LIMIT ANALYSIS OF UNIFORMLY LOADED BEAMS

by

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ACKNOWLEDGEMENT

The author wishes to express his indebtedness to Professor Walid H. Rimawi for his suggestion of the topic and his continuous encouragement and advice throughout the preparation of this thesis.

The author also wishes to express his deep gratitude to his family for their understanding and encouragement throughout his stay at Robert College.

SYNOPSIS

The actual behavior of structures cannot be predicted by the elastic solutions alone. The inelastic behavior of the material should be taken into consideration. To determine the carrying capacity of a structure the safety factor should be found. However, this is not available except for very simple cases. For actual engineering structures upper and lower bounds of the safety factor are found.

In this work Limit Analysis by Direct Method of Variation is applied to simply supported and fixed ended, uniformly loaded rectangular beams including beams of small length to depth ratios. In the analysis a state of plane stress is assumed. Mises Yield Criterion and its associated flow law are used.

Solutions are found and results are plotted for the purpose of comparison with the simple one dimensional theory.

NOTATION

A	area of domain
b	depth of the beam
F, F^o, F^*	functionals for the collapse load, lower bound and upper bound respectively
F_i	body force
f	yield function
K	yield stress in pure shear
L	length of the beam
L_T	part of boundary on which load is specified(plane stress)
L_V	part of boundary on which velocity is prescribed to vanish (plane stress)
m, $\overline{m}$	multiplier and safety factor respectively
m^o, m^*	lower and upper bounds respectively
n_j	normal vector on surface S
n_x, n_y	components of unit normal vector in x and y directions
p	ultimate tensile stress divided by m at impending plastic flow
R_i	reaction of surface S_V
S	boundary of the domain
S_T	part of boundary on which tractions are specified
S_V	part of the boundary on which the velocity is prescribed to vanish
S_{ij}	stress deviator tensor
T_i, T_x, T_y	component of given load in i, x or y direction
u_i	displacement vector

V	volume of domain
v_i	velocity field
α°	constant defined by Eq.(2.37)
α^*	point function defined by Eq.(2.40)
γ	distance to the neutral axis from the top of the beam
δ_{ij}	Kronecker delta
ε_{ij}	strain tensor
$d\varepsilon_{ij}^p$	plastic strain increments
$d\lambda$	nondimensional parameter
μ	Scaler factor in the plastic potential law
σ_{ij}	stress tensor
σ_x , σ_y	stress components in the x and y direction
τ	shearing stress in xy plane
φ	point function related to the yield condition

TABLE OF CONTENTS

CHAPTER I

HISTORICAL REVIEW

1.1 Elastic Analysis of Beams

(a) Ordinary Beams

In the Seventeenth century Galileo (1)[x] tested a cantilever beam and concluded that it failed at the support. Later on, Hooke stated the proportionality of stress and strain and published his famous Law, "Ut tensio sic vis", in 1678 (1,2). Mariotte in 1680 used Hooke's Law to determine the strength of cantilever beams. J. Bernoulli (1645-1705) stated that the curvature of the deflection curve at each point is proportional to the bending moment at that point. Later Euler used this concept in his investigation of elastic curves. Navier (1) in 1821 gave the general equations of equilibrium. Coulomb (1736-1806) worked on the position of the neutral axis and drew attention to the shearing stresses in a cantilever and mentioned that they became important only in short beams (3). Saint Venant (1797-1886) was concerned with the torsion and flexure of cylinders. He was the first to examine the accuracy of the fundamental assumptions regarding bending (3). Saint Venant proposed the semi-inverse method for the solution of elastic problems and applied it to

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[x] The number in brackets refers to the listing in the Bibliography.

the bending of a cantilever by a force applied at the end. In 1862 Airy presented the stress function for the solution of the two dimensional problems. Later Maxwell and Morera handled the same problem for the three dimensional case (3). In the twentieth century Timoshenko worked on the bending of prismatic bars using Airy stress function.

(b) Deep Beams

Deep beams with periodic loading, have been studied by Craemer and Dischinger taking the boundary loadings being represented by Fourier series. The case of nonperiodic loading was treated by Bay (4) using stress functions and by Conway and Chow and Morgan (5) using strain energy principles. Later, Chow, Conway and Winter (6) used finite difference method to solve the differential equation of the stress function. They assumed a simply supported beam loaded uniformly or with a concentrated load and applied the results to reinforced concrete design. Continuous deep beams were analysed by Pei (7) for various loadings. In 1966 Coull (8),(9) treated both deep beams and walls using energy methods and Fourier Series for general loading, including the effect of the gravitational forces.

1.2 Plastic Analysis of Beams

(a) Early Developments

Tresca, in 1864, stated that a metal yields plastically when the maximum shear stress attains a critical value. Saint Venant applied Tresca's yield conditions in determining the stresses in a partly plastic cylinder subjected to torsion or bending. Von Mises in 1913 proposed a yield criteria stating that yielding occurs when distortion energy reaches a certain value. Hencky and Prandtl in 1923 worked on the geometry of shear lines in problems of plane plastic strain.[x]

Robertson and Cook in 1913 and Roderick in 1948 developed the simple plastic theory for the elastic - plastic bending of prismatic members subjected to terminal couples (11).

The method of limit design is believed to have been introduced by Kist in 1917. Further development in this area was carried on by Maier - Leibnitz, Bleich, Van den Broek, Baker and Johnston.

(b) Classical Theory of Limit Analysis

Formal proof of the theorems of Limit Analysis for beams and frames was first presented by Greenberg and Prager in 1951 (12) The limit analysis of space frames was treated by Heyman (13). Drucker, Prager and Greenberg (14) extended the theorems for problems of plain strain and applied it to continuous media of perfectly plastic material under any history of loading.

In 1951 Hill obtained upper and lower bounds for the yield point load, using the principle of maximum plastic work and its

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[x] This and other pertinent information are found in Hill (10).

complimentary minimum principle. This was the first formulation in three dimensions for the rigid perfectly plastic material.

(c) Variational Principles of Limit Analysis

In 1963 Mura and Lee (15) proposed a functional, whose stationary value is the safety factor. Later, Lee, Mura, Bryant and Rimawi (16) subjected this functional to two conjugated systems of constrained conditions and derived two new functionals which are used in finding the lower and the upper bounds.

CHAPTER II

INTRODUCTION

2.1 Definition

Beams are members in engineering structure which generally resist forces applied laterally or transversely to their axes. In obtaining solution to beams, either the elastic or the plastic approach is used. In understanding the behavior of the structures and for more reliable safety factor plastic analysis is preferred.

2.2 Elastic Solution

(a) Approximate Methods

If a body is not strained beyond a certain limiting value called the elastic limit, the deformations disappear with the removal of the forces, and such a body is regarded as perfectly elastic. For an elastic body Hooke's Law which states the proportionality between the stress and the strain holds. In solving simple beams, that is, a beam in which one dimension is long compared to the other two dimensions, the Elementary Theory of Beams is used and approximate results are derived. In this theory, certain assumptions which goes back to Bernoulli are made;

i) The beam is prismatic and straight.

ii) The loading and hence the bending moment M is applied in a plane containing one of the principal axis.

iii) Plane cross-sections before bending remain plane after bending.

iv) Deflections are small.

v) Shearing stresses are uniform across the width of the beam.

In the simple theory the normal stress perpendicular to the axis of the beam and the effect of shearing stresses on strains are usually neglected. When the depth of the beam is comparable to its span it is called a deep beam, and the simple theory can no longer be used.

Other approximate methods have been developed to get better results compared to the simple theory when rigorous solutions cannot be readily obtained. For example experimental methods such as the photoelastic method is used for solving two-dimensional problems. Soap-film method for determining stresses in torsion and bending of prismatic bars is also an application of experiments. In some cases the energy methods are used where minimum conditions of certain integrals are investigated.

(b) Exact Methods

In elasticity, the exact solutions are obtained by satisfying the following equations.

Equations of equilibrium

$$\sigma_{ij,j} = -F_i \tag{2.1}$$

Boundary conditions

$$\sigma_{ij} n_j = T_i \tag{2.2}$$

Equations of Compatibility

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{2.3}$$

Generalized Hooke's Law, stress-strain relation

$$\sigma_{ij} = \lambda \delta_{ij} \theta_1 + 2\mu \varepsilon_{ij} \tag{2.4}$$

$$\theta_1 = \varepsilon_{ii}$$

Strain - displacement relation

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.5}$$

where σ_{ij} is the stress component, F_i the body force, n_j the normal vector on the surface, T_i the surface traction, ε_{ij} the strain components. u_i the displacement vector, δ_{ij} the Kronecker delta and λ and μ are Lamé constants.

