ON THE COSMOLOGICAL IMPACT OF A QUANTIZED SCALAR FIELD

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Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Graduate Program in Physics
Boğaziçi University
2015

ACKNOWLEDGEMENTS

First, I would like to express my sincere gratitude to my thesis supervisor Prof. Ali Kaya for his perfect guidance, patience and never ending support during the preparation of this thesis. I gained insight into different topics in theoretical physics by his lectures.

I also want to thank my dear friends Emine Ertuğrul, Merve Uzun and Emine Şeyma Kutluk. They were always with me while I was writting my thesis.

Finally, I would like to thank my husband and my mother for all their support and encouragement that motivated me and made me feel confident.

ABSTRACT

ON THE COSMOLOGICAL IMPACT OF A QUANTIZED SCALAR FIELD

We explicitly compute the leading order contribution to the stress-energy- momentum tensor of a scalar field, which propagates in a Friedmann-Robertson-Walker (FRW) universe and placed in the infinite adiabatic vacuum. Since the adiabatic regularization requires removing the adiabatic zeroth, second and fourth order terms, one ends up with the sixth order expressions as the leading order contribution to the stress-energy-momentum tensor. This is a good approximation for all modes of a massive field when the mass m of the field is larger than the Hubble parameter H and only for the high energy (subhorizon) modes of a massless field satisfying $k > k_*$, where k_* is a fixed comoving momentum scale obeying $k_* \gg aH$. To determine the magnitude of the vacuum energy density, we consider the spacetimes undergoing power-law expansion and discuss the implications for different cosmologically relevant backgrounds. We next consider the backreaction effects of a quantized massless real scalar field propogating in a FRW universe. This can be viewed as a semiclassical approach, where gravity is treated classically. We manage to obtain a new first order differential equation, which approximately determines the evolution of the vacuum energy density of the massless scalar field. In this procedure, we utilize an approximation that uses the fact that the subhorizon modes evolve nearly adiabatically and the superhorizon modes freeze out. To check the validity of our method, we take fixed backgrounds of cosmological interest such as de Sitter space and our findings are shown to agree with the known results in the literature. We also analyse the possible implications of the backreaction effects for the slow-roll inflationary models. We find out that although these are negligible during slow-roll regime, the vacuum energy density created might have a cosmological significance at subsequent stages, since it decreases slower than radiation or dust.

ÖZET

KUANTUM SKALER ALANIN KOZMOLOJİK ETKİLERİ

Bu tezde, ilk olarak Friedmann-Robertson-Walker (FRW) uzayında yayılan ve sonsuz adyabatik vakumda tanımlı bir skaler alanın stres-enerji-momentum tensörüne başlıca ölçülebilir katkıyı açık bir şekilde hesapladık. Adyabatik regularizasyon, adyabatik sıfırıncı, ikinci ve dördüncü düzeydeki terimleri çıkartmayı gerektirdiğinden, adyabatik altıncı düzeydeki ifadeler stres-enerji-momentum tensörüne başlıca ölçülebilir katkıyı sağlayan terimlerdir. Bu, kütlesi olan alanların eğer kütlesi Hubble parametresi H ' den büyükse, her modu için, kütlesiz alanların ise sadece $k > k_*$ koşulunu sağlayan modları için iyi bir yöntemdir (k_*, $k_* \gg aH$ koşulunu sağlayan sabit momentum ölçeğidir). Zamanın kuvvetiyle orantılı olarak genişleyen uzaylarda, vakum enerji yoğunluğunun büyüklüğünü hesapladık ve farklı kozmolojik uzaylarda bulduğumuz sonuçları tartıştık.

İkinci olarak, FRW uzayında yayılan kütlesiz kuantum skaler bir alanın geri-reaksiyon etkilerini araştırdık. Bu, yerçekiminin klasik olarak ele alındığı, yarı-klasik bir yaklaşımdır. Kütlesiz skaler alanın vakum enerji yoğunluğunun zamana bağlı nasıl değiştiğini yaklaşık olarak belirleyen 1. dereceden yeni bir diferansiyel denklem elde ettik. Bu yöntemde, ufuk altı modların yaklaşık olarak adyabatik değiştiği, ufuk üstü modların ise sabit kaldığı gerçeğinden yararlandık. Yöntemimizin geçerliliğini ölçmek için de Sitter uzayı gibi kozmolojik uzaylarda bulduğumuz sonuçlari literatürdeki sonuçlarla kıyasladık ve bulgularımızın birebir örtüştüğünü gördük. Ayrıca, geri-reaksiyon etkilerinin bazı enflasyon modellerinde olası sonuçlarını inceledik. Enflasyon sırasında geri-reaksiyon etkileri ihmal edilecek kadar küçük olmasına rağmen, enflasyon sonrasındaki evrelerde bu etkilerin kozmolojik önemi olabileciğini gördük.

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LIST OF SYMBOLS

a	Scale factor
$a_{\mathbf{k}}$	Creation operator
$a_{\mathbf{k}}^{\dagger}$	Annihilation operator
$g_{\mu\nu}$	Metric tensor
h	Hubble parameter in conformal time
H	Hubble parameter in proper time
$\mathbf{k}$	Comoving momentum
m	Mass of the scalar field
M_p	Planck mass
n	Expansion power
N_k	Number operator
P	Pressure
S	Action
t	Proper time
$T_{\mu\nu}$	Stress-energy-momentum tensor
α	Parameter quantifying subhorizon/superhorizon border
ϵ_{φ}	Slow-roll parameter
η	Conformal time
η_{φ}	Slow-roll parameter
μ	Field redefinition
μ_k	Mode function
ξ	Coupling to gravity
ρ	Energy density
ϕ	Scalar filed
φ	Inflaton field
ψ	Di-gamma function

LIST OF ACRONYMS/ABBREVIATIONS

BD	Bunch-Davies
CMB	Cosmic Microwave Background
FRW	Friedmann-Robertson-Walker
IR	Infrared
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
UV	Ultraviolet
WKB	Wentzel–Kramers–Brillouin

1. INTRODUCTION

Quantum field theory (QFT) in flat space time is a successful synthesis of quantum mechanics and special relativity (for more information see e.g. [1–3]). Here, one quantizes the field itself instead of quantizing a single particle and the particles are defined as the excitations of that field. QFT is basically used to describe the elementary particles, but it has also widespread applications in different branches of the modern research. For instance, in condensed matter physics, one can consider the excitations in a solid as the quanta of the fields. On the other hand, the path integral methods and Feynmann diagrams turn out to be very useful tools in statistical mechanics. Although there is a finite number of exactly solvable models, QFT has unignorable successes in perturbation theory where the observations and the theoretical predictions are in agreement with high accuracy. Some of the well known examples are the hyperfine splitting of the hydrogen ground state, the Lamb shift in the hydrogen atom, the determination of the gyromagnetic ratio of the electron, the running of the electromagnetic coupling α in quantum electrodynamics (QED) and the derivation of Fermi's theory for beta decay of the neutron in the electroweak theory.

As QFT has many successful applications in flat spacetime, a natural generalization is to formulate the theory in curved spacetime. Among other things, this can be viewed as an interesting problem in pure mathematics. The Poincare symmetry plays a very crucial role in flat space QFT, like it implies the uniqueness of the vacuum state. It is an important problem to determine how QFT changes in the absence of the special properties of the flat spacetime. For instance, the non-uniqueness of the vacuum is one such feature and we will discuss how this can be dealt with. Not surprisingly, QFT in curved spacetime has a key part to play in understanding the early time cosmology. In this thesis we consider its application to a specific problem.

The recent discovery that the universe is accelerating (see e.g. [4]) indicates the existence of a non-zero cosmological constant. It is clear that, the ordinary matter can not creat such an influence and mimic a cosmological constant. This pushes for searching new candidates like quantum fields that can apply negative pressure which may lead to possitive acceleration.

It is tempting to identify cosmological constant as the vacuum energy corresponding to (a set of) quantum fields. However, there is a huge order of magnitude discrepancy between the estimated and observed values of the cosmological constant if it is attributed as a vacuum energy (see e.g. [5]). This is called the cosmological constant problem. Our findings support that, this problem is an infrared issue rather than being an ultraviolet one in QFT.

Once a quantum field is given on a curved background, a scalar field in a special case of us, one can define a natural stress-energy-momentum tensor $T_{\mu\nu}$ of that field. Following the quantization procedure, $T_{\mu\nu}$ becomes an operator and in a semiclassical approximation one deals with its vacuum expectation value. As we will see, even in the free theory (in the absence of any interactions) these expressions include divergences. Since $T_{\mu\nu}$ forms the right hand side of the Einstein equations and affects the geometry, it should be a finite quantity to be compatible with the physical system. So one should cure these divergences in a systematical manner to relate the theory with physical observables.

In flat spacetime in free field theory, one can get rid of such divergences by appliying normal ordering. On the other hand, due to the non-trivial dependence on the background, regularization in cosmology is not so obvious and easy as compared to flat spacetime. To make sense of physical quantities, various tecniques have been developed to remove those divergent terms such as proper-time, point splitting, zeta-function, dimensional and adiabatic regularizations (for details see e.g. [6]). In this thesis we will concentrate on adiabatic regularization.

One can make the analogy between the well known Casimir effect and adiabatic regularization. In a physical system consisting of a quantum field between two infinite parallel plates, the finite Casimir energy associated with the plates is determined by subtracting the contribution of the infinite space itself. This effect can be thought of as the shift of the zero-point energy of the electromagnetic field due to the presence of the conductors. In the curved spacetime scheme, adiabatic regularization can be viewed as extracting out the mode by mode contributions of the suitably defined adibatic basis by a systematical procedure.

Adiabatic regularization was used to determine the stress-energy-momentum tensor of scalar fields propagating in various curved spacetimes. As a review one may see [7], where adiabatic regularization of the stress-energy-momentum tensor of a quantized massive scalar field minimally coupled to the gravitational field was carried out in detail for the three types of FRW metric (closed, flat, and open). In [8], one can get idea about obtaining the regularized energy density and pressure in de Sitter space and in a radiation dominated FRW universe. The effects of free quantized scalar fields that propagate in cosmological spacetimes with Big Rip singularities were investigated in [9]. It was shown that for the regularized massless minimally coupled field an attractor state exists , while for the massive field the energy density is always seen to asymptotically approach the energy density of the corresponding massless field. As another example, it was discussed in [10] that a massive quantized field can avoid a cosmic singularity by preventing a cosmological collapse and converting it to an expansion in a closed FRW geometry. Finally, it was shown in [11] that at sufficiently early cosmological epochs, the quantum vacuum effects dominate due to trace anomaly and the strong energy condition is violated which induces acceleration.
In this thesis, we add to this numerious set of examples a new one and examine the leading order contribution to the energy density and pressure for a quantized scalar field in the adiabatic vacuum state [12].

In the above examples, the background is assumed to be fixed and the stress-energy-momentum tensor of the quantum fields are calculated in a given metric. However, the cosmic evolution of the scale factor is not known if the universe is dominated by a quantum field itself. In this thesis, we would like to study the backreaction problem of the quantum fields on the geometry, where the background is not fixed [13]. To elucidate on the problem, note that in a cosmological spacetime, the quantum field, a massless scalar field in our case, can be conveniently decomposed into modes in Fourier space. To determine the cosmic evolution, one must deal with a coupled integro-differential equation system which involves both the mode functions and the scale factor of the FRW universe. In other words, the mode functions are affected by the expansion via the Klein Gordon equation on the background. Simultaneously, the stress-energy-momentum tensor is determined by the integrals depending on the mode functions. Due to the particle creation effects, the stress-energy-momentum tensor feeds the expansion and fixes the evolution of the scale factor.

In fact, the coupled equation system is very hard to tackle exactly (see e.g. [14–18] for some attempts). But, by treating the subhorizon and superhorizon modes separately, their approximate contributions to the stress-energy-momentum tensor can be evaluated. The crucial point is that the subhorizon modes evolve nearly adiabatically, so their contributions after regularization turn out to be negligible. On the other hand, the stress-energy-momentum tensor of the superhorizon modes obeys an equation of state with state parameter $w = -1/3$. This separation can be made safely for the accelerating backgrounds, which we deal with. In that way, we obtain a set of simple equations consisting of two unknown functions, the scale factor of the universe and the energy density of the vacuum fluctuations. We also take care of the adiabatic subtraction terms that survive in the massless limit of the generalized expressions and treat the infrared divergences carefully. We apply our findings to different fixed backgrounds like de Sitter space. They are in very good agreement with the known results in the literature.

