UNRAVELING IN TWO SIDED MATCHING MARKETS

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UNRAVELING IN TWO SIDED MATCHING MARKETS

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Thesis Abstract

Umut Mert Dur, "Unraveling in Two Sided Matching Markets"

This thesis aims to investigate the role of the restriction on workers preferences, that not all firms remain acceptable when all information about market is clear, on unravelling in two-sided matching markets. The model builds upon Halaburda (2007) by introducing an important modification. In this paper, some of the firms are not considered as acceptable in the second round. In the literature risk aversion, similarity of preference and costly search are revealed as the possible reasons of unravelling. Even if these reasons are not present, I have found that restriction on preferences of workers can lead to unravelling in some markets.

Tez Özeti

Umut Mert Dur, "Çift Taraflı Piyasalarda Erken Eşleşme"

Bu tez çalışanların firmalar üzerindeki tercihlerinde, her firmanın bütün bilgiler elde edildiğinde tercih edilememesi kısıtının çift taraflı piyasalarda erken eşleşme üzerindeki rolünü incelemeyi amaçlamaktadır. Ele alınan piyasa, Halaburda (2007) makalesinde kullanılan modelin üzerinde önemli bir değişiklik yapılarak modellenmiştir. Bu tezde, fimalardan bazıları ikinci periyotta tercih edilebilir olarak alınmıyor. Literatürde riskten kaçınma, tercihlerin benzerliği, masraflı araştırma erken eşleşmenin olası sebepleri olarak gösterilmiştir. Bu çalışmada yukarıdaki sebepler piyasada bulunmasa bile, çalışanların tercihlerindeki kısıtın erken eşleşmeye neden olacağını gösteriyorum.

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I dedicate this Master's thesis of mine to my family.

CONTENTS

CHAPTER 1

INTRODUCTION

Timing of hiring is very crucial in labor markets, where both firms and workers aim to maximize their utilities. To attain this goal, sometimes both firms and workers contract before the full information about both sides is available. However, early contracting with incomplete information about the characteristics of the agents may cause inefficient assignments (Roth and Xing, 1994) and reduces the mobility in the labor market (Niederle and Roth, 2003). Despite inefficient assignments, early contracting occurs in some markets, especially in entry-level labor markets, such as for graduating law students and gastroenterologists (Roth, 1991), NBA drafts (Groothuis, Hill and Perri, 2007). Moreover, this phenomena can be observed in other markets: Some colleges in the U.S. have early admission programs (Mumcu and Saglam, 2007), post-season college football bowl selections (Frechette, Roth and Unver, 2007) and soccer transfer market. This phenomena, called unravelling, can be defined as a case in which contracting occurs before the full information about both sides of the market is available.

The existing literature studied the effect of stability (Roth, 1991), worker's risk aversion (Li and Suen, 2000 and Suen, 2000), costly search (Damiano, Li and Suen, 2005) and similarity of preferences (Halaburda, 2007) on unravelling. This paper, motivated from the early contracting observed in the European soccer transfer market, studies the effect of elimination resulting from preliminary knockout qualifying rounds on unravelling. In the 2008-2009 season, 122 out of 131 transfers of participating teams from Belgium, Bulgaria, Greece, Romania and Turkey were realized before the preliminary knockout qualifying rounds.

Halaburda (2007) investigates the effect of similarity of preferences in two sided matching markets when workers' ranking over firms are identical, while the firms' ranking (preferences) over workers can vary between identical and independent. She concludes that in a two period model in which all unmatched agents can participate in the second period, unravelling cannot be observed if the firms' ranking over workers are independent, and unravelling becomes more likely as firms' ranking over workers grow more similar. However none of the points mentioned in the literature completely explains the reasons of unravelling observed in some markets like European soccer transfer market. My paper builds upon Halaburda (2007) and I focus on the existence of unraveling in an economy where none of the points mentioned in the literature is present.

In my paper, two sided matching market is composed of firms on the one side and workers on the other side. Firms and workers correspond to teams and players in the European soccer transfer market, respectively. Both agents are assumed to be risk neutral. Workers and firms have preferences over agents on the other side of the market. All workers have identical preferences over the firms, whereas firms are allowed to have independent preferences over workers. To examine the unravelling, the model consists of two periods. It is allowed to contract in both periods. In the first period workers' preferences are known by all agents. In the second period, firms learn their own preferences privately and workers realize which firm is worth contracting.

Re-entry is banned so if the agents contract in the first period, they have to leave the market. The remaining workers and remaining firms who are considered worth contracting by workers are matched by a mechanism. Unravelling is observed when a firm makes an offer to a worker in the first period and the offer is accepted by that worker. This can be reasonable when the expected payoff of matching in the first period is greater than the expected payoff of second period mechanism for both sides of the market. Like Halaburda (2007) there exists an *ex-post* stable mechanism in the second period. In this mechanism, the most preferred firm among the participants of second period is matched with its first choice and the next-most preferred firm is matched with its most-preferred worker among the remaining workers.

I introduce a major modification to Halaburda (2007). In this paper, some of the firms can be unacceptable for the workers in the second period although they were acceptable in the first period.[1] In Halaburda (2007) all the remaining agents who have not contracted in the first period can participate in the second period. However, as all information about the firms is available, workers may consider some firms are not worth for matching with in the second period. Workers consider these firms as having a value equal to zero. So in my model all the workers who have not matched with a firm in the first period could enter to the second period but not all unmatched firms could participate in the second period.

As a consequence of the modification explained above, this paper and Halaburda (2007) differentiate in the following way. When there is no restriction on workers

[1] Departing from truncation of the preference list, in this paper there is no restriction that the acceptable firms in the second period are more prefered firms in the first period.

preferences in the second period as in Halaburda (2007), workers who receive an offer in the first period have to compare the utility gained from that offer to the utility gained from participating in *ex-post* stable matching mechanism in the second period. If there exists restriction on workers preferences, however, workers also face a zero utility if they accept the offer of a firm who turns out not to be acceptable in the second period. Without restriction, firms offer early contract if the average matching values of the workers is greater than the expected utility gained from the ex-post stable matching mechanism. With restriction, they have to consider the possibility of not participating in the second period or the chance to be matched with better workers if they participate.

This paper indicates that one of the reasons behind the unravelling observed in the two-sided matching market is the existence of the restriction on the workers preferences that not all firms remain acceptable for the workers when full information about the market is clear. Intuitively restriction brings the risk of being unmatched for the firms and may lead all firms to offer an early contract to insure against this risk if it dominates the chance to be matched with better workers in the second period. This can be observed in the European soccer transfer market where 93% of transfers in 2008-2009 season were done before the knock-out stage.

The next chapter introduces some basic knowledge about transfer markets. The following chapter is the literature review. Chapter four presents the model. In chapter five and six results are introduced. The last chapter discusses the result of the analysis.

CHAPTER 2

BACKGROUND

In this section I will give brief information about the soccer transfer market and European Cups including the qualification criteria and competition format. I will also emphasize the achievement of soccer clubs who started to the compete from the preliminary qualifying rounds.

The soccer transfer market is composed of soccer teams on the one side and players on the other. For the scope of this paper only the teams playing in the European Cups are considered. Some teams have to participate in the preliminary knockout qualifying rounds before the group stages. The group stages are a financial source for teams and opportunity for players to be recognized worldwide. These teams offer contracts to international players with limited information of their performance in the previous season and do not have the chance to test their performance before the preliminary knockout qualifying rounds. However if the teams prefer not to offer a contract before the preliminary knockout qualifying rounds and fail to succeed in the elimination they will not be considered as acceptable by the players. International players who consider this opportunity may sign a contract with these teams before the preliminary knockout qualifying rounds. On the other hand, players risk their chance to play in European Cups if they sign a contract before the preliminary knockout qualifying rounds, since if the team that the player contracts with fails he cannot play in another team which has qualified for the further stages in European Cups.

Every year the UEFA (Union of European Football Associations) organizes two international soccer tournaments; the UEFA Champions League and the UEFA Cup. The UEFA Champions League is the UEFA's most prestigious club competition. Formerly, it was known as the European Champion Clubs' Cup. In 1992 its format and name were changed. The UEFA Champions League has several stages. It starts with three preliminary knockout qualifying rounds. A group stage, with thirty-two teams and eight groups, follows the preliminary knockout qualifying rounds. The first two teams in every group play in the first knockout round. After the knockout round quarter-final, the semi-final and the final is played by the successful teams of the previous rounds. Losing teams in the third preliminary knockout qualifying round enter the first round of the UEFA Cup, and teams who placed third in the group stage enter the third round of the UEFA Cup.

Every country has different quotas according to their place in the UEFA coefficient ranking list (a ranking list based on the five year performance of national teams and achievements of soccer clubs in the international arena). For example Spain, England, and Italy will have four teams in the 2008-2009 competition whereas France, Germany, and Portugal will have three teams.

Different teams start in different rounds, according to their position in the domestic league and the UEFA coefficients of their league. Twenty-eight champions from countries twenty-five to fifty-three will play in the first preliminary knockout qualifying round of 2008-2009 competition. Fourteen winners from the First Qualifying Round, eight champions from countries seventeen to twenty-four and six runners-up from countries ten to fifteen will play in the second preliminary knockout qualifying round. Fourteen winners from the Second Qualifying Round, six champions from countries eleven to sixteen, three runners-up from countries seven to nine, six third-place finishers from countries one to six and three fourth-place finishers from countries one to three will play in third preliminary knockout qualifying round. Sixteen winners from the Third Qualifying Round, ten champions from countries one to ten, six runners-up from countries one to six will play in group stage.

The UEFA Cup is the other international competition for European soccer clubs. Like in the Champions League, teams qualify for the UEFA Cup based on their performance in national leagues and cup competitions. Today's format was first adopted for the 2004-05 season. The UEFA Cup starts with two knockout qualifying rounds. Eighteen and lower ranked countries enter the first qualifying round. In addition, three places in the first qualifying round are reserved for the Fair Play winners. Teams from countries nine to eighteen, survivors of the first qualifying round and eleven UEFA Intertoto Cup winners play in the second qualifying round. Winners of the qualifying rounds join teams from the countries ranked one to thirteen in the first round proper. In addition, losers in the third qualifying round of the Champions League also enter this round, and another place is reserved for the title-holders. Fourty survivors of first round plays in eight groups. The top three teams in each group and the teams who are ranked in third place in the champions league qualify for the round of thirty-two. After the round of thirty-two, the round of sixteen, the quarter-final, semi-final and final is played by the successful teams of the previous rounds.

