PATH INTEGRALS IN QUANTUM THEORY
AND
SOME APPLICATIONS TO HOMOGENEOUS SPACES

by

Kamil Mehmet Yenice

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PATH INTEGRALS IN QUANTUM THEORY
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In this work we consider the path integrals in quantum theory and some of their applications to homogeneous spaces from the semiclassical point of view. We begin with the basic principles underlying the Feynman path integral formulation of quantum mechanics and show the domain where the method is useful and powerful. We present a method for calculating the path integrals for quadratic Lagrangians and apply it to the example of harmonic oscillator. Semiclassical propagator given by the Van Vleck-Pauli formula is also discussed. We next handle the Hamiltonian derivation of path integral. The starting-point for this derivation is the usual operator formalism of quantum mechanics.

Later on we consider path integrals on

homogeneous spaces concentrating on the motion on group manifolds. It turns out that for the free motion on the group manifold the semiclassical approximation gives the exact solution. We thus study the path integrals for U(1) and SU(2) groups. In these cases the propagator is calculated directly by two methods : the sum over classical paths and the stationary state expansion, which are shown explicitly to be equivalent. We finally give some remarks for motion on the hyperspheres S^{2n+1} and S^{2n}. Deriving a recursion relation for the propagator of S^{2n+1} we claim that there is the possibility of getting the exact solution from some kind of semiclassical propagator for S^{2n+1} .

KUANTUM MEKANİĞİNDE YOL İNTEGRALLERİ

VE

HOMOGEN UZAYLARA BAZI UYGULAMALARI

Bu çalışmada, kuantum mekaniğinde yol integrallerini ve onların homogen uzaylara bazı uygulamalarını yarı klasik açıdan gözönüne aldık. Kuantum mekaniğinin Feynman formülasyonunun temelini oluşturan esas prensipler ile başlayarak, methodun etkili ve kullanışlı olduğu alanları gösterdik. İkinci dereceden Lagranjiyenlere tekabül eden yol integralleri için bir hesaplama yöntemi sunarak, harmonik osilatör örneğine uyguladık. Van Vleck-Pauli formülü ile verilen yarı klasik propagatör de ayrıca incelendi. Bundan sonra, Hamiltonyen yol integrallerinin türetilmesini ele aldık. Kuantum mekaniğinin geleneksel operatör formalizmi bu formülasyonun türetilmesinde hareket noktasıdır.

Bunları takiben, özellikle grup uzayları üzerinde hareketi inceleyerek homogen uzaylarda yol integrallerini

inceledik. Grup uzaylarında serbest hareket için yarı klasik yaklaşım doğru çözümü verir. Bundan dolayı, U(1) ve SU(2) grupları için yol integrallerini inceledik. Bu hallerde, propagatör birbirine özdeş olan klasik yollar üzerinden toplama ve durağan haller açılımı yöntemleri ile hesaplanır. Son olarak S^{2n+1} ve S^{2n} hiperküreleri hakkında bazı düşüncelere yer verdik. S^{2n+1} küresinin propagatörü için bir indirgeme bağıntısı türeterek, doğru çözümün S^{2n+1} küresinin bir çeşit yarı klasik propagatöründen elde edilebileceğini iddia ettik.

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A	:	Normalization factor
a,b	:	End point labels
c	:	Intermediate point label
D	:	Van Vleck determinant
$\int Dx(t)$	:	Integration over paths in coordinate space
$\int DqDp$	:	Integration over paths in phase space
$E(\varphi,\theta,\psi)$	:	Euler angles
G	:	Quantum Mechanical propagator
g	:	Metric
H	:	Hamiltonian
h	:	Planck constant
I	:	Moment of inertia
j,k,l,m	:	Angular momentum eigenvalues
K	:	Path integral propagator
L	:	Lagrangian
m	:	Mass
N	:	Number of intermediate steps in a time interval
p	:	Momentum
q	:	Phase space coordinate
$\bar{q}$	:	Unit vector
R	:	Curvature
S	:	Action
T	:	Time interval
t	:	Time coordinate
t_a,t_b	:	End point times
U	:	SU(2) group element

x	: Space coordinate
x_a, x_b	: End point coordinates
$x(t)$	: A path
$\bar{x}(t)$	: Classical path
$y(t)$	: Deviation from the classical path
$\alpha, \beta, \gamma, \delta$	: Matrix indices
δ	: Angle between the unit vectors $\vec{q}_1$ and $\vec{q}_2$
Γ	: Geodesic on a unit sphere
$\Delta t, \epsilon$	: Small time - interval
$\sigma_{x,y,z}$	: Pauli Matrices
Ψ	: Wave function
θ_3	: Theta function of the third kind

I. INTRODUCTION

The path integral formulation of quantum theory was suggested by some of Dirac's remarks[(1),(2)] concerning the relation of classical action to quantum mechanics. Later, the formulation was developed and shown to be equivalent to the more usual formulations by Feynman[(3),(4)]. In contrast to the Schrödinger formulation, which stems from Hamiltonian mechanics, the Feynman formulation is tied to the Lagrangian formulation of mechanics. This new approach introduced a new mathematical concept into quantum theory, namely the function space integral or "sum over paths", so that one directly computes the propagator of the Schrödinger wave function rather than the wave function itself, expressing this propagator as an integral over all possible paths from one given point to another.

In recent years, more attention has been placed on the Hamiltonian path integral formulated in phase space[(5)], whose equivalence to the Lagrangian path integral has been shown only in Cartesian coordinates. Since there is no general proof for the canonical invariance of the Hamiltonian path integral, it is as yet unknown to what extent the Hamiltonian formulation is valid. This formulation of path integral differs from Feynman's work in that it requires the description of the classical system in terms of canonical variables and in the classical limit reduces to a Hamiltonian variational principle. An advantage of the Hamiltonian formulation is that if the path integral is defined in the simplest and most obvious way there is no need to introduce special normalization constants to maintain the unitarity of the propagator. In the conversion from the Hamiltonian formulation to Feynman's formulation the correct normalization constants automatically appear.

A representation of the propagator in the closed form of a path integral is useful for physical applications, and is especially appropriate for the perturbative expansion and for the semiclassical approximation. Nevertheless, the original form of Feynman's path integral is applicable only to a limited class of problems, where the Lagrangian is quadratic in the variables x and $\dot{x}$. For a more complicated Lagrangian, for instance, the one which describes the motion in a Riemannian space, the original form of the path integral needs a modification involving an additional term in the Lagrangian proportional to $\hbar$. This approach was developed by De Witt(6).

The semiclassical approximation in the path integral theory has paramount importance. Since we know that at small times quantum effects are not very substantial the short-time propagator is given correctly by the semiclassical approximation. In the framework of the path integral representation, the semiclassical expression arises quite naturally. The Planck constant is a small parameter in a certain sense and the propagator is given by the classical action in the phase with an overall normalizing factor, which can be calculated, for instance, by the Van Vleck determinant. Then the finite time propagator is obtained by integration of a product of N short-time propagators. The limit $N \to \infty$ leads to the functional integral overall the intermediate coordinates. This is interpreted as the path integral.

Studies with the semiclassical approximation in group manifolds have also quite nice features. As a matter of fact, the semiclassical amplitude for the free motion on a group manifold coincides with the exact solution, apart from the fact that there may be, and there usually is more than one classical path connecting two given points. The simplest examples of such situation are the motion on a circle or the motion of a particle in a box with elastic walls. The compactness of the group manifold is evidently the reason of appearance of an infinite set of classical trajectories with fixed end points. The total amplitude is then given by linearly superimposing the corresponding contributions. If the manifold is simply connected we just straightforwardly add the amplitudes. Whereas if the classical paths divide into homotopically distinct classes, then there is the possibility of relative phase factors(7).

It is clear that Feynman path integrals for quadratic Lagrangians such as free motion in Euclidean space is given exactly by the semiclassical approximation, because in these cases one has a multiple Gaussian integral, that is calculated exactly by the stationary phase method. In the literature, however, there appears no such simple explanation why the free quantal motion in group manifolds is also semiclassical. As a matter of fact we have shown that the propagator for a free motion on the hyperspheres S^{2n+1} can be put in a form of a summation over classical trajectories showing some form of the quasiclassical approximation. A direct calculation with this propagator shows that for S^1 and S^3 semiclassical results are exact.

In the context of the path integrals on group manifolds numerous works appear in the literature. Motion in the SU(N) group manifold was considered by Dowker[8]. He showed that the exact solution in this case is given by semiclassical formula. Marinov and Terentyev[9] considered the free motion on a sphere of arbitrary dimension. They write the path integral on S^n and expand the quantum mechanical propagator in terms of Gegenbauer polynomials. In more recent works I.H.Duru[10,11] studied the path integral of some potentials which are related to the path integral on SU(2) group manifold.

In the present work we first give the basis of Path integral in the context of Feynman's postulational formulation as well as some quantum mechanical properties of the propagator. We consider the equivalence of this formulation to the Schrödinger equation only for a particle moving in a potential V(x,t) in one dimension. The form of Feynman Path integral displays that it is especially a useful technique for systems with quadratic Lagrangians. From this point of view we show the calculation method of path integral with an application in the example of harmonic oscillator. The pre-exponential factor, namely the Van Vleck determinant not only underlies an over all normalizing factor, but also an expansion of it contributes to effective Lagrangian for the case of Riemannian spaces. Hence it takes an important role from the semiclassical point of view. We then give the derivation of the Van Vleck determinant in some detail. We conclude path integral formulation of quantum mechanics with the Hamiltonian derivation.

Finally, we study the path integral on homogeneous spaces. We especially concentrate our attention on the equivalence of semiclassical results to the exact solutions for free quantal motion on U(1) and SU(2) group manifolds. Some remarks about the path integral on the hyperspheres S^{2n+1} and S^{2n} conclude this work.

2.1. The Feynman Formulation of Path Integral

In quantum mechanics given the initial state of a system one can always fully solve the quantum mechanical problem if the propagator is known. At the present work we will find the propagator without considering the eigenvalue problem of the corresponding Hamiltonian as in the more usual formulations. We will first consider the motion of a particle with one degree of freedom. The generalization to higher dimensions will become clear.

Let us specify the position of a particle moving in one dimension by a coordinate x depending on time t, then x(t) describes a path. The particle starts to move from a point x_a at time t_a to a point x_b at t_b, the function x(t) describing the path takes the values $x_a = x(t_a)$ and $x_b = x(t_b)$ at the end points. In quantum mechanics, there is a probability amplitude, the propagator, such that it takes the particle from the point x_a to the point x_b. Now at this stage we have two postulates: The first one is that, when a particle's position is observed in some region of space-time then the probability amplitude that the particle has a path lying in this region is the sum of complex contributions, one from each path in the region, that is $P = |K|^2$ where

$$K = \sum_{\substack{\text{over all paths}\\ \text{in the region}}} \phi[x(t)] \quad . \tag{2.1.1}$$

Here, ϕ is the contribution from each path. The second Postulate is that each path has a contribution equal in magnitude, and a phase factor equal to the classical action in units of $\hbar$:

$$\Phi[x(t)] = A\, e^{\frac{i}{\hbar} S[x(t)]} , \qquad (2.1.2)$$

where A is an overall normalization factor.

At this point a question arises in minds immediately. If each path does not have any superiority to any other, how does the classical path become most important in the appropriate limit? The answer to this question comes in the following way. In the classical approximation the action S is much greater than $\hbar$; therefore, each path has a very large phase angle. When the summation Eq.(2.1.1) over paths is concerned, since each path has a different phase, contributions from the paths essentially cancel each other. Only near the classical path the paths start having coherent contributions, since S is stationary in this region. Then, coherence is lost again once the phase differs from the stationary value

$$\frac{1}{\hbar} S[x(t)] \equiv S_{cl}/\hbar . \qquad \text{by about } \pi.$$

As a result, in the classical limit not only the classical path dominates the propagator by itself, but also the paths whithin about $\hbar$ of S_{cl} make contributions coherently.

After having discussed the classical limit we can conclude with the semiclassical approximation to the propagator. Depending on the condition that the Planck constant h is a small parameter in the classical scale, the propagator is represented by $\exp\left(S_0 + i\hbar S_1 + (i\hbar)^2 S_2 + \cdots\right)$.

The first term S_0 is just the classical action function, S_1 is also determined by the classical motion only, higher terms are more complicated. Hence, the action function, the role of which is to generate solutions in the classical mechanics, is the solution itself for the quantum theory :

$$K = A' e^{\frac{i}{\hbar} S_{cl}} \qquad (2.1.3)$$

is just the semiclassical expression for the propagator. Here A' is some normalizing factor which measures the number of paths in the coherent range.

