PRIME STATISTICS IN PARTICLE ALGEBRAS

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ABSTRACT

PRIME STATISTICS IN PARTICLE ALGEBRAS

The importance of the fermion algebra extends to all branches of physics. It is characterized by the important property that at most one particle can be present in a quantum state with otherwise same quantum numbers. In this thesis we will deal with algebras $\mathcal{A}_d$ where at most $d-1$ particles can be present in a quantum state with otherwise same quantum numbers. $\mathcal{A}_2$ is thus the fermion algebra. In the limit where d goes to infinity the algebra becomes the boson algebra. Thus, the particles obeying $\mathcal{A}_d$ can be considered as a generalization of bosons and fermions. Algebras $\mathcal{A}_d$ have some important properties. They are constructed in terms of a single annihilation operator a and a single creation operator a^* satisfying certain relations. $\mathcal{A}_d$ has a unique d dimensional representation. In this thesis we will prove another important property of these algebras that the tensor product of two algebras $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ is isomorphic to $\mathcal{A}_d$ where $d = d_1 d_2$. This property brings in the idea that the particle algebras of prime dimensions are fundamental. We use this valuable property to constitute the idea of prime statistics.

ÖZET

PARÇACIK CEBİRLERİNDE ASAL SAYI İSTATİSTİĞİ

Fermion cebirinin önemi fiziğin bütün dallarına yayılmıştır. Bu cebirin temel özelliği bir kuantum durumunda aynı kuantum numaralarına sahip olan en fazla bir parçacığın bulunabilmesidir. Bu tezde, bir kuantum durumunda aynı kuantum numaralarına sahip en fazla $d-1$ parçacığın bulunabildiği $\mathcal{A}_d$ olarak adlandırdığımız cebirlerle ilgileneceğiz. Bu tanımla $\mathcal{A}_2$ fermion cebiri olur. d'nin sonsuza gittiği durum ise bozon cebiri olarak düşünülebilir. $\mathcal{A}_d$ cebirleri bazı önemli özelliklere sahiptirler. Yok edici operatör a ve oluşturucu operatör a^* ile kurulurlar ve yegâne ve indirgenemez bir d boyutlu bir temsilleri vardır. Bu tezde bu cebirlerin önemli başka bir özelliğini göstereceğiz. İki parçacık cebiri $\mathcal{A}_{d_1}$ ve $\mathcal{A}_{d_2}$' nin tensör çarpımlarının $d = d_1 d_2$ olması koşuluyla $\mathcal{A}_d$ cebirine izomorfik olduğundan yola çıkarak asal sayı istatistiği fikrine ulaşacağız.

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LIST OF SYMBOLS/ABBREVIATIONS

$\mathcal{A}_d$	The particle algebra of order d
$\mathcal{A}_2$	The fermion algebra
$\mathcal{A}$	The boson algebra
a	Annihilation operator of $\mathcal{A}$ or $\mathcal{A}_2$ or $\mathcal{A}_d$
a^*	Creation operator of $\mathcal{A}$ or $\mathcal{A}_2$ or $\mathcal{A}_d$
a_d	Annihilation operator of $\mathcal{A}_d$
a_d^*	Creation operator of $\mathcal{A}_d$
c_i	i^{th} annihilation operator of the orthofermion algebra
c_i^*	i^{th} creation operator of the orthofermion algebra
e_{ij}	Generator of the elementary algebra of complex matrices
$\mathcal{F}_p$	The orthofermion algebra of order p
M	aa^*
M_d	The elementary algebra of $d \times d$ complex matrices
m	The eigenvalue of the operator M
N	a^*a, the number operator
n	The eigenvalue of N
Spec N	The set of eigenvalues of N
δ_{ij}	The Kronecker delta function

1. INTRODUCTION

1.1. The Orthofermion Algebra $\mathcal{F}_p$ and The Elementary Algebra $d \times d$ of Complex Matrices M_d

It is well known that the Pauli exclusion principle forbids two fermions to occupy the same quantum state simultaneously. For example electrons in a single atom have quantum numbers labeling their spin state and orbital state. Consequently, the electrons sharing the same orbital state should have opposite spins. Thus there can be at most two electrons in the same orbital state one being spin-up and one being spin-down. In some cases the electrons are strongly correlated it can be observed that even electrons with opposite spins cannot be in the same orbital state. This more exclusive situation is an example of super exclusion principle[1]. More generally the definition of the super exclusion principle is that there can be at most one particle in a state with otherwise any given positional quantum numbers. Orthofermions are fermions which obey the super exclusion principle. The algebra of orthofermions $\mathcal{F}_p$ is generated by annihilation and creation operators denoted by c_i and c_i^* respectively which satisfy the equations[2]:

$$c_i\ c_j^* + \delta_{ij} \sum_{k=1}^{p} c_k^*\ c_k = \delta_{ij}, \quad c_i\ c_j = 0, \quad c_i^*\ c_j^* = 0, \qquad i, j\ \epsilon\ \{1, 2, ..., p\}, \tag{1.1}$$

where δ_{ij} is the Kronecker delta function. There is a matrix representation of the orthofermion algebra which can be constructed by denoting c_i as

$$[c_i]_{jk} = \delta_{j1}\delta_{ki+1}, \quad i, j, k\ \epsilon\ \{1, 2, 3, ..., p+1\},\ (i \neq p+1). \tag{1.2}$$

The above definition of the matrix representation tells us that $\mathcal{F}_p$ has a $(p+1)$-dimensional irreducible representation[4]. The number operator of this algebra of order

p can be defined as:

$$N = c_1^* c_1 + c_2^* c_2 + ... + c_p^* c_p \tag{1.3}$$

The eigenvectors of this operator represent the states of orthofermions. For the example of electrons in the orbit of an atom with the super exclusion principle the states can be written as $\mid 0,0 \rangle$, $\mid 1,0 \rangle$ and $\mid 0,1 \rangle$. The former means there is no electron in that orbit, the second one means there is one electron with spin up and the last one means there is one electron with spin down. This was an example of orthofermions of order 2. The matrix form of N has 1 in all of the diagonal elements except for the uppermost one and zero elsewhere. Thus, the spectrum of the number operator is shown by $Spec\ N = \{0, 1\}$ for any dimension. The weakness of this algebra is that the eigenvalues are the same for all eigenvectors except for 0. Hence the eigenvalues are not distinguishable labels of the eigenvectors.