For the scope of this paper I mainly focus on the teams of the countries ranked below seven. The best seven leagues are; Spain, Italy, England, Germany, France, the Netherlands and Portugal. Also the top twenty teams of Europe belong to these countries (five teams from Spain and England, three teams from Italy, two teams from Germany, Portugal and the Netherlands and one team from France). Teams of the countries ranked below seven have many outstanding successes in both the UEFA Champions League and the UEFA Cup even if they have started from the early qualifying rounds. These successes lead these teams to be more attractive to many international players. In 2008 the winner of the UEFA Cup, FC Zenit Saint Petersburg joined the competition from the second qualifying round where the other runner-up Rangers joined the competition from the second qualifying round of the Champions League. CSKA Moscow (2005-2006) and Galatasaray(1999-2000) are other UEFA cup winners who belong to countries ranked below seven. In the champions league Galatasaray (2000-2001), Panathinaikos (2001-2002) and Fenerbahce (2007-2008) played in the Champions league quarter-final.

It is not surprising that the players of the successful teams are transferred to the most elite leagues (England, Spain, Italy, Germany, Portugal, France and the Netherlands). This can be considered as the opportunity that players are seeking. For example, after the success of Artmedia (Slovakian team) in the Champions league, who came in third in the group stage after starting from the first qualifying round, its players were transferred to Russian, English and Italian teams.

CHAPTER 3

LITERATURE REVIEW

In this section, I discuss the findings of previous literature on stability (Roth, 1991), worker's risk aversion (Li and Suen, 2000 and Suen, 2000) and costly search (Damiano, Li and Suen, 2005) as a cause of unravelling. According to stability hypothesis implementation of ex-post stable matching after the arrival of new information precludes early contracting. Stability hypothesis can be seen as an obstacle for the unravelling. Stability hypothesis is introduced and discussed in Roth (1991). Roth (1984) proved that the algorithm employed in the American medical internship market produces a stable matching and this centralized matching mechanism prevents unraveling. Roth (1991) analyzes the entry-level labor markets of the newly graduated medical students in the United Kingdom as a unique natural experiment for the validity of stability hypothesis markets. In the 1960s centralized matching procedures were introduced in the United Kingdom for preventing early contracting in the entry-level labor markets of the newly graduated medical students. Different regions used different algorithms. Roth (1991) focused on the fact that only a few of the algorithms have survived to date and the rest of the algorithms could not succeed in preventing unraveling. Roth (1991) analyzes the common features of the algorithms which have survived and those which failed to survive. In this paper it is concluded that the algorithms which have survived lead to ex-post stable matching. Moreover, after the unsuccessful algorithms were changed by the stable matching algorithms the existence of unraveling was prevented. As a result, Roth (1991) argues that unraveling can be observed in a market if the centralized matching mechanism produce unstable matchings. However the gastroenterology market in the US, with clearinghouse mechanism producing ex-post stable matching, failed to prevent unravelling (Niederle and Roth, 2003).

According to the view of risk averse workers, early contracting provides insurance for the workers. Workers prefer to accept the offer in the first period in order to avoid unemployment. It is mentioned that, if the market is composed of non-risk averse workers unravelling cannot be observed. The risk aversion characteristic of workers is showed as the cause of unravelling by Li and Suen (2000) and Suen (2000). Li and Suen (2000) analyzes the assignment markets where money is not used in the contracts. Departing from the assignment models with complementarities used in Koopmans and Beckmann (1957) the abilities of workers and firms are unknown at the beginning of the game. Li and Suen (2000) focus on the role of this uncertainty on the contracting of the workers and firms before the uncertainty is resolved in order to reduce the risk being unmatched. Li and Suen (2000) conclude that in a competitive framework, risk aversion of workers is needed for early contracting and workers have to be more risk averse than the firms. As a consequence of risk aversion, insurance gained by early contracting easily dominates the outcome of efficient matching which is done after the resolution of the uncertainty.

Suen (2000) introduces a similar model to Li and Suen (2000) while there exists only uncertainty on the abilities of workers. In this model when both sides are risk neutral an equilibrium involving unraveling cannot be observed. Moreover only the mediocere firms offer early contract before the abilities of the workers are revealed. As a result only the top and low ranked firms participate in the stable ex-post matching mechanism. This distribution of firms benefit the firms and hurt the workers. Suen (2000) argues that a risk averse worker unravels in order to prohibit the loss due to the distribution of firms in period 2. However Halaburda (2007) shows that unraveling can be observed when the workers are risk neutral.

According to the costly search view, when the payoff values of the workers are not significantly different, firms do not want to spend resources on learning the characteristics of workers and differentiate between them. Damiano, Li and Suen (2005) consider costly search as a key to unravelling. In this article an assignment market with complementarity is examined in two period model. This market consists of heterogeneous types of workers and firms. The quality of the pool of workers and firms changes as an unraveling occurs in the first period and agents leave the market. Damiano, Li and Suen (2005) show that when search is costless an equilibrium involving unraveling is not observed. However, when an arbitrarily small cost exists in order to participate in the rounds unraveling can be observed and firms match with the first worker they meet. However, Halaburda (2007) shows that unravelling can be observed in an economy composed of risk neutral workers when the search is costless.

In this thesis I aim to show that similarity of preferences, which is the cause of unravelling according to Halaburda (2007), is not needed to observe an equilibrium involving unraveling.

CHAPTER 4

THE MODEL

The basic model draws on Halaburda (2007) which consists of a two period game in a labor market. The market is composed of non-empty, finite, disjoint sets of m firms and n workers, $F = \{f_1, f_2, ..., f_m\}$ and $W = \{w_1, w_2, ..., w_n\}$, respectively. It is assumed that the number of workers is greater than or equal to the number of firms and there are at least two firms in the market, $n \geq m \geq 2$. Each firm has exactly one position to fill, and each worker can take at most one job. Firms and workers can early contract in the first period. Those who contract in the first period leave the market. Remaining agents are matched in the second period by a stable mechanism.

The list of preference relations of the agents is represented as $R = (R_{f_1}, ..., R_{f_m}, R^1_{w_1}, ..., R^1_{w_n}, R^2_{w_1}, ..., R^2_{w_n})$ where R_f is the preference relation of firm f, R^1_w is the preference relation of worker w in period 1 and R^2_w is the preference relation of worker w in period 2. I distinguish between workers' first and second period preferences to take into account the possibility of updating preference in the second period with the arrival of new information.[2]

For any firm $f \in F$, R_f is a binary preference relation which is a linear order on $\Sigma_f = \{\{w_1\}, \{w_2\}, ..., \{w_n\}\}$.[3] Firms find all workers acceptable. In a similar manner, in the first period R^1_w is a binary preference relation which is a linear order on

[2]For simplicity this informational incompleteness is only imposed on workers preferences. In a more general model one could have firms' preferences in both periods to possibly be different.

[3]A binary relation R on Σ is linear as long as it is reflexive (xRx for all $x \in \Sigma$), transitive (for all $x, x', x'' \in \Sigma$ if xRx' and $x'Rx''$ then xRx'') and complete (for all $x, x' \in \Sigma$ with $x \neq x'$ it is either xRx' or $x'Rx$ buth not both).

$\Sigma_w^1 = \{\{f_1\}, \{f_2\}, ..., \{f_m\}\}$ for any $w \in W$. Likewise, in the first period workers find all firms acceptable. In the second period, R_w^2 is a restriction of R_w^1 to Σ_w^2 which is a subset of Σ_w^1, $\Sigma_w^2 \subseteq \Sigma_w^1$. For any firm f and f' in Σ_w^2 if f is preferred to f' in the first period than it is also preferred to f' in the second period. Let the number of acceptable firms in the second period be denoted by $|\Sigma_w^2| = m'$ where $m' \leq m$.[4] Being unassigned is represented by $\emptyset$.[5]

Let the set of all preference relations for firm f and worker w be R_f and $R_w = (R_w^1, R_w^2)$, respectively. Define also $R = \times_{k \in F \cup W} R_k$. Let P_k denote the strict preference relation for the R_k for all $k \in F \cup W$ with $P_w = (P_w^1, P_w^2)$. Then, worker w is acceptable to firm f if he is strictly preferred to $\emptyset$, $w P_f \{\emptyset\}$. Also firm f is acceptable to worker w in period $t \in \{1, 2\}$ if he is strictly preferred to $\emptyset$, $f P_w^t \{\emptyset\}$. I write $f_1 P_w^2 f_2 P_w^2 \emptyset$ to denote f_1 is preferred to f_2, f_2 is preferred to being unmatched and no other firm is acceptable in the second period for the worker w. Note that only m' firms among m firms are acceptable for the workers in the second period.

For any finite set X, any linear order $\hat{R}$ over X, let $Top(X; l)$ denote the l^{th} ranked element of X under the preference order $\hat{R}$, where l is a positive integer number and satisfies $l \leq |X|$.[6] For instance, firm f's top ranked worker is represented as $Top(\Sigma_f; 1)$ for any linear order R_f over Σ_f.

It is assumed that workers have identical preferences over the firms, i.e. for any

[4] Note that m' is common to all workers.

[5] An example will be helpful at this point. Let there be 5 firms in the market and workers consider all firms acceptable in the first period, $\Sigma_w^1 = \{\{f_1\}, \{f_2\}, \{f_3\}, \{f_4\}, \{f_5\}\}$. Only 3 of the firms will be acceptable in the second round, $\Sigma_w^2 = \{\{f_2\}, \{f_3\}, \{f_5\}\}$. If the the preferences of the workers in the first period is given as $f_1 \succ_w f_2 \succ_w f_3 \succ_w f_4 \succ_w f_5 \succ_w \emptyset$ than the preferences of the workers in the second period will be $f_2 \succ_w f_3 \succ_w f_5 \succ_w \emptyset$, with $\emptyset \succ_w f_1$ and $\emptyset \succ_w f_4$.

[6] For the rankings of the agents I used a pointer function like Mumcu and Saglam (2007).

$w \in W$, $w' \in W$, $R_w^1 \equiv R_{w'}^1$ and $R_w^2 \equiv R_{w'}^2$ always hold. However firms may have identical or independent preferences over workers. In the first period there exist $m!$ possible combinations of workers' ranking over firms. It is assumed that all firms have an equal chance to be in the preference lists of the workers in the second period. Let the probability of any firm f to participate in the second period be denoted by $\alpha = \frac{m'}{m}$. There are $n!$ possible combinations of firms' ranking over workers. The ranking for any firm f is drawn from a joint distribution over R_f. When all the firms have identical preference, every ranking in R_f is drawn with equal probability of $\frac{1}{n!}$ and for any $f \in F$, $f' \in F$, $R_f \equiv R_{f'}$ is always true. If the preferences of firms are independent then there are $(n!)^m$ possible combination of firms' ranking over workers and any firm f's ranking is drawn independently from the rest of the firms. That is the probability of all firms having identical ranking equal to $(1/n!)^m$. Let R^{in} denote the preference relations when firms have independent preferences and R^{id} denote the preference relations when firms have identical preferences.