In these equations subscript notation is used, where the subscripts take the values 1, 2, 3 (x, y, z). The repeated indices mean summation and a comma indicates differentiation with respect to the space coordinates.

Indirect methods are also available. In the Semi-inverse method certain assumptions are made about the components of stress or strain or displacement. Then a solution is sought satisfying the elasticity equations.

Sometimes equilibrium equations are combined with the compatibility equations to give equations which satisfy both. This makes it possible to express the stress components in terms of independent functions of position (17).

When the displacements are specified on the boundary the equations of equilibrium may be written in terms of displacements and then solved. For the deep beams approximate solutions are available by the finite difference method. The area enclosed by the four edges of a deep beam is taken as the x - y plane and this plane is divided into equal divisions having lengths of h and k in the x and y directions respectively thus forming a network. $Z_{x,y}$ denotes the ordinate at each net point (x, y) to a curved surface representing Airy's stress function $z = f(x, y)$. Then, the biharmoric differential equation that Airy's stress function should satisfy is equivalent to a linear equation in terms of Z. The unknown Z values are determined by solving the set of simultaneous linear equations obtained from each net-point (6). Solutions are obtained also by superimposing two stress functions. The first stress function is in the form of a trigonometric series which satisfies all but one of the boundary conditions, that the normal stress on the ends of the beam is zero. Then a second stress function is introduced by the use of least work to give the distribution of normal stresses on the ends. By superimposing the two solutions the boundary conditions are satisfied.

2.3 Plastic Solution

Plasticity is the property that enables a material to be deformed continuously and permanently without rupture during the application of stresses exceeding those necessary to cause yielding of the material. When the initial yield stress is exceeded, deformations reach large amounts under stresses and the final deformation depends not just upon the final state of stress (as is the case of elasticity) but upon the series of stress states from the beginning.

After passing the elastic limit the stress is a monotonically increasing function of the strain, this is known as work - hardening in one dimensional case. In the genaral case when more than one dimension is considered work-hardening concept is more involved depending on the history of loading and the information concerned can be found in Hill (10) or Mendelson (18).

The theory of Plasticity generally assumes perfectly plastic material, that is a material which does not exhibit work hardening but flows plastically under constant stress. When the material is rigid it means that no elastic strains occur. In this thesis homogenous, isotropic and perfectly plastic material will be assumed.

(a) Yield Criteria

For each material there exists a state of stress at which the material will begin to deform plastically. The criteria for deciding which combination of stresses will cause yielding are called yield criteria. In plastic flow analysis the first step is to decide on a yield criterion. Any yield criterion can be expressed in the form

$$f(I_1, I_2, I_3) = 0 \tag{2.6}$$

where I_1, I_2 and I_3 are the three invariants of the stress tensor σ_{ij}. In terms of the principal components of stress $\sigma_1, \sigma_2, \sigma_3$ the invariants are defined as

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{2.7}$$

A stress tensor is a combination of the spherical stress tensor (hydrostatic pressure) and the stress deviator tensor.

$$\sigma_{ij} = S_{ij} + \sigma_m \delta_{ij} \tag{2.8}$$

where σ_m is the mean stress, or,

$$\sigma_m = \frac{1}{3}\sigma_{ii}$$

and S_{ij} is the stress deviator tensor. It was found that the hydrostatic pressure has no effect on yielding or plastic flow. The criterion can be expressed in terms of the stress deviator invariants J_1, J_2 and J_3.

$$f(J_1, J_2, J_3) = 0 \tag{2.9}$$

where

$$J_1 = 0$$
$$J_2 = -(S_1S_2 + S_2S_3 + S_3S_1) \qquad (2.10)$$
$$J_3 = (S_1S_2S_3)$$

The two common yield criteria are Tresca's and Mises'.

Tresca yield criterion assumes that yielding occurs when the maximum shear stress reaches the value of the maximum shear stress under simple tension (18). The maximum shear stress is half the difference between the maximum and minimum principal stresses. This theory requires that the yield stress in tension and compression be equal.

Tresca criterion can be formulated as follows

$$\sigma_1 - \sigma_2 = \pm \sigma_o$$
$$\sigma_2 - \sigma_3 = \pm \sigma_o \qquad (2.11)$$
$$\sigma_3 - \sigma_1 = \pm \sigma_o$$

where σ_o is the yield stress in simple tension.

For the biaxial case ($\sigma_3 = 0$) a graphical representation is given in Fig.1.

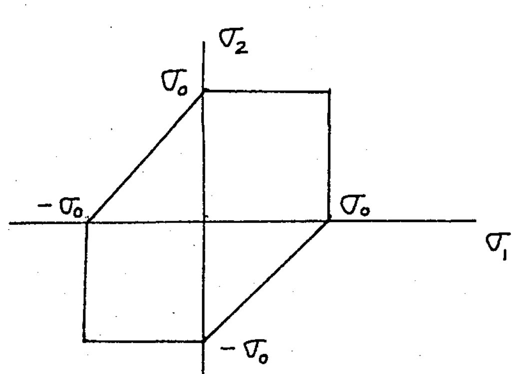

FIG. 1 Tresca Yield Criterion

For the case of pure shear $\sigma_1 = -\sigma_2 = k$ and $\sigma_3 = 0$ substituting these stresses in Equation (2.11) results in

$$k = \sigma_0/2$$

Mises Yield Criterion, Distortion Energy Theory, on the other hand assumes that yielding will occur when the distortion energy equals the distortion energy at yield in simple tension. Mathematically it takes the form

$$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \sigma_0^2 \qquad (2.12)$$

for the plane stress case ($\sigma_3 = 0$) and Eq. (2.12) reduces to

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_0^2 \qquad (2.13)$$

This equation represents an ellipse as shown in Fig. 2.

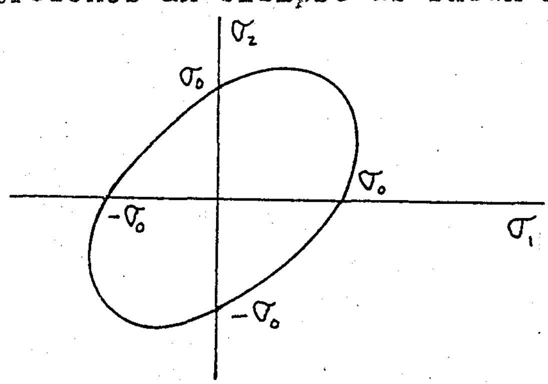

FIG. 2 Mises Yield Criterion

For the case of pure shear $\sigma_1 = -\sigma_2 = k$ and substituting these in Eq. (2.13) results into

$$k = \frac{\sigma_0}{\sqrt{3}}$$

Mises yield criterion has the advantage that the relative magnitudes of the principal stresses are not needed.

Experiments (10) had shown that the Von Mises yield criterion is in excellent aggreement with experiment for many ductile materials like copper, nickel, aluminun, iron, cold-worked mild steel, medium carbon and alloy steels.

For the upper yield-point of annealed mild steel Tresca's Law fits the data better than Mises'. However, the sensitivity of the upper yield point is questionable due to such conditions as the eccentricity of loading, non-uniformity of specimen and the stress concentration in fillets.

(b) Stress - Strain Rate Relationships

The mechanical behavior of a perfectly plastic material cannot be characterized by the yield condition alone. The stress strain relations for the plastic range are also needed. In the plastic range the total strain is the combination of the elastic and the plastic (permanent strain).

During plastic flow it is assumed that the rate of change of the plastic strain at any instant is proportional to the instantaneous stress deviation.

The stress-strain relations in the plastic range were given by Levy and then independently by von Mises as

$$d\varepsilon_{ij} = S_{ij}\, d\lambda \tag{2.14}$$

where $d\varepsilon_{ij}$ is the strain increment and $d\lambda$ is a non-negative constant. In Eq. (2.14) elastic strains are ignored and the total strain increments are assumed to be equal to the plastic strain increments. Later on Prandtl and Reuss generalized Eq. (2.14) to the following form

$$d\varepsilon_{ij}^{p} = S_{ij}\, d\lambda \tag{2.15}$$

where $d\varepsilon_{ij}^{p}$ is the plastic strain increment. Dividing both sides of Eq. (2.15) by dt and noting that $d\varepsilon_{ij}^{p} / dt = v_{i,j}$ results in a relationship between the velocity vector and stress deviator tensor

$$\dot{\varepsilon}_{ij}^{p} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) = \mu S_{ij} \tag{2.16}$$

where μ is another constant.

Prandtl - Reuss relations have been tested experimentally by Lode and then Taylor and Quinney. Although the results are not very accurate, they are assumed to be satisfactory.