We also use the elementary algebra of $d \times d$ complex matrices M_d. M_d is generated by the operator e_{ij} whose elements in the matrix representation have one in the ij^{th} place and zero everywhere else. This $d \times d$ irreducible matrix representation is unique as the definition of the algebra requires it. e_{ij} should satisfy the following relations:

$$e_{ij}\ e_{kl} = \delta_{jk}\ e_{il}, \quad e_{ij}^* = e_{ji}, \qquad\qquad i, j \ \epsilon \ \{1, 2, ..., d\}. \tag{1.4}$$

1.2. The Particle Algebra $\mathcal{A}_d$

1.2.1. The Fermion Algebra $\mathcal{A}_2$ and The Boson Algebra $\mathcal{A}$

To emphasize the importance of the particle algebra let us look at the basic properties of the fermion algebra and the boson algebra. The fermion algebra is the algebra physically corresponding to fermions obeying the Pauli exclusion principle. It is defined by the relations:

$$aa^* + a^*a = 1, \quad a^2 = 0 \tag{1.5}$$

This algebra has a unique two-dimensional irreducible representation which can be written in matrix form as:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with the eigenvectors

$$|\,0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |\,1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The spectrum of N is given by $Spec\ N = \{0,1\}$. $|\,0\rangle$ and $|\,1\rangle$ mean that there is no particle and there is one particle in the quantum state, respectively. The boson algebra describes bosons which obey Bose-Einstein Statistics. Bosons can share the same quantum state. The algebra is defined by the relation:

$$aa^* - a^*a = 1 \tag{1.6}$$

This algebra has a unique infinite-dimensional irreducible representation which can be written in matrix form as

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & \sqrt{2} & 0 & . & . & . \\ 0 & 0 & 0 & \sqrt{3} & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$a^* = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 0 & \sqrt{2} & 0 & 0 & . & . & . \\ 0 & 0 & \sqrt{3} & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & 3 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & 3 & 0 & . & . & . \\ 0 & 0 & 0 & 4 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

As there is no limit for the number of particles in a state the eigenvectors of N are shown as $\mid 0\rangle$ $\mid 1\rangle$ $\mid 2\rangle$ ... and the numbers in the notation mean how many particles are there in the state. Thus $Spec\ N = \{0, 1, 2, ...\}$. In matrix notation they are shown as

$$|\,0\rangle = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ . \\ . \end{pmatrix}, \ |\,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ . \end{pmatrix}, \ |\,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ . \\ . \\ . \end{pmatrix}, \ |\,3\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ . \\ . \end{pmatrix} \ \ldots \tag{1.7}$$

One could ask that if there exists a particle algebra with the number operator such that $Spec\ N = \{0, 1, ..., d-1\}$ with a unique irreducible d-dimensional representation. As a particle algebra the orthofermion algebra has a d-dimensional representation. But it does not have a number operator with the desired spectrum. If there exists such an algebra it could interpret particles such that there can be at most $d-1$ particles in a state with otherwise same quantum numbers. It has been found that the particle algebra $\mathcal{A}_d$ has these properties. The particle algebra is defined by the following relations[3]:

$$aa^* - a^*a = 1 - \frac{d}{(d-1)!}\ a^{*d-1}a^{d-1}, \qquad a^d = 0. \tag{1.8}$$

It can be seen the algebra is a generalization of the fermion and the boson algebra when the limit d goes to infinity and $d = 2$ cases are checked. In the limit where d goes to infinity the right hand side becomes 1 and the algebra becomes the boson algebra $\mathcal{A}$. In the case $d = 2$ the right hand side becomes $1 - 2a^*a$ and the fermion algebra $\mathcal{A}_2$ is obtained. $\mathcal{A}_d$ is isomorphic $\mathcal{F}_p$ where $d = p+1$. Thus, when dealing with orthofermions one can use $\mathcal{A}_d$ instead of $\mathcal{F}_p$ to benefit its properties. The isomorphism is given in the appendix.

1.2.2. Construction Of $\mathcal{A}_d$

The desired properties of the algebra to be constructed are as follows. Firstly, the algebra should become $\mathcal{A}$ and $\mathcal{A}_2$ in the cases mentioned in the previous subsection. Secondly, the number operator should have the spectrum $Spec\ N = \{0, 1, ..., d-1\}$. Thus let us start with the algebra $\mathcal{A}_d$ satisfying the relations:

$$aa^* - a^*a = 1 - \lambda\ a^{*d-1}a^{d-1}, \qquad a^d = 0,\ d \in Z^+ \tag{1.9}$$

where λ may depend on d. This looks like a generalization of the fermion algebra and the boson algebra. We dig the equation above to find λ and further properties.

$$aa^*a - a^*aa = a \tag{1.10}$$

which is obtained by multiplying (1.9) with a from the left. Defining the number operator $N = a^*a$ this becomes

$$aN - Na = a, \quad aN = (N+1)a, \quad a(N-1) = Na \tag{1.11}$$

using these properties one can calculate the following

$$a^{*k}a^k = N(N-1)(N-2)...(N-k+1), \quad k = 1, 2, ..., d-1 \tag{1.12}$$

where k is limited by $d-1$ as $a^d = 0$. By using this property one can say that

$$(N-d+1)a^{*d-1}a^{d-1} = 0 \tag{1.13}$$

because the left hand side is equal to $a^{*d}a^d$. By using this equation one can obtain the equation

$$(a^{*d-1}a^{d-1})^2 = (d-1)!\ a^{*d-1}a^{d-1} \tag{1.14}$$

By multiplying (1.10) with a^* from the left and defining $M = aa^*$ we get

$$M^2 - NM = M, \quad M^2 - MN = M \tag{1.15}$$

where the second relation is the conjugate of the first one. Now let us use these properties to reach at $\lambda = \frac{d}{(d-1)!}$. The starting equation can be rewritten as

$$(M - N) = 1 - \lambda a^{*d-1} a^{d-1} \tag{1.16}$$

multiplying this with M from the left and opening the term $a^{*d-1}a^{d-1}$ gives

$$M^2 - MN = M - \lambda MN(N-1)(N-2)...(N-d+1) \tag{1.17}$$

where the left hand side is equal to M. Thus

$$\lambda MN(N-1)(N-2)...(N-d+1) = 0. \tag{1.18}$$

This gives two possibilities that either $\lambda = 0$ or $MN(N-1)(N-2)...(N-d+1) = 0$. $\lambda = 0$ cannot be valid because in that case the assumption $a^d = 0$ contradicts with $aa^* - a^*a = 1$. Thus we are left with $MN(N-1)(N-2)...(N-d+1) = 0$ which also mean $Ma^{*d-1}a^{d-1} = 0$. Then multiplying the starting relation with $a^{*d-1}a^{d-1}$ from the right:

$$(M - N)a^{*d-1}a^{d-1} = a^{*d-1}a^{d-1} - \lambda(a^{*d-1}a^{d-1})^2. \tag{1.19}$$

Using the facts $Ma^{*d-1}a^{d-1} = 0$, $Na^{*d-1}a^{d-1} = (d-1)a^{*d-1}a^{d-1}$ and $(a^{*d-1}a^{d-1})^2 = (d-1)!\ a^{*d-1}a^{d-1}$ we get

$$-(d-1)a^{*d-1}a^{d-1} = a^{*d-1}a^{d-1} - \lambda(d-1)!\ a^{*d-1}a^{d-1}. \tag{1.20}$$

This brings in that

$$-d+1=1-\lambda(d-1)! \tag{1.21}$$

which means $\lambda=\frac{d}{(d-1)!}$.