For any linear order R_f over Σ_f firm f gains utility v_j by matching with the worker at $Top(\Sigma_f; j)$, where $j \leq n$. Being unassigned provides utility zero. Since all the workers in Σ_f are acceptable, $v_j > 0$ for all $j \in \{1, 2, ..., n\}$. The matching value vector of firms, which is defined as $v = [v_1, v_2, ..., v_{n-1}, v_n]$ is commonly known at the beginning of the game. I assume that for any $j \in \{1, 2, ..., n-1\}$, the value of matching with a j^{th} ranked worker is greater than the value of matching with a $(j+1)^{th}$ ranked worker, i.e.,

$$v_1 > v_2 > ... > v_{n-1} > v_n > 0. \tag{1}$$

Similarly, for any linear order R_w^2 over Σ_w^2 worker w gains utility u_k by matching with the firm at $Top(\Sigma_w^2; k)$, where $k \leq m$. Like for firms, being unassigned provides utility zero. All firms in Σ_w^2 are considered acceptable and having a value greater than zero, i.e. $u_k > 0$ for all $k \in \{1, 2, ..., m'\}$ and $u_k = 0$ for all $k \in \{m', ..., m\}$. The matching value vector of workers, which is defined as $u = [u_1, u_2, ..., u_{m'-1}, u_{m'}]$ is commonly known at the beginning of the game. I assume for any
$k \in \{1, 2, ..., m' - 1\}$, the value of matching with a k^{th} ranked firm is greater than the value of matching with a $(k + 1)^{th}$ ranked firm:

$$u_1 > u_2 > ... > u_{m'-1} > u_{m'} > 0. \tag{2}$$

The matching market is described as (F, W, u, v, R, m'). Here F and W are the set of firms and workers who participate in the first period respectively. Matching value vectors of workers and firms are denoted by u and v respectively. The preference profile of all agents is represented by R and the number of firms which are considered acceptable in the second period is m'.

Definition 1. Matching between F and W is defined as a function $\mu : F \rightarrow W \cup \{\emptyset\}$ which assigns a unique worker to a firm. That is, for any two different firms f and f', where $f \in F$ and $f' \in F$, either $\mu(f) \neq \mu(f')$ or $\mu(f) = \mu(f') = \emptyset$ is true. If $\mu(f) = \emptyset$ then firm f is unmatched. If $\mu(f) = w$, where $w \in W$, then firm f is matched with worker w.

The game is divided into two periods, period one and period two. At the beginning, the number of firms (m), the number of workers (n), the utility value vectors (u, v) and workers' preferences in the first period (R_w^1) and the number of firms which can participate in the second period, m', are commonly known. However firms do not know their own ranking of workers. In this period workers find all firms acceptable. Firms decide whether to offer an early contract or not in the first period. The firms which decide to offer simultaneously make early offers. Each firm can offer a contract to only one worker. Workers, who receive an offer, can either accept or reject the offer. Upon acceptance of the offer, worker and firm is matched and they leave the market.

At the beginning of the second period new information arrives which firms are found to be acceptable in the second period. Also at the beginning of the second period each firm learns its own ranking of workers as its own private information. The unmatched workers and unmatched firms among acceptable ones participate in the period two market. A stable matching mechanism is applied to the agents who participate in the second period.

According to Gale and Shapley (1962) a matching is ex-post stable if none of the firms and workers prefer to be matched with another agent than their current mates. Roth and Sotomayor (1990) showed that if workers have identical ranking there exists a unique stable matching outcome which is obtained by the firm's serial dictatorship. Since, in this model, workers have identical preferences, the existence of unique stable matching outcome is guaranteed. The firm's serial dictatorship works as follows: the top ranked firm according to the workers second period preferences, R^2_w, is assigned her top choice, the firm ranked second according to R^2_w is assigned his top choice among the remaining workers, and so on. The serial dictatorship is also used in Balinski and Sonmez (1999) and Abdulkadiroglu and Sonmez (1998).

All agents make strategic decisions only in the first period of the game. Firms decide to offer an early contract to a worker or not and workers either accept one of the offers or reject all, if they receive any. The strategy of firm f is $\sigma_f \in W \cup \emptyset$ and σ_{-f} represents the strategy profiles of all agents except firm f.[7] The strategy of the worker w depends on whether he/she has received an early contract or not. Let $\Omega_w \subseteq F$ be a set of firms that make early contract offers to worker w. Note that Ω_w can be empty. The strategy of worker w is $\sigma_w(\Omega_w) \in \Omega_w \cup \emptyset$ and σ_{-w} represents the strategy profiles of all agents except worker w. Let σ_F and σ_W denote the strategy profile of all firms and all workers, respectively. Let $\sigma = (\sigma_F, \sigma_W)$ be the strategy profiles of all agents.

Firm f believes that other agents will play σ_{-f} with probability $\beta_f(\sigma_{-f})$ where $\beta_f(\sigma_{-f}) \in [0, 1]$. Any worker w believes that other agents will play σ_{-w} with probability equals to $\beta_w(\sigma_{-w})$ where $\beta_w(\sigma_{-w}) \in [0, 1]$. Let β denote the beliefs of all agents.

The expected utility of the firm f by playing a strategy based on its beliefs about the rest of the agents' strategies is $EU_f(\sigma_f | \beta_f(\sigma_{-f}))$ and the expected utility of

[7] Definitions of strategies and beliefs are adopted from Halaburda (2007).

worker w by playing a strategy based on its beliefs about the rest of the agents' strategies and its offer set is $EU_w(\sigma_w|\Omega_w, \beta_w(\sigma_{-w}))$.

For strategies to be sequentially rational every worker and firm play strategy σ^* which maximizes their expected utilities given their information sets. A sequential equilibrium is formed by strategies, σ^*, and beliefs, β^*, when strategies are sequentially rational given the belief and beliefs are consistent with the strategies played.

Definition 2. (σ^*, β^*) constitutes a sequential equilibrium such that

1. Every firm f plays σ^*_f as long as

$$EU_f(\sigma^*_f|\beta^*_f(\sigma_{-f})) \geq EU_f(\sigma_f|\beta^*_f(\sigma_{-f})) \tag{3}$$

for all $\sigma_f \in W \cup \emptyset$.

2. Every worker w plays σ^*_w as long as

$$EU_w(\sigma^*_w|\Omega_w, \beta^*_w(\sigma_{-f})) \geq EU_w(\sigma_w|\Omega_w, \beta^*_w(\sigma_{-w})) \tag{4}$$

for all $\sigma_f \in \Omega_w \cup \emptyset$.

3. For the beliefs of the firm f to be consistent it must satisfy the following conditions. For any agent $i \in (W \cup F)/f$,

$$\beta^*_f(\sigma_i) = \begin{cases} 1 & \text{if } \sigma_i = \sigma^*_i \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

4. For the beliefs of the worker w to be consistent it must satisfy the following conditions. For any firm who offers an early contract to w, $f \in \mathbf{\Omega}_w$,

$$\beta^*_w(\sigma_f|\Omega_w) = \begin{cases} 1 & \text{if } \sigma_f = w \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For the other agents, $i \in (W \cup F)/(w \cup \mathbf{\Omega}_w)$,

$$\beta^*_w(\sigma_f|\Omega_w) = \begin{cases} 1 & \text{if } \sigma_f = \sigma^*_f \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Finally, I define unravelling.

Definition 3. Unravelling is a case in which some firms and workers contract in the first period, before firms know their own ranking and workers know the set of acceptable firms in the second period.

For the rest of the paper $Top(\Sigma^t_w; k)$, a k^{th} ranked firm by the worker w in period $t \in \{1, 2\}$, is represented by f^t_k. If R^1_w and R^2_w are the same, $Top(\Sigma^t_w; k)$ is denoted by f^1_k for both periods. When firms have identical preferences $Top(\Sigma_f; k)$ is represented by w_k and $Top(\Sigma_f; k)$ is denoted by $w_{k,f}$ if the preferences of firms are independent.

CHAPTER 5

EQUILIBRIA WITHOUT UNRAVELING

In this section, I study conditions under which an equilibrium without unraveling exists. In particular I analyze whether the uncertainty on the preference of the workers in terms of the set of acceptable firms affects the tendency towards early contracting under the ex-post stable matching mechanism.

In the following subsections I look for the cases when all firms are acceptable at period two, $m = m'$, and some firms are found unacceptable at period two, $m' < m$. When $m = m'$ the model corresponds to the Halaburda (2007). In this case, as Halaburda (2007) I show that in the second period, unravelling cannot be observed if the preference of the firms are independent. However, when $m' < m$, an equilibrium involving unraveling can be observed with firms having independent preferences.

No Restriction on the Preferences of Workers

In this case, as of date one, there is no uncertainty on the preferences of workers and all firms and workers who have not unravelled in the first period participate in the *ex-post* stable matching mechanism in the second period.

As it is assumed, workers have identical preferences over the firms and firms may have identical or independent preferences over the workers.

If both firms and workers have identical preferences over the other side of the market firm f_i^1 will unravel if the number of workers is between the of sum of relative values of workers with respect to the i^{th} ranked worker value and the sum of relative values of firms with respect to the value of i^{th} ranked firm. The most preferred firm never prefers to offer an early contract.[8]

When firms have independent preferences and the number of firms is strictly less than the number of workers, firms do not prefer to deviate from the ex-post stable matching outcome. If $n = m$ only, firm f_m^1 prefers to deviate but its offer is not accepted by any worker. That is, unravelling is not observed when firms have independent preferences over workers. The formal results of this case can be found in the Appendix.

In the next section, I will show that when the preferences of the workers for the second period is restricted, unravelling can be observed when firms have both identical and independent preferences.

Restriction on the Preferences of Workers

If there is an uncertainty on preference of the workers for the second period, not all firms will participate in the ex-post stable matching mechanism in the second period. Only $m' < m$ firms will be acceptable for the workers in the second period.

As a result of the restriction on the preferences of the workers in the second period, the rank of each firm, among the participants of the second period, will be considered.