Now we come to the point of calculation of sum over paths in Eq.(2.1.1). To do this let us first consider the Riemann integral. In the definition of the ordinary Riemann Integral, to find the area A, under a curve the procedure is this : First , a subset of ordinates Fig.(2.1.1) spaced at equal segments k are taken. Clearly, k times the sum of the ordinates gives an approximate value of A. To recover this approximation k is chosen in the limit $k \longrightarrow 0$:

$$A = \lim_{k \to 0} \left\{ k \sum_{j} g(x_j) \right\} \qquad (2.1.4)$$

In analogy with the Riemann integral Eq.(2.1.4), we first divide the time interval $t_b - t_a$ into N intermediate segments each of width ϵ , by introducing N-1 times $t_n = \epsilon n + t_0$, n = 0,1,2,...N. For each t_n we next introduce an assigned coordinate x_n ; therefore, a path x(t) is fully specified by an infinity of numbers

$x(t_0) \ldots x(t_N)$, namely, the values of the function $x(t)$ at every point t in the interval $t_0 = t_a$ to $t_N = t_b$, Fig.(2.1.2). We contruct a path by interpolating the gaps in the discrete function. To sum over all paths we must integrate over all possible values of infinite values $x_1, \ldots, x_{N-1}$, except of course $x(t_0)$ and $x(t_N)$ which will be kept fixed at $x_0 = x_a$ and $x_N = x_b$, respectively,

$$K(b,a) \sim \iint \cdots\cdots \int \phi[x(t)]\, dx_1 \cdots\cdots dx_{N-1}. \qquad (2.1.5)$$

We hope that if we take the limit $N \longrightarrow \infty$ at the end we will obtain a more representative sample of the complete set of all possible paths between a and b. Finally, introducing a normalizing factor whose value depends upon the particular problem we write

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A} \iint \cdots \int e^{\frac{i}{\hbar} S[b,a]} \frac{dx_1}{A} \cdots\cdots \frac{dx_{N-1}}{A}, \qquad (2.1.6)$$

where

$$S[b,a] = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt, \qquad (2.1.7)$$

is a line integral taken over the trajectory passing through the straight sections in between. In a more compact notation

$$K(b,a) = \int_{x_a}^{x_b} e^{\frac{i}{\hbar} S[b,a]}\, Dx(t). \qquad (2.1.8)$$

This is called a path integral.

2.2. Some Properties of the Propagator

Now note that if t_c is some time between t_a and t_b due to the facts that the action is an integral in time and L does not depend upon any higher derivatives of the position than the first the action along any path between the points x_a and x_b can be written as

$$\mathcal{S}[b,a] = \mathcal{S}[b,c] + \mathcal{S}[c,a] \qquad (2.2.1)$$

where c denotes a point on the path on which $S[b,a]$ is evaluated. Carrying this action into Eq.(2.1.8) we get

$$K(b,a) = \int e^{\frac{i}{\hbar}\{\mathcal{S}[b,a] + \mathcal{S}[c,a]\}} \, Dx(t). \qquad (2.2.2)$$

The integration over any path may be split into two parts; therefore, it is possible to integrate over all paths from x_a to x_b in two successive steps. In the first step we integrate over all paths from x_a to x_c keeping $S[b,c]$ costant, and also integrating over all possible values of x_c

$$K(b,a) = \int_{x_c} \int_{c}^{b} e^{\frac{i}{\hbar}S[b,c]} \, K(c,a)\, Dx(t)\, dx_c. \qquad (2.2.3)$$

In the second step we just integrate over all paths from x_c to x_b and get the result

$$K(b,a) = \int_{x_c} K(b,c)\, K(c,a)\, dx_c . \tag{2.2.4}$$

This means that amplitudes for events occurring in succession in time multiply which is an important rule for the propagators.

So far we have dealt with the amplitude, the propagator, for a particle to reach a particular point in space and time by closely following its motion in getting there. Nevertheless, it is generaly helpful to deal with the amplitude which does not give any previous history. Thus instead of K consider Ψ (x,t) as the total amplitude to arrive at (x,t) from some past situation. In fact, this kind of amplitude has also the same probability characteristic as those we have so far studied. As usual we will call this kind of amplitude a wave function or the state of a system. Therefore, we see that the propagator $K(x',t';x,t)$ having the same probabilistic characteristics is actually a wave function :

$$K(x',t';x,t) = \Psi(x',t'), \tag{2.2.5}$$

which is just the amplitude of getting to (x',t'), but $K(x',t';x,t)$ itself gives the past history of the system, i.e. , where it came from. Hence, our rule for the propagators gets the form

$$\Psi(x'',t'') = \int_{-\infty}^{\infty} K(x'',t''; x',t')\, \Psi(x',t')\, dx' . \tag{2.2.6}$$

This result can be interpreted physically as the amplitude to arrive at (x'',t'') is given by the integral over all possible values of x' of the total amplitude to arrive at the point (x',t') multiplied by the amplitude to go from (x',t') to (x'',t''), which includes all the past history of the system.

Now we will use all of these properties of the propagator and the wave function to derive the path integral form of the propagator in a fancy alternative way. We know that at small times the quantum effects are not very substantial, so that the small-time asymptotics of the propagator is given correctly by the semiclassical approximation. Thus we can build up the finite time propagator by iteration, dividing the time interval into small segments and using the semiclassical expression at any segment. According to superposition principle the complete wave function arriving at x_2 at time $t_0+2\epsilon$ is given in terms of states $\Psi(x_1,t_0+\epsilon)$ as

$$\Psi(x_2,t_0+2\epsilon) = \int dx_1\, K(x_2,t_0+2\epsilon; x_1,t_0+\epsilon)\,\Psi(x_1,t_0+\epsilon) \tag{2.2.7}$$

to yield a quantum mechanical Huygens' principle[3] :
If the amplitude of the wave Ψ is known on a given

surface, in particular, the surface consisting of all x at time t, its value at a particular nearby point at time $t + \epsilon$, is a sum of contributions from all points of the surface at t. Each contribution is delayed in phase by an amount proportional to the action it would require to get from the surface to the point along the path of least action of classical mechanics. Thus the full propagator from (x_a, t_a) to $(x_2, t_0 + 2\epsilon)$ is given as

$$K(x_2, t_0+2\epsilon; x_a, t_a) = \int K(x_2, t_0+2\epsilon; x_1, t_0+\epsilon) K(x_1, t_0+\epsilon; x_a, t_a) dx_1. \tag{2.2.8}$$

Proceeding in the same manner we obtain the result

$$K(x_b, t_b; x_a, t_a) = \int_{x_1}\int_{x_2} \cdots\cdots \int_{x_{N-1}} K(b, N-1) \cdots$$

$$\cdots K(i+1, i) \cdots K(1, a) dx_1 \cdots dx_{N-1}, \tag{2.2.9}$$

where

$$K(i+1, i) = \frac{1}{A} \exp\left[\frac{i}{\hbar} S_{cl}(x_{i+1}, t_{i+1}, x_i, t_i)\right]. \tag{2.2.10}$$

Note here that although the paths linking two successive points are the minimizing classical paths the sum of stationary paths for all subintervals is not neccessarily a stationary path for the full interval. Thus as soon as we go beyond $\Delta t = \epsilon$ we have to start including non-stationary paths. Nevertheless the phase for each of these non-stationary paths is still the classical action for that path, since for the path via $x_1, x_2, \ldots, x_{N-1}$, for instance

$$S_{cl}(x_b, t_b; x_a, t_a) = S_{cl}(x_b, t_b; x_{N-1}, t_{N-1}) + \sum_{i=1}^{N-2} S_{cl}(x_{i+1}, t_{i+1}; x_i, t_i) + S_{cl}(x_1, t_1; x_0, t_0). \tag{2.2.11}$$

The path is still classical, it is just no longer stationary between its end points. Here S_{cl} which is given by

$$S_{cl}(x_{i+1}, t_{i+1}; x_i, t_i) = \mathrm{Min} \int_{t_i}^{t_{i+1}} L(\dot{x}(t), x(t), t)\,dt, \tag{2.2.12}$$

and L is supposed to be a quadratic function of the velocities. Then by the rule for multiplying amplitude of events which occur successively in time, we have

$$\Phi[x(t)] = \lim_{\epsilon \to 0} \prod_{i=0}^{N-1} K(i+1, i) = \frac{1}{A^N} \exp\left[\frac{i}{\hbar} S_{cl}(x_b, t_b; x_a, t_a)\right]. \tag{2.2.13}$$

Consequently, using these facts we reach the result of

Eq.(2.1.8) for the standard finite time path integral :

$$K(x_b,t_b;x_a,t_a)=\int e^{\frac{i}{\hbar}S[x_b,x_a]}\,Dx(t). \quad (2.1.8)$$

2.3. Equivalence to the Schrödinger Equation

In this subsection we will derive the Schrödinger equation from Feynman's formulation of path integral which we have dealt with so far. For this aim we will consider the special case of a particle moving in a potential V(x,t) in one dimension i.e., for which $L = m\dot{x}^2/2 - V(x,t)$.

Let us first introduce the equation :

$$\Psi(x_2,t_2)=\int_{-\infty}^{\infty} K(x_2,t_2;x_1,t_1)\Psi(x_1,t_1)\,dx_1. \quad (2.3.1)$$

which gives the wave function at a time t_2 in terms of the wave function at time t_1. In order to obtain a differential equation we go to an infinitesimal time interval such that

$$\Psi(x,t+\epsilon)=\int_{-\infty}^{\infty} K(x,t+\epsilon;x',t)\Psi(x',t)\,dx', \quad (2.3.2)$$

where the propagator is given by the semiclassical formula for that infinitesimal time interval. Using the fact that there is no need to do any integrations over intermediate x's to calculate K, since there is just one slice of time ϵ between the start and finish. We have

$$K(x,t+\epsilon;x',t) = \frac{1}{A}\exp\left[\epsilon\frac{i}{\hbar}L\left(\frac{x-x'}{\epsilon},\frac{x+x'}{2}\right)\right]. \qquad (2.3.3)$$

Thus, keeping in mind $L = m\dot{x}^2/2 - V(x,t)$. Eq.(2.3.2) becomes

$$\Psi(x,t+\epsilon) = \int_{-\infty}^{\infty}\frac{1}{A}\left\{\exp\left[\frac{i}{\hbar}\frac{m(x-x')^2}{2\epsilon}\right]\right\}$$

$$\cdot\exp\left\{\left[-\frac{i}{\hbar}\epsilon V\left(\frac{x+x'}{2},t\right)\right]\right\}\Psi(x',t)\,dx'. \qquad (2.3.4)$$

Here, notice that the factor $\exp\left[\frac{im}{2\hbar\epsilon}(x-x')^2\right]$ oscillates very rapidly as $(x-x')$ varies since ϵ is infinitesimal, $\hbar$ is small and $m/\hbar\epsilon$ is large. On the other hand $\Psi(x',t)$ is a smooth function; therefore, when this first factor in the exponent is multiplied by $\Psi(x',t)$, the integral vanishes for the most part due to the random phase of the exponential. Hence, only contributions within a distance η of the stationary point $x = x'$, where the phase has the minimum value of zero, are substantial. In terms of $\eta = x'-x$, the region of coherence range is

$$|\eta| \lesssim \left(\frac{2\epsilon\hbar\pi}{m}\right)^{1/2} \qquad (2.3.5)$$

Making the substitution $x' = x + \eta$ we obtain

$$\Psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} e^{im\eta^2/2\hbar\epsilon} \cdot e^{-\frac{i\epsilon}{\hbar} V[(x+\eta/2,t)]} \Psi(x+\eta,t)\,d\eta. \qquad (2.3.6)$$

We may expand Ψ in a power series keeping terms of order ϵ and by the relation Eq.(2.3.5) second-order terms in η. Furthermore we can replace $\epsilon V\left[\left(\frac{x+\eta}{2},t\right)\right]$ by $\epsilon V(x,t)$ since by doing so we make an error only of higher order than ϵ. Expanding the left hand side to first order in ϵ and the right-hand side to first order in η, we get

$$\Psi(x,t) + \epsilon\frac{\partial\Psi}{\partial t} = \frac{1}{A}\int_{-\infty}^{\infty} e^{im\eta^2/2\hbar\epsilon}\left[1 - \frac{i\epsilon}{\hbar}V(x,t)\right]$$

$$\cdot\left[\Psi(x,t) + \eta\frac{\partial\Psi}{\partial x} + \frac{1}{2}\eta^2\frac{\partial^2\Psi}{\partial x^2}\right]d\eta. \qquad (2.3.7)$$

Now if we take the terms to zero order in ϵ and η on both sides

$$\Psi(x,t) = \left(\frac{1}{A}\int_{-\infty}^{\infty} e^{im\eta^2/2\hbar\epsilon}\, d\eta\right)\Psi(x,t)$$

$$= \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}\Psi(x,t)\ . \qquad (2.3.8)$$

In order that both sides agree in the limit ϵ approaches zero A needs to be so chosen that the expression of Eq.(2.3.8) equals to one :

$$A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}.$$

Now evaluating the Gaussian integrals

$$\int_{-\infty}^{\infty}\frac{1}{A}\exp\left[im\eta^2/2\hbar\epsilon\right]\eta\, d\eta = 0, \qquad (2.3.9)$$

and

$$\int_{-\infty}^{\infty}\frac{1}{A}\exp\left[im\eta^2/2\hbar\epsilon\right]\eta^2 d\eta = i\hbar\epsilon/m\,. \qquad (2.3.10)$$

Therefore Eq.(2.3.7) becomes

$$\Psi + \epsilon\frac{\partial\Psi}{\partial t} = \Psi - \frac{i\epsilon}{\hbar}V\Psi - \frac{\hbar\epsilon}{2im}\frac{\partial^2\Psi}{\partial x^2}\ . \qquad (2.3.11)$$

This will be true to order ϵ if V satisfies the differential equation :

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi \qquad (2.3.12)$$

giving the Schrödinger equation for a particle moving in one dimension.