To obtain the total strain components, the incremental strain components must be integrated over the whole history of loading. Hencky suggested total stress-strain relations where the total strain components are related to the current stress, and not to the history of loading, which can be shown to be true for the case of proportional loading.

Drucker, by assuming that the rate of work must be zero or positive concluded that the yield surface must be convex and the plastic strain rate $\dot{\varepsilon}_{ij}^{p}$ must be normal to the yield surface and is related to the stress deviator S_{ij} by

$$\dot{\varepsilon}_{ij}^{p} = \mu \frac{\partial f}{\partial S_{ij}} \tag{2.17}$$

where f is the yield function.

2.4 Approximate Solution

(a) Slip Lines and Shear Lines

Solutions to rigid - perfectly plastic material can be obtained under conditions of plain strain, using the theory of slip lines. Plain strain is the condition wherein the displacements all occur in parallel planes in the body, say, planes parallel to the xy plane, and all stresses and strains are independent of the z direction. In a body, having determined the principal stresses and directions, the maximum and minimum shearing stresses can readily be determined. The shearing stresses act on the planes bisecting the principal directions. If curves are drawn in the plane such that at every point of each curve the tangent coincides with one of the maximum shear directions then two families of curves called shear lines or slip lines are obtained. The shear lines are characterized by the stress field and the slip lines by the velocity field. They are orthogonal and for the plane strain case they coincide.

(b) Limit Analysis

If the surface tractions on a body is increased monotonically a state of impending plastic flow will take place causing increase of plastic strain under constant surface tractions.

The ratio of the surface traction at the instant of impending plastic flow to the given value of the surface traction is the safety factor.

The aim of the limit analysis is to find the safety factor $\overline{m}$. When this is not possible then, to find the lower and the upper bounds of the safety factor under increasing surface tractions. The lower bound theorem can be stated as follows: A load which produces a statically admissible stress field will be equal to or less than the true load that will produce plastic flow. A statically admissible stress field is the one that satisfies the equations of equilibrium Eq. (2.1), the boundary conditions Eq. (2.2) and a yield inequality,

$$f \leq 0 \tag{2.18}$$

The upper bound theorem on the other hand is stated as: A load which produces a kinematically admissible velocity field will be equal or greater than the true load. A kinematically admissible velocity field is the one that satisfies the following conditions.

$$v_i = 0 \quad \text{on} \quad S_v \tag{2.19}$$

$$\delta_{ij} v_{i,j} = 0 \quad \text{in} \quad V \tag{2.20}$$

$$\int_{S_T} T_i v_i \, ds > 0 \tag{2.21}$$

$$f = 0 \tag{2.22}$$

In these equations v_i is the velocity vector, T_i is the traction acting on a body of volume V whose surface is divided into two portions S_T and S_v. On S_T the tractions are specified and on S_v the velocity is assumed to vanish.

2.5 Limit Analysis by Direct Method of Variations

A body of volume V is considered under increasing surface tractions mT_i, supported along a surface S_v, where velocity is prescribed. The surface under tractions is denoted by S_T as shown in Fig.3.

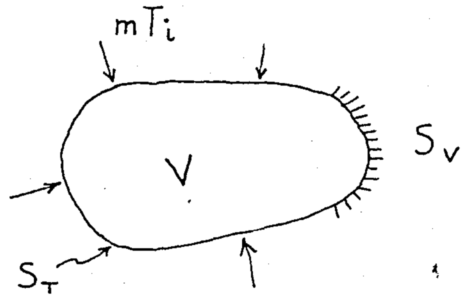

FIG. 3 A Body Under Increasing mT_i

The functional proposed by Lee and Mura which applies to Fig. 3 is the following,

$$F = \int_V S_{ij} \frac{1}{2} (v_{i,j} + v_{j,i}) + \int_V \sigma \delta_{ij} v_{i,j} \, dV - \int_{S_V} R_i v_i \, dS - m \left(\int_{S_T} T_i v_i \, dS - 1 \right) - \int_V \mu (f + \varphi)^2 \, dV \tag{2.23}$$

Subjected to the constraint condition

$$\mu \geq 0 \tag{2.24}$$

μ is the scalar function in the plastic potential flow law defined earlier.

The arguments of F are velocity v_i, deviatoric stress S_{ij} and the Lagrangian multipliers; the hydrostatic pressure $\sigma = \frac{1}{3} \sigma_{kk}$ μ, mT_i, and φ. δ_{ij} is the Kronecker delta. When we set the first variation of F equal to zero we get the following equations.

$$(s_{ij} + \delta_{ij}\sigma)_{,j} = 0 \qquad \text{in } V \qquad (2.25)$$

$$(s_{ij} + \delta_{ij}\sigma)\, n_j = m T_i \qquad \text{on } S_T \qquad (2.26)$$

$$(s_{ij} + \delta_{ij}\sigma)\, n_j = R_i \qquad \text{on } S_V \qquad (2.27)$$

$$f + \varphi^2 = 0 \qquad \text{in } V \qquad (2.28)$$

$$\mu\varphi = 0 \qquad \text{in } V \qquad (2.29)$$

$$\frac{1}{2}(v_{i,j} + v_{j,i}) = \mu \frac{\partial f}{\partial s_{ij}} \qquad \text{in } V \qquad (2.30)$$

$$\delta_{ij} v_{i,j} = 0 \qquad \text{in } V \qquad (2.31)$$

$$v_i = 0 \qquad \text{on } S_V \qquad (2.32)$$

and

$$\int_{S_T} T_i v_i \, dS = 1 \qquad (2.33)$$

where n_j is a normal vector on the surface, R_i the reaction on S_V. Equations (2.25) to (2.27) are the equilibrium conditions. Eq. (2.30) is the plastic potential flow low. Equation (2.28) is the yield condition. Eq. (2.28) and (2.29) define the admissible domain of the stress space. When φ is non-zero, μ is zero and $f \leq 0$. When φ is zero, μ is non-zero and $f = 0$. Eqs. (2.31) to (2.33) define a kinematically admissible velocity field. Eqs. (2.25) to (2.33) represent the conditions of plastic flow.

The value of the functional F under the above conditions is the safety factor $\overline{m}$.

The yield function, f, in Mises Yield Criterion which will be used later may be written in the form

$$f = \frac{1}{2} s_{ij} s_{ij} - k^2 \qquad (2.34)$$

(a) Lower Bound Theorem

When the functional F Eq. (2.23) is integrated under constraint conditions, Eqs. (2.24) to (2.26), it yields

$$F^{o} = m - \int_{V} \mu \left(\frac{1}{2} S_{ij} S_{ij} - k^{2} + \varphi^{2} \right) dV \tag{2.35}$$

By optimizing F^{o} the values of m, φ and S_{ij} are determined and then used, to find a lower bound on the safety factor as follows:

$$m^{o} = \frac{m}{\alpha^{o}} \tag{2.36}$$

Where α^{o} is determined by

$$\max \left(\frac{1}{2} S_{ij} S_{ij} \right) = (\alpha^{o})^{2} k^{2} \tag{2.37}$$

It should be noted that m^{o}, thus defined, is a classical lower bound (16).

(b) Upper Bound Theorem

When the functional F, Eq. (2.23) is integrated under constraint conditions, Eqs. (2.24) and Eqs. (2.29) to (2.33), it yields

$$F^{*} = 2k^{2} \int_{V} \mu \, dV + \int_{V} \mu \left(\frac{1}{2} S_{ij} S_{ij} - k^{2} - \varphi^{2} \right) dV \tag{2.38}$$

F^{*} is optimized to yield μ, φ and S_{ij} which are used to determine an upper bound on the safety factor as follows:

$$m^{*} = 2k^{2} \int_{V} \mu \, \alpha^{*} \, dV \tag{2.39}$$

Where α^{*} is determined by

$$\frac{1}{2} S_{ij} S_{ij} = (\alpha^{*})^{2} k^{2} \tag{2.40}$$

In determining S_{ij} the flow law given by Eq. (2.30) must be used.

m^* defined by Eq. (2.39) can be shown to be the classical upper bound (16).

CHAPTER III

LIMIT ANALYSIS OF PLANE STRESS PROBLEMS

3.1 Introduction

If a thin plate is loaded along its boundary parallel to its plane with uniformly distributed forces, then the stress components on both faces are zero and they are, also, assumed to be zero within the plate. The state of stress is specified by the other two directions and is called plane stress. In plasticity much has been done on plane strain problems.

However, in plane stress due to difficulties encountered less problems are solved. For example, Tresca and Mises yield conditions are different in plane stress where as they agree in plane strain. Furthermore, the characteristics of the stress and velocity equations may or may not agree and they do not need to be orthogonal (10).

