1}/10^n$) is less than $1/10^n$ because I know that r_{n+1} is less than 1 but greater than 0.

R: Ok.

S8: For this reason (shows Figure 6.78)

Together with the interview, in both cases, he focused on the values of r_2 and r_{n+1}. Referring to the proof, since they were both between 0 and 1, $r_2/10$ had to be


3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

Because x is approach to $D+\frac{d_1}{10}+\frac{1}{10}$ but it cannot be equal to $D+\frac{d_1}{10}+\frac{1}{10}$ since $0\le r_2<1$ s.t $x=D+\frac{d_1}{10}+\frac{r_2}{10}$

b. the rigth side of the inequality expressed in line 16 holds?

Similarly, x is approach to $D+\frac{d_1}{10}+\dots+\frac{d_n}{10^n}+\frac{1}{10^n}$ but it cannot be equal to $D+\frac{d_1}{10}+\dots+\frac{d_n}{10^n}+\frac{1}{10^n}$ since $0\le r_{n+1}<1$ s.t. $x=D+\frac{d_1}{10}+\dots+\frac{d_n}{10^n}+\frac{r_{n+1}}{10^n}$.

Figure 6.76. S8's answer to question 3 in the post-test

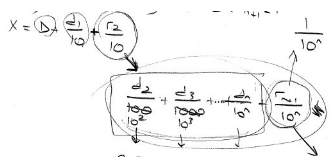


Figure 6.77. S8's representation 7

smaller than 1/10 and $r_{n+1}/10^n$ is smaller than $1/10^n$. What is interesting is that as shown in the Figure 6.77, he knew that he was able to continue the division from $r_2/10$. His statement given below showed that he knew that there would at most 9 times 1/100 th s in $r_2/10$.

> "That (referring to $d_2/100$) would never be bigger than 1/10. that is in fact it could not be bigger than 10/100 because this is 1/100 th s digit. The maximum value that it would take could only be 9/100. Therefore that (referring to 9/100) could not pass this(1/10), and x has to be less than this (referring to $D+d_1/10+d_2/10^2+.......+d_n/10^n+1/10^n$)"

That means he sought for the number of 1/100 th s in $r_2/10$. This also suggested

Figure 6.78. S8's representation 8

that the left over part $r_3/100$ had to be smaller than 1/100 because there would be no more 1/100 th s in $r_2/10$ (i.e d_2 cannot be equal to 10). Continuing the reasoning in this way he was able to write that $r_2/10$ was equal to $d_3/10+....+d_n/10^n+r_{n+1}/10^n$. That is, he knew that once continued there would only be d_n times $1/10^n$ in $r_n/10^{n-1}$ such that $r_{n+1}/10^n$ would be smaller than $1/10^n$. In this way, he concluded that the right side of the inequalities held in steps 8 and 16.

For question 4, he could not write any answer in the pre-test. In the post test, however he explained his reasoning even by drawing a diagram as shown in the figure (Figure 6.79).

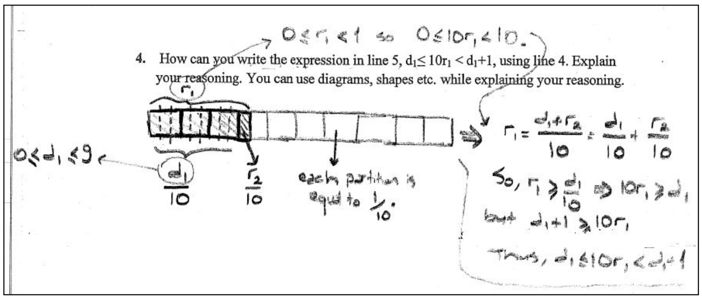


Figure 6.79. S8's answer to question 4 in the post-test

The data in the figure seems to suggest that he reasoned in the following way. S8 continued reasoning on the previous example, 10/3, in question 1. In order to show 10/30, he thought through quantities such that he first partitioned a whole in to 10 equal pieces. Then, he repartitioned the equal piece into 3 more equal pieces. Thinking 10 of those 1/30 th pieces, he obtained 10/30 (r_1 in the drawing). S8's thinking process so far seemed to suggest that he thought of the rational number, 10/30, from a partitive interpretation of division. This allowed him to determine the number of 1/10th s in 1/3 such that he grouped the pieces (each being equal to 1/30) by 3 and then counted how many of those 1/10 th there are in 1/3 rd. That is he measured 10/30 th with 1/10 th (i.e, quotative division). In this way, he was able to find $d_1/10$ as being equal to 3/10 such that there were 3 times 1/10 th in 10/30 and $r_2/10$ was equal to 1/30. During the interview, he further explained his reasoning:

S8: Now r_1 could be between zero and 1. I told you. For this reason, $10r_1$ is between 0 and 10. now d_1 is , $0 \leq d_1 \leq 10$ in this way. What will happen if this increases by 1..it will go to natural numbers (d_1 explains why it couldn't be 10) Now it is easier to define over the figure. d_1+r_2 when we collect all of this what do we get? We reach r_1. $d_1/10$ $r_2/10$. r_1, is greater than $d_1/10$. why? we said here that from the piece of 10/30 that is 1/3 there are some remaining parts 1/30 portion, I mean. The non-terminating part I mean $r_2/10$. Here too . r_1 is greater than $d_1/10$. Therefore, $10r_1$ is bigger than d_1. d_1+1 is bigger than $10r_1$ because it is $r_1<(d_1+1)/10$. (He indicates the expression). That is d_1+1 means that. it exceeds this part (shows $r_2/10$ part). For this reason it exceeds the portion 1/30.

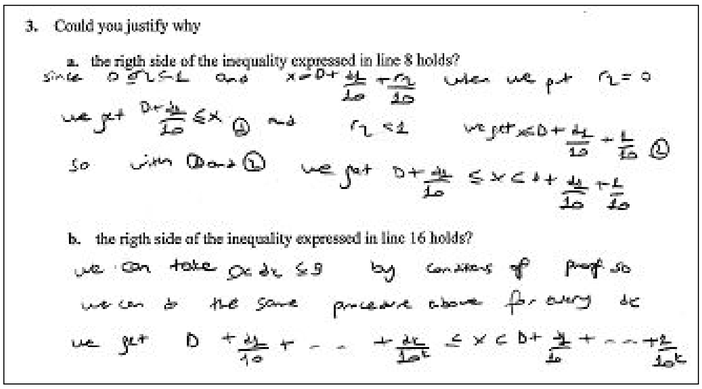
As the data indicated S8 thought that since r_1 was equal to $d_1/10$ and $r_2/10$, he was able to reason that r_1 had to be bigger than $d_1/10$. In this way, he reasoned that $10r_1$ was bigger than d_1. For the other side of the inequality, i.e. $10r_1$ is smaller than d_1+1, he again thought through the diagram such that he knew that the amount referring to $d_1/10+1/10$ exceeded the portion $r_2/10$. His reasoning follows: as the earlier data suggested for him d_1 representing the number of 1/10 th s could not be equal to 4. This was again because once d_1 equaled to 4, it would mean that there would be one more 3 groups of 1/30 th however he only had one 1/30 th. Therefore, he knew that $d_1/10+1/10$ would exceed $d_1/10+r_2/10$. In this way, he was able to justify that the inequality held.

To conclude, for the case 'From partially correct answer to correct answer', S8 was chosen as a representative. In the pre-test, he answered only Q3 correctly and in the post-test he was able to answer all the questions in dimension 2 correctly. For Q3, he referred to proof directly and explained the inequalities numerically. However, in Q4 along with the proof, he used quantities to justify the inequalities asked in the question.

- Case III: From partially correct answer to no/incorrect answer

S5 was the only student representing case III. She answered Q3 partially and Q4 incorrectly in the pre-test and answered both questions incorrectly in the post test.

Her answer to Q3 in the pre-test were shared below (Figure 6.80):


3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

since $0 \le r_2 < 1$ and $x = D + \frac{d_1}{10} + \frac{r_2}{10}$ when we put $r_2 = 0$ we get $D + \frac{d_1}{10} \le x$ (2) and $r_2 < 1$ we get $x < D + \frac{d_1}{10} + \frac{1}{10}$ (1)

so with (2) and (1) we get $D + \frac{d_1}{10} \le x < D + \frac{d_1}{10} + \frac{1}{10}$

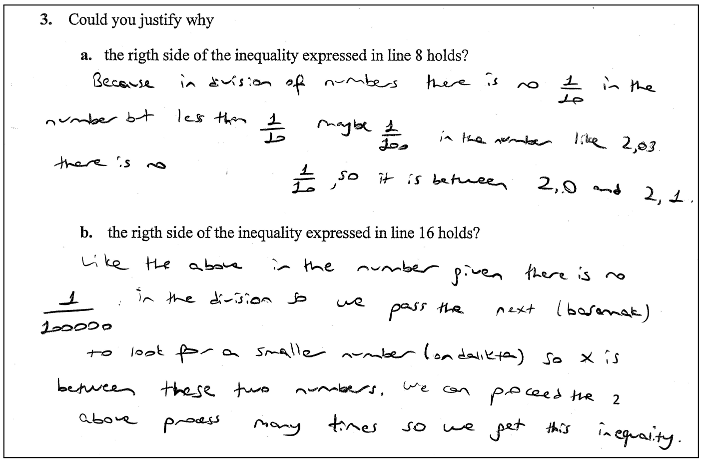
b. the rigth side of the inequality expressed in line 16 holds?

we can take $0 \le d_k \le 9$ by conditions of proof so we can do the same procedure above for every d_k we get $D + \frac{d_1}{10} + \dots + \frac{d_k}{10^k} \le x < D + \frac{d_1}{10} + \dots + \frac{1}{10^k}$

Figure 6.80. S5's answer to question 3 in the pre-test

In the pre-test for Q3 part (a), she considered the different values of r_2 given in the proof and justified why the right side of the inequality held. She knew that $x=D+d_1/10+r_2/10$ and once r_2 was equal to zero, then x would be equal to $D+d_1/10$. Therefore, x would be bigger than or equal to $D+d_1/10$. By the same token, she thought of the case $r_2 <1$ such that $r_2/10$ would be less than 1/10. Therefore, x would be smaller than $D+d_1/10+1/10$. For part (b), different than part (a) she started by emphasizing the values for d_k and said that 'we can do the same procedure'. However, she thought of the values of d_k rather than r_k seemed to indicate that she did not provide her reasoning clearly focusing on r_k's. Also, it seemed that she wrote the inequality based on the previous inequality given in the proof for part(a). Therefore, the data were not conclusive enough to reveal her reasoning. In this regard her answer to part (b) was considered as incorrect. In the post-test she answered Q3 in the following way (Figure 6.81):

In the post-test it seems that S5 tried to express her reasoning in both parts, thinking through the digits of the decimal expansion. She gave the example, 2,03 and stated in part(a) that it was between 2,0 and 2,1 since it had no 1/10 th digit. In part (b) she continued referring to part (a) such that x had to be smaller than the numbers given in the proof. Her focusing on 1/10 th and the 1/100000 seemed to indicate that she thought of the inequality through focusing on d_k's. However, she



3. Could you justify why

a. the rigth side of the inequality expressed in line 8 holds?

Because in division of numbers there is no $\frac{1}{10}$ in the number but less than $\frac{1}{10}$ maybe $\frac{1}{100}$ in the number like 2,03 there is no $\frac{1}{10}$, so it is between 2,0 and 2,1.

b. the rigth side of the inequality expressed in line 16 holds?

Like the above in the number given there is no $\frac{1}{100000}$ in the division so we pass the next (basamak) to look for a smaller number (ondalikta) so x is between these two numbers. We can proceed the 2 above process many times so we get this inequality.

Figure 6.81. S5's answer to question 3 in the post-test

did not directly address the inequalities referring to the values of r_n's in the proof. therefore, her answer was considered as incorrect.

Although S5 could not explain her answer satisfactorily in the post-test, she was able to do it during the interview. The excerpt below shows her reasoning:

> S5: r_2 between zero and 1 here means that we have no longer 1/10, no 1/10.
>
> R: Uh-Huh. OK. So what does that mean? Can you establish a relationship between $r_2/10$ and 1/10?
>
> S5: 1/10 is bigger, r_2 is smaller..
>
> R: $r_2/10$
>
> S5: Huh $r_2/10$ sorry..$r_2/10$ is smaller. It is the new remainder. It is no longer 1/10, it is the remainder when we begin to search for 1/100 th s.Therefore, I think we know that x can be squeezed to two numbers.
>
> R: OK. If we look at part b ... (I'm reading the question) It should be here,How did you make it possible that x is smaller than the right side of the inequality at step 16?
>
> S5: This is how I think. We move this k steps to get closer.
>
> R: Yes. What happened to my X?
>
> S5: My x. . . $1/10^2$ we write from all types. 10^k, sorry, we approach to n. Ds are between 0 and 9 here. .. $r_{n+1}/10^n$ is smaller than $1/10^n$. That is how it went forward.

The data showed that She explained the relationship between $r_2/10$ and 1/10 and between $r_{n+1}/10^n$ and $1/10^n$ such that $r_2/10$ and $r_{n+1}/10^n$ would be smaller than

$1/10$ and $1/10^n$ respectively. Her stating "$r_2/10$ is smaller. It is the new remainder. It is no longer $1/10$, it is the remainder when we begin to search for $1/100$.." suggested that for her the term $r_2/10$ represented the left over part. she knew that $r_2/10$ indicated that there would not be $1/10$ anymore and she had to seek for finding how many $1/100$'s there would be in the left over part. Although she did not explain her reasoning drawing diagrams, data indicated that S5 focused on quantities (such as, $r_2/10$ referring to some amount). In this way, she obtained the inequality.

For Q4, S5 could not give correct answer in both tests. In the pre-test she answered in the following way (Figure 6.82):

4. How can you write the expression in line 5, $\underline{d_1 \leq 10r_1 < d_1+1}$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.

we know $D \leq x < D+1$ and we also know $0 \leq x - D < 1$

and called called $r_1 = x - D$ so we get

$0 \leq 10(x-D) < 10$

$\downarrow$

$0 \leq 10r_1 < 10$

Figure 6.82. S5's answer to question 4 in the pre-test

Data indicated that she was able to explain how the inequality $0 \leq 10r_1 < 10$ held. She knew that r_1 was between 0 and 1 and multiplying the inequality with 10 gave her $0 \leq 10r_1 < 10$. However, the question was about explaining the inequality $d_1 \leq 10r_1 < d_1+1$. The inequality she obtained has already been known from line 4. Therefore, her answer was not acceptable.

4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.

$0 \leq d_1 \leq 9$

$1 < d_1+1 \leq 10$

So d_1 and d_1+1 is between $10r_1$

The remeinder r_1 can not bigger then 1 $\quad$ d_1 is between 0 and 9

So $0 \leq 10r_1 - d_1 < 1$.

Figure 6.83. S5's answer to question 4 in the post-test

In the post-test (Figure 6.83), instead of explaining why $d_1 \leq 10r_1 < d_1+1$ holds, she used the inequalities $0 \leq d_1 \leq 9$ and $1 \leq d_1+1 \leq 10$. Then, she wrote that $10r_1$was between d_1 and d_1+1, however, she did not explain how she obtained that 10 $r_1$was between d_1 and d_1+1.

Although S5 could not answer during the tests, in the interview, interestingly, she was able to explain the inequality by thinking of an example, 13/7 asked in Q10, and referring to the diagrams. First as shown in the figure, she stated that D would be equal to 1, r_1 would be 6/7 and d_1 would be equal to 8.

> S5: Now my r_1 is 6/7 for 13/7. What is 10r_1? 60/7. d_1 was 8 and d_1+1 was 9. 60/7 was between 8 and 9 so inequality holds.That is I am trying to squeeze the remaining portion into two numbers. I am doing re-partitioning because this time I am searching for 1/70 th s. That is I am looking for the number of the groups of 7 pieces in 60 pieces. therefore, it is squeezed between d_1 and d_1+1.
>
> R: How did you squeeze in two numbers?
>
> S5: As making smaller partitioning.
>
> R: Can you find consecutive numbers for all r_1?
>
> S5: Yes because I do re-partitioning again. I do the same partitioning. It has to be between two consecutive numbers. because there were 8 many 7*1/70 th s in 6/7 making 8 1/10 but there was no 9 many for r_1. It will be the same for r_2, r_3, r_4. If the division were over, then there was no need to do approximation.

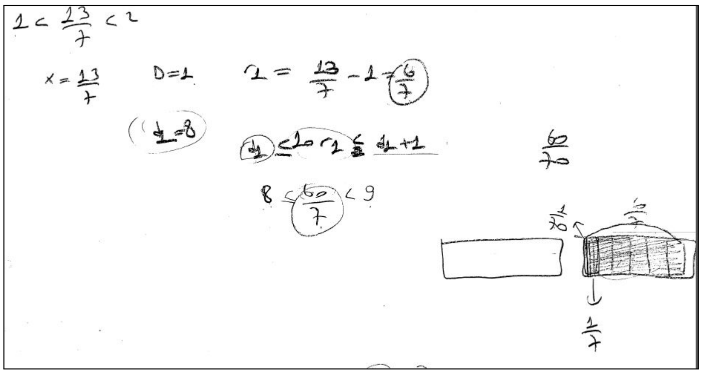


Figure 6.84. S5's representation 1

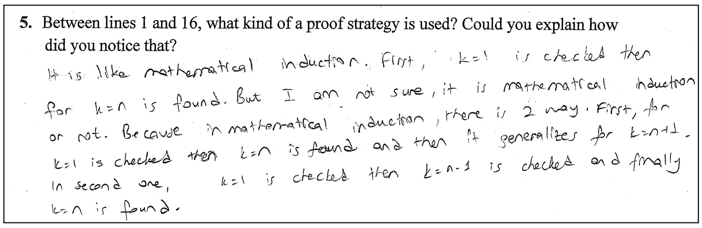
First, she found the number of 1/10 th's 6/7. For this reason, she only focused on the piece 6/7 as she drew initially in the figure. In order to find the consecutive number that 10r_1 lies between, she said that she needed to re-partition each piece into 10 obtaining the smallest piece as 1/70. As she stated in the interview she looked for 7 groups of 1/70 in 6/7. She was aware that 7 of 1/70 th s made 1/10. Since there were 8 many 1/10 th s in 6/7 and there was no more, she squeezed $10r_1$ between 8 and 9 saying that she did not have 9 many 1/10 th s in the diagram. Thus, she concluded that for any r_1 she could find an interval with bounds of consecutive numbers that $10r_1$ lies in. The diagram helped her focus on the quantities such that r_1 and d_1 referred to the amounts, such as the left over part and the number of 1/10 th s in the left over part respectively. Therefore, she could justify why the inequality, $d_1 \leq 10r_1 < d_1+1$, holds. Initially, S5 reasoned through an example. However, her stating that via re-partitioning the left-over parts such as r_2, r_3, etc. no matter how small they are and finding smaller decimals through grouping would enable her to find the interval with bounds of consecutive numbers for $10r_2$, $10r_3$, etc. suggested that she was able to think for every r_k.

To conclude, the case 'from partially correct answer to no/incorrect answer' was the only case among all the proof comprehension dimensions. S5's work was considered in this case. Although she was able to give partial answer to Q3 and Q4 in dimension 2 for the pre-test, she could not give correct answers in the post-test. However, though her answers lacked the reasoning in the tests, during the interview she was able to reason quantitatively in some cases and to explain the reasoning correctly.