In this thesis we shall proceed as follows: In chapter II, we give a brief review of the standard canonical quantization. Fixing the vacuum is a subtle issue in a curved background and we discuss some physically motivated prescriptions used to define the vacuum state such as the Bunch-Davies (BD) vacuum and the adiabatic vacuum. In chapter III, using the definition of the stress-energy-momentum tensor for a scalar field, we redrive the general expressions for the expectation values of the vacuum energy density and the pressure. As expected, these expressions are plagued by infinities. Among the well known regularization techniques, we concentrate on adiabatic regularization which involves a mode by mode subtraction procedure. We work out its application to the stress-energy-momentum tensor. In chapter IV, we explicitly compute the sixth order finite contribution to the stress-energy-momentum tensor of a scalar field which propagates in a FRW universe. The field is assumed to be placed in the adiabatic vacuum of order infinite. We first obtain the general expressions for an arbitrary scale factor and then concentrate on the power law expansion to have a better understanding of these results. Next in chapter V, different than the earlier treatments, we try to figure out the backreaction effects of a quantized massless real scalar propogating in a FRW universe. This can be viewed as a semiclassical approach, where gravity is treated classically. Following the guidelines of an approximation, we manage to determine the evolution of the vacuum energy density by a first order differential equation. We also compare our findings with the known results in the literature for fixed backgrounds. Finally in the conclusion part, we stress on the crucial points that we realise in our work and discuss the significance of our findings from the cosmological point of view.

2. QUANTIZATION OF THE SCALAR FIELD IN CURVED SPACETIME

2.1. The Canonical Quantization

The action of a massive non-minimally coupled real scalar field in a curved background can be written as

$$S = \int \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \xi R \phi^2 \right), \tag{2.1}$$

where ∇ is the covariant derivative associated with the background metric and ξ governs the coupling to the curvature scalar R. Varying the action with respect to ϕ gives the equation of motion for the scalar field

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi - \xi R \phi = 0. \tag{2.2}$$

As usual the canonically conjugate momentum is defined as

$$\Pi = \frac{\partial L}{\partial(\nabla_0 \phi)} \tag{2.3}$$

and (2.1) gives

$$\Pi = -\sqrt{-g}\; g^{0\mu} \partial_\mu \phi. \tag{2.4}$$

For quantization, the canonical commutation relations may be imposed as

$$[\phi(\eta,\mathbf{x}),\phi(\eta,\mathbf{x}')]=0,$$
$$[\Pi(\eta,\mathbf{x}),\Pi(\eta,\mathbf{x}')]=0,$$
$$[\phi(\eta,\mathbf{x}),\Pi(\eta,\mathbf{x}')]=i\delta^3(\mathbf{x}-\mathbf{x}'), \tag{2.5}$$

where $x^\mu \equiv (\eta, x^i)$ and $\mathbf{x} \equiv x^i$.

We are interested in a scalar propogating in a FRW spacetime with the metric

$$ds^2 = a(\eta)^2\left(-d\eta^2 + \ \mathbf{dx.dx}\right), \tag{2.6}$$

where a is the scale factor of the universe and η is the conformal time. For convenience, we introduce another field by

$$\mu = a\phi.$$

Since the action is quadratic and the field equation is linear, it is useful to Fourier expand the field as

$$\mu = \int \frac{d^3k}{(2\pi)^{3/2}}\left[\mu_k(\eta)a_{\mathbf{k}}e^{i\mathbf{k.x}} + \mu_k^*(\eta)a_{\mathbf{k}}^\dagger e^{-i\mathbf{k.x}}\right]. \tag{2.7}$$

After replacing ϕ with μ in (2.2) one can see that the mode function μ_k obeys the following equation

$$\mu_k'' + \left[k^2 + m^2a^2 + (6\xi - 1)\frac{a''}{a}\right]\mu_k = 0, \tag{2.8}$$

where each prime denotes an η derivative. If one imposes the standard relation for the ladder operators

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] = \delta^3(\mathbf{k}-\mathbf{k}'),$$

the canonical commutation equations (2.5) can be satisfied by imposing the Wronskian condition

$$\mu_k \mu_k^{'*} - \mu_k^* \mu_k^{'} = i. \tag{2.9}$$

Given the initial conditions $\mu_k(\eta_0)$ & $\mu_k^{'}(\eta_0)$ obeying (2.9), the system is uniquely determined. As usual the ground state is defined by imposing

$$a_{\mathbf{k}}|0 >= 0.$$

The Fock space of states can be built by applying the creation operators $a_{\mathbf{k}}^{\dagger}$ to $|0>$. This is as far as one can proceed in canonical quantization.

2.2. Non-Uniqueness of the Vacuum

In the absence of any other symmetries, canonical quantization does not fix the mode functions completely, thus there does not exist a unique vacuum state in a curved spacetime. Unlike the flat space case, any prescription of the vacuum state will rely on the observer's coordinate system. The presence of gravity makes the concepts of vacuum and particle ambigious. Namely, the vacuum state of one observer can be seen as a multi-particle state by another observer. Therefore, any set of initial conditions obeying (2.9) defines a viable vacuum of the theory. To specify the vacuum, one then needs a physically motivated boundary condition or a prescription to fix the mode function μ_k in (2.7).

One possible prescription is to define the vacuum as the lowest energy state. The mode functions of the vacuum are specified by minimizing the expectation value of the Hamiltonian operator $\hat{H}$ at a particular moment of time. As an example consider a massive scalar in a FRW universe. The expectation value of $\hat{H}$ at an instantaneous time η_0 is given by

$$< 0|\hat{H}(\eta_0)|0 >= \int d^3k \left[|\mu_k'|^2 + \omega_k^2(\eta_0)|\mu_k|^2\right] = \int d^3k E_k,$$

where $\omega_k^2 = \left[k^2 + m^2a^2 - \frac{a''}{a}\right]$. For the modes satisfying $\omega_k^2 > 0$, the minimization procedure with respect to all possible values of μ_k gives the desired initial conditions for the mode function:

$$\mu_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}} \qquad \mu_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)}.$$

On the other hand, for the modes with $\omega_k^2 < 0$, there exists no minimum for E_k, i .e. no instantaneous lowest energy vacuum state appears. For these modes, although one can still use a set of normalized mode functions to define the mode expansion and determine the creation annihilation operators, it would not be justifiable to consider that state as the physical vacuum state. Thus, the lowest energy eigenstate prescription for vacuum breaks down when $\omega_k^2 < 0$.

In an asymptotically flat spacetime another way of defining the vacuum and the particle states is to use the so called free asymptotic states that can be obtained by treating the interactions perturbatively. However, the backgrounds of cosmological interest do not have such a property, therefore, one may try to define the vacuum state in the absence of the free asymptotic states. Moreover, a time-dependent gravitational field (i.e. a nonstatic spacetime) induces particle production. Thus, if a state is defined as vacuum at a specific time t_o, it ceases to be a vacuum state at a later time t_1, instead it becomes an excited multiparticle state.

Due to these issues, the vacuum is usually specified by a physically viable condition, while there can still be ambiguities inherent in some situations (see [21] and references therein). The most commonly used vacuum states are the BD vacuum and the adiabatic vacuum. Both of these involve an asymptotic definition of the vacuum and now, we will give a short review of both prescriptions.

2.2.1. The Bunch-Davies Vacuum

In homogeneous and isotropic spacetimes, which are relevant to cosmology, a natural choice of the vacuum state can be obtained by demanding that the positive frequency solution to the mode equation (2.8) should match asypmtotically to the Minkowski spacetime solution [19]:

$$\lim_{k/aH\to\infty} \mu_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}. \tag{2.10}$$

We note that k/a is the physical momentum and H is the Hubble parameter in the proper time; $H = \frac{\dot{a}}{a}$, $\dot{a} = \frac{da}{dt}$ and $dt = a(\eta)d\eta$. Physically this enforces that the modes with sufficiently small wavelengths do not feel the expansion of the universe and propagate as in flat spacetime. As an example of this prescription, consider de Sitter spacetime with the scale factor given by

$$a(\eta) = -\frac{\eta_0}{\eta}.$$

In the simple case of a massless, minimally coupled scalar field the mode equation (2.8) becomes

$$\mu_k^{''} + \left[k^2 - \frac{2}{\eta^2}\right] \mu_k = 0. \tag{2.11}$$

Using the condition (2.10) together with (2.9), the BD vacuum can be uniquely specified by

$$\mu_k^{BD} = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 + \frac{iH}{k/a}\right). \tag{2.12}$$

As it can be seen, in the limit $k/aH \to \infty$ (i.e $(-k\eta) \to \infty$) the solution asymptotically approaches the Minkowski mode function

$$\frac{1}{\sqrt{2k}} e^{-ik\eta}. \tag{2.13}$$

In the early-time limit $\eta \to -\infty$, the BD vacuum state is determined by (2.12) for all k modes. However, in a realistic case, for instance during inflation, one has $\eta \geq \eta_i$ for some moment at early times. Then, the BD vacuum state can only be defined for the modes μ_k satisfying $k\eta_i \gg 1$, which are subhorizon at $\eta = \eta_i$, and the prescription fails for the superhorizon modes with $k\eta_i \leq 1$. Nevertheless, the absence of an adequate vacuum state for the superhorizon modes is not so important, since in inflationary theories all observationally relevant modes are subhorizon at $\eta = \eta_i$.

2.2.2. The Adiabatic Vacuum

This prescription of vacuum is based on the minimalization postulate (see e.g. [20]). If one considers the number operator

$$N_k \equiv a_k^\dagger a_k,$$

which counts the particles with momentum k, the postulate states that if the Hubble parameter is changing adiabatically (i.e. very slowly), the particle number in a given mode should remain unchanged. Thus, the definition of vacuum relies on keeping the number of particles as constant as possible. Most of the time this prescription is compatible with the BD vacuum. For example in de Sitter space, the adiabatic vacuum of order infinite is identical to the BD vacuum. The adiabatic vacuum of a certain order is usually defined by identifying initial values of the exact mode functions with the approximate adiabatic mode functions determined to that order. Namely, for a given momentum k and at a fixed time η_0, the value and the derivative of the exact mode function μ_k are specified by matching them to a mode function solution expanded asymptotically in the adiabatic basis. The procedure is as follows. One writes the mode function μ_k in the adiabatic basis (using WKB form) as

$$\mu_k^{ad} = \frac{1}{\sqrt{2\Omega_k}} e^{-i\int \Omega_k d\eta}, \tag{2.14}$$

where the Wronskian condition (2.9) is satisfied identically. Inserting (2.14) into the mode equation (2.8), the new variable Ω_k can be seen to obey the equation

$$\Omega_k^2 = \left[k^2 + m^2 a^2\right] + (6\xi - 1)\frac{a''}{a} + \frac{3}{4}\frac{\Omega_k'^2}{\Omega_k^2} - \frac{1}{2}\frac{\Omega_k''}{\Omega_k}. \tag{2.15}$$

In this scheme, equation (2.15) can be solved iteratively. Here, the number of derivatives with respect to η acts like a perturbation parameter. Each successive itteration can be obtained from

$$\Omega_k^{[n]} = \left(k^2 + m^2 a^2 + 6\left(\xi - \frac{1}{6}\right)\frac{a''}{a} - \frac{1}{2}\left[\frac{\Omega_k^{[n-2]''}}{\Omega_k^{[n-2]}} - \frac{3}{2}\frac{\Omega_k^{[n-2]'}}{\Omega_k^{[n-2]}}\right]\right)^{\frac{1}{2}}. \tag{2.16}$$

For instance, the zeroth order solution is

$$\Omega_k^{[0]} = \sqrt{k^2 + m^2 a^2}, \tag{2.17}$$

which contains no time derivatives. This solution can be used in the right hand side of (2.15) to obtain the second order solution

$$\Omega_k^{[2]} = \sqrt{k^2 + m^2 a^2}\left[1 + \frac{(6\xi - 1)a''}{2a(k^2 + m^2 a^2)} - \frac{m^2(a'^2 + a a'')}{4(k^2 + m^2 a^2)^2} + \frac{5m^4 a^2 a'^2}{8(k^2 + m^2 a^2)^3}\right], \tag{2.18}$$

which contains terms up to 2 derivatives.

Continuing this procedure iteratively, a series solution can be obtained for Ω_k[1]. The adiabatic mode function μ_k^{ad} of order n is then given by

$$\mu_k^{ad[n]} = \frac{1}{\sqrt{2\Omega_k^{[n]}}} e^{-i\int \Omega_k^{[n]} d\eta}. \tag{2.19}$$

Eq. (2.19) is actually an approximation to the exact mode functions μ_k.