2.4. Calculation of Path Integrals

In this subsection we will introduce a technique which was given by Feynman. This technique will be valid only for the case that the integrand of the path integral is an exponential of a quadratic form in the variables $\dot{x}$ and x, and all of the variables appear up to the second order. For this purpose let us introduce the Lagrangian of the form :

$$L = a(t)\,\dot{x}^2 + b(t)\,\dot{x}x + c(t)\,x^2 + d(t)\,\dot{x} + e(t)\,x + f(t). \qquad (2.4.1)$$

The action of course will be time integral of this Lagrangian between two end points. We now wish to compute

$$K(x_b,t_b;x_a,t_a) = \int_{x_a}^{x_b} \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(\dot{x},x,t)\,dt\right] Dx(t). \qquad (2.4.2)$$

Let us write every path as

$$x(t) = \bar{x}(t) + y(t) \qquad (2.4.3)$$

where $\bar{x}(t)$ is the classical path between the specified end points and y is a new variable, namely the deviation from the classical path Fig.(2.4.1). Since all the paths agree at the end points $y(t_a) = y(t_b) = 0$. The classical path is stationary so that any variation in the alternative path x(t) amounts to the associated variation in the deviation y(t). Hence we can write

$$\int_{x_a}^{x_b} Dx(t) = \int_0^0 Dy(t) \tag{2.4.4}$$

by means of which Eq.(2.4.2) takes the form :

$$K(x_b, t_b ; x_a, t_a) = \int_0^0 \exp\left\{\frac{i}{\hbar} S\left[\bar{x}(t) + y(t)\right]\right\} Dy(t). \tag{2.4.5}$$

Expanding the action S in a Taylor series about $\bar{x}$:

$$S[\bar{x}+y] = \int_{t_a}^{t_b} L(\dot{\bar{x}}+\dot{y}, \bar{x}+y, t)\, dt$$

$$= \int_{t_a}^{t_b} \left[L(\dot{\bar{x}}, \bar{x}, t) + \left(\frac{\partial L}{\partial x}\Big|_{\bar{x}} y + \frac{\partial L}{\partial \dot{x}}\Big|_{\bar{x}} \dot{y}\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^2}\Big|_{\bar{x}} y^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\Big|_{\bar{x}} y\dot{y} + \frac{\partial^2 L}{\partial \dot{x}^2}\Big|_{\bar{x}} \dot{y}^2\right)\right] dt. \tag{2.4.6}$$

The first piece $L(\dot{\bar{x}},\bar{x},t)$ integrates to give $S[\bar{x}]=S_{cl}$. The second piece linear in y and $\dot{y}$ vanishes due to the classical equation of motion. All that remain are the second order terms in y

$$S[x(t)] = S_{cl}[b,a] + \int_{t_a}^{t_b} \left[a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\right] dt. \tag{2.4.7}$$

Since the integral over paths does not depend upon the classical path, the propagator can be written as

$$K(b,a) = e^{i/\hbar\, S_{cl}[b,a]} \int_0^0 \left(\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\right]dt\right\}\right)\mathcal{D}y(t). \tag{2.4.8}$$

Due to the fact that the path integral has no memory of $\bar{x}$, it can only depend upon t. Hence

$$K(x_b,t_b;x_a,t_a) = e^{i/\hbar\, S_{cl}[b,a]}\; F(t_b,t_a), \tag{2.4.9}$$

where $F(t_b,t_a)$ is some function of t_a and t_b. For the case where the coefficients in Eq.(2.4.1) are time independent F can depend only on the difference $t = t_b - t_a$.

Now we can get further information about F from Eq.(2.2.4). This we demonstrate for the case of a free particle where Eq.(2.2.4) yields.

$$F(t+s)\sqrt{t+s} = F(t)\sqrt{t}\; F(s)\sqrt{s}\; A \tag{2.4.10}$$

Now, if we define a new function f(t)

$$f(t) \equiv A\, F(t)\sqrt{t} \tag{2.4.11}$$

Inserting this into Eq.(2.4.10) we get

$$f(t+s) = f(t)f(s)\ . \tag{2.4.12}$$

This relation simply says that f(t) must be of the form $f(t) = e^{at}$. Now choosing $t = \epsilon$, an expansion in a Taylor series to zeroth order in ϵ yields

$$f(\epsilon) = 1 \tag{2.4.13}$$

This tells us to set $a = 1$, so that F(t) becomes

$$F(t) = \frac{1}{A\sqrt{t}} = A' \tag{2.4.14}$$

giving the normalizing factor. The technique given here will be clear when we consider the harmonic oscillator example in a later subsection.

To summarize : For a quadratic Lagrangian the propagator is evaluated by Feynman's theorem which involves essentially the computation of the classical action S_{cl}, and it takes the form

$$K(x_b,t_b;x_a,t_a) =$$

$$F(t_b,t_a)\exp\left[\frac{i}{\hbar} S_{cl}\left(x_b,t_b;x_a,t_a\right)\right], \qquad (2.4.9)$$

where F is an entirely time dependent function described as a conditional path integral. As seen above we did not evaluate this conditional path integral but gave another procedure for obtaining F. Papadapoulos(12) has recently evaluated this conditional path integral for a general quadratic Lagrangian.

However it is also possible to obtain the normalizing factor F by using the Van Vleck-Pauli formula. In this formula $F(t_b,t_a)$ is given by

$$F(t_b,t_a) = \left[\frac{i}{\hbar 2\pi}\frac{\partial^2}{\partial x_b \partial x_a} S_{cl}\left(x_b,t_b;x_a,t_a\right)\right]^{1/2}. \qquad (2.4.15)$$

In the next subsection we will give a complete derivation of this formula.

2.5. Derivation of the Van-Vleck Determinant.

As discussed in the previous sections, in the limit $\hbar \to 0$, the propagator is given by the semiclassical formula :

$$K(x_b,t_b;x_a,t_a) = A(x_b,t_b;x_a,t_a)\, e^{i/\hbar\, S_{cl}[b,a]} \quad . \tag{2.5.1}$$

We have shown that this expression is exact for the cases where the Lagrangian is quadratic in x and $\dot{x}$. Then the preexponential factor depends only on time. In this subsection we will calculate the preexponential factor in an alternative way and for more general cases.

Let the motion be considered as a result of two subsequent processes, such that

$$S_{cl}(x_b,t_b;x_a,t_a) = S_{cl}(x_b,t_b;x_c,t_c) + S_{cl}(x_c,t_c;x_a,t_a), \tag{2.5.2}$$

under the condition

$$\frac{\partial S_{cl}(x_b,x_c)}{\partial x_c} + \frac{\partial S_{cl}(x_c,x_a)}{\partial x_c} = 0 \tag{2.5.3}$$

which means that x_c is a point on the classical trajectory from x_a to x_b. It is also understood from the last equation that the intermediate point x_c on the trajectory is obtained as an extremum of the action at fixed end points. Writing the fundamental property of the propagator Eq.(2.2.4), in the semiclassical approximation

$$A\exp\left(\frac{i}{\hbar}S_{cl}\right)=\int A_1A_2\exp\frac{i}{\hbar}\left(S_{cl_1}+S_{cl_2}\right)dx_c' \quad , \qquad (2.5.4)$$

where S_{cl} and A depend on $(x_b,t_b;x_a,t_a)$, S_{cl_1} and A_1 depend on $(x_b,t_b;x_c',t_c')$, and S_{cl_2} and A_2 depend on $(x_c',t_c';x_a,t_a)$. Since we are working in the limit $\hbar\longrightarrow 0$ the calculation of the integral in Eq.(2.5.4) corresponds to finding the asymtotic value of this integral of a rapidly oscillating function, which is called the stationary phase method. In Eq.(2.5.4) $A_1.A_2$ is slowly varying and $1/\hbar$ is large. Now let us write Eq.(2.5.4) as

$$A\exp\left(\frac{i}{\hbar}S_{cl}\right)=\int\exp\left\{\frac{i}{\hbar}\left(S_{cl_1}+S_{cl_2}\right)+\right.$$
$$\left.+\ln A_1+\ln A_2\right\}dx_c' . \qquad (2.5.5)$$

The stationary point of the integrand follows from the expression

$$\frac{\partial S_{cl_1}}{\partial x_c}+\frac{\partial S_{cl_2}}{\partial x_c}+\hbar\left(\frac{1}{A_1}\frac{\partial A_1}{\partial x_c}\right.$$
$$\left.+\frac{1}{A_2}\frac{\partial A_2}{\partial x_c}\right)=0 . \qquad (2.5.6)$$

The semiclassical approximation involves neglecting the term in the parenthesis, since it is multiplied by $\hbar$ which is small and $A_1 . A_2$ is slowly varying. In this approximation Eq.(2.5.6) is identical to Eq.(2.5.3) and x_c' becomes x_c. Furthermore, expanding the integrand of the integral in the Eq.(2.5.5) in a Taylor series it becomes

$$A \exp\left(\frac{i}{\hbar} S_{cl}\right) = \int dx_c' \exp\left\{ \frac{i}{\hbar}\left(S_{cl_1} + S_{cl_2}\right)\Big|_{x_c} + \ln A_1\Big|_{x_c} + \ln A_2\Big|_{x_c} + \frac{i}{\hbar}\frac{\partial^2}{\partial x_c'^{\alpha}\,\partial x'^{\beta}} \cdot \left(S_{cl_1} + S_{cl_2}\right)\Big|_{x_c'=x_c} \left(x_c'^{\alpha} - x_c^{\alpha}\right)\left(x_c'^{\beta} - x_c^{\beta}\right)\right\} . \qquad (2.5.7)$$

Recalling Eq.(2.5.2) this relation becomes

$$A\left(x_b, t_b; x_a, t_a\right) = \left[(2\pi i\hbar)^f / \det B \right] A\left(x_b, t_b; x\ , t\ \right) A\left(x_c, t_c; x_a, t_a\right), \qquad (2.5.8)$$

where B is given by

$$B_{\alpha\beta} = \frac{\partial^2 S_{cl}(x_b, x_c)}{\partial x_c^{\alpha}\,\partial x_c^{\beta}} + \frac{\partial^2 S_{cl}(x_c, x_a)}{\partial x_c^{\alpha}\,\partial x_c^{\beta}} \qquad (2.5.9)$$

Here the intermediate coordinate x_c is a function of $x_a, x_b; t_b - t_c,\ t_c - t_a$ given by Eq.(2.5.2). To find a better expression for the matrix B, apply the operator $\partial^2/\partial x_a^{\alpha}\,\partial x_b^{\beta}$ to Eq.(2.5.2) and the operators $\partial/\partial x_a^{\alpha}$ and $\partial/\partial x_b^{\alpha}$ to Eq.(2.5.3). Therefore, we get three equations :

$$\frac{\partial^2 S_{cl}(x_b,x_a)}{\partial x_a^\alpha\,\partial x_b^\beta} = \frac{\partial^2 S_{cl}(x_c,x_a)}{\partial x_a^\alpha\,\partial x_c^\gamma}\,\frac{\partial x_c^\gamma}{\partial x_b^\beta} + \frac{\partial x_c^\gamma}{\partial x_a^\alpha}\,\frac{\partial^2 S_{cl}(x_b,x_c)}{\partial x_c^\gamma\,\partial x_b^\beta}$$

$$+\frac{\partial x_c^\gamma}{\partial x_a^\alpha}\left(\frac{\partial^2 S_{cl}(x_b,x_c)}{\partial x_c^\gamma\,\partial x_c^\delta}+\frac{\partial^2 S_{cl}(x_c,x_a)}{\partial x_c^\gamma\,\partial x_c^\delta}\right)\frac{\partial x_c^\delta}{\partial x_b^\beta}$$

$$\frac{\partial^2 S_{cl}(x_c,x_a)}{\partial x_a^\alpha\,\partial x_c^\beta} + \frac{\partial x_c^\gamma}{\partial x_a^\alpha}\left(\frac{\partial^2 S_{cl}(x_b,x_c)}{\partial x_c^\gamma\,\partial x_c^\beta}+\frac{\partial^2 S_{cl}(x_c,x_a)}{\partial x_c^\gamma\,\partial x_c^\beta}\right)=0$$

$$\frac{\partial^2 S_{cl}(x_b,x_c)}{\partial x_c^\alpha\,\partial x_b^\beta} + \left(\frac{\partial^2 S_{cl}(x_b,x_c)}{\partial x_c^\alpha\,\partial x_c^\gamma}+\frac{\partial^2 S_{cl}(x_c,x_a)}{\partial x_c^\alpha\,\partial x_c^\gamma}\right)\frac{\partial x_c^\gamma}{\partial x_b^\beta}=0 \tag{2.5.10}$$

or using the notation

$$Q_\alpha^{(1)\beta} = \partial x_c^\beta/\partial x_a^\alpha\ , \qquad Q_\alpha^{(2)\beta} = \partial x_c^\beta/\partial x_b^\alpha$$

$$D_{\alpha\beta} = \partial^2 S_{cl}(x_b,x_a)/\partial x_a^\alpha\,\partial x_b^\beta$$

$$D_{\alpha\beta}^{(1)} = \partial^2 S_{cl_1}/\partial x_a^\alpha\,\partial x_c^\beta\ , \quad D_{\alpha\beta}^{(2)} = \partial^2 S_{cl_2}/\partial x_c^\alpha\,\partial x_b^\beta . \tag{2.5.11}$$