<u>6.2.2.3. Dimension 3: Logical status of statements and proof framework.</u> Q5 and Q6 addressed the dimension 'logical status of statements and proof framework'. Q5 was about the method of the given proof and Q6 was about the general structure about the proof. For this dimension, as the pre-test is considered, 1 of the interviewees' performance were classified as correct answer,1 of the interviewees' performance were labeled as partially correct answer, 4 of them were classified as no/incorrect answer. In the post-test, 2 students' performances were considered as correct answer, 2 of them were partially correct answer and 2 of them were no/incorrect answer. The transitions between performances from pre to post test were shared as follows:

- Case I: From partially correct answer to correct answer

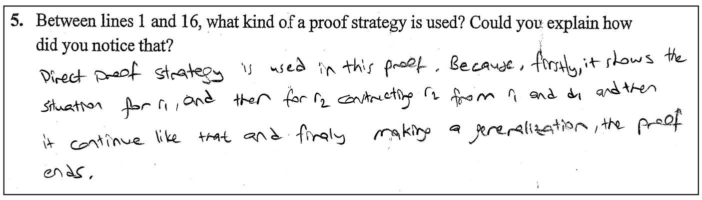
S9 was the only representative for this case. Initially, she only answered the Q5 about the method of proof wrongly stating "it is like mathematical induction". The pre-test answer for Q5 is given below (Figure 6.85):


5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?

It is like mathematical induction. First, k=1 is checked then
for k=n is found. But I am not sure, it is mathematical induction
or not. Because in mathematical induction, there is 2 way. First, for
k=1 is checked then k=n is found and then it generalizes for k=n+1.
In second one, k=1 is checked then k=n-1 is checked and finally
k=n is found.

Figure 6.85. S9's answer to question 5 in the pre-test

In her answer, she also emphasized that she was not sure whether the proof was really a mathematical induction or not. In spite of not being sure about the method, she could not explain what was missing or how it was not appropriate for mathematical induction. For instance, she wrote that in mathematical induction k=n was found and the proof was generalized to k=n+1. However, the proof did not continue like her explanation. Though, in the post-test her mind changed and stated that it was a direct proof. Her answer in the post test was given below (Figure 6.86):


5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?

Direct proof strategy is used in this proof. Because, firstly, it shows the
situation for r_1, and then for r_2 contracting r_2 from r_1 and d_1 and then
it continue like that and finaly making a generalization, the proof
ends.

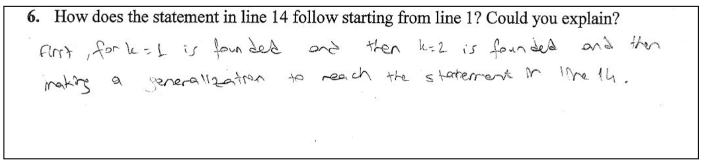
Figure 6.86. S9's answer to question 5 in the post-test

In the post test, she explained her reasoning on why the proof was a direct proof by focusing on the construction of r_2 from r_1 and d_1. Since in 'direct proof' the consecutive steps is constructed from the previous steps, her answer was acceptable. Since she did not explain the reason for not being mathematical induction, in the interview researcher asked about the reason for changing her mind:

R: Well, in the pre-test you said induction. Why did you turn to direct proof? Why not induction?
S9:Becuase induction means that you show n=1.
R:Yes.
S9:Then, after showing for n, you go to the n+1 but here we do not go to n+1. We go to n.
R: We go to n but which steps were missing so that it was not induction? Or was there any surplus?
S9: There were 3 steps for induction. There is no inductive step here. Since it is not mathematical induction, I decided to call direct proof.

In the interview, she explained her reason behind changing the answer from mathematical induction to direct proof. For her in mathematical induction there are 3 steps. However, in this proof since there was no inductive step she concluded that it was not mathematical induction.

For the question 6, S9 also gave correct answers both in the pre-test and the post-test. In this question, mainly the general structure of the proof was asked. She wrote the following in the pre-test (Figure 6.87):


6. How does the statement in line 14 follow starting from line 1? Could you explain?

First, for k=1 is founded and then k=2 is founded and then making a generalization to reach the statement in line 14.

Figure 6.87. S9's answer to question 6 in the pre-test

She thougth that first the case k=1 was proceeded and followed by the cases k=2 and k=3 to reach the general representation.

In the post test, she gave the following answer (Figure 6.88):

In her answer she focused again on the procedure maintained until line 13 for the case k=2. She also emphasized that the procedure continued by writing decimal

6. How does the statement in line 14 follow starting from line 1? Could you explain?

when we continue the procedure from line 13, we will get x for k=3 and it will continue in same way. From the beginning r/q rational number and irrational number is actually writing as an equality in terms of its decimal places (bu sayıları ondalık esitlik seklinde yazıyoruz basından beri. Örneğin; $\frac{7}{3} = 1 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{1/3}{10^3}$). So, we can generalize the situation as line in 14.

Figure 6.88. S9's answer to question 6 in the post-test

representation of rational and irrational numbers. In the interview, she explained:

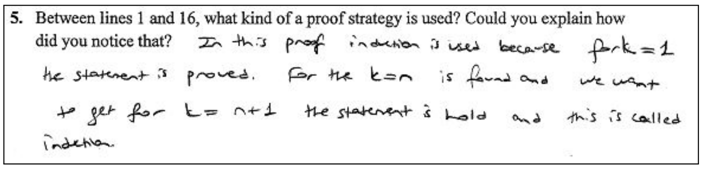
> R: What did you say in question 6? How does the statement in line 14 follows starting from line 1? Could you explain?
>
> S9: I can say without looking this(referring to her answer). Here we showed for k=1 and later for k=2.
>
> R: Yes.
>
> S9: I am showing for k=3. Until this point the same procedure continues therefore after k=3 the same conclusion was obtained using the same procedure. There is a definite pattern. Therefore, we generalize it to k=n as continuing this pattern.

She asserted that in the proof the inequalities were shown for k=1, k=2 and k=3. Then by following the same procedure she thought that it was generalized to k=n. From her explanation it can be inferred that she was aware of the structure of the proof.

To conclude, S9 was chosen as a representative for the case 'From partially correct answer to correct answer' since she gave partial answer in the pre-test for this dimension that is she only answered Q6 correctly. However, in post test she was able to answer both Q5 and Q6.

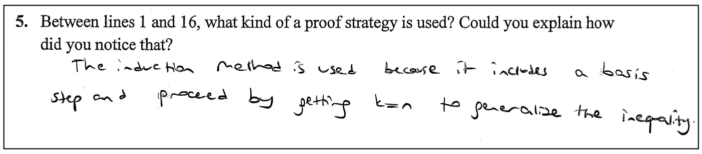
- Case II: From no /incorrect answer to Partially correct answer

S5 and S8's performances were considered in this category S5 was chosen to illustrate the case based on her performance in the pre and the post tests. In the pre-test, she could not give correct answers to questions (Figure 6.89 and Figure 6.91). However, in the post test she gave an explanatory answer for Q6 (Figure 6.92) but failed to determine the method of proof in Q5 (Figure 6.90).



5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that? In this proof induction is used because for k = 1 the statement is proved. For the k=n is found and we want to get for k= n+1 the statement is hold and this is called induction.

Figure 6.89. S5's answer to question 5 in the pre-test



5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?
The induction method is used because it includes a basis step and proceed by getting k=n to generalize the inequality.

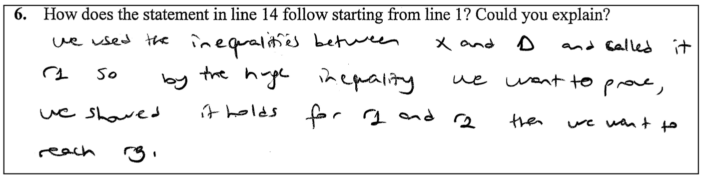
Figure 6.90. S5's answer to question 5 in the post-test

When both pre-test and the post-test answers of S5 were considered, there seems a difference in answers although both indicated that her choice was mathematical induction. In the pre-test she stated after showing k=1 and k=n, the proof included the step k=n+1. In this case, she did not support her idea for k=n+1 referring to the proof. In the post-test, she just considered the basis step and k=n for induction. her not emphasizing the inductive step in the post test, could be the reason for her hesitation about the method of proof during the interview. She stated:

> S5: It is not exactly induction, but I think it is some kind of induction. Because we accept as k=k and do not show for k=k+1 , but we try to reach a generalization. That is. Because after the basis step, the 8 th line, as assuming k=k we do not try to reach k=k+1 but we use induction such a way that we generalize the findings. After the first step, we continue with the similar operations and try to generalize them to 10^k. Therefore, I thought that it is induction. that is, it not exactly the mathematical induction but a kind of induction. It is not contradiction or contrapositive, I know that. What do we have as proof methods? Imm.. Direct proof. But here it is assuming something and prove something? If it were one step then it could be a direct proof but there is no such a thing. It goes to a generalization. Until k th step. I think it is induction. I could not associate with another method.

From the above excerpt, it can be inferred that S5 was aware that the inductive step was missing in the proof. However, she did not accept that it could be a direct proof either. Her realizing that in direct proof, there should be an assumption from which a generalization is made but her not being able to realize that the assumption in the proof was x being a positive real number suggested that she was not aware of the structure of the proof.

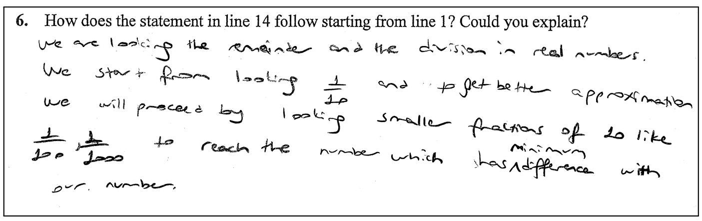
For the Q6, in the pre-test she gave incorrect answer and in the post test she gave correct answer. Her pre-test answer was as follows (Figure 6.91):


6. How does the statement in line 14 follow starting from line 1? Could you explain?

we used the inequalities between x and Δ and called it r1 so by the huge inequality we want to prove, we showed it holds for r1 and r2 then we want to reach r3.

Figure 6.91. S5's answer to question 6 in the pre-test

S5 gave a limited answer as focusing on finding r_1 and r_2 and at the end obtaining r_3. However, finding r's were not the main purpose in the proof. They just leaded to justify why inequalities in each step holds so that real numbers were squeezed in an interval for all k. S5's answer to post-test (Figure 6.92) and her reasoning during the interview were as follows:


6. How does the statement in line 14 follow starting from line 1? Could you explain?

we are looking the remainder and the division in real numbers. We start from looking $\frac{1}{10}$ and to get better approximation we will proceed by looking smaller fractions of 10 like $\frac{1}{100}$, $\frac{1}{1000}$ to reach the number which has minimum difference with our number.

Figure 6.92. S5's answer to question 6 in the post-test

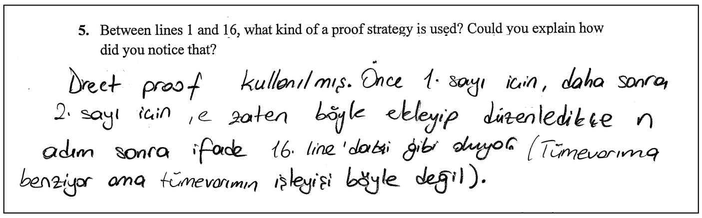
S5: For better approximation we start from 1/10. after finding 1/10's in the number we search for 1/100 th s and$_1$/1000's. So up to 10^k, we found the decimals and found a number which represents our number better. Hence, Lower bound and upper bound are found.

In the post test, she focused on finding 1/10 th s, 1/100 th s, 1/1000 th s etc. for the cases k=1, k=2, k=3 respectively for better approximation such that she could determine an interval for which both the upper and lower bounds were very close to the real number. Therefore, she was able to explain the general structure of the proof.

To conclude, S5 was chosen as a representative for the case 'From no /incorrect answer to partially correct answer ' since she gave no/incorrect answer in the pre-test for this dimension. However, in post test she was able to answer Q6.

- Case III: No change:

S1, S7 and S15 were the representatives for this case. There was no change in their performances in this dimension from the pre-test to the post test. S1 was taken as an example for this case. In both tests his answers to Q5 and Q6 were considered as correct. For Q5 for both the pre-test (Figure 6.93) and the post-test (Figure 6.94) he wrote:



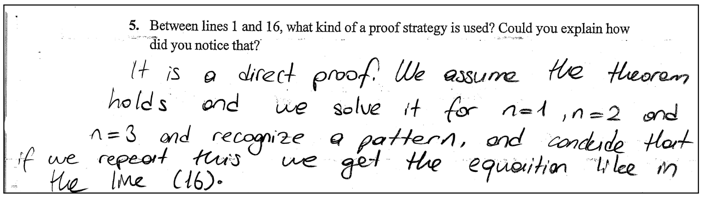
5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?

Drect proof kullanılmış. Önce 1. sayı için, daha sonra 2. sayı için, e zaten böyle ekleyip düzenledikçe n adım sonra ifade 16. line'daki gibi oluyor (Tümevarıma benziyor ama tümevarımın işleyişi böyle değil).

Figure 6.93. S1's answer to question 5 in the pre-test

> "Direct proof was used. First for the first number and then for the second number, then adding like that and arranging after n the step the expression in the 16 th line was obtained. (It looks like induction but the process of induction is not like this."

S1 thought in both cases that the proof was direct proof by focusing on the construction of the main idea. However, in the pre-test he made a comment that the proof looked like mathematical induction but it was not since the process of the mathematical induction was not similar to this one. The researcher further questioned



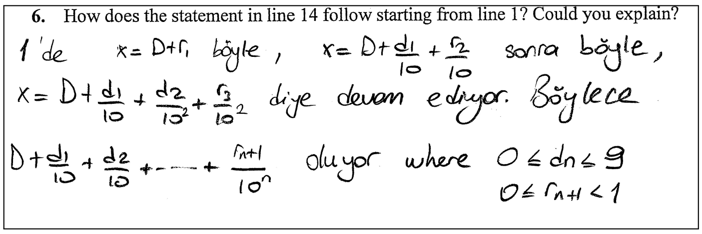
5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?

It is a direct proof. We assume the theorem holds and we solve it for $n=1$, $n=2$ and $n=3$ and recognize a pattern, and conclude that if we repeat this we get the equaition like in the line (16).

Figure 6.94. S1's answer to question 5 in the post-test

him in the interview. He stated that there was no inductive step. He emphasized he did not assume the truthness of the n th step and show the truthness of n+1. This suggested that he knew the steps of mathematical induction and did not consider the proof as an example of mathematical induction.

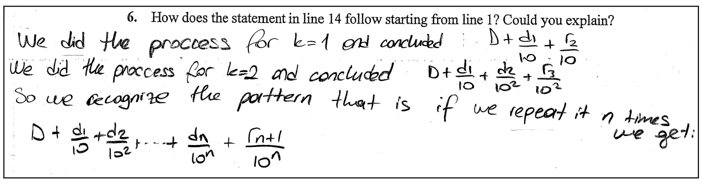
For the Q6, his answers in both test were considered as correct because he gave the general structure of the proof and explained how the proof proceeded as follows (Figure 6.95 and Figure 6.96):



6. How does the statement in line 14 follow starting from line 1? Could you explain?

1'de $x = D + r_1$ böyle, $x = D + \frac{d_1}{10} + \frac{r_2}{10}$ sonra böyle,
$x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{r_3}{10^2}$ diye devam ediyor. Böylece
$D + \frac{d_1}{10} + \frac{d_2}{10} + \dots + \frac{r_{n+1}}{10^n}$ oluyor where $0 \le d_n \le 9$
$0 \le r_{n+1} < 1$

Figure 6.95. S1's answer to question 6 in the pre-test



6. How does the statement in line 14 follow starting from line 1? Could you explain?

We did the process for $k=1$ and concluded $D + \frac{d_1}{10} + \frac{r_2}{10}$
We did the process for $k=2$ and concluded $D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{r_3}{10^2}$
So we recognize the pattern that is if we repeat it n times we get:
$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n} + \frac{r_{n+1}}{10^n}$

Figure 6.96. S1's answer to question 6 in the post-test

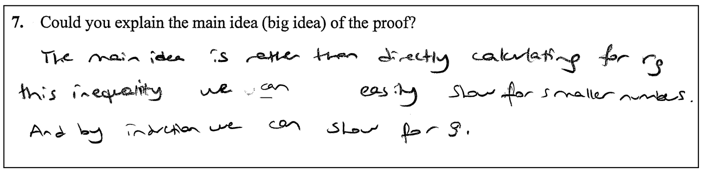
In both answers, he wrote the findings for k=1, k=2 and k=n. This showed that he was aware of the structure.

To conclude, S1 was the representative for the case 'no change' since he was able to give correct answers to Q5 and Q6. Additionally, he was able to explain why he chose the method of proof as 'direct proof' by thinking the difference between direct proof and mathematical induction.

6.2.2.4. Dimension 4: Summarizing via high-level ideas. Q7 and Q8 were addressed in this dimension. Q7 was about the main idea of the proof and Q8 was about the most important lines of the proof. For this dimension, as the pre-test is considered, 1 of the interviewees' performance were considered as correct answer, 3 of the interviewees' performance were considered no/incorrect answer, 2 of them were classified as partially correct answer. In the post-test, 4 students' performances were considered as correct answer, 2 students' performances were considered partially correct. The transitions between performances from pre to post test were shared as follows:

- Case I: From no/incorrect answer to correct answer

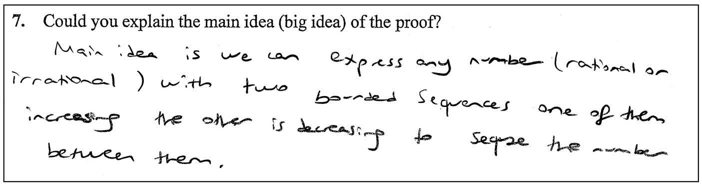
S5, S7 and S15 were the representatives for this case. S5 was chosen for illustrating the case with the support of her answers in the pre and post-tests and the interview. S5 answered Q7 in the following way (Figure 6.97):


7. Could you explain the main idea (big idea) of the proof?

The main idea is rather than directly calculating for r_9 this inequality we can easily show for smaller numbers. And by induction we can show for 9.

Figure 6.97. S5's answer to question 7 in the pre-test

She thought that the main idea was related to r_n's. However, the proof was about decimal representation of positive real numbers. The number r_n was used as a transition to determine the intervals of any positive real number while finding the decimals. Therefore, she could not give an explanatory answer. In the post test, her answer was changed as follows (Figure 6.98):



7. Could you explain the main idea (big idea) of the proof?

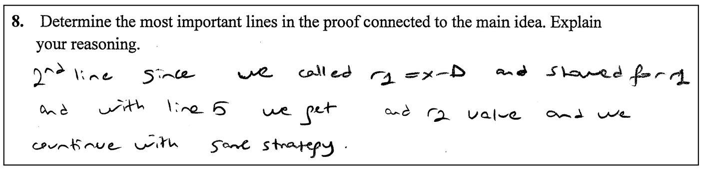
Main idea is we can express any number (rational or irrational) with two bounded sequences one of them increasing the other is decreasing to sequze the number between them.

