PATH ALGEBRAS

by

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To the memory of my father...

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ABSTRACT

PATH ALGEBRAS

In this thesis, we investigate path algebras and Leavitt path algebras. Some properties of path and Leavitt path algebras are given. Our aim is to see what conditions on the graph E makes Leavitt path algebras simple. We also study the relations between Incidence algebras and path algebras.

ÖZET

YOL CEBİRLERİ

Bu tezde, yol cebirleri ve Leavitt yol cebirleri incelenmiştir. Yol cebirlerinin ve Leavitt yol cebirlerinin bazı özellikleri verilmiştir. Amaç E grafı üzerindeki hangi koşulların Leavitt yol cebirlerini basit yapacağını görmektir. Aynı zamanda, çakışma cebirleri ve yol cebirleri arasındaki ilişkiler çalışılmıştır.

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LIST OF SYMBOLS

E^0	The vertex set of the graph E
E^1	The edge set of the graph E
E^*	The set of all paths of E
KE	The path K-algebra over E
$L_K(E)$	The Leavitt path K-algebra over E
$CP(v)$	The set of all closed paths based at v
$CSP(v)$	The set of all closed simple paths based at v

1. INTRODUCTION

Wherever in the world you study mathematics, you see that the first examples of rings includes fields, $\mathbb{Z}$, $n \times n$ matrix rings, and polynomial rings. These are fundamental examples and any one of these rings R has the 'Invariant Basis Number' property:

If m and m' are integers with the property that the free left modules ${}_RR^m$ and ${}_RR^{m'}$ are isomorphic, then $m = m'$.

In words, the IBN property says that any two bases for a free left R-module have the same number of elements. But the IBN does not hold for all rings. In 1962, Leavitt proved that for each positive integer $n > m$ and field K there exists a K-algebra A with ${}_KA^m \cong_K A^n$ in [1]. These algebras later will be denoted as Leavitt algebra of type (m, n) and A is denoted by $L_K(m, n)$.

In 2000's Path algebras and Leavitt Path algebras are defined on a directed graph E. Leavitt path algebras, which are natural generalizations of Leavitt algebras, are the algebraic versions of the Cuntz-Krieger algebras $C^*(E)$ of directed graph E described in [2] at the same time generalize Cuntz Algebras. The introduction of Leavitt path algebras in [3] has recently attracted the interest of algebraists as well as of analysts working on C^*-algebras.

First of all, we read Leavitt's article [4] published in 1965. As his ideas were definitely the roadmap for the simplicity of Leavitt path algebras, the proof of the theorem, simplicity of the Leavitt algebras, is studied to see how the original proof proceeds. After getting Leavitt's idea behind his proof, some properties of path and Leavitt path algebras are given. Our aim is to see what conditions on the graph E makes Leavitt path algebras simple. We read [3], but the proof of the simplicity theorem was really complicated and was hard to follow since it contained too many case analysis. On the other hand, the two-sided ideals of Leavitt path algebras are studied in [5]. Using some results of [5] we reproved the simplicity theorem of Leavitt algebras shortening the proof given in [3].

Moreover, we studied the relations between Incidence algebras and path algebras. The algebra homomorphism defined in [6] from path algebras on a row-finite graph E into incidence algebra is studied. We still try to generalize this case for arbitrary graphs, i.e. we try to see for which conditions on E, we still have such an algebra homomorphism.

2. PATH AND LEAVITT PATH ALGEBRAS

We begin this chapter by reminding the reader basic definitions in graph theory. We continue with the construction of the standard path algebra and Leavitt path algebra of a graph. Finally, we investigate their algebraic structures.

A directed graph $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For each edge e, $s(e)$ is the *source* of e and $r(e)$ is the range of e. If $s(e) = v$ and $r(e) = w$, then we also say that v *emits* e and that w *receives* e. A vertex which does not receive any edges is called a *source*. A vertex which emits no edges is called a *sink*.A graph is called *row-finite* if $s^{-1}(v)$ is a finite set for each vertex v. A graph is called *finite* if it is row-finite and E^0 is a finite set.

A *path* μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the source of μ and $r(\mu) := r(\mu_n)$ is the range of μ. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq r(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*. We denote by E^* the set of all paths of E. The *length* of a path μ, denoted by $|\mu|$, is the number of edges it contains. The length of $v \in E^0$ is 0.

Definition 2.1. Let K be a field and E be a graph. The *path K-algebra* over E is defined as the free K-algebra $K[E^0 \cup E^1]$ with the relations:

(1) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$.

(2) $e = er(e) = s(e)e$ for all $e \in E^1$.

This algebra is denoted by KE.

Definition 2.2. Given a graph E we define *the extended graph of E* as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r'

and s' are defined as

$$r'|_{E^1} = r, \qquad s'|_{E^1} = s, \qquad r'(e^*) = s(e) \qquad \text{and} \qquad s'(e^*) = r(e)$$

.

Definition 2.3. Let K be a field and E be a row-finite graph. The *Leavitt path algebra* of E with coefficients in K is defined as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e^*f = \delta_{e,f}r(e)$ for every $f \in E^1$ and $e^* \in (E^1)^*$.
(CK2) $v = \sum_{\{e \in E^1 : s(e)=v\}} ee^*$

This algebra is denoted by $L_K(E)$ (or simply by $L(E)$ when the field K is understood).

The conditions (CK1) and (CK2) are called the *Cuntz-Krieger* relations. In particular condition (CK2) is the *Cuntz-Krieger* relation at v_i. If v_i is a sink, we do not have a (CK2) relation at v_i . Note that the condition of row-finiteness is needed in order to define the equation (CK2).

The elements of E^1 are called real edges, while for $e \in E^1$ we call e^* a ghost edge. The set $\{e^* \mid e \in E^1\}$ is denoted by $(E^1)^*$. We say that a path in $L_K(E)$ is a real path (resp., a ghost path) if it contains no terms of the form e^* (resp., e). For a path $\alpha = e_1 \cdots e_n$, we denote by α^* the ghost path $e_n^* \cdots e_1^*$.

Examples of Leavitt path algebras

- Matrix algebras $M_n(K)$: Consider the 'oriented n-line' graph A_n defined by $(A_n)^0 = \{v_1, \ldots, v_n\}$, $(A_n)^1 = \{e_1, \ldots, e_n\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for

$i = 1, \ldots, n-1$. i.e.

$$A_n = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Then $L_K(A_n) \cong M_n(K)$: Let $E_{(i,j)}$ be the standard (i,j)-matrix unit in $M_n(K)$. i.e $E_{i,j}$ is the matrix which has 1 in the (i,j)-entry and 0 else. Now define the map on the generators of $L_K(A_n)$, $\phi : L_K(A_n) \to M_n(K)$ by setting $\phi(v_i) = E_{(i,i)}$, $\phi(e_i) = E(i, i+1)$ and $\phi(e_i^*) = E(i+1, i)$. ϕ is clearly one-to-one and onto. In order to show ϕ is a K-algebra homomorphism it is enough to show that ϕ factors through the appropriate relations in $L_K(A_n)$ i.e. we need to check:

(i) $\langle \{uv - \delta_{u,v}u : u, v \in E^0\} \cup \{e - er(e), e - s(e)e : e \in E^1\} \rangle \subset Ker(\phi)$

(ii) $\langle \{e^* f - \delta_{e,f} r(e) : e, f \in E^1\} \cup \{v - \sum_{\{e \in E^1 : s(e)=v\}} ee^* : v \in s(E^1)\} \rangle \subset Ker(\phi)$ But this is a straightforward calculation if we note that:

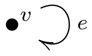
$$E_{(i,i)}E_{(j,j)} = \begin{cases} E_{(i,i)} & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$$E_{(i,i+1)}E_{(j+1,j)} = \begin{cases} E_{(i,i)} & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$$E_{(i+1,i)}E_{(j,j+1)} = \begin{cases} E_{(i+1,i+1)} & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

- Laurent polynomial algebras $K[x, x^{-1}]$: Consider the 'one vertex, one loop' graph C_1 defined by $(C_1)^0 = \{v\}$, $(C_1)^1 = \{e\}$.

$C_1 =$



Then clearly $K[x,x^{-1}] \cong L_K(C_1)$, via the map ϕ defined by $\phi(v)=1$, $\phi(e)=x$ and $\phi(e^*)=x^{-1}$.

- Leavitt algebras $A = L_K(1,n)$ for $n \geq 2$:

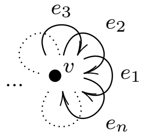
Definition 2.4. The Leavitt algebra of type $(1,n)$ is the free associative $K-algebra$ with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

(1) $x_i y_i = \delta_{ij} 1_R$ for all $1 \leq i, j \leq n$
(2) $\sum_{k=0}^{n} y_i x_i = 1_R$

This algebra is denoted by $L_K(1,n)$.

Now consider the 'rose with n-petals'graph R_n for $n \geq 2$ defined by $(R_n)^0 = \{v\}$, $(R_n)^1 = \{e_1, \ldots e_n\}$.

$R_n =$



Then $L_K(1,n) \cong L_K(R_n)$ for $n \geq 2$. If we define the map ϕ by $\phi(v)=1$, $\phi(e_i)=y_i$ and $\phi(e_i^*)=x_i$ and extend it K-linearly to $L_K(R_n)$, we get the desired algebra isomorphism (by definition of $L_K(1,n)$, ϕ factors through the appropriate relations in $L_K(R_n)$).

Lemma 2.5. *If α, β be paths in E then*

$$\alpha^*\beta = \begin{cases} \beta' & \text{if } \beta = \alpha\beta' \text{ for some } \beta' \in E^* \\ (\alpha')^* & \text{if } \alpha = \beta\alpha' \text{ for some } \alpha' \in E^* \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for some $\alpha', \beta' \in E^*$.

Proof. Write $\alpha = e_1 \cdots e_m$ and $\beta = f_1 \cdots f_n$ and assume $0 \neq \alpha^*\beta = e_m^* \cdots e_1^* f_1 \cdots f_n$. Then $e_1^* f_1 \neq 0$ and we get $e_1 = f_1$ and $e_1^* e_1 = r(e_1)$ by CK1. Similarly,

$$e_m^* \cdots e_2^* \underbrace{e_1^* f_1}_{r(e_1)=r(f_1)=s(f_2)} f_2 \cdots f_n = e_m^* \cdots \underbrace{e_2^* f_2}_{\neq 0} \cdots f_n.$$

Proceeding like this we get $\alpha^*\beta = f_{m+1} \cdots f_n$ (if $m < n$) or $\alpha^*\beta = e_m^* \cdots e_n^*$ (if $m > n$). ☐

Lemma 2.6. *If μ, ν, α, β are paths in E then*

$$\mu\nu^*\alpha\beta^* = \begin{cases} \mu\alpha'\beta^* & \text{if } \alpha = \nu\beta' \text{ for some } \beta' \in E^* \\ \mu(\nu')^*\beta^* & \text{if } \nu = \alpha\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Proof. Follows from previous lemma. ☐

Lemma 2.7. *Every monomial in $L_K(E)$ is of the following form:*

(i) kv with $k \in K$ and $v \in E^0$,

(ii) $ke_1 \cdots e_a f_1^* \cdots f_b^*$ where $k \in K$; $a, b \geq 0$, $a + b > 0$, $e_1, \cdots, e_a, f_1^* \cdots f_b* \in E^1$.