In what follows the material will be assumed isotropic, homogeneous and in a state of plane stress. The Mises yield criterion and its associated flow laws will be employed.

3.2 Mathematical Formulation

The theorems of Chapter II for the plane stress case, take the following form:

(a) Lower Bound

Before defining a lower bound the following equations are needed

Equations of equilibrium,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0$$
$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad \text{in A} \qquad (3.1)$$

Boundary condition,

$$\sigma_x n_x + \tau n_y = m T_x$$
$$\tau n_x + \sigma_y n_y = m T_y \qquad \text{on } L_T \qquad (3.2)$$

Mises Yield Criterion,

$$f = \frac{1}{3}\left[(\sigma_x)^2 - \sigma_x \sigma_y + (\sigma_y)^2 + 3(\tau)^2\right] - k^2 \qquad (3.3)$$

The functional,

$$F^\circ = m - \int_A \mu \left\{ f + (\varphi)^2 \right\} dA \qquad (3.4)$$

$$\mu \geq 0 \qquad (3.5)$$

The lower bound m° associated with a stress field which satisfies Eqs. (3.1) and (3.2) is defined as

$$m^\circ = \frac{m}{\alpha^\circ} \qquad (3.6)$$

where α° is determined by

$$\max\left\{\frac{1}{3}\left[(\sigma_x)^2 - \sigma_x \sigma_y + (\sigma_y)^2 + 3(\tau)^2\right]\right\} = (\alpha^\circ)^2 k^2 \qquad (3.7)$$

Eq. (3.4) is optimized to determine the stress field parameters and m, φ and μ.

In the above equations σ_x , σ_y and τ are the stress components. T_x and T_y are the given load components on L_T, the boundary where load is specified, n_x and n_y the components of the unit normal vector in the x and y direction and "A" the area of the domain:

(b) Upper Bound Theorem

Before defining an upper bound, the following equations are needed.

Kinematic constraint,

$$v_x = 0 \qquad \text{on } L_v \tag{3.8}$$
$$v_y = 0 \qquad \text{"}$$

Incompressibility,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{in } A \tag{3.9}$$

The rate of work done on the boundary is positive

$$\int_{L_T} \left(T_x v_x + T_y v_y \right) dL = 1 \tag{3.10}$$

Flow low,

$$\mu \sigma_x = \frac{2\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$
$$\mu \sigma_y = \frac{2\partial v_y}{\partial y} + \frac{\partial v_x}{\partial x} \tag{3.11}$$
$$\mu \tau = \frac{1}{4}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

The functional,

$$F^* = 2k^2 \int_A \mu \, dA + \int_A \mu \left\{ \frac{1}{3}\left[(\sigma_x)^2 - \sigma_x \sigma_y + (\sigma_y)^2 \right] - k^2 - \varphi^2 \right\} dA \tag{3.12}$$

$$\mu \geqslant 0 \tag{3.13}$$

The upper bound m^* obtained from a velocity field that satisfies eqs. (3.8) to (3.11) is given by

$$m^* = 2k^2 \int_A \mu \alpha^* dA \tag{3.14}$$

where α^* is determined from

$$\frac{1}{3}\left(\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau^2 \right) = (\alpha^*)^2 k^2 \tag{3.15}$$

where v_x and v_y are the components of the velocity vector, L_v the part of the boundary on which velocity is prescribed to vanish. The functional, eq.(3.12) is used to determine the parameters in the velocity field, μ and φ.

3.3 Application to Simply Supported Beams

(a) Lower Bound

A rectangular, simply supported beam having a uniform loading of mp is considered. The length of the beam is L, its depth is b and its thickness is unity. The beam and the chosen coordinate axes are shown in Fig.4.

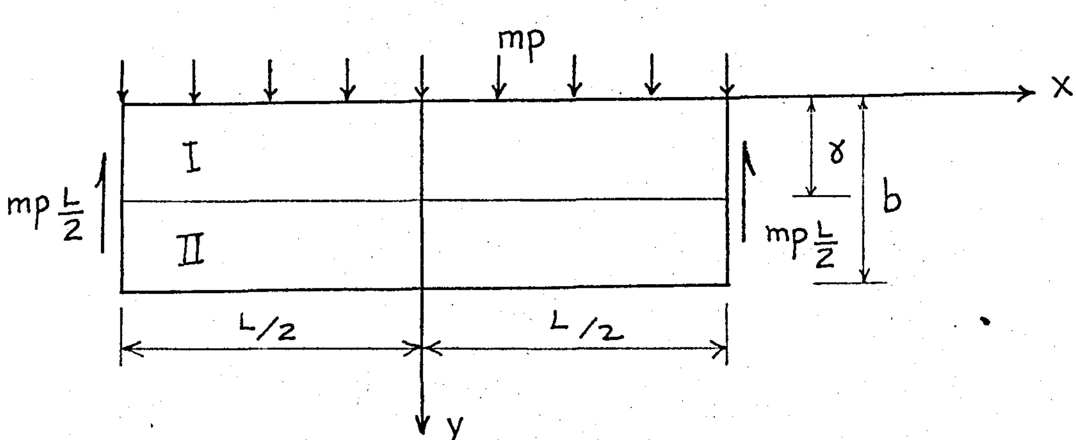

FIG.4 Regions of Stress Field

A stress field consisting of two region of stress components, separated by a line of discontinuity, is assumed. The distance from the top of the beam to the line of discontinuity is denoted by γ. The line of discontinuity, in this case, is the neutral axis. The shear stress, and the normal stress perpendicular to that line are in equilibrium, where as the normal stress parallel the line need not be continuous.

The assumed stress components are,

In Region I

$$\begin{aligned} \sigma_x &= (mp)\left(x^2 - \frac{L^2}{4}\right) B_1 \\ \sigma_y &= (mp)\left(B_1 y^2 + B_2 y + B_3\right) \\ \tau &= -x(mp)\left(2B_1 y + B_2\right) \end{aligned} \tag{3.16}$$

In Region II

$$\begin{aligned} \sigma_x &= (mp)\left(x^2 - \frac{L^2}{4}\right) B_4 \\ \sigma_y &= (mp)\left(B_4 y^2 + B_5 y + B_6\right) \\ \tau &= -x(mp)\left(2B_4 y + B_5\right) \end{aligned} \tag{3.17}$$

These stress components satisfy the equations of equilibrium, Eq. (3.1). B_1 to B_6 are the stress parameters to be determined.

Boundary conditions,

$$\begin{aligned} \sigma_y(x,0) &= -mp \\ \tau(x,0) &= 0 \\ \sigma_y(x,b) &= 0 \\ \tau(x,b) &= 0 \end{aligned} \tag{3.18}$$

$$\sigma_x\left(\pm\frac{L}{2}, y\right) = 0$$
$$\int_0^b \tau\left(\pm\frac{L}{2}, y\right) dy = \mp\, mp\frac{L}{2}$$

The last two boundry conditions are redundant when the first four are satisfied.

The requirement of continuity at the line of discontinuity,

$$\sigma_{y\,I} = \sigma_{y\,II} \qquad \tau_I = \tau_{II} \tag{3.19}$$

and the first four of Eqs. (3.18) provide six equations which results in the solution of the unknown stress parameters as follows,

$$B_1 = \frac{1}{b\gamma} \qquad B_2 = 0 \qquad B_3 = -1$$
$$B_4 = \frac{1}{b(\gamma-b)} \qquad B_5 = -\frac{2}{\gamma-b} \qquad B_6 = \frac{b}{\gamma-b} \tag{3.20}$$

The other unknowns m, φ, μ and γ are determined by taking the first variation of F^o.