Figure 6.98. S5's answer to question 7 in the post-test

S5: The number can be rational or irrational. we can express the number between two bounded sequences. One of them increasing and the other one decreasing rational number sequences. The lower bound always increases but the upper bound decrease because we add smaller numbers in each step compared to previous ones. Therefore, we can express an irrational or rational number between two rational number sequence that the lower bound is increasing and upper bound is decreasing. The big idea is this.

Associating the post test and the interview, data suggested that she used the proof and combined it with the definition at the end of the proof. In the proof the positive real number was squeezed in intervals as the number of decimals increased and the bounds formed rational number sequences one of which was increasing and the other was decreasing. Her answer showed that she was able to grasp the idea of proof and also inferred the definition from it by stating that the real number either irrational or rational can be represented by two rational number sequences one of which was increasing and the other was decreasing.

For Q8, she gave the following answer in the pre-test (Figure 6.99):

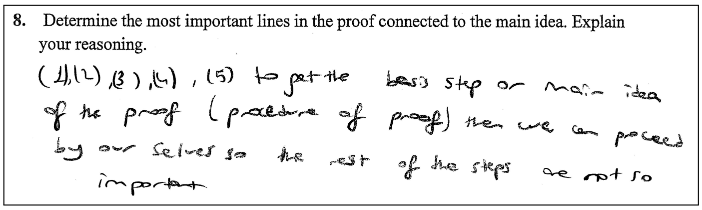


8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

2nd line since we called $r_1 = x - D$ and solved for r_1 and with line 5 we get and r_2 value and we continue with same strategy.

Figure 6.99. S5's answer to question 8 in the pre-test

In her answer she focused on finding r_1 and thought that the second line was the most important line. Here, since just one line was not the basis for the construction, her answer was considered as incomplete. However, in the post-test she gave a different answer to the question (Figure 6.100) and also supported with the interview:


8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

(1),(2),(3),(4), (5) to get the basis step or main idea of the proof (procedure of proof) then we can proceed by our selves so the rest of the steps are not so important

Figure 6.100. S5's answer to question 8 in the post-test

> S5:I said 1 2 3 4 5. Someone who understands the first part... r_1 actually starts to make sense to us, ourselves, starting from r_1 , we are able to find r_2, d_2 and d_3. But without this part we can have a difficulty. I think the most important part is this part of the proof is the initial steps.

Along with her answer in the post test, she emphasized that comprehending the first 5 steps of the proof were important since it allowed to continue with the proof. She thought that it became meaningful to use r_1 to find r_2, d_2 and r_3 using the steps 1, 2, 3, 4, 5. Her answer showed that she was able to determine the most important lines in the construction to obtain the main idea of the proof.

To conclude, for the case 'From no /incorrect answer to correct answer', S5's answers to Q7 and Q8 in both tests were shared. In the pre-test she could not give correct answers but in the post-test she was able to answer the questions correctly.

- Case II: No change

S1, S8 and S9's performances were considered as no change in this dimension. They all gave partial answers in the test. S1 was chosen as illustration of the case. In the pre-test and the post-test, he answered Q7 as follows (Figure 6.101 and Figure 6.102):

7. Could you explain the main idea (big idea) of the proof?

x'i olabildiğince precise yapmaya çalışmış.
D zaten doğal sayı ve D'ye $\frac{r_1}{10}$'lu şeyler ekliyoruz.
D=3 için
$3,245678913 \leq x < 3,245678914$
Yani x'in decimallerini buluyoruz.

Figure 6.101. S1's answer to question 7 in the pre-test

"It tried to make x precise as far as possible. D is natural number and we add D to r_1's. For D=3 $3,245678913 \leq x < 3,245678914$. That is we are finding decimals of x." (S1's answer to Q7 in the pre-test-Figure 6.101)

7. Could you explain the main idea (big idea) of the proof?

We are finding the $(\frac{1}{10})^{th}, (\frac{1}{10^2})^{th}, (\frac{1}{10^3})^{th} \cdots (\frac{1}{10^4})^{th}$ digits of the rational numbers.

$$\frac{10}{7} = 1,4285$$

$$1+\frac{4}{10}+\frac{2}{10^2}+\frac{8}{10^3}+\frac{5}{10^4} \leq \frac{10}{7} < 1+\frac{4}{10}+\frac{2}{10^2}+\frac{8}{10^3}+\frac{5}{10^4}+\frac{1}{10^4}$$

Figure 6.102. S1's answer to question 7 in the post-test

He thought that the main idea was making the number x more precise and he added that we found the decimals of x. Finding decimals were not the main idea of the proof but it leaded to reach the idea that considering each decimal place, an interval could be found for any positive real number and hence a real number could be defined by rational number sequences. Therefore, his answer was considered as incorrect.

For Q8, in the pre-test S1 wrote (Figure 6.103):

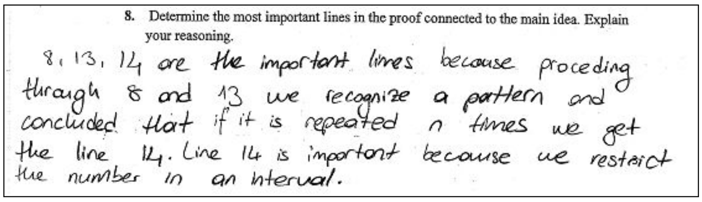
8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

The first 6 lines of the proof is the most important lines I think. Since when you comprehend these lines, line 12 would be easier to understand and thus line 14 follows easier.

Figure 6.103. S1's answer to question 8 in the pre-test

For him the first six lines were important. These lines were showing the process of construction and as he said they were the leading steps to understand how the conclusion follows. Therefore, his answer were considered as correct.

In the post-test, he answered as follows (Figure 6.104):


8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

8, 13, 14 are the important lines because proceding through 8 and 13 we recognize a pattern and concluded that if it is repeated n times we get the line 14. Line 14 is important because we restrict the number in an interval.

Figure 6.104. S1's answer to question 8 in the post-test

Although he focused on the lines that the inequalities were expressed in the proof, he also mentioned that proceeding through the 8 th step a pattern was recognized. Therefore, this might show that he also give importance to the lines preceding the 8 th step which was similar to his answer in the pre-test. In the interview, he also mentioned that the first 6 steps were important, but at the same time he considered the steps such as lines 13 and 14 which related to finding the intervals.

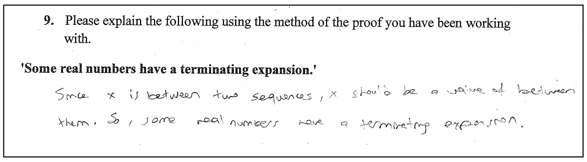
To conclude, for the case 'No change', S1's answers to Q7 and Q8 in both tests were shared. In the pre-test and post-test, he only answered Q8 correctly. Therefore, his performance for proof comprehension dimension 4 were considered partial and there were no change from the pre-test to the post-test.

6.2.2.5. Dimension 5: Transferring the general ideas or methods to another context.
Q9 and Q13 addressed the dimension 'Transferring the general ideas or methods to another context'. Q9 was about justification of the statement that some real numbers have terminating decimal expansion. Q13 was about the explanation of a definition about representing real numbers using rational number sequences. For this dimension, as the pre-test is considered, 4 of the interviewees' performance were labelled as no/incorrect answer, 2 of them were classified as partially correct answer.

In the post-test, all the performances were labeled as partially correct answer. 2 performance remained the same from the pre-test to the post-test as partially correct answer. The transitions between performances from pre to post test were shared as follows:

- Case I: From no /incorrect answer to partially correct answer

S1, S7, S8 and S9's performances were classified in this case. For illustration S9's performance was shared. In the pre-test she could not answer the question correctly (Figure 6.105):


9. Please explain the following using the method of the proof you have been working with.

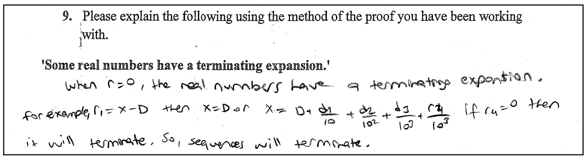
'Some real numbers have a terminating expansion.'

Since x is between two sequences, x should be a value of between them. So, some real numbers have a terminating expansion.

Figure 6.105. S9's answer to question 9 in the pre-test

In the pre-test, she thought of sequences and since x was between two sequences, it had to have a value between them. Her saying was correct but she did not explain in which condition terminating expansion could be obtained referring to the proof. Her statement was correct for all the cases of decimal expansions (terminating, non-terminating or repeating).

In the post test, she gave the following answer which was considered as correct (Figure 6.106):


9. Please explain the following using the method of the proof you have been working with.

'Some real numbers have a terminating expansion.'

When $r=0$, the real numbers have a terminating expantion. for example, $r_1 = x - D$ then $x = D$ or $x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{r_4}{10^3}$ If $r_4 = 0$ then it will terminate. So, sequences will terminate.

Figure 6.106. S9's answer to question 9 in the post-test

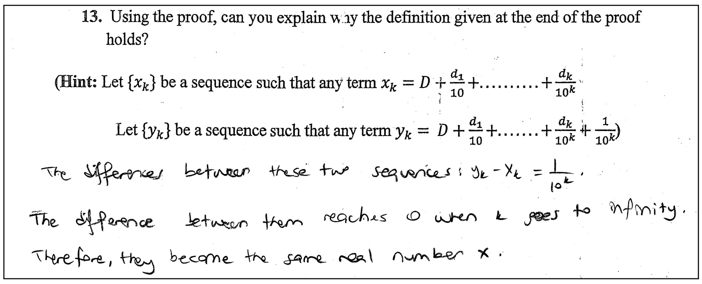
She stated that for r being zero, the terminating expansion will be obtained. In the interview she also considered the equal partitioning and grouping together with her statement r being zero.

> S9: What does it mean that r is zero? It means that the remaining part does not exist. That is as we find the decimal places, as we grouped the pieces, there is no piece out of the groups.

Here she referred to grouping as finding the decimals via diagram, As an example she thought of grouping by 7 independent of her example in the post-test. She meant that as she obtained the exact number of groups of 7, that is if there is no remaining part subjected to subdivision again, then the terminating expansion is obtained.

For her r being zero, it meant that there was no remaining part after grouping tha parts equally partitioned into 10 pieces. This showed that she reasoned through quantities.

Similarly, she could not provide any answer in the pre test for Q13. S9 gave partial answer to Q13 in the post test (Figure 6.107).



13. Using the proof, can you explain why the definition given at the end of the proof holds?

(**Hint:** Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + \dots\dots\dots + \frac{d_k}{10^k}$

Let $\{y_k\}$ be a sequence such that any term $y_k = D + \frac{d_1}{10} + \dots\dots + \frac{d_k}{10^k} + \frac{1}{10^k}$)

The differences between these two sequences: $y_k - x_k = \frac{1}{10^k}$.

The difference between them reaches 0 when k goes to infinity.

Therefore, they become the same real number x.

Figure 6.107. S9's answer to question 13 in the post-test

In the post-test her answer was limited because she just expressed the difference between sequences y_k-x_k=$1/10^k$ and explained that once k goes to infinity, the difference becomes zero which meant her that the sequences represented the same real number. Since she did not express the properties of the sequences, her answer was considered as partially correct answer.

Although S9 did not consider the properties of the sequences in the post-test, in the interview she was able to explain it:

S9: What it says in the definition. Every increasing and bounded rational number sequence represents a real number. Two sequences which have a difference approaching to zero represent the same real number. I explained it as. Because there are two sequences, the difference between them is always $1/10^n$. Of those two. That is why every increasing and bounded rational number sequence represents a real number. They meet at a real number.

R: Well, how about the x_k sequence?What is the first term of x_k?

S9: If we look at the first term of x_k, $D+d_1/10$.

R: x2?

S9: $D+d_1/10+d_2/100$. x_3 is $D+d_1/10+d_2/100+d_3/1000$. It goes like that.

R: What kind of a sequence?

S9: Increasing.

R: Why ?

S9: Increasing and bounded sequence. Because it cannot be more than x. I would say D. Between D and x.

R: What can you say about y_k? What is the first term?

S9: $D+d_1/10+1/10$. The second $D+d_1/10+d_2/100+ 1/100$. This is a decreasing sequence.

R: Why ?

S9: Yes it is... Because the value of y_1 is larger than that (y_2). Because the y1 is $D+d_1/10$ the other $D+d_1/10+d_2/100+1/100$. Yes, now let's get out of here.. if you look at them they're the same. We will compare 1/10 with $(d_2+1)/10$ and 1/10 is bigger.. This because consider, such as 10/100. $(d_2+1)/100$. d_2 was between zero and what?

R: Between 0 and 9, it can be can also 0 or 9.

S9: It could be 9. Then a greater than equal to value. So ...

R: Which one is bigger? 10/100 or $(d_2+1)/100$?

S9: That's 10/100 greater than or equal to $(d_2+1)/100$. Because our d_2 will be up to 9 , because the maximum value may be 10/100. so it is decreasing. Also it is bounded.

R: Between what and what?

S9: Here again we have x on the left. Here's what our value is the value of X? Let's think about it. D+1 is then. Between D and D+1.

R: In this case.. if you explain again the definition, Did you check the difference of the sequences?

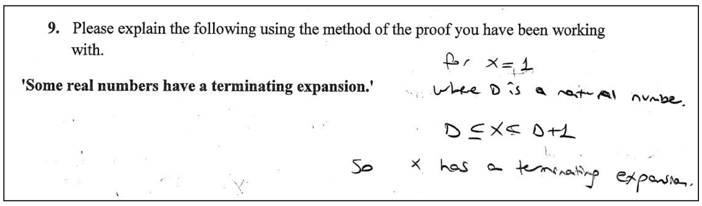
S9: Yes, in this case both of them is bounded and one is increasing and the other is decreasing. They approach to a point at the infinity. 10^k is infinite. $1/10^k$, 1/Infinity is zero. Then reached out to a real number.

First, she started with analyzing the difference between sequences $1/10^k$ when k becomes bigger and asserted that the difference becomes zero as k goes to infinity. After that the researcher asked about the properties of sequences since S9 did not mention them in the post-test. First, she stated the terms of the sequence x_k and reasoned that since its terms could be squeezed between D and x it would be bounded. Then, she moved to the other sequence y_k and followed the same steps as in the case of x_k, stating the terms, boundedness and decreasing properties of y_k. She analyzed the first two terms to obtain decreasing property. She compared 1/10 with $(d_2+1)/100$ by inferring to the property of d_2 such that it was a natural number between 0(including) and 9(including) and obtained that $(d_2+1)/100$ could be at most 10/100. Therefore, the sequence was decreasing. Since the terms of y_k were between x and D+1, she stated that y_k was also bounded. By the help of these properties, again she reached the conclusion that the difference between terms of the sequences become zero as k goes to infinity. Therefore, the sequences represented the same number, x.

To conclude,for the case of 'From no /incorrect answer to partially correct answer', S9's work was shared. From the pre-test to the post-test she showed an improvement however, her answers remained partial in the post test. During the interview, for Q9 she used a different way for explanation and thought of quantities. In Q13, she stated the properties of the sequences correctly and conclude the definition given at the end of the proof.

- Case II: No change (Partial)

S5 represents this case. In both tests she answered partially in this dimension. For Q9 she gave incorrect answers in both tests. For Q13 she gave partial answers.


9. Please explain the following using the method of the proof you have been working with.

'Some real numbers have a terminating expansion.'

for $x=1$

where D is a natural number.

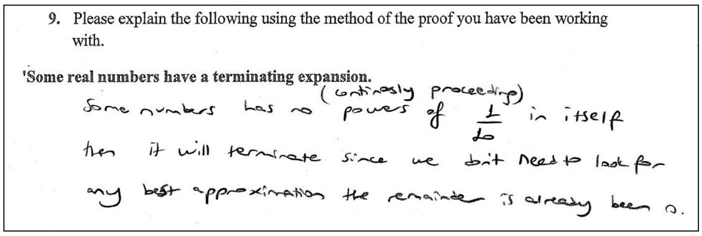
$D \leq x \leq D+1$

So x has a terminating expansion.

Figure 6.108. S5's answer to question 9 in the pre-test

In the pre-test (Figure 6.108) for the Q9 she thought that if x is between D and D+1 then the terminating expansion is obtained.However, such reasoning was not acceptable because all real numbers could be represented between two consecutive natural numbers. Therefore, her answer was considered as incorrect.

In the post test, her answer was also considered as incorrect (Figure 6.109).


9. Please explain the following using the method of the proof you have been working with.

'Some real numbers have a terminating expansion.

Some numbers has no (continously proceeding) powers of $\frac{1}{10}$ in itself then it will terminate since we don't need to look for any best approximation the remainder is already been 0.

Figure 6.109. S5's answer to question 9 in the post-test

Although she thought that the remainder was already zero when the terminating expansion occurs, since she did not consider the proof, her answer was limited. However, in the interview, she was able to answer the question thinking through the proof:

> S5: As saying terminating expansion I am thinking this: after some point for instance after 1/100 th s digit there is nothing. The digits 1/10000, 1/100000 are not written. Then we obtain terminating expansion.
> R: When you get terminating expansion what happens to x? x was written as $D+d_1/10+r_2/10$
> S5: x can be represented by $D+d_1/10$ as without having $r_2/10$.
> R: What does it mean without having $r_2/10$.
> S5: Iıı there is nothing expressed in terms of 1/10. How can I say? For example, the number is something like that 2,0003. Then, the other digits after 3 are zero. We don't need to look at the digits after 3.In the number there is no 1/10, 1/100 and 1/1000 but there is 1/10000. Then the rest is zero. We do not look at r_2's here. Sorry r_4 or r_5.
> R: If you generalize what happens?
> S5: If r_n is zero, If we expand until some step and r becomes zero after this step, then there is terminating expansion.

She remarked that if r=0 after some steps then terminating expansion occurs. She used 2.0003 as an example and expanded the number according to the decimal

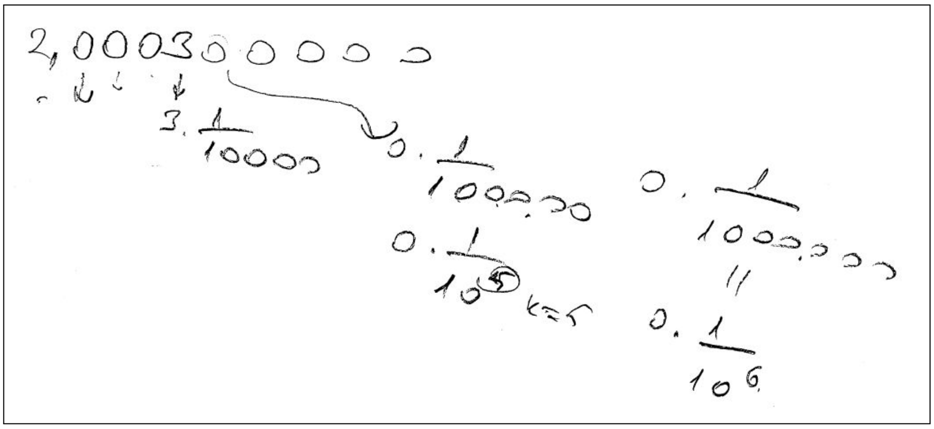
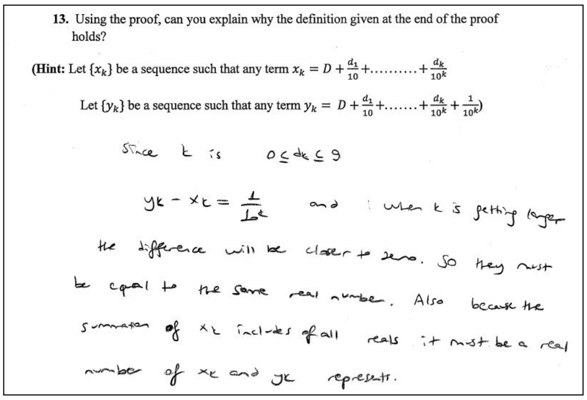


Figure 6.110. S5's representation 2

expansion as in the figure.Therefore, she added that for the example she gave r_5 becomes zero so it does not continue. She also generalized that the terminating decimal expansion occurs for any r_n equal to zero.