Eqs.(2.5.10) become

$$D_{\alpha\beta} = D^{(1)}_{\alpha\gamma}\, Q^{(2)\gamma}_{\beta} + Q^{(1)\gamma}_{\alpha}\, D^{(2)}_{\gamma\beta} + Q^{(1)\gamma}_{\alpha}\, B_{\gamma\delta}\, Q^{(2)\delta}_{\beta} ,$$

$$D^{(1)}_{\alpha\beta} \quad Q^{(1)\gamma}_{\alpha}\, B_{\gamma\beta} = 0 , \quad D^{(2)}_{\alpha\beta} + B_{\alpha\gamma}\, Q^{(2)\gamma}_{\beta} = 0 . \tag{2.5.12}$$

To exclude the matrices $Q^{(1)}$ and $Q^{(2)}$, it is appropriate to use matrix notation. Thus

$$\begin{aligned} D &= D_1 Q_2^T + Q_1 D_2 + Q_1 B Q_2^T \\ D_1 &+ Q_1 B = 0 \\ D_2 &+ B Q_2^T = 0 . \end{aligned} \tag{2.5.13}$$

From the last two equations

$$\begin{aligned} Q_1 &= - D_1 B^{-1} \\ Q_2 &= - B^{-1} D_2 \end{aligned} \tag{2.5.14}$$

and substituting these into the first equation of Eqs.(2.5.13)

$$D^{-1} = - D_2^{-1}\, B\, D_1^{-1} \tag{2.5.15}$$

This means that

$$B = - D_2 D^{-1} D_1 = - D_2 \left(- D^{-1}\right)\left(- D_1\right) \tag{2.5.16}$$

Now, let us take this expression for B into Eq.(2.5.8) and group the related terms.

$$\frac{(2\pi i\hbar)^{f/2} A}{\sqrt{\det(-D)}} = \frac{(2\pi i\hbar)^{f/2} A_2}{\sqrt{\det(-D_2)}} \frac{(2\pi i\hbar)^{f/2} A_1}{\sqrt{\det(-D_1)}} . \qquad (2.5.17)$$

The solution is evidently

$$A = \left[\det(-D) / (2\pi i\hbar)^f\right]^{1/2} .$$

This expression for the preexponential factor in the semiclassical approximation was found, using the Schrödinger equation, in an early work by Van Vleck. **Note** here that if the action is bilinear in x_a and x_b for any bilinear Hamiltonian, the matrix D is independent of x_a and x_b. In this case the exact solution is given by the semiclassical approximation.

2.6. Harmonic Oscillator

In this subsection we will apply the method given in the subsection(2.4) to calculate the path integral of a linear harmonic oscillator. For a harmonic oscillator the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2} x^2 . \qquad (2.6.1)$$

Thus the path integral is given by

$$K(b,a) = \int_a^b \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) dt\right\} Dx(t). \qquad (2.6.2)$$

Now, let us expand each path about the path that makes the largest contribution to the path integral. Thus for any path we write

$$x = \bar{x} + y, \qquad Dx(t) = Dy(t).$$

Upon introducing this into the classical action.

$$S[\bar{x}+y] = \frac{m}{2}\int_{t_a}^{t_b}\left[\dot{\bar{x}}^2 - \omega^2\bar{x}^2\right]dt + {} + \frac{m}{2}\int_{t_a}^{t_b}\left[\dot{y}^2 - \omega^2 y^2\right]dt. \qquad (2.6.3)$$

The first integral gives the minimizing action S_{cl}. Thus we may write

$$K(b,a) = F(T)\, e^{i/\hbar\, S_{cl}}, \qquad (2.6.4)$$

where

$$F(T) = \int_0^0 \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m}{2}\left(\dot{y}^2 - \omega y^2\right)dt\right\} Dy(t),$$

and $T = t_b - t_a$ with $y(t_a) = y(t_b) = 0$, is a factor independent of the end points x_a and x_b. Thus expansion about the classical path allows us to separate the path integral into a factor dependent upon T and the end points, and another path integral dependent upon T alone. Thus the important dependence on x_a and x_b can be found by simply solving the minimizing differential equations subject to the end-point conditions $x(t_a) = x_a$ and $x(t_b) = x_b$ and calculating the integral

$$\int_{t_a}^{t_b} \frac{m}{2}\left(\dot{\bar{x}}^2 - \omega^2 \bar{x}\right) dt \, .$$

The minimizing differential equation yields the solution

$$X(t) = C\cos(\omega t + \varphi) \tag{2.6.5}$$

where C is a constant which will be determined by the initial and final conditions. Upon substituting this solution into the above integrand, we get

$$S_{cl} = \int_{t_a}^{t_b} \frac{m}{2}\omega^2 C^2\left[\sin^2(\omega t+\varphi) - \cos^2(\omega t+\varphi)\right] dt$$

$$= -\frac{m\omega C^2}{2}\Big[\sin(\omega t_b+\varphi)\cos(\omega t_b+\varphi)$$

$$- \sin(\omega t_a+\varphi)\cos(\omega t_a+\varphi)\Big]. \tag{2.6.6}$$

Considering the solution x(t) at t_a and t_b

$$S_{cl} = -\frac{m\omega}{2}\left[\sqrt{C^2 - X_b^2}\; X_b - \sqrt{C^2 - X_a^2}\; X_a\right]. \tag{2.6.7}$$

Notice that

$$\sin\omega T = \sqrt{1 - \frac{X_b^2}{C^2}}\;\frac{X_a}{C} - \sqrt{1 - \frac{X_a^2}{C^2}}\;\frac{X_b}{C}$$

$$\cos\omega T = \frac{X_b}{C}\frac{X_a}{C} + \sqrt{1-\frac{X_b^2}{C^2}}\sqrt{1-\frac{X_a^2}{C^2}}, \qquad (2.6.8)$$

the simultaneous solution of which yields C

$$C^2 = \frac{1}{\sin^2\omega T}\left[X_a^2 + X_b^2 - 2X_aX_b\cos\omega T\right]. \qquad (2.6.9)$$

Therefore, S_{cl} becomes :

$$S_{cl} = \frac{m\omega}{2\sin\omega T}\left[\left(X_a^2+X_b^2\right)\cos\omega T - 2X_aX_b\right]. \qquad (2.6.10)$$

Now let us come to the calculation of F(T).
Consider the property Eq.(2.2.4) :

$$K(b,a) = \int_{x_c} K(b,c)K(c,a)\,dx_c$$

Let $t = t_b - t_c$ and $s = t_c - t_a$, so that $t + s = t_b - t_a = T$.
Employing Eq(2.2.4)

$$K(b,a) = F(t+s)\exp\left\{\frac{im\omega}{2\hbar\sin\omega(t+s)}\left[\left(X_a^2+X_b^2\right)\cos\omega(t+s) - 2X_aX_b\right]\right\}$$

$$= F(t)F(s)\int_{-\infty}^{\infty}\exp\left\{\frac{im\omega}{2\hbar\sin\omega t}\left[\left(X_c^2+X_b^2\right)\cos\omega t -\right.\right.$$

$$- 2x_c x_b\Big] + \frac{im\omega}{2\hbar \sin\omega s}\Big[(x_a^2 + x_c^2)\cos\omega s - 2x_c x_a\Big]\Big\}\, dx_c. \qquad (2.6.11)$$

It can be easily shown by algebraic manipulations that,

$$K(b,a) = F(t+s)\, e^{\frac{i}{\hbar} S_{cl}}$$

$$= F(t)F(s)\, e^{\frac{i}{\hbar} S_{cl}} \int_{-\infty}^{\infty} \exp\Big[\frac{im\omega \sin\omega(t+s)}{2\hbar \sin\omega t \sin\omega s}\Big]$$

$$\cdot \Big\{x_c - \frac{x_b \sin\omega s + x_a \sin\omega t}{\sin\omega(t+s)}\Big\}^2 dx_c . \qquad (2.6.12)$$

On the each side the factors $e^{iS_{cl}/\hbar}$ cancel each other, and the integral on the right hand side is of the form

$$\int_{-\infty}^{\infty} e^{-\alpha\xi^2}\, d\xi = \sqrt{\frac{\pi}{\alpha}}$$

Therefore,

$$F(t+s) = F(t)F(s)\sqrt{\frac{2\pi i\hbar \sin\omega t \sin\omega s}{m\omega \sin\omega(t+s)}}, \qquad (2.6.13)$$

and

$$F(t+s)\sqrt{\sin\omega(t+s)} =$$

$$= F(t)\sqrt{\sin\omega t}\; F(s)\sqrt{\sin\omega s}\sqrt{\frac{2\pi i\hbar}{m\omega}}. \qquad (2.6.14)$$

Introducing a new function,

$$f(t) = \sqrt{\frac{2\pi i\hbar \operatorname{Sin}\omega t}{m\omega}}\; F(t),$$

Eq.(2.6.14) becomes

$$f(t+s) = f(t)\, f(s). \tag{2.6.15}$$

which implies that f(t) must be of the form : $f = e^{at}$.
Using the same arguments that lead to Eq.(2.4.13), $f(t)=1$, F(t) becomes

$$F(t) = \left(m\omega / 2\pi i\hbar \operatorname{Sin}\omega t\right)^{1/2} \tag{2.6.16}$$

and this completes the solution of the path integral of a harmonic oscillator.

2.7. Hamiltonian Derivation of Path Integral

We will now derive the propagator as integrals over trajectories in phase space. To do so, we will begin with the general principles of quantum mechanics and then deducude the Feynman path integral from them.

Let us introduce Hilbert-space operator Q_H acting on the state vector with eigenvalue q in the Heisenberg picture

$$Q_H(t)|q,t\rangle_H = q|q,t\rangle_H . \tag{2.7.1}$$

In this picture, one freezes the complete time dependence of the state vector, and the operators are given by

$$Q_H(t) = e^{\frac{i}{\hbar}Ht} \; Q_S \; e^{-\frac{i}{\hbar}Ht} , \qquad (2.7.2)$$

where Q_S is the time-independent position operator in the Schrödinger picture, and H in the exponent is the Hamiltonian. The time evolution is then given by the equation :

$$|q,t\rangle_H = e^{\frac{i}{\hbar}Ht} \; |q\rangle , \qquad (2.7.3)$$

where $|q\rangle = e^{-iHt/\hbar} \; |q,t\rangle_H$ is an eigenstate of Q_S with eigenvalue q :

$$Q_S|q\rangle = q|q\rangle$$

The finite-time propagator is then given by

$$G(q',t';q,t) = {}_H\langle q',t'|q,t\rangle_H$$
$$= \langle q'|\exp\left\{-\frac{i}{\hbar}H(t'-t)\right\}|q\rangle \qquad (2.7.4)$$

which measures the overlap of eigenstates of the position operator at different times. It studies the whole global motion of the system i.e., specify the dynamics of the system completely. We are going to express this transition amplitude as a path integral in terms of a classical Hamiltonian, H(p,q), without reference to operators and states in Hilbert space.