Proof. Take a monomial $\mu \in L_K(E)$ and write $\mu = x_1 \cdots x_n$ with $x_i \in E^0 \cup E^1 \cup (E^1)^*$. We will prove by induction on n. For $n = 1$ the monomial is of form kx_1 and we are done. Assume for $k \in \mathbb{N}$ we have $kx_1 \cdots x_k$ has the form kpq^* with $p = e_1 \cdots e_a$ and $q = f_1 \cdots f_b$ then $kx_1 \cdots x_k x_{k+1} = kpq^* x_{k+1}$ where $x_{k+1} \in E^0 \cup E^1 \cup (E^1)^*$. Then by lemma 2.5 $q^* x_{k+1} = \begin{cases} (q'^*) & \text{if} q = x_{k+1}q' \\ 0 & \text{otherwise} \end{cases}$ then $kx_1 \cdots x_{k+1} = kp(q')^*$ and we are done. ☐

Corollary 2.8. $L_K(E) = span\{pq^* \mid p, q \quad \text{are paths in } E \text{ for which } r(p) = r(q)\}$

Proof. $span\{pq^* \mid p, q$ are paths in E for which $r(p) = r(q)\} \subset L_K(E)$ and is a vector space closed under multiplication by lemma 2.6 so is a subalgebra of $L_K(E)$.

Since its vertices are paths of length 0 they are contained in this spanning set and also since it contains the generators ($e = es(e)^*, v = vv^*, e^* = r(e)e^*$), it is the algebra $L_K(E)$. □

Lemma 2.9. *If E^0 is finite then $L_K(E)$ is a unital K-algebra. If E^0 is infinite, then $L_K(E)$ is an algebra with local units(specifically, the set generated by finite sums of distinct elements of E^0).*

Proof. Assume E^0 is finite. We will show $\sum\limits_{i=1}^{n} v_i$ is the unit element.
If $v_j \in E^0$

$$(\sum_{i=1}^{n} v_i)v_j = \sum_{i=1}^{n} v_i v_j = \sum_{i=1}^{n} \delta_{ij} v_j = v_j \quad \text{and similarly} \quad v_j(\sum_{i=1}^{n} v_i) = v_j$$

If $e_j \in E^1$

$$(\sum_{i=1}^{n} v_i)e_j = (\sum_{i=1}^{n} v_i)s(e_j)e_j = (\sum_{i=1}^{n} v_i s(e_j))e_j = s(e_j)e_j = e_j,$$

similarly $e_j(\sum_{i=1}^{n} v_i) = e_j$. If $e_j^* \in (E^1)^*$

$$(\sum_{i=1}^{n} v_i)e_j^* = (\sum_{i=1}^{n} v_i)(e_j r(e_j))^* = \sum_{i=1}^{n} v_i r(e_j)e_j^* = r(e_j)e_j^* = s'(e_j^*)e_j^* = e_j^*,$$

similarly $e_j^*(\sum\limits_{i=1}^{n} v_i) = e_j^*$. Since $L_K(E)$ generated by $E^0 \cup E^1 \cup (E^1)^*$, then for any $\alpha \in L_K(E)$ we have $\alpha(\sum\limits_{i=1}^{n} v_i) = (\sum\limits_{i=1}^{n} v_i)\alpha = \alpha$.

Now assume E^0 is infinite. Consider a finite subset $\{a_j\}_{j=1}^{t}$ of $L_K(E)$. Then by Corollary 2.8 $\sum\limits_{s=1}^{n_i} k_s^i v_s^i + \sum\limits_{l=1}^{m_i} c_l^i p_l^i$ where $k_s^i, c_l^i \in K \setminus \{0\}$ and p_l^i are monomials of form $e_1 \cdots e_a f_1^* \cdots f_b^*$. Define $\cup_{i=1}^{n}\{v_s^i, s(p_l^i), r(p_l^i) : s = 1, \dots n_i, l = 1, \dots m_i\}$ then $\alpha = \sum\limits_{v \in V} v$ is a finite sum of vertices. By a similar argument above it can be shown that $\alpha a_i = a_i \alpha = a_i$ for every $1 \leq i \leq t$. □

Lemma 2.10. *$L_K(E)$ is a $\mathbb{Z}$-graded algebra with grading induced by setting $deg(v) = 0$ for all $v \in E^0$, $deg(e) = deg(e^*) = 1$ for all $e \in E^1$. That is, $L_K(E) = \underset{n\in\mathbb{N}}{\oplus} L_K(E)_n$, where $L_K(E)_n$ is generated as a vector space by monomials of the form pq^* having $deg(p) - deg(q) = n$.*

Proof. (i)$L_K(E) = \sum\limits_{n\in\mathbb{Z}} L_K(E)_n$ follows from Corollary 2.8.
(ii)We will show $L_K(E)_n L_K(E)_m \subset L_K(E)_{m+n}$.First note that $L_K(E)nL_K(E)_m = span\{\mu\nu^*\alpha\beta^* : \mu\nu^* \in L_K(E)_n, \alpha\beta^* \in L_K(E)_m\}$. Now let $\mu\nu^*\alpha\beta^* \in L_K(E)nL_K(E)_m$ Then $\mu\nu^* \in L_K(E)_n$ and $\alpha\beta^* \in L_K(E)_m$ gives $deg(\mu) - deg(\nu) = n$ and $deg(\alpha) - deg(\beta) = m$. Now by Lemma 2.6

$$\mu\nu^*\alpha\beta^* = \begin{cases} \mu\alpha'\beta^* & \text{if } \alpha = \nu\beta' \text{ for some } \beta' \in E^* \\ \mu(\nu')^*\beta^* & \text{if } \nu = \alpha\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise} \end{cases}$$

If $\mu\nu^*\alpha\beta^* = 0 \in L_K(E)_{m+n}$ and we are done. If $\alpha = \nu\alpha'$ for some $\alpha' \in E^*, deg(\alpha) = deg(\nu\alpha')$ and we get $deg(\alpha') = deg(\beta) + m - deg(\nu)$. Now $\mu\nu^*\alpha\beta^* = \mu\alpha'\beta^*$, and $deg(\mu\alpha') = deg(\mu) + deg(\alpha') = deg(\mu) + deg(\beta) + m - deg(\nu)$ so $deg(\mu\alpha') - deg(\beta) = m + n$ and we are done.
The case $\nu = \alpha\nu'$ can be shown by similar way. ☐

By this lemma we can define the degree of an arbitrary polynomial in $L_K(E)$ as the maximum of the degrees of its monomials.

Lemma 2.11. *$L_K(E)$ can be equipped with an involution $x \mapsto \overline{x}$ defined in the monomials by:*

(i) $\overline{kv} = kv$ *with* $k \in K$ *and* $v \in V$
(ii) $\overline{ke_1 \cdots e_n f_1^* \cdots f_m^*} = kf_m \cdots f_1 e_n^* \cdots e_1^*$ *where* $k \in K; m, n \geq 0, m + n > 0, e_i, f_j \in E^1$,

and extending linearly to $L_K(E)$.

Proof. The proposed map is well-defined and by lemma 2.7 and linear by definition. It is easy to show that $\overline{xy} = \overline{y}\overline{x}$ and $\overline{\overline{x}} = x$ for every $x, y \in L_K(E)$ and it is straightforward to see that the map is compatible with the relations defining $L_K(E)$. □

Remark 2.12.

- The involution transforms a polynomial in only real edges into a polynomial in only ghost edges and vice versa.
- If J is an ideal of $L_K(E)$ then so is $\overline{J}$.

3. CLOSED PATHS

In this chapter, we investigate some certain paths in the graph E which play a central role in the structure of Leavitt path algebra $L_K(E)$.

Definition 3.1. An edge e is an *exit* to the path $\mu = \mu_1 \cdots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$.

Definition 3.2. A path $\mu = \mu_1 \cdots \mu_n$ is a *cycle* if $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for all $i \neq j$.

Definition 3.3. A *closed path based at* v is a path $\mu = \mu_1 \cdots \mu_n$ with $\mu_j \in E^1$ and $s(\mu) = r(\mu) = v$. Denote by $CP(v)$ the set of all such paths.

Definition 3.4. A *closed simple path based at* v is a closed path $\mu = \mu_1 \cdots \mu_n$ based at v such that $s(\mu_j) \neq v$ for all $1 < j < n$. Denote by $CSP(v)$ the set of all such paths.

Remark 3.5.

- Every cycle based at v is also a closed simple path based at v but converse is not true. A closed simple path based at v may visit some of its vertices other than v more than once.
- Every closed simple path based at v is a closed path based at v but converse is not true. A closed path based at v may visit v more than once.

Lemma 3.6. *Let $\mu, \nu \in CSP(v)$. Then $\mu^* \nu = \delta_{\mu,\mu} v$.*

Proof. Let$\mu, \nu \in CSP(v)$. If $\mu = \nu$ then by lemma 2.5 $\mu^* \nu = v$. If $deg(\mu) < deg(\nu)$ write $\nu = \nu_1 \nu_2$ where $deg(\nu_1) = deg(\mu)$ and $deg(\nu_2) > 0$. Now if $\mu = \nu_1$ then we have that $v = r(\mu) = r(\nu_1) = s(\nu_2)$, contradicting that $\nu \in CSP(v)$, so $\mu \neq \nu_1$ and thus by lemma 2.5 $\mu^* \nu = 0$. The case $deg(\mu) > deg(\nu)$ is similar. $\square$

Lemma 3.7. *For every $p \in CP(v)$ there exist unique $c_1, \cdots, c_m \in CSP(v)$ such that $p = c_1 \cdots c_m$.*

Proof. Write $p = e_1 \cdots e_n$. Let $T = \{t \in \{1, \cdots, n\} : r(e_t) = v\}$ and list $t_1 < \cdots < t_m = n$ all elements of T. Then $c_1 = e_1 \cdots e_{t_1}$ and $c_j = e_{t_j+1} \cdots e_{t_j}$ for $j > 1$ give the desired decomposition.