For Q13, S5 could give partial answer in both pre and post test (Figure 6.111 and Figure 6.112). However, there was a slight change in her answers.



13. Using the proof, can you explain why the definition given at the end of the proof holds?

(**Hint:** Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + ... + \frac{d_k}{10^k}$

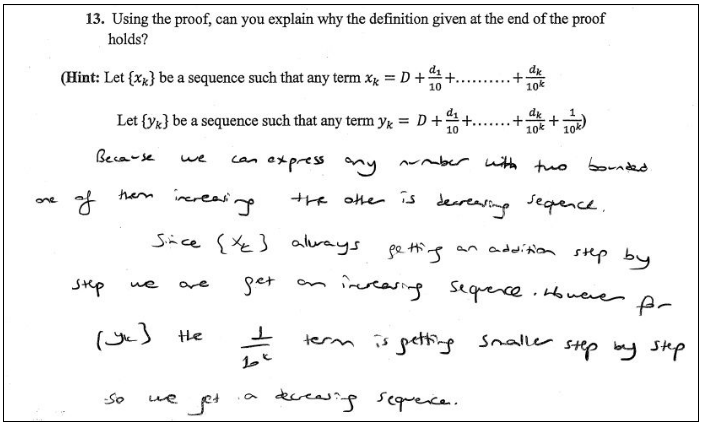
Let $\{y_k\}$ be a sequence such that any term $y_k = D + \frac{d_1}{10} + ... + \frac{d_k}{10^k} + \frac{1}{10^k}$)

Since k is $0 \leq d_k \leq 9$

$y_k - x_k = \frac{1}{10^k}$ and when k is getting larger the difference will be closer to zero. So they must be equal to the same real number. Also because the summation of x_k includes of all reals it must be a real number of x_k and y_k represents.

Figure 6.111. S5's answer to question 13 in the pre-test

For instance, in the pre-test she just focused on the difference between sequences and checked the value of $1/10^k$ when k becomes larger. However, in the post-test she slightly inferred to the sequences x_k and y_k and stated that one of them is increasing and the other one is decreasing.



13. Using the proof, can you explain why the definition given at the end of the proof holds?

(Hint: Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + \ldots\ldots\ldots + \frac{d_k}{10^k}$

Let $\{y_k\}$ be a sequence such that any term $y_k = D + \frac{d_1}{10} + \ldots\ldots + \frac{d_k}{10^k} + \frac{1}{10^k})$

Because we can express any number with two bounded one of them increasing the other is decreasing sequence.

Since $\{x_k\}$ always getting an addition step by step we are get an increasing sequence. However for $\{y_k\}$ the $\frac{1}{10^k}$ term is getting smaller step by step so we get a decreasing sequence.

Figure 6.112. S5's answer to question 13 in the post-test

In the interview, she elaborated on her answer in the post test:

R: What you said initially ..two bounded one of them increasing and the other is decreasing sequence.. which one increasing? which one decreasing?

S5: This one (x_k) is increasing because I am adding something every time. even a small number, I am adding a small number.

R: What is the first term of the sequence?

S5: D $+d_1/10$.

R:What is the second term?

S5: $D+d_1/10+d_2/10^2$

R: What you said what kind of a sequence is y_k?

S5: I said y_k is decreasing because the things added become smaller. Since we try to squeeze the real number between two numbers the number ta the right initially was big but in order to make a better approximation we make the number at the right smaller. We try to approximate the real number x better.

R: What do you think what is the first term of the sequence(y_k)?

S5: First term is $D+d_1/10$ and we need to add 1/10.

R: Second term?

S5: $D+d_1/10+d_2/100+1/100$. y_k is gradually decreasing since we try to approach the number more.

R:Ok.

S5: Now, because I'm starting to look for smaller numbers 1/10 th s 1/100 th s 1/1000 th s.

R: What is said in the definition? What does it mean two sequences which have a difference approaching to zero?

S5: As saying approaching to zero. The difference is $1/10^k$. $1/10^k$ become smaller because k becomes larger. The difference decreases and approaches to zero.

R: What does approaching to zero mean?

S5: It means that they represent one number.

R: Which number?

> S5: The number which is between them. They represent the number that we are approximating. It means that they become similar with the number x.

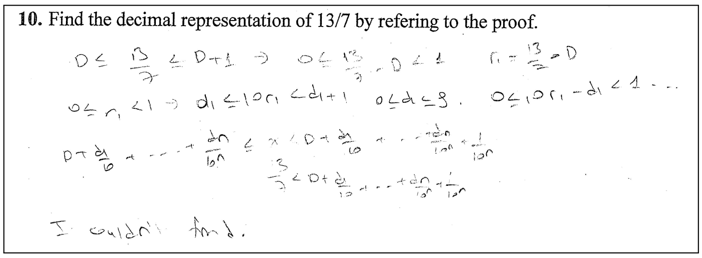
As in the case of S9, S5 was also able to give the terms of the sequences. She explained that x_k was increasing since at each step small portions were added. She also knew that y_k was decreasing because the sequence approached the number x from right. Then, she focused on the difference stating that it became zero as k becomes larger and the sequences would represent the number x.

To conclude S5's performance was considered in the case 'no change', in the pre-test and the post-test she could not answer Q9 but for Q13 she anwered partially in both cases. During the interview, she was elaborated in her answers to Q9 and Q13 and reasoned correctly.

6.2.2.6. Dimension 6: Illustrating with examples. Q10 and Q11 addressed the dimension 'Illustrating with examples'. Q10 was about the decimal representation 13/7 and Q11 was about finding an interval for $\sqrt{2}$ with bounds having 6 digits. For this dimension, as pre-test is considered, 3 of the interviewees' performance were labeled as partially correct answer, 4 of them were classified as no/incorrect answer. In the post-test, 2 students' performances were coded as correct answer and 5 of them were coded partial answer. 3 performance remained the same from the pre-test to the post-test partially correct answer. For this dimension two cases (From no/incorrect answer to correct answer and from no/incorrect answer to partially correct answer) were considered. For the case of no change, all the students only wrote correctly the interval for $\sqrt{2}$ in the Q11 but for the Q10 they proceeded the usual long division without referring to the proof. Therefore, for the case of no change, students' answers and interview excerpts were not shared. The transitions between performances from pre to post test were shared as follows:

- Case I: From no/incorrect answer to correct answer

S9's answers can be considered as representative for this case. In the pre-test for the Q10 (Figure 6.113), she tried to follow proof steps but since she could not find the corresponding D, d_1 and r_1 for 13/7, she could not proceed anymore. Additionally she wrote that 'I could not find'. This saying can tell us that she could not comprehend the proof to apply it to an example. She could not assign numerical value for the letters used in the proof.



10. Find the decimal representation of 13/7 by refering to the proof.

$D \leq \frac{13}{7} \leq D+1 \rightarrow 0 \leq \frac{13}{7} - D < 1 \quad r_1 = \frac{13}{7} - D$

$0 \leq r_1 < 1 \rightarrow d_1 \leq 10r_1 < d_1 + 1 \quad 0 \leq d \leq 9. \quad 0 \leq 10r_1 - d_1 < 1 \ldots$

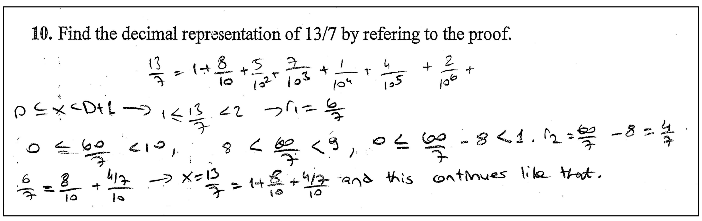
$D + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n} \leq x < D + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n} + \frac{1}{10^n}$

$\frac{13}{7} < D + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n} + \frac{1}{10^n}$

I couldn't find.

Figure 6.113. S9's answer to question 10 in the pre-test

In the post- test (Figure 6.114), compared to the pre-test performance, she could match the steps of the proof in the example and assigned the corresponding numerical values for the D d_1 and r_1 and r_2.



10. Find the decimal representation of 13/7 by refering to the proof.

$\frac{13}{7} = 1 + \frac{8}{10} + \frac{5}{10^2} + \frac{7}{10^3} + \frac{1}{10^4} + \frac{4}{10^5} + \frac{2}{10^6} + \ldots$

$D \leq x < D+1 \rightarrow 1 \leq \frac{13}{7} < 2 \rightarrow r_1 = \frac{6}{7}$

$0 \leq \frac{60}{7} < 10, \quad 8 < \frac{60}{7} < 9, \quad 0 \leq \frac{60}{7} - 8 < 1. \quad r_2 = \frac{60}{7} - 8 = \frac{4}{7}$

$\frac{6}{7} = \frac{8}{10} + \frac{4/7}{10} \rightarrow x = \frac{13}{7} = 1 + \frac{8}{10} + \frac{4/7}{10}$ and this continues like that.

Figure 6.114. S9's answer to question 10 in the post-test

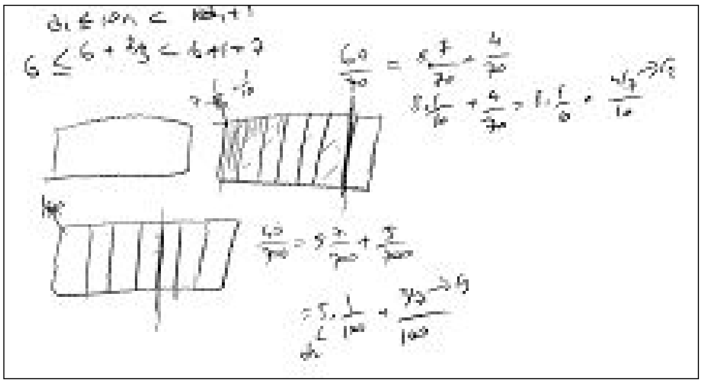
In the interview, in addition to her explanation about the answer, she also showed to find decimal representation of 13/7 as considering diagrams. The question of explaining via diagrams asked for this question because it requires application. During the interview, she correctly express her ideas via drawing diagrams. while finding the 1/10 th digit she did repartitioning on 6/7 and grouping by 7. At the end she found 8 many 1/10 and continued from the portion 4/70 to find 1/100 th s. This event shows us that she was able to use both diagrams by thinking quantitatively and proof of the statement. The below excerpt shows her use of diagrams while finding the decimal representation of 13/7.

R: Well, if you think through diagram, what can you say?

S9: Now... let's think through diagrams. 13/7. We have 1 Whole. What is left?

R: Explain by drawing, it is better.

S9:



We have 1 whole. Second one is 6/7. The remaining parts of 6/7 of us are here now.

R: OK.

S9: Then we divide each of them a piece 10. Let us say we divided into 10.

R: OK:

S9: What does one piece become. 1/70 because each are divided into 10 pieces , we have totally 70 pieces. One of them is 1/70. 7 of them will form 1/10. How many 1/10 do we have? 8 many 1/10 + 4/70. This one 8/10 +(4/7)/10. This is our r_2 (4/7).

R: Okay. Our r_2. What is d_1?

S9: Our d_1 is 8.

R: Once again, if you show one more step..

S9: 4/7 is r_2.. we will do the same this time. Again, we will divide each into 10 pieces.

R: Yes.

S9: We have 10 parts 40/70. It..

R: 40/70? The remaining part here, what was it?

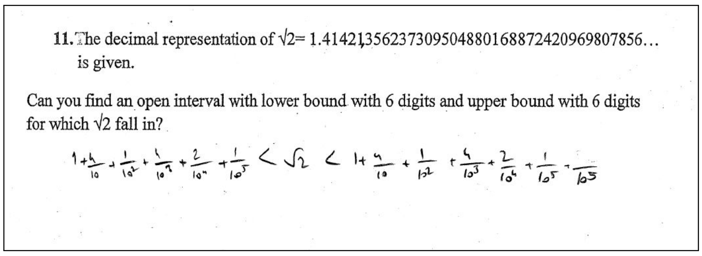
S9: OK. 40/700. Now this time each part became 1/700. Because we divided them into 10. Then we have 40/700.

R: Yes.

S9: We are searching for the number of 1/100 th s. How many? 5. 35/700. We now have 5 many 1/100 and the piece 5/7/100 remains.

R: OK. 5/7 what's here?
S9: Now our 5/7 is r_3.

For question 11, for the pre-test S9 wrote I can't find so there was no answer for this question. In the post test (Figure 6.115), she was able to determine the lower bound but in the upper bound there was some missing information that she forgot to write.


11. The decimal representation of √2= 1.414213562373095048801688724209698078 56... is given.

Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which √2 fall in?

$1+\frac{4}{10}+\frac{1}{10^2}+\frac{4}{10^3}+\frac{2}{10^4}+\frac{1}{10^5} < \sqrt{2} < 1+\frac{4}{10}+\frac{1}{10^2}+\frac{4}{10^3}+\frac{2}{10^4}+\frac{1}{10^5}+\frac{}{10^5}$

Figure 6.115. S9's answer to question 11 in the post-test

In the interview, she realized the missing part and added that the missing part should be $1/10^5$. Additionally, she explained how she thought. The below excerpt depicts her idea.

R: Let's look at question 11. In the question decimal expansion of $\sqrt{2}$ was given? It was asked to find lower and upper bounds with 6 digits.
S9: Yes. For $\sqrt{2}$, it is less than this (1.41422) but bigger than this (1.41421) Therefore, I am looking to the digit$1/10^6$. (for upper bound). I looked at $3/10^6$. I wrote here (showing the upper bound, the last added part) $1/10^5$. Because this one is bigger than this.
R: Which one?
S9: $1/10^5$ is bigger than $3/10^6$.
R: Therefore you sqeezed in this interval (showing her answer)
S9:Yes.

She considered the $1/10^6$ th digit of $\sqrt{2}$, since she need to find a value that is greater than $1/10^6$ digit. Therefore, she thought of $3/10^6$ and compared it with $1/10^5$ to determine the interval of $\sqrt{2}$. Then she concluded that $3/10^6$ is less than $1/10^5$ so she added an extra $1/10^5$ to the lower bound of $\sqrt{2}$ with 6 digits and obtained 1,41422.

To conclude, S9's performance were considered in this case. In the pre-test, she could not give correct answers to Q10 and Q11. However, in the post-test her answers turned to correct answers. Additionally, she showed that she was able to think quantitatively especially for the decimal representation of 13/7 by shifting between the proof procedure and the diagrams.

- Case II: From no /incorrect answer to Partially correct answer

S15 is the representative for this case. For the Q10 initially he used long division to find the decimal representation of 13/7 although it is expected that students should have proceeded according to the proof. Therefore, his answer was considered as incorrect.

In the post test he gave the following answer (Figure 6.116):

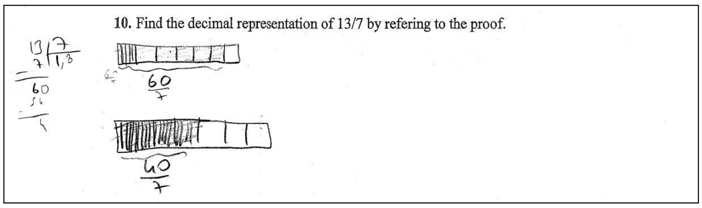


Figure 6.116. S15's answer to question 10 in the post-test

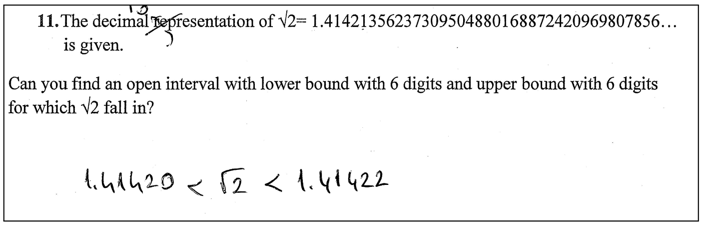
He tried to squeeze the number in an interval in one side and he also draw some diagrams looks like he did partitioning however he could not show all the steps he followed so his answer was not accepted. The intervals that he wrote were incorrect because the bounds did not represent decimal representation as considered in the proof. For instance, instead of writing the lower bound as 1+8/10+4/7, he should write it as 1+8/10 because the part 4/7 was actually the quantity of 4/70, the part that was used to find the 1/100 th s in 6/7. Besides his wrong interpretation of bounds, he could not express the decimal representation of 13/7. Therefore, his answer was not accepted.

In the interview, he first used proof as he started to explain the answer however, after some time he slightly returned to diagrams and continue thinking through quantities. At the same time at some stages he was able to illustrate the steps matching with the proof steps. The interview excerpt shows how he explained the question.

> S15: I can divide 13 by 7 in order to find D the big D. It is D=1. The remainder is 6.
>
> R: How did you follow the proof?
>
> S15: I found D as 1. Then, I said $1 \leq 13/7 < 2$. Therefore, the number is between 1 and 2.
>
> R: What is r_1?
>
> S15: r_1 is x-D. Then, r_1 is 6/7 from 13/7-1. The remainder is 6 and divisor is 7 so r_1 is 6/7.
>
> R: Ok. Then, what did you do?
>
> S15: Then, I partitioned this into 10 pieces. (Showing his drawing of 6/7). First, I shaded 6/7 by taking 6 pieces of 7 pieces. Then, I partitioned all of them (each piece of 1/7) by 10. I need to group 7 of them. Because I am dividing by 7.
>
> R: How many pieces do you have in 6/7?
>
> S15: 60 many.
>
> R: What do you do with 60 pieces?
>
> S15: I will group them by 7.
>
> R: Then, what do you find?
>
> S15: The number of 1/10 th s.
>
> R: The operation you did matches which part of the proof?
>
> S15:It matches with the 4 th step. I looked at 10 times 6/7.
>
> R: What will be d_1?
>
> S15: The results when I divide 60 by 7. It should be 8. d_1 is 8. Then the remainder is 4. When I grouped 7 of them 4 pieces remain.
>
> R: What is the amount of 4 pieces in the whole piece?
>
> S15: 4/70. Then, I partitioned each piece (each piece of 4/70) again by 10. Then the amount become 40/700. I can write it like this 40/700=40/7*1/100. I am looking for the number of 1/100 th s. I will divide 40 by 7 and find d_2. It is 5. and r_3 is 5/700. Sorry r_3 is 5/7.

After shifting to the diagram he divided each piece of 6/7 to 10 pieces. Then, he grouped 7 pieces in 60/70 and stated that he found the number of 1/10's in 6/7 as 8. However, there is the remaining portion of 4/70. Later, he continued from that portion and did the re-partitioning. First he measured 6/7 with 1/70's. Then he measured 4/70 with 1/700's. In some stages he was also able to follow the proof. For instance, determining the number of 1/10's he also follow the proof and stated that now he is in step 4. It seems that he was able to combine proof and diagram.