The stress components with the stress parameters substituted yields

In Region I

$$\sigma_x = mp\left(x^2 - \frac{L^2}{4}\right)\frac{1}{b\gamma}$$
$$\sigma_y = mp\left(\frac{1}{b\gamma}y^2 - 1\right) \tag{3.21}$$
$$\tau = \frac{-2}{b\gamma}mp\,xy$$

In Region II

$$\sigma_x = mp\left(x^2 - \frac{L^2}{4}\right)\frac{1}{b(\gamma-b)}$$
$$\sigma_y = \frac{mp}{\gamma-b}\left(\frac{y^2}{b} - 2y + b\right) \tag{3.22}$$

$$\tau = -\frac{2mp}{\gamma - b}\left(\frac{y}{b} - 1\right)$$

Substituting Eqs. (3.21) and (3.22) into the functional, F^o, Eq. (3.4) and assuming that μ and φ are constants, lead to

$$F^o = m - \frac{\mu (mp)^2}{3}\left\{\frac{L^5}{30b}\left[\frac{1}{\gamma(b-\gamma)}\right] + \frac{2L^3}{9b} + L\left[\frac{3b^3 + 3b^2\gamma - 7b\gamma^2 + \gamma^3}{15b(b-\gamma)}\right]\right\} + \mu k^2 bL - \varphi^2 \mu bL \quad (3.33)$$

Taking the first variation of F^o with respect to the arguments m , φ , μ and γ and then, setting each one to zero, the following equations are obtained.

$$\begin{aligned} & 1 - \frac{2}{3} m \mu p^2 \left[\epsilon\right] = 0 \\ & \varphi = 0 \\ & -\frac{(mp)^2}{3}\left[\epsilon\right] + k^2 bL = 0 \\ & -4\gamma^5 + 20b\gamma^4 - 28b^2\gamma^3 + 12b^3\gamma^2 + 2L^4\gamma - L^4 b = 0 \end{aligned} \quad (3.34)$$

where

$$\epsilon = \frac{L^5}{30b}\left[\frac{1}{\gamma(b-\gamma)}\right] + \frac{2L^3}{9b} + L\left[\frac{3b^3 + 3b^2\gamma - 7b\gamma^2 + \gamma^3}{15b(b-\gamma)}\right]$$

It can be shown that for

$$\gamma \leq b \text{ , } \mu > 0$$

The lower bound, m^o, is calculated from Eq. (3.6), where α^o is determined by solving

$$\max\left\{\frac{1}{3}\left[(\sigma_x)^2 - \sigma_x \sigma_y + (\sigma_y)^2 + 3(\tau)^2\right]\right\} = (\alpha^o)^2 k^2 \quad (3.35)$$

The maximum value of the bracketed quantity in Eq. (3.35) was found to occur on the boundary of the beam.

The values of m^o.p/k are plotted in Fig. 5 (See page 31).

(b) Upper Bound

Because of symmmetry about the y axis, only half of the beam is considered for finding an upper bound. Four zones are considered with different velocity components as shown in Fig.6. They are

$$\text{Zone 1} \quad \left[0 \le y \le \gamma \qquad 0 \le x \le \omega_1(\gamma - y) \right]$$
$$v_x = -\frac{\delta}{\omega_1} x$$
$$v_y = \delta \left[\frac{L}{2} - \omega_1(\gamma - y) \right]$$
$$\text{for Zone 2} \quad \left[\gamma \le y \le b \qquad 0 \le x \le \omega_2(y - \gamma) \right] \tag{3.36}$$
$$v_x = -\frac{\delta}{\omega_2} x$$
$$v_y = \delta \left[\frac{L}{2} - \omega_2(y - \gamma) \right]$$

for Zones 3 and 4
$$\left[0 \le y \le \gamma \qquad \omega_1(\gamma - y) \le x \le \frac{L}{2} \right]$$
$$\left[\gamma \le y \le b \qquad \omega_2(y - \gamma) \le x \le \frac{L}{2} \right]$$
$$v_x = \delta(y - \gamma)$$
$$v_y = \delta\left(\frac{L}{2} - x\right)$$

where v_x and v_y are the components of the velocity vector in the x and y directions. γ locates the neutral axis and δ is a parameter for the angular velocity, ω_1 and ω_2 are the independent variables whose arc tangent denote the angles.

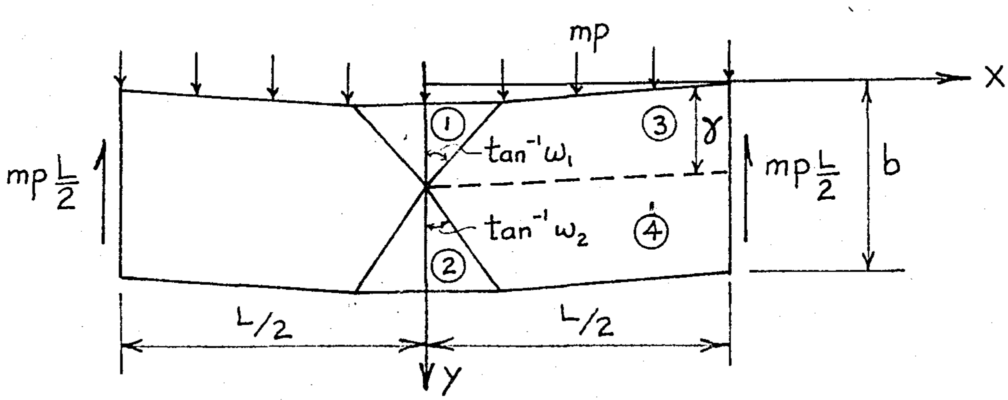

FIG.6 Zones of Velocity Field

Zones 3 and 4 are rigid and $\mu_3 = \mu_4$, $\varphi_3 = \varphi_4$. The velocity field is continuous on the boundaries between the zones.

The velocity field chosen satisfies the kinematic constraint Eq. (3.8), and the incompressibility requirement Eq. (3.9) can be satisfied by adjusting v_z. Substituting Eqs. (3.36) into Eq. (3.10) leads to

$$\delta = \frac{1}{\left\{mp\left[\frac{L^2}{4} - \gamma^2\omega_1^2\right]\right\}} \tag{3.37}$$

Upon substituting the velocity field into the flow law Eq. (3.11) the stress components for each zone are determined as follows.

Zone 1

$$\mu_1\sigma_x = -2\delta/\omega_1 + \delta\omega_1$$
$$\mu_1\sigma_y = 2\delta\omega_1 - \delta/\omega_1$$
$$\mu_1\tau = 0$$

Zone 2

$$\mu_2\sigma_x = 2\delta/\omega_2 - \delta\omega_2$$
$$\mu_2\sigma_y = -2\delta\omega_2 + \delta\omega_2 \tag{3.38}$$
$$\mu_2\tau = 0$$

Zones 3 and 4

$$\mu_3\sigma_x = \mu_3\sigma_y = \mu_3\tau = 0$$

When Eqs. (3.38) are substituted into the F^* Eq. (3.12), assuming constant μ and φ for each zone, the latter takes the form

$$F^* = k^2\left\{\mu_1\gamma^2\omega_1 + \mu_2(b-\gamma)^2\omega_2 + \mu_3\left[Lb - \gamma^2\omega_1 - (b-\gamma)^2\omega_2\right]\right\}$$
$$+ \frac{\gamma^2}{\mu_1(mp)^2\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)^2}\left(\frac{1}{\omega_1} - \omega_1 + \omega_1^3\right) + \frac{(b-\gamma)^2\left(1/\omega_2 - \omega_2 + \omega_2^3\right)}{\mu_2(mp)^2\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)^2} \tag{3.39}$$
$$-\mu_1(\varphi_1)^2\gamma^2\omega_1 - \mu_2(\varphi_2)^2(b-\gamma)^2\omega_2$$
$$-\mu_3(\varphi_3)^2 \quad Lb - \gamma^2\omega_1 - (b-\gamma)^2\omega_2$$

The unknowns $\mu_1, \mu_2, \mu_3, \varphi_1, \varphi_2, \varphi_3, \gamma, \omega_1$ and ω_2 are determined by setting the first variation of F^* given by Eq.(3.39) with respect to the unknowns individually to zero, yields the following equations

$$\mu_1 = \frac{\left(\frac{1}{\omega_1^2} - 1 + \omega_1^2\right)^{1/2}}{k\,mp\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)} \quad , \quad \frac{L^2}{4} > \gamma^2\omega_1^2 \tag{3.40}$$

$$\mu_2 = \frac{\left(\frac{1}{\omega_2^2} - 1 + \omega_2^2\right)^{1/2}}{k\,mp\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)} \quad , \quad \frac{L^2}{4} > \gamma^2\omega_1^2$$

$$\mu_3 = 0 \qquad \varphi_1 = \varphi_2 = 0 \qquad \varphi_3 = k$$

$$2\sqrt{3}\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)\left(-1 + 2\omega_1^2\right) + 4\sqrt{3}\,\gamma^2\left(1-\omega_1^2+\omega_1^4\right) + 6(b-\gamma)^2\left(1-\omega_1^2+\omega_1^4\right)^{1/2} = 0$$

$$\omega_2^2 = 1/2$$

$$L^2\gamma\left(1-\omega_1^2+\omega_1^4\right)^{1/2} + 2\sqrt{3}(b-\gamma)(\gamma\omega_1^2 b) - 2\sqrt{3}\,(b-\gamma)\frac{L^2}{4} = 0$$

γ and ω_1 are determined by solving the above equations.