1.2.3. Representations of $\mathcal{A}_d$

Representations of (1.8) can be constructed by starting from the simultaneous eigenvectors of the commuting hermitian operators $M=aa^*$ and $N=a^*a$. Denoting these eigenvectors by $\mid n,m\rangle$ where m and n are the eigenvalues of M and N respectively. In notation:

$$N\mid n,m\rangle=n\mid n,m\rangle,\quad M\mid n,m\rangle=m\mid n,m\rangle. \tag{1.22}$$

Let us search for the properties of the representation. Using $M^2-NM=M$ we get

$$M(M-N)\mid n,m\rangle=M\mid n,m\rangle \tag{1.23}$$

which concludes in

$$m(m-n)=m \tag{1.24}$$

Hence $m=0$ or $m=n+1$. Then applying aN to the eigenvector and using $aN=(N+1)a$ we obtain

$$a(N\mid n,m\rangle)=(N+1)(a\mid n,m\rangle) \tag{1.25}$$

which means that $a\mid n,m\rangle\sim\mid n-1,m\rangle$. Thus we can define $a\mid n,m\rangle=\alpha(n)\mid n-1,m\rangle$ since m is zero or a function of n. Taking the conjugate of this equation and multiplying with itself gives:

$$\langle n,m \mid a^{*}a \mid n,m\rangle = \langle n-1,m \mid \alpha(n)^{*}\alpha(n) \mid n-1,m\rangle. \tag{1.26}$$

$$\langle n,m \mid N \mid n,m\rangle = \langle n-1,m \mid \alpha(n)^{*}\alpha(n) \mid n-1,m\rangle. \tag{1.27}$$

Left hand side is equal to n. We are free to choose $\alpha(n)$ as real. The coefficients $\alpha(n)$ can be determined such that these vectors have unit norm. Then $\alpha(n) = \sqrt{n}$. To wit, a is a ladder operator and it decreases the eigenvalue of N. Since N cannot have negative eigenvalue, n should be integer and the limit for decreasing process is defined by $a \mid 0,m\rangle = 0$. Now let us consider a^{*}'s function. Consider the eigenvalue equation

$$a^{*}a \mid n,m\rangle = n \mid n,m\rangle \Rightarrow a^{*}\sqrt{n} \mid n-1,m\rangle = n \mid n,m\rangle, \tag{1.28}$$

which holds for $m \neq 0$. Therefore a^{*} is also a ladder operator which increases the eigenvalue of N with a multiplication factor of $\sqrt{n+1}$ on the eigenvector $\mid n,m\rangle$. One also must take into account the fact that $a^{*d} = 0$ which tells us that there is a upper limit for the increasing process. Indeed this accounts for that eigenvalue of M is not trivial. The limit is given by the relation $a^{*d} \mid 0,m\rangle = 0$. The last step requires $a^{*} \mid d-1,m\rangle = 0$. This implies that $M \mid d-1,m\rangle = 0$ as a^{*} hits the vector first. Thus $m = 0$ for $n = d-1$. It is also clear that M cannot have zero eigenvalue for other values for n by the same way. Thus, for other cases $m = n+1$. The set of eigenvectors is given by

$$S_d = \{\mid 0,1\rangle, \mid 1,2\rangle, ..., \mid d-2,d-1\rangle \mid d-1,0\} \tag{1.29}$$

Now we have all we need to summarize the characteristic relations of the representation:

$$a \mid n,n+1\rangle = \sqrt{n} \mid n-1,n\rangle, \quad n = 0,1,...,d-2,$$

$$a \mid d-1,0\rangle = \sqrt{d-1} \mid d-2,d-1\rangle,$$

$$a^* \mid n,n+1\rangle = \sqrt{n+1} \mid n+1,n+2\rangle, \quad n = 0,1,...,d-3 \ , \tag{1.30}$$

$$a^* \mid d-2,d-1\rangle = \sqrt{d-1} \mid d-1,0\rangle,$$

$$a^* \mid d-1,0\rangle = 0.$$

These equations quite looks like the boson algebra with the difference of existence of an upper limit for the eigenvectors of N. The physical meaning of this is that there is an upper limit for the number of particles in a state with otherwise same quantum numbers. Namely, $Spec(N) = \{0,1,2,..,d-1\}$. It can be seen that the cases $d = 2$ and in the limit d goes to the infinity we get boson algebra and the fermion algebra. Thus $\mathcal{A}_d$ is the generalization of the boson algebra and fermion algebra. Having done the representations of the algebra we can present them in matrix form. There are d eigenvectors which constitute an orthonormal set. Thus they can be written in $d \times 1$ matrices as

$$\mid 0,1\rangle = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}, \mid 1,2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, . . . \mid d-2,d-1\rangle = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mid d-1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}. \tag{1.31}$$

With this choice of the eigenvectors the operators can be written as

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & \sqrt{d-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & \sqrt{d-2} & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & \sqrt{d-1} \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a^* = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & \sqrt{d-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & \sqrt{d-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & \sqrt{d-1} & 0 \end{pmatrix}$$

$$N = a^*a = \begin{pmatrix} 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & . \; . \; . & 0 & 0 & 0 & 0 \\ . & . & . & . & . \; . \; . & . & . & . & . \\ . & . & . & . & . \; . \; . & . & . & . & . \\ . & . & . & . & . \; . \; . & . & . & . & . \\ 0 & 0 & 0 & 0 & . \; . \; . & d-4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & d-3 & 0 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & d-2 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & d-1 \end{pmatrix}$$

$$M = aa^* = \begin{pmatrix} 1 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & . \; . \; . & 0 & 0 & 0 & 0 \\ . & . & . & . & . \; . \; . & . & . & . & . \\ . & . & . & . & . \; . \; . & . & . & . & . \\ . & . & . & . & . \; . \; . & . & . & . & . \\ 0 & 0 & 0 & 0 & . \; . \; . & d-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & d-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & d-1 & 0 \\ 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Tensor products of $\mathcal{A}_d$