For the Q11, in the pre-test S15 directly tried to write the bounds (Figure 6.117):


11. The decimal representation of √2= 1.41421356237309504880168872420969807856…
is given.

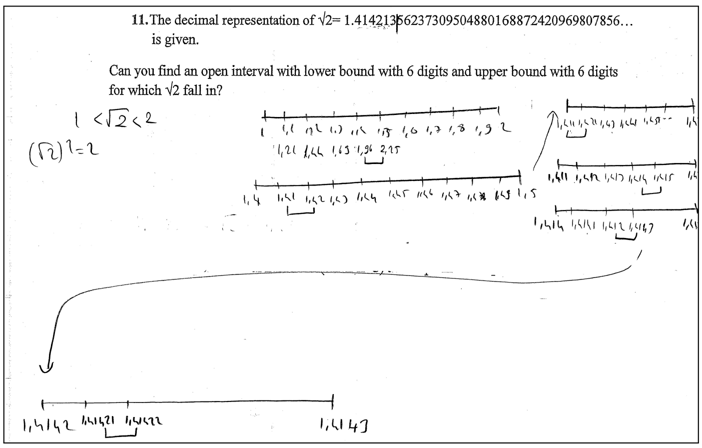
Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which √2 fall in?

1.41420 < √2 < 1.41422

Figure 6.117. S15's answer to question 11 in the pre-test

For the lower he did not consider the number of digits. It is true that $\sqrt{2}$ is between the numbers that he wrote as bounds. However, for this question just one answer is considered as correct which represents the smallest interval with bounds having 6 digits. The interval expressed by the student was not the smallest interval that can be written.

In the post-test he tried to write the interval step by step and explained his procedure in the interview (Figure 6.118):


11. The decimal representation of √2= 1.41421356237309504880168872420969807856…
is given.

Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which √2 fall in?

1 < √2 < 2

(√2)² = 2

Figure 6.118. S15's answer to question 11 in the post-test

S15: The first thing I did, I checked between which numbers is $\sqrt{2}$ located. between$\sqrt{1}$ and $\sqrt{4}$ I said. I took the square each one. Then I said that is between 1 and 2. Then I divided into the interval of 1 and 2 into 10 equal parts. 1.1 1.2. I took its square. square of $\sqrt{}$ 2 is 2. I checked the place of 2, I mean in which interval it stays. I found that2 is between square of 1.5 and

> square of 1.4 appeared. Therefore, $\sqrt{2}$ is between 1.4 and 1.5. I divided the interval of 1.4 and 1.5 into 10 equal pieces. 1.41 1.42 the interval goes like this. $\sqrt{2}$ is between 1.41 and 1.42. Then I opened it again. (Divided into 10 equal pieces). I took the squares. Between 1.414 1.415 with this appeared. (Then again, equal-partitioning, and doing the same procedure to find 6-digit) 6 digits in this way I continued to say.

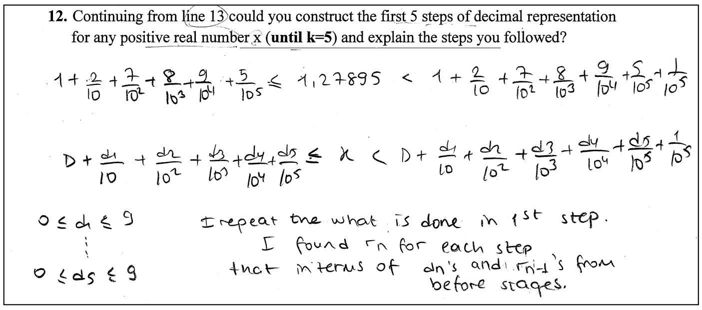
He did partitioning and he found the interval for $\sqrt{2}$ at the end of each partition. In this way, he shrank the number $\sqrt{2}$. In the interview except he explained his thinking process and gave an emphasis on repartitioning the last interval by ten to find a new interval for $\sqrt{2}$. He actually followed the idea discussed in the instruction period for the case of irrational numbers.

To conclude, the case of 'from no /incorrect answer to partially correct answer' was illustrated with S15's performances in Q10 and Q11. In the pre-test he could not answer the questions correctly but in the post-test he answered Q11 correctly as thinking differently compared to the other students. He was able to find the interval of $\sqrt{2}$ as squeezing the number and narrowing down the intervals. Therefore, he did re-partitioning in the number line. During the interview, for the Q10, he shifted between the proof and the diagram to explain the decimal representation of 13/7.

6.2.2.7. Dimension 7: Identifying the modular structure. Q12 addressed the dimension 'Identifying the modular structure', asking to proceed the proof until k=5. For this dimension, as the pre-test is considered, 2 of the interviewees' performance were labeled as correct answer, 5 of them were classified as no/incorrect answer. In the post-test, 4 students' performances were coded as correct answer. However, 3 performance remained the same from the pre-test to the post-test no/incorrect answer. For this dimension only one case (From no/incorrect answer to correct answer) were shared. The other case (no change) was not shared because the students' correct answers were similar to the example given in the following case and some of the student gave incorrect answers because they just wrote the intervals and did not express how the transition between intervals occurred.

- Case I: From no/incorrect answer to correct answer

S7 answered in the pre-test wrongly (Figure 6.119). She just wrote the interval for k=5 and stated that she repeated the operations done in step 1. However, since she did not state how she found r_n in each step, her answer was not completed. This answer did not show her comprehension about the construction of the other steps of the proof.



12. Continuing from line 13 could you construct the first 5 steps of decimal representation for any positive real number x **(until k=5)** and explain the steps you followed?

$1+\frac{2}{10}+\frac{7}{10^2}+\frac{8}{10^3}+\frac{9}{10^4}+\frac{5}{10^5}\le 1{,}27895 < 1+\frac{2}{10}+\frac{7}{10^2}+\frac{8}{10^3}+\frac{9}{10^4}+\frac{5}{10^5}+\frac{1}{10^5}$

$D+\frac{d_1}{10}+\frac{d_2}{10^2}+\frac{d_3}{10^3}+\frac{d_4}{10^4}+\frac{d_5}{10^5}\le x < D+\frac{d_1}{10}+\frac{d_2}{10^2}+\frac{d_3}{10^3}+\frac{d_4}{10^4}+\frac{d_5}{10^5}+\frac{1}{10^5}$

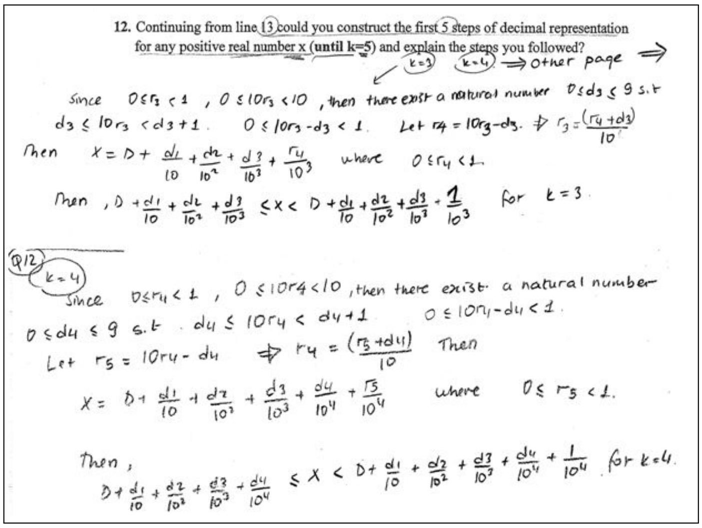
$0\le d_1\le 9$
$\vdots$
$0\le d_5\le 9$

I repeat the what is done in 1st step.
I found r_n for each step
that in terms of d_n's and r_{n-1}'s from before stages.

Figure 6.119. S7's answer to question 12 in the post-test

On the other hand, in the post test, as in the case of S9, she also constructed each step by deducing r_n's and d_n's (Figure 6.120). Her answer in the post test was supported by the interview that she explained the steps she followed and how she got the numbers r_n's and d_n's and combined them.

> S7:Now I'm getting a r_3 between 0 and 1. When I multiply r_3 with 10, the number is between zero and 10 that is what I'm saying. and I can find a number between 0(including) and 9(including) such that $10r_3$ falls between d_3 and d_3+1.
> R: Yes.
> S7: Then I subtracted d_3 from both sides of the inequality and got $0 \le 10r_3-d_3<1$. Then I named the expression as $r_4=10r_3-d_3$. Then again I wrote r_3 is in the form of remaining part and decimal part as $(r_4+d_3)/10$.
> R: You expressed r_3 in terms of d_3 and r_4?
> S7: Exactly.
> R: OK.
> S7: This is the whole part and then comes the remaining part. Then X is equal to. $D+d_1/10+d_2/10^2$. Here, for $r_3/10^2$ I wrote $(d_3+r_4)/10$.As a result, any remaining r_3, I found ,it was smaller than the other. $d_3/10^3+r_4/10^3$.
> R: Does it have to be smaller than r_3?
> S7: When I say small I mean not numerically, but smaller in this way $(r_4/10^3)$.



12. Continuing from line 13 could you construct the first 5 steps of decimal representation for any positive real number x (until k=5) and explain the steps you followed?

k=3   k=4 $\Rightarrow$ other page $\Rightarrow$

Since $0 \le r_3 < 1$, $0 \le 10r_3 < 10$, then there exist a natural number $0 \le d_3 \le 9$ s.t $d_3 \le 10r_3 < d_3+1$. $0 \le 10r_3 - d_3 < 1$. Let $r_4 = 10r_3 - d_3$. $\Rightarrow r_3 = \frac{(r_4+d_3)}{10}$

Then $x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{r_4}{10^3}$ where $0 \le r_4 < 1$.

Then, $D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \le x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{1}{10^3}$ for $k=3$.

Q12   k=4

Since $0 \le r_4 < 1$, $0 \le 10r_4 < 10$, then there exist a natural number $0 \le d_4 \le 9$ s.t. $d_4 \le 10r_4 < d_4+1$ $0 \le 10r_4 - d_4 < 1$.

Let $r_5 = 10r_4 - d_4$ $\Rightarrow r_4 = \frac{(r_5+d_4)}{10}$ Then

$x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \frac{r_5}{10^4}$ where $0 \le r_5 < 1$.

Then,

$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} \le x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \frac{1}{10^4}$ for $k=4$.

Figure 6.120. S9's answer to question 12 in the post-test

R: As a quantity.
S7: The quantity is smaller. Here, too, that I have shown that r_4 is between zero and 1.

In the interview she differently compared r_3 and r_4 and as a quantity she concluded that r_4 represents a smaller quantity compared to r_3.

To conclude, although S7 could not answer Q12 in the pre-test, she was able to give correct answer in the post-test and also explain her answer in the interview. Therefore, the data showed that she was aware of how the proof continues. When all the data from the interviews were considered, it can be said that students were able to explain their answers in the post-test. Even in the dimensions 1, 2, 5 and 6, students were able to think through quantities and support their answers through them in the post-test. There were also some important cases. Although some students could not give correct answers in the post-test, they were able to give correct reasoning during the interview. Therefore, this may indicate that students' comprehension of the given proof was evolved. In the next section, the results will be discussed under the light of the literature.

7. CONCLUSION AND DISCUSSION

This research study investigated prospective mathematics teachers (N=19)'s quantitative reasoning while developing the decimal representation of real numbers and its effect on their performance on a proof comprehension test about the proof of a statement on decimal representation of real numbers. Since one aspect of the study was related to prospective teachers' quantitative reasoning, the researcher focused on the data that revealed their reasoning through quantities both in teaching sessions and interviews. The other aspect of the study was about prospective teachers' performance on a proof comprehension test. Therefore, the researcher focused both on the test results from the pre and post test and the interviewees' reasoning during the interviews.

In this regard, in this chapter, the results of the research study will be discussed in two parts taking into consideration the previous research results. In the first part the teaching of decimal representation of real numbers based on quantitative reasoning was addressed. Then, conclusions related to the results from the students' proof comprehension performances in both the pre and post test and the interviews were discussed.

7.1. Decimal Representation of Real Number

Voskoglou and Kosyvas (2012) argued that the difficulties related to the transition from the set of rational numbers to the set of real numbers are connected to students' incomplete understanding of rational number and irrational numbers. They further argued that the incomplete understanding of rational numbers affects students' comprehension of irrational numbers. Incomplete understanding of rational number could be due to students' thinking of different representations of a rational number representing the different numbers (Khoury and Zarkis, 1994; O'Connor, 2001) such that they might not be able to relate decimals and fractions to each other (Voskoglou and Kosyvas 2012). That is, students might think that the decimal representation was another representation of a rational number, rather than representing the same number (Voskoglou and Kosyvas 2012). For instance, students had a difficulty in understanding the multiple representations such as 1/2=2/4=...=5/10 (Voskoglou and Kosyvas 2012). In this regard, "multiple representations of real numbers (rational numbers written as fractions and periodic decimals, irrational numbers considered as non rational ones and as incommensurable decimals which are limits of sequences of rational numbers, geometric representations, etc) and on flexible transformations among them" (Voskoglou, 2013, p.41) was suggested as a way to teach irrational numbers. By the same token Schwanke (2008) argued that division algorithm leads students to relate decimal representation and fractions. Under the light of these findings, the instruction focused on quantitative reasoning (Thompson, 1994) in the following way: In the Phase 1 of instruction, students were asked to think about positive rational number examples, i.e fractions, representing amounts as quantities: Lee (2012) explained "a quantitative approach to fractions means that fractions are treated as lengths. For example, thinking about 3/4 does not mean solely a symbol of three over four, but rather taking three equal parts out of four equal parts in a rectangle or segment" (p.2). Similarly, Smith III and Thompson (2007) claimed "because the central goal is to focus on quantities and how they relate in situations, and because this represents a major obstacle for many students, it is important for teachers to open discussions with questions that lead to discussions of quantities, not numbers" (p.36). Based on the aforementioned reasons, in this study students were asked to use rectangular diagrams to express the fractions such as 7/2, 11/3 and 8/7 so that these rational numbers would represent an amount rather than just a symbol for them (Lee, 2012). Similarly, thinking through fractions as quantities via diagrams and relating them to the long division (Usiskin *et al.*, 2003), their awareness of the fact that decimals represented the same number as fractions were emerged.

In fact, the results from the pre-test for Q1-ii for which 78,9% of the students answered incorrectly showed that they did not know that the numerator of r_1 represented the remainder in long division. Similar to the earlier research results (Voskoglou and

Kosyvas 2012, Khoury and Zarkis, 1994; O'Connor, 2001) this suggested that most of the students could not relate the rational number example with its decimal representation. This was also evident in the pre-test results from the Q10 for which 89,5% of students' answers relating 13/7 with its decimal representation was incorrect. Similarly at the beginning of the teaching session, none of the students was able to explain the reasoning behind continuing the long division of 11/3 by the addition of comma next to the quotient and zero next to the remainder. Some even stated that they had not thought of the reason at all. All these suggested that these students were not able to relate fractions and decimals (Schwanke, 2008).

However, data from the teaching sessions showed that thinking through quantities depicted in diagrams, students were able to justify the reasoning behind the addition of comma next to the quotient and zero next to the remainder. This in turn enabled them to represent fractions in decimal form. In any of the examples such as 7/2, 11/3 and 8/7 students' quantitative reasoning evolved in the following way: While thinking through diagrams, calling on their knowledge of equivalent fractions which was activated at the beginning of the instruction, they reasoned that the long division could be continued by getting equivalent fractions. In particular, during the discussion of equivalent fractions, students mostly explained by multiplying or dividing the numerator and the denominator of the fraction by the same number they could obtain equivalent fractions. When they were asked to show their reasoning using a diagram, they expressed the equivalent fractions using two separate diagrams but they knew that equivalent fractions represented the same quantity. Then, they were suggested to consider the equivalent fractions in a single diagram. This enabled them to do repartitioning in the same diagram. This was important because using the shaded part as the remainder after discarding the whole parts of the fractions, they were able to re-partition each piece of the remainder in to 10 equal pieces. They knew that they needed to find the 1/10 th digit because they did not have any whole number of groups of divisor in the dividend anymore. Depending on the divisor, in order to find the number of 1/10 th s in the shaded part as the remainder, they also grouped each equal piece in the remainder repartitioned into 10 equal pieces. After grouping and counting the number of 1/10 th s, students also determined what was left in the remainder. They knew that the left over part in the remainder meant that they did not have whole number of groups of 1/10 th s in the remainder. Therefore, they realized that they needed to find the number of 1/100 th s in the left over part in the remainder. By re-partitioning the left over part in the remainder into 10 more equal pieces getting 1/100 th digit, they were able to group and count the number of 1/100 th s. In this way students were able to relate the long division with the diagrams in such a way that they represented the fraction and its decimal representation in a single diagram. Relating long division with the diagrams through the actions of equal partitioning, grouping and counting led them to think that the different representations of rational numbers, i.e. the fractional and decimal representations, corresponded to the same number (the same quantity).

After students relating fractions and periodic decimals through diagrams in Phase 1 of the instruction, as mentioned earlier, based upon the suggestions by Voskoglou (2013), prior to students' considering irrational numbers as non rational ones and as incommensurable decimals which are limits of sequences of rational numbers (Voskoglou, 2013), in Phase 2 of instruction, students reasoned on positive rational numbers i.e fractions, by squeezing them in two rational number sequences. This was important because one of the common confusions among students was that different decimals could not correspond to the same real number (Tall and Schwarzenberger, 1978).

Data from the teaching sessions showed that students developed the idea that any positive rational number could be squeezed and represented by rational number sequences based upon what they currently knew. They were able to justify the reasoning behind the fact that any positive rational number could be squeezed and represented by different rational number sequences. Not only that but also their realization of the irrational numbers as limits of sequences of rational numbers emerged. Students' reasoning evolved in the following way: They first represented the positive rational number (i.e 11/3) in an interval considering decimal representations of it. That is first students found upper and lower bounds as squeezing the number according to the 1/10 th digit of the decimal representation of it and then 1/100 th digit of the decimal representation of it. While finding the lower bounds students just discarded the remaining part ($r_k/10^{k+1}$). While finding the upper bounds students used a different approach. For instance, as squeezing the number up to the 1/10 th digit, students agreed upon the idea that the remainder in the decimal form could not complete itself to 1/10 because they knew that it did not include whole number groups of 1/10 th anymore. This suggested that while reasoning about the upper and lower bounds they called upon their construction of the decimal representation of rational numbers using diagrams through quantitative reasoning. In this way, they expressed the number in nested intervals up to $1/10^k$ th digit of the decimal representation of it. Later they analyzed the lower bounds and upper bounds of the intervals as terms of the sequences. By comparing the differences between each term of the sequences they approximated the decimal representation of the number such that they concluded that as the number of the terms of sequences increases, both sequences which formed by lower bounds and upper bounds tend to approach the same number (i.e. 11/3). Upon the prompt by the researcher if they were able to generalize their findings for any positive rational number x, they extended their findings to any positive real number using the concepts of sequences and limit. They reasoned that they could represent any rational number as decimals using diagrams. Knowing this they could squeeze such decimal representation nested intervals up to $1/10^k$ digit. Comparing the differences between the terms of the sequences, they concluded that the difference approaches to zero which meant for them that these sequences represented the same positive rational number x. Upon their consideration of different rational number sequences representing the same rational number, they investigated a case of irrational number in Part 3 of the phase 2 of the instruction.