Now we will prove uniqueness. Assume $p = c_1 \cdots c_r = d_1 \cdots d_s$ with $c_i, d_j \in CSP(v)$. Now multiply by c_1^* on the left then $c_1^* c_1 \cdots c_r = c_1^* d_1 \cdots d_s$ and use lemma 3.6 to get $0 \neq vc_2 \cdots c_r = c_1^* d_1 \cdots d_s$. Now if $c_1 \neq d_1$ then by lemma 3.6 again the product will be zero hence $c_1 = d_1$. By an inductive process we get $r = s$ and $c_i = d_i$ for every $1 \leq i \leq r$. $\square$

Definition 3.8. For $p \in CP(v)$ we define *the return degree (at v) of p* to be the number $m \geq 1$ in the decomposition above.(So in particular, $CSP(v)$ is the subset of $CP(v)$ having return degree equal one.) We denote it by $RD(p) = RD_v(p) = m$. This notation is extended to vertices by setting $RD_v(v) = 0$, and to nonzero linear combinations of the form $\sum k_s p_S$, with $p_s \in CP(v) \cup v$ and $k_s \in K \setminus 0$ by setting $RD(\sum k_s p_s) = max\{RD(p_s)\}$.

Lemma 3.9. *Every closed path contains a cycle.*

Proof. By lemma 3.7 it is enough to show that every simple closed path contains a cycle. Let μ be a closed simple path based at v. $\square$

Lemma 3.10. *For a graph E the following conditions are equivalent:*

(i) Every cycle has an exit.

(ii) Every closed path has an exit.

(iii) Every closed simple path has an exit.

(iv) For every $v \in E^0$,if $CSP(v) \neq \emptyset$, then there exists $c \in CSP(v)$ having an exit.

Proof. (ii)$\Rightarrow$ (iii) $\Rightarrow$ (i) is trivial by definition, and (iii) $\Rightarrow$ (iv) is obvious.

(i)$\Rightarrow$ (ii). Consider $\mu \in CP(v_i)$. By lemma 3.7 we can factor $\mu = c^{(1)} \cdots c^{(m)}$, where $c^{(j)} \in CSP(v_i)$, and we examine $c^{(m)}$. If it is a cycle then we can find an exit for it, and therefore for μ, by assumption. If not, $c^{(m)}$ visits a vertex (different from v_i) more than once. Write $c^{(m)} = e_1^{(m)} \cdots e_s^{(m)}$ with each $e_i^{(m)} \in E^1$ and let $e_{s_0}^{(m)}$ be the last edge such that who shares the same source with an edge different than itself (i.e. it is the last edge for which $s(e_j^{(m)}) = s(e_i^{(m)})$ for $i \neq j$). Thus there exists $s_1 < s_0$ such that $s(e_{s_0}^{(m)}) = s(e_{s_1}^{(m)})$. we have three cases:

- Case (1) $e_{s_0}^{(m)} = e_{s_1}^{(m)}$ and $s_0 < s$. Then $r(e_{s_0}^{(m)}) = r(e_{s_1}^{(m)})$; that is $s(e_{s_0+1}^{(}m)) = s(e_{s_1+1}^{(m)})$, but this contradicts with the choice of $e_{s_0}^{(m)}$. Thus this case is impossible.
- Case (2) $e_{s_0}^{(m)} = e_{s_1}^{(m)}$ and $s_0 = s$. Then $r(e_{s_0}^{(m)}) = r(e_{s_1}^{(m)}) = v_i$ implies $s(c_{s_1+1}^{(m)}) = v_i$, which is again impossible because $c^{(m)} \in CSP(v_i)$.
- Case (3) $e_{s_0}^{(m)} \neq e_{s_1}^{(m)}$. So this is the only possible case. In this case since $s(e_{s_0}^{(m)}) = s(e_{s_1}^{(m)})$ and they are different $e_{s_1}^{(m)}$ is an exit for $c^{(m)}$, and then for μ.

(iv)$\Rightarrow$ (iii). Consider $c \in CSP(v)$. By hypothesis we find $\tilde{c} \in CSP(v)$ having an exit. If $c = \tilde{c}$ we are done. If not write $c = e_1 \cdots e_n, \tilde{c} = f_1 \cdots f_m$ and proceed by steps:

- Step 1. If $e_1 \neq f_1$, since $s(e_1) = s(f_1) = v$, then f_1 is an exit for c.
- Step 2. If $e_1 = f_1$ then $r(e_1) = r(f_1)$; that is, $s(e_2) = s(f_2)$.
- Step 3. If $e_2 \neq f_2$, then as in Step 1, e_2 is an exit for c.
- Step 4. If $e_2 = f_2$, then continue as in Step 2.

With this process, we either find an exit or we run out of edges in one path but not in the other since the paths $c, \tilde{c}$ are different. Thus:

Case 1. $c = \tilde{c} g_1 \cdots g_s$ where $g_i \in E^1$ for $1 \leq i \leq s$. But then we get $s(g_1) = r(\tilde{c}) = v$ which contradicts $c \in CSP(v)$. So this case is impossible.

Case 2. $\tilde{c} = c h_1 \cdots h_r$ where $h_i \in E^1$ for $1 \leq i \leq r$. But similarly this is impossible.

Thus in our process we find an exit and this finishes the proof. $\square$

4. TWO-SIDED IDEALS IN LEAVITT PATH ALGEBRAS

In this chapter, we explicitly describe two-sided ideals in Leavitt path algebras associated with a row-finite graph.

Lemma 4.1. *Let I be a two-sided ideal of $L_K(E)$. Every $0 \neq \mu \in I$ can be written as $\mu = \alpha_1 + \cdots \alpha_m$ where $\alpha_i \in I$ and, for each i, α_i is a sum of paths all have the same source and have the same range.*

Proof. Let $\mu \in I$ and $\mu = \mu_1 + \cdots \mu_n$, where $\mu_i^{,}s$ are monomials in $L_K(E)$, then $\alpha_i = s(\mu_i)\mu r(\mu_i)$ is the sum of those μ_j whose sources are all the same and whose ranges are all the same; specifically, the sum of those μ_j for which $s(\mu_j) = s(\mu_i)$and $r(\mu_j) = r(\mu_i)$. Moreover, $\alpha_i \in I$. Thus we may write $\mu = \alpha_1 + \cdots + \alpha_m$ where each α_i with above properties. $\square$

Notation: Let $L_K(E)_R$ (resp., $L_K(E)_G$) denote the subring of elements in $L_K(E)$ whose terms involve only real edges (resp., ghost edges).

Lemma 4.2. *Let I be a two-sided ideal of $L_K(E)$ and $I_{real} = I \cap L_K(E)_R$. Then I_{real} is the two-sided ideal of $L_K(E)_R$ generated by elements of I_{real} having the form $v + \sum_{i=1}^{n} \lambda_i g^i$, where $v \in E^0$, g is a cycle based at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Proof. Let J be the ideal of $L_K(E)_R$ generated by elements in I_{real} of the indicated form. Our claim is $J = I_{real}$. Assume for a contradiction $I_{real} \setminus J \neq \emptyset$; choose $\mu \in I_{real} \setminus J$ of minimal length. By lemma 4.1, we can write $\mu = \tau_1 \cdots \tau_m$ with each τ_i is in I_{real} and is the sum of those paths whose sources are all the same and whose ranges are all the same. Since $\mu \notin J$, one of $\tau_i \notin J$. Replacing μ by τ_i, we may assume that $\mu = \lambda_1\mu_1 + \cdots \lambda_n\mu_n \in Ireal$ where all the μ_i have the same source and the same range, and $\lambda_i \in K$ for $1 \leq i \leq n$. First we claim that one of the μ_i must have length 0, i.e. $\mu_i = v$ for some vertex $v \in E^0$. Suppose not. Then for each i we can write $\mu_i = e_i\nu_i$ where $e_i \in E^1$ and $\nu_i \in L_K(E)_R$(otherwise μ_i can not be in $L_K(E)_R$ and therefore μ can not be in I_{real}). So $\mu = \sum_{i=1}^{n} \lambda_i e_i \nu_i$. Now

$$e_i^*\mu = \sum_{\{j|e_j=e_i\}} \lambda_j \nu_j \in I \cap L_K(E)_R = I_{real}$$

and has smaller length than μ. Since $\mu \in I_{real} \setminus J$ is of minimal length we get $e_i^*\mu \in J$ and hence $e_ie_i^*\mu \in J$ (J is a two-sided ideal of $L_K(E_R)$). Then

$$\sum_{\text{distinct} e_i} e_ie_i^*\mu = \sum_{\{j|e_j=e_i\}} = \mu \in J,$$

a contradiction. So we can assume without loss of generality that $\mu_1 = v$, with v a vertex.Since all the terms in μ have the same source and the same range, each μ_i is a closed path based at v. Multiplying by a scalar if necessary we write $\mu = v + \lambda_2\mu_2 + \cdots + \lambda_n\mu_n$.

Case (I): There exists no, or exactly one, closed simple path at v. If there are no closed simple paths at v then we get $\mu = v \in j$ by definition of J, a contradiction. If there is exactly one closed simple path g based at v then necessarily g must be a cycle. (Otherwise g visits some of its vertices other than v more than once. So it must contain a cycle but since g itself is not a cycle we have found another closed simple path, a contradiction.) Moreover, the only paths in E which have source and range equal to v are powers of g. Then $\mu = v + \sum_{i=2}^{n} \lambda_i g^i \in J$ again by definition of J, a contradiction.

Case(II): There exist at least two distinct closed simple paths g_1 and g_2 based at v. As $g_1 \neq g_2$ and by lemma 3.6, $g_2^*g_1 = 0 = g_1^*g_2$. Without loss of generality assume $|\mu_2| \geq \cdots |\mu_n| \geq 1$. Then for some $k \in \mathbb{N}$, $|g_1^k| > |\mu_2|$. Multiplying by $(g_1^*)^k$

on the left and g_1^k on the right, we get

$$\mu' = (g_1^*)^k \mu (g_1)^k = v + \sum_{i=2}^{n} \lambda_i (g_1^*)^k \mu_i (g_1)^k$$

. If $\lambda_i (g_1^*)^k \mu_i (g_1)^k = 0$ for every i, then we get $\mu' = (g_1^*)^k \mu (g_1)^k = v \in J$. Then $\mu = \mu v \in I$, and therefore $v \in I \cap L_K(E)_R = I_{real}$. But $\mu \in I_{real} \setminus J$ is of minimal length so v necessarily must be in J. Then $\mu = \mu v \in J$, a contradiction. Note that $0 \neq (g_1^*)^k \mu_i (g_1)^k$, then $(g_1^*)^k \mu_i \neq 0$. Since $|g_1^k| > |\mu_i|$, by lemma 2.5 we get $g_1^k = \mu_i \mu_i'$ for some path μ_i'. Since the μ_i are closed paths based at the vertex v, we get from the equation $(g_1)^k = \mu_i \mu_i'$ that $\mu_i = (g_1)^r$ for some integer $r \leq k$. So μ_i commutes with $(g_1)^k$ and thus each non-zero term $(g_1^*)^k \mu_i (g_1)^k = \mu_i$.