Consider two particle algebras $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ which may describe two different state spaces of a physical system. That is, there are two labels such that for one of them at most $d_1 - 1$ particles can be in the same state and for the other label there can be at most $d_2 - 1$ particles in the same state. One can consider these state spaces as a single state space. There must be $d_1 d_2$ states for this single state space. Also one can use tensor products of the algebras $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ to act on this state space. Now consider another particle algebra $\mathcal{A}_d$ where $d = d_1 d_2$. This algebra also have $d_1 d_2$ eigenvectors. One can make a mapping between these state spaces which is one to one and onto. We will search for such a mapping using the matrix representations of the algebras $\mathcal{A}_{d_1}$, $\mathcal{A}_{d_2}$ and $\mathcal{A}_d$. The algebras $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ are defined in ${d_1}^2$ and ${d_2}^2$ dimensional vector spaces respectively. Tensor products of these algebras belongs to a $d^2 = (d_1 d_2)^2$ dimensional vector space. This can be understood by looking at their matrix representations where tensor product of $d_1 \times d_1$ and $d_2 \times d_2$ matrices results in $d \times d$ matrices where $d = d_1 d_2$. Also $\mathcal{A}_d$ is defined in d^2 dimensional vector space. Thus a morphism which is one to one and onto between $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ and $\mathcal{A}_d$ is automatically an isomorphism because they are algebras of endomorphisms of the same dimensional vector spaces. Having analyzed the matrix representations of the algebras we have found that the annihilation operator a of the algebra $\mathcal{A}_d$ can be written in terms of the generators of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ as

$$a = \frac{1}{(d_1-1)!(d_2-1)!} \sum_{i=1,j=1}^{d_1,d_2-1} \sqrt{\frac{j+d_2(i-1)}{j}}\; a^{d_1-i}\; a^{*d_1-1}\; a^{i-1} \otimes a^{d_2-j}\; a^{*d_2-1}\; a^j$$

$$+ \frac{d_2}{(d_1-1)!\sqrt{d_2!}} \sum_{k=1}^{d_1} a^{d_1-k}\; a^{*d_1-1}\; a^k \otimes a^{*d_2-1}, \quad a^0 \equiv 1 \tag{2.1}$$

where the generators on the left of the tensor products are elements of $\mathcal{A}_{d_1}$ and the generators on the right of the tensor products are elements of $\mathcal{A}_{d_2}$. It is important to keep in mind that d_1 and d_2 must be chosen such that $d_1 \leq d_2$. If d_1 and d_2 could be freely chosen one would not be able to write a general equation like (2.1). From now on we will not label which algebra the generators belong to as it can be understood by looking at their places in the expressions. The tensor products of $\frac{1}{(d_1-1)!}\ a^{d_1-i}\ a^{*d_1-1}\ a^{i-1}$, $\frac{1}{(d_2-1)!\sqrt{j}}\ a^{d_2-j}\ a^{*d_2-1}\ a^{j}$, $\frac{1}{(d_1-1)!\sqrt{k}}\ a^{d_1-k}\ a^{*d_1-1}\ a^{k}$ and $\frac{1}{\sqrt{(d_2-1)!}}\ a^{*d_2-1}$ constitute a set of orthonormal vectors in terms of which a can be constructed. In order to get rid of complexity we will replace them with the appropriate elements of M_d as they are isomorphic. We speak to this fact in the Appendix A. Therefore the equation becomes

$$a = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sqrt{j + d_2(i-1)}\ e_{ii} \otimes e_{jj+1} + \sum_{k=1}^{d_1-1} \sqrt{kd_2}\ e_{kk+1} \otimes e_{d_2 1}. \tag{2.2}$$

Taking the conjugate of this equation one finds

$$a^* = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sqrt{j + d_2(i-1)}\ e_{ii} \otimes e_{j+1j} + \sum_{k=1}^{d_1-1} \sqrt{kd_2}\ e_{k+1k} \otimes e_{1d_2}. \tag{2.3}$$

We will check whether this equations satisfies (1.8). Therefore we need to derive the expressions $aa^*,\ a^*a,\ a^{d-1}, a^{*d-1}a^d$ and a^{*d}. One gets

$$aa^* = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} (j + d_2(i-1))(1 - \delta_{id_1}\delta_{jd_2})\ e_{ii} \otimes e_{jj},$$

$$a^*a = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} (j - 1 + d_2(i-1))\ e_{ii} \otimes e_{jj} \tag{2.4}$$

After canceling the terms which become zero one obtains

$$a^{d-1} = \sqrt{(d-1)!}\ e_{1d_1} \otimes e_{1d_2} \tag{2.5}$$

$$a^{*d-1} = \sqrt{(d-1)!}\ e_{d_1 1} \otimes e_{d_2 1}. \tag{2.6}$$

These are hard to show by direct calculation because the number of terms giving zero increase as d_1 and d_2 get bigger. An example is given for $d_1 = 2$ and $d_2 = 3$ in the appendix. The left hand side of (1.8) becomes

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} [(j + d_2(i-1))(1 - \delta_{id_1}\delta_{jd_2})\ e_{ii} \otimes e_{jj} - (j - 1 + d_2(i-1))\ e_{ii} \otimes e_{jj}]$$

$$(\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\ e_{ii} \otimes e_{jj}) - (\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\ (j + d_2(i-1))(1 - \delta_{id_1}\ \delta_{jd_2})\ e_{ii} \otimes e_{jj})$$

$$(\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\ e_{ii} \otimes e_{jj}) - d_1 d_2\ e_{d_1 d_1} \otimes e_{d_2 d_2}. \tag{2.7}$$

The right hand side of (1.8) becomes

$$1 - \ d_1 d_2\ e_{d_1 d_1} \otimes e_{d_2 d_2}\ \ - \frac{d}{(d-1)!}\sqrt{(d-1)!}\ e_{d_1 1} \otimes e_{d_2 1}\ \sqrt{(d-1)!}\ e_{1 d_1} \otimes e_{1 d_2}. \tag{2.8}$$

writing 1 in the basis of $M_{d_1} \otimes M_{d_2}$ we obtain

$$(\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\ e_{ii} \otimes e_{jj}) - \ d\ e_{d_1 d_1} \otimes e_{d_2 d_2} \tag{2.9}$$

which is identical with (2.7). The second relation in (1.8) can be satisfied in a step by multiplying (2.5) with a one more time. Thus we have shown that $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ and $\mathcal{A}_d$ are isomorphic. Now we will express the generators of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ in terms of $\mathcal{A}_d$ and see whether they satisfy (1.8) for d_1 and d_2 dimensions respectively. We were able to describe $\mathcal{A}_d$ in terms of the $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ as we could take their tensor products to get higher dimensions. Now we need to use higher dimensional algebra $\mathcal{A}_d$ to express lower dimensional algebras $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$. To do this we can expand $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ to d dimensional vector space by taking tensor product with 1. It can be shown that the product has the same properties and is isomorphic to its factor(other than one).