Let q_i denote the possible worst rank of f_i^1 in the second period preference list, R_w^2, if it is acceptable for the workers. Similarly, let r_i denote the possible best rank of f_i^1 in the second period preference list, R_w^2. The values of r_i and q_i depend on m, m' and i and defined as

[8] See Appendix for details.

$$r_i = \begin{cases} 1 & \text{if } i \leq m - m' \\ i - m + m' & \text{if } m - m' < i \end{cases} \tag{8}$$

$$q_i = \begin{cases} i & \text{if } i < m' \\ m' & \text{if } m' \leq i \end{cases} . \tag{9}$$

For instance, consider an economy with four firms and $m\prime = 2$, (i.e. $\alpha = 0.5$).

Table 1: Possible Ranking of firms in period 2

f_1^1 ranked 1^{st}	f_2^1 ranked 1^{st}	f_3^1 ranked 1^{st}
$f_1^1 P_w f_2^1 P_w \emptyset$	$f_2^1 P_w f_3^1 P_w \emptyset$	$f_3^1 P_w f_4^1 P_w \emptyset$
$f_1^1 P_w f_3^1 P_w \emptyset$	$f_2^1 P_w f_4^1 P_w \emptyset$	
$f_1^1 P_w f_4^1 P_w \emptyset$		

As it can be seen in Table 1 above, firm f_1^1 can be ranked only in the first place, f_2^1 and f_3^1 can be ranked in first or second place and f_4^1 can be ranked in second place, if they are acceptable for the workers in the second period. Therefore $r_1 = q_1 = 1$, $r_2 = 1$, $q_2 = 2$, $r_3 = 1$, $q_3 = 2$ and $r_4 = q_4 = 2$. Moreover, every firm succeeds in being in the second period 3 times which is half of the possible 6 ranking of firms. In the second period, f_1^1 can be only ranked in first place, f_2^1 can be ranked twice at first place and once at second place, f_3^1 can be ranked once at first place and twice at second place, f_4^1 can be ranked only at second place.

Firms with Identical Preferences over Workers

Consider a candidate equilibrium in the first period such that $\sigma^*_f = \{\emptyset\}$ for all $f \in F$, $\sigma^*_w = \{\emptyset\}$ for all $w \in W$ and the equilibrium beliefs are defined as in equations $(3-7)$. Consider a possible deviation for f_i^1. Firm f_i^1 will offer an early contract in the first period if the expected utility gained by unravelling is greater or equal to the utility gained from the outcome of the ex-post stable matching mechanism. This condition is shown below.

$$\frac{1}{n}\sum_{j=1}^{n} \upsilon_j \geq \frac{1}{\binom{m}{m'}} \sum_{j=r_i}^{q_i} \left(\binom{i-1}{j-1} \binom{m-i}{m'-j} \upsilon_j \right) \tag{10}$$

The left hand side of the inequality (10) represents the expected utility gained from unravelling where the number of workers is equal to n. The right hand side of the inequality (10) is the expected utility of f_i^1 gained from ex-post stable matching mechanism and will be denoted as Ψ_i where i is the rank of the firm in the first period. Note that $\binom{m}{m'}$ term is the total number of possible ranking of firms in the second period. The $\binom{i-1}{j-1}$ term is the number of possible ranking such that only $(j-1)$ firms ranked better than f_i^1 in the first period qualify for the second period. The $\binom{m-i}{m'-j}$ term is the number of possible ranking such that only $(m'-j)$ firms ranked worse than f_i^1 qualify for the second period. Thus $\binom{i-1}{j-1}\binom{m-i}{m'-j}$ term is the total number of times such that firm f_i^1 is ranked in j^{th} place among the firms who have succeeded to be qualified for the second period.[9]

[9] In this model we assume that if a worker unravels with a firm which turns out to be unacceptable receive 0 utility in the second period, while the firm enjoys the exact utility of υ_j. However, one could think of a situation where a firm suffers a loss in utility when it finds out that it has matched with a

Note that the left hand side of the equation (10) does not depend on the rank of the firms in the first period and it is the same for all the firms. However Ψ_i depends on the rank of firm f in the first period. Thus strategy of the firms can be compared by looking at Ψ_i. In the following lemma it is shown that all firms prefer to deviate if it is profitable for f_1^1 to offer an early contract.

Lemma 1. In two sided matching market (F, W, u, v, R^{id}, m') if

$$\frac{1}{n}\sum_{j=1}^{n} v_j \geq \alpha v_1 \tag{11}$$

then it is profitable for all firms to offer an early contract in the first period.

Proof: I will first show that Ψ_i is a decreasing function with respect to i by comparing the probabilities of firm f_i^1 and f_{i+1}^1 being ranked in the j^{th} place in the second period. In the second period the probability of firm f_i^1 to be ranked in the j^{th} place for $j \in \{r_{i+1}, ..., q_i\}$ is

$$\frac{(m-m')!m'!}{m!}\frac{(i-1)!}{(i-j)!(j-1)!}\frac{(m-i)!}{(m-i-m'+j)!(m'-j)!}. \tag{12}$$

The probability of firm f_{i+1}^1 to be ranked in the j^{th} place for $j \in \{r_{i+1}, ..., q_i\}$ is

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worker which considers it unacceptable. To capture this one could easily introduce a new parameter γ into the model where $\gamma \in [0, 1]$. Here γ may represent percentage loss in utility if the firm unravels with a worker who finds it unacceptable in the second period. In this case the new condition for f_i to offer early contract is: $\frac{\gamma+\alpha(1-\gamma)}{n}\sum_{j=1}^{n} v_j \geq \frac{1}{\binom{m}{m'}}\sum_{j=r_i}^{q_i}\left(\binom{i-1}{j-1}\binom{m-i}{m'-j}v_j\right)$, but introducing γ would not change our results qualitatively.

$$\frac{(m-m')!m'!}{m!}\frac{(i)!}{(i-j+1)!(j-1)!}\frac{(m-i-1)!}{(m-i-1-m'+j)!(m'-j)!}. \tag{13}$$

The ratio of expressions (12) and (13) is

$$\frac{(i+1-j)(m-i)}{i(m-m'-i+j)}. \tag{14}$$

The derivative of this ratio with respect to j is

$$\left(\frac{m-i}{i}\right)\left(\frac{-m+m'-1}{(m-m'-i+j)^2}\right). \tag{15}$$

Expression (15) is always negative for $i \in \{1, ..., m-1\}$. Hence expression (14) is a decreasing function with respect to j. The firms' serial dictatorship allows a j^{th} ranked firm in the second period to match with j^{th} ranked worker in the ex-post stable matching mechanism. Therefore, it is more probable for the firm f_i^1 to be matched with its more preferred workers than f_{i+1}^1 to be matched with those workers.[10] Since $v_j > v_{j+1}$ for $j \in \{1, ..., n-1\}$, Ψ_i is greater than Ψ_{i+1}. This implies that if firm f_i^1 has an incentive to deviate from the equilibrium then any firm whose rank is worse than i will have an incentive to deviate as well. Hence Ψ_i is a decreasing function with respect to i.

Note that Ψ_i takes its largest value when $i = 1$. By letting $i = 1$, the inequality (10) turns into inequality (11) which is the necessary condition for f_1^1 to deviate and offer an early contract in the first period. Thus if inequality (11) is satisfied then it is also

[10]Note that $r_i \leq r_{i+1}$ and $q_i \leq q_{i+1}$. This situation is also true when $r_i \neq r_{i+1}$ and $q_i \neq q_{i+1}$.

profitable for all firms to deviate.

There are two forces that derive the monotonicity of Ψ_i in the model. First, elimination of some firms in the second period is done in such a way that the first period ranking of firms are not disturbed, i.e. the second period rank of a firm will never be worse than its first period rank. Second, the use of firm serial dictatorship as a ex-post stable mechanism in the second period ensures that better ranked firms match with better ranked workers.

Now I study worker w's incentive to deviate from the candidate equilibrium. Worker w will accept the early contract offer of firm f_i^1 if the following condition is satisfied

$$\frac{1}{\binom{m}{m'}}\sum_{j=r_i}^{q_i}\left(\binom{i-1}{j-1}\binom{m-i}{m'-j}u_j\right) \geq \frac{1}{n}\sum_{j=1}^{m'}u_j. \tag{16}$$

The left hand side of the inequality (16) is the expected utility gained from early contracting with firm f_i^1 which can be ranked in the j^{th} place with probability $\frac{\binom{i-1}{j-1}\binom{m-i}{m'-j}}{\binom{m}{m'}}$ in the second period and it will be denoted by Φ_i. The right hand side is the expected utility of matching with j^{th} ranked firm among m' firms in the second period.

The right hand side of the equation (16) does not depend on the rank of the firm who offers an early contract in the first period and it is the same for all firms. However Φ_i depends on the first period rank the of firm that offers an early contract to worker w. Similar to lemma 1 the following lemma shows that any firm's early contract offer will be accepted by the workers if it is profitable for firms to accept the least preferred firm's early contract offer.

Lemma 2. In two sided matching market (F, W, u, v, R^{id}, m') if

$$\alpha u_{m'} \geq \frac{1}{n} \sum_{j=1}^{m'} u_j \tag{17}$$

then it is profitable for any worker w to accept any firm's early contract offer in the first period.

Proof: Proved similar to lemma 1.

As a consequence of lemma 1 and lemma 2 every agent in the market is willing to deviate from ex-post stable matching outcome if both inequality (11) and (17) hold. Let $A' = \sum_{k=1}^{m'} u_k$ and $B = \sum_{j=1}^{n} v_j$ then I have the following proposition.

Proposition 1. In two sided matching market (F, W, u, v, R^{id}, m') if

$$\frac{B}{\alpha v_1} \geq n \geq \frac{A'}{\alpha u_{m'}} \tag{18}$$

then there is no equilibrium without unraveling.

Proof: I have shown in lemma 1 that if inequality (11) holds every firm offers an early contract. Inequality (11) can be rewritten as

$$\frac{B}{\alpha v_1} \geq n. \tag{19}$$

Also in lemma 2 I have proved that if inequality (17) is true every firm's early contract offer is accepted by the workers. I can rewrite inequality (17) as

$$n \geq \frac{A'}{\alpha u_{m'}} \tag{20}$$

Combining inequality (19) and inequality (20) gives inequality (18).

Note that in proposition 1 the condition for all workers and firms to be willing to deviate from the ex-post stable outcome is stated. However this condition can be weakened if I can show a profitable deviation for one firm and one worker. Note that in definition 3 unraveling arises at least there is one firm and one worker that contract in the first period. The necessary and sufficient condition for the existence of an equilibrium involving unraveling is stated in the following proposition.