Similarly, in order to find the decimal representation of an irrational number, they initially took the interval of two consecutive numbers and partitioned that interval into 10 equal pieces. At each finding of an interval for the irrational number, they again did partitioning in the current interval. This showed that students transformed their reasoning in the case of rational number to the case of irrational number. That is, they accomplished the development of the decimal representation of irrational numbers via partitioning of the intervals. After they analyzed the lower and upper bounds as sequences, they came up with the same conclusion that the irrational numbers could also be represented by rational number sequences.

In this regard, results from this study showed that reasoning on quantities depicted in diagrams, once students related long division with multiple representations of rational numbers such as fractions, equivalent fractions and decimals through the actions of equal partitioning, grouping and counting, they were able to deduce that all these representations corresponded to the same number (Schwanke, 2008; Voskoglou and Kosyvas, 2012). Similarly, squeezing the decimal representation of positive rational numbers, they were able to deduce that rational number could be represented by rational number sequences. Also, through partitioning and re-partitioning of the intervals an example of an irrational number could be squeezed into, they were able to think of the bounds of the intervals as rational number sequences. Therefore, they could conclude that not only rational numbers but also irrational numbers could be thought of as the limits of sequences of rational numbers (Voskoglou, 2013). In this way, they were able to think that different decimals could correspond to the same real number (Tall and Schwarzenberger, 1978).

7.2. Proof Comprehension

This section will first summarize the findings of quantitative and qualitative data. Then, pointing to the previous research on students' difficulties in proving and the effect of using examples on students' comprehension of proof, possible reasons on the effect of students' quantitative reasoning for their performance on the proof comprehension test will be elaborated.

In this study, students' performances of proof comprehension about decimal representation of real numbers were assessed before and after the instruction via proof comprehension test which included 7 dimensions of proof comprehension (Mejia-Ramos et al., 2012). The results showed that there was a significant difference between students' performances in the pre-test and the post-test. The percentages of partially correct answers and incorrect answers decreased from the pre-test to the post-test. Also, the percentages of correct answers increased from the pre-test to the post-test. Since there was an increase from the pre-test to the post-test for all the proof comprehension dimensions, it can be concluded that students' performances increased from the pre-test to the post-test.

When the results from the pre-test and the post-test were coded for 6 students, who were interviewed after the post-test, as no/incorrect answer, partially correct answer and correct answer, 5 different transitions from the pre-test to the post-test occurred: No answer → Partially correct answer, Partially correct answer → No answer, No answer → Correct answer, Partially correct answer → Correct answer, No change. However, all the students whose answers were categorized as Partially correct answer → No answer and No change, were able to reason correctly during the interview. These results together with the data from the interviews also indicated that students' performance increased from the pre-test to the post-test.

The development of students' performances from the pre-test to the post-test on all of the dimensions can be associated with several reasons. Firstly, as suggested by Ball *et al.* (2003) for the teaching of proof, during the instruction, students deduced mathematical expressions through examining some examples. They used different representations such as fractions, diagrams and long-division for the decimal representation of real numbers and were able to shift between representations. Also, they drew diagrams and thought through them to obtain generalizations. Therefore, students' activities during the instruction might help them to comprehend the proof about the decimal representation of real numbers.

Secondly, the development of students' performances could be due to the elimination some of the reasons for the difficulties in proving. Previous research showed that one source of students' difficulties in proving regarded their lack of knowledge about definitions of terms and statements and how to use them in the proof (Moore, 1994). Similarly, Samkoff and Weber (2015) asserted that understanding the definitions of all the terms in the proof might contribute to the students' proof comprehension. In this study, based on the suggestions by Samkoff and Weber (2015) students focused on not only the definitions of some mathematical terms such as rational numbers, fractions, equivalent fractions, functions, sequences, bounded sequences but also their relations to each other. For instance, fractions as quantities representing amounts are considered as positive rational numbers. By the same token, rational number sequences are considered as functions from which real numbers were formed. In the proof comprehension test, examining the meaning of the terms and statements (Dimension 1-Q2), results showed that percentages of both partially correct answers and correct answers were increased from the pre-test to the post-test. Additionally, although for Q13 (Dimension 5) in the post-test, students could not give totally correct answers, more than half of the students were able to explain how the definition might be derived from the proof with the use of Q2. Also, during the interview, students were able to elaborate on their ideas regarding Q13. For instance, although they did not mention the characteristics of the sequences given in Q13 in the interview they were able to explain the characteristics of the sequences and supported their answers. Apart from definitions of mathematical terms, students also dealt with the meaning of the notations used in the proof. In Q1 (Dimension 1), students were asked to give rational and irrational examples about the value of r_1 in the proof and its components for the case of a rational number. Although in the pre-test, the majority of the students gave incorrect answer to the Q1, in the post-test more than 70% of the students were able to answer the questions correctly. Contrary to the pre-test results, students were able to reason that the numerator and the denominator of r_1 for the case of rational number referred to the remainder and the dividend respectively. Also they knew that for the irrational number case, r_1 referred to the number after the integer part. Therefore, they were able to reason on the theorem. This finding suggested that the emphasis on the concepts and definition of mathematical terms and notations during the instruction might have contributed to the improvement of students' performance from the pre-test to the post-test.

Moore (1994) pointed to a second difficulty associated with proving: the lack of understanding about the concepts. In particular, research suggested that, in order to construct irrational numbers, students should have the understanding of rational numbers including multiple representations of them (Voskoglou and Kosyvas, 2012) such as the relationship between fractional and decimal representations (Weller *et al.*, 2009). During the instruction, students reasoned on rational numbers through diagrams such that they were able to relate decimal representation with fractions via long division. They also reasoned on the nested intervals which were determined by the decimal representation of the rational numbers and defined the decimal representation by rational number sequences formed by the bounds of nested intervals. They further proceeded to irrational numbers defining them as the limits of the rational number sequences from which they could deduce real numbers. In this way they were able to generalize their findings from positive rational numbers to positive irrational numbers and hence to positive real numbers. This might have contributed to the increase in their performances on the dimension 5-Q13 from the pre-test to the post-test. Particularly, the percentage of partially correct answers in the pre-test was 26,3% though the percentage of partially correct answers in the post-test was 57,9%. Yet, none of the students answered this question correctly. The reason was that students did not explain the properties of sequences neither in the pre-test nor in the post-test. However, when elaborated on their answers during the interview, all the students emphasized the properties of the sequences and supported how the definition holds.

Another difficulty associated with proving was reported as the failure of determining the method of proof (Stylianides *et al.*, 2004; Stylianides *et al.*, 2005; Güler and Dikici, 2014). Results of this study supported earlier research findings. For the Q5 (Dimension 3), assessing the method of proof, students either wrote direct proof or mathematical induction. Although when the correct and partially correct percentages were compared in itself, there was an increase in the correct answer percentages and a decrease in partially correct percentages, the total percentage of correct answers and partially correct answers were remained the same from the pre-test to the post-test. The reason for the increase in correct answers could be that students who chose the correct method as direct proof was able to explain their reasoning in the post-test. On the other hand, the percentages of incorrect answers remained the same from the pre-test to the post-test. However, during the interview students who wrote the method as mathematical induction showed a hesitation. They were aware of the steps of the mathematical induction such that there was no inductive step in the proof. However, since the proof was constructed up to n th step, they were not able to determine the proof as direct proof. On the contrary, students choosing the method of proof as direct proof correctly stated that the method was not mathematical induction since there was no inductive step. Students' difficulty in determining the method of proof might also have affected their answers to Dimension 3-Q6, Dimension 4-Q7 and Q8. These questions were about the general structure of the proof as in the Dimension 3-Q5. Although there was an increase in students' performances to these questions from the pre-test to the pos-test, still the increase was not much high.

The other aspect of students' difficulties in proving was reported as the failure to generate or use students' own examples (Moore,1994; Raman, 2003; Weber, 2001). Samkoff and Weber, (2015) suggested that illustrating the theorem with an example might trigger students' proof comprehension. Also, some other studies indicated that examples can be used as a transition to proof because it may help students' to relate their ideas with verbal/symbolic representations (Gibson, 1998). For instance, in the studies of Movshovitz-Hadar (1997) and Malek and Movshovitz-Hadar (2011), students were given transparent proofs the proof of a particular case describing with an example. Results of these studies showed that transparent proofs had a positive impact on students' proving and proof comprehension performances. Also some other studies done with mathematicians' about their use of examples while proving suggested that examples can be beneficial for students to make them critically think and explore the proof (Lockwood *et al.*, 2016) and develop deductive arguments (Ellis *et al.*, 2013). In this study the examples investigated during the instruction might have had an effect on students' performances. While dealing with rational numbers and irrational numbers via examples students were able to make a generalization about an alternative definition of real numbers. Therefore, examples might have led them to reason on the connections between the steps of the proof and how the definition in the proof was reached.

However, how examples are used has an important role (Stylianides *et al.*, 2016). In this study, the main focus was on quantitative reasoning. Results showed that both during the teaching sessions and the post-test, students did not think about the examples as just numerical entities but they attributed them as quantities thinking through diagrams. It can be concluded that through quantitative reasoning students were able to make sense of the proof. During the teaching sessions, students did repartitioning in the same diagram considering the positive rational numbers as amounts. In particular, while doing long division, representing the remainder through diagrams, they were able to re-partition each piece of the remainder in to ten equal pieces. They knew that they needed to find the 1/10 th digit because they did not have any whole number of groups of divisor in the dividend anymore. Depending on the divisor, in order to find the number of 1/10 th s in the shaded part as the remainder, they also grouped each equal piece in the remainder repartitioned into 10 equal pieces. After grouping and counting the number of 1/10 th s, students also determined what was left in the remainder. They knew that the left over part in the remainder meant that they did not have whole number of groups of 1/10 th s in the remainder. Therefore, they realized that they needed to find the number of 1/100 th s in the left over part in the remainder. By re-partitioning the left over part in the remainder into 10 more equal pieces getting 1/100 th digit, they were able to group and count the number of 1/100 th s. Relating long division with the diagrams through the actions of equal partitioning, grouping and counting led them to think that the different representations of rational numbers, i.e the fractional and decimal representations, corresponded to the same number (the same quantity). In the post-test, the effect of students' quantitative reasoning was also evident. Students had a tendency to answer questions using diagrams and thinking through quantities. For instance, for Q4, Q10, Q11 which were related to the application of the proof, students used quantities in the post-test as giving correct answers whereas more than 70% of the students could not give correct answers in the pre-test.

Also during the interview, there were instances that students used quantitative reasoning for explaining their answers for dimension 1 (Meaning of the terms and statements), dimension 2 (Justification of claims), dimension 5 (General method) and dimension 6 (Applications to examples).

For example, in dimension 1 for Q1, a student explained the function of r_1 as saying that it was the remaining portion and he was able to infer the values d_1 and r_2 using r_1. At this stage he turned to the diagrams focusing on quantities via equal partitioning and grouping, he was able to explain how he obtained r_1 and how the process continued using r_1. His explanations suggested that since he interiorized the meaning of r_1, he was able to proceed to explain the procedure even it was not asked.

Another example was that, in Dimension 2, for Q4 in the post test, a student used quantities to explain the inequality, $d_1 \leq 10r_1 < d_1+1$ and justified his reasoning during the interview. Although he did not write any number in the answer sheet but used notations, it was evident from his drawing that while answering the question, he used re-partitioning and grouping. During the interview, he explained the inequality referring to the example of 10/3. He considered the $r_1=1/3$ and did equal-partitioning and grouping by 3 in the left over 1/3. Then, he turned to the notations and explained how both sides of the inequality held. His reasoning through quantities helped him to give meaning to what d_1 and r_1 were. Therefore, by comparing the quantities rather than thinking of them as solely numbers or notations, he was able to reason why the inequality held. He attributed quantities to mathematical notations.

Also again in the same dimension for Q3, although a student answered by referring to the proof in the pre-test and the post-test as expected, during the interview he elaborated his answer by thinking through diagrams and quantities which helped him to explain the meaning of the inequalities given in the proof. He took the example of 10/7 as considering a quantity he represented it in a diagram. Later he did re-partitioning and grouping activity in the remaining part 6/7 to find the decimals. Finding decimals helped him to define intervals for 13/7. By using the remaining portions after finding 1/10 th digit, 1/100 th digit and further digits, he was able to justify why the inequality asked in Q3 holds as explaining that the remaining portion meant that there were no more 1/10 th s, no more 1/100 th s. From the example, he generalized his finding to any number focusing on the same reasoning as he thought through quantities.

For Q9 in dimension 5, which asked about the terminating decimal expansion referring to the proof, a student in the interview thought about the terminating decimal representation referring to equal partitioning and grouping but in her answer in the post-test she clang to the proof which was also expected. It can be concluded that by both using the quantitative reasoning through and proof, she was able to deduce the meaning of $\mathrm{r_k}/10^{\mathrm{k}+1}$. One other think to infer was that she was able to make sense of the proof by the reasoning through diagrams quantitatively.

Another example about the use of quantitative reasoning occurred in Dimension 6 for Q10 and Q11. Although for Q10 the percentage of correct answer was not high, some students used multiple representations to explain their reasoning referring to the proof. Similarly, for Q11 (Dimension 6), some students reasoned via partitioning. one of the students gave a different solution about representing $\sqrt{2}$ in an interval which has bounds of digit. He was able to write the bounds of $\sqrt{2}$. His writing stemmed from thinking through quantities because he was able to transfer the reasoning in the case of rational numbers to irrational case. He was able to write nested intervals for each case of the decimal digits as proceeded in the proof.

Therefore, quantitative reasoning skills might help students to grasp the idea of the proof and internalize the process of proving. This alligns with Weber, and his colleagues such that quantitative reasoning was essential for developing students' thinking processes on proof. Also, Ellis (2007) argued that students who focused on generalizations by quantitative reasoning could perform better in constructing proofs than students who focus on empirical examples as obtaining generalizations. Therefore, alligned with the earlier research results (Weber et.al, 2014), this research showed that students were able to make generalizations through quantities and even explain the abstract expressions in the proof comprehension test. The results from quantitative data along with the qualitative data suggest that quantitative reasoning might develop students' thinking processes on proof comprehension.

7.3. Limitations and Implications for Future Research and Teaching

Although the results showed that students performances improved in the post-proof comprehension test compared to the pre- proof comprehension test, there are several limitations of the study. First, the research was pursued with only one experimental group. There were no control group due to the number of participants. Because of this reason, the results were supported with qualitative data. However, if the study were conducted with both experimental and control groups, the results could have been more accurate and the results could have been generalizable to a large scale group. Therefore, a further study might be done with experimental and control groups with a larger number of participants. Also, since the research studies about the quantitative reasoning and proof comprehension relationship are scarce, there is a need to further study this relationship. In particular, the current study was done focusing on decimal representation of real numbers. Further research investigating the effect of students' quantitative reasoning on both proving and proof comprehension could be done on different topics. In this way a general view about the connections between quantitative reasoning and proof comprehension could be determined.

Another limitation was that the interventions was held after 5 p.m. on the weekdays. That is, students participated in the study after their regular courses finished. Therefore, students could have been tired and had a difficulty in following the instruction. This might have affected their performance during the instruction and during the post-test, too. Though, the results showed a significant difference between the pre-test and the post-test. In this regard, I suggest that teachers and teacher educators could integrate quantitative reasoning and proof comprehension in teaching mathematics courses. Also, the teaching of decimal representation of real numbers in this study could be a guide to both teachers and mathematics teacher educators. The instruction stimulated students to think through different concepts and relate them such that they were able to come up with a generalization. This teaching strategy might shed light on teachers' teaching of rational numbers, fractions, decimal representations and real numbers and how to relate them to each other.

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APPENDIX A: STATEMENT AND PROOF

In this part, the statement about the decimal representation of real numbers and its proof which were used in this study were shared.

Statement: Prove that if x is a positive real number, then for each natural number k there is an integer d_k, $0 \le d_k \le 9$, such that

$$D + \frac{d_1}{10} + \ldots + \frac{d_k}{10^k} \le x < D + \frac{d_1}{10} + \ldots + \frac{d_k}{10^k} + \frac{1}{10^k}$$ where D is a natural number.

Proof:	**Line Number:**
Let $x \in R^+$, then there exists a natural number D such that	**(1)**
$D \le x < D + 1$. Then, $0 \le x - D < 1$. Let $r_1 = x - D$	**(2)**
Then, $0 \le r_1 < 1$, and $x = D + r_1$	**(3)**
Since $0 \le 10r_1 < 10$, there exists a natural number $0 \le d_1 \le 9$ such that	**(4)**
$d_1 \le 10r_1 < d_1 + 1$. Then, $0 \le 10r_1 - d_1 < 1$. Let $r_2 = 10r_1 - d_1$	**(5)**
$r_1 = \frac{(d_1 + r_2)}{10} \rightarrow r_1 = \frac{d_1}{10} + \frac{r_2}{10}$	**(6)**
Lines 3 and 6 implies that $x = D + \frac{d_1}{10} + \frac{r_2}{10}$, where $0 \le r_2 < 1$	**(7)**
Then, it can be concluded that $D + \frac{d_1}{10} \le x < D + \frac{d_1}{10} + \frac{1}{10}$ for k=1	**(8)**

Figure A.1. Proof-Part1

Since $0 \leq r_2 < 1$, $0 \leq 10r_2 < 10$. Then, there exists a natural number **(9)**

$0 \leq d_2 \leq 9$ such that $d_2 \leq 10r_2 < d_2 + 1$, $0 \leq 10r_2 - d_2 < 1$ Let $r_3 = 10r_2 - d_2$ **(10)**

This implies, $r_2 = \frac{(r_3 + d_2)}{10}$. Then, $x = D + \frac{d_1}{10} + \frac{r_2}{10} = D + \frac{d_1}{10} + \frac{[\frac{(r_3+d_2)}{10}]}{10}$ **(11)**

$x = D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{r_3}{10^2}$, where $0 \leq r_3 < 1$ **(12)**

Then, $D + \frac{d_1}{10} + \frac{d_2}{10^2} \leq x < D + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{1}{10^2}$ for $k = 2$ **(13)**

Continuing this procedure, we get

$D + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots \ldots\ldots + \frac{d_n}{10^n} + \frac{r_{n+1}}{10^n}$, for $k = n$ where $0 \leq d_n \leq 9$ and $0 \leq r_{n+1} < 1$ **(14)**

Then,

$D + \frac{d_1}{10} + \ldots\ldots\ldots\ldots + \frac{d_n}{10^n} \leq x < D + \frac{d_1}{10} + \ldots\ldots\ldots\ldots + \frac{d_n}{10^n} + \frac{1}{10^n}$. **(15)**

Thus, for any natural number k there exists a natural number $0 \leq d_k \leq 9$ such that

$\boldsymbol{D + \frac{d_1}{10} + \ldots\ldots\ldots\ldots + \frac{d_k}{10^k} \leq x < D + \frac{d_1}{10} + \ldots\ldots\ldots\ldots + \frac{d_k}{10^k} + \frac{1}{10^k}}$ **(16)**

Figure A.2. Proof-Part2

Thus, we can conclude the following definition:

Definition: *Every increasing and bounded rational number sequence represents a real number. Two sequences which have a difference approaching to zero are called sequences representing the same real number.*

Figure A.3. Proof-Part3

APPENDIX B: PROOF COMPREHENSION TEST

In this part, the proof comprehension test used in this study was shared.