That is, one can take the tensor product of the equations (1.8) with the identity of any dimension and the new equations still hold. Hereby the inverse of the mapping can be defined as from $\mathcal{A}_d$ to $1_{d_1} \otimes \mathcal{A}_{d_2}$ and $\mathcal{A}_{d_1} \otimes 1_{d_2}$. Now we can write the generators of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ in terms of $\mathcal{A}_d$ as

$$1 \otimes a = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sqrt{j}\; e_{[(i-1)d_2+j][(i-1)d_2+j+1]} \tag{2.10}$$

$$a \otimes 1 = \sum_{i=1}^{d_1-1} \sum_{j=1}^{d_2} \sqrt{i}\; e_{[(i-1)d_2+j][id_2+j]} \tag{2.11}$$

where the subscripts are presented in brackets. Starting with (2.10) we need to find $1 \otimes a^{d-1}$ in order to check whether (1.8) holds or not. The square of (2.10) gives

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sum_{m=1}^{d_1} \sum_{n=1}^{d_2-1} \sqrt{jn}\; e_{[(i-1)d_2+j][(m-1)d_2+n+1]}\; \delta_{[(i-1)d_2+j+1][(m-1)d_2+n]} \tag{2.12}$$

$\delta_{[(i-1)d_2+j+1][(m-1)d_2+n]}$ is nonzero only when i=m and thus j=n+1. We get

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sum_{m=1}^{d_1} \sum_{n=1}^{d_2-1} \sqrt{jn}\; e_{[(i-1)d_2+j][(m-1)d_2+n+1]}\; \delta_{[i][m]}\delta_{[n][j+1]} \tag{2.13}$$

$\delta_{[n][j+1]}$ kills the term with $j = d_2 - 1$ and we end up with

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-2} \sqrt{j(j+1)}\; e_{[(i-1)d_2+j][(i-1)d_2+j+2]}. \tag{2.14}$$

By this way, the $(d_2-1)^{th}$ power of (2.10) gives

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-(d_2-1)} \sqrt{j(j+1)(j+2)....(j+d_2-1)}\; e_{[(i-1)d_2+j][(i-1)d_2+j+d_2-1]} \tag{2.15}$$

which means that the only value for j is 1. Thus we have

$$1 \otimes a^{d_2-1} = \sum_{i=1}^{d_1} \sqrt{(d_2-1)!}\ e_{[(i-1)d_2+1][id_2]}. \tag{2.16}$$

Also

$$1 \otimes a^{*d_2-1} = \sum_{i=1}^{d_1} \sqrt{(d_2-1)!}\ e_{[id_2][(i-1)d_2+1]} \tag{2.17}$$

by taking the conjugate of (2.16). For showing the second relation of (1.8) we multiply (2.16) with (2.10) and get $\delta_{[(i-1)d_2+1][(i-1)d_2+j+1]}$ which vanishes the equation within the boundary of the values i and j can take. Substituting the product of (2.16) and (2.19) into (1.8) right hand side we get

$$1 - \ d_2 \sum_{i=1}^{d_1} e_{[id_2][id_2]} \tag{2.18}$$

Looking at the left hand side we calculate $(1 \otimes a)(1 \otimes a^*) = 1 \otimes aa^*$ as

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} \sqrt{j}\ e_{[(i-1)d_2+j][(i-1)d_2+j+1]} \sum_{m=1}^{d_1} \sum_{n=1}^{d_2-1} \sqrt{n}\ e_{[(m-1)d_2+n+1][(m-1)d_2+n]}$$

which results in

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} j\ e_{[(i-1)d_2+j][(i-1)d_2+j]}. \tag{2.19}$$

Similarly $1 \otimes a^*a$ is

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2-1} j\ e_{[(i-1)d_2+j+1][(i-1)d_2+j+1]}. \tag{2.20}$$

Thus on the left hand side of (1.8) we have

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2-1} j\ e_{[(i-1)d_2+j][(i-1)d_2+j]}\ -\ \sum_{i=1}^{d_1}\sum_{j=1}^{d_2-1} j\ e_{[(i-1)d_2+j+1][(i-1)d_2+j+1]}. \tag{2.21}$$

After changing the range of the j in the second term from 2 to d_2 we are able to subtract it from the first term. After a few steps we get

$$\sum_{i=1}^{d_1}\sum_{j=1}^{d_2-1} e_{[(i-1)d_2+j][(i-1)d_2+j]}\ -\ \sum_{i=1}^{d_1} d_2 e_{[id_2][id_2]}. \tag{2.22}$$

It can be shown that the first term is identity. Thus the equation is identical with (2.19). The second relation in (1.8) can be verified by multiplying (2.15) with $1 \otimes a$ again. One should follow the same steps for $\mathcal{A}_{d_1} \otimes 1$ to complete the proof. $aa^* \otimes 1$, $a^*a \otimes 1$, $a^{d_1} \otimes 1$ and $a^{*d_1} \otimes 1$. From (2.11) we obtain

$$\begin{aligned} aa^* \otimes 1 &= \sum_{i=1}^{d_1-1}\sum_{j=1}^{d_2} i\ e_{[(i-1)d_2+j][(i-1)d_2+j]} \\ a^*a \otimes 1 &= \sum_{i=1}^{d_1-1}\sum_{j=1}^{d_2} i\ e_{[id_2+j][id_2+j]} \end{aligned} \tag{2.23}$$

$a^{d_1} \otimes 1$ and $a^{*d_1} \otimes 1$ can be derived like we did before:

$$a^2 \otimes 1 = \sum_{i=1}^{d_1-1}\sum_{j=1}^{d_2}\sum_{m=1}^{d_1-1}\sum_{n=1}^{d_2} \sqrt{im}\ e_{[(i-1)d_2+j][md_2+n]}\delta_{[id_2+j][(m-1)d_2+n]}. \tag{2.24}$$

Expanding the Kronecker delta function like $\delta_{[id_2+j][(m-1)d_2+n]} = \delta_{[i+1][m]}\delta_{[j][n]}$ one obtains

$$a^2 \otimes 1 = \sum_{i=1}^{d_1-1}\sum_{j=1}^{d_2} \sqrt{i(i+1)}\ e_{[(i-1)d_2+j][(i+1)d_2+j]}. \tag{2.25}$$