Proposition 2. In two sided matching market $(F, W, u, \upsilon, R^{id}, m')$, no equilibrium exists without unraveling if only if there exists $i \in \{1, .., m\}$ such that

$$(\Psi_i)^{-1} \sum_{j=1}^{n} \upsilon_k \geq n \geq (\Phi_i)^{-1} \sum_{k=1}^{m'} u_k. \tag{21}$$

Proof: The necessary condition for firm f_i^1 to offer an early contract (inequality 10) can be written as

$$(\Psi_i)^{-1} \sum_{j=1}^{n} \upsilon_k \geq n. \tag{22}$$

For the offer of f_i^1 to be accepted inequality (16) must be true and inequality (16) can be rewritten as

$$n \geq (\Phi_i)^{-1} \sum_{k=1}^{m'} u_k. \tag{23}$$

Combining inequality (22) and inequality (23) gives inequality (21).

In an economy with firms having identical preferences, regardless of the value of α (i.e. $\alpha = 1$ or $\alpha < 1$) I can find a domain of cardinal preferences (value vectors v and u) such that there is no equilibrium without unraveling. While in Halaburda (2007) there is no equilibrium with unraveling that involves the most preferred firm to unravel, in my case there can be such an equilibrium. In the following proposition it is shown that for the most preferred firm, f_1^1, the inequality (21) boils down to inequality (10).

Proposition 3. In two sided matching market (F, W, u, v, R^{id}, m'), the most preferred firm unravels with a worker if and only if

$$(\Psi_1)^{-1} \sum_{j=1}^{n} v_k \geq n.$$

Proof: I want to show that right hand side of the inequality (21) is always true, that is

$$\alpha u_1 \geq \frac{1}{n} \sum_{i=1}^{m'} u_i. \tag{24}$$

The assumption of $u_1 > u_i$ for $i \in \{2, ..., m\}$ leads to

$$m' u_1 > \sum_{i=1}^{m'} u_i. \tag{25}$$

Dividing both sides by m gives

$$\frac{m'}{m}u_1 > \frac{1}{m}\sum_{i=1}^{m'} u_i. \tag{26}$$

Since $n \geq m$ it is true that

$$\frac{1}{m}\sum_{i=1}^{m'} u_i \geq \frac{1}{n}\sum_{i=1}^{m'} u_i. \tag{27}$$

If the two inequality (26) and (27) are combined I get

$$\frac{m'}{m}u_1 > \frac{1}{n}\sum_{i=1}^{m'} u_i. \tag{28}$$

Inequality (28) is the necessary condition for accepting the early contract offer.

According to proposition 3 if it is profitable for firm f_1^1 to offer an early contract it will always be accepted by the worker who receives the offer. In this case existence of unravelling is independent from utility value vector u. However, when $\alpha = 1$ the most preferred firm never offers an early contract and workers strategies on the early contract offer of the other firms always depend on the utility value vector u.

It is worth pointing out that in my model the incentive for a firm to offer an early contract differs from the model of Halaburda (2007). In the latter, f_i^1 will always match with the i^{th} ranked worker and receive the utility v_i in the second period. Hence it will offer an early contract if it expects to get utility higher than v_i. However, in my model, there is no guarantee that firm f_i^1 will match with the i^{th} ranked worker in the second period since it may be eliminated in the second period and get 0 utility or it may match with a better worker and get higher utility, otherwise. Hence it is the trade off between these two effects that drives the incentive of firm f_i^1 to offer an early contract.

Similarly different incentives are at work for workers as well. In the model of Halaburda (2007) workers can match with i^{th} ranked firm when it is ranked in the i^{th} place or be unmatched in ex-post stable matching mechanism. His decision to unravel if he gets an offer from a firm whose value is at least as good as the expected utility that he will get from the ex-post stable matching mechanism. In my model if the worker participates in the ex-post stable matching mechanism, as in Halaburda (2007), he may be either unassingned and receive a utility zero or assigned to a firm with value u_i. However if he unravels with firm f_i^1 he will either get utility at least u_i or zero.

The following examples highlight the difference between the two model for a given economy. Example 1 exhibits an economy where unraveling is observed only when $\alpha \neq 1$.

Example 1. Consider an economy with four firms and six workers. Workers and firms have identical preferences. Workers' ranking of firms is

$$f_1^1 P_w^1 f_2^1 P_w^1 f_3^1 P_w^1 f_4^1 P_w^1 \emptyset$$

for all $w \in W$ and firms' ranking of workers is

$$w_1 P_f w_2 P_f w_3 P_f w_4 P_f w_5 P_f w_6 P_f \emptyset$$

for all $f \in F$. The value vectors are given as $v = [12, 10, 9, 8.7, 5, 4]$ and $u = [10, 8, 7, 6]$. The sum of the values are $B = \sum_{k=1}^{n} v_k = 48.7$ and $A = \sum_{k=1}^{m} u_k = 31$. When $\alpha = 1$ for any of the firms the inequality (21) is not satisfied so they do not unravel. Hence there is no equilibrium with unravelling. In fact there exists a unique equilibrium in the first period such that $\sigma_f^* = \{\emptyset\}$ for all $f \in F$, $\sigma_w^* = \{\emptyset\}$ for all $w \in W$ and the equilibrium beliefs are defined as in equations $(3 - 7)$. Now assume that $\alpha = 0.5$, i.e. in the second period only half of the firms will be considered worth matching. The value vector of firms is updated as $u' = [10, 8, 0, 0]$. The sum of the values are: $B = \sum_{k=1}^{n} v_k = 51.7$ and $A' = \sum_{k=1}^{m'} u_k = 18$. The inequality (18) is satisfied so all firms and workers will prefer to deviate from ex-post stable matching.

Example 2 exhibits an economy where unraveling is observed only when $\alpha = 1$.

Example 2. Example (1) is modified by only changing the value vector v : $v = [11.5, 11, 3, 2.5, 2, 1.5]$. When $\alpha = 1$ without restriction on preferences in the second period, firm f_3^1 and firm f_4^1 prefer early contract and their offer will be accepted since inequality (21) is satisfied for $i = 3$ and $i = 4$. However when $\alpha = 0.5$, the least preferred firm f_4^1 will not offer an early contract since inequality (10) is not satisfied. So none of the firms will offer early contract. Hence there is no equilibrium without unraveling.

Firms with Independent Preferences over Workers

Now I study the case when the preferences of firms are independent. When preferences of firms are independent, there are $(n!)^m$ possible combinations of firms' ranking over workers and any firm f's ranking is drawn independently from the rest of the firms. As in section 5.2.1 consider the same candidate equilibrium in the first period such that $\sigma_f^* = \{\emptyset\}$ for all $f \in F$, $\sigma_w^* = \{\emptyset\}$ for all $w \in W$ and the equilibrium beliefs are defined as in equations $(3-7)$. When the firms' preferences are independent f_i^1 will deviate and offer an early contract if

$$\frac{1}{n}\sum_{j=1}^{n} v_j \geq \frac{1}{\binom{m}{m'}}\sum_{j=r_i}^{q_i}\left(\binom{i-1}{j-1}\binom{m-i}{m'-j}\eta_j\right), \tag{29}$$

where η_j is defined as

$$\eta_j = \sum_{k=1}^{j}\frac{(j-1)!}{(j-k)!}\frac{(n-k)!}{n!}(n-j+1)v_k. \tag{30}$$

The left hand side of inequality (29) is the expected utility that firm f_i^1 gains from unraveling. The right hand side of inequality (29) is the expected utility gained in the ex-post stable matching mechanism and will be denoted by Ψ_i^{in}.

Note that Ψ_i^{in} is similar to Ψ_i. The only difference between Ψ_i and Ψ_i^{in} is that expectation is taken over η_j in calculating Ψ_i^{in}, while it is taken over v_j in calculating Ψ_i. In fact when the firms have identical preferences, due to the firms' serial dictatorship, a j^{th} ranked firm in the second period gets a utility v_j. However when firms have independent preferences firms have a chance to get a higher utility than v_j.

When firms have independent preferences the j^{th} ranked firm f in the second period can match with its k^{th} ranked worker with probability $\frac{(j-1)!}{(j-k)!}\frac{(n-k)!}{n!}(n-j+1)$ if top $(j-1)$ ranked firms match with the firm f's top $(k-1)$ workers for $k \leq j \leq m'$. Hence $\Psi_i^{in} \geq \Psi_i$.

As in inequality (10) the left hand side of inequality (29) is same for all firms and Ψ_i^{in} depends on the rank of the firm f in the first period. Similar to lemma 1, lemma 3 states that all firms offer an early contract if it is profitable for f_1^1 to offer an early contract.

Lemma 3. In two sided matching market (F, W, u, v, R^{in}, m') if

$$\frac{1}{n}\sum_{j=1}^{n} v_j \geq \alpha v_1 \tag{31}$$

then it is profitable for all firms to offer an early contract in the first period.

Proof: As stated above the only difference between conditions for firms to offer an early contract when preferences are identical and independent is that expectation is taken over η_j in calculating Ψ_i^{in}, while it is taken over v_j in calculating Ψ_i. If I can show that $\boldsymbol{\eta} = [\eta_1, ..., \eta_{m'}]$ is a monotone decreasing vector like the value vector v the proof is done. I can show the monotonicity $\boldsymbol{\eta}$ like in the proof of Lemma 2 in Halaburda (2007). Let the probability of the j^{th} ranked firm f in the second period matching with its k^{th} ranked worker is $P_{j,k}$ and is defined by

$$P_{j,k} = \frac{(j-1)!}{(j-k)!}\frac{(n-k)!}{n!}(n-j+1). \tag{32}$$

for $k \leq j$ and 0 otherwise. Probability that firm f_{j+1}^2 gets its worker $k < j+1$ is

$$P_{j+1,k} = \frac{(j)!}{(j+1-k)!} \frac{(n-k)!}{n!} (n-j). \tag{33}$$

Since j and n are fixed, the ratio of expression (32) to (33) decreases as k increases. It is more probable for the better firm to be matched with its better workers. I assume that v_j decreases as j increases. Hence η_j is decreasing as j increases like v_j.

Now I study worker w's incentive to deviate from the candidate equilibrium when firms have independent preferences. Change in the preference structure of firms from identical to independent does not affect the necessary condition for workers to accept the offer of f_i^1. As a result, inequality (16) is also the necessary condition for workers to accept early contract offer of f_i^1 when firms have independent preferences. Hence the result of lemma 2 in section 5.2.1 is also valid for this case. That is, if the early contract offers of the least preferred firm is profitable than workers will accept any firms' offer in the first period. This result is formally stated in lemma 4 below.

Lemma 4. In two sided matching market (F, W, u, v, R^{in}, m') if

$$\alpha u_{m'} \geq \frac{1}{n} \sum_{j=1}^{m'} u_j \tag{34}$$

then it is profitable for any worker w to accept any firm's early contract offer in the first period.

Proof: Proved similar to lemma 1.