1. Let x be a positive real number.
    a. Give an example for the case where x is a rational number.
        i. Find r_1.
        ii. What is the meaning of components of the numerator and denominator of r_1 in terms of division algorithm?
            a) Dividend-Divisor
            b) Quotient-Divisor
            c) Remainder-Divisor
            d)Remainder- Dividend
    b. Give an example for the case where x is an irrational number. Find r_1.
2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used.
    a. What does sequence mean?
    b. What does bounded sequence mean?
    c. Can you give an example of 'bounded sequence'?
    d. What does "rational number sequence" mean?
    e. Can you give an example of a rational number sequence?
3. Could you justify why
    a. the rigth side of the inequality expressed in line 8 holds?
    b. the rigth side of the inequality expressed in line 16 holds?
4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.
5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?
6. How does the statement in line 14 follow starting from line 1? Could you explain?
7. Could you explain the main idea (big idea) of the proof?
8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.

Figure B.1. Proof Comprehension Test

9. Please explain the following using the method of the proof you have been working with.
'Some real numbers have a terminating expansion.'

10. Find the decimal representation of 13/7 by refering to the proof.

11. The decimal representation of $\sqrt{2}=$ 1.4142135623730950488016887242096980785 6… is given. Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which $\sqrt{2}$ fall in?

12. Continuing from line 13 could you construct the first 5 steps of decimal representation for any positive real number x (**until k=5**) and explain the steps you followed?

13. Using the proof, can you explain why the definition given at the end of the proof holds?
(**Hint:** Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + \ldots\ldots\ldots + \frac{d_k}{10^k}$

Let $\{y_k\}$ be a sequence such that any term $y_k = D + \frac{d_1}{10} + \ldots\ldots + \frac{d_k}{10^k} + \frac{1}{10^k}$)

Figure B.2. Proof Comprehension Test-Continued

APPENDIX C: RUBRIC FOR PROOF COMPREHENSION TEST

In this part, in the following pages, a table of proof comprehension test dimensions, corresponding comprehension test questions and the rubric was shared.

Dimension*	Explanation*	Question	Possible Correct Answers	Reason for asking these questions	Scores
D1: Meaning of the terms and statements	**Understanding the meaning of symbols, terms and definitions.**	**1.** Let x be a positive real number. **a.** Give an example for the case where x is a rational number. **i.** Find r_1. **ii.** What is the meaning of components of the numerator and denominator of r_1, respectively in terms of division algorithm? a) Dividend-Divisor b) Quotient-Divisor c) Remainder-Divisor d)Remainder- Dividend **b.** Give an example for the case where x is an irrational number. Find r_1.	**1.** If r_i is a rational number, in the proof it assumed that $0 \leq r_i < 1$. Then, $r_i = a/b$ where a is a positive integer or equals to zero and b is a positive integer. the meanings of the components a and b are the following: As an example, participantss may give 3/2 and epress the number as 3/2=1+1/2. In this case, r_1 is ½ and in terms of components 1 is remainder and 2 is divisor. **(Participants may give different examples. The expected answer need not to be the example given above.)** **b)**The example for the irrational case can be the following: ∏=3,14159265... and ∏=3+(0,141159265....)/10 therefore here r_1=0,141159265.... is an irrational number since it has infinite decimal.	In order to understand the proof participants need to make sense of the number r_i why it is between 0 and 1 and how it is obtained in the proof intrinsically. Understanding the meaning of r_i will help participants to undertstand the definiton given after the proof which includes convergence, boundedness. Therefore, this question can be asked.	For part (a): i) Each correct example will be scored as **1 point** and for the part ii, correct answer will be scored as **1 point.** For part (b): any example with decimal part showing r_1 **1 point**. **(Total = 3 points) Otherwise 0 point.** If there is a case as tha folowing: ∏=3,14, conduct an interview.

Figure C.1. Rubric-Page 1

		2. At the end of the proof, in the definition given at the end, 'bounded sequence' is used. **a.** What does sequence mean? **b.** What does bounded sequence mean? **c.** Can you give an example of 'bounded sequence'? **d.** What does "rational number sequence" mean? **e.** Can you give an example of a rational number sequence?	**2.** A sequence is function from the set of natural numbers to real numbers. A sequence is bounded if it is bounded above and below. For example, {1/n}, n∈N, is a bounded sequence since it takes values between 0 and 1. **Participants may give different examples. The expected answer need not to be the example given above.** **d.** Rational number sequence can be defined as the following: It is a function from natural numbers to rational numbers. It means that terms of this type of sequence are rational numbers. **e.** Example: $a_1=1$, $a_2=1+1/2$, $a_3=1+1/2+1/3$, $a_4=1+1/2+1/3+1/4$.	This question is asked because in order to assess participants' understanding of the definition given after the proof, it is essential to determine participants' pre-conceptions. If participants cannot define the sequence, bounded sequence, it can be hard to detrmine their understanding of the definition.	For parts a) **2 points** for defining sequence as a function and **1 point** for defining sequence as ordering elements, **1 point** for each correct answer (for b, c, d, e). **(Total= 6 points or 5 points)**
D2: Justification of claims	**Understanding how new assertions in the proof follow from previous ones.**	**3.** Could you justify why **a.** the rigth side of the inequality expressed in line 8 holds? **b.** the rigth side of the inequality expressed in line 16 holds?	**3.** **a.** The inequality holds because $0\leq r_2<1$ and $r_2/10<1/10$ **b.** The inequality holds because $0\leq r_{\alpha+1}<1$ and $r_{\alpha+1}/10^{\alpha}<1/10^{\alpha}$.	To assess whether participants are aware that $r_{\alpha+1}/10^{\alpha}<1/10^{\alpha}$ because it is the crucial step to conclude the proof of the statement. By understanding that $r_{\alpha+1}/10^{\alpha}<1/10^{\alpha}$, they can say that since $r_{\alpha+1}$ is between 0 and 1 and if we divide this number to a large number	For parts a) and b), **1 point for each correct answer.** **(Total=2 points)**

Figure C.2. Rubric-Page 2

				10^{a}, we can get smaller number compared to $1/10^{a}$.	
		4. How can you write the expression in line 5, $d_1 \leq 10r_1 < d_1+1$, using line 4. Explain your reasoning. You can use diagrams, shapes etc. while explaining your reasoning.	4. Since every real number falls into the interval of two integers. For $10r_1$, which is between 0 and 10, we can find two consecutive natural numbers that $10\, r_1$ is between them. Also,explanations with diagrams and relationships between inequalities are acceptable.	The line 5 is the first necessary step to write the positive real number x as a decimal representation. Therefore, misunderstanding in this step may result in misunderstanding in the whole proof.	**1 point.**
D3: Logical Structure	**Understanding the logical relationship between lines or components of a proof.**	5. Between lines 1 and 16, what kind of a proof strategy is used? Could you explain how did you notice that?	5. The proof strategy used is direct proof.	The reason is to determine whether participants notice the type of proof and the logical structure in the proof. undertstanding the logical structure requires the understanding the proof strategy.	**1 point** for giving the correct strategy and **1 point** for explaining the reason. **(Total=2 points)**
		6. How does the statement in line 14 follow starting from line 1? Could you explain?	6. We are doing direct proof but we try to construct the x as a decimal representation for any natural number k. Through the construction of x for k=1, we construct the case for k=2. Therefore, we need to use the previous step that is k=n-1 to construct the step k=n.	Construction is a part of logical structure in this proof. If participants understand how x is written as decimal expansion then they may understand the proof.	**1 point.**

Figure C.3. Rubric-Page 3

D4: Higher Level Ideas	**Identifying a good summary of the overarching approach of the proof.**	**7.** Could you explain the main idea (big idea) of the proof?	**7.** Each '+' real number x can be represented in an interval in which the bounds are real numbers with terminating decimal expansion and if the difference of bounds tend to zero, then they represent the '+' real number x.	The reason for asking these questions is whether participants understand the aim of proof as they read the proof.	**1 point.** (The emphasis on the definition will also be considered as 1 point.)
		8. Determine the most important lines in the proof connected to the main idea. Explain your reasoning.	**8.** The most important lines in the proof is the construction of the '+' real number x as a decimal expansion. Therefore, first 6-8 lines will be important.		**1 point** (just stating the steps without explanation will be considered as **0 point.**)
D5: General Method	**Applying the methods within the proof to a different context.**	**9.** Please explain the following using the method of the proof you have been working with. **'Some real numbers have a terminating expansion.'**	**9.** Let x= $D+d_1/10+.........+d_n/10^n+r_{n+1}/10^n$, be a real number if $r_{n+1}=0$, x will be terminating.	To see whether participants use the proof for a different but related mathematical statement.	**1 point.**
		13. Using the proof, can you explain why the definition given at the end of the proof holds? (Hint: Let $\{x_k\}$ be a sequence such that any term $x_k = D + \frac{d_1}{10} + ...+ \frac{d_k}{10^k}$Let $\{y_k\}$ be a sequence such that any term	**13.** Firstly increasing-decreasing and boundedness properties of the sequences $\{x_k\}$ and $\{y_k\}$ will be shown. Since any term of $\{x_k\}$ is $x_n = D + \frac{d_1}{10} + ...+ \frac{d_n}{10^n}$ and for $x_{n+1} = D + \frac{d_1}{10} + ...+ \frac{d_n}{10^n} + \frac{d_n}{10}$ Then, for any natural number n,		Showing the increasing-decreasing **(1 point each)**, boundedness **(1 point)** and rational number properties **(1 point)** of $\{x_k\}$ and $\{y_k\}$

Figure C.4. Rubric-Page 4

		$y_k = D + \frac{d_1}{10} + \ldots\ldots + \frac{d_k}{10^k} + \frac{1}{10^k})$	$x_a < x_{a+1}$. Therefore, $\{x_k\}$ is increasing. Also, it bounded since the terms are between D and D+1. However, $\{y_k\}$ is decreasing and bounded. Additionally, they are rational number seqeunces. Now, if the difference of two terms of the sequence is taken that is, $y_k - x_k = \frac{1}{10^k}$. Then, $\lim_{n\to\infty} \frac{1}{10^k} = 0$. For any real number $x = D + \frac{d_1}{10} + \ldots\ldots\ldots + \frac{d_n}{10^n} + \frac{r_{n+1}}{10^n}$.		Taking the difference of the nth terms of the sequences is **1 point** and calculating the limit for the difference is **1 point**. **(Total=6 points)**
D6: Application to examples	**Using the ideas in the proof in terms of a specific example.**	**10.** Find the decimal representation of 13/7 by refering to the proof.	**10.** 1< 13/7<2 0< 13/7-1<1 0< 6/7<1 → 0<60/7<10. Then 60/7 is between 8 and 9. Therefore, 8<60/7<9 → 0<60/7- 8<1 → 0<4/7=60/7- 8<1. Then, 4/7+8=60/7 (4/7)/10+8/10=6/7 ----> 13/7= 1+ 8/10+ (4/7)/10. 0<4/7<1→0<40/7<10. 5<40/7<6→ 0<40/7-5<1 0<5/7+5=40/7<1→(5/7)/10+5/10=4/7. 13/7= 1+ 8/10+ 5/10^2+(5/7)/10^2. Then, 0<5/7<1→0<50/7<10.	Undertstanding a proof requires to apply it to specific cases. Therefore, if participants can apply this theorem to numerical examples it can give a clue about participants understandings of theorem and proof.	**1 point**

Figure C.5. Rubric-Page 5

			$7<50/7<8 \rightarrow 0<50/7-7<1$ $0<1/7+7=50/7<1 \rightarrow (1/7)/10+7/10=5/7$. $13/7= 1+ 8/10+ 5/10^2+7/10^3+(1/7)/10^3$. Then, $0<1/7<1 \rightarrow 0<10/7<10$. $1<10/7<2 \rightarrow 0<10/7-1<1$ $0<3/7+1=10/7<1 \rightarrow (3/7)/10+1/10=1/7$. $13/7= 1+ 8/10+ 5/10^2+6/10^3+1/10^4+(3/7)/10^4$.		
		11. The decimal representation of $\sqrt{2}=$ 1.41421356237309504880168872420969807856… is given. Can you find an open interval with lower bound with 6 digits and upper bound with 6 digits for which $\sqrt{2}$ fall in?	**11.** $1+ 4/10 + 1/10^2 + 4/10^3 + 2/10^4 + 1/10^5 + 3/10^6 + < \mathbf{\sqrt{2}} < 1+ 4/10 + 1/10^2 + 4/10^3 + 2/10^4 + 1/10^5 + 3/10^6 + 1/10^6$.		**1 point**

Figure C.6. Rubric-Page 6

D7: Identifying modular structure	**Understanding the main components and modules within a proof and the logical relationship between them.**	**12.** Continuing from line 13 could you construct the first 5 steps of decimal representation for any positive real number x (until k=5) and explain the steps you followed?	**12.** In the proof up to k=2 the decimal representation of the positive real number x and the interval that x is in were expressed. $x= D+ d_1/10+d_2/10^2+ r_3/10^2$ where $0\leq r_3<1$. Then, $D+ d_1/10+d_2/10^2 \leq x< D+ d_1/10+d_2/10^2+ 1/10^2$ for k=2 Now, since $0\leq r_3<1$, $0\leq 10r_3\leq 10$. Therefore, there exists a natural number $d_3, 0\leq d_3\leq 9$ such that $d_3\leq 10r_3< d_3+1$. Then, $0\leq r_4=10r_3- d_3< 1$. $r_3= (r_4+d_3)/10$. $x= D+ d_1/10+d_2/10^2+d_3/10^3+ r_4/10^3$ $D+ d_1/10+d_2/10^2+ d_3/103<x< D+ d_1/10+d_2/10^2+ d_3/10^3+ 1/10^3$. Now, since $0\leq r_4<1$, $0\leq 10r_4\leq 10$ Therefore, there exists a natural number $d_4, 0\leq d_4\leq 9$ such that $d_4\leq 10r_4< d_4+1$. Then, $0\leq r_5=10r_4- d_4< 1$. $r_4= (r_5+d_4)/10$. $x= D+ d_1/10+d_2/10^2+d_3/10^3+ d_4/10^4+ r_5/10^4$. Then, $D+ d_1/10+d_2/10^2+ d_3/10^3+ d_4/10^4<x< D+ d_1/10+d_2/10^2+ d_3/10^3+ d_4/10^4+1/10^4$.	The reason for asking question 1 is to assess whether participants understand how x is constructed as a decimal expansion because the proof depends on the construction of x as decimal expansion .	**1 point**

Figure C.7. Rubric-Page 7

			Now, since since $0 \leq r_5 < 1$, $0 \leq 10r_5 \leq 10$ Therefore, there exists a natural number $d_5, 0 \leq d_5 \leq 9$ such that $d_5 \leq 10r_5 < d_5+1$. Then, $0 \leq r_6 = 10r_5 - d_5 < 1$. $r_5 = (r_6+d_5)/10$. $x = D + d_1/10 + d_2/10^2 + d_3/10^3 + d_4/10^4 + d_5/10^5 + r_6/10^5$. Then, $D + d_1/10 + d_2/10^2 + d_3/10^3 + d_4/10^4 + d_5/10^5 < x < D + d_1/10 + d_2/10^2 + d_3/10^3 +$ $d_4/10^4 + d_5/10^5 + 1/10^5$, for $k=5$.		

***Note: Dimensions and explanations are referenced from the following article:**

Hodds, M., Alcock, L. & Inglis, M. (2014). Self-Explanation Training Improves Proof Comprehension. *Journal for Research in Mathematics Education*, 45(1), 62-101.

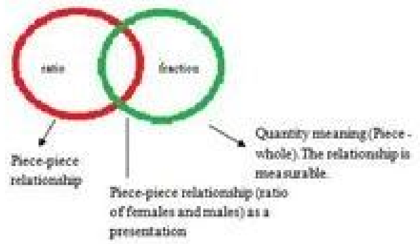
No answer will be considered as 0 mark.

Totally, 27 Points.

Figure C.8. Rubric-Page 8

APPENDIX D: LESSON PLAN

In this part, in the following pages, the lesson plan used in this study was shared.

<table>
<tr><th colspan="3">Lesson Plan for the Decimal Representation of Real Numbers (QR)</th></tr>
<tr><th colspan="3">Introduction</th></tr>
<tr><th>Researcher's Questions</th><th>Participants' Activity</th><th>Duration</th></tr>
<tr><td>At the beginning researcher asks the mathematical definition of rational number and the difference between rational number and fraction. (The reason for asking about fractions is that in the lesson just positive rational numbers will be dealt and the term fraction will be used since we use partitioning as representing rationals with decimal expansion.)</td><td>The expected answer for the definition of rational numbers will be the following: for a and b integers, a/b is a rational number if b≠0. $\frac{x}{y}$ is a fraction which shows equal parts of a whole.
If participants cannot state the difference between fractions and rational numbers, researcher wants them give examples of fractions and rational numbers and try to make them see the differences and similarities. Every positive rational nember is a fraction. However, for the case of negative integers this does not hold.
As a special case, if they define fraction as a ratio, the following explanation will be given by the researcher. Also, the different meaning of rational numbers will be introduced and quantity meaning will be emphasized.


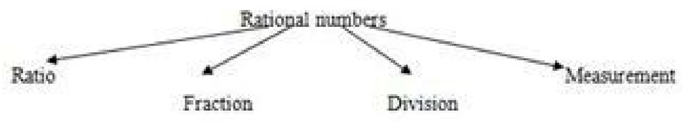

</td><td>30 min.</td></tr>
</table>

Figure D.1. Lesson Plan-Page 1

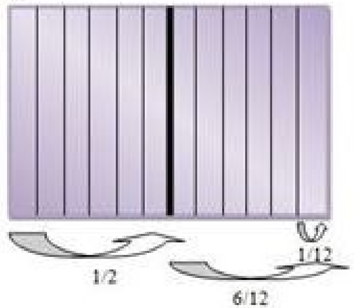
asks 'what does equivalent fraction mean?'	First, participants will write the answer to the paper and researcher will check each participant's answers. One possible answer could be that if we enlarge both numerator and denominator by the same number we get equivalent fractions. The expected answer is the following: two or more fractions are equivalent if they represent the same amount of a whole object. If participants cannot give this answer, researcher wants them give examples of equivalent fractions. Lets assume that participants say that ½ and 6/12 are equivalent fractions. Then, researcher will ask 'why?'. If they say that the numerator and denominator of ½ is enlarged by 6. The next question will be can you explain the meaning of 6 using diagrams. They first do this one by one on their paper and researcher observes each participant. If they draw two diagrams, say them to show the meaning of 6 in just one diagram. After that group discussion will be conducted. 	