Applying this process $d_1 - 1$ times we get

$$a^{d_1-1} \otimes 1 = \sum_{i=1}^{d_1-1} \sum_{j=1}^{d_2} \sqrt{i(i+1)(i+2)...(i+d_1-2)}\; e_{[(i-1)d_2+j][(i+d_1-2)d_2+j]}$$
$$= \sum_{j=1}^{d_2} \sqrt{(d_1-1)!}\; e_{[j][(d_1-1)d_2+j]} \tag{2.26}$$

where i can only take the value 1 to make $(i + d_1 - 2)d_2 + j$ lower than $d = d_1 d_2$. Similarly

$$a^{*d_1-1} \otimes 1 = \sum_{j=1}^{d_2} \sqrt{(d_1-1)!}\; e_{[(d_1-1)d_2+j][j]}. \tag{2.27}$$

Thus we have

$$1 - d_1 \sum_{j=1}^{d_2} e_{[(d_1-1)d_2+j][(d_1-1)d_2+j]} \tag{2.28}$$

on the right hand side. Looking at the left hand side we get

$$aa^* \otimes 1 - a^* a \otimes 1 = \sum_{i=0}^{d_1-2} \sum_{j=1}^{d_2} (i+1)\; e_{[id_2+j][id_2+j]} - \sum_{i=1}^{d_1-1} \sum_{j=1}^{d_2} i\; e_{[id_2+j][id_2+j]}$$
$$= \sum_{i=0}^{d_1-2} \sum_{j=1}^{d_2} e_{[id_2+j][id_2+j]} - \sum_{j=1}^{d_2} (d_1-1)\; e_{[d_1d_2-d_2+j][d_1d_2-d_2+j]}$$
$$= \sum_{i=0}^{d_1-1} \sum_{j=1}^{d_2} e_{[id_2+j][id_2+j]} - d_1 \sum_{j=1}^{d_2} e_{[(d_1-1)d_2+j][(d_1-1)d_2+j]} \tag{2.29}$$

which equals to (2.28) because the first term is an expression of identity. Multiplying (2.26) with $a \otimes 1$ again we get $\delta_{[id_2+j][j]}$ to be summed over i and j. Since i cannot take the value of zero the expression vanishes. This completes the proof. We tried a mapping between the generators and checked whether this mapping holds (1.8). We achieved this goal and have seen that $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ and $\mathcal{A}_d$ are isomorphic. Now let us have a look at how the eigenvectors of $N = aa^*$ map to each other as we can express a_d in terms of tensor products involving a_{d_1} and a_{d_2}. One can realize that the eigenvectors can also be mapped by taking tensor product. Denoting the eigenvectors of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ by just the eigenvalue of N we write their tensor product as

$$\mid n_1\rangle\otimes\mid n_2\rangle=\mid n_1;n_2\rangle. \tag{2.30}$$

It can be recognized that when hit by a:

$$a\mid n_1;n_2\rangle\sim\mid n_1;n_2-1\rangle,\qquad n_2\neq 0,$$

$$a\mid n_1;0\rangle\sim\mid n_1-1;d_2-1\rangle,$$

$$a^*\mid n_1;n_2\rangle\sim\mid n_1;n_2+1\rangle,\qquad n_2\neq d_2-1,$$

$$a^*\mid n_1;d_2-1\rangle\sim\mid n_1+1;0\rangle \tag{2.31}$$

with the exceptions that

$$a\mid 0;0\rangle=0,\quad a^*\mid d_1-1;d_2-1\rangle=0. \tag{2.32}$$

These relations reveal the map between the eigenvectors of $\mathcal{A}_d$ and the eigenvectors of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ indeed. We can show it as

$$\begin{pmatrix} \mid 0;0\rangle & \mid 0;1\rangle & ... & \mid 0;d_2-2\rangle & \mid 0;d_2-1\rangle \\ \mid 1;0\rangle & \mid 1;1\rangle & ... & \mid 1;d_2-2\rangle & \mid 1;d_2-1\rangle \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \mid d_1-2;0\rangle & \mid d_1-2;1\rangle & ... & \mid d_1-2;d_2-2\rangle & \mid d_1-2;d_2-1\rangle \\ \mid d_1-1;0\rangle & \mid d_1-1;1\rangle & ... & \mid d_1-1;d_2-2\rangle & \mid d_1-1;d_2-1\rangle \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} \mid 0\rangle & \mid 1\rangle & ... & \mid d_2-2\rangle & \mid d_2-1\rangle \\ \mid d_2\rangle & \mid d_2+1\rangle & ... & \mid 2d_2-2\rangle & \mid 2d_2-1\rangle \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \mid (d_1-2)d_2\rangle & \mid (d_1-2)d_2+1\rangle & ... & \mid (d_1-1)d_2-2\rangle & \mid (d_1-1)d_2-1\rangle \\ \mid (d_1-1)d_2\rangle & \mid (d_1-1)d_2+1\rangle & ... & \mid d_1d_2-2\rangle & \mid d_1d_2-1\rangle \end{pmatrix}.$$

3. CONCLUSIONS

In this thesis we used the particle algebra $\mathcal{A}_d$, the orthofermion algebra $\mathcal{F}_p$ and the algebra of $d \times d$ complex matrices M_d to prove that $\mathcal{A}_d$ is isomorphic to $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ provided that $d = d_1 d_2$. Although these algebras are isomorphic their physical meaning is quite different. $\mathcal{A}_d$ is generated by a single pair of creation and annihilation operators a^* and a so that the number operator $N = a^* a$ has spectrum $0, 1, 2, ..., d-1$ providing with at most $d-1$ particles can be present in a state with otherwise same quantum numbers. The orthofermion algebra, on the other hand, possesses $p = d-1$ pairs of creation and annihilation operators describing $d-1$ orthofermions. Only one orthofermion can be placed in a spatial state with otherwise same quantum numbers. Here it is important that orthofermions obey the super exclusion principle so that even different orthofermions can not be placed together in the same state.

The fact that $\mathcal{A}_d$ is isomorphic to $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ is important since it implies that $\mathcal{A}_d$ can be "factorized" to its prime factors, i.e. $\mathcal{A}_d$ is isomorphic to $\mathcal{A}_{p_1} \otimes \mathcal{A}_{p_2} \otimes ... \otimes \mathcal{A}_{p_n}$ where p_i are prime and $d = p_1 p_2 ... p_n$. Physically this implies that $\mathcal{A}_p$ with prime p, i.e. prime statistics is important. The case $p = 2$ describes ordinary fermions and its importance is manifest. The cases $p \geq 3$ may be applied to the anyonic theories. Thus, anyons providing the property that there can be prime numbers of particles in a state with otherwise same quantum numbers can be considered "fundamental". Therefore one can consider an anyon as a composition of fundamental anyons. Separately, case $p = 3$ may be applied to quark color and may lead to a theory which is different from the standard model. For this purpose the model should be further developed. Another interesting possible application of the particle algebras could be preonic models where quarks and leptons are described in terms of their constituted preonic particles.