The n'th order adiabatic vacuum state at time η_0 is determined by identifying

[1]In general, the series is not convergent, it is rather an asymptotic expansion.

the initial values of the exact mode functions with the approximate adiabatic mode functions of order n which is given by the conditions

$$\mu_k(\eta_0) = \mu_k^{ad[n]}(\eta_0) + \mathcal{O}(a^{(n+1)}),$$
$$\mu_k'(\eta_0) = \mu_k^{'ad[n]}(\eta_0) + \mathcal{O}(a^{(n+1)}), \quad (2.20)$$

where the left hand side gives the initial value of the exact solution to equation (2.8). Note that in the large k limit, one recovers the usual Minkowski positive frequency solution. As an example, consider again a massless, minimally coupled scalar field in de Sitter space. The exact mode solution to the equation (2.11) is

$$\mu_k = \alpha_k \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) + \beta_k \frac{e^{+ik\eta}}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right). \quad (2.21)$$

On the other hand, in the limit $n \to \infty$, the series for $\Omega_k^{[\infty]}$ converges and the adiabatic mode function μ_k^{ad} becomes (for details see [21])

$$\mu_k^{ad[\infty]} = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right). \quad (2.22)$$

Following the route of the initial conditions (2.20), the infinite adiabatic vacuum is specified by setting $\alpha_k = 1$ and $\beta_k = 0$ in (2.21), which is identical to the BD vacuum.

3. THE STRESS-ENERGY-MOMENTUM TENSOR & ADIABATIC REGULARIZATION

3.1. Definition of $T_{\mu\nu}$

The standard stress-energy-momentum tensor $T_{\mu\nu}$ is defined by the variation of the action (1) with respect to the metric $g_{\mu\nu}$, which gives

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \xi\nabla_\mu\nabla_\nu\phi^2 - \frac{1}{2}g_{\mu\nu}[(\nabla\phi)^2 + (m^2+\xi R)\phi^2 - 2\xi\nabla^2(\phi^2) + \xi\phi^2 R_{\mu\nu}. \quad (3.1)$$

In a FRW background, the corresponding energy density ρ and the pressure P are given by $\rho = T_{00}$ and $P = T_{ii}$[2]. Applying the canonical quantization procedure, $T_{\mu\nu}$ becomes an operator. Using (2.7), one can compute the vacuum expectation values of ρ and P in terms of the mode functions μ_k, which yields

$$\langle 0|\rho|0\rangle = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[|\mu_k' - h\mu_k|^2 + \left(k^2 + m^2a^2 - 6\xi h^2\right)|\mu_k|^2 + 6\xi h\left(|\mu_k|^2\right)'\right]k^2dk, \quad (3.2)$$

$$\langle 0|P|0\rangle = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[|\mu_k' - h\mu_k|^2 - \left(\frac{k^2}{3} + m^2a^2 + 6\xi h^2\right)|\mu_k|^2 + 6\xi h\left(|\mu_k|^2\right)' - 2\xi\left(|\mu_k|^2\right)''\right]k^2dk, \quad (3.3)$$

where h is the Hubble parameter with respect to the conformal time, i.e. $h = \frac{a'}{a}$.

Despite being a free theory, the formal expressions above diverge. In a semiclassical approximation, one uses the vacuum expectation value $\langle T_{\mu\nu}\rangle$ in the right hand side of the Einstein equations. Therefore, it is necessary to use one of the regularization methods to get rid of these divergences. In cosmological spacetimes adiabatic regularization is a very convenient method which we prefer to utilize.

[2] $i = (1, 2, 3)$ and there is no summation in the repeated indices.

3.2. Adiabatic Regularization

In Minkowski space, the infinite zero-point energy density of the vacuum state is subtracted away from the energy of a system by postulating that it does not gravitate. A similar issue arises in curved spacetimes however, it is not easy to get rid of the infinite vacuum energy density by a simple subtraction and a more sophisticated procedure should be introduced to make the the energy density finite. In homogeneous spacetimes, adiabatic regularization is a well known method curing the divergences in the quantities that are quadratic in fields. It includes a mode by mode subtraction procedure, thus it is applicaple to any metric of sufficient symmetry that allows a mode decomposition of the quantized field. The tecnique was first applied to regularize the expectation values of particle number in an expanding universe, then extended to the stress-energy-momentum tensor (see e.g. [19, 20, 22–25]). In [8, 26] it is demonstrated that adiabatic regularization is equivalent to point-splitting regularization in cosmological spacetimes. Remarkably the famous trace anomaly can also be derived using adiabatic regularization [24], which supports the validity of this technique.

To motivate the method, assume that a generalized positive frequency solution of (2.8) is given by

$$\mu_k^{ad} = \frac{1}{\sqrt{2\Omega_k}} e^{-i\int \Omega_k d\eta}, \tag{3.4}$$

where Ω_k obeys

$$\Omega_k^2 = \left[k^2 + m^2 a^2\right] + (6\xi - 1)\,\frac{a''}{a} + \frac{3}{4}\frac{\Omega_k'^2}{\Omega_k^2} - \frac{1}{2}\frac{\Omega_k''}{\Omega_k}. \tag{3.5}$$

In Minkowski space, where $a(\eta)$ is constant, (3.4) determines the physical vacuum state. Now, for an operator that is quadratic in fields, normal ordering that makes the operator finite is equivalent to the subtraction of the operator's expectation value in the vacuum. Parker & Fulling suggest that even when $a(\eta)$ is not constant, a similar subtraction may remove the infinities in the stress-energy-momentum tensor.

However, to eliminate all the divergences, one must subtract the terms of sufficiently high adiabatic order, not just the $0'$th order flat space ones.

The integrals in (3.2) & (3.3) unavoidably contain quartic, quadratic and logarithmic divergences, which can be identified as the unobservable vacuum contributions. To cancel them out, the necessary subtraction terms are given by the $4'$th order adiabatic mode functions and the corresponding expectation values can be computed as

$$< 0|\rho|0 >^{ad[4]} = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[\frac{\Omega_k^{[4]}}{2} + \frac{1}{2\Omega_k^{[4]}}\left(h + \frac{(\Omega_k^{[4]})'}{2\Omega_k^{[4]}}\right)^2 + \frac{1}{2\Omega_k^{[4]}}(k^2 + m^2a^2 - 6\xi h^2) \right. \\ \left. -3\xi h\frac{(\Omega_k^{[4]})'}{(\Omega_k^2)^{[4]}}\right] k^2 dk, \quad (3.6)$$

$$< 0|P|0 >^{ad[4]} = \frac{1}{4\pi^2 a^4}\int_0^\infty \left[\frac{\Omega_k^{[4]}}{2} + \frac{1}{2\Omega_k^{[4]}}\left(h + \frac{(\Omega_k^{[4]})'}{2\Omega_k^{[4]}}\right)^2 - \frac{1}{2\Omega_k^{[4]}}\left(\frac{k^2}{3} + m^2a^2 + 6\xi h^2\right) \right. \\ \left. -3\xi h\frac{(\Omega_k^{[4]})'}{(\Omega_k^2)^{[4]})} + \xi\frac{(\Omega_k^{[4]})''}{(\Omega_k^2)^{[4]}} - 2\xi\frac{(\Omega_k^{[4]})'^2}{(\Omega_k^3)^{[4]}}\right] k^2 dk. \quad (3.7)$$

Namely, the following regularized expressions

$$\langle\rho\rangle \equiv < 0|\rho|0 > - < 0|\rho|0 >^{ad[4]}$$
$$\langle P\rangle \equiv < 0|P|0 > - < 0|P|0 >^{ad[4]}$$

become ultraviolet (UV) finite. One important feature of adiabatic regularization is that the regularized stress-energy-momentum tensor is guaranteed to be conserved.

4. STRESS-ENERGY TENSOR OF ADIABATIC VACUUM IN FRIEDMANN-ROBERTSON-WALKER SPACETIMES

As mentioned in the introduction, adiabatic regularization has been used to determine the stress-energy-momentum tensor of scalar fields in various curved spaces, in various vacua. In this section, we also consider a scalar field coupled to gravity having the action (2.1) and propogating in a fixed FRW space-time with the metric introduced in (2.6). However different than earlier treatments, we concentrate on the adiabatic vacuum of infinite order, which is determined by the conditions

$$\mu_k(\eta_0) = \mu_k^{ad[\infty]}(\eta_0)$$

$$\mu'_k(\eta_0) = \mu_k^{\prime ad[\infty]}(\eta_0).$$

Note that these conditions are equivalent to using (2.14) as the mode function with $\Omega_k^{[\infty]}$. We will analyse the adiabatic regularization of the stress-energy tensor for the massive and the massless cases separately.

4.1. Massive Case

If one examines the second order solution to (2.15), which is given by

$$\Omega_k^{[2]} = \sqrt{k^2+m^2a^2}\left[1+\frac{(6\xi-1)a''}{2a(k^2+m^2a^2)}-\frac{m^2(a'^2+aa'')}{4(k^2+m^2a^2)^2}+\frac{5m^4a^2a'^2}{8(k^2+m^2a^2)^3}\right],$$

it can be seen that it has the largest magnitude H^2/m^2 when $k=0$[3] , (recall that H is the Hubble parameter with respect to the proper time $H=\frac{a'}{a^2}$). Thus this term is always smaller than the zeroth order term $\Omega_k^{[0]} = \sqrt{k^2+m^2a^2}$ if $m \gg H$. By inspection, it is not difficult to see that the n'th order term is proportional to $(H/m)^n$

[3]Note that the terms are more suppressed for larger k.

for $k = 0$. Therefore, the series solution for the massive scalar field turns out to be valid as long as $m \gg H$ and the adiabatic vacuum is physically viable[4] for all momentum modes.

Inserting the WKB mode function (2.14) in (3.2) and (3.3), the vacuum expectation values can be found as

$$\langle \rho \rangle = \frac{1}{4\pi^2 a^4} \int_0^\infty \left[\frac{\Omega_k}{2} + \frac{1}{2\Omega_k} \left(h + \frac{\Omega'_k}{2\Omega_k} \right)^2 + \frac{1}{2\Omega_k} \left(k^2 + m^2 a^2 - 6\xi h^2 \right) - 3\xi h \frac{\Omega'_k}{\Omega_k^2} \right] k^2 dk, \tag{4.1}$$

$$\langle P \rangle = \frac{1}{4\pi^2 a^4} \int_0^\infty \left[\frac{\Omega_k}{2} + \frac{1}{2\Omega_k} \left(h + \frac{\Omega'_k}{2\Omega_k} \right)^2 - \frac{1}{2\Omega_k} \left(\frac{k^2}{3} + m^2 a^2 + 6\xi h^2 \right) - 3\xi h \frac{\Omega'_k}{\Omega_k^2} + \xi \frac{\Omega''_k}{\Omega_k^2} - 2\xi \frac{\Omega'^2_k}{\Omega_k^3} \right] k^2 dk. \tag{4.2}$$

For the adiabatic vacuum, one should use $\Omega_k^{[\infty]}$ in (4.1) and (4.2) to obtain $\langle \rho \rangle$ and $\langle P \rangle$, which are necessarily divergent.

To proceed with adiabatic regularization, one should subtract the zeroth, second and fourth order adiabatic terms by using $\Omega_k = \Omega_k^{[4]}$ in (4.1) and (4.2). In an order by order computation, these subtraction terms are exactly cancelled out by the terms coming from $\Omega_k^{[\infty]}$. Therefore, the finite leading order contribution to the stess-energy-momentum tensor can be obtained by using the sixth order solution $\Omega_k = \Omega_k^{[6]}$ in (4.1) and (4.2). Although the stress-energy-momentum tensor expressions for $\Omega_k^{[6]}$ are very complicated, after performing the elementary and convergent momentum integrals we

[4]One can worry about the existence of $\Omega^{[\infty]}$, since the infinite series may fail to converge in time. However, this should not come up for very massive fields and for the high energy modes of massless fields [21].

obtain relatively simple expressions for the vacuum expectation values[5] such that

$$\begin{aligned}
\rho(m) \simeq & \frac{1}{40320 m^2 \pi^2 a^{12}} \big[(844 - 3528\xi) a'^6 - 72(173 - 1596\xi + 3780\xi^2) a a'^4 a'' \\
& + 12(683 - 6636\xi + 16380\xi^2) a^2 a'^3 a''' \\
& + 9(1237 - 14924\xi + 59220\xi^2 - 75600\xi^3) a^2 a'^2 a''^2 \\
& + 9(17 - 168\xi + 420\xi^2)(-16 a^3 a'^2 a^{(4)} + a^4 a^{(3)^2} - 2a^4 a'' a^{(4)} + 2a^4 a' a^{(5)}) \\
& + (673 - 8316\xi + 34020\xi^2 - 45360\xi^3)(2a^3 a''^3 - 6a^3 a' a'' a^{(3)})\big] , \\
P(m) \simeq & \frac{1}{40320 m^2 \pi^2 a^{12}} \big[12(211 - 882\xi) a'^6 - 8(4363 - 39186\xi + 90720\xi^2) a a'^4 a'' \\
& + 4(5819 - 56028\xi + 137340\xi^2) a^2 a'^3 a^{(3)} \\
& + 15(2389 - 31108\xi + 107100\xi^2 - 105840\xi^3) a^2 a'^2 a''^2 \\
& + 10(-473 + 5628\xi - 21924\xi^2 + 27216\xi^3) a^3 a''^3 \\
& + 4(617 - 6930\xi + 23940\xi^2 - 22680\xi^3) a^4 a'' a^{(4)} \\
& + (1601 - 19152\xi + 74340\xi^2 - 90720\xi^3) a^4 a^{(3)^2} - 6(17 - 168\xi + 420\xi^2) a^5 a^{(6)} \\
& + 6(-3949 + 44828\xi - 160020\xi^2 + 166320\xi^3) a^3 a' a'' a^{(3)} \\
& + 4(1907 - 18732\xi + 46620\xi^2) a^3 a'^2 a^{(4)} \\
& + 78(17 - 168\xi + 420\xi^2) a^4 a' a^{(5)}\big] , \qquad (4.3)
\end{aligned}$$

where $a^{(n)}$ denotes n'th order η derivative of $a(\eta)$. It is important to note that (4.3) obeys the conservation equation $\rho' + 3h(\rho + P) = 0$. As the total number of derivatives acting on the scale factor is six for each term, the magnitudes of ρ and P are fixed by H^6/m^2. Since these expressions are not very illuminating, in the following sections we evaluate them for some cosmologically relevant spaces.