Since $g_2^* g_1 = 0$, $g_2^* \mu_i = g_2^* g_1^k = 0$ for every $i \in \{2, \cdots n\}$ such that $(g_1^{)} * \mu_i (g_1)^k \neq 0$ and so we get $g_2^* \mu' g_2 = g_2 (v + \sum_{i=2}^{n} \mu_i) g_2 = g_2^* v g_2 = v \in I \cap L_K(E)_R = I_{real}$. But $\mu \in I_{real} \setminus J$ is of minimal length so v necessarily must be in J. Then $\mu = \mu v \in J$, a contradiction. □

Lemma 4.3. *Let I be a two-sided ideal of $L_K(E)$ and $I_{ghost} = I \cap L_K(E)_G$. Then I_{ghost} is the two-sided ideal of $L_K(E)_g$ generated by elements of I_{real} having the form $v + \sum_{i=1}^{n} \lambda_i (g^*)^i$, where $v \in E^0$, g is a cycle based at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Proof. This can be shown by a similar argument with the previous lemma, but I think I have shown this by using involution.So I will write the proof later. □

Theorem 4.4. *Let E be a row-finite graph. Let I be any two-sided ideal of $L_K(E)$. Then I is generated by elements of the form $v + \sum_{i=1}^{n} \lambda_i g^i$, where $v \in E^0$, g is a cycle based at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Proof. Let J be the two-sided ideal of $L_K(E)$ generated by I_{real}. By lemma 4.2, it is enough to show that $I = J$. Suppose not. Choose $x = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^*$ in $I \setminus J$, where d is minimal and $\mu_1, \cdots i \mu_d, \nu_1, \cdots, \nu_d$ are real paths in $L_K(E)R$. By lemma 4.1,

$x = \alpha_1 + \cdots + \alpha_m$, where each $alpha_j \in I$ and is a sum of monomials all having the same source and the same range. Since $x \notin J, \alpha_j \notin J$ for some j. By the minimality of d, $x = \alpha_j$. ($x = \alpha_1 + \cdots + \alpha_m$ and x is the sum of d monomials but α_j is the sum of $\tilde{d}$ monomials where $\tilde{d} \leq d$). Thus we can assume $x = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^*$ where for all i, j $s(\mu_i \nu_i^*) = s(\mu_j \nu_j^*)$ and $r(\mu_i \nu_i^*) = r(\mu_j \nu_j^*)$ i.e. $s(\mu_i) = s(\mu_j)$ and $r(\nu_i^*) = r(\nu_j^*) = w \in E^0$. Among all such $x = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^* \in I \setminus J$ with minimal d,select one for which $(|\nu_1|, \cdots, |\nu_d|)$ is the smallest in the lexicographic order (dictionary ordering) of $\mathbb{Z}^d$. First note that we have $|\nu_i| > 0$ for some i otherwise x is in $I_{real} \subset J$, a contradiction. Let e be in E^1. Then note that

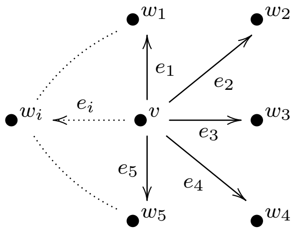
$$xe = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^* e = \sum_{i=1}^{d'} \lambda_i \mu_i' (\nu_i')^*$$

either has fewer terms ($d' < d$), or $d = d'$ and $(|\nu_1'|, \cdots, |\nu_d'|)$ is smaller than $(|\nu_1|, \cdots, |\nu_d|)$. Then by minimality, we get xe is in J for every $e \in E^1$, and then $xee^* \in J$ for every $e \in E^1$. Since $|\nu_i| > 0$ for some i, and $w = r(\nu_i^*) = s(\nu_i)$ w is not a sink and emits finitely many edges. Hence we have

$$x = xw = x \sum_{\{e \in E^1 : s(e) = w\}} ee^* = \sum_{\{e \in E^1 : s(e) = w\}} (xe)e^* \in J.$$

We assumed that $x \in I \setminus J$, hence we get a contradiction, so the result follows. $\square$

Remark 4.5. We note that the theorem 4.4 does not hold for arbitrary graphs. The theorem is false if E is not row-finite. An example is the "infinite clock" graph: $E^0 = \{v, w_1, w_2, \ldots,\}$ and $E^1 = \{e_1, e_2, \ldots,\}$ with $s(e_i) = v$ and $r(e_i) = w_i$ i.e

$E =$ 

Then the two-sided ideal generated by $v - e_1e_1^*$ is not generated by elements of the desired form. Let $I = \langle v - e_1e_1^*\rangle$. If theorem 4.4 is true, we have no cycle then $v + \sum \lambda_i g^i = v \in I$ i.e. I contains a vertex. If $0 \neq x \in I$, $x = \sum k_i\alpha_i\beta_i^*(v - e_1e_1^*)\gamma_i\delta_i^*$. Then $0 \neq k_i\alpha_i\beta_i^*(v - e_1e_1^*)\gamma_i\delta_i^*$ for some i implies $\alpha_i, \beta_i, \gamma_i, \delta_i$ are edges with source v. Also $\alpha_i\beta_i^* \neq 0$ implies $r(\alpha_i) = s(\beta_i^*) = r(\beta_i)$ so $\alpha_i = \beta_i = e_j$ and similarly, $\gamma_i = \delta_i = e_k$ for some $j, k \in \mathbb{N}$. If $\gamma_i = e_1$ then $0 \neq k_i\alpha_i\beta_i^*(v - e_1e_1^*)e_1\delta_i^* = k_i\alpha_i\beta_i^*0\delta_i^* = 0$ is a contradiction. If $\beta_i = e_1$ then $0 \neq k_i\alpha_ie_i^*(v - e_1e_1^*)e_1\delta_i^* = k_i\alpha_i0\gamma_i\delta_i^* = 0$ is a contradiction. Now, $\alpha_i = \beta_i = e_j \neq e_1$ and $\gamma_i = \delta_i = e_k \neq e_1$ gives $0 \neq k_i\alpha_i\beta_i^*(v - e_1e_1^*)\gamma_i\delta_i^* = k_ie_je_j^*(v - e_1e_1^*)e_ke_k^* = k_ie_je_j^*\underbrace{ve_k}_{=e_k}e_k^* - k_ie_je_j^*e_1\underbrace{e_1^*e_j}_{=0}e_j^* = k_ie_je_j^*e_ke_k^*$. Thus we get $k = j$ since otherwise $k_ie_je_j^*e_ke_k^* = 0$. Hence $e_j = e_k$ and $x = \sum_{i=1}^{n} k_ie_ie_i^*$.

No vertex is of this form, otherwise if w_j is of this form for some j then $w_j = \sum_{i=1}^{n} k_ie_ie_i^*$ and $w_j^2 = \sum_{i=1}^{n} k_i\underbrace{w_je_i}_{=0}e_i^* = 0$ is a contradiction. If v is of this form then $v = \sum_{i=1}^{n} \lambda_ie_ie_i^*$ then for all f in $E^1 \setminus \{e_1, \dots, e_n\}$, $f = vf = (\sum_{i=1}^{n} \lambda_ie_ie_i^*)f = 0$ is a contradiction.

Since E is acyclic, I must be generated by vertices but I contains no vertex. So theorem 4.4 does not hold for arbitrary graphs.

Theorem 4.6. *Let E be a row-finite graph. Let I be any two sided ideal of $L_K(E)$. Then $I$is generated by $(I \cap E^0) \cup Y$, where*

$$Y = \{v + \sum_{i=1}^{n} \lambda_i g^i : v \in E^0 \setminus I, g \textit{ is a unique non-trivial cycle based at } v\ \} \subset I$$

Proof. Let $x \in I$ then by Theorem 4.4 $x = v + \sum \lambda_i g^i$ where g is a cycle based at v. Suppose $h \neq g$ is another cycle based at v. Then by lemma 3.6 $h^*g = 0 = g^*h$. Now $h^*xh = h^*vh + \sum \lambda_i h^*g^ih = h^*h = v \in I$ □

5. SIMPLICITY OF $L_K(E)$

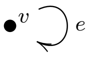
In this chapter our aim is to see what conditions on the graph E makes Leavitt path algebras simple. Here the important point is that the conditions which yield the simplicity of $L_K(E)$ are independent of the field K.

First of all we will investigate the simplicity of examples of the Leavitt path algebras given in the chapter 2.

- Matrix algebra $M_n(K)$ which arises as $L_K(A_n)$ for the 'oriented n-line' graph A_n defined by $(A_n)^0 = \{v_1, \ldots, v_n\}$, $(A_n)^1 = \{e_1, \ldots, e_n\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \ldots, n-1$. i.e.

  $A_n \quad =$ 

  is simple: For a commutative ring R, if I is an ideal in R then $M_n(I)$ is an ideal in $M_n(R)$. Moreover, any ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I in R. Therefore $M_n(R)$ is simple if and only if R is simple. Since K is a field $M_n(K)$ is simple. Therefore, $L_K(A_n)$ is simple.
- Laurent polynomial algebra $K[x, x^{-1}]$ which arise as $L_K(C_1)$ for the 'one vertex, one loop' graph C_1

  $C_1 \quad =$ 

  is not simple: We will show that $\langle 1+x \rangle$ is a proper ideal in $K[x, x^{-1}]$ since $1 \notin \langle 1+x \rangle$. Assume for a contradiction $1 \in \langle 1+x \rangle$ then

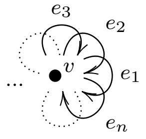
  $$1 = (1+x)(a_{-n}x^{-n} + a_{-n+1}x^{-n+1} \ldots + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) \Rightarrow$$

  $$1 = a_{-n}x^{-n} + (a_{-n} + a_{-n+1})x^{-n+1} + \ldots + (a_{-2} + a_{-1})x^{-1} + (a_{-1} + a_0) + (a_0 +$$

$a_1)x + \ldots + (a_{n-1} + a_n)x^n + a_nx^{n+1}$

gives $a_n = 0$, $a_{n-1} + a_n = 0$ so $a_{n-1} = 0$. Continuing this way we get $a_i = 0$ for all $i \in \{-n, -n+1, \ldots, 0, 1, \ldots, n-1, n\}$. But then $1 = 0$ is a contradiction.