To proceed further, let us subdivide the time interval t'-t into N equal segments and take the limit $N \longrightarrow \infty$ later. Let

$$\epsilon \equiv \Delta t \equiv (t' - t)/N, \tag{2.7.5}$$

So that $t_i = i\epsilon + t$, $i=0,1...$. Now using the completeness of the state vectors $|q_i, t_i\rangle$ we can write Eq.(2.7.4) as

$$\langle q', t' | q, t\rangle = \langle q' | e^{-\frac{i}{\hbar} HN\epsilon} | q\rangle$$

$$= \langle q' | \left(1 - \frac{i}{\hbar}\epsilon H\right)^N | q\rangle$$

$$= \int dq_1 \cdots dq_{N-1} \langle q' | \left(1 - \frac{i\epsilon}{\hbar} H\right) | q_{N-1}\rangle \cdots \langle q_1 | \left(1 - \frac{i\epsilon}{\hbar} H\right) | q\rangle, \tag{2.7.6}$$

since we consider large N, so that the step in time ϵ is small, and $\exp(-i\epsilon/\hbar\ H) \approx 1 - \frac{i\epsilon}{\hbar} H$. Therefore, we may write each term in the integrand as

$$\langle q_2 | \left(1 - \frac{i}{\hbar}\epsilon H\right) | q_1\rangle =$$

$$\int \frac{dp_1}{2\pi\hbar} \langle q_2 | p_1\rangle \langle p_1 | \left(1 - \frac{i\epsilon}{\hbar} H\right) | q_1\rangle, \tag{2.7.7}$$

and define the classical Hamiltonian H(p,q) by

$$\langle p | H | q\rangle = H(p,q) \langle p | q\rangle \tag{2.7.8}$$

Thus Eq.(2.7.7) can be written as

$$\langle q_2|\left(1-\frac{i\epsilon}{\hbar}H\right)|q_1\rangle =$$

$$\int\frac{dp_1}{2\pi\hbar}\, e^{ip_1(q_2-q_1)/\hbar}\left[1-\frac{i\epsilon}{\hbar}H(p_1,q_1)\right]. \tag{2.7.9}$$

Therefore, G(q',t';q,t) becomes

$$\langle q',t'|q,t\rangle = \lim_{N\to\infty}\int\!\!\int\cdots\int\frac{dp_0}{(2\pi\hbar)^f}\prod_{n=1}^{N-1}\frac{dp_n\,dq_n}{(2\pi\hbar)^f}$$

$$\cdot\exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}p_n(q_{n+1}-q_n)\right]$$

$$\cdot\prod_{n=0}^{N-1}\left[1-\frac{i\epsilon}{\hbar}H(p_n,q_n)\right], \tag{2.7.10}$$

with the conditions $q_0=q$, $q_N=q'$

At small ϵ we have

$$\left[1-\frac{i\epsilon}{\hbar}H(p,q)\right]\langle p|q\rangle \approx \exp\left[-\frac{i\epsilon}{\hbar}H\right]\langle p|q\rangle. \tag{2.7.11}$$

Then it becomes possible to write the amplitude Eq.(2.7.10) over unitary amplitudes replacing

$\left(1-i\epsilon/\hbar\, H\right)$ by exp $\left(-i\epsilon/\hbar\, H\right)$

$$\langle q',t'|q,t\rangle = \lim_{N\to\infty} \iint \cdots\cdots \int \frac{dp_0}{(2\pi\hbar)^f} \prod_{n=1}^{N-1} \frac{dp_n\, dq_n}{(2\pi\hbar)^f}$$

$$\cdot \exp\left[\frac{i}{\hbar} \Delta t \sum_{n=0}^{N-1} \frac{p_n(q_{n+1}-q_n)}{\Delta t} - H(p_n, q_n) \right], \qquad (2.7.12)$$

with $\Delta t \equiv \epsilon$.

Suppose now the set of values $\{p_0, q_1, p_2 \cdots \cdots q_i, p_i, q_{i+1} \cdots\cdots q_{N-1}, p_{N-1}\}$ as to be successive values of certain functions q(t) and p(t), which may be discontinuous functions, such that using the notation

$$t_n = t + \Delta t$$

$$q_n = q(t_n) \qquad (2.7.13)$$

$$p_n = p(t_n)$$

in the limit $N \longrightarrow \infty$ (or $\Delta t \longrightarrow 0$) one can write

$$(q_{n+1} - q_n)/\Delta t \xrightarrow[\Delta t \to 0]{} \dot{q}(t_n)$$

$$\sum_{n=0}^{N-1} f(t_n)\Delta t \xrightarrow[\Delta t \to 0]{} \int_t^{t'} f(\tau) d\tau . \qquad (2.7.14)$$

Consequently, under the light of these facts the finite-time propagator becomes

$$G(q',t';q,t)=$$

$$\int Dp\, Dq \exp\left\{\frac{i}{\hbar}\int_t^{t'}\left[p\dot{q} - H(p,q)\right]d\tau\right\}, \quad (2.7.15)$$

where $\int$ Dq Dp is the volume element of the phase space given by

$$\int Dq\, Dp = \lim_{N\to\infty}\int \prod^{N-1} \frac{dp(\tau)\, dq(\tau)}{(2\pi\hbar)^t} \qquad (2.7.16)$$

Notice here that at the beginning the number of p integrals was one more than the number of q integrals, but in the limit $N \longrightarrow \infty$, it does not matter at all.

If the classical Hamiltonian is in the form

$$H(p,q)= \frac{p^2}{2m} + V(q), \qquad (2.7.17)$$

one can easily obtain the original form the Feynman path integral by just performing the p-integration in the Eq.(2.7.12) by means of the formula :

$$\int \frac{dp}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\left[p\dot{q} - \frac{1}{2m}p^2\right]\right\} =$$

$$\left(\frac{m}{2\pi i\hbar}\right)^{1/2} \exp \frac{i}{\hbar}\frac{m}{2}\dot{q}^2 \qquad (2.7.18)$$

Thus Eq.(2.7.12) becomes

$$G(q',t';q,t)=\lim_{N\to\infty}\int\prod_{n=1}^{N-1}\frac{dq_n}{(m/2\pi i\hbar\epsilon)^f}\exp\left\{\frac{i}{\hbar}\sum_{n=0}^{N-1}\epsilon\right.$$

$$\left.\cdot\left[\frac{m}{2}\left(\frac{q_{n+1}-q_n}{\epsilon}\right)^2-V\left(\frac{q_{n+1}-q_n}{2}\right)\right]\right\}$$

$$=\int Dq\exp\left\{\frac{i}{\hbar}\int_t^{t'}L(q,\dot{q})dt\right\}.$$

(2.7.19)

where L is the Lagrangian

$$L=\frac{m}{2}\dot{q}^2-V(q), \qquad (2.7.20)$$

with $q_0=q(t)$ and $q_N=q'(t')$

Eq.(2.7.19) is just the Feynman integral over paths in coordinate space with the volume element

$$Dq=\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{-f/2}\lim_{N\to\infty}\prod_{n=1}^{N-1}\left\{dq_n\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{-f/2}\right\},$$

$$\epsilon=t'-t/N. \qquad (2.7.21)$$

III. PATH INTEGRALS ON HOMOGENEOUS SPACES

The Feynman path integral as a method for quantization is quite appropriate in a standard situation when the coordinate space is flat and the Hamiltonian is of the form : $H = \frac{p^2}{2m} + V(q)$. However, it needs to be considered more carefully for a study of compact coordinate spaces. For instance, the property that to apply the Feynman method one should have a unique classical trajectory at sufficiently small times for fixed end coordinates q_i and q_f , breaks down, if the coordinate space has a boundary or is closed. In this case one has not only to integrate over q, but also to sum over all possible trajectories which may contribute with different phases. In the next subsection we will study the motion of a free particle on a circle, the U(1) group manifold, in which these kind of problems arise.

3.1. Path Integral for the Group U(I)

Free motion of a material point on a circle corresponds to the Lagrangian :

$$L = \frac{1}{2} I \varphi^2 \tag{3.1.1}$$

with one coordinate φ , $0 \leqslant \varphi \leqslant 2\pi$. This space is the group manifold of the group U(1) and with the metric corresponding to Eq.(3.1.1), it is flat. The action for a curve φ (t) between $\varphi_1 = \varphi(t_1)$ and

$\varphi_2 = \varphi(t_2)$ is

$$S[\varphi(t)] = \int L[\varphi(t)]\, dt. \qquad (3.1.2)$$

Thus the propagator K from (φ_1 ,t_1) to (φ_2 ,t_2) is the sum over the paths of exp $[iS/\hbar]$. In an ordinary case given paths $\varphi(t)$ and $\psi(t)$ with the proper end points there arises no question of relative phase between their contributions to the propagator, since if one deforms $\psi(t)$ continuously into $\varphi(t)$, the contribution due to ψ must continuously go over into that due to φ .

For a particle moving on a circle there are paths between given end points which are not deformable into one another. For instance the paths which loop around a circle different numbers of times are in different homotopy classes, i.e., they are not continuously deformable into one another. Therefore the classical action has the form

$$S_n(\varphi_2, t_2; \varphi_1, t_1) = \frac{(\varphi_2 - \varphi_1 - 2\pi n)^2}{2(t_2 - t_1)}, \qquad (3.1.3)$$

which depends not only the boundary points of the trajectory but also on the number of revolutions in process of the motion, where n= $0, \pm 1, \pm 2, \ldots\ldots$ represents the number of revolutions in positive or negative directions. Thus the propagator is

$$K(\varphi_2, t_2 ; \varphi_1, t_1) = \sum_{n=1}^{\infty} a_n k_n(\varphi, t),$$

$$\varphi = \varphi_2 - \varphi_1 ; \quad t = t_2 - t_1 , \qquad (3.1.4)$$

where

$$k_n(\varphi, t) = \left(\frac{I}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{iI}{2\hbar t}(\varphi - 2\pi n)^2\right],$$

$$|a_n| = 1 . \qquad (3.1.5)$$

The most general a_n' s which are possible are given by $a_n = e^{in\delta}$ so that K $\longrightarrow$ $e^{i\delta} K$ when $\varphi_2 \longrightarrow \varphi + 2\pi$. Using the notation $\varphi_2 = \varphi + \varphi_1$ and $t_2 = t + t_1 \equiv T + t_1$ Eq.(3.1.4) becomes

$$K(\varphi + \varphi_1, T + t_1 ; \varphi_1, t_1) = K(\varphi, T) = K_\delta(\varphi, T), \qquad (3.1.6)$$

$$\gamma = I/\hbar T ,$$

where

$$K_\delta(\varphi, T) = \left(\frac{\gamma}{2\pi i}\right)^{1/2} \sum_{n=-\infty}^{\infty} e^{in\delta} \, e^{i\gamma(\varphi - 2\pi n)^2/2} .$$

Now recall the definition of the Jacobi theta function[13] :

$$\Theta_3(z,t) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 t} \; e^{2inz} , \tag{3.1.7}$$

and its fundamental identity

$$\Theta_3(z,t) = (-it)^{-1/2} \, e^{z^2/i\pi t} \; \Theta_3(z/t, -1/t) . \tag{3.1.8}$$

Θ_3 is analytic in z in the half-plane $\mathrm{Im}(t) > 0$. Obviously K_δ is a theta function and noticing the fact that $\Theta_3(z,t) = \Theta_3(-z,t)$, K_δ becomes

$$K_\delta(\varphi,\tau) = \left(\frac{\gamma}{2\pi i}\right)^{1/2} e^{i\gamma\varphi^2/2} \; \Theta_3\left(\gamma\pi\varphi - \tfrac{1}{2}\delta, 2\pi\delta\right). \tag{3.1.9}$$

Notice that $\mathrm{Im}\,\gamma = 0$ which depriving the series of its absolute convergence x . In spite of this fact it is possible to restore analyticity by taking the moment of inertia I or (γ) to have a small positive imaginary part. Now let us handle this problem with conventional quantum mechanics. In that case we will call the propagator G for just notational convention, and we will show that the path integral (i.e. semiclassical approximation) propagator K is equal to the propagator G of ordinary quantum mechanics. In general the propagator can be written as a sum over stationary states.

$$G(\varphi_2, t_2; \varphi_1, t_1) =$$

$$\sum_m \Psi_m(\varphi_2)\,\Psi_m^*(\varphi_1)\, e^{-iE_m(t_2-t_1)/\hbar}, \qquad (3.1.10)$$

where Ψ_m's and E_m's are the eigenfunctions and eigenvalues of the quantal Hamiltonian $H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\varphi^2}$ respectively :

$$\Psi_m(\varphi) = \frac{1}{(2\pi)^{1/2}}\, e^{i(m+\alpha)\varphi}, \qquad m = 0, \pm 1, \ldots \qquad (3.1.11)$$

and

$$E_m = \frac{\hbar}{2I}(m+\alpha)^2, \qquad 0 \le \alpha < 1 . \qquad (3.1.12)$$