4.1.1. Power Law Expansion

Let us first concentrate on the power law expansion by setting

$$a = \left(\frac{\eta_0}{\eta}\right)^n . \qquad (4.4)$$

[5] In [27] by means of the quantum effective action, which is regularized by point splitting, the stress-energy-momentum tensor of a massive scalar field is calculated, which must agree with (4.3).

In terms of the proper time defined by

$$dt = ad\eta,$$

the scale factor becomes

$$a = \left(\frac{t}{t_0}\right)^{\alpha}, \qquad \alpha = \frac{n}{n-1}.$$

As $n = \alpha/(\alpha - 1)$, we will analyse $\alpha = 1$ case separately.

In the background (4.4) the expectation values in (4.3) become

$$\rho(m) = C_n \frac{H^6}{m^2}, \qquad P(m) = \frac{n-2}{n}\rho(m), \tag{4.5}$$

where the constant C_n is given by

$$\begin{aligned} C_n = & \frac{1}{20160\pi^2 n^4}[3060 - 4742n - 4029n^2 + 4716n^3 + 1365n^4 \\ & - 126(240 - 438n - 389n^2 + 5n^3(92 + 37n))\xi \\ & + 3780(n+1)(20 - 74n + 25n^2 + 35n^3)\xi^2 \\ & - 22680n(n+1)^2(11n - 10)\xi^3]. \end{aligned} \tag{4.6}$$

From the equation of state (4.5), we see that the pressure and the energy density have opposite signs for $0 < n < 2$. In terms of α this corresponds to the range $\alpha > 2$ or $\alpha < 0$. While the background is accelerating for $\alpha > 2$, there is a big-crunch singularity at finite proper time for $\alpha < 0$.

It is interesting to compare the vacuum contribution to the energy density with the background energy density driving the metric (4.4). We know from the Friedmann equation that $H^2 \sim \rho_B/M_p^2$, where ρ_B and M_p denote the background energy density and the Plank mass, respectively. Using (4.5), the ratio becomes

$$\frac{\rho(m)}{\rho_B} \sim \frac{H^4}{M_p^2 m^2}. \tag{4.7}$$

Since the adiabacity requires $m \gg H$, the background energy density is much larger than the vacuum energy density unless $H > M_p$, which shows that backreaction effects are negligible in this setup.

For $\alpha < 0$ (or equivalently $0 < n < 1$), H is an increasing function of time. If one observes the ratio H/m, it will grow to become order one at some time before the big-crunch. Thus, the adiabacity will eventually be lost. On the other hand, the vacuum energy density will also increase and the ratio (4.7) becomes H^2/M_p^2 when the adiabatic expansion breaks down, i.e. $H/m \sim \mathcal{O}(1)$. $\rho(m)$ can be still negligible as long as the condition $H < M_p$ holds. One can try to figure out the effects of the vacuum energy density in that case and analyse if this can prevent the big-rip singularity, in view of [9].

The coefficient C_n is much simpler in the cases $\xi = 0$ and $\xi = 1/6$, which correspond to the minimally coupled and the conformally coupled scalar fields, respectively:

$$C_n = \begin{cases} \frac{1}{5040\pi^2 n^4}[30 - 41n - 15n^2 + 24n^3] & \xi = \frac{1}{6}, \\ \frac{1}{20160\pi^2 n^4}[3060 - 4742n - 4029n^2 + 4716n^3 + 1365n^4] & \xi = 0. \end{cases} \tag{4.8}$$

In figure 4.1, the plots of C_n are given for $\xi = 0$ and $\xi = 1/6$. As it can be analyzed from the graphs, C_n (and correspondingly the vacuum energy density) can be positive, negative or even zero for different values of the power n.

Now we focus on some important backgrounds represented by different expansion powers. For instance $n = 1$ corresponds to de Sitter space. In this background C_n

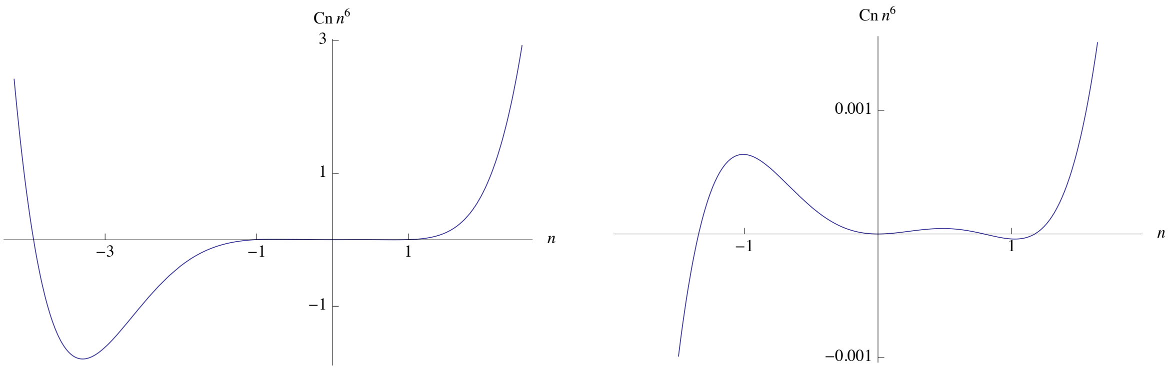

Figure 4.1: The graphs of $n^6 C_n$ for $\xi = 0$ (left) and $\xi = 1/6$ (right). H includes a factor of n coming from a', thus C_n is multiplied by n^6. Although it cannot be realized in the graph for $\xi = 0$, the curve actually oscillates in the $(-1, 1)$ interval, intersecting with the n-axis three times.

becomes

$$C_1 = \frac{270 - 7308\xi + 45360\xi^2 - 90720\xi^3}{20160\pi^2},$$

while the equation of state takes the form

$$P = -\rho(m).$$

It follows that the adiabatic vacuum energy density is equivalent to a cosmological constant.

For $\xi = 0$, $C_1 > 0$ showing that the vacuum energy density is positive for minimally coupled scalar. For $\xi = \frac{1}{6}$, $C_1 < 0$ and the vacuum energy density is negative. So the contributions of the scalar field can be interpretted as a modification of the background cosmological constant. One can compare these results with a massive scalar field defined in the BD vacuum. The vacuum energy density in this vacuum is given by $\rho_{BD} \simeq -m^4$(see [28]). One can conclude that according to the chosen vacuum, the vacuum energy densities turn out to alter both in magnitude and sign .

If one analyses $n = -1$ case corresponding to the radiation dominated background, C_n is found as

$$C_{-1} = \frac{211 - 882\xi}{10080\pi^2}.$$

The scalar mimics a very stiff matter with an equation of state $P(m) = 3\rho(m)$.

Finally, $n = -2$ corresponds to the matter dominated background and C_n becomes

$$C_{-2} = \frac{36(3\xi - 1)(29 + 96\xi)\xi - 139}{10080\pi^2}.$$

The equation of state $P(m) = 2\rho(m)$ represents again a very stiff matter.

4.1.2. The Linear Expansion, $\alpha = 1$

For $\alpha = 1$, the universe is expanding linearly:

$$a = t/t_0.$$

In the conformal time, this corresponds to

$$a = \exp\left[\frac{\eta}{\eta_0}\right].$$

Using (4.3), one obtains the equation of state as $P(m) = \rho(m)$ with

$$\rho(m) = \frac{(1 - 6\xi)^2(13 - 66\xi)}{192m^2\pi^2}H^6. \tag{4.9}$$

For $\xi = \frac{1}{6}$, ρ and P of a conformally coupled scalar vanish identically, while for $\xi = 0$, they become positive for a minimally coupled scalar. The equation of state $P(m) = \rho(m)$ can also be obtained by taking the $n \to \infty$ limit of (4.5), which is equivalent to

$\alpha = 1$.

4.2. Massless Case

Setting $m = 0$ for a massless scalar field, the second order solution to Ω_k becomes

$$\Omega_k^{[2]} = k \left[1 + \frac{(6\xi - 1)a''}{2ak^2} \right]. \qquad (4.10)$$

The higher order adiabatic contributions are proportional to the powers of Ha/k. Thus, the adiabatic expansion loses its validity for superhorizon modes (i.e for $k/a < H$). However, in the regime $k/a \gg H$, i.e. for subhorizon modes, the expansion can be still proposed to identify the mode functions. For a fixed wavenumber k_* satisfying $k_* \gg aH$, the adibatic vacuum is physically viable for the UV modes obeying $k \geq k_* \gg aH$. One can calculate $T_{\mu\nu}(k_*)$ for the modes in this range $k \geq k_*$ in the adiabatic vacuum.

To apply adiabatic regularization to $T_{\mu\nu}$, one can follow the same steps that are used in the massive case. The main difference is that the integral limits in the vacuum expectation values in (4.1) and (4.2) should be changed from $(0, \infty)$ to (k_*, ∞), where the adiabatic expansion can safely be used. After subtracting the zeroth, second and fourth order adiabatic terms, the leading order contribution to $T_{\mu\nu}(k_*)$ can be obtained by using the sixth order solution $\Omega_k^{[6]}$ in

$$\langle \rho \rangle (k_*) = \frac{1}{4\pi^2 a^4} \int_{k_*}^{\infty} \left[\frac{\Omega_k}{2} + \frac{1}{2\Omega_k} \left(h + \frac{\Omega_k'}{2\Omega_k} \right)^2 + \frac{1}{2\Omega_k} \left(k^2 + m^2 a^2 - 6\xi h^2 \right) - 3\xi h \frac{\Omega_k'}{\Omega_k^2} \right] k^2 dk, \qquad (4.11)$$

$$\langle P \rangle (k_*) = \frac{1}{4\pi^2 a^4} \int_{k_*}^{\infty} \left[\frac{\Omega_k}{2} + \frac{1}{2\Omega_k} \left(h + \frac{\Omega_k'}{2\Omega_k} \right)^2 - \frac{1}{2\Omega_k} \left(\frac{k^2}{3} + m^2 a^2 + 6\xi h^2 \right) - 3\xi h \frac{\Omega_k'}{\Omega_k^2} + \xi \frac{\Omega_k''}{\Omega_k^2} - 2\xi \frac{\Omega_k'^2}{\Omega_k^3} \right] k^2 dk. \qquad (4.12)$$

Performing the integrals and taking $m \to 0$ limit, we find

$$\begin{aligned}
\rho(k_*) &\simeq \frac{(1-6\xi)^2}{256 k_*^2 \pi^2 a^9}\left[-16a'^4a'' + 16aa'^3a^{(3)} + (29-108\xi)aa'^2a''^2 - 8a^2a'^2a^{(4)}\right.\\
&\left. +6(1-4\xi)a^2a''^3 + a^3 a^{(3)^2} - 2a^3a''a^{(4)} + 18(4\xi-1)a^2a'a''a^{(3)} + 2a^3a'a^{(5)}\right],\\
P(k_*) &\simeq -\frac{(1-6\xi)^2}{768 k_*^2 \pi^2 a^9}\left[96a'^4a'' - 96aa'^3a^{(3)} - (209-540\xi)aa'^2a''^2 + 48a^2a'^2a^{(4)}\right.\\
& +(34-120\xi)a^2a''^3 + 3(24\xi-7)a^3a^{(3)^2} + 4(18\xi-7)a^3a''a^{(4)}\\
&\left. +2(89-252\xi)a^2a'a''a^{(3)} - 14a^3a'a^{(5)} + 2a^4a^{(6)}\right]. \qquad (4.13)
\end{aligned}$$

In the expressions above, the total number of time derivatives on each term is always six and the magnitudes are fixed by H^6a^2/k_*^2. This partial $T_{\mu\nu}(k_*)$ totally decouples from the rest of the modes. As a result, ρ and P in (4.13) are conserved, i.e they obey the conservation equation $\rho' + 3h(\rho + P) = 0$. In the following sections we evaluate (4.13) for different cosmological spacetimes.