Figure D.2. Lesson Plan-Page 2

Main Body (1st part)		
Researcher's Activity	**Participants' Activity**	**Duration**
In this part, the relationship between decimal representation and equivalent fractions will be emphasized. First researcher will ask participants 'How can you explain the decimal representation of fractions?' After participants' explanation, they will be asked 'Is there a relationship between decimal representation and equivalent fractions? Please explain your answer.'	Participants may give examples and say that we try to express the denominator as 10 or a power of ten. Then, we can express the fraction in decimal form. Participants may give examples to the second question. For example, in order to express 4/5 in decimal form we enlarge numerator and denominator by 2 and the fraction will be 8/10=0,8. In this case, researcher will ask what does enlarge refer to in our case? They are expected to say that we partition every piece (5 piece) by 3 so we get equivalent fraction so decimal representation and equivalent fraction are related.	
If participants gave examples of fractions such as 1/2 and 3/5, researcher askes do they have any common property? Do their denominator have any in common?	Expected answer from participants is that when the denominator is 2 or 5 or any combination of 2 or 5, the denominator will be a multiple of 10. Then it can be represented as a terminating decimal representation.	
Researcher ask what if the denominator is not a power of 2 or 5, what kind of a difference does occur in decimal expansion? For example for 1/3.	Expected answer is that the decimal representation is repeating.	
Researcher again askes the relationship between equivalent fractions and decimal representation? The reason for asking the same question as before, is that participants need to internalize the answer because through decimal expansion they will use this relationship.		**45 min.**
1. Using the long division could you explain the relation between mixed fractions (and improper fractions) and division algorithm? 2. After relating mixed fractions and long division, researcher will ask the meaning of numerator and denominator of ½ in terms of division algorithm	If participants give examples of improper fractions ***, researcher will continue using participants' examples. Otherwise, researcher will ask about a mixed fraction for instance, 7/2. For this example, 7 | 2 7=2×3+1 → 7/2=3+1/2 -6 3 1 2. The expected answer is that in the representation of a mixed fraction (which are another representation of improper fractions using division algorithm), integer part denotes the quotient, numerator denotes the remainder and denominator denotes divisor in terms of division algorithm.	

Figure D.3. Lesson Plan-Page 3

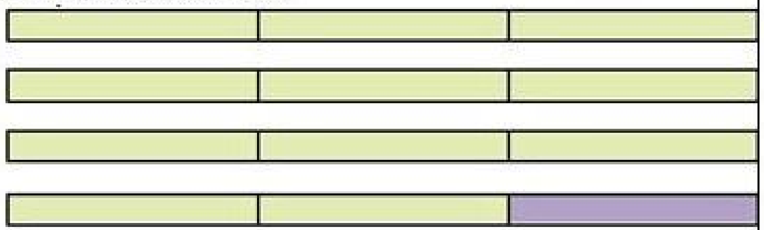
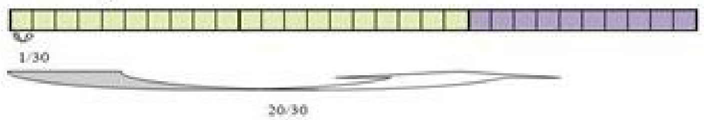
Can you express the improper fraction 11/3 using division algorithm? Can you continue with the long division?	Participants are expected to express in the following way: 11=3×3+2 --> 11/3=3+2/3. 11|3 -9 3 2 If they continue with long division, they will add and an ',' after 3 and a zero near to 2 and divide 20 by 3.
Then, researcher will ask them to explain why they put a zero near to the remainder and continue in this way?	They may answer that we use decimal system so we put a zero.
Researcher askes participants the following: Could you explain the reason for putting zero in the long division using equivalent fractions? **(Participants answer this question individually and write their answers on the paper. Researcher will check their answers)** If participants cannot give answers, researcher suggest to use diagrams. **How can 2/3 be represented in decimal form using equivalent fractions?	The representatiton of 11/3 is;  Using 4th diagram (2/3), we continue with the long division, **2/3=20/30=20/3×1/10. Expanding the fraction by means subdividing each portion to 10 pieces.  The question to consider is the following: How many 3 pieces are there in 20 pieces? The answer to this question is that there are 6 group of 3 pieces and the reamining is 2/3pieces.

Figure D.4. Lesson Plan-Page 4

(Participants will answer this question individually in their papers and researcher checks their answers one by one)

Then, the next question is how can each small portion be represented by fractions?
Since we partition 3 pieces by 10. The total number of pieces becomes 30. Therefore, each small piece represents 1/30.

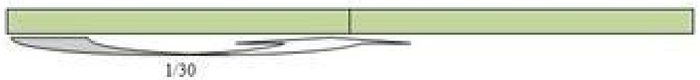
Researcher askes the fractional representation of 3 pieces. The answer is that since each small piece is 1/30. 3 groups will be 3×1/30=1/10. Then, researcher askes what does 20/3 mean in the expression 20/3×1/10. The expected answer is that the quantity of 1/10 in a whole.

Therefore, 2/3=20/30=20/3×1/10=(6+2/3)×1/10=6/10+2/30.

The researcher askes the meaning of 2/30. The expected answer is that it is the remaining portion. However, if participants cannot answer in this way researcher suggest them to think again using the diagram.

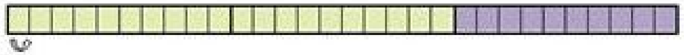
Then, researcher askes how can we continue to division?
The expected answer is that we need to continue with 2/30.

Then, the portion of 2/30 will be drawn (zooming) as below.



Researcher askes what to do for division. If participants cannot answer the question, she will ask if it is possible to continue as in the previous case.
2/30=20/300=20/3×1/100 (Again, the meaning of this expression will be asked.)
Researcher askes them to express this using diagrams.

Figure D.5. Lesson Plan-Page 5



1/300

How many 3 pieces are there in 20 pieces?
The answer to this question is that there are 6 group of 3 pieces and the reamining is 2/3pieces. **(Participants will answer this question individually on their papers and researcher checks participants' answers one by one)**

Then, the next question is how can each small portion be represented by fractions?
Since we partition 3 pieces by 10. The total number of pieces becomes 300. Therefore, each small piece represents 1/300.

Researcher askes the fractional representation of 3 pieces. The answer is that since each small piece is 1/300. 3 groups will be $3\times 1/300=1/100$. Then, researcher askes what does 20/3 mean in the expression $20/3\times 1/10$. The expected answer is that the quantity of 1/10 in a whole.

Therefore, $2/30=20/300=20/3\times 1/100=(6+2/3)\times 1/100=6/10+2/300$.

After completing the discussion about 2/30. Researcher asks participants to express 11/3 using the results obtained. **First, participants will write individually on their papers and researcher check their solution.**
The expected answer is that

$$\mathbf{11/3=3+2/3 = 3+6/10+2/30 = 3+6/10+6/100+2/300}$$
$$\mathbf{=3+2/3 = 3+6/10+(2/3)/10 = 3+6/10+6/10^2+(2/3)/10^2.}$$

As a last exercise, students will work on the fraction 8/7 and find the decimal representation by combining long division, diagram and division algorithm.

Figure D.6. Lesson Plan-Page 6

Main Body (2nd Part)		
Researcher's Activity	**Participants' Activity**	**Duration**
In the second session, researcher will continue with the same fraction, 11/3. Researcher will remind participants that they found decimal expansion of 11/3 in the last session and ask them to explain how it was done. Also, s/he wants them to express 11/3 in decimal form.	Participants should emphasize equivalent fraction, division algorithm and their relation with the decimal expansion. $11/3=3+6/10+6/10^2+(2/3)/10^2$	
After rewieving the first lesson, the researcher will ask participants to squezze 11/3 between to numbers? Firstly, she will ask what is the interval of 11/3 in the number line? The second question is that considering the expression $3+6/10+2/3/10$ can we write a smaller interval? The third question is that considering the expression, $3+6/10+6/10^2+(2/3)/10^2$, can we write a smaller interval for 11/3. Then, the researcher will want participants to find the intervals for the nth step.	The expected answer is that 11/3 is between 3 and 4. The aim of the second question is to force students to change both upper and lower bounds. Expected answer is that 11/3 is between 3+6/10 and 3+6/10+1/10. That is, $3+6/10<3+6/10+2/3/10<3+6/10+1/10$ The expected answer for the third question is that $3+6/10+6/10^2<3+6/10+6/10^2+(2/3)/10^2<3+6/10+6/10^2+1/10^2$ Participants may answer the last question as follows: $3+6/10+......+6/10^n<3+6/10+......+6/10^n+(2/3)/10^n<3+6/10+.....+6/10^n+1/10^n$	**75 min.**
After determining the upper and lower bounds, the researcher will write the following expression (considering lower bounds): $f(n)=3+6/10+......+6/10^n$ and ask the class whether the expression is a mapping from positive natural numbers to real numbers or not?	At this stage, participants need to be aware of that mapping is a function. The expected answer is that f(n) is a mapping since the properties of the mapping hold that is each element in the domain matches with one and only one element in the range and there is no element in the domain such that it has not any correspondent element in the range. If participants have a difficulty in answering the question, researcher will ask probing questions about the definition of mapping and want participants to analyze the properties of f(n) and to determine whether it is a mapping or not.	

Figure D.7. Lesson Plan-Page 7

After defining f(n) as a mapping from positive natural numbers to real numbers, researcher will ask what is the name of this special kind of mapping?	Participants may answer that it is a sequence. If they cannot, the researcher will say that the functions whose domain is positive natural numbers are named as sequences.
Participant will ask the other question that is $g(n)=3+6/10+.....+6/10^n+1/10^n$ a mapping?	Participants may say that it is also a sequence as in the case of f(n).
After defining lower bounds and upper bounds as seqeunces, their properties will be examined. f(n) sequence will be denoted as xn and g(n) sequence will be denoted as yn. The researcher will ask that when you consider the terms of the seqeunces which are formed by lower bounds and upper bounds, what can you say about the type of the sequences.	The expected answers: The seqeunce formed by lower bounds is an increasing seqeunce because each term is greater than the prevous term that first term is $3+6/10$ and the second term is $3+6/10+6/10^2$. The second term is bigger than the first term and it goes similar in the other terms. However, the seqeunce formed by upper bounds is decreasing because the first term is $3+6/10+1/10=3,7$ and the second term is $3+6/10+6/10^2+1/10^2=3,67$. The second term is less than the first term and the relationship between other consecutive terms is similar. If participants cannot answer the question, researcher will want them to write the terms and try to share what they observed.
The class then focuses on the boundedness of these two seqeunces and the researcher will ask whether the sequences are bounded or not. Also, participants will be asked whether they rational number sequences or not.	They may say that they are bounded because all the terms of the sequences are always between 3 and 4. Also, they are rational number seuqences since the terms are rational numbers. In the case that they cannot answer the boundedness question. researcher will ask them to give the definition of bounded seqeuence and then want them to chech whether the sequences meets the properties of the boundedness or not.
The lesson continues with the discussion about the differences of the terms of sequences. Participants will be asked to find the difference between the first terms. Then,they will be asked about the differences of second third and n th terms.	The expected answer is that the difference of first terms is $1/10$, the difference of second terms is $1/10^2$, the difference of third terms is $1/10^3$. For the nth terms it is $1/10^n$.
The researcher will ask about their observations.	They may say that the difference become smaller when n becomes larger. At this instant researcher may ask that what happens if n goes to infinity? The answer is that if n goes to infinity difference between the terms goes to zero.
Then participants will be asked that what does it mean that the difference goes to infinity?	It means that these to sequences may represent the number 11/3. We have the best approximation of decimal representation of 11/3.

Figure D.8. Lesson Plan-Page 8

After that researcher will ask participants to generalize the form of decimal representation refering to the decimal representation of 11/3. In the first place they will try generalize this finding to a positive rational number x. At this instant, the researcher will help the students while doing generalization and by asking questions they will try to obtain a decimal representation for any positive rational number X. Also participants will explain what kind of numbers that the symbols may denote.	The expected answer is that D=3 d1=6 d2=6 d3=6 r3=2/3 Then, any positive rational number x can be represented as follows: $X=D+d_1/10+d_2/10^2+d_3/10^3+r_3/10^3$ The researcher will also want them to write the expressions until the first decimal place, the second decimal place and decimal place. $X=D+d_1/10+r_1/10$ $X=D+d_1/10+d_2/10^2+r_2/10^2$ $X=D+d_1/10+d_2/10^2+\ldots\ldots\ldots\ldots+d_n/10^n+r_n/10^n$ Participants are expected to be aware that for this case where X is a postive natural number, the number D denotes the whole part and it is a positive natural number. d_n s are the natural numbers from 0 to 9 and r_n s are real numbers between 0 and 1 (including 0)
The next step is to represent X in an interval. Therefore, the researcher will ask What can be the interval that the number X falls in while the bounds are natural numbers.	Participants may answer as X is between D and D+1 because we add some numbers to D but it can never exceed D+1 due to adding small numbers.
Then, researcher promote participants to think about narrowing down the interval of X. As a clue she will say that they may think through the decimal representations of X. Also, participants will be asked what happens if for any natural number, r_n is zero? Then the researcher will ask if this is the case what happens in the inequalities?	The possible answer is the following: $D+d_1/10\leq X=D+d_1/10+r_1/10< D+d_1/10+1/10$ $D+d_1/10+d_2/10^2\leq X=D+d_1/10+d_2/10^2+r_2/10^2< X=D+d_1/10+d_2/10^2+1/10^2$ $D+d_1/10+d_2/10^2+d_3/10^3\leq X=D+d_1/10+d_2/10^2+d_3/10^3+r_3/10^3<$ $D+d_1/10+d_2/10^2+d_3/10^3+1/10^3$ $D+d_1/10+d_2/10^2+\ldots\ldots\ldots\ldots+d_n/10^n\leq D+d_1/10+d_2/10^2+\ldots\ldots\ldots\ldots+d_n/10^n+r_n/10^n<$ $D+d_1/10+d_2/10^2+\ldots\ldots\ldots\ldots+d_n/10^n+1/10^n$ The participants may say that it means that at some stage in the long division, the remainder will be zero. We need to add an equal sign to the lower bound (shown above)

Figure D.9. Lesson Plan-Page 9

After determining the upper and lower bounds, the researcher will write the following expression (considering lower bounds): $f(n)=D+d_1/10+d_2/10^2+.........+d_n/10^n$ and ask the class whether the expression is a mapping from positive natural numbers to real numbers or not?	At this stage, participants need to be aware of that mapping is a function. The expected answer is that f(n) is a mapping since the properties of the mapping hold that is each element in the domain matches with one and only one element in the range and there is no element in the domain such that it has not any correspondent element in the range. Most probably, participants may say that it is a sequence.
Researcher will ask the other question that is $g(n)= D+d_1/10+d_2/10^2+..........+d_n/10^n+1/10^n$ a mapping?	Participants may say that it is also a sequence as in the case of f(n).
After defining lower bounds and upper bounds as seqeunces, their properties will be examined. f(n) sequence will be denoted as xn and g(n) sequence will be denoted as yn. The researcher will ask that when you consider the terms of the seqeunces which are formed by lower bounds and upper bounds, what can you say about the type of the sequences.	The expected answers: The seqeunce formed by lower bounds is an increasing seqeunce because each term is greater than the previous term that first term is $D+d_1/10$ and the second term is $D+d_1/10+d_2/10^2$. The second term is bigger than the first term and it goes similar in the other terms. However, the seqeunce formed by upper bounds is decreasing because the first term is $D+d_1/10+1/10=D,(d_1+1)$ and the second term is $D+d_1/10+d_2/10^2+1/10^2=D,d_1(d_2+1)$. The second term is less than the first term and the relationship between other consecutive terms is similar. If participants cannot answer the question, researcher will want them to write the terms and try to share what they observed.
The class then focuses on the boundedness of these two seqeunces and the researcher will ask whether the sequences are bounded or not. Also, participants will be asked whether they rational number sequences or not.	They may say that they are bounded because all the terms of the sequences are always between D and D+1. Also, they are rational number seuqences since the terms are rational numbers. In the case that they cannot answer the boundedness question. researcher will ask them to give the definition of bounded seqeuence and then want them to chech whether the sequences meets the properties of the boundedness or not.
The lesson continues with the discussion about the differences of the terms of sequences. Participants will be asked to find the difference between the first terms. Then, they will be asked about the differences of second third and n th terms.	The expected answer is that the difference of first terms is 1/10, the difference of second terms is $1/10^2$, the difference of third terms is $1/10^3$. For the nth terms it is $1/10^n$.

Figure D.10. Lesson Plan-Page 10

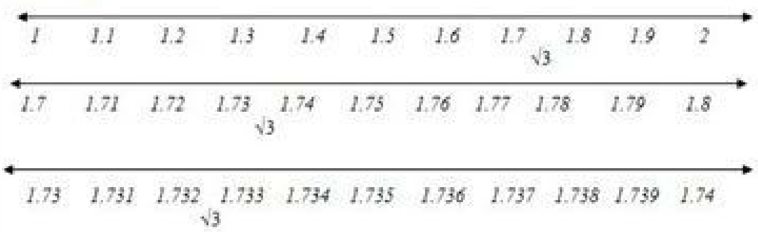
The researcher will ask about their observations.	They may say that the difference become smaller when n becomes larger. At this instant researcher may ask that what happens if n goes to infinity? The answer is that if n goes to infinity difference between the terms goes to zero.	
Then participants will be asked that what does it mean that the difference goes to infinity?	It means that these two sequences (one is increasing bounded rational number sequence and the other is decreasing bounded rational number sequence) may represent the positive rational number X. We have the best approximation of decimal representation of the positive rational number X.	
Researcher will ask whether this process can be obtained for all real numbers or not and participants answer is yes, they are expected to generalize it. If participants cannot generalize this to every real number because we will deal with rational numbers, researcher will ask them to squeeze the irrational number $\sqrt{3}$. For this aim, researcher will ask first on the number line in which interval does $\sqrt{3}$ lie? (hangi 2 doğal sayı arasında demek istiyoruz) Then, she will continue by partitioning the interval by 10. $\sqrt{3}$ will be in an interval with both ends represented as one digit decimal. Then divide the last interval by 10 to squeeze the number between 2 digit decimals and continue that way to squeeze. The researcher will want participants to write the bounds in decimal form as follows: $1+7/10<\sqrt{3}<1+7/10+1/10$ $1+7/10+3/10^2<\sqrt{3}<1+7/10+3/10^2+1/10^2$ $1+7/10+3/10^2+2/10^3\leq\sqrt{3}< 1+7/10+3/10^2+2/10^3+1/10^3$. After that they will again examine the upper and lower bounds in terms of sequences and properties of sequences.	$\sqrt{3}$ is between 1 and 2. If we partition this interval by 10. It will be as the following  Expected answer for the case of bound of irrational numbers interms of sequence concept, is that lower bounds form increasing bounded rational number seqeunce and upper bounds form decreasing bounded rational number sequence.	

Figure D.11. Lesson Plan-Page 11

The lesson again continues with the discussion about the differences of the terms of sequences. Participants will be asked to find the difference between the first terms. Then,they will be asked about the differences of second third and n th terms.	The expected answer is that the difference of first terms is 1/10, the difference of second terms is $1/10^2$, the difference of third terms is $1/10^3$. For the nth terms it is $1/10^n$.	
The researcher will ask about their observations.	They may say that the difference become smaller when n becomes larger. At this instant researcher may ask that what happens if n goes to infinity? The answer is that if n goes to infinity difference between the terms goes to zero.	
Then participants will be asked that what does it mean that the difference goes to infinity?	It means that these two sequences (one is increasing bounded rational number sequence and the other is decreasing bounded rational number sequence) may represent the positive rational number √3. We have the best approximation of decimal representation of the positive rational number √3.	
The researcher asks what does this information give us?	We can express positive real numbers with increasing bounded rational number and decreasing bounded rational number sequences. Therefore, "every increasing and bounded rational number sequence represents a real number. Two sequences which have a difference approaching to zero are called sequences representing the same real number." (Taner, 1991, p. 3)	

Figure D.12. Lesson Plan-Page 12