- Leavitt algebra $A = L_K(1, n)$ which arise as $L_K(R_n)$ for the 'rose with n-petals'graph R_n for $n \geq 2$

$R_n =$ 

is simple by Leavitt's theorem.

Theorem 5.1. *Simplicity of the Leavitt algebras For any field K, $L_K(1, n)$ is simple for $n \geq 2$.*

Proof. First we will prove if an ideal I of $L_K(1, n)$ contains a non-zero polynomial in $\{y_i\}$ (or $\{x_i\}$) alone, then $I = L_K(1, n)$.

If I contains a non-zero polynomial in $\{y_i\}$ alone, it must contain such polynomials of minimal degree say m. Let $\alpha \in I$ be such polynomial with $deg(\alpha) = m$. If $m = 0$,then we are done.
So assume $m \geq 1$ then

$$\alpha = \sum_{1}^{n} y_i\alpha_i^{(1)} + c_0$$

where $deg(\alpha_i^{(1)}) \leq m - 1$ and at least one $\alpha_i^{(1)} \neq 0$. Here $c_0 \neq 0$ otherwise $x_i\alpha = \alpha_i^{(1)} \in I$ of degree $< m$. Now $x_1\alpha = \alpha_1^{(1)} + c_0x_1$. If $\alpha_1^{(1)} = 0$, then $x_1\alpha y_1 = c_0 \in I$ and we are done. Thus assume $\alpha_1^{(1)} \neq 0$. If $deg(\alpha_1^{(1)}) = 0$ then $\alpha_1^{(1)} = c_1 \in K$ and $x_1\alpha = c_1 + c_0x_1$. We multiply by x_2 on the left, and y_2 on the right, and get

$$x_2(x_1\alpha)y_2 = x_2(c_1 + c_0x_1)y_2 = c_1 + 0 = c_1$$

But $c_1 \neq 0$, and $x_2(x_1\alpha)y_2 \in I$,so are are done in this case. Note that here we use the fact that $n \geq 2$, so that we have an element y_2 which is orthogonal on the right to x_1. If $deg(\alpha_1^{(1)}) \geq 0$ then $\alpha_1^{(1)} = y_1\alpha_1^{(2)} + \cdots + y_n\alpha_n^{(2)} + c_1$

$$\Rightarrow \qquad x_1\alpha = \alpha_1^{(1)} + c_0x_1 = y_1\alpha_1^{(2)} + \cdots + y_n\alpha_n^{(2)} + c_1 + c_0x_1$$

$$\Rightarrow \qquad x_1^2 = \alpha_1'(2) + c_1x_1 + c_0x_1^2$$

We continue this way (for at most m steps) to get $deg(\alpha_1^{(m)}) = 0$

Now

$$x_1^m = \alpha_1^{(m)} + c_{m-1}x_1 + c_{m-2}x_1^2 + \cdots + c_1x_1^m - 1 + c_0x_1^m$$

If $\alpha_1^{(m)} = 0$ then $x_1^m\alpha y_1^m = c_{m-1}y_1^{m-1} + c_{m-2}y_1^{m-2} + \cdots + c_1y_1 + c_0 \in I$ of degree $\leq m$ contradicts with the minimality of m. So $0 \neq \alpha_1^{(m)} \in K$ then we get $x_2x_1^m\alpha y_2 = \alpha_1^{(m)} \in K \cap I$. So we are done. The proof for the case in $\{x_i\}$ only is similar.

Now we will show $L_K(1,n)$ is simple. Suppose I is non-zero ideal with $I \neq L_K(1,n)$. Since $\alpha \in I$ can not be a polynomial in $\{x_i\}$ (or $\{y_i\}$) alone (otherwise $I = L_K(1,n)$), we write $\alpha = \sum y_i\alpha_i + \beta$ where β is a (possibly zero) a polynomial in $\{x_i\}$ alone and at least one $\alpha_i \neq 0$. Let $d_y(\alpha)$ be the degree of α in $\{y_i\}$, and assume we have chosen an $\alpha \in I$ with $d_y(\alpha)$ minimal. Then $x_1\alpha = \alpha_1 + x_1\beta \in I$ with $d_y(x_1\alpha) < d_y(\alpha)$. But this contradicts with the minimality of $d_y(\alpha)$ unless $x_1\alpha = 0$, so that $\alpha_1 = -x_1\beta$. If $\beta \neq 0$, then all terms of α_1 begin with x_1. But then α would have terms beginning with y_1x_1 and in definition of $L_K(1,n)$ it was assumed all such terms have been eliminated, using $y_1x_1 = \sum_{k=0}^{n} y_ix_i$. So we conclude that $\beta = 0$ so that $x_i\alpha = \alpha_i \in I$ with $d_y(\alpha_i) < d_y(\alpha)$. Therefore, $I = L_K(1,n)$. □

Definition 5.2. For a graph E we define a preorder $\geq$ on the vertex set E^0 given by: $v \geq w$ if and only if $v = w$ or there is a path μ such that $s(\mu) = v$ and $r(\mu) = w$.

Definition 5.3. We say that a subset $H \subset E^0$ is *hereditary* if $v \in H$ and $v \geq w$ imply $w \in H$.

Definition 5.4. We say that a subset $H \subset E^0$ is *saturated* if whenever $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subset H$, then $v \in H$.

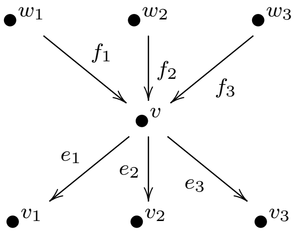
Definition 5.5. The *the hereditary saturated closure* of a set $x \subset E^0$ is defined as the smallest hereditary and saturated subset of E^0 containing X. For the hereditary saturated closure of X we use the notation $\bar{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$$\Lambda_0(X) := \{v \in E^0 \mid x \geq v \text{ for some } x \in X\}, \text{ and for } n \geq 1,$$

$$\Lambda_n(X) := \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$$

Example 5.6. Let E be the following graph:
$E =$

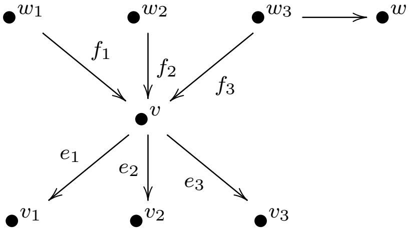
$E =$



- $H_1 = \{v, v_1, v_2\} \subset E^0$ is not hereditary -since $v \in H_1$ and $v \geq v_3$ but $v_3 \notin H_1$- and not saturated -since $s^{-1}(w_1) = f_1 \neq \emptyset$ and $r(s^{-1}(w_1)) = v \subset H_1$ but $w_1 \notin H_1$.
- $H_2 = \{w_1, w_2, w_3, v\} \subset E^0$ is saturated but not hereditary since $v \in H_2$ and $v \geq v_3$ but $v_3 \notin H_2$.
- $H_3 = \{w_1, w_2, v, v_1, v_2, v_3\} \subset E^0$ is hereditary but not saturated since $s^{-1}(w_3)$ $= f_3 \neq \emptyset$ and $r(s^{-1}(w_3)) = v \subset H_1$ but $w_3 \notin H_3$.

  Here, for the following graph E',

  $E' =$

  the same set would be saturated since $r(s^{-1}(w_3)) = v \not\subseteq H_3$.

Lemma 5.7. *If J is an ideal of $L_K(E)$, then $J \cap E^0$ is a hereditary and saturated* *subset of* E^0*.*

Proof. We first show $J \cap E^0$ is hereditary. Consider, $v, w \in E^0$ such that $v \in J$ and $v \geq w$. By definition of preorder we can find a path $\mu = \mu_1 \dots \mu_n$ such that $s(\mu_1) = v$ and $r(\mu_n) = w$. Since J is an ideal $\mu_1^* v \mu_1 = \mu_1^* \mu_1 = r(\mu_1) = s(\mu_2) \in J$, similarly $\mu_2^* s(\mu_2) \mu_2 = \mu_2^* \mu_2 = r(\mu_2) = s(\mu_3) \in J$. Repeating this argument n times, we get $r(\mu_n) = w \in J$.

Now we show that $J \cap E^0$ is saturated. Consider a vertex v with $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq J$. The first condition implies that v is not a sink, so by applying $(CK2)$ we obtain $v = \sum_{\{e \in E^1 : s(e) = v\}} ee^*$. If we take e such that $s(e) = v$, then by hypothesis we have that $r(e) \in J$. Now $e = er(e) \in J$ and thus we conclude that$\sum_{\{e \in E^1 : s(e) = v\}} ee^* = v \in J$ since J is an ideal. □

Theorem 5.8 (Simplicity Theorem)**.** *Let* E *be a row-finite graph. Then the Leavitt path algebra* $L_K(E)$ *is simple if and only if* E *satisfies the following conditions:*

i The only hereditary and saturated subsets of E^0 *are* $\emptyset$ *and* E^0*;*

ii Every cycle in E *has an exit.*

Proof. First we will show that if E satisfies (i) then $L_K(E)$ is simple if and only if every nonzero two sided ideal contains a vertex. Assume E satisfies (i) and $L_K(E)$ is simple. Then any ideal I is $L_K(E)$ itself so it contains a vertex. Conversely assume E satisfies (i) and any ideal I of $L_K(E)$ contains a vertex, then $I \cap E^0 \neq \emptyset$ will be E^0 by (i). Therefore I contains a set of local units by Lemma 1.6 and hence $I = L_K(E)$. □

Now assume E satisfies (i) and (ii), by above proof it is enough to show that any ideal contains a vertex. Let I be an ideal of $L_K(E)$, now by theorem 10 I is generated by elements of the form $v + \sum_{k=1}^{m} \lambda_k g^k$, where $v \in E^0$, g is a cycle at v and $\lambda_i \in K$ for all $1 \leq i \leq m$.

Write $g = e_1 \cdots e_\sigma$, by (i) there exists an exit e_0 for g such that $s(e_0) = s(e_j)$ for some j and $e_0 \neq e_j$. Now construct the path $z = e_1 \cdots e_{j-1}e_0$. This path has $z^*g = 0$ since $z^*g = e_0^*e_{j-1}^* \cdots e_1^*e_1 \cdots e_\sigma = e_0^*e_j \cdots e_\sigma = 0$.