On the other hand, α^* is determined from,

$$\frac{1}{3}\left[(\sigma_x)^2 - \sigma_x\sigma_y + (\sigma_y)^2 + 3(\tau)^2\right] = (\alpha^*)^2 k^2$$

which results in

$$\alpha^* = k \tag{3.41}$$

The upper bound m^*, Eq.(3.14), may be written as

$$m^* = \frac{2k\left[\left(1-\omega_1^2+\omega_1^4\right)^{1/2}\gamma^2 + \left(1-\omega_2^2+\omega_2^4\right)^{1/2}(b-\gamma)^2\right]}{p\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)} \tag{3.42}$$

The upper bound m^* is calculated from Eq. (3.42) after substituting the values of γ and ω_1. The results of m^*p/k are plotted against the length to depth ratio, L/b in Fig. 5.

(c) Comparison with the Simple Theory

The safety factor obtained from the simple theory, in which the effect of σ_y and τ are neglected, is given by the equation

$$\frac{m\,p}{k} = 2\sqrt{3}\left(\frac{b}{L}\right)^2 \qquad (3.43)$$

The values obtained are plotted in Fig. 5, for comparison.

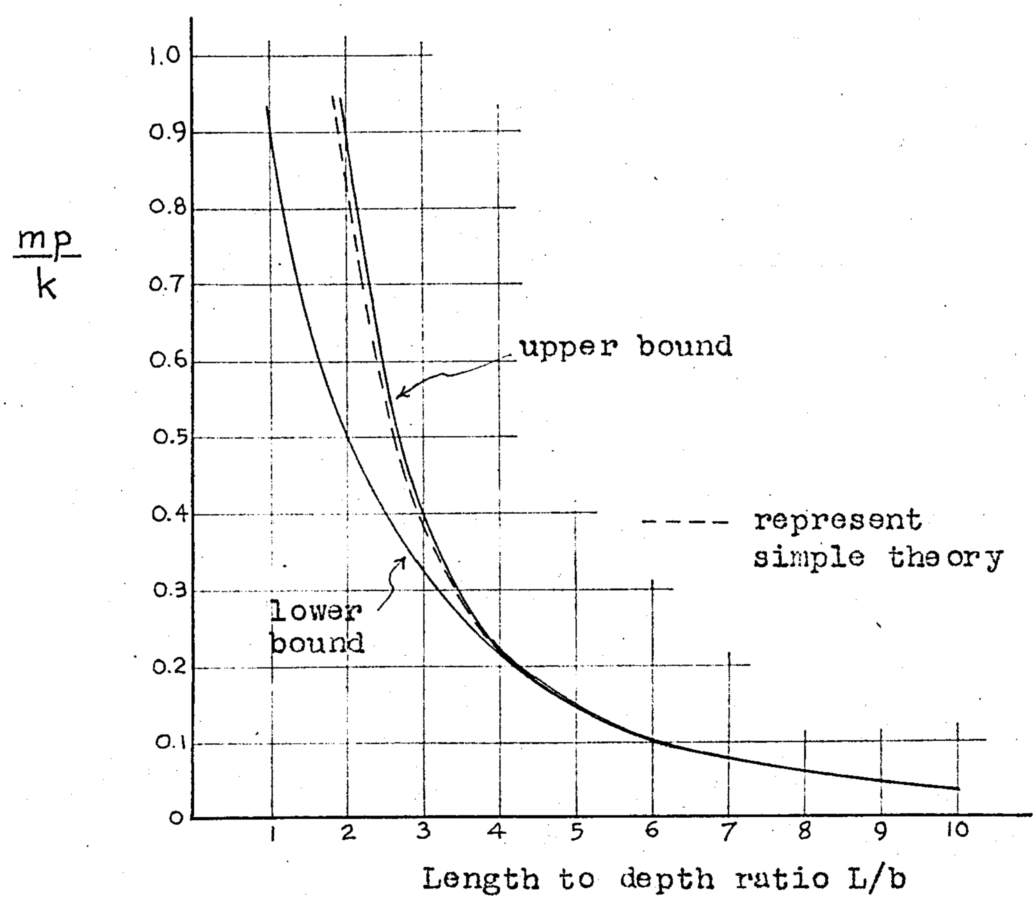

FIG. 5 Safety Factor of Simply Supported Beam

3.4 Application to Fixed Ended Beams

(a) Lower Bound

The beam of Fig.4 is now assumed to be fixed ended and reproduced in Fig.7. The same stress field as in the case of a simply supported beam is used except that σ_x is not equal to zero at $x = L/2$. Therefore, only the σ_x component is changed as follows.

In Region I, $\sigma_x = \dfrac{mpx^2}{b\gamma} + B$

In Region II $\sigma_x = \dfrac{mpx^2}{b(\gamma-b)} + D$ (3.44)

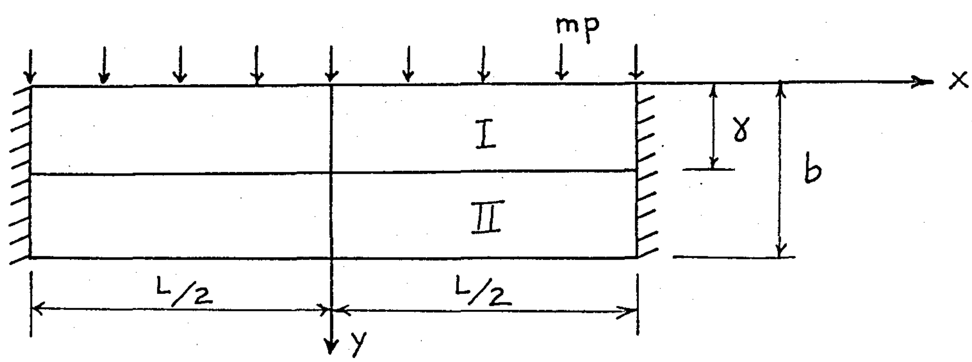

FIG.7 Regions of Stress Field

Assuming constant μ and φ the stress components are substituted into the functional F^o Eq.(3.4) and resulting is

$$F^\circ = m - \frac{2}{3}\mu\Big\{-\frac{L^5(mp)^2}{160\,b\gamma(\gamma-b)} + \frac{L^3 mp}{24b}\Big[2B-2D+\frac{mp}{b}\Big(\frac{8}{3}\gamma+2b\Big)\Big]$$
$$+\frac{L}{2}\Big[B^2\gamma + Bmp\gamma\Big(-\frac{\gamma}{3b}+1\Big) + D^2(b-\gamma) + \frac{Dmp}{3b}(b-\gamma)^2$$
$$+\frac{(mp)^2}{15b}\big(3b^2+6b\gamma-\gamma^2\big)\Big]\Big\} + \mu k^2 bL - \mu\varphi^2 bL \qquad (3.45)$$

The extra unknowns B,D and also γ are obtained by setting the first variation of the functional, F^o, Eq.(3.45) with respect to the unknowns individually to zero, leads to the following equations

$$\frac{L^2}{6b^2} + \frac{2B\gamma}{b} + \frac{\gamma}{b}\left(-\frac{\gamma}{3b} + 1\right) = 0$$

$$-\frac{L^2}{6b^2} + 2D\left(1 - \frac{\gamma}{b}\right) + \frac{1}{3}\left(1 - \frac{\gamma}{b}\right)^2 = 0$$

$$\frac{L^4(2\gamma - b)}{80\, b\gamma^2(\gamma - b)^2} + \frac{2}{9}\left(\frac{L}{b}\right)^2 + B^2 - \frac{2B\gamma}{3b} + B \qquad (3.46)$$

$$-D^2 - \frac{2D}{3}\left(1 - \frac{\gamma}{b}\right) + \frac{2}{15}\left(3 - \frac{\gamma}{b}\right) = 0$$

However, it should be noted that from $\frac{\partial F}{\partial \mu} = 0$ it can be shown that $\mu > 0$ for $\gamma \leq b$ which is a required condition.

To solve for B, D and γ from Eqs. (3.46); for a certain L/b ratio, a value for γ is chosen then B and D values are determined from the first two equations Eq. (3.46). Afterwards the results are checked by the third equation. By trial and error a satisfactory result is found.

Then all stress components are obtained. When the quantity $\frac{1}{3}\left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau^2\right)$ is maximized α° is determined from Eq. (3.7). In this case, also, the maximum values occur on the boundaries.

The lower bound, m^{o}, is given by Eq. (3.6). The results are plotted in Fig. 8, (See page 38).

(b) Upper Bound

The upper bound solution for a fixed end beam is similar to the simply supported case, except that at the fixed end two more yielded zones are assumed as shown in Fig. 9.