Similar to section 5.2.1, by combining the results of lemma 3 and lemma 4 one can conclude that every agent in the market prefers to deviate from ex-post stable matching outcome if both inequalities (31) and (34) hold. Hence one can find that the condition for all agents wanting to deviate from ex-post stable matching is exactly the same as the condition when firms have identical preferences. Let $A' = \sum_{k=1}^{m'} u_k$ and $B = \sum_{j=1}^{n} v_j$ then I have the following proposition.

Proposition 4. In two sided matching market (F, W, u, v, R^{in}, m') if

$$\frac{B}{\alpha v_1} \geq n \geq \frac{A'}{\alpha u_{m'}} \tag{35}$$

then there is no equilibrium without unraveling.

Proof: Proved same as proposition 1.

As stated in section 5.2.1 the condition for the existence of an equilibrium involving unravelling in equation (35) is strong. I can weaken this condition by showing a profitable deviation for a single firm and a single worker. That is, I have to show that there exists at least one firm who prefers to offer an early contract and that firm is acceptable for the worker to whom it offers. Combining inequality (29) and inequality (16) necessary and sufficient condition for unraveling is expressed in the following proposition.

Proposition 5. In two sided matching market (F, W, u, v, R^{in}, m') there does not exist an equilibrium without unraveling if only if there exists $i \in \{1, .., m\}$ such that

$$\left(\Psi_i^{in}\right)^{-1} \sum_{i=1}^{n} v_i \geq n \geq (\Phi_i)^{-1} \sum_{i=1}^{m'} u_i. \tag{36}$$

Proof: Proved similar to proposition 2.

The right hand side of the inequality (36) is the same as when firms have identical preferences. However due to the independent preferences of the firms the left hand side of the equation (36) is less than or equal to the case where firms have identical preference. Hence, existence of unravelling when firms have independent preferences implies the existence of unravelling when firms have identical preferences.

Note that from equation (36) changing the preference of firms from identical to independent does not affect the necessary conditions for the most preferred firm to unravel. That is the condition stated in proposition 3, also valid when firms have independent preferences. This is formally stated in the following proposition.

Proposition 6. In two sided matching market (F, W, u, v, R^{in}, m'), the most preferred firm unravels with a worker if and only if

$$\left(\Psi_1^{in}\right)^{-1} \sum_{j=1}^{n} v_k \geq n.$$

Proof: Proved similar to proposition 3.

As a consequence of proposition 6 if the number of workers is less than $\frac{B}{\alpha v_1}$ (condition for the most preferred firm to unravel) existence of unravelling can not be affected by the similarity in preferences of firms. Another conclusion which is the same as the case when firms have identical preferences is that the existence of unravelling is independent from the utility value vector u when it is profitable for the most preferred firm in the first period since f_1^1's offer will always be accepted by all workers.

As in the case of firms with identical preferences, in my model, when firms have independent preferences, the incentive for a firm to offer an early contract differs from the model of Halaburda (2007). In Halaburda (2007), f_i^1 will match with the j^{th} ranked worker and receive an expected utility η_j in the second period and, where $j \leq i$. Hence it will offer an early contract if it expects to get utility higher than η_j. It is proved in Halaburda (2007) that η_j is always grater than the expected utility in the first period and unravelling can not be observed when $m = m'$. However, in my model, there is no guarantee that firm f_i^1 will match with the j^{th} ranked worker in the second period since it may be eliminated in the second period and get 0 utility or it may match with the j^{th} ranked worker with higher probability and get higher expected utility. Hence it is the trade off between these two situations that drives the incentive of firm f_i^1 to offer an early contract. Depending on the parameter values if the former situation, 0 utility, dominates the second one unraveling is observed when $m \neq m'$.

With respect to workers' incentive to deviate, the comparison of Halaburda (2007) and my model when firms preferences are independent is exactly the same as when firms have identical preferences in section 5.2.1

Existence of unravelling when $\alpha \neq 1$ will be illustrated in the following example.

Example 3. Consider an economy with six workers and four firms. The firms have independent preferences. Assume that the values of firms to workers is discrete uniformly distributed between $\frac{1}{m}$ to one and the values of workers to firms is discrete uniformly distributed between $\frac{1}{n}$ to one. That is $v = [1, \frac{5}{6}, \frac{4}{6}, \frac{3}{6}, \frac{2}{6}, \frac{1}{6}]$ and $u = [1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}]$.

When $\alpha = 1$, inequality (36) does not hold for any firm f_i^1 as stated in Halaburda (2007).

Now assume $\alpha = 0.5$. The value vector of firms is updated as $u = [1, \frac{3}{4}, 0, 0]$. From inequality (29) the necessary condition for f_1^1 to offer an early contract is $\frac{1}{2} + \frac{1}{n} \geq \frac{1}{2}$. For all finite values of n the left hand side is greater than 0.5. That is f_1^1 will always prefer to deviate from the ex-post stable matching outcome. So all the firms prefer to offer early contract in the first period. Workers will accept the offer of the f_i if inequality (16) is true.

It has been proved that f_1's offer is always accepted by the workers. Workers accept all the other firms offer since inequality (16) is satisfied for all the firms. Hence all the agents are willing to unravel in this economy when $\alpha = 0.5$. However when $\alpha = 1$ there exists a unique equilibrium in the first period such that $\sigma^*_f = \{\emptyset\}$ for all $f \in F$, $\sigma^*_w = \{\emptyset\}$ for all $w \in W$ and the equilibrium beliefs are defined as in equations $(3-7)$.

CHAPTER 6

EQUILIBRIA WITH UNRAVELING

In the previous section I have shown that, regardless of the value of α, for some domain of cardinal preferences early contracting is more profitable for both firms and workers than the *ex-post* stable matching outcome. Now, in this section, I study pure strategy equilibria that involves unravelling.

In every equilibrium with unravelling, let $\mathcal{U}$ denote the set of firms that unravel with a worker and is defined as

$$\mathcal{U} = \{f \mid \sigma^*_w = f \text{ for } \forall w \in W\} \tag{37}$$

where $\mathcal{U} \subseteq F$. Let $\mathcal{V}$ denote the set of workers that unravel with a firm $f \in \mathcal{U}$ and is expressed as

$$\mathcal{V} = \{w \mid \sigma^*_f = w \text{ for } \forall f \in \mathcal{U}\} \tag{38}$$

where $\mathcal{V} \subseteq W$. Let $\mathcal{V}^c$ and $\mathcal{U}^c$ be the complements of $\mathcal{V}$ and $\mathcal{U}$, respectively. The firms and workers who do not unravel, $f \in \mathcal{U}^c$ and $w \in \mathcal{V}^c$, participate in the *ex-post* stable matching mechanism in the second period.[11] For a given $\mathcal{U}$ let f^1_h be the least preferred firm in $\mathcal{U}$ where[12]

[11] Note that only the acceptable firms participate in the *ex-post* stable matching mechanism.

[12] Recall that the most preferred agent is represented by the lowest subscript, i.e. f^1_1 is the most preferred firm.

$$h = \sup\{j \mid f_j^1 \in \mathcal{U}\} \tag{39}$$

and f_l^1 be the most preferred firm in $\mathcal{U}$ where

$$l = \inf\{j \mid f_j^1 \in \mathcal{U}\}. \tag{40}$$

For the given $\mathcal{U}$ and $\mathcal{V}$, let $EU^w(t = 2|\mathcal{U})$ be the expected utility of the worker $w \in \mathcal{V}^c$ in the second period and $EU_i^w(t = 1|\mathcal{U})$ be the expected utility of the worker $w \in \mathcal{V}^c$ from accepting the offer of firm $f_i^1 \in \mathcal{U}^c$ in the first period. For firms, let $EU_i(t = 2|\mathcal{U})$ be the expected utility of firm $f_i^1 \in \mathcal{U}^c$ in second period and $EU(t = 1|\mathcal{U})$ be the expected utility of any firm $f_i^1 \in \mathcal{U}^c$ from early contracting with worker $w \in \mathcal{V}^c$.[13]

After introducing the notation that is used in this section, I will give some properties of the set $\mathcal{U} = \{f_l^1, ..., f_h^1\}$. As I have shown in lemma 1 and 3 if firm f_l^1 has an incentive to early contract then any firm whose rank is worse than l will have an incentive to early contract as well. Moreover, as in lemma 2 and 4 if the early contract offer of f_h^1 is acceptable by the workers then the early contract offer of any firm whose rank is better than h is acceptable as well. That is, for $i \in \{l, ..., h\}$, firm f_i^1 is an element of set $\mathcal{U}$. In the following proposition the convexity of $\mathcal{U}$ is stated and proved.

Proposition 7. In two sided matching market (F, W, u, v, R, m') in any equilibrium,

[13]Note that the expected utilities of all firms in both periods are same.

the equilibrium set, $\mathcal{U}$, includes firm f_i^1, where $i \in \{l, ..., h\}$.[14]

Proof: Assume that the statement is not true. Then there exists f_i^1 such that $h \geq i \geq l$ and $f_i^1 \notin \mathcal{U}$. That is, f_i^1 either prefers not to offer an early contract or its offer is not accepted by the worker to whom it offers. First assume that f_i^1's offer is not accepted in the first period. Given all the other firms in $\mathcal{U}$ contracted earlier, $f_h^{1\prime}$s offer is accepted in period 1 if

$$EU^w(t = 2|\mathcal{U}\backslash f_h^1) \leq EU_h^w(t = 1|\mathcal{U}\backslash f_h^1). \tag{41}$$

Since f_i^1's offer is not accepted given all the other firms in $\mathcal{U}$ except f_h^1 contracted earlier, then

$$EU^w(t = 2|\mathcal{U}\backslash f_h^1) \geq EU_i^w(t = 1|\mathcal{U}\backslash f_h^1). \tag{42}$$

However I have proved implicitly in lemma 2 and 4 that

$$EU_i^w(t = 1|\mathcal{U}\backslash f_h^1) \geq EU_h^w(t = 1|\mathcal{U}\backslash f_h^1). \tag{43}$$

Inequality (43) contradicts with inequality (41) and (42). Now assume that f_i^1 prefers not to offer an early contract. Firm f_l^1 prefers contracting early given all the other firms in $\mathcal{U}$ contracted earlier if

$$EU_l(t = 2|\mathcal{U}\backslash f_l^1) \leq EU(t = 1|\mathcal{U}\backslash f_l^1). \tag{44}$$

[14] Note that this proposition is valid for both R^{in} and R^{id}.