Since $v + \sum_{k=1}^m \lambda_k g^k \in I$, then
$z^*\left(v + \sum_{k=1}^m \lambda_k g^k\right) z = z^*vz = z^*z = e_0^*e_{j-1}^* \cdots e_1^*e_1 \cdots e_{j-1}e_0 = r(e_0) \in E$.

Thus we have shown that is E satisfies the two indicated properties, then $L_K(E)$ is simple.

For the converse, we must show that $L_K(E)$ is not simple when either of these conditions hold: (1) E contains a cycle p having no exit, and (2) there exists a nontrivial hereditary and saturated subset of E^0.

For the first situation , suppose that there is a cycle p having no exit. We will prove that $L_K(E)$ cannot be simple. Let v be the base of that cycle. We will show that for $\alpha = v + p$, $< \alpha >$ is a non-trivial ideal of $L_K(E)$ since $v \notin < \alpha >$. Write $p = e_1 \cdots e_\sigma$. Since this cycle has no exit, for every e_i there is no edge with source $s(e_i)$ other than e_i itself. This easily implies that $CSP(v) = \{p\}$. Also by CK2 relation we get $s(e_i) = e_ie_i^*$ since e_i is the only edge having the vertex $s(e_i)$ as its source, so we get $pp^* = v$ (here recall that $p^*p = v$ always holds).

Now suppose that $v \in < \alpha >$. So there exist nonzero monic monomials $a_n, b_n \in L_K(E)$ and $k_n \in K$ with $v = \sum_{n=1}^m k_n a_n \alpha b_n$. Since $v\alpha v = \alpha$, by multiplying v if necessary we may assume that $va_nv = a_n$ and $vb_nv = b_n$ for all $1 \leq n \leq m$.

We claim for each a_n (resp. b_n) there exists an integer $u_{a_n} \geq 0$ (resp. $u_{b_n} \geq 0$) such that $a_n = p^{u_{a_n}}$ or $a_n = (p^*)^{u_{a_n}}$(resp. $b_n = p^{u_{b_n}}$ or $b_n = (p^*)^{u_{b_n}}$).

Now since it is a monomial a_1 is of the form $f_1 \cdots f_c g_1^* \cdots g_d^*$ where $c, d \geq 0$. But here we may assume that $c, d \geq 1$ since the cases $c = d$ or $d = 0$ will be contained in what follows. Since a_1 starts and ends in v the sets $\{t : r(g_t^*) = v\}$ and $\{t : s(f_t) = v\}$

are not empty so we can consider the elements: $k = max\{t : s(f_t) = v\}$ and $l = min\{t : r(g_t^*) = v\}$. We will focus on $a_1' = f_k \cdots f_c g_1^* \cdots g_l^*$.

First of all, since $v = r(g_l^*) = s(g_l)$ and e_1 is the only edge coming from v (remember that p has no exit), then $g_l = e_1$. Now, $s(g_{l-1}) = r(g_{l-1}^*) = s(g_l^*) = r(g_l) = r(e_1) = s(e_2)$, and again the only edge coming from $s(e_2)$ is e_2 and therefore $g_{l-1} = e_2$. Continuing this way we must stop before we run out of edges of p because by our choice of l we have that $v \notin \{r(g_t^*) : t \leq l\}$. So in the end we get there exists $\gamma \leq \sigma$ such that $g_1^* \cdots g_l^* = e_\alpha^* \cdots e_1^*$.

With the same (reversed) ideas in the paragraph above we can find $\delta \leq \sigma$ such that $f_1 \cdots f_c = e_1 \cdots e_\delta$. Thus $a_1' = e_1 \cdots e_\delta e_\gamma^* \cdots e_1^*$.

Now we claim that $\delta = \gamma$. Assume not, i.e. $\delta \neq \gamma$. Since p is a cycle we know that $r(e_\delta) \neq r(e_\sigma) = s(e_\gamma^*)$, so $e_\delta e_0^*$, which is a contradiction since $a_1 \neq 0$. So $a_1' = e_1 \cdots e_\gamma e_\gamma^* \cdots e_1^*$ i.e. $a_1' = p_0 p_0^*$ for a certain subpath p_0 of p. By using again the argument of the CK2 relation (remember that $s(e_i) = e_i e_i^*$), we obtain $p_0 p_0^* = v$. Thus, $a_1' = v$.

Hence, we get $a_1 = f_1 \cdots f_{k-1} a_1' g_{l+1}^* \cdots g_d^* = a_1 = f_1 \cdots f_{k-1} g_{l+1}^* \cdots g_d^* = xy^*$, with $x, y \in CP(v)$. (Obviously, the case $c \geq 1, d = 0$ yields $a_1 = x$, the case $c = 0, d \geq 1$ yields $a_1 = y^*$ and the case $c = d = 0$ yields $a_1 = v$.) By lemma 3.7 we have $x = c_1 \cdots c_\nu$ for some $c_\mu \in CSP(v) = \{p\}$, and the same happens with y. Therefore we get $a_1 = p^u (p^*)^w$ for some $u, w \geq 0$, and using the fact that $pp^* = v$ we finally obtain that a_1 is of the form p^u or $(p^*)^u$ for some $u \geq 0$ as claimed. This argument holds for the other coefficients a_n and b_n.

Now since both p and p^* commute with p, p^* and α, by using the conclusion of the previous paragraph we write the sum $v = \sum_{n=1}^{m} k_n a_n \alpha b_n$ as $v = \alpha P(p, p^*)$ where $P(p, p^*) = k_{-m}(p^*)^m + \cdots + k_0 v + \cdots k_n p^n$, where $m, n \geq 0$. First we claim that $k_{-i} = 0$ for every $i > 0$, as follows. If not, let m_0 be the maximum i having $k_{-i} \neq 0$. Then $v = \alpha P(p, p^*) = (v + p) P(p, p^*) = k_{-m_0}(p^*)^{m_0} +$ terms of greater degree$=v$, and since $m_0>0$ we get that $k_{m_0}=0$, which is absurd. In a similar way we obtain $k_i=0$ for every $i>0$, and therefore $P(p,p^*)=k_0v$. But this would yield $v=\alpha P(p,p^*)=\alpha k_0v=k_0\alpha=k_0(v+p)$, which is impossible.

Thus we have shown that if E contains a cycle which has no exit,then $L_K(E)$ is not simple. So the first part of the converse is done.

Now we will consider the second part of the converse, the situation where E^0 contains a nontrivial hereditary and saturated subset H, and we show that $L_K(E)$ is not simple.

We construct a new graph

$$F=(F^0,F^1,r_F,s_F)=(E^0\setminus H,r^{-1}(E^0\setminus H),r|_{E^0\setminus H},s|_{E^0\setminus H}).$$

In other words, F is the graph consisting of all vertices not in H, together with all edges whose range is not in H. To check F is well-defined, we have to show that $s_F(F^1)\subseteq F^0$ and $r_F(F^1)\subseteq F^0$. It is clear that $r_F(F^1)\subseteq F^0$. On the other hand, if $e\in F^1$ then $s(e)\in F^0$, since otherwise we have $s(e)\in H$; but since $s(e)\geq r(e)$ and H is hereditary, we get $r(e)\in H$, which contradicts $e\in F^1$. So F is a well defined graph.

We now produce a K-algebra homomorphism $\Psi:L_K(E)\to L_K(F)$. First we define Φ on the generators of $L_K(E)$ by setting $\Phi(v)=\chi_{F^0}(v)v$, $\Phi(e)=\chi_{F^1}(e)e$ and $\Phi(e^*)=\chi_{(F^1)^*}(e^*)e^*$ where χ_X denotes the characteristic function of a set X. Then we get Ψ by extending Φ linearly to $L_K(E)$. In order to show that Φ factors through the appropriate relations in $L_K(E)$ we need to check:

(i) $\langle\{uv-\delta_{u,v}u:u,v\in E^0\}\cup\{e-er(e),e-s(e):e\in E^1\}\rangle\subseteq Ker(\Phi)$

(ii) $\langle\{e^*f\delta_{e,f}r(e):e,f\in E^1\}\cup\{v-\sum_{\{e\in E^1:s(e)=v\}}ee^*:v\in s(E^1)\}\rangle\subseteq Ker(\phi)$

(i) This is a straightforward computation done by cases, and we only consider the nontrivial case arising when $e \in F^1$. But then $r(e) \notin H$, and therefore $\Phi(e - er(e)) = e - er(e) = 0$ in $L_K(F)$. Now, since $s(e) \geq r(e) \notin H$ and H is hereditary, we have $s(e) \notin H$, so that $\Phi(e - s(e)e) = e - s(e)e = 0$ in $L_K(F)$.

(ii) $\Phi(e^*f - \delta_{e,f}r(e)) = 0$ in $L_K(F)$ is straightforward. So now consider $v \in s(E^1)$; i.e., consider a vertex v which is not a sink in E. We have 3 cases:
Case 1. Suppose $v \in H$. Then for every $e \in E^1$ with $s(e) = v$ we have that $e \notin F^1$ (otherwise $e \in F^1$ implies $r(e) \notin H$ and by hereditariness $s(e) = v \notin H$). So, $\Phi(v - \sum_{\{e \in E^1 : s(e)=v\}} ee^*) = 0 - v - \sum_{\{e \in E^1 : s(e)=v\}} 0.0 = 0$. Case 2. Suppose $v \notin H$ and $v \notin s(F^1)$. Since $v \in s(E^1)$ we have that $s^{-1}(v) \neq \emptyset$. But since H is saturated there must exit $e \in E^1$ such that $s(e) = v$, but $r(e) \notin H$. That means $e \in F^1$ with $s(e) = v$, which contradicts the hypothesis that $v \in s(F^1)$. Thus the saturated condition on H implies that this case cannot occur. Case 3. Suppose $v \notin H$ but $v \in s(F^1)$. Then we have a CK2 relation in $L_K(F)$ at v:

$$v = \sum_{\{e \in F^1 : s(e)=v\}} ee^*$$

Consider $e \in E^1$ such that $s(e) = v$. If $e \in F^1$ then $\Phi(ee^*) = ee^*$. If $e \notin F^1$ then $\Phi(ee^*) = 0$. Thus we get

$$\Phi(v - \sum_{\{e \in E^1 : s(e)=v\}} ee^*) = v - \sum_{\{e \in F^1 : s(e)=v\}} ee^* = 0$$

by the CK2 relation in $L_K(F)$ shown above.

Thus we have shown that there exists a K-algebra homomorphism

$$\Psi : L_K(E) \to L_K(F).$$

Now consider $Ker(\Psi) \trianglelefteq L_K(E)$. Since $H \neq \emptyset$ there exists $v \in H$, so $0 \neq v \in Ker(\Psi)$. Since $H \neq E^0$ there exists $w \in E^0 \setminus H$ and in this case $\Psi(w) = w \neq 0$ so $\Psi \neq 0$. In other words, $0 \neq Ker(\Psi) \neq L_K(E)$, so that $L_K(E)$ is not simple.