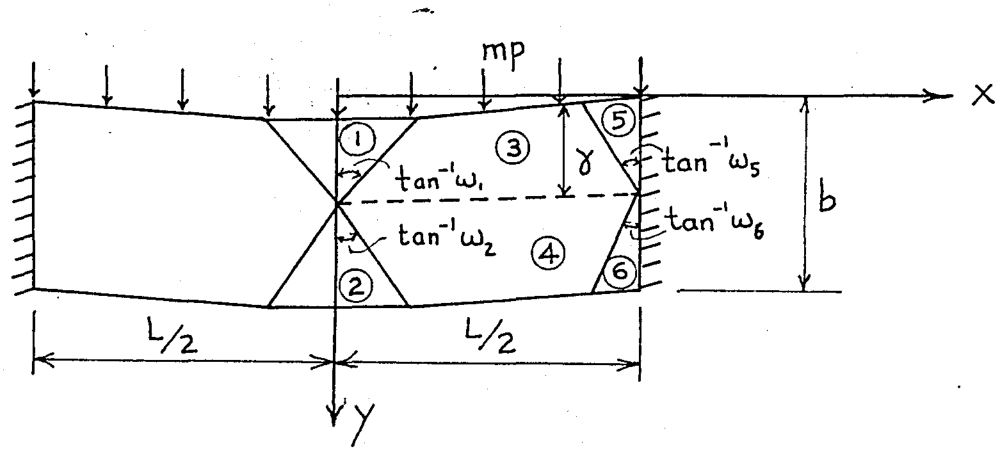

FIG.9 Zones of Velocity Field

The velocities in zones 1 to 4 are the same as given by Eq.(3.36) only the range of x is different for zones 3 and 4.

for zone 3 $$\left[\omega_1(\gamma-y) \leq x \leq \frac{L}{2} - \omega_5(\gamma-y)\right]$$

for zone 4 $$\left[\omega_2(y-\gamma) \leq x \leq \frac{L}{2} - \omega_6(y-\gamma)\right] \qquad (3.47)$$

zone 5 $$\left[L/2 - \omega_5(\gamma-y) \leq x \leq L/2\right]$$
$$v_x = -\frac{\delta}{\omega_5}\left(\frac{L}{2} - x\right)$$
$$v_y = \delta\left(\frac{L}{2} - x\right)$$

zone 6 $$\left[L/2 - \omega_6(y-\gamma) \leq x \leq L/2\right]$$
$$v_x = \frac{\delta}{\omega_6}\left(\frac{L}{2} - x\right)$$
$$v_y = \delta\left(\frac{L}{2} - x\right)$$

The velocity field chosen satisfies the kinematic constraint Eq.(3.8) $v_x = 0$ and $v_y = 0$. The incompressibility requirement Eq.(3.9) can be satisfied by adjusting v_z. Substituting the velocity field into Eq.(3.10) leads to

$$\delta = \frac{1}{\left[mp\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)\right]} \qquad (3.48)$$

From the flow law the following is obtained

for zone 5

$$\mu_5 \sigma_x = \frac{2\delta}{\omega_5}$$
$$\mu_5 \sigma_y = \frac{\delta}{\omega_5} \qquad (3.49)$$
$$\mu_5 \tau = -\frac{\delta}{4}$$

for zone 6

$$\mu_6 \sigma_x = -\frac{2\delta}{\omega_6}$$
$$\mu_6 \sigma_y = -\frac{\delta}{\omega_6}$$
$$\mu_6 \tau = -\frac{\delta}{4}$$

Substituting the stress components Eqs. (3.38) and (3.49) into the functional F^* Eq. (3.12), assuming constant μ and φ for each zone results in

$$\begin{aligned} F^* = k^2 \Big\{ & \mu_1 \gamma^2 \omega_1 + \mu_2 (b-\gamma)^2 \omega_2 + \mu_5 \gamma^2 \omega_5 + \mu_6 (b-\gamma)^2 \omega_6 \\ & + \mu_3 \left[Lb - \gamma^2 \omega_1 - (b-\gamma)^2 \omega_2 - \gamma^2 \omega_5 - (b-\gamma)^2 \omega_6 \right] \Big\} \\ & + \frac{\gamma^2}{\mu_1 (mp)^2 \left(\frac{L^2}{4} - \gamma^2 \omega_1^2 \right)^2} \left(\frac{1}{\omega_1} - \omega_1 + \omega_1^3 \right) + \frac{(b-\gamma)^2}{\mu_2 (mp)^2 \left(\frac{L^2}{4} - \gamma^2 \omega_1^2 \right)^2} \left(\frac{1}{\omega_2} - \omega_2 + \omega_2^3 \right) \\ & + \frac{\gamma^2}{\mu_5 (mp)^2 \left(\frac{L^2}{4} - \gamma^2 \omega_1^2 \right)^2} \left(\frac{1}{\omega_5} + \frac{1}{16} \omega_5 \right) + \frac{(b-\gamma)^2}{\mu_6 (mp)^2 \left(\frac{L^2}{4} - \gamma^2 \omega_1^2 \right)^2} \left(\frac{1}{\omega_6} + \frac{1}{16} \omega_6 \right) \\ & - \mu_1 \varphi_1^2 \gamma^2 \omega_1 - \mu_2 \varphi_2^2 (b-\gamma)^2 \omega_2 - \mu_5 \varphi_5^2 \gamma^2 \omega_5 - \mu_6 \varphi_6^2 (b-\gamma)^2 \omega_6 \\ & - \mu_3 \varphi_3^2 \left[Lb - \gamma^2 \omega_1 - (b-\gamma)^2 \omega_2 - \gamma^2 \omega_5 - (b-\gamma)^2 \omega_5 \right] \end{aligned} \qquad (3.50)$$

The unknowns μ_1 to μ_6, φ_1 to φ_6, ω_1, ω_2 and γ are determined by setting the first variation of F^* Eq. (3.50) with respect to the unknowns individually to zero, the following eqs. are obtained.

$$\mu_1 = \frac{\left(\frac{1}{\omega_1^2} - 1 + \omega_1^2 \right)^{1/2}}{kmp \;\; L^2/4 - \gamma^2 \omega_1^2} \quad , \quad \frac{L^2}{4} > \gamma^2 \omega_1^2$$

$$\mu_2 = \frac{\left(\frac{1}{\omega_2^2} - 1 + \omega_2^2 \right)^{1/2}}{kmp \;\; L^2/4 - \gamma^2 \omega_1^2} \quad , \quad \frac{L^2}{4} > \gamma^2 \omega_1^2$$

$$\mu_3 = 0$$

$$\mu_5 = \frac{\left(1/\omega_5^2 + 1/16\right)^{1/2}}{kmp\left(L^2/4 - \gamma^2\omega_1^2\right)} \quad , \quad \frac{L^2}{4} > \gamma^2\omega_1^2$$

$$\mu_6 = \frac{\left(1/\omega_6^2 + 1/16\right)^{1/2}}{kmp\left(L^2/4 - \gamma^2\omega_1^2\right)} \quad , \quad \frac{L^2}{4} > \gamma^2\omega_1^2$$

$$\varphi_1 = \varphi_2 = \varphi_5 = \varphi_6 = 0 \qquad (3.51)$$

$$\varphi_3 = k$$

$$\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)\left(\frac{1}{\omega_1^2} - 1 + \omega_1^2\right)^{1/2} + \frac{\left(\frac{L^2}{4} - \gamma^2\omega_1^2\right)\left(-\frac{1}{\omega_1^2} - 1 + 3\omega_1^2\right) + 4\gamma^2\left(1 - \omega_1^2 + \omega_1^4\right)}{\left(\frac{1}{\omega_1^2} - 1 + \omega_1^2\right)^{1/2}}$$

$$\frac{4(b-\gamma)^2\left(\frac{1}{\omega_2} - \omega_2 + \omega_2^3\right)\omega_1}{\left(\frac{1}{\omega_2^2} - 1 + \omega_2^2\right)^{1/2}} + \frac{4\left(\frac{1}{\omega_5} + \frac{\omega_5}{16}\right)\gamma^2\omega_1}{\left(\frac{1}{\omega_5^2} + \frac{1}{16}\right)^{1/2}} + \frac{4(b-\gamma)\left(\frac{1}{\omega_6} + \frac{\omega_6}{16}\right)\omega_1}{\left(\frac{1}{\omega_6^2} + \frac{1}{16}\right)^{1/2}} = 0$$

$$\omega_2^2 = \frac{1}{2}$$

$$L^2\gamma\left(1 - \omega_1^2 + \omega_1^4\right)^{1/2} + L^2\gamma\left(1 + \frac{\omega_5^2}{16}\right)^{1/2} - 2\sqrt{3}\,(b-\gamma)\left(\frac{L^2}{4} - \gamma b\omega_1^2\right)$$
$$- 4\left(1 + \frac{\omega_6^2}{16}\right)^{1/2}(b-\gamma)\left(\frac{L^2}{4} - \gamma b\omega_1^2\right) = 0$$