Since f_i^1 prefers not to offer an early contract given all the other firms in $\mathcal{U}$ except f_l^1 contracted earlier, then

$$EU_i(t = 2|\mathcal{U}\backslash f_l^1) \geq EU(t = 1|\mathcal{U}\backslash f_l^1). \tag{45}$$

However I have proved implicitly in lemma 1 and 3 that

$$EU_l(t = 2|\mathcal{U}\backslash f_l^1) \geq EU_i(t = 2|\mathcal{U}\backslash f_l^1). \tag{46}$$

Inequality (46) contradicts inequality (44) and (45).

As I have shown that $\mathcal{U}$ is a convex set that includes all the firms ranked between f_l^1 and f_h^1, the expected utility of the worker $w \in \mathcal{V}^c$ in the second period, $EU^w(t = 2|\mathcal{U})$, can be defined as

$$EU^w(t = 2|\mathcal{U}) = \frac{1}{n - h + l - 1}(\sum_{j=1}^{m'} u_j - \sum_{i=l}^{h} \Phi_i) \tag{47}$$

Now I can define an equilibrium involving unraveling such that $\mathcal{U} \neq \emptyset$ and $\mathcal{V} \neq \emptyset$. Given f_l^1, the least preferred firm, f_h^1, that is accepted in the first period is characterized by

$$EU_h^w(t = 1|\mathcal{U}\backslash f_h^1) \geq \frac{1}{n - h + l - 1}(\sum_{j=1}^{m'} u_j - \sum_{i=l}^{h} \Phi_i) \tag{48}$$

and

$$EU^{w}_{h+1}(t=1|\mathcal{U}) < \frac{1}{n-h+l-1}(\sum_{j=1}^{m'} u_j - \sum_{i=l}^{h} \Phi_i). \tag{49}$$

The inequality (48), given $\mathcal{U}\backslash f^1_h$, is the condition for the firm f^1_h's offer to be acceptable by a worker $w \in \mathcal{V}^c$ while the inequality (49), given $\mathcal{U}$, is the condition for the firm f^1_{h+1}'s offer not to be acceptable by a worker $w \in \mathcal{V}^c$.[15] Similarly given f^1_h, the most preferred firm, f^1_l, early contracting in the first period is characterized by

$$EU_l(t=2|\mathcal{U}\backslash f^1_l) \leq EU(t=1|\mathcal{U}\backslash f^1_l) \tag{50}$$

and

$$EU_{l-1}(t=2|\mathcal{U}) > EU(t=1|\mathcal{U}). \tag{51}$$

The inequality (50), given $\mathcal{U}\backslash f^1_l$, is the condition for the firm f^1_l to prefer unravelling by a worker $w \in \mathcal{V}^c$ while the inequality (51), given $\mathcal{U}$, is the condition for the firm f^1_{l-1} not to prefer unraveling by a worker $w \in \mathcal{V}^c$. In the following lemma, by combining the inequalities $(48-51)$ I define necessary and sufficient condition for the existence of an equilibrium with $\mathcal{U} = \{f^1_l, ..., f^1_h\}$.

Lemma 5. In two sided matching market (F, W, u, v, R, m') there exists an equilibrium with nonempty $\mathcal{U} = \{f^1_l, ..., f^1_h\}$ if and only if inequalities (48) and (49) are satisfied for given f^1_l and inequalities (50) and (51) are satisfied for given f^1_h.

In the following lemma it is shown that for the most preferred firm in any $\mathcal{U}$ there

[15] Inequalities (48) and (49) are also true for Φ^{in}_i.

exists a unique least preferred firm $f_h^1 \in \mathcal{U}$.

Lemma 6. In two sided matching market (F, W, u, v, R, m'), for any f_l^1, if there exists f_h^1 that satisfies inequalities (48) and (49), it is unique.

Proof: Assume that the lemma is not true. That is, there exist two different $f_{h'}^1$ and $f_{h''}^1$ satisfying inequalities (48) and (49). Without loss of generality assume $h' > h''$. First consider the equilibrium $\mathcal{U} = \{f_l^1, ..., f_{h''}^1\}$. From inequality (49) one can write

$$EU_{h''+1}^{w}(t = 1|\mathcal{U}) < \frac{1}{n - h'' + l - 1}(\sum_{j=1}^{m'} u_j - \sum_{i=l}^{h''+1} \Phi_i). \quad (52)$$

That is firm $f_{h''+1}^1$'s offer is not acceptable in the first period. However when the equilibrium $\mathcal{U} = \{f_l^1, ..., f_{h'}^1\}$ is considered, as I have showed in proposition 7, firm $f_{h''+1}^1$ unravels with a worker since $l < h'' + 1 \leq h'$. This contradicts with inequality (52).

Similar to lemma 6, lemma 7 states that given the least preferred firm in any $\mathcal{U}$, there exists a unique most preferred firm $f_l^1 \in \mathcal{U}$.

Lemma 7. In two sided matching market (F, W, u, v, R, m'), for any f_h^1, if there exists f_l^1 that satisfied inequalities (50) and (51), it is unique.

Proof: I prove by contradiction. Assume that there exist two different firms $f_{l'}^1$ and $f_{l''}^1$ satisfying inequalities (50) and (51). Without loss of generality assume $l'' > l'$. Consider $\mathcal{U} = \{f_{l''}^1, ..., f_h^1\}$, from inequality (51) one can write

$$EU_{l''-1}(t = 2|\mathcal{U}) > EU(t = 1|\mathcal{U}). \quad (53)$$

That is firm $f^1_{l''-1}$ does not prefer to offer an early contract. However when the equilibrium U={$f^1_{l'}$, ..., f^1_h} is considered, from proposition 7, firm $f^1_{l''-1}$ unravels with a worker since for $l' \leq i \leq h$. This contradicts with inequality (53).

In section 5.2.1 I have shown that when firms have identical preferences if the inequality (21) is not satisfied for any $i \in \{1, ..., m\}$ then unraveling is not observed and the only equilibrium is $\mathcal{U} = \emptyset$. The same situation occurs when firms have independent preferences if the inequality (36) is not true for any i. If inequality (21) or (36) holds for any i in the related firms' preference structure then firm f^1_i unravels. By unraveling firm f^1_i can match with a worker, that would have been matched with the better firms in the *ex-post* stable matching mechanism, with a positive probability. Hence better firms match with worse workers if they wait for the second period in case they are found to be acceptable. This loss in the expected payoff of better firms in the second period may induce them to unravel in the first period. As top ranked firms unravel the expected utility of workers decreases in the second period. Hence workers accept firms in the first period that previously would not be accepted. Eventually this process induces all firms to unravel in the first period or stop before the worst firm. Hence there exists at least one equilibrium in this model. The existence of pure strategy equilibrium is stated in the following proposition as either $\mathcal{U} = \emptyset$ or $\mathcal{U} \subseteq F$.

Proposition 8. In two sided matching market (F, W, u, v, R, m') there exists at least one equilibrium in pure strategies.

Proof: I know that if inequality (21) or (36) is not satisfied for any i in the related firms' preference structure then the unique equilibrium is $\mathcal{U} = \emptyset$. If the inequality (21) or (36) is satisfied, then I have proved in lemma 6 and 7 that there exists a unique f_h^1 for given f_l^1 where h is increasing as l decreases and there exists a unique f_l^1 for given f_h^1 where l is decreasing as h increases. As a result of this monotonicity l goes to 1 and h goes to m. Hence an equilibrium exists, $\mathcal{U} \subseteq F$.

In proposition 8, I prove the existence of an equilibrium such that $\mathcal{U} \neq \emptyset$. Moreover in some markets there may exist more than one equilibrium. A multiple equilibrium case will be illustrated in the following example.

Example 4. Example (1) is modified by only changing the value vector υ : $\upsilon = [18, 10, 8, 6, 5, 3]$. Assume $\alpha = 0.5$. In this market there are two possible unravelling sets in pure strategy equilibrium: $\mathcal{U} = F$ and $\mathcal{U} = \{f_2^1, f_3^1, f_4^1\}$. These two equilibrium arise in the following way. Given value vectors, the inequality (17) holds hence every firms' early contract offer is acceptable in the first period. Knowing all the firms will be accepted in the first period, except f_1^1 all the other firms prefer unraveling. If f_1^1 believes that the other firms offer early contract to w_4, w_5 and w_6 then $\mathcal{U} = \{f_2^1, f_3^1, f_4^1\}$ arises as an equilibrium. However if f_1^1 believes that any other firm offers to w_1 then $\mathcal{U} = F$ arises as an equilibrium. However, as stated in Halaburda (2007), when $\alpha = 1$, $\mathcal{U} = F$ can not be an equilibrium.

In an economy with multiple equilibrium, one equilibrium is fully included by another. In particular if there exist two equilibrium, one of the equilibriums must fully include the other one. In the following proposition this property will be stated.

Proposition 9. In two sided matching market (F, W, u, v, R, m') if there exist two nonempty distinct equilibrium denoted by $\mathcal{U}'=\{f^1_{l'}, ..., f^1_{h'}\}$ and $\mathcal{U}''=\{f^1_{l''}, ..., f^1_{h''}\}$ then

$$l' < l'' \Longleftrightarrow h'' < h'.$$

Proof: Assume the statement is not true. That is $l' < l''$ and $h' < h''$. First assume that $l'' - l' < h'' - h'$. That is the number of firms in set $\mathcal{U}'$, $h' - l'$, is less than $\mathcal{U}''$, $h'' - l''$. As I discussed before proposition 8, if it is profitable for firm $f^1_{l'}$ to offer an early contract given $(h' - l')$ worse ranked firms unravel then it must be the case that $f^1_{l'}$ offers an early contract and unravel when $(h'' - l'')$ worse ranked firms unravel. Hence $\mathcal{U}''$ cannot be an equilibrium. Now assume that $l'' - l' > h'' - h'$. As I discussed before proposition 8, if it is profitable for workers to accept firm $f^1_{h'+1}$'s offer given that $(h' - l'')$ better ranked firms unravel then it must be the case that $f^1_{h'+1}$'s offer is considered acceptable by the workers when $(h' - l')$ better ranked firms unravel, where $h' - l' > h' - l''$. Hence $\mathcal{U}'$ cannot be an equilibrium.

It is worth pointing out that in my model the expected utilities of agents when the firms' preferences are identical have the same monotonicity properties as when the firms' preferences are independent. Since every result in this section mainly depends on the monotonicity of expected utilities there is no need for stating the results separately. Moreover the results of this model differ from the results of Halaburda (2007) in two ways. Firstly, Halaburda (2007) does not include equilibrium with unraveling when firms have independent preferences whereas in this model equilibrium with unraveling may exist depending on the value vectors. Secondly, Halaburda (2007) does not include an equilibrium involving the most preferred firm. However in Example 1 the existence of such an equilibrium is illustrated.