Under rotation by 2π the change in phase of Ψ is $e^{2\pi i\alpha}$. Using Eq.(3.1.10) G can be formed as

$$G_\delta(\varphi, T) =$$

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \exp\left[i\varphi\left(n+\frac{\delta}{2\pi}\right)\right]\exp\left[-\frac{i}{2\gamma}\left(n+\frac{\delta}{2\pi}\right)^2\right], \qquad (3.1.13)$$

where $\delta = 2\pi\alpha$ and $\varphi = \varphi_2 - \varphi_1$, $\gamma = I/\hbar T$ as before. Once again using the definition theta function G_δ becomes

$$G_\delta(\varphi, T) =$$

$$\frac{1}{2\pi} \exp\left[\frac{i\delta\varphi}{2\pi} - \frac{i\delta^2}{8\pi^2\gamma}\right] \theta_3\left(\frac{1}{2}\varphi - \frac{\delta}{4\gamma\pi} , \frac{-1}{2\gamma\pi}\right). \tag{3.1.14}$$

using the identity Eq.(3.1.8)

$$\theta_3\left(\frac{1}{2}\varphi - \frac{\delta}{8\pi\gamma}, \frac{-1}{2\gamma\pi}\right) =$$

$$\left(-i2\gamma\pi\right)^{1/2} e^{i2\gamma\left(\frac{1}{2}\varphi - \frac{\delta}{4\pi\delta}\right)^2} \theta_3\left(\frac{\delta}{2} - \varphi\gamma\pi, 2\gamma\pi\right). \tag{3.1.15}$$

Inserting this relation into the expression for G Eq.(3.1.14) we obtain

$$G_\delta(\varphi, T) = \left(\frac{\gamma}{2\pi i}\right)^{1/2} \exp i\gamma\varphi^2/2$$

$$\cdot\ \theta_3\left(\frac{1}{2}\delta - \gamma\pi\varphi,\ 2\gamma\pi\right). \tag{3.1.16}$$

Noticing the fact $\theta_3(z, t) = \theta_3(-z, t)$, G becomes

$$G_\delta(\varphi, T) = \frac{\gamma}{2\pi i}\, e^{i\gamma\varphi^2/2}\, \theta_3\left(\gamma\pi\varphi - \tfrac{1}{2}\delta, 2\pi\gamma\right), \quad (3.1.17)$$

which is equal to $K_\delta(\varphi, T)$; accordingly, we have shown that the exact solution of the motion in U(1) group manifold is given by the semiclassical series.

3.2. Path Integral on SU(2) Group Manifold

In this subsection we are going to write the path integral for the infinitesimal propagator on SU(2) group manifold by using the semiclassical formula and check the result by the standard quantum mechanical outcome. However we should remark here that our semiclassical formula including the Van Vleck determinant does not hold for this case, since SU(2) is curved i.e., the corresponding metric is non-trivial. For curved spaces and complicated mechanical systems, the normalization of the propagator is a function of the coordinates, which is not the case for, e.g. the harmonic oscillator. Therefore, the short-time propagator needs some modifications in this case.

De Witt[6] suggested in his work in the Riemannian spaces that a modification involving an additional term in the Lagrangian proportional to $\hbar$ is necessary for the propagator to satisfy the Schrödinger equation. The details of this study is outside the scope of this work, hence we will only give the result which is related to our problem. In our case the curvature R of the space is constant and the only modification which arises is a time dependent phase $\exp\left(i\hbar RT/12\right)$ which multiplies the propagator :

$$K\left(x_b,t_b;x_a,t_a\right)=g''^{-1/4}\,D^{1/2}\,g'^{-1/4}$$

$$\cdot\, e^{i\hbar RT/12}\exp\left[\frac{i}{\hbar}S\left(x_b,t_b;x_a,t_a\right)\right], \qquad (3.2.1)'$$

where $g''=g(x_b)$ and $g'=g(x_a)$, and $D=D(x_b,t_b; x_a,t_a)$ is the Van Vleck determinant.

Let us first write the SU(2) group element by means of Pauli matrices

$$U(\varphi,\theta,\psi)=$$

$$\exp\left(-i\varphi\sigma_3/2\right)\exp\left(-i\theta\sigma_y/2\right)\exp\left(-i\psi\sigma_3/2\right), \qquad (3.2.1)$$

where φ, θ and ψ are the Euler angles assigned to the system.

A left multiplication of $U(\varphi,\theta,\psi)$ with the matrix

$$\exp\left(-i\alpha\hat{n}.\vec{\sigma}/2\right) \qquad (3.2.2)$$

results in an active rotation of the system through angle α about the axis $\hat{n}$. On the other hand, passive rotation of the observer is effected by the same multiplication but with α replacing $-\alpha$.

The metric tensor for SU(2) or (SO(3)) is

$$g_{\theta\theta} = g_{\varphi\varphi} = g_{\psi\psi} = 1, \qquad g_{\varphi\psi} = g_{\psi\varphi} = \cos\theta ,$$

the other components are zero. Let us show Euler angles by E =(φ, θ, ψ) for convenience. Then,

$$\begin{aligned} ds^2 &= g_{ij}\, dE_i\, dE_j \\ &= (d\theta)^2 + (d\varphi)^2 + (d\psi)^2 + 2\cos\theta\, d\varphi\, d\psi. \end{aligned} \tag{3.2.3}$$

Consequently, the relation between metric and the Lagrangian corresponding the motion on SU(2) manifold is

$$\begin{aligned} L &= \frac{I}{2}\frac{(ds)^2}{(dt)^2} \\ &= \frac{1}{2} I \left(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\cos \dot{\varphi}\, \psi \right), \end{aligned} \tag{3.2.4}$$

where I is the moment of inertia. Now what we need is this : given the initial and final configurations of the system, what is the action computed along the classical path connecting them? It is clear that the action S from E_a to E_b is a function of U_a and U_b and evidently $T = t_b - t_a$ and $S = f(U_a, U_b)$. At this point, an important argument can be given. Since the external world is rotationally symmetric, it requires f to be invariant under left multiplication of its arguments. The natural lie group metric is also invariant under both left and right multiplication, so that for any $A \in SU(2)$

$$f(B,C) = f(BA, CA) = f(AB, AC).$$

In the same manner, let us multiply the argument of $S = f(U_a, U_b)$ by A from the left and by $U_a^{-1}A$ from the right :

$$S = f\left(1, A U_b U_a^{-1} A\right). \tag{3.2.5}$$

For diagonalizable matrices with which we deal, only the eigenvalues, their multiplicity, and functions of these remain invariant under a similarity transformation. In the case of SU(2) only one independent invariant remains after the application of the determinant and unitarity conditions. Let us call this invariant Γ and for convenience define

$$\cos\frac{\Gamma}{2} = \frac{1}{2}\, Tr\, U_b U_a^{-1}. \tag{3.2.6}$$

S is some function of Γ . In terms of Euler angles

$$\cos\frac{\Gamma}{2} = \cos\frac{1}{2}(\theta_b - \theta_a)\cos\frac{1}{2}(\varphi_b - \varphi_a)$$

$$.\cos\frac{1}{2}(\psi_b - \psi_a) - \cos\frac{1}{2}(\theta_b + \theta_a)$$

$$.\sin\frac{1}{2}(\varphi_b - \varphi_a) - \sin\frac{1}{2}(\psi_b - \psi_a). \tag{3.2.7}$$

To specify the explicit dependence of Γ in S consider the boundary conditions $\theta_a = \theta_b = 0$, $\varphi_b = \varphi_a$ So that the motion can be taken as uniform rotation with the figure axis pointing in the $\Theta = 0$ direction. The resulting ambiguity in φ and ψ is resolved by taking φ = constant; accordingly, constant $= \dot{\psi} = \frac{(\psi_b - \psi_a + 2n\pi)}{(t_b - t_a)}$ where n is the number of times ψ passes through ψ_b for $t < t_b$. Taking this information into Eq.(3.2.4), and integrating between end points

$$S = \frac{I}{2T}\left(\psi_b - \psi_a + 2n\pi\right)^2. \tag{3.2.8}$$

From Eq.(3.2.7) we get only

$$\Gamma \equiv \psi_b - \psi_a + 2n\pi . \tag{3.2.9}$$

Consequently,

$$S = \frac{I}{2T}\,\Gamma^2 . \tag{3.2.10}$$

Obviously, Γ is just the arc length of the geodesic in SU(2) connecting U_a and U_b. In the Eq.(3.2.6) the inverse of the cosine function is multi-valued. This corresponds to the discrete set of geodesics connecting points in SU(2) manifold, each of which is only a local minimum. In the case of SO(3) group, the relevant paths correspond to the solutions of

$$\operatorname{Cos}\frac{1}{2}\Gamma = \pm\frac{1}{2}\operatorname{Tr} U_b U_a^{-1} . \tag{3.2.11}$$

The homotopy group of SO(3) has two elements, i.e., it is doubly connected. Hence between given end points in SO(3) there are two classes of paths.

Now notice that we could determine the classical action for SU(2) in an alternative and more straightforward way. We know that any SU(2) matrix can be written in the form

$$\begin{pmatrix} q_1 + i q_2 & q_3 + i q_4 \\ -q_3 + i q_4 & q_1 - i q_2 \end{pmatrix}, \tag{3.2.12}$$

satisfying the condition $|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2 = 1$

The space of points that obey this last condition is the three-dimensional sphere S^3; consequently, any SU(2) matrix is isomorphic to the sphere S^3 i.e., the parameter space of SU(2) is the sphere S^3 which is parametrized by the components of a four dimensional unit vector. Therefore we can think the motion on SU(2) group manifold equivalently as a free particle moving on the parametrizing sphere. The corresponding Lagrangian is

$$L = \frac{1}{2} \dot{\bar{q}}^2 + \frac{\lambda}{2} \left(\bar{q}^2 - 1 \right) \tag{3.2.13}$$

where $\bar{q} = \bar{q}\,(q_1 , q_2 , q_3 , q_4)$. The Euler-Lagrange equations of motion immediately follow

$$\ddot{\bar{q}} = \lambda \bar{q}, \tag{3.2.14}$$

together with the constraint that the norm of $\bar{q}$ is unity we get

$$\begin{aligned} \ddot{\bar{q}} \cdot \bar{q} &= \lambda \\ \dot{\bar{q}} \cdot \bar{q} &= 0. \end{aligned} \tag{3.2.15}$$

Simultaneously consideration of the last equation with the equations of motion leads us to

$$\ddot{\bar{q}} \cdot \dot{\bar{q}} = \frac{d}{dt}\left(\frac{1}{2}\dot{\bar{q}}^2\right) = 0, \tag{3.2.16}$$

with the result

$$\frac{1}{2}\dot{\bar{q}}^2 = \epsilon = \text{Constant.} \tag{3.2.17}$$

λ is identified by means of this expression and Eq.(3.2.15)

$$\begin{aligned}\lambda = \ddot{\bar{q}} \cdot \bar{q} &= \frac{d}{dt}(\dot{\bar{q}} \cdot \bar{q}) - \dot{\bar{q}}^2 \\ &= -2\epsilon = \text{Constant.}\end{aligned} \tag{3.2.18}$$

Upon inserting this value for λ into the equations of motion we obtain

$$\ddot{\bar{q}} + 2\epsilon\bar{q} = 0 \tag{3.2.19}$$

which is nothing more than a harmonic oscillator equation. The solution to this equation is immediate

$$\bar{q}(t) = \bar{a}\operatorname{Cos}\omega t + \bar{b}\operatorname{Sin}\omega t \tag{3.2.20}$$

where $\omega = \sqrt{2\epsilon}$ and $\bar{a}$ and $\bar{b}$ are constant vectors. To identify $\bar{a}$ and $\bar{b}$ we refer to the constraint $\bar{q}^2 = 1$; accordingly,

$$a^2 = b^2 = 1 \quad \text{and} \quad \bar{a} \cdot \bar{b} = 0 \tag{3.2.21}$$

The first of these immediately verify our previous result Eq.(3.2.17).

Now under the light of these facts, the computation of S_{cl} follows

$$S_{cl} = \int_0^T L\,dt = \epsilon T . \qquad (3.2.22)$$

At this stage we need the physical meaning of S_{cl}. To achieve this we consider the initial and final conditions

$$\bar{q}_1 = \bar{q}(0) = \bar{a}$$
$$\bar{q}_2 = \bar{q}(T) = \bar{a}\, \mathrm{Cos}\,\omega T + \bar{b}\, \mathrm{Sin}\,\omega T. \qquad (3.2.23)$$

The scalar product of these two vectors gives

$$\bar{q}_1 \cdot \bar{q}_2 = \mathrm{Cos}\,\omega T,$$
$$\mathrm{arc\,Cos}\left(\bar{q}_1 \cdot \bar{q}_2\right) = \omega T . \qquad (3.2.24)$$

Let us call this angle ωT between the vectors $\bar{q}_1$ and $\bar{q}_2$, δ . Then

$$\epsilon = \delta^2 / 2T , \qquad (3.2.25)$$

which is the arc length of the geodesic between given points on the sphere S^3. In fact the only classical path is not δ , but there are the classical paths winding around the sphere S^3 a number of times i.e., defining a new angle $\Gamma = \delta + 2n\pi$ where n is the number of revolution, we obtain

$$S_{cl} = \Gamma^2 / 2T . \qquad (3.2.26)$$

This expression is the same as the one we have found by using a different argument.