The parameters ω_5 and ω_6 were given the value of unity each since the variation did not yield their values. ω_1 and γ are solved using the above equations. The last three equations of Eqs. (3.51), with $\omega_5 = \omega_6 = 1$, yield,

$$L^2\gamma(1-\omega_1^2+\omega_1^4)^{1/2} + 1.03\,L^2\gamma - 7.59\,(b-\gamma)\left(L^2/4 - \gamma b\omega_1^2\right) = 0$$

$$\left(\frac{L^2}{4}-\gamma^2\omega_1^2\right)\left(\frac{1}{\omega_1^2}-1+\omega_1^2\right)^{1/2} + \frac{\left(\frac{L^2}{4}-\gamma^2\omega_1^2\right)\left(-\frac{1}{\omega_1^2}-1+3\omega_1^2\right)}{\left(\frac{1}{\omega_1^2}-1+\omega_1^2\right)^{1/2}} \qquad (3.52)$$

$$+\frac{4\gamma^2(1-\omega_1^2+\omega_1^4)}{\left(\frac{1}{\omega_1^2}-1+\omega_1^2\right)^{1/2}} + 4.12\,\gamma^2\omega_1 + 7.59\,(b-\gamma)^2\,\omega_1 = 0$$

These two equations are solved by trial and error. The sensitivity of γ and ω_1 causes fluctuations in the results. However, the second equation is less sensitive to γ. Therefore, ω_1 is determined from the latter and then substituted into the former as a check. Solving for α^* from

$$\frac{1}{3}\left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau^2\right) = (\alpha^*)^2 k^2$$

results into

$$\alpha^* = k^2$$

The upper bound m^* Eq.(3.14) leads, then, to

$$m^* = \frac{2k}{P\left(L^2/4-\gamma^2\omega_1^2\right)}\Bigg[\left(\frac{1}{\omega_1^2}-1+\omega_1^2\right)^{1/2}\gamma^2\omega_1 + \left(\frac{1}{\omega_2^2}-1+\omega_2^2\right)^{1/2}(b-\gamma)\,\omega_2 \qquad (3.53)$$

$$\left(\frac{1}{\omega_5^2}+\frac{1}{16}\right)^{1/2}\gamma^2\omega_5 + \left(\frac{1}{\omega_6^2}+\frac{1}{16}\right)^{1/2}(b-\gamma)\,\omega_6\Bigg]$$

The results are plotted in Fig.8 for various L/b ratios.

(c) Comparison with the Simple Theory

Simple plastic Theory yields

$$\frac{m_p}{k} = 4\sqrt{3}\left(\frac{b}{L}\right)^2 \qquad (3.54)$$

The values obtained from Eq. (3.54) are also plotted in Fig.8 for comparison.

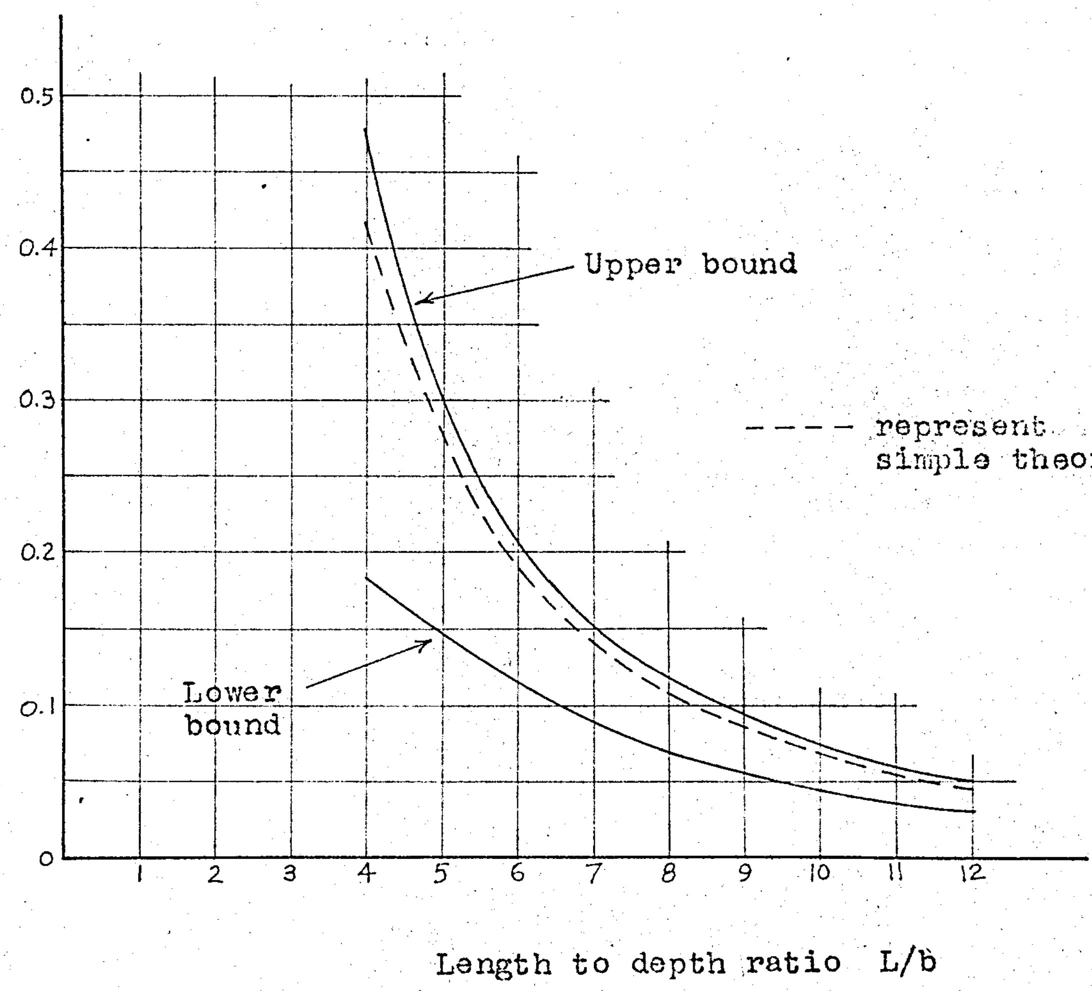

FIG.8 Safety Factor of Fixed Ended Beams

CHAPTER IV

DISCUSSION AND CONCLUSION

In the case of simply supported beam satisfactory results are obtained for L/b ratios as low as 3.5. For smaller ratios bounds begin to deviate as shown in Fig. (5).

Ball and Lee (19) solved the same problem using the classical method of limit analysis and obtained results for L/b ratios larger than 3.55. For values lower than 3.55 their solution is not applicable.

In this thesis bounds for L/b ratios less than 3.55 are available and one may choose the values of the lower bound determined and be on the safe side.

The simple plastic theory yields values close to those of the upper bound in this work. This is in agreement with Hodges findings (20) for simply supported beams subjected to a concentrated load at its midspan.

In the lower bound solution the maximum of

$$\frac{1}{3}\left(\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau^2 \right)$$

for large values of L/b ratio was found to be in the region II, Fig.4 for L/b ratio smaller than 3.5 the maximum point shifts to region I.

The case of fixed ended beams is treated here for the first time as far as could be determined from the literature survey. The results are not as good as those of the simple beam case and until better bounds are found the present solution can be used if desired.

In the upper bound solution first variation of the functional did not yield values for ω_5 and ω_6. Therefore a value was assigned to them to obtain the upper bound. The analysis of the fixed ended beam was similar to that of the simply supported beam, except in the lower bound case σ_x at the supports was not equal to zero and in the upper bound solution additional yield zones at the ends of the beam are considered. The simple theory gives values close to the upper bound.

To improve the lower bounds other stress fields which may be more involved should be considered. Series representation may be assumed and also, the regions of constant stress fields could be chosen smaller thus finite difference or finite element method may be applied. In the case of the upper bounds smaller zones can be chosen and the associated velocity fields be adjusted and the finite difference or the finite elements method may be used.

In the classical method of limit analysis, to compute a lower bound the point at which max $(1/2\ S_{ij}\ S_{ij})$ occurs has to be assumed first in order to satisfy the yield condition. When the assumption is not correct the computations have to be repeated. In the limit analysis by the Direct Method of Variation this point is located after satisfying all the constraint conditions. Therefore no such assumption is needed.

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