Thus we conclude that the negation of either condition (i) or condition (ii) yields that $L_K(E)$ is not simple, which completes the proof of the simplicity theorem.

Example 5.9. We re-establish the simplicity (or non-simplicity) of the algebras given at the beginning of this chapter.

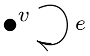
- Matrix algebra $M_n(K)$ which arises as $L_K(A_n)$ for the 'oriented n-line' graph A_n defined by $(A_n)^0 = \{v_1, \dots, v_n\}$, $(A_n)^1 = \{e_1, \dots, e_n\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \dots, n-1$. i.e.

  A_n =

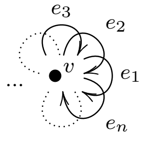
  is simple: Since there are clearly no cycles in A_n, we need only verify condition (i) in the simplicity theorem (theorem 5.8) i.e. we need to show that A_n^0 has only trivial hereditary and saturated subsets. If $H \neq \emptyset$ is a set of vertices which is hereditary and saturated, let $v_i \in H$. By hereditariness we have that $v_{i+1}, \dots, v_n \in H$. Now if we use the condition of being saturated at v_{i_1} we get that $v_{i_1} \in H$, and inductively $v_{i-1}, \dots, v_1 \in H$ and therefore $H = A_n^0$. Now simplicity theorem implies to conclude that $M_n(K) \cong L_K(A_n)$ is simple.
- Laurent polynomial algebra $K[x, x^{-1}]$ which arise as $L_K(C_1)$ for the 'one vertex, one loop' graph C_1

  C_1 =

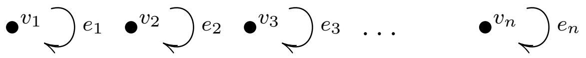
  is not simple: The cycle e does not have an exit, so by the simplicity theorem $K[x, x^{-1}] \cong L_K(C_1)$ is not simple.
- Leavitt algebra $A = L_K(1, n)$ which arise as $L_K(R_n)$ for the 'rose with n-petals'graph R_n for $n \geq 2$

  R_n =

is simple: The conditions in the simplicity theorem are clearly satisfied here, so $L_K(1,n) \cong L_K(R_n)$ is simple.

Example 5.10. Let C_n denote the graph having n vertices and n edges, where the edges form a single cycle (the graph described in example 5.9 (2) is the graph C_1.):

$C_n \ =$ 

Then $L_K(C_n)$ is not simple for all $n \in \mathbb{N}$, since the single cycles e_i have no exit for all $i \in \mathbb{N}$.

6. INCIDENCE AND PATH ALGEBRAS

In this chapter we investigate the relations between Incidence algebras and path algebras. In [6] the algebra homomorphism is defined from path algebras on a row-finite graph E into incidence algebra. We try to generalize this case for arbitrary graphs, i.e. we try to see for which conditions on E, we still have such an algebra homomorphism.

Definition 6.1. Let χ be a set with a binary relation $\leq$. $(\chi, \leq)$ is a pre-ordered set if $\leq$ is reflexive, and transitive.

Definition 6.2. A pre-ordered set $(\chi, \leq)$ is called a partially ordered set (poset) if $\leq$ is also anti-symmetric.

Example 6.3.

- The integers $\mathbb{Z}$ under usual ordering.
- The rational numbers in $[0, 1]$ under usual ordering.
- The set of positive integers in $\mathbb{N}$ under divisibility($a \leq b$ if a divides b).
- The power set of a given set under inclusion.

If $(\chi, \leq)$ is a pre-ordered set, we can define an equivalence relation $\sim$ on X (canonical equivalence relation on χ) by

$$x \sim y \Leftrightarrow x \leq y \text{ and } y \leq x,$$

for $x, y \in \chi$. Let $[x]$ be the equivalence class of $x \in \chi$ under this equivalence relation and consider the set $\widetilde{\chi} = \{[x] \mid x \in \chi\}$. Define $\preccurlyeq$ on $\widetilde{\chi}$ by

$$[x] \preccurlyeq [y] \Leftrightarrow x \leq y.$$

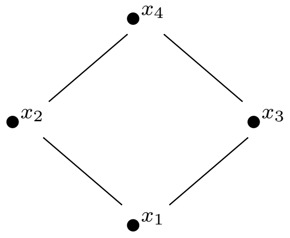
Thus, $(\widetilde{\chi}, \preccurlyeq)$ is a partially ordered set. We say that $(\widetilde{\chi}, \preccurlyeq)$ is the *poset associated* to $(\chi, \leq)$.

In a poset $(\chi, \leq)$ we put $x < y$ in case $x \leq y$ but $x \neq y$.

Definition 6.4 (Hasse Diagram of a poset)**.** Some partially ordered sets can be represented visually by Hasse diagrams.In order to construct the *Hasse Diagram*,we represent each element of S as a vertex in the plane and draw a line segment or curve that goes upward from x to y whenever y covers x (that is, whenever $x < y$ and there is no z such that $x < z < y$)

Example 6.5. Let χ be the partially ordered set $\chi = \{x_1, x_2, x_3, x_4\}$ and relations generated by $\{x_1 \leq x_2, x_1 \leq x_3, x_2 \leq x_4, \}$. We represent the partially ordered set χ by the following Hasse diagram:

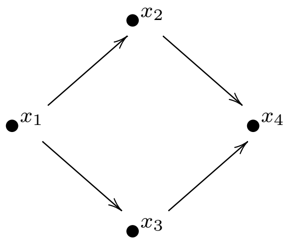
$\chi =$



For a given poset χ we construct a directed graph E_χ with χ as a set of vertices and with an arrow from v to w and there is no $u \in \chi$ such that $v < u < w$.

Example 6.6. Let χ be the poset in the previous example. Then the associated directed graph E_χ is :

$E_\chi =$



Definition 6.7. Let $(\chi, \leq)$ be a poset. $x \in \chi$ is called a

- *maximal element* if $x \leq y$, then $x = y$;
- *minimal element* if $y \leq x$, then $x = y$.

Definition 6.8. Let $(\chi, \leq)$ be a poset. An element $x \in \chi$ is

- the maximum element of χ if $\forall y \in \chi, y \leq x$, which is denoted by 1;
- the minimum element of χ if $\forall y \in \chi, x \leq y$, which is denoted by 0.

Definition 6.9. Given $x, z \in \chi$, a pre-ordered set, the interval (or segment) from x to z, denoted by $[x, z]$ is $\{y \in \chi : x \leq y \leq z\}$.

Definition 6.10. If every interval of χ is finite, then χ is locally finite.

Definition 6.11. The *incidence algebra* $I(\chi, R)$ of the locally finite partially ordered set χ over the commutative ring with identity R, is

$$I(\chi, R) = \{f : \chi \times \chi \to R : f(x, y) = 0 \text{ if } x \nleq y\}$$

with the operations given by

$$\begin{aligned}(f + g)(x, y) &= f(x, y) + g(x, y),\\ (f.g)(x, y) &= \sum_{x \leq z \leq y} f(x, z)g(z, y) \text{ and}\\ (r.f)(x, y) &= r.f(x, y),\end{aligned}$$

for $f, g \in I(\chi, R)$ with $r \in R$ and $x, y, z \in \chi$.

Definition 6.12. For a graph E we define a preorder $\geq$ on the vertex set E^0 given by:

$$v \geq w \quad \Leftrightarrow \quad v = w \text{ or there is a path } \mu \text{ such that } s(\mu) = v \text{ and } r(\mu) = w.$$

Definition 6.13. On the set E^*, we define the following order: For any two paths

$p, q \in E^*$,

$$p \leq q \quad \Leftrightarrow \quad \text{there exists } \gamma, \eta \in E^* \text{ such that } \gamma p \eta = q$$

Lemma 6.14. *$(E^*, \leq)$ is a pre-ordered set.*

Proof. For any $p \in E^*$, we have $p = s(p)pr(p)$ so $p \leq p$. Now let $p, q, r \in E^*$ with $p \leq q$ and $q \leq r$. Then $\gamma p \eta = q$ and $\alpha q \beta = r$ for some $\gamma, \eta, \alpha, \beta \in E^*$. Then $\alpha \gamma p \eta \beta = r$ thus $p \leq r$ as required. □

Lemma 6.15. *Two non trivial paths $p = e_1 \cdots e_n$ and $q = f_1 \cdots f_m$ where $e_i, f_j \in E^1$ are equal if and only if $n = m$ and $e_i = f_i$ for every $i \in 1, \ldots, n$.*

Lemma 6.16. *For any graph E, $(E^*, \leq)$ is a partially ordered set.*

Proof. By lemma 6.14 $(E^*, \leq)$ is a pre-ordered set. It is enough to show $\leq$ is anti-symmetric. Let $p, q \in E^*$ with $p \leq q$ and $q \leq p$, then there exist $\alpha, \beta, \gamma, \eta \in E^*$ such that $\alpha p \beta = q$ and $\gamma q \eta = p$. Thus we get $\gamma \alpha p \beta \eta = p$. By lemma 6.15 $\alpha = s(p)$ and $\beta = r(p)$ thus $p = q$. □

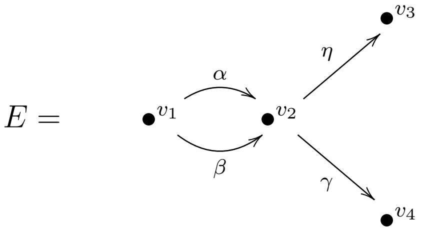
Lemma 6.17. *For any graph E, $(E^*, \leq)$ is locally finite.*

Proof. We want to show that for any $\alpha, \beta \in E^*$ with $\alpha \leq \beta$ (in other words α is a subpath of β) $|[\alpha, \beta]|$ is finite. But since α, β paths in E, they are of finite length so there are finitely many subpaths of β that contain α as subpath. □

Definition 6.18. A vertex $v \in E^0$ is said to be a *source* if $r^{-1}(v) = \emptyset$. We denote by E^0_{so} the set of all sources of E. A vertex $w \in E^0$ is said to be a *sink* if $s^{-1}(w) = \emptyset$. We denote by E^0_{si} the set of all sinks of E. Moreover, we define the sets $E^*_{so} = \{p \in E^* \mid s(p) \in E^0_{so}\}$ and $E^*_{si} = \{q \in E^* \mid r(q) \in E^0_{si}\}$.