Now let us return to our previous discussion. The position of the system on the SU(2) group manifold as a function of time can be easly obtained by considering the connection between geodesics and one parameter subgroups

$$U(t) = \exp\left(-i\Gamma t\hat{n}.\vec{\sigma}/2T\right) U_a , \tag{3.2.27}$$

from which the relation of Γ and $\hat{n}$ to U_b and U_a is evident

$$U_b U_a^{-1} = \exp\left(-i\Gamma\hat{n}.\vec{\sigma}/2\right). \tag{3.2.28}$$

Finally, the computation of the factor $g''^{-1/4} D^{1/2} g'^{-1/4}$ is necessary for the evaluation of Eq.(3.2.1'), the expression for the short -time propagator. The calculation of this factor is tedious and the result can be found in the paper by Schulman[15]

$$(g'')^{-1/4} D^{1/2} (g)^{-1/4} = \left(\frac{I}{2\pi i\hbar T}\right)^{3/2} \frac{\Gamma}{2\sin\frac{1}{2}\Gamma} . \tag{3.2.29}$$

Consequently, carrying all these information into Eq.(3.2.11) and noticing that the curvature R of SU(2) is $R = \frac{3}{2}$, the path integral expression for the short-time propagator is written as

$$K = \left(\frac{I}{2\pi i\hbar T}\right)^{3/2} \frac{\Gamma}{2\sin\frac{1}{2}\Gamma} \exp\left(\frac{i\hbar T}{8I}\right) . \exp\left(iI\Gamma^2/2\hbar T\right), \tag{3.2.30}$$

where $T = t_b - t_a$, and Γ is the smallest solution of Eq.(3.2.6). Let us call this Γ_0 , then $\Gamma = \Gamma_0 + 2n\pi$, $n = \pm 1, \pm 2, \ldots$.

Now we are going to check this result by the Green function which will be constructed by standard Quantum mechanical technique. However, we will first do the same thing for the sphere S^2 to be more instructive. The quantal Hamiltonian is

$$H = -\frac{\hbar^2}{2I}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \right.$$

$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right], \qquad (3.2.31)$$

its eigenfunctions and eigenvalues are

$$\Psi_{\ell m} = Y_{\ell m}(\theta, \phi),$$

$$E_{\ell m} = \ell(\ell+1), \qquad (3.2.32)$$

where Y_{lm}'s are the spherical harmonics.

Then using the formula, the spectral expansion for the Green function

$$G(\theta_2, \phi_2, t_2; \theta_1, \phi_1, t_1) = \sum_{\ell m} \Psi_{\ell m}(\theta_2, \phi_2)\, \Psi^*_{\ell m}(\theta_1, \phi_1)\, e^{-iE_{\ell m}(t_2 - t_1)/\hbar}$$

we obtain

$$G(\theta_2, \phi_2, t_2; \theta_1, \phi_1, t_1) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_b, \phi_b)\, Y^*_{\ell m}(\theta_a, \phi_a)\, e^{\frac{-i}{\hbar}\ell(\ell+1)T}, \qquad (3.2.33)$$

where

$$Y_{\ell m}(\theta,\phi)=\left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right]^{1/2}$$

$$\cdot\,(-1)^m e^{im\phi} P_\ell^m(\cos\theta)\,. \tag{3.2.34}$$

It is possible to take the summation over m by using the addition formula for spherical harmonics.

$$P_\ell(\cos\theta)=\sum_{m=-\ell}^{+\ell} e^{im(\phi_b-\phi_a)}\frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}$$

$$\cdot\, P_\ell^m(\cos\theta_a)\,P_\ell^m(\cos\theta_b)\,. \qquad (.2.35)$$

where

$$\cos\theta=\cos\theta_a\cos\theta_b$$

$$+\sin\theta_a\sin\theta_b\cos(\phi_a-\phi_b). \tag{3.2.36}$$

It is evident that the angle θ given by Eq.(3.2.36) is the angle between the vectors $\vec{q}_a$ and $\vec{q}_b$ that specify the end points on the sphere S^2, since S^2 is parametrized by a unit vector whose components are

$$\vec{q} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix} \qquad (3.2.37)$$

Consequently Green's function is written as

$$G(\theta,T) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \, e^{-i\frac{T}{\hbar}\ell(\ell+1)} \, P_\ell(\cos\theta)$$

(3.2.38)

Now let us turn to our original problem. Laplace operator on SU(2) group manifold is given by

$$\nabla^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2} - 2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi}\right), \qquad (3.2.39)$$

with this Laplacian the Schrödinger equation is

$$-\frac{\hbar^2}{2I}\nabla^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \qquad (3.2.40)$$

The stationary states are labeled by j,m,k, which are related respectively to the eigenvalues of

$J^2, J_z, J_\zeta = \hat{n}_\zeta . \vec{J}$, where $\hat{n}_\zeta$ points along the figure axis.

The normalized eigenfunctions having the appropriate rotational properties together with their eigenvalues are

$$\left(\frac{2j+1}{8\pi^2}\right)^{1/2} D^{j*}_{mk}(\varphi,\theta,\psi),$$

$$E_{jmk} = \frac{\hbar^2}{2I} j(j+1) , \tag{3.2.41}$$

D^{j}_{mk} is the representation matrix of the rotation group and given by the expression[16].

$$D^{j}_{mk}(\varphi,\theta,\psi) = e^{im\varphi} d^{j}_{mk}(\theta) e^{im\psi} , \tag{3.2.42}$$

where the matrix element $d^{j}_{mk}(\theta)$ is given by in terms of the Jacobi polynomial :

$$d^{j}_{mk}(\theta) = \left[\frac{(j+m)!\,(j-m)!}{(j+k)!\,(j-k)!}\right]^{1/2} \left(\cos\frac{\theta}{2}\right)^{m+k}$$

$$\cdot \left(\sin\frac{\theta}{2}\right)^{m-k} P^{(m-k,\,m+k)}_{j-m} . \tag{3.2.43}$$

This relation is strictly speaking only valid for non negative values of m-k and m+k. Nevertheless all the results to be derived from it are true for **the** general case.

Consequently, for the Green's function we have

$$G(E_b,t_b;E_a,t_a)=\sum_{jmk}\frac{2j+1}{8\pi^2}\,D^{j*}_{mk}(E_b)\,D^{j}_{mk}(E_a)$$
$$\cdot\exp\left(-\frac{i\hbar T}{2I}\,j(j+1)\right), \tag{3.2.44}$$

here the summation index j runs over all nonnegative integer and half-integer values, and $-j\leq m,k\leq j$

Notice that the m and k dependence is only in the D matrices. To take **the** summation over m and k, we will use the unitarity property of U. Hence consider the argument of D to be $U(E)\in SU(2)$, rather than just E. Then $D^{j*}_{mk}(U)=D^{j}_{km}(U^{-1})$, and for any j

$$\sum_{mk}D^{j*}_{mk}(U_b)\,D^{j}_{mk}(U_a)$$
$$=\sum_{mk}D^{j*}_{mk}(U_b)\,D^{j*}_{mk}(U_a^{-1})$$
$$=\sum_{m}D^{j*}_{mm}(U_bU_a^{-1})=\mathrm{Tr}\,D^{j*}(U_bU_a^{-1}). \tag{3.2.45}$$

Here, D^j can be taken diagonal, because the trace is invariant under a similarity transform, and this will be

the case if the z axis is taken along the direction $\hat{n}$ defined by U_b and U_a through Eq.(3.2.28).
Then, with the Γ of Eq.(3.2.6)

$$D^{j}_{mm}\left(U_b U_a^{-1}\right) = \exp\left(im\Gamma\right), \tag{3.2.46}$$

so that

$$\begin{aligned} \mathrm{Tr}\, D^{j*}(U_b U_a^{-1}) &= \sum_{m=-j}^{j} e^{im\Gamma} \\ &= e^{-ij\Gamma}\left[1 + e^{i\Gamma} + \dots + e^{2ij\Gamma}\right] \\ &= e^{-ij\Gamma}\,\frac{\left[1 - e^{i(2j+1)\Gamma}\right]}{1 - e^{i\Gamma}} \\ &= \frac{\sin\,(j+\frac{1}{2})\Gamma}{\sin\Gamma/2}\,. \end{aligned} \tag{3.2.47}$$

Together with this expression G follows

$$G = \frac{-2}{\sin\Gamma/2}\,\frac{1}{8\pi^2}\,\frac{\partial}{\partial\Gamma}\sum_{j}\cos\left[(j+\tfrac{1}{2})\Gamma\right] e^{-\frac{i\hbar T}{2I}j(j+1)}. \tag{3.2.48}$$

To get rid of the summation over half-integers let us transform j to $\ell/2$ where ℓ can take integer values

$$G = \frac{-2}{\sin\Gamma/2}\,\frac{1}{8\pi^2}\,\frac{\partial}{\partial\Gamma}\sum_{\ell=0}^{\infty}\cos\tfrac{1}{2}(\ell+1)\Gamma\; e^{-\frac{i\hbar T}{8I}\ell(\ell+2)}. \tag{3.2.49}$$

Now let us take ℓ to k-1, such that

$$G = \frac{-2}{\sin\Gamma/_2}\,\frac{e^{i\hbar T/8I}}{8\pi^2}\,\frac{\partial}{\partial\Gamma}\left\{\sum_{k=1}^{\infty}\cos k\Gamma/_2\; e^{-\frac{i\hbar T}{8I}k^2}\right\}$$

$$= \frac{-1}{\sin\Gamma/_2}\,\frac{e^{i\hbar T/8I}}{8\pi^2}\,\frac{\partial}{\partial\Gamma}\left\{\sum_{k=-\infty}^{\infty} e^{ik\frac{\Gamma}{2}} \cdot e^{-\frac{i\hbar T}{8I}k^2} - 1\right\}. \tag{3.2.50}$$

We will now employ the eq.(A.8)

$$\sum_{n=-\infty}^{\infty} e^{in\theta}\, e^{-in^2 T} = \sqrt{\frac{\pi}{iT}}\sum_{n=-\infty}^{\infty} e^{i/_{4T}\,(\theta+2n\pi)^2}, \tag{A.8}$$

the formula which is derived in the Appendix on considering the Jacobi imaginary transform(13). Then Eq.(3.2.50) becomes

$$G = \frac{-1}{\sin\Gamma/_2}\,\frac{e^{\frac{i\hbar T}{8I}}}{8\pi^2}\left(\frac{8I\pi}{i\hbar T}\right)^{1/2}\frac{\partial}{\partial\Gamma}$$

$$\cdot\left\{\sum_{k=-\infty}^{\infty} e^{iI/_{2\hbar T}(\Gamma+4n\pi)^2} - 1\right\}. \tag{3.2.51}$$

on taking the Γ -derivative we obtain

$$G = \left(\frac{I}{2\pi i\hbar T}\right)^{3/2} \sum_{n=-\infty}^{\infty} \frac{(\Gamma + 4n\pi)}{\sin\frac{1}{2}\Gamma}$$

$$\cdot\, e^{\frac{i\hbar T}{8I}} \exp\left[\frac{iI(\Gamma + 4n\pi)^2}{2\hbar T}\right], \qquad (3.2.52)$$

as infinitesimal propagator on SU(2) group manifold. For sufficiently small T, the only term contributing n = 0. Hence

$$G = \left(\frac{I}{2\pi i\hbar T}\right)^{3/2} \frac{\Gamma}{\sin\Gamma/2}\, e^{\frac{i\hbar T}{8I}}\, e^{\frac{iI}{2\hbar T}\Gamma^2}$$

$$(3.2.53)$$

which agrees with the semiclassical result Eq.(3.2.30).

3.3. Some Remarks About the Path Integral on the Hyperspheres S^{2n+1} and S^{2n}

In this subsection we will consider the exact solution to the free quantal motion on the spheres S^n and discuss the validity of semiclassical approximation in these cases.