4.2.1. Power Law Expansion

Using the background (4.4) in (4.13) , the energy density and the pressure in the UV range becomes

$$\rho(k_*) = D_n \frac{H^6a^2}{k_*^2}, \qquad P(k_*) = \frac{n-6}{3n}\rho(k_*), \qquad (4.14)$$

where the constant D_n is given by

$$D_n = \frac{1}{128\pi^2 n^6}\left[5(1-6\xi)^2 n^2 (n+1)(2-n)(2+n(n+1)(6\xi-1))\right]. \qquad (4.15)$$

For $0 < n < 6$, the vacuum pressure has the opposite sign compared to the vacuum energy density. This correspods to $\alpha < 0$ or $\alpha > \frac{6}{5}$. For $0 < n < \frac{3}{2}$, the adiabacity will be lost in time, since Ha/k_* grows. An interesting case is the de Sitter space with $n = 1$. For $\xi = 0$ and $\xi = 1/6$, $\rho(k_*)$ and $P(k_*)$ vanish identically. On the other hand, for a generic ξ the energy density grows, showing a different behaviour compared to a cosmological constant. But in that case, the vacuum energy density is still much smaller than the background energy density which drives the metric.

It is remarkable to point out the equivalence between BD vacuum and the adiabatic vacuum in the range $k \gg aH$. Let us recall that BD vacuum has the mode function

$$\mu_k^{BD} = \eta\sqrt{\frac{k}{2}}h_u(k\eta), \tag{4.16}$$

where h_u is the spherical Henkel function of first kind and u is a function of n given by

$$u = -\frac{1}{2} + \sqrt{\frac{1}{2} + n(n+1)(1-6\xi)}. \tag{4.17}$$

As $k \to \infty$, (4.16) takes the asymptotic form

$$\mu_k^{BD} \to \frac{(-i)^{n+1}}{\sqrt{2k}}e^{-ik\eta}(1-6\xi)\left(1 - i\frac{n(n+1)}{2k\eta} - \frac{n(n+1)\left[(1-6\xi)n(n+1)-2\right]}{8k^2\eta^2}...\right). \tag{4.18}$$

It is easy to verify that this expression exactly coincides with the adiabatic expansion (up to a phase). Thus, for the high energy modes with $k \gg aH$, the BD vacuum is equivalent to the adiabatic vacuum. One should note that there are some special values of n, like $n = -1$, for which the series terminates and become finite. Another point is that although the adiabatic expansion is not valid for the infrared (IR) modes, (4.16) can be still used for the BD vacuum (but the expectation values are disturbed by IR divergences see [29]).

Finally, we focus on some special values of n. For the minimally coupled scalar with $\xi = 0$ the coefficient D_n becomes

$$D_n = \frac{5(n^4 - 5n^2 + 4)}{128\pi^2 n^4}. \tag{4.19}$$

The stress-energy-momentum tensor vanishes for $n = 1$, $n = -1$ and $n = -2$ cor-

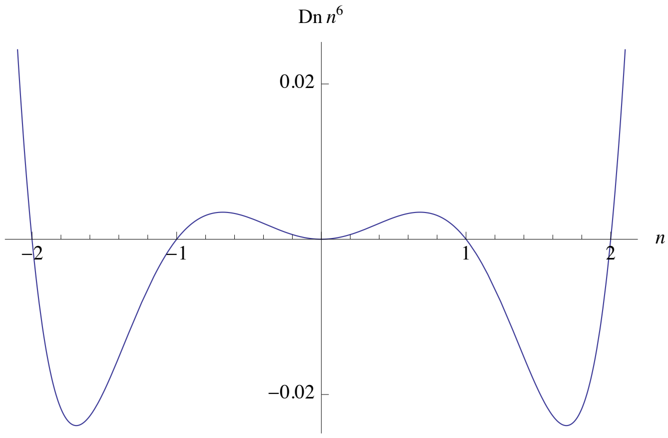

Figure 4.2: The graph of $n^6 D_n$ for $\xi = 0$. D_n is multiplied by n^6 as H includes a factor of n coming from a'.

responding to de Sitter space, the radiation dominated and the matter dominated universes, respectively. In figure (4.2) it can be observed that the energy density can be positive or negative according to different values of n.

4.2.2. The Linear Expansion, $\alpha = 1$

In the linear expanding background $a = t/t_0$, (4.13) becomes

$$\rho(k_*) = \frac{5(1-6\xi)^3}{128 k_*^2 \pi^2}\, a^2\, H^6, \tag{4.20}$$

which obeys the equation of state $P(k_*) = \rho(k_*)/3$. If the scalar is conformally coupled, one ends up with vanishing ρ and P. On the other hand, they turn out to be positive for the minimally coupled scalar.

5. BACKREACTION EFFECTS OF A QUANTIZED MASSLESS SCALAR FIELD

5.1. The Backreaction Problem

Quantization in a fixed background is a well understood issue, which has been handled out in many discussions in the literature. In this section, different than the earlier approaches, we concentrate on the backreaction effects of a quantum field and try to figure out its influence on the cosmic evolution. Here, the background geometry is derived by the quantum field itself. To start with, we apply adiabatic regularization to remove all the infinities in the stress-energy-momentum tensor. Then, we analyse the mode functions for the subhorizon and superhorizon modes separately and utilize an approximation which leads us to the superhorizon modes only. As we will see, in this problem one actually faces an integro-differential equation system involving the scale factor $a(t)$ and the mode function μ_k. By our approximation this system reduces to an ordinary differential equation set that determines the backreaction effects. Unfortunately the relation between the scale factor and the vacuum energy density is still complicated. Although analytic solutions are difficult to obtain, one can still get quantitative information about the impacts of the vacuum density on the cosmic evolution.

Let us first introduce a massless scalar field ϕ, minimally coupled to gravity obeying the equation

$$\nabla_\mu \nabla^\mu \phi = 0. \tag{5.1}$$

The scalar is assumed to propagate in a FRW universe with the metric (2.6). Following the guidelines in the standard canonical quantization (i.e. the steps in chapter II for

$m \to 0$), it is convenient to decompose the field in Fourier space as

$$\mu = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\mu_k(\eta) a_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} + \mu_k^*(\eta) a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}.\mathbf{x}}\right], \tag{5.2}$$

where we perform a field redefinition of the form $\mu = a\phi$ and $\mathbf{k}$ is the comoving momentum variable. The mode functions obey a second order differential equation

$$\mu_k'' + \left[k^2 - \frac{a''}{a}\right] \mu_k = 0, \tag{5.3}$$

together with the Wronskian condition

$$\mu_k \mu_k'^* - \mu_k^* \mu_k' = i. \tag{5.4}$$

As usual, the ground state is identified by imposing

$$a_{\mathbf{k}} \, |0>= 0. \tag{5.5}$$

The vacuum expectation values can be calculated using the definition of the stress-energy-momentum tensor $T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2$. Computing $< T_{\mu\nu} >$ in the background (2.6), the expectation values of the energy density ρ and the pressure P become

$$\begin{aligned} < 0|\rho|0 >&= \frac{1}{4\pi^2 a^4} \int_0^\infty k^2 \left[a^2 \left|\left(\frac{\mu_k}{a}\right)'\right|^2 + k^2|\mu_k|^2\right] dk, \\ < 0|P|0 >&= \frac{1}{4\pi^2 a^4} \int_0^\infty k^2 \left[a^2 \left|\left(\frac{\mu_k}{a}\right)'\right|^2 - \frac{k^2}{3}|\mu_k|^2\right] dk. \end{aligned} \tag{5.6}$$

As indicated before, these expressions contain quartic, quadratic and logarithmic divergences and adiabatic regularization removes all of these contributions by means of a systematic technique. To perform regularization, we start by writing the mode function μ_k in the WKB form μ_k^{ad} defined by (2.14). For the massless case, the variable Ω_k

obeys

$$\Omega_k^2 = k^2 - \frac{a''}{a} + \frac{3}{4}\frac{\Omega_k'^2}{\Omega_k^2} - \frac{1}{2}\frac{\Omega_k''}{\Omega_k}. \tag{5.7}$$

By expanding Ω_k up to adiabatic fourth order we recompute (5.6) using $\mu_k^{ad[4]}$. This procedure gives the finite regularized vacuum expectation values as

$$\rho_V = \frac{1}{4\pi^2 a^4}\int_0^\infty k^2 \left[a^2 \left| \left(\frac{\mu_k}{a}\right)' \right|^2 + k^2|\mu_k|^2 - k - \frac{h^2}{2k} - \frac{1}{8k^3}(3h^4 + h'^2 - 2hh'') \right] dk, \tag{5.8}$$

$$P_V = \frac{1}{4\pi^2 a^4}\int_0^\infty k^2 \left[a^2 \left| \left(\frac{\mu_k}{a}\right)' \right|^2 - \frac{k^2}{3}|\mu_k|^2 - \frac{k}{3} - \frac{h^2}{6k} + \frac{h'}{3k} - \frac{1}{24k^3}(3h^4 + h'^2 - 2hh'' - 12h^2h' + 2h''') \right] dk. \tag{5.9}$$

One may check that ρ_V and P_V satisfy the conservation equation $\rho_V' + 3h(\rho_V + P_V) = 0$.

For the massless fields, adiabatic regularization should be performed very carefully. For instance, for a conformally coupled massless scalar, if one naively applies adiabatic regularization, it is not possible to obtain the conformal anomaly since the regularized stress-energy-momentum tensor vanishes identically. The correct procedure is to start the theory with a massive field, then keep all the surviving terms in the *mass* $\to 0$ limit [24]. Similarly, to regularize the stress-energy-momentum tensor of a massless minimally coupled scalar, one should take into account the subtraction terms that survives the *mass* $\to 0$ limit. The fourth order adiabatic subtraction terms for the energy density are calculated as

$$\int_0^\infty \frac{k^2\,dk}{512\pi^2 a^7\omega^{11}} \left[56m^6\omega^2 a^6 a'^2 a'' - 105m^8 a^7 a'^4 + 144m^4\omega^4 a^4 a'^2 a'' + 64\omega^{10} a a'^2 + 16m^2\omega^6 a^3(a''^2 - 2a'a''') + 64\omega^8 a'^2 a'' - 224m^6\omega^2 a^5 a'^4 + 4m^4\omega^4 a^5(a''^2 - 2a'a''') + 16m^2\omega^6 a a'^4 + 128\omega^{12} a^3 + 16m^4\omega^6 a^5 a'^2 + 16\omega^8 a(a''^2 - 2a'a''') + 64m^2\omega^8 a^3 a'^2 + 124m^4\omega^4 a^3 a'^4 + 96m^2\omega^6 a^2 a'^2 a'' \right],$$

where $\omega = \sqrt{k^2 + m^2 a^2}$. After scaling $k \to mk$, one can see that the terms with m^8, $m^6\omega^2$, $m^4\omega^4$ and $m^2\omega^6$ are finite and m-independent. Performing the integrals the finite adiabatic subtraction terms are calculated as

$$\rho_F = \frac{1}{960\pi^2 a^4}\left[92h^4 + 60h^2h' + 11h'^2 - 22hh''\right]. \tag{5.10}$$

The stress-energy conservation $\rho_F' + 3h(\rho_F + P_F) = 0$ gives the finite contribution for the pressure

$$P_F = \frac{1}{2880\pi^2 a^4}\left[92h^4 - 308h^2h' - 109h'^2 - 82hh'' + 22h'''\right]. \tag{5.11}$$

The final expressions for the regularized stress-energy-momentum tensor can be obtained by subtracting these extra terms from (5.8) and (5.9)

$$\rho_V \to \rho_V - \rho_F, \tag{5.12}$$

$$P_V \to P_V - P_F. \tag{5.13}$$

To work out the backreaction effects, (5.12) and (5.13) can be used in the right hand side of the Einstein equations:

$$h^2 = \frac{8\pi a^2}{3M_p^2}\,\rho_V, \tag{5.14}$$

$$h' = -\frac{4\pi a^2}{3M_p^2}\,(\rho_V + 3P_V). \tag{5.15}$$

As a result, one gets an integro-differential equation system: The integrals of the mode function μ_k expressed in (5.8) and (5.9) can be treated as sources which will determine the dynamics of the scale factor $a(t)$ in the Einstein equations. One also has (5.3) to fix the evolution of μ_k, which involves $a(t)$. Once the initial conditions for μ_k and $a(t)$ are given, the evolution of the system is fully specified. It is important to note that with suitably selected initial conditions, (5.8) and (5.9) are safe from any ultraviolet divergences. On the other hand, they are not guaranteed to be infrared finite for the massless fields [29]. This problem can be overcome by introducing an infrared cutoff to make sense of physics. We note that an IR cutoff is also needed in point-splitting regularization for the massless fields.