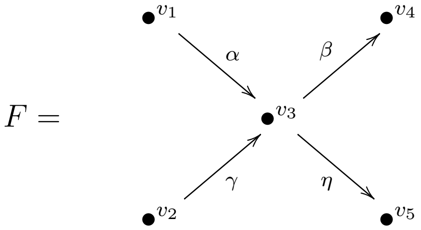
Example 6.19.

- Let E be the following graph



  Then $E^0_{so} = \{v_1\}$, $E^0_{si} = \{v_3, v_4\}$, $E^*_{so} = \{v_1, \alpha, \alpha\eta, \alpha\gamma, \beta, \beta\gamma, \beta\eta\}$, $E^*_{si} = \{v_3, v_4, \eta, \gamma, \alpha\eta, \beta\eta, \alpha\gamma, \beta\gamma\}$

- Let F be the following graph



  Then $E^0_{so} = \{v_1, v_2\}$, $E^0_{si} = \{v_4, v_5\}$, $E^*_{so} = \{v_1, \alpha, \alpha\beta, \alpha\eta, v_2, \gamma, \gamma\beta, \gamma\eta\}$, $E^*_{si} = \{v_4, \beta, \alpha\beta, \gamma\beta, v_5, \eta, \gamma\eta, \alpha\eta\}$

Definition 6.20. We define the following partially ordered set E^*_{so} with the induced order from E^*. Observe that, equivalently, for every $p, q \in E^*_{so}$, we have that

$$p \leq q \quad \Leftrightarrow \quad \text{there exist } \eta \in E^* \text{ such that } p\eta = q$$

Lemma 6.21. *Let E be a finite graph without any cycle. If p is a path with maximal length then $s(p)$ is a source.*

Proof. Assume $s(p)$ is not a source, then there exists a path $\alpha \in E^*$ such that $r(\alpha) = s(p)$. Now αp is path with length greater than p, which contradicts the maximality of the length of p. □

Lemma 6.22. *Let E be a finite graph without no cycle. Then any path in E is connected to a source, i.e for any path q in E there exists a path p such that $r(p) = s(q)$ and $s(p)$ is source.*

Proof. Follows from lemma 6.21 since any path in E is a subpath of a maximal path. □

Theorem 6.23. *Let E be any graph. Define*

$$\Phi : KE \longrightarrow I(E^*, K)$$
$$\alpha \mapsto f_\alpha(u,v) = \begin{cases} 1, & \textit{if } v = u\alpha \\ 0, & \textit{else} \end{cases}$$

Φ is an injective K-algebra homomorphism.

Proof. We will show that $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$. For any $(u, v) \in E^* \times E^*$ we have

$$\Phi(\alpha\beta) = f_{\alpha\beta}(u,v) = \begin{cases} 1, & \text{if } v = u(\alpha\beta) \\ 0, & \text{else} \end{cases} .$$

Now

$$\begin{aligned} \Phi(\alpha)\Phi(\beta) = f_\alpha f_\beta(u,v) &= \sum_{u \le z \le v} f_\alpha(u,z) f_\beta(z,v) \\ &= f_\alpha(u, u\alpha) f_\beta(u\alpha, u\alpha\beta) \\ &= \begin{cases} 1, & \text{if } v = u\alpha\beta \\ 0, & \text{else} \end{cases} \end{aligned}$$

We extend this map linearly to get an algebra homomorphism.

Now, we will show Φ is injective.

$$\begin{aligned} Ker\Phi &= \{\sum_\alpha c_\alpha \alpha \mid \Phi(\sum_\alpha c_\alpha \alpha) = 0\} \\ &= \{\sum_\alpha c_\alpha \alpha \mid \sum_\alpha c_\alpha \Phi(\alpha) = 0\} \\ &= \{\sum_\alpha c_\alpha \alpha \mid \sum_\alpha c_\alpha (f_\alpha(u,v)) = 0 \text{ for all } u, v \in E^*\} \end{aligned}$$

Take $(u, u\alpha_0) \in E^* \times E^*$, then

$$\begin{aligned} &\sum_\alpha c_\alpha f_\alpha(u, u\alpha_0)) = 0 \\ &\Rightarrow \sum_{\alpha \neq \alpha_0} c_\alpha f_\alpha(u, u\alpha_0) + c_{\alpha_0} f_{\alpha_0}(u, u\alpha_0) = 0 \\ &\Rightarrow c_{\alpha_0} f_{\alpha_0}(u, u\alpha_0) = 0 \\ &\Rightarrow c_{\alpha_0} = 0 \end{aligned}$$

This can be done for any α hence we get $c_\alpha = 0$ for any α. □

Theorem 6.24. *Let E be a graph such that $E^*_{so} \neq \emptyset$. Define*

$$\begin{aligned} \Phi : KE &\longrightarrow I(E^*_{so}, K) \\ \alpha \mapsto f_\alpha(u,v) &= \begin{cases} 1, & \textit{if } v = u\alpha \\ 0, & \textit{else} \end{cases} \end{aligned}$$

Φ is one-to-one if $T(E^0_{so}) = \bigcup_{u \in E^0_{so}} T(w) = E^0$ *where* $T(w) = \{t \in E^0 \mid w \geq t\}$.

Proof. $Ker\phi = \{\sum_\alpha c_\alpha \alpha \mid \sum_\alpha c_\alpha (f_\alpha(u,v)) = 0$ for all $u, v \in E^*_{so}\}$.

Since $T(E^0_{so}) = \bigcup_{w \in E^0_{so}} T(w) = E^0$, for every path α in there exists u in E^*_{so} such that $r(u) = s(\alpha)$. Now consider $(u, u\alpha_0) \in E^0_{so} \times E^0_{so}$, then as in the previous proof $\sum_\alpha c_\alpha f_\alpha(u, u\alpha_0) = 0$ gives $c_\alpha = 0$ for any α. □

Definition 6.25. The support of a function is the set of points where the function is not zero. The support of f is denoted by *supp*(f).

Definition 6.26. The finite support of $I(X, R)$ denoted by $[I(E^*_{so}, K)]^{fs}$ consists of the functions whose support is finite i.e.

$$[I(E^*_{so}, K)]^{fs} = \{f \in I(X, R) : supp(f) \text{ is finite } \}.$$

Theorem 6.27. *Let E be a graph such that $E^*_{so} \neq \emptyset$. Define*

$$\begin{aligned} \Phi : KE &\longrightarrow I(E^*_{so}, K) \\ \alpha &\mapsto f_\alpha(u, v) = \begin{cases} 1, & \text{if } v = u\alpha \\ 0, & \text{else} \end{cases} \end{aligned}$$

$Im\Phi = [I(E^*_{so}, K)]^{fs}$ *if and only if* $|r^{-1}(v)| \leq 1$ *for every* $v \in T(E^0_{so})$.

Proof. First we assume $|r^{-1}(v)| \leq 1$ for every $v \in T(E^0_{so})$ and we will show that $Im\Phi = [I(E^*_{so}, K)]^{fs}$.

Take $p_i, q_i \in E^0_{so}$. Consider $e_{p_i q_i} \in I(E^*_{so}, K)$ with $p_i \leq q_i$, then there exists $\eta \in E^*$ such that $p_i\eta = q_i$, sor$(p_i) = s(\eta) \in E^0$. We have $\Phi(\eta) = f_\eta$ where $f_\eta(\alpha\beta) = \begin{cases} 1, & \text{if } \beta = \alpha\eta \\ 0, & \text{else} \end{cases}$. Now we will show that $f_\eta = e_{p_i q_i}$. We have three cases:

i If $\alpha = p_i, \beta = q_i$: Then $f_n(p_i, q_i) = f_\eta(p_i, p_i\eta) = 1$.

$$\text{ii If } \alpha \neq p_i : \begin{cases} \text{If } \beta \neq \alpha\eta \text{ then } f_\eta(\alpha,\beta) = 0 \\ \text{If } \beta = \alpha\eta \text{ then } r(\alpha) = s(\eta) = v_\eta = r(p_i) \\ \text{which is a contradiction since } |r^{-1}(v_\eta)| \leq 1 \end{cases} .$$

iii If $\alpha = p_i$ but $\beta \neq q_i$: Then $\beta \neq p_i\eta$ gives $f_\eta(p_i,\beta) = 0$.

Hence we get $f_\eta = e_{p_iq_i}$. Now take $h \in [I(E^*_{so},K)]^{fs}$ then

$$h = \sum_{i=1}^{n} k_i e_{p_iq_i} = \sum_{i=1}^{n} k_i f_{\eta_i} = f(\sum_{i=1}^{n} k_i\eta_i).$$

For the converse suppose $Im\Phi = [I(E^*_{so},K)]^{fs}$. Consider $w \in T(E^0_{so})$. Since $E^0_{so} \neq \emptyset$ there exists p in E^*_{so} such that $r(p) = w$. Take $e_{pp} \in I(E^*_{so},K)$. Since $Im\Phi = [I(E^*_{so},K)]^{fs}$, there exists an element $\sum_{i=1}^{n} k_i\alpha_i \in KE$ such that $\Phi(\sum_{i=1}^{n} k_i\alpha_i) = e_{pp}$. Then we have $\sum_{i=1}^{n} k_i f_{\alpha_i} = e_{pp}$ i.e. $\sum_{i=1}^{n} k_i f_{\alpha_i}(u,v) = e_{pp}(u,v)$ for all $u,v \in KE$. Thus $\sum_{i=1}^{n} k_i f_{\alpha_i}(p,p) = 1$. Here if $p \neq p\alpha_i$ for all i then the sum will be zero. So we have $p = p\alpha_{i_0}$ for some i_0. Then $\alpha_{i_0} = w$ and the corresponding coefficient $k_{i_0} = 1$.

Now assume there exists $\tilde{p} \neq p \in E^*_{so}$ such that $r(\tilde{p}) = w$. Then

$$0 = e_{pp}(\tilde{p},\tilde{p}) = \sum_{i=1}^{n} k_i f_{\alpha_i}(\tilde{p},\tilde{p}) = \underbrace{f_w(\tilde{p},\tilde{p})}_{\substack{=1 \\ \text{since } \tilde{p} = \tilde{p}w}} + \underbrace{\sum_{\alpha_i \neq w} k_i f_{\alpha_i}(\tilde{p},\tilde{p})}_{\substack{=0 \\ \text{since } \tilde{p} = \tilde{p}w \neq \tilde{p}\alpha_i}} = 1$$

is a contradiction. Therefore $|r^{-1}(v)| \leq 1$ for every $v \in T(E^0_{so})$. □

Corollary 6.28. *If E is finite and $|r^{-1}(v)| \leq 1$ for every $v \in T(E^0_{so})$ then f is onto.*

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