Using the spectral expansion, the exact solution for S^n is given by[9],

$$K(\theta, \tau) = \sum_{\ell=0}^{\infty} \frac{\ell+\nu}{\nu} \, C_\ell^\nu(\cos\theta) \, e^{\frac{-i\hbar\tau}{2} \ell(\ell+2\nu)} \qquad (3.3.1)$$

where $C_\ell^\nu(\cos\theta)$ is the Gegenbauer polynomial, $\nu=(n-1)/2$

Now let us remember what we have done before. We know that for the cases of $U(1) = S^1$ and $SU(2) = S^3$ the semiclassical approximation to the path integral leads to Eq.(3.3.1). Hence for these cases the semiclassical formula gives the exact result. The equivalence of the semiclassical formula to the exact result Eq.(3.3.1) is obtained by using the Poisson summation formula or equivalently the Jacobi imaginary transformation for Θ - function in the last step. As we have seen the use of this transformation contains an infinite summation, however, in contrast to Eq.(3.3.1) where the summation extends over energy eigenvalues, the new summation is over homotopy classes of classical paths which wind around the sphere (S^1 or S^3) different number of times. An immediate question is whether Eq.(3.3.1) which contains a sum over energy eigenvalues can be transformed into such a form for other spheres S^n. Here we will show that this is indeed possible for odd n, but we have not been able to find such a formula for even n.

We start by differentiating Eq.(3.3.1) with respect to $\cos\Theta$

$$\frac{d}{d(\cos\theta)} K^{\nu}(\theta, T) =$$

$$\sum_{\ell=0}^{\infty} 2(\ell+\nu)\, C_{\ell-1}^{\nu+1}(\cos\theta)\, e^{-\frac{i\hbar T}{2}\ell(\ell+2\nu)} . \tag{3.3.2}$$

The differentiation rule[17] of the Gegenbauer polynomial is

$$\frac{d\, C_{\ell}^{\nu}(\cos\theta)}{d(\cos\theta)} = 2\nu\, C_{\ell-1}^{\nu+1}(\cos\theta) . \tag{3.3.3}$$

In Eq.(3.3.2) we use the superscript ν on K to mean that K^{ν} is the propagator for S^n. Consider now the index transformation $\ell \longrightarrow \ell+1$ so that

$$\ell(\ell+2\nu) \longrightarrow \ell(\ell+2\nu+2) + (2\nu+1)$$

Then

$$\frac{d\, K^{\nu}(\theta,T)}{d(\cos\theta)} = 2\nu\, C_{-1}^{\nu+1}(\cos\theta) \cdot e^{-i\hbar T(2\nu+1)/2}$$

$$+ \sum_{\ell=0}^{\infty} 2(\ell+\nu+1)\, C_{\ell}^{\nu+1}(\cos\theta)\; e^{-i\hbar T(2\nu+1)/2}$$

$$\cdot \exp\left[-\frac{i\hbar T}{2}\,\ell(\ell+2\nu+2)\right] \tag{3.3.4}$$

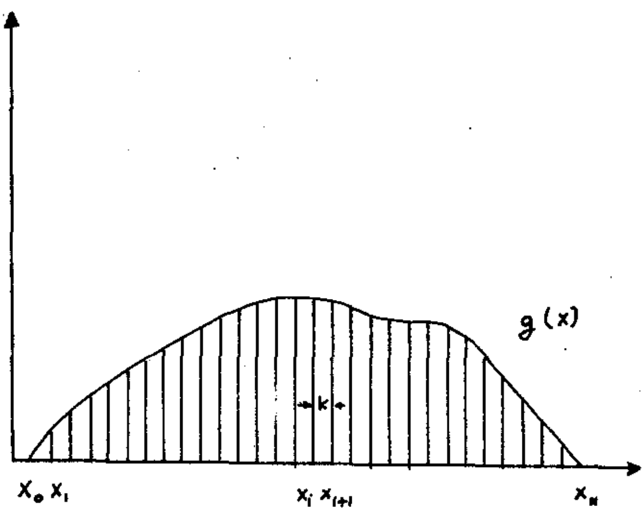
Notice that $C_{-1} = 0$ and the summation on the right in the above equation is related to $K^{\nu+1}(\theta, T)$; therefore,

$$\frac{d}{d(\cos\theta)} K^{\nu}(\theta,T) = 2(\nu+1)\, e^{-\frac{i\hbar T}{2}(2\nu+1)} K^{\nu+1}(\theta,T). \tag{3.3.5}$$

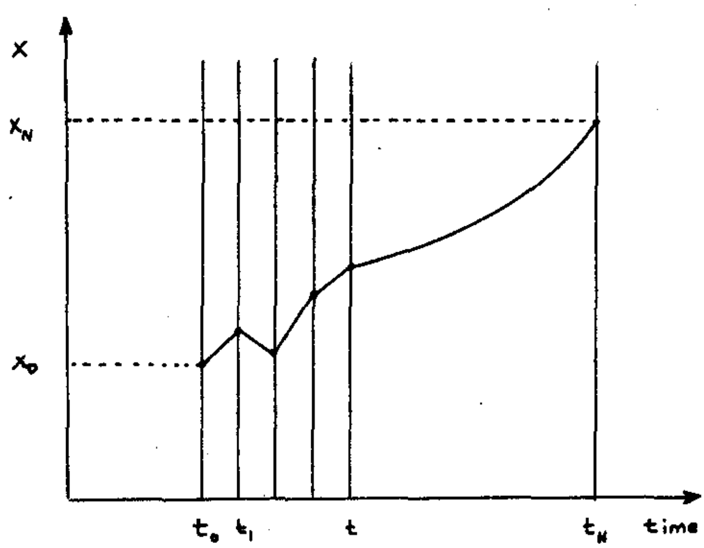
This formula gives $K^{\nu+1}$ in terms of K^{ν}.

$$K^{\nu+1}(\theta,T) = \frac{e^{\frac{i\hbar T}{2}(2\nu+1)}}{2(\nu+1)} \frac{d}{d(\cos\theta)} K^{\nu}(\theta,T). \tag{3.3.6}$$

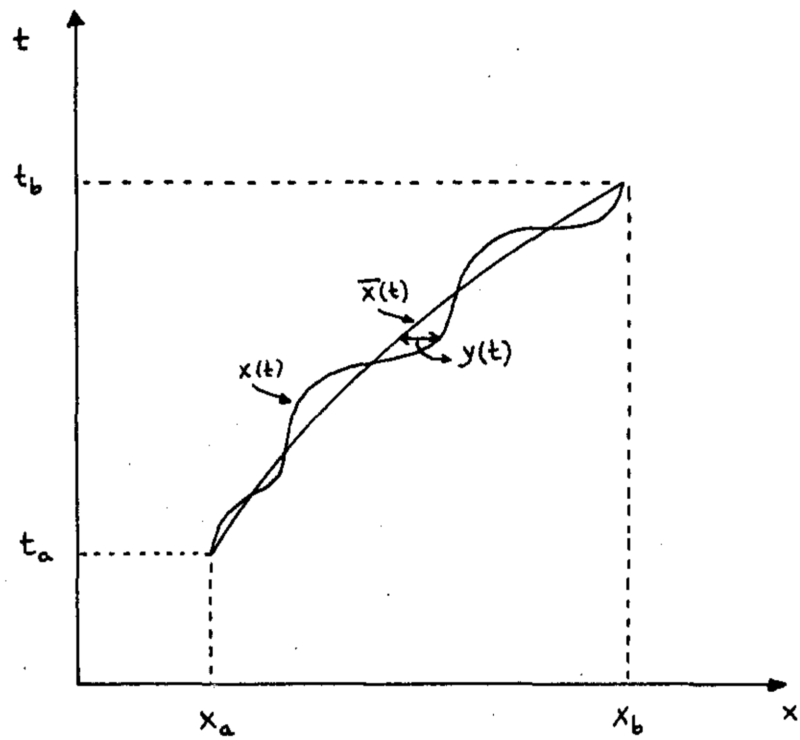
This nice expression makes it possible to calculate the propagator for $S^3, S^5, \ldots, S^{2n+1}$ given K^0 for S^1. Notice that it is applicable both the exact result and to the path integral result. When applied to the path integral it coincides with the semiclassical result for $U(1) = S^1$ and $SU(2) = S^3$. For other S^{2n+1}'s ($n = 2,3,\ldots$) it does not coincide with the semiclassical result, showing that the semiclassical result is not exact for these cases. However, the formula is still of the form of a summation over classical trajectories and indicates some form of a quasi - classical approximation for these cases may also give the exact result.



Fig(2.1.1)



Fig(2.1.2)



Fig(2.4.1)

SUMMARY

The Feynman path integral formulation of quantum mechanics carrying a global approach in contrast to the local Schrödinger equation provides a powerfull technique that a significant portion of today's theoretical calculations rely on.The path integral takes an important role not only for its modern applications in quantum field theory but also forms a useful basis for the development of approximation methods. It is especially adequate for semiclassical considerations in some cases.

In this work we have given the fundamental principles underlying the Feynman path integral approach to quantum mechanics, showing its equivalence to the Schrödinger equation for a simple special case. It turns out that despite its intuitive appeal the applicability of this approach has been limited because of analytical difficulties and expressions for path integrals are available only for a few cases.From this point of view, we have shown a calculation method for systems with quadratic Lagrangians in the variables $\dot{x}$ and x and later on applied it to a special example, namely the harmonic oscillator in one dimension. In the calculation technique we have just mentioned above evaluation of path integral reduces to the computation of exponential of classical action with an overall normalizing factor depending on time only. This preexponential factor is given alternatively by Van Vleck determinant. Furthermore, this expression related to the second derivatives of the classical action is more appropriate in the case of curved spaces since in this case the preexponential factor depends also on the coordinates. We have then given the complete derivation of the Van Vleck determinant. Our next task has been the Hamiltonian derivation of path integral. We have begun with the usual operator formalism and shown that the Feynman original form of path integral is equivalent to this form only for the case where the Hamiltonian is given in the quadratic form.

In the context of the semiclassical approximation the path integral propagator for free quantal motion on group manifolds is of prime interest, since it turns out that the exact result is given correctly by the semiclassical result. In these cases the most interesting part of the problem is that. The classical trajectory with fixed end points at sufficiently small times is not unique. We have first considered the example of U(1) group. Writing the path integral in semiclassical approximation we have seen that a sum over an infinite number of classical paths, arising from the multiply connectedness of U(1) group, is necessary. We have shown the equivalence of this result to the exact solution relating the sum over paths to a theta function : Our second example has been the free quantal motion on SU(2) group manifold where we have also used the semiclassical considerations, but it has been necessary to modify the semiclassical formula since SU(2) is curved. We have then obtained the classical action by using two different arguments. Finally, we have written the stationary state expansion and using the Jacobi imaginary transformation shown that the semiclassical solution coincides with the exact solution.

It turns out that the semiclassical approximation is not generally exact on the spheres. While the semiclassical result coincides with the exact solutions for the spheres S^1 and S^3 it does not hold for the other spheres. However, in our present work we have shown that the propagator for the spheres S^{2n+1} can be written by means of a recursion relation giving the propagator in the form of a summation over classical paths which implies that there is the possibility of giving the exact solution by starting solely from the path integral.

APPENDIX

THE POISSON SUMMATION FORMULA AND THE JACOBI IMAGINARY TRANSFORM FOR THETA FUNCTION

Consider the summation

$$\sum_{m=-\infty}^{\infty} e^{im\theta} \; e^{-im^2 t} \tag{A.1}$$

We would like to linearize the m^2 term in the exponent. To do so let us employ the Jacobi imaginary transform which stems from the definition of a Gaussian integral :

$$\int_{-\infty}^{\infty} e^{-a(x+ib)^2} \, dx = \sqrt{\frac{\pi}{a}} \; , \tag{A.2}$$

giving the transform

$$e^{-ab^2} = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - 2iabx} \, dx \, . \tag{A.3}$$

Let us apply this transform to the $e^{-im^2 t}$ term in the summation eq(A.1)

$$\sum_{m=-\infty}^{\infty} e^{im\theta} \, e^{-im^2 t} =$$

$$\sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \sum_{m=-\infty}^{\infty} e^{im(\theta - 2ax)} \, dx \, , \tag{A.4}$$

with $a = it$. Now by virtue of the one dimensional Poisson summation formula :

$$\sum_{n=-\infty}^{\infty} e^{in\theta} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\theta + 2n\pi) \tag{A.5}$$

the eq(A.4) becomes

$$\sum_{m=-\infty}^{\infty} e^{im\theta} e^{-im^2 t} =$$

$$\sqrt{\frac{a}{\pi}}\, 2\pi \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ax^2} \delta(\theta + 2m\pi - 2ax)\, dx \tag{A.6}$$

$$= \sqrt{\frac{\pi}{a}} \sum_{m=-\infty}^{\infty} e^{-a\left(\frac{\theta + 2m\pi}{2a}\right)^2}, \tag{A.7}$$

on using the property of δ function : $\delta(\alpha x) = \alpha^{-1}\delta(x)$.
Therefore the summation eq(A.1) takes the form

$$\sum_{m=-\infty}^{\infty} e^{im\theta} e^{-im^2 t} =$$

$$\sqrt{\frac{\pi}{it}} \sum_{m=-\infty}^{\infty} e^{i\frac{(\theta + 2m\pi)^2}{4t}} \tag{A.8}$$

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