5.2. An Approximation

In this section we take the advantage of an approximation to simplify the expressions in (5.8) and (5.9). First, we choose the appropriate initial conditions to determine the vacuum state. The field is assumed to be placed in the BD vacuum or the l'th order adiabatic vacuum with $l \geq 4$. In both cases the solution will approach the usual mode functions of Minkowski space in the $k \to \infty$ limit. As one can decompose the field into modes, the evolution of the mode functions can be studied in two different regions separately. In the subhorizon regime, i.e for $k \gg h$, the mode functions evolve nearly adiabatically as $\mu_k \simeq e^{ik\eta}/\sqrt{2k}$. As discussed in the previous section, the difference between the actual mode function μ_k and the adiabatic mode function μ_k^{ad} is tiny, thus the contribution will be negligible. On the other hand, in the superhorizon regime, i.e for $k \ll h$, the mode functions obey $(\mu_k/a)' \simeq 0$. In other words, the superhorizon modes freeze-out. We can make a better approximation by introducing a new parameter α that quantifies the border of the subhorizon and superhorizon modes. To be more explicit, for $k \gg \alpha h$, we disregard the contributions of the integrals in (5.8) and (5.9), because the difference between the actual mode function μ_k and the adiabatic mode function μ_k^{ad} is in fact small. For $k \ll \alpha h$, the mode functions are assumed to satisfy exactly $(\mu_k/a)' = 0$. The necessity of introducing the α parameter can be justified as follows. Focusing on the differential equation (5.3), the subhorizon/superhorizon regimes are determined by comparing k with $\sqrt{a''/a}$, which can be different than h in most cases. Additionally, there are some errors because of taking exactly $(\mu_k/a)' = 0$ and also due to the small difference between the actual and the adiabatic mode functions, which we ignore. The role of the parameter α is to compensate these errors. Its magnitude is predicted to be of order unity[6] .

One can obtain the regularized expressions for the stress-energy-momentum ten-

[6]Our α parameter is similar to the ϵ parameter of Starobinsky [31].

sor (5.8) and (5.9) approximately as follows. In the $[\alpha h, \infty]$ range, the contribution is negligible by the above argument. To perform the integrals in the $[0, \alpha h]$ range, we define two variables

$$\rho_S = \frac{1}{4\pi^2 a^4}\int_0^{\alpha h} k^2 \left[a^2 \left| \left(\frac{\mu_k}{a}\right)' \right|^2 + k^2 |\mu_k|^2 \right] dk, \quad (5.16)$$

$$P_S = \frac{1}{4\pi^2 a^4}\int_0^{\alpha h} k^2 \left[a^2 \left| \left(\frac{\mu_k}{a}\right)' \right|^2 - \frac{k^2}{3} |\mu_k|^2 \right] dk. \quad (5.17)$$

These satisfy

$$P_S = -\frac{1}{3}\rho_S, \quad (5.18)$$

where we have used the approximation[7] $(\mu_k/a)' = 0$ for the superhorizon modes. Evaluating the integrals in (5.8) and (5.9) in the $[0, \alpha h]$ range, one obtains

$$\rho_V = \rho_S - \frac{1}{4\pi^2 a^4}\left[\frac{(\alpha^4+\alpha^2)}{4} h^4 + \frac{1}{8}\ln\left(\frac{\alpha h}{h_0}\right)(3h^4 + h'^2 - 2hh'') \right] - \rho_F, \quad (5.19)$$

$$\begin{aligned} P_V = {} & P_S - \frac{1}{4\pi^2 a^4}\left[\frac{(\alpha^4+\alpha^2)}{12} h^4 - \frac{\alpha^2}{6} h^2 h' \right. \\ & \left. + \frac{1}{24}\ln\left(\frac{\alpha h}{h_0}\right)(3h^4 + h'^2 - 2hh'' - 12h^2h' + 2h''') \right] - P_F. \end{aligned} \quad (5.20)$$

Here, a comoving infrared cutoff appears in the denominator of the arguments in the logarithmic functions. The infrared divergences may depend on how the actual mode function μ_k behave and they are absorbed in ρ_S and P_S. Namely, the expectation values are determined up to an unknown function ρ_S.

To solve the acceleration equation (5.15), one needs $\rho_V + 3P_V$ in the right hand

[7] There exists two solutions to (5.3) for the superhorizon modes: $\mu_k(\eta) \simeq c_1(k)a(\eta) + c_2(k)a(\eta)\int_{\eta_0}^{\eta} d\eta'/a^2(\eta')$, where (5.6) implies $c_1c_2^* - c_1^*c_2 = i$. The second piece is the decaying mode and it will be negligible in time. In fact, $(\mu_k/a)' \simeq c_2/a^2$ therefore first terms in (5.16) and (5.17) decrease as $(1/a^2)^2$, which explains the equation of state (5.18).

side:

$$\rho_V + 3P_V = -\frac{1}{4\pi^2 a^4}\left[\frac{(\alpha^4+\alpha^2)}{2}h^4 - \frac{\alpha^2}{2}h^2h' + \frac{1}{8}\ln\left(\frac{\alpha h}{h_0}\right)(6h^4 + 2h'^2 - 4hh'' - 12h^2h' + 2h''')\right] - (\rho_F + 3P_F). \quad (5.21)$$

As it can be seen, this combination does not depend on ρ_S as a result of the equation of state (5.18), rather it involves only the scale factor $a(\eta)$ and its time derivatives. Thus one can work out (5.15) to determine the evolution of the scale factor $a(\eta)$.

It is important to indicate that ρ_V and P_V obeys the stress-energy conservation equation $\rho_V' + 3h(\rho_V + P_V) = 0$, which can be imposed to pin down the evolution of the vacuum energy density. First, we insert (5.19) and (5.20) in the right hand side of $\rho_V' = -3h(\rho_V + P_V)$. Subsequently, we use the equation of the state $P_S = -\frac{1}{3}\rho_S$ and further substitude ρ_S with ρ_V using (5.19). This gives[8]

$$\rho_V' + 2h\rho_V = \frac{1}{4\pi^2 a^4}\left[\frac{(\alpha^4+\alpha^2)}{2}h^5 - \frac{\alpha^2}{2}h^3h' + \frac{1}{8}\ln\left(\frac{\alpha h}{h_0}\right)(6h^5 + 2hh'^2 - 4h^2h'' + 2hh''' - 12h^3h')\right] + h(\rho_F + 3P_F), \quad (5.22)$$

which is the final and the simplest expression for the vacuum energy density. On one hand we have two unknowns, the vacuum energy density ρ_V and the scale factor of the universe $a(\eta)$. On the other hand we have two equations to work out the backreaction effects: The Friedmann equation (5.14) and the conservation equation (5.22). The accelaration equation (5.15) is automatically satisfied as long as (5.14) and (5.22) hold.

ρ_V obeys a first order differential equation. Therefore, its initial value must be specified by the vacuum chosen. One can decompose the solution of (5.22), which

[8] Remember that ρ_F and P_F are given by the expressions (5.10) and (5.11). Since they are conserved quantities, we use the conservation equation to replace ρ_F'.

involves a homogeneous and a particular piece in the form $\rho_V = \rho_H + \rho_P$. Eq. (5.22) signals that the homogeneous solution ρ_H should decay as $\rho_H = C_0/a^2$. Here, the constant C_0 can be fixed by the initial value of ρ_V. ρ_H mimics a perfect fluid with an equation of state parameter $\omega = -1/3$.

Although α parameter is assumed to be time dependent in general, this dependence is expected to be weak. This is actually because α is itself of unit order . In any case, one can not fix ρ_V exactly by integrating (5.22). But, it is possible to estimate the order of magnitude of the vacuum energy density and determine its evolution.

One physical reason for introducing an infrared cutoff is to assume that there exist no superhorizon modes initially. The comoving infrared cutoff h_0 can be specified as the initial Hubble radius in the conformal time η.

One important detail about our approximation is that it can be safely used if there is not any flow of the superhorizon modes into the subhorizon regime. Otherwise, the difference between the actual and the adiabatic modes will not be negligible due to the presense of the modes which are born as superhorizon and at a later time become subhorizon. Thus, for the accelerating cosmologies, the formulas above can be safely used. On the other hand, for the decelerating cosmologies the method also gives results, which are in good agreement with the ones in the literature. Here the key factor is the α parameter that compensates errors.

5.3. Applications To Power Law Expansion

In this section we compare our findings with the known results in the literature by applying our formula (5.22) to fixed backgrounds with specified scale factors $a(\eta)$. Let us start with de Sitter space by taking

$$a = \frac{\eta_0}{\eta}. \tag{5.23}$$

In [28] (also in the book [19]), the vacuum energy density of a minimally coupled scalar in the BD vacuum is given as

$$\rho_{BD} = \frac{-29h^4}{960\pi^2 a^4}. \tag{5.24}$$

By integrating (5.22) we obtain

$$\rho_V = \frac{(-119 + 60\alpha^4)h^4}{960\pi^2 a^4} + \frac{C_0}{a^2}. \tag{5.25}$$

For the values $C_0 = 0$ and $\alpha = (3/2)^{1/4}$, our result exactly coincide with the result in de Sitter space. The α parameter is seen to be of order one as expected. Moreover, as a result of de Sitter invariance, the energy density must be a constant which explains the choice of C_0.

Table 5.1: Classification of the universes undergoing power law expansion in the conformal time. The scale factor (5.26) is given by $a = (t/t_0)^m$ in the proper time, where $m = n/(n-1)$.

$-\infty < n < 0$	Decelerating
$n = 0$	Minkowski
$0 < n < 1$	Big-crunch
$n = 1$	de Sitter
$1 < n$	Accelerating

Next, we apply our formula to a background with a general power-law expansion

$$a = \left(\frac{\eta_0}{\eta}\right)^n. \tag{5.26}$$

In [32] the stress-energy-momentum of a minimally coupled massless quantum scalar field in the BD vacuum has been calculated by performing point-splitting regularization, where the vacuum energy density is found as

$$
\begin{aligned}
\rho_{BD} = & \frac{1}{2880\pi^2}\left[-\frac{1}{6}{}^{(1)}H_0{}^0 + {}^{(3)}H_0{}^0\right] \\
& - \frac{1}{1152\pi^2}{}^{(1)}H_0{}^0\left[\ln\left(\frac{R}{\mu^2}\right) + \psi(2+n) + \psi(1-n) + \frac{4}{3}\right] \\
& + \frac{1}{13824\pi^2}\left[-24\Box R + 24RR_0{}^0 + 3R^2\right] - \frac{R}{96\pi^2 a^2\eta^2}.
\end{aligned} \tag{5.27}
$$

In the above equation, the parameter μ in the logaritmic term is an arbitrary constant and ψ denotes the di-gamma function. $H_{\mu\nu}$ involves the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R, which is defined as

$$
{}^{(1)}H_{\mu\nu} = 2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}, \tag{5.28}
$$

$$
{}^{(3)}H_{\mu\nu} = R_\mu{}^\rho R_{\rho\nu} - \frac{3}{2}RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} + \frac{1}{4}R^2 g_{\mu\nu}. \tag{5.29}
$$

In (5.27) some extra divergences may pop up when the argument of the di-gamma function is zero or an integer. These terms can be cancelled out by using an infinite renormalization. It is well-known that every infinite renormalization inherits some finite ambiguities. Scaling μ is equivalent to a finite renormalization in the quantum effective action. As a result, one can add multiples of ${}^{(1)}H_0{}^0$ to ρ_{BD} in (5.27) (see e.g. comments below eq. (7.52) in [19]). To obtain the vacuum energy density in the power-law expansion for $n \neq -1$ we use (5.26) and compute (5.27) as

$$
\begin{aligned}
\rho_{BD} = & \frac{h^4}{1920\pi^2 n^2 a^4}\Big[81 - 30n - 109n^2 \\
& -90(n^2-1)\left(\ln\left[\frac{6(n+1)h^2}{n\mu^2}\right] + \psi(1-n) + \psi(n+2)\right)\Big], \qquad n \neq -1.
\end{aligned} \tag{5.30}
$$

For $n = -1$, which represents the radiation dominated universe, ${}^{(1)}H_0{}^0$ vanishes identically, therefore ρ_{BD} is safe from any divergences that may arise in the logarithm or the di-gamma functions. In that case, the vacuum energy density becomes

$$
\rho_{BD} = \frac{h^4}{960\pi^2 a^4}, \qquad n = -1. \tag{5.31}
$$

For the radiation dominated background it is possible to solve the mode equation (5.3) exactly as $\mu_k = e^{-ik\eta}/\sqrt{2k}$. The coefficient is fixed by the proper normalization of the BD vacuum. During the adiabatic regularization process, (5.8) and (5.9) vanish identically. Therefore, one ends up with the finite subtraction term $\rho_F = -h^4/(960\pi^2 a^4)$, which determines the regularized vacuum density as

$$\rho_V = \frac{h^4}{960\pi^2 a^4}. \tag{5.32}$$

Eq. (5.31) matches exactly with (5.32). This explains the importance of keeping the finite adiabatic subtraction terms for the massless fields. In view of our approximation, (5.32) can be obtained by setting $\alpha = 0$, which can be regarded as a singular case.