CHAPTER 7

CONCLUSION

In this paper I investigate unravelling in two-sided matching markets composed of two periods where some of the firms are not considered worth matching in the second period. Consideration of some firms as unacceptable in the second period is a restriction on the preferences of workers over firms.

To compare the role of restriction on the preference lists of workers in the second period with other reasons mentioned in the literature (stability, risk aversion and similarity in preferences) the agents are assumed to be risk neutral. In the second period the matchings are arranged by ex-post stable matching mechanism. Firms' preferences over the workers can be identical or independent while workers have identical preferences. To see the effect of the restriction on workers' preferences, the model is studied in two sections. In the second period (1) workers consider all the firms on their preference lists and (2) workers do not consider some firms as acceptable.

This study shows that without restriction, unravelling cannot be seen when the preferences of firms over workers are independent. However, if all firms are not acceptable in the second period unravelling can be observed even if firms have independent preferences. The existence of an equilibrium involving unravelling is summarized in table 2 below.

Intuitively, restrictions on the workers' preferences bring uncertainty to utility

Table 2: Existence of unravelling

	Firms with iden. pref.	*Firms with indep. pref.*
Without restriction pref.	Possible	Impossible
With restriction pref.	Possible	Possible

gained by the firms if they do not contract in the first period. Without restrictions on the workers' preferences firms gain utility strictly greater than 0. However with restriction firms may gain 0 utility if they are not acceptable in the second period or more utility than they earn when there is no restriction on the workers preferences if they are acceptable in the second period. Note that since the agents are risk averse any change in the variance of value vectors does not affect the strategies of agents. However, means of the value vectors are not preserved and the lack of mean preserving spread is the source of the change in the agents' strategies. If the variance of the matching values of workers is low (nearly homogenous workers market) firms will not want to take the risk of being unmatched and will prefer to unravel. Even the most preferred firm in the first round may offer an early contract. For workers, the case with restriction on preferences is similar to the case without restriction but having less firms in both periods. Given the number of workers is more than the number of firms, by accepting the high ranked firms' offers workers insure against not unmatching.

Also not mentioned in the literature, unravelling due to the existence of restriction on the workers' preferences is observed in the European soccer transfer market, where teams are on the one side and players are on other side. Teams may have independent preference since all the teams need different kinds of players to strengthen their squad. However all players' preferences are based on the publicly known ranking of the teams. Teams try to transfer players before the all information about the agents are clear. The teams who are qualified for the following group matches and the performance of the players are learnt publicly.

To conclude in two sided matching markets restriction on the workers' preference may lead to unravelling no matter whether firms have identical or independent preferences over workers.

APPENDICES

Appendix A. Firms with Identical Preferences over Workers

When there is no uncertainty in the workers preference, all firms and workers prefer to be matched than be unmatched.[16]

Firm f_i^1 offers early contract if the expected utility gained in the first period is higher or equal to the utility that will be gained from the *ex-post* stable matching mechanism. Since the workers are *ex-ante* identical an offer made to any worker in period 1 yields the same expected payoff which is the mean of the values of all workers. Firm f_i^1 will match with w_i as the outcome of the *ex-post* stable matching mechanism and utility of matching with w_i is v_i.[17] The necessary condition for f_i^1 to offer an early contract is

$$\frac{1}{n}\sum_{k=1}^{n} v_k \geq v_i. \tag{54}$$

For any worker accepting an early contract offer is rational if the utility gained from matching with f_i^1 in the first period is greater or equal to the expected utility gained in second period. Otherwise, the worker will reject the offer. Utility gained from matching with firm f_i^1 in the first period is u_i. The expected utility gained from ex-post stable matching in the second period is $\frac{1}{n}\sum_{k=1}^{m} u_k$. So the necessary condition for any worker to accept the offer is

[16] Without restriction on the preferences of workers $R_w^1 = R_w^2$.
[17] This is a result of firms' serial dictatorship.

$$u_i \geq \frac{1}{n} \sum_{k=1}^{m} u_k. \tag{55}$$

For unravelling to occur, both inequalities (54) and (55) must be true for any $i \in \{1, ..., m\}$. Let the sum of the utility values of firms is denoted by $A = \sum_{k=1}^{m} u_k$ and the sum of the utility values of workers is denoted by $B = \sum_{k=1}^{n} v_k$. The necessary condition for firm f_i^1 to offer an early contract is $\frac{B}{v_i} \geq n$. For its offer to be accepted it must be the case that $n \geq \frac{A}{u_i}$. The following proposition summarizes the findings of the preceding discussion.

Proposition 10. In two sided matching market (F, W, u, v, R^{id}, m'), when $m = m'$ there does not exist an equilibrium without unraveling if only if there exists $i \in \{1, .., m\}$ such that

$$\frac{B}{v_i} \geq n \geq \frac{A}{u_i}. \tag{56}$$

Note from expression (54) that the most preferred firm, f_1^1, will never unravel. This statement is a consequence of the assumption that $v_1 > v_k$ where $k \in \{2, ..., n\}$. However, it is possible that other firms may find early contracting profitable. All the firms except the most preferred will offer early contract as long as

$$\frac{1}{n} \sum_{k=1}^{n} v_k \geq v_2. \tag{57}$$

This result is obvious because the left hand side of inequality (54) is the same for all firms and the right hand side gets its largest value when $i = 2$ for $i \in \{2, ..., m\}$. All the firms' early contract offer will be accepted if the inequality (54) is true for the least preferred firm

$$u_m \geq \frac{1}{n}\sum_{k=1}^{m} u_k. \tag{58}$$

In inequality (55), the right hand side is the same for all workers and the left hand side can gets its lowest value for $i = m$ for $i \in \{1, ..., m\}$. Combining inequality (57) and (58) gives the necessary condition for all agents except f_1^1 will prefer to deviate from the ex-post stable outcome.

Proposition 11. In two sided matching market (F, W, u, v, R^{id}, m') when $m' = m$ if

$$\frac{B}{v_2} \geq n \geq \frac{A}{u_m} \tag{59}$$

then there is no equilibrium without unraveling.

To illustrate the case in which all agents except f_1^1 prefer to deviate from ex-post stable mechanism Example 1 is modified.

Example 5. Consider Example 1 with the following modification where $v_1 = 24$ instead of 12. The new value vector is given as $v = [24, 10, 9, 6, 5, 4, 3]$. The sum of the values are: $B = \sum_{k=1}^{n} v_k = 61$ and $A = \sum_{k=1}^{m} u_k = 31$. For all firms, except f_1^1, inequality (56) is satisfied, and all three will unravel.

To illustrate the case in which none of the agents prefers deviating from ex-post stable mechanism the value vectors in Example (1) is modified.

Example 6. Now consider the value vectors $\upsilon = [12, 10, 9, 6, 5, 4, 3]$ and $u = [30, 8, 7, 6]$. The sum of the values are $B = \sum_{k=1}^{n} \upsilon_k = 49$ and $A = \sum_{k=1}^{m} u_k = 51$. For all firms the inequality (56) is not satisfied hence they do not unravel.

Appendix B. Firms with Independent Preferences over Workers

When the firms have independent preferences, realized ranking of the firm f and worker w are given as

$$w_{1,f} P_f w_{2,f} P_f w_{3,f} P_f ... w_{(n-1),f} P_f w_{n,f} P_f \emptyset \tag{60}$$

$$f_1^1 P_w f_2^1 P_w f_3^1 P_w ... f_{m-1}^1 P_w f_m^1 P_w \emptyset$$

where $w_{j,f}$ is the firm f's j^{th} ranked worker.

As a result of the ex-post stable matching mechanism, in period 2, f_i^1 can match with w_{j,f_i^1} where $j \in \{1, ..., i\}$. Firm f_i matches with w_{i,f_i^1} with probability

$$\frac{(n-j)!}{n!}(n-i+1)\frac{(i-1)!}{(i-j)!}. \tag{61}$$

The expected utility of stable matching mechanism for firm f_i^1 can be written as

$$\sum_{j=1}^{i} \upsilon_j \frac{(n-j)!}{n!}(n-i+1)\frac{(i-1)!}{(i-j)!}. \tag{62}$$

As in Section 7.2.1 the expected utility of unravelling for firms f_i^1 is

$$\frac{1}{n}\sum_{k=1}^{n} v_k. \tag{63}$$

For f_i^1 to offer an early contract the condition below must be satisfied

$$\frac{1}{n}\sum_{k=1}^{n} v_k \geq \sum_{j=1}^{i} v_j \frac{(n-j)!}{n!}(n-i+1)\frac{(i-1)!}{(i-j)!}. \tag{64}$$

The following proposition shows that equation (64) can be satisfied only when the number of workers is equal to the number of firms.

Proposition 12. In two sided matching market (F, W, u, v, R^{in}, m') when $m = m'$, only the least preferred firm, f_m^1, will offer an early contract if only $m = n$. When $m < n$ no firm will prefer to offer early contract.

Proof: It is obvious that better firms match with better workers with higher probability than less preferred firms. As a result the lowest value of the right hand side of the inequality (64) can be attained when i is as large as possible. In this set up i can be at most n and right hand side of the inequality (64) turns into

$$\frac{1}{n}\sum_{j=1}^{n} v_j. \tag{65}$$

The lowest value of the right hand side of the inequality (64) is equal to the left hand side of the inequality (64). That means only f_n^1 will offer an early contract when $n = m$. No firm will prefer to offer an early contract if $m < n$ since the right side will be greater than $\frac{1}{n}\sum_{j=1}^{n} v_j$ which violates the necessary condition for unravelling.

The workers will behave just like in Section 7.2.1. Workers will accept the offer of f_i^1 if

$$u_i \geq \frac{1}{n}\sum_{k=1}^{m} u_k. \tag{66}$$

In Proposition 8 it was proved that only the least preferred firm offer an early contract. In Proposition 13 it will be proved that this offer will not be accepted and unraveling is not observed when firms have independent preferences.

Proposition 13. In two sided matching market $(F, W, u, v\ , R^{in}, m')$ when $m = m'$, there will be no deviation from ex-post stable matching.

Proof: Since early contract is only offered by f_n^1 when $n = m$ I want to show that inequality (66) is not true for $i = m$. As the firms' ranking of the workers is given in strict order and $u_k > u_m$ for any $k \in \{1, ..., m-1\}$ it is true that

$$\sum_{k=1}^{m} u_k > mu_m. \tag{67}$$

Dividing both sides of the inequality (67) by m yields

$$\frac{1}{m}\sum_{k=1}^{m} u_k > u_m, \tag{68}$$

which is a contradiction to inequality (66).

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