APPENDIX A: Isomorphisms among $\mathcal{A}_d$, $\mathcal{F}_{d-1}$ and M_d

$\mathcal{A}_d$, $\mathcal{F}_{d-1}$ and M_d are all irreducibly represented by $d \times d$ matrices. This implies that using the generators of one of them one can write a basis for other two. Thus they are isomorphic automatically as in the case of $\mathcal{A}_d$ and $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ where $d = d_1 d_2$. We find it useful to write how the generators of one algebra can be expressed in terms of the others and thus show the isomorphism in another way.

The generators of $\mathcal{A}_d$ can be written in terms of the generators of $\mathcal{F}_{d-1}$ as:

$$a = c_1 + \sum_{i=2}^{d-1} \sqrt{i}\; c_{i-1}^* c_i \tag{A.1}$$

$$a^* = c_1^* + \sum_{i=2}^{d-1} \sqrt{i}\; c_i^* c_{i-1} \tag{A.2}$$

Now we can check whether these formulas satisfy (1.8). a^{d-1} and a^{*d-1} can be found by starting from a^2:

$$\begin{aligned} a^2 &= (c_1 + \sum_{i=2}^{d-1} \sqrt{i}\; c_{i-1}^* c_i)(c_1 + \sum_{j=2}^{d-1} \sqrt{j}\; c_{j-1}^* c_j) && \text{(A.3)} \\ &= \sum_{i=2}^{d-1} \sum_{j=2}^{d-1} \sqrt{ij}\; c_{i-1}^* c_i c_{j-1}^* c_j + \sum_{j=2}^{d-1} \sqrt{j}\; c_i c_{j-1}^* c_j. && \text{(A.4)} \end{aligned}$$

where the other cross terms give zero. Using the defining relations of $\mathcal{F}_{d-1}$ one gets

$$a^2 = \sqrt{2}\; c_2 + \sum_{i=2}^{d-2} \sqrt{i(i+1)}\; c_{i-1}^* c_{i+1} \tag{A.5}$$

Applying this process $d-1$ times one ends up with

$$a^{d-1} = \sqrt{(d-1)!}\; c_{d-1} \tag{A.6}$$

One can observe that $a^d = 0$ multiplying (A.6) with a once more. Thus the right hand side of (1.8) becomes

$$1 - d\ c^*_{d-1}c_{d-1} \tag{A.7}$$

For the left hand side

$$\begin{aligned} aa^* &= (c_1 + \sum_{i=2}^{d-1} \sqrt{i}\ c^*_{i-1}c_i)(c^*_1 + \sum_{j=2}^{d-1} \sqrt{j}\ c^*_j c_{j-1}) && \text{(A.8)} \\ &= c_1 c^*_1 + \sum_{i=2}^{d-1} i\ c^*_{i-1}c_{i-1}. && \text{(A.9)} \end{aligned}$$

Similarly

$$\begin{aligned} a^*a &= (c^*_1 + \sum_{j=2}^{d-1} \sqrt{j}\ c^*_j c_{j-1})(c_1 + \sum_{i=2}^{d-1} \sqrt{i}\ c^*_{i-1}c_i) && \text{(A.10)} \\ &= c^*_1 c_1 + \sum_{i=2}^{d-1} i\ c^*_i c_i && \text{(A.11)} \end{aligned}$$

The left hand side is then

$$\begin{aligned} aa^* - a^*a &= c_1c^*_1 + \sum_{i=1}^{d-2}(i+1)\ c^*_i c_i - c^*_1 c_1 + \sum_{i=2}^{d-1} i\ c^*_i c_i && \text{(A.12)} \\ &= c_1c^*_1 - c^*_1c_1 + 2c^*_1c_1 - (d-1)c^*_{d-1}c_{d-1} + \sum_{i=2}^{d-2} c^*_i c_i && \text{(A.13)} \\ &= c_1c^*_1 + \sum_{i=1}^{d-2} c^*_i c_i - (d-1)c^*_{d-1}c_{d-1} && \text{(A.14)} \\ &= c_1c^*_1 + \sum_{i=1}^{d_1} c^*_i c_i - dc^*_{d-1}c^*_{d-1} && \text{(A.15)} \end{aligned}$$

Which equals to the right hand side since the first two terms sum up to identity as a result of (1.1). We showed that one can obtain $\mathcal{A}_d$ using the generators of $\mathcal{F}_{d-1}$ and showed that they are isomorphic. The inverse of this process requires much more work. It is done in detail in one of our references[3] and we will not review it here. The generators of $\mathcal{F}_{d-1}$ can be rewritten using e_{ij} :

$$c_k = e_{1k+1}, \quad c_k^* = e_{k+11}, \quad k \ \in \ \{1,2,...,d-1\}. \tag{A.16}$$

We will obtain the orthofermion algebra using the definition of the operators above and show that they are isomorphic. To achieve this we will satisfy the equations in (1.1) in the order they are presented. The first equation yields:

$$\begin{aligned}
e_{1i+1}e_{j+11} + \delta_{ij}\sum_{k=1}^{d-1} e_{k+11}e_{1k+1} &= \delta_{ij} && \text{(A.17)}\\
e_{11}\delta_{ij} + \delta_{ij}\sum_{k=1}^{d-1} e_{k+1k+1} &= \delta_{ij} && \text{(A.18)}\\
e_{11} + \sum_{k=2}^{d} e_{kk} &= 1 && \text{(A.19)}\\
\sum_{k=1}^{d} e_{kk} &= 1 && \text{(A.20)}
\end{aligned}$$

We need to show that the resulting equation is an expression of identity:

$$(\sum_{k=1}^{d} e_{kk})e_{ij} = \sum_{k=1}^{d} e_{kj}\delta_{ki} = e_{ij} \tag{A.21}$$

$$e_{ij}(\sum_{k=1}^{d} e_{kk}) = \sum_{k=1}^{d} e_{ik}\delta_{jk} = e_{ij}. \tag{A.22}$$