To obtain an expression for the vacuum energy density for $n \neq -1$, we evaluate our approximate expression (5.22) for the background (5.26), which gives

$$\begin{aligned}\rho_V =& \frac{h^4}{1920\pi^2 n^2 a^4}\left[\frac{2}{n-2}\left(-21+153n+(79+60\alpha^2)n^2-(92+60\alpha^2+60\alpha^4)n^3\right)\right.\\ & \left.-90(n^2-1)\ln\left[\frac{\alpha^2 h^2}{h_0^2}\right]\right]+\frac{C_0}{a^2}. \end{aligned} \tag{5.33}$$

Comparing (5.33) with (5.30), we see that setting $C_0 = 0$ corresponds to BD vacuum. It is important to emphasize that in (5.33) the coefficient of $\ln(h^2)$, which is α-independent, exactly matches with the one in (5.30). This has a great significance since this agreement can not be obtained by adjusting free parameters. One can in fact match the other terms by adjusting the value of α and by choosing an appropriate value for the ratio μ/h_0 (recall that this may be understood as a finite renormalization of a coupling constant in the effective action).

First, one may demand to set

$$\mu = h_0, \tag{5.34}$$

as these parameters are both infrared cutoffs. To make the vacuum energy densities

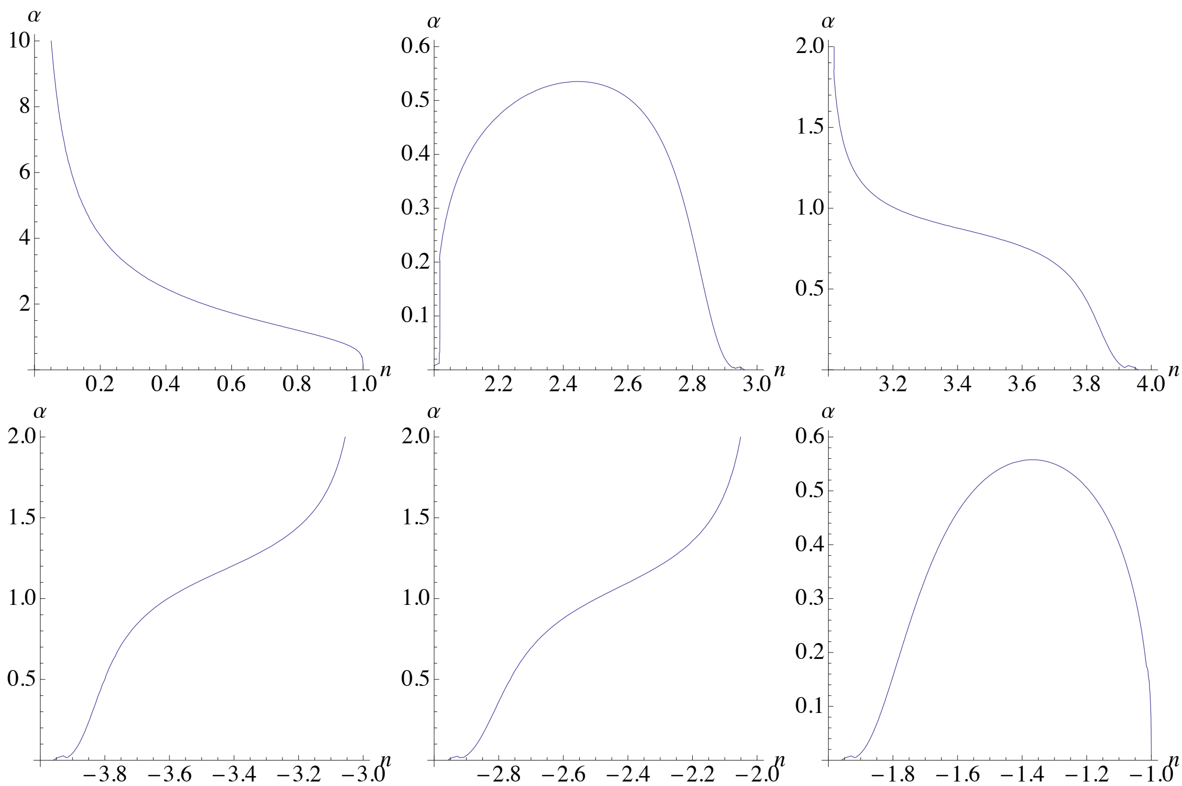

Figure 5.1: The numerical plots of α-parameter as a function of the power n from (5.35). For larger $|n|$ the plots resemble $(3,4)$ or $(-4,-3)$ intervals for positive and negative values, respectively.

agree, i.e. $\rho_V = \rho_{BD}$, α must satisfy the following equation:

$$\ln\left[\frac{36(n+1)^2}{n^2\alpha^4}\right]+2\psi(1-n)+2\psi(n+2)=\frac{(5+8\alpha^2+8\alpha^4)n^3+(2-8\alpha^2)n^2-11n-8}{3(n^2-1)(n-2)}. \tag{5.35}$$

This equation does not allow analytic solutions for α. However we can gain an insight by analysing numerical plots some of which are presented in figure (5.1). In the plots, the behaviour of α can be viewed for different intervals of unit size. Note that, the integer values of n generate singularities in the di-gamma function and in that case, to make sense of ρ_{BD}, one requires an infinite renormalization. Therefore, it is safe to take n as non-integer. In the limit $n \to \pm\infty$, (5.35) becomes

$$6\ln(\alpha^2/6)+(5+8\alpha^2+8\alpha^4)=2\psi(1-n)+2\psi(n+2). \tag{5.36}$$

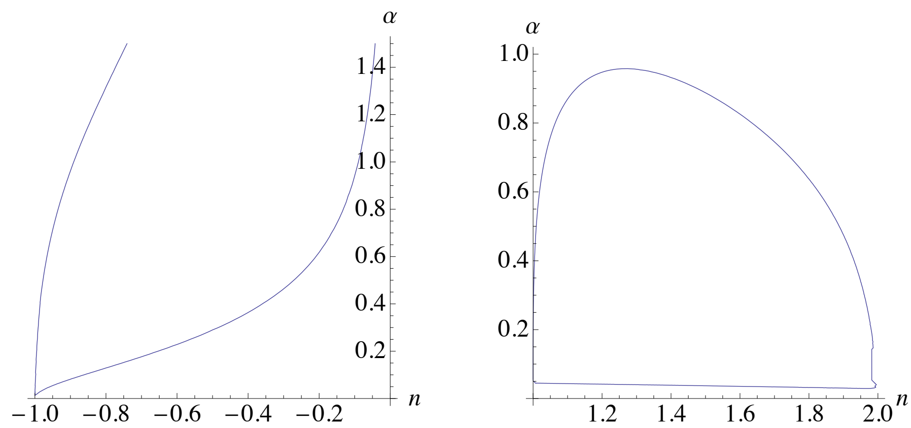

Figure 5.2: The numerical plots of α-parameter as a function of the power n for the intervals $(-1, 0)$ and $(1, 2)$. The infrared cutoff are chosen as $\mu = 10h_0$ and $\mu = 100h_0$, respectively.

The property $\psi(x) - \psi(x-1) = 1/(x-1)$ of the di-gamma function implies that, for large values of $|n|$, α is independent of n in the interval $(n, n+1)$. The graphs show that α continues to be of unit order for generic values, which supports our approximation.

For $n \in (1, 2)$ and $n \in (-1, 0)$, there does not exist any solution of (5.35) for α. Moreover, as n approaches an integer, α grows such that in the limit $\alpha \gg 1$. These cases can be handled by making a finite renormalization such that $\mu \neq h_0$. For instance, by demanding $\mu = 10h_0$, α becomes $\alpha \to 10\alpha$ in (5.22) and one can find an order one solution of α for $n \in (-1, 0)$. In the same way, if one sets $\mu = 100h_0$, α becomes $\alpha \to 100\alpha$ in (5.22), there exists a solution for α in the range $n \in (1, 2)$. Figure (5.2) shows the corresponding numerical plots. Note that there arises two solutions for α, for an arbitrary n in these intervals.

We conclude that our approximate expression can generate the known results in the literature. In de Sitter space ($n = 1$), it is enough to pick out $\alpha = (3/2)^{1/4}$. The radiation dominated universe ($n = -1$) is singular, which is equivalent to selecting $\alpha = 0$. In this case, only the finite subtraction terms contribute to the result. For all other integer values of n, (5.27) requires an infinite renormalization. The logarithmic term in (5.30) exactly agrees with the one in (5.33), which is very non-trivial, since finite renormalizations do not have any restriction on this term. We see that the other terms can agree by performing a finite renormalization, i.e. by scaling μ or adjusting α.

As pointed out above, our approximation can be safely used for accelerating cosmologies. However, it still gives the correct results for the decelerating cosmologies with negative n. Actually, in a decelerating model, the flow of the modes from superhorizon to subhorizon regime must be considered in the calculations. Fortunately, the α parameter cancels out the errors caused by disregarding this flow.

5.4. Application to Slow-Roll Inflation

In this section, we try to pin-down the backreaction effects for a simple inflationary model, where the background is driven by a single scalar field φ with an appropriate potential $V(\varphi)$. Using our method we want to see whether the backreaction effects of a massless scalar field has any consequence on the standard results of inflation. To begin with, let us review the well-known results in the standard inflationary scenario excluding any backreaction effects. For the inflaton field $T_{\mu\nu}$ is given by

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{5.37}$$

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{5.38}$$

and the inflaton obeys a second order differential equation

$$\ddot{\varphi} + 3H\dot{\varphi} + \partial_\varphi V = 0. \tag{5.39}$$

Remember that the dot denote the derivative with respect to the proper time t, which is identified as $dt = ad\eta$. In order to stay in the slow-roll regime, one must impose the following conditions

$$\dot{\varphi}^2 \ll V, \qquad \ddot{\varphi} \ll 3H\dot{\varphi}. \tag{5.40}$$

These can be satisfied by taking the slow-roll parameters ϵ_φ and η_φ very small:

$$\epsilon_\varphi \equiv \frac{M_p^2}{16\pi}\left(\frac{\partial_\varphi V}{V}\right)^2 \ll 1, \tag{5.41}$$

$$\eta_\varphi \equiv \frac{M_p^2}{8\pi}\left(\frac{\partial_\varphi^2 V}{V}\right) \ll 1. \tag{5.42}$$

Going back to the field equations, the inflaton approximately obeys

$$\dot{\varphi} \simeq -\frac{\partial_\varphi V}{3H}, \tag{5.43}$$

while the Friedmann equation becomes

$$H^2 \simeq \frac{8\pi}{3M_p^2}V. \tag{5.44}$$

The change in the Hubble parameter and the inflaton are prescribed by the slow-roll parameters as

$$\dot{H} \simeq -\epsilon H^2, \qquad \ddot{\varphi} \simeq -(\eta - \epsilon)H\dot{\varphi}. \tag{5.45}$$

Of course, when the backreaction effects are neglected, it can be verified that

$$\epsilon = \epsilon_\varphi, \qquad \eta = \eta_\varphi. \tag{5.46}$$

Note that the slow-roll parameters ϵ and η have great significance in the observational data.

Let us now calculate the backreaction of the quantum field ϕ in light of (5.22).

One sees that the vacuum energy density has the order of magnitude given by

$$\rho_V = \mathcal{O}(H^4), \tag{5.47}$$

which is much smaller than the background energy density $H^2 M_p^2$.