The other relations of (1.1) yield:

$$e_{1i+1}e_{1j+1} = e_{1j+1} = \delta_{i+11} = 0, \quad e_{i+11}e_{j+11} = e_{i+11}\delta_{1j+1} = 0 \tag{A.23}$$

which are evident since i and j cannot take the value 0. Hence we showed that $\mathcal{F}_{d-1}$ and M_d are isomorphic. The generators of M_d are expressed in terms of the generators of $\mathcal{F}_{d-1}$ as:

$$e_{11} = 1 - \sum_{k=1}^{d-1} c_k^* c_k \qquad e_{1i+1} = c_i,$$

$$e_{i+11} = c_i^*, \quad e_{i+1j+1} = c_i^* c_j, \quad i, j \in \{1, 2, ..., d-1\}. \tag{A.24}$$

One has to write sixteen equations in order to satisfy (1.4) using the above expressions. We will not show it here as we do not need to do it. Now we will show that M_d and $\mathcal{A}_d$ are isomorphic. To achieve this goal we will use the definition which is used to construct generalized parafermions[6]:

$$a = \sum_{k=1}^{d-1} \sqrt{k} \;\; e_{kk+1}, a^* = \sum_{k=1}^{d-1} \sqrt{k} \;\; e_{k+1k} \tag{A.25}$$

to satisfy (1.8). For the left hand side of (1.8)

$$\begin{aligned} aa^* - a^*a &= \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \sqrt{ij} \;\; e_{ii+1}\; e_{j+1j} - \sum_{k=1}^{d-1}\sum_{l=1}^{d-1} \sqrt{kl} \;\; e_{k+1k}\; e_{ll+1} && \text{(A.26)} \\ &= \sum_{i=1}^{d-1} i \; e_{ii} - \sum_{k=1}^{d-1} k \; e_{k+1k+1} && \text{(A.27)} \\ &= \sum_{i=1}^{d-1} \; e_{ii} - (d-1)\; e_{dd} && \text{(A.28)} \\ &= \sum_{i=1}^{d} \; e_{ii} - d \; e_{dd} && \text{(A.29)} \\ &= 1 - d\; e_{dd}. && \text{(A.30)} \end{aligned}$$

For the right hand side we start with a^2:

$$\begin{aligned} a^2 &= \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \sqrt{ij} \;\; e_{ii+1}\; e_{jj+1} && \text{(A.31)} \\ &= \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \sqrt{ij} \;\; e_{ij+1}\; \delta_{i+1j} && \text{(A.32)} \\ &= \sum_{i=1}^{d-2} i(i+1)\; e_{ii+1} && \text{(A.33)} \end{aligned}$$

It can be seen that δ_{i+1j} cancels the term with $i = d-1$. One can obtain a^{d-1} applying the same procedure $d-1$ times:

$$a^{d-1} = \sum_{i=1}^{d-(d-1)} \sum_{j=1}^{d-1} \sqrt{i(i+1)(i+2)...(i+d-2)(i+(d-1))} \;\; e_{i(d-1)+1} \quad \text{(A.34)}$$

$$= \sqrt{(d-1)!} \;\; e_{1d}. \quad \text{(A.35)}$$

By the same process $a^{*d-1} = \sqrt{(d-1)!} \;\; e_{d1}$. Thus the right hand side becomes

$$1 - a^{*d-1}a^{d-1} = 1 - \frac{d}{(d-1)!}\sqrt{(d-1)!} \;\; e_{d-11}\sqrt{(d-1)!} \;\; e_{1d-1} \quad \text{(A.36)}$$

$$= 1 - de_{dd} \quad \text{(A.37)}$$

which is equal to the left hand side. Hence we showed that M_d and $\mathcal{A}_d$ are isomorphic. The generators of M_d can be expressed using the generators of $\mathcal{A}_d$ as

$$e_{ij} = \frac{1}{(d-1)!}\sqrt{\frac{(i-1)!}{(j-1)!}} \; a^{d-i} \;\; a^{*d-1} \;\; a^{j-1}. \quad \text{(A.38)}$$

We used this formula to show that the algebras $\mathcal{A}_d$ and $\mathcal{A}_{d_1} \otimes \mathcal{A}_{d_2}$ are isomorphic.

APPENDIX B: $\mathcal{A}_2 \otimes \mathcal{A}_3$

We can examine (2.2) by taking $d_1 = 2$, $d_2 = 3$ and thus $d = 6$ for which

$$\begin{aligned} a &= \sum_{i=1}^{2}\sum_{j=1}^{2}\sqrt{j+3(i-1)}\; e_{ii}\otimes e_{jj+1} + \sum_{k=1}^{1}\sqrt{k3}\; e_{kk+1}\otimes e_{31} \\ &= e_{11}\otimes e_{12} + \sqrt{2}e_{11}\otimes e_{23} + \sqrt{4}e_{22}\otimes e_{12} + \sqrt{5}e_{22}\otimes e_{23} + \sqrt{3}e_{12}\otimes e_{31}. \end{aligned} \tag{B.1}$$

The expressions for $a^*, aa^*,\ a^*a,\ a^5, a^{*5}$ are

$$a^* = e_{11}\otimes e_{21} + \sqrt{2}e_{11}\otimes e_{32} + \sqrt{4}e_{22}\otimes e_{21} + \sqrt{5}e_{22}\otimes e_{32} + \sqrt{3}e_{21}\otimes e_{13},$$

$$aa^* = e_{11}\otimes e_{11} + 2e_{11}\otimes e_{22} + 4e_{22}\otimes e_{11} + 5e_{22}\otimes e_{22} + 3e_{11}\otimes e_{33},$$

$$a^*a = e_{11}\otimes e_{22} + 2e_{11}\otimes e_{33} + 4e_{22}\otimes e_{22} + 5e_{22}\otimes e_{33} + 3e_{22}\otimes e_{11},$$

$$a^5 = \sqrt{120}e_{12}\otimes e_{13},$$

$$a^{*5} = \sqrt{120}e_{21}\otimes e_{31}.$$

It can be seen that a^6 and a^{*6} give zero from the expressions. The right hand side of (1.8) is

$$1 - \frac{6}{5!}a^{*5}a^5 \;=\; 1 - 6e_{22}\otimes e_{33}$$

$$= e_{11} \otimes e_{11} + e_{11} \otimes e_{22} + e_{11} \otimes e_{33} + e_{22} \otimes e_{11} + e_{22} \otimes e_{22} - 5e_{22} \otimes e_{33}.$$

The left hand side becomes

$$aa^* - a^*a = e_{11} \otimes e_{11} + e_{11} \otimes e_{22} + e_{11} \otimes e_{33} + e_{22} \otimes e_{11} + e_{22} \otimes e_{22} - 5e_{22} \otimes e_{33}$$

which is identical with the right hand side.

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