In the following, we will carry out the calculations assuming that ϵ_φ, η_φ and

$$\gamma \equiv \frac{H^2}{M_p^2} \tag{5.48}$$

are small and we will keep only the leading order terms. First, we look at the inflaton equation. It remains unchanged since there is no coupling between ϕ and φ in this model:

$$\dot{\varphi} \simeq -\frac{\partial_\varphi V}{3H}. \tag{5.49}$$

On the other hand, we have two sources ρ_V and ρ_φ that contribute to the right hand side of the Friedmann equation:

$$H^2 = \frac{8\pi}{3M_p^2}(\rho_V + \rho_\varphi). \tag{5.50}$$

From this equation one realizes that $\rho_\varphi \sim H^2 M_p^2$ and conclude with $\rho_V \sim \gamma \rho_\varphi$. Using (5.21), (5.10) and (5.11) in Einstein equations and neglecting all the higher order terms in the slow-roll parameters, the acceleration equation becomes

$$\dot{H} = -\frac{8\pi}{3M_p^2}\rho_V - \left[\frac{119}{360\pi} - \frac{\alpha^4}{6\pi}\right]\frac{H^4}{M_p^2} - \frac{4\pi}{M_p^2}\dot{\varphi}^2. \tag{5.51}$$

Moreover, combining (5.49) and (5.50) we obtain

$$\dot{\varphi}^2 = \frac{M_p^2}{4\pi}\epsilon_\varphi H^2. \tag{5.52}$$

Eq. (5.52) also holds even in the absence of a quantum field, since the presense of ρ_V induces a modification of order $\epsilon_\varphi\gamma$ in (5.52), which we neglect. Inserting (5.52) in (5.51), the ϵ parameter, which is defined in (5.45), turns out to be

$$\epsilon = \epsilon_\varphi + \left[\frac{119}{360\pi} - \frac{\alpha^4}{6\pi} + \frac{8\pi\rho_V}{3H^4}\right]\gamma. \tag{5.53}$$

Likewise, we differentiate (5.49) with respect to the proper time t and use the definition in (5.45) to obtain

$$\eta = \eta_\varphi. \tag{5.54}$$

One can conclude that in the presence of a quantum scalar field, the slow-roll parameter ϵ can not be fixed just by the potential of the inflaton, rather it is modified by the backreaction effects. Let us estimate the magnitude of this correction in the chaotic inflationary model, where the inflaton has a potential $V = \frac{1}{2}m^2\varphi^2$. We know that the cosmologically relevant perturbations leave the horizon as $\varphi \sim 3M_p - 4M_p$ and a viable value of the mass of the inflaton is $m \sim 10^{-6}M_p$. It turns out that $\epsilon_\varphi \sim 10^{-3}$ and $\gamma \sim 10^{-11}$, therefore, from (5.53) the correction due to the backreaction effects are negligible.

These results show that the backreaction of a quantum scalar field does not have a significant impact on the cosmic evolution during the slow-roll inflation. However, it might play a major role in the later epocs. Eq. (5.22) shows that the vacuum energy density evolves as H^4, which is always very small compared to the background energy density. But there is also the homogeneous piece scaling as $1/a^2$. In a dust dominated or radiation dominated stage, this piece will decrease slower than H^4. Thus, during inflation, ρ_V is expected to become H_0^4 (H_0 denotes the inflationary Hubble parameter) and begin to decrease like $\rho_V \sim H_0^4/a^2$ just after inflation (an analytic expression can be found in [33]). On the other hand, one can approximately calculate the initial energy density of radiation ρ_R as $H_0^2M_p^2$. So the radiation energy density falls off like $H_0^2M_p^2/a^4$. As a result of comparing these two values, we can see that when the universe grows M_p/H_0 times after inflation, ρ_V reaches ρ_R and the vacuum energy starts to dominate the universe. In the chaotic model considered above, the magnitude of M_p/H_0 can be estimated as $M_p/H_0 \sim 10^6$, which shows that the time of equivalence corresponds to a very early cosmic stage, before nucleosynthesis. The only loophole in the above discussion is that (5.22) is exactly applicable to accelerating cosmologies. But, as we saw in the previous section, (5.22) also gives viable results even for the decelerating backgrounds. So our results indicate that the vacuum energy density induced during inflation must be considered in the subsequent epochs in the cosmic evolution. Certainly, in a realistic case one should work out the evolution of the vacuum energy density rigorously by taking into account different epochs in the cosmic evolution. For instance, in the reheating (or preheating) stage, a possible coupling of our scalar field to the inflaton field can affect the evolution of the vacuum energy density. To conclude, such modifications may give an evolution which is still consistent with the current standard cosmic scenario.

6. CONCLUSION

The existence of quantum fields has significant theoretical and observational consequences that enrich our understanding of physics. In flat spacetime, a crucial example is the experimentally verified Casimir effect. This effect can be menifested as a force of attraction between two uncharged conducting plates, which can be explained by a shift of the vacuum energy density of the electromagnetic field. Moreover, quantum filed theory in flat spacetime successfully describes the behaviour of elementary particles over a widespread range of energy. This motivates one to study the impacts of quantum fields in curved backgrounds. The quantum field theory in curved spacetime has also remarkable successes, the primary example being the Hawking radiation from blackholes. Quantum fields are also expected to play a key role in cosmology, especially at very early times when the universe experiences a rapid expansion, which leave observable signatures today like the density inhomogeneities, structure formation and anisotropies in the CMB. After the discovery of the current accelerated expansion of the universe, determining the vacuum energy density of the quantized fields becomes an important problem, since it is supposed to contribute to the cosmological constant and lead to positive acceleration. This crucial problem can be worked out in a semi-classical approximation, where gravity is handled classically. In this thesis, we have studied different aspects of this problem for a quantized scalar field in cosmological spaces.

In chapter II, we start off by reviewing the standard canonical quantization procedure and discuss the difficulties of fixing the vacuum state in a curved background. The non-uniqueness of the vacuum is the primary handicap, which is a consequence of the absence of the Poincare symmetry in an expanding universe. To define the vacuum state, time dependence of the mode function must be fixed by using physically motivated prescriptions. A preferred choice is the BD vacuum, which coincides with the Minkowski vacuum for the short wavelength modes. Another alternative is to introduce a well defined series in the adiabatic basis to obtain a solution for the mode function in a series form. The vacuum obtained in this way is called the adiabatic vacuum.

Terminating the series at the n'th order corresponds to an approximate state, which is called the n'th order adiabatic vacuum. It is also possible to get an exact solution, i.e. the infinite order adiabatic vacuum. In chapter IV, as a new result, we compute the leading sixth order contribution to the regularized stress-energy-momentum tensor for a scalar placed in the infinite adibatic vacuum.

The stress-energy-momentum tensor of a quantized scalar in a FRW background becomes an operator that can naturally be found in terms of the field itself. It is not surprising that the vacuum expectation values of the energy density and the pressure contain divergences, which must be regularized. Compared to the flat space case, the regularization in a curved background is more complicated, since it depends on the geometry in a non-trivial way. In chapter III, we discuss one plausible tecnique, i.e. adiabatic regularization, to eliminate the divergences. This is a convenient method when one deals with a FRW background. After explaining the underlying physical motivation, we give the technical details in the procedure, which are necessary in chapter IV and V.

In chapter IV, we explicitly compute the sixth order contribution to the stress-energy-momentum tensor of a scalar field, which propagates in a FRW universe and placed in the infinite adiabatic vacuum. Since the corresponding stress-energy-momentum tensor can be computed as a series and as adiabatic regularization requires removing the adiabatic zeroth, second and fourth order terms, one ends up with the sixth order expressions as the leading order contribution to the stress-energy-momentum tensor. This is a good approximation for all modes of a massive field as long as $m \gg H$ holds. On the other hand, only the high energy (subhorizon) modes satisfying $k_* \gg aH$ for a fixed k_* can be worked out for a massless scalar field. The sixth order expressions are seen to be complicated. To understand the magnitude of the vacuum energy density in different cosmologies, we consider the spacetimes with the scale factor in the form of a simple power $a(\eta) = \left(\frac{1}{\eta}\right)^n$. We find out that the magnitude of the vacuum energy densities are proportional to H^6/m^2 and $H^6 a^2/k_*^2$ for the massive and the UV modes of the massless scalar, respectively. Note that the higher order terms contribute as additional powers of $(H/m)^2$ and $(Ha/k_*)^2$, thus they are suppressed. By analysing the implications for different cosmologically relevant spacetimes we realise that the vacuum energy density can become positive, negative or zero depending on the expansion power n. Additionally, it can be a decreasing or increasing function of time or remain constant according to the chosen cosmological background.

For the massive fields satisfying $m \gg H$, the adiabatic vacuum is a viable choice. As an example consider the stabilized moduli at early times, which appear in the string theory motivated models. Not to change the standard cosmological scenario, these are required to satisfy this adiabacity condition. On the other hand, all the known massive fields at late times (except neutrinos) also satisfy this condition. Therefore, considering these massive fields in the adiabaic vacuum is reasonable. For the massless fields it turns out that only the (subhorizon) modes with sufficiently large wavenumbers $k_* \gg aH$ can be studied adiabatically, where the subhorizon regime is determined by the Hubble parameter H. We recognize that the adiabatic vacuum matches with the BD vacuum, at least when the universe experiences a simple power law expansion (it is reasonably valid for any scale factor, although we are not able to provide any proof).

One of the important results that we appreciate is that when placed in an adiabatic vacuum, the energy densities of the massive field and the UV modes of the massless field become tiny compared to the energy density that drives the background. Interestingly for the values of the expansion power $0 < n < 1$ (for massive fields) and $0 < n < 3/2$ (for massless fields), the vacuum energy density increases in time and the adiabacity is eventually lost, i.e. H/m and Ha/k_* become order one. Even in these cases one comes up with negligible vacuum energy densities which will not affect the background. In any case, it can be an interesting research problem to determine the evolution of the vacuum energy density and inverstigate whether its contribution can avoid the big crunch. Supported by these arguments, we think that the cosmological constant problem is an IR issue, rather than a UV problem. This view is different than the traditional notion, which underlines the big discrepancy between the estimated and observed values of the energy density of the cosmological constant. To be more explicit, following an appropriate regularization, the contribution of the modes with large wavenumbers (UV modes) turn out to be negligible. The situation is similar to the

Casimir effect, where the vacuum energy density is determined by the distance between the plates, namely by the IR scale instead of the UV scale. Moreover, the massive fields have no contribution to the Casimir effect, and can be disregarded. Since the theoretical calculations for the Casimir energy agree with what is observed experimentally, the previous statement is worthy to be taken into consideration, namely the IR physics can be a key to overcome the cosmological constant problem.

In chapter V, we try to pin down the backreaction of a quantized massless, real scalar field propogating in a FRW universe. The usual treatments in the literature involve quantization in a fixed background, which prompts to study the effects of the quantum fields perturbatively. Compared to this, determining the backreaction effects is more challenging since one is not able to solve the mode functions and consequently there does not appear an explicit expression for the stress-energy-momentum tensor. By considering the subhorizon and superhorizon modes separately, we show that these contributions can be obtained approximately. Specifically, for massless fields, the subhorizon modes with wavelengths smaller than the Hubble radius evolve nearly adiabatically, thus their contributions to the stress-energy-momentum tensor are killed by the subtraction terms in the adiabatic regularization process (even the leading order contribution is found to be negligible, which is calculated in detail in the previous chapter). On the other hand, the contribution of the superhorizon modes is seen to obey a simple equation of state. Utilizing this approximation, we achieve to obtain a first order equation for the vacuum energy density, which together with the Friedmann equation determines the backreaction effects.

While developing our formulas we consider different aspects of the problem carefully. For regularization we keep the subtraction terms up to fourth order to remove UV divergences. Moreover, we also regard the finite subtraction terms that survive in the massless limit (the necessity of keeping them becomes clear when compared to the known results in the literature). We fine tune the subhorizon&superhorizon border by introducing a new parameter α. In addition, supported by a physical argument, we put a cutoff to remove the IR divergence.

To check the validity of our approximation, we take fixed backgrounds of cosmological interest to compare our estimations with the known results in the literature such as de Sitter space and spacetimes which experience power law expansion. We see that our results are in complete agreement by adjusting the α parameter. We also analyse the possible implications on the slow-roll inflationary picture. We find out that during the near exponential expansion, the background evolution does not alter and the backreaction effects just give rise to a modification in one of the slow-roll parameters. However, the vacuum energy density created during inflation might have a cosmological significance at subsequent stages, since it decreases as $1/a^2$, which is slower than radiation or dust. One can realise that this decrease describes an equation of state $P_V = -\frac{1}{3}\rho_V$. Thus, the vacuum energy density and pressure can not give rise to acceleration in the epochs following inflation.

As our findings are in complete agreement with the known results, this encourages us to improve this work in different ways. One of them is to investigate the implications for decelerating cosmologies. To be more accurate, one should estimate the contributions of the modes, which were first superhorizon and later become subhorizon. In that way, the precise evolution of the vacuum energy density induced during inflation can be determined at later stages. Another possible research problem is to study the backreaction effects of a massive scalar field. For this case, the situation is more complicated, since there are already two different scales in the problem; the mass of the field and the Hubble parameter. Moreover, it does not seem to be possible to express the contributions of the superhorizon modes as a simple equation of state. Therefore one may demand to solve the integro-differential equation by means of numerical analysis.

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