CLASSIFICATION OF CALIBRATIONS ON $\mathbb{R}^n$ FOR $n \leq 7$

by

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ABSTRACT

CLASSIFICATION OF CALIBRATIONS ON $\mathbb{R}^n$ FOR $n \leq 7$

In this thesis, our aim is to understand all possible calibrated submanifolds of $\mathbb{R}^n$ for $n \leq 7$. We start with basic definitions and tools that are used throughout this thesis. Then we study the general theory of calibrated geometries and give some interesting examples of them, following the historical development of the subject. After examining calibrations on $\mathbb{R}^n$ with dimensions one, two, $n-1$, and $n-2$ for arbitrary n, we focus on the nontrivial cases: classification of 3-dimensional calibrations on $\mathbb{R}^6$ and $\mathbb{R}^7$. On $\mathbb{R}^6$, we have four types of calibrations, namely special Lagrangian, Kähler, single point and double point calibrations. on $\mathbb{R}^7$ in addition to these four cases, we get six new types, which are called as associative, $\mathbb{CP}^2$, double $\mathbb{CP}^1$, S^1, S^2 and S^3 calibrations.

ÖZET

$\mathbb{R}^n$ ÜZERİNDEKİ KALİBRASYONLARIN $n \leq 7$ İÇİN SINIFLANDIRILMASI

Bu tezdeki asıl amacımız $n \leq 7$ için $\mathbb{R}^n$'nin olası bütün kalibre edilmiş altmanifoldlarını anlamaktır. Tezin genelinde kullanılacak olan konuyla ilgili temel çalışmayı yaparak başlayacağız. Sonrasında kalibre edilmiş geotmerilerin genel teoresini çalışıp bazı ilginç örneklerini vereceğiz. $\mathbb{R}^n$ üzerindeki bir, iki, n ve $n-1$ boyutlu kalibrasyonları inceledikten sonra, çalışılması kolay olmayan ilk iki duruma yoğunlaşacağız: $\mathbb{R}^6$ ve $\mathbb{R}^7$ üzerindeki üç-boyutlu kalibrasyonların sınıflandırılması. $\mathbb{R}^6$ üzerinde özel Lagrangian, Kähler, tek nokta ve çift nokta isimlere sahip dört farklı kalibrasyona sahip olacağız. $\mathbb{R}^7$ üzerinde ise bunlara ek olarak birleşmeli, $\mathbb{CP}^2$, çift $\mathbb{CP}^1$, S^1, S^2 ve S^3 olarak adlandırdığımız altı farklı kalibrasyon elde edeceğiz.

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LIST OF SYMBOLS

$card$	Cardinality
$\mathbb{CP}^n$	The n-dimensional complex projective plane
dim	Dimension
e_i	A unit basis vector
e^i	The basis one-form dual to e_i
$e_{i_1 i_2 \ldots i_k}$	A basis k-vector in $\Lambda_k \mathbb{R}^n$
$e^{i_1 i_2 \ldots i_k}$	The basis k-form dual to $e_{i_1 i_2 \ldots i_k}$ in $\Lambda^k \mathbb{R}^n$
$\exp$	Exponential map
$G(k, \mathbb{R}^n)$	The set of oriented k-planes through origin in $\mathbb{R}^n$
$G(\phi)$	The face of the Grassmannian exposed by ϕ
id	Identity matrix
Im	Imaginary part
J	Complex structure
M	A manifold
$\mathbb{O}$	The set of Octonions
$O(n)$	The group of orthogonal matrices $A_{n\times n}$, $A^T A = AA^T = I$
Re	Real part
$SO(n)$	The group of orthogonal matrices $A_{n\times n}$ with $\det A = 1$
$SU(n)$	The group of unitary matrices $A_{n\times n}$ with $\det A = 1$
S^k_r	The k-sphere with radius r
T^3	Three-dimensional torus $S^1 \times S^1 \times S^1$
$T_p M$	Tangent space at the point $p \in M$
$U(n)$	The group of unitary matrices $A_{n\times n}$, $A^H A = AA^H = I$
V^*	The dual space corresponding to vector space V
vol_N	The volume form on N
$\Lambda_k \mathbb{R}^n$	The k-th exterior algebra
$\Lambda^k \mathbb{R}^n$	The dual space of $\Lambda_k \mathbb{R}^n$

∂X	Boundary of X
ω	Kähler form
ω^k	ω wedged k-times with itself
$(\cdot, \cdot)$	The Hermitian inner product on $\mathbb{C}^n$
$\langle \cdot, \cdot \rangle$	The standard Euclidean inner product
$\lvert \cdot \rvert$	L^2-norm
$\lVert \cdot \rVert$	The operator norm
$\lVert \cdot \rVert^*$	The comass norm
$\wedge$	Wedge product
$\lrcorner$	Interior product

LIST OF ACRONYMS/ABBREVIATIONS

ASSOC	Associative plane
LAG	Lagrangian plane
SLAG	Special Lagrangian plane

1. INTRODUCTION

Calibrated geometries, which were introduced by Harvey and Lawson in their foundational paper [1], are geometries of distinguished submanifolds determined by a fixed, closed differential form ϕ on a Riemannian manifold M with metric g. Let N be a k-dimensional compact, oriented immersed submanifold of (M, g). By pull-back of g under the immersion $i : N \hookrightarrow M$ together with the orientation on N, we have the volume form vol_N on N. If the restriction of ϕ onto N is equal to the volume form of N, that is $\phi|_N = vol_N$, then N is said to be a calibrated submanifold of M associated with ϕ. Lecture notes [2–4] are explanatory references and the paper [5] is a good summary for learning this subject.

The most important feature of a calibrated submanifold is that it is minimal, proven in [1]. A submanifold N is minimal if its volume is stationary under all variations of N with compact support or equivalently its mean curvature H vanishes (see for example [6]). If N is an immersed submanifold with the immersion $i : N \hookrightarrow M$, then the mean curvature H of N depends on i and its first and second derivatives, so the condition that N is minimal is a second order partial differential equation on i. Such equations are difficult to solve in general. On the other hand, a calibrated submanifold N is determined only by i and its first derivative, which give a first order partial differential equation, which is an easier problem to deal with. Hence calibrated submanifolds play an important role for finding minimal submanifolds. Besides its importance in mathematics, specifically in calculus of variation problems, it also has applications in string and supergravity theories, see [7].

A calibration is a closed k-form ϕ on a Riemannian manifold M^n satisfying $\phi(\xi) \leq 1$ where ξ is a k-dimensional oriented plane lying in a tangent space T_pM at $p \in M$. This condition is based on oriented planes in tangent spaces, which leads us to consider the *Grassmannian*, the set of all such planes through origin. Then the definition of calibrated submanifold, which is stated above can be reformulated as $\phi(\xi) = 1$ for all tangent spaces $\xi = T_pN$ of N. If a calibration ϕ does not satisfy the equality for any oriented planes, then it is not interesting to study since there can not be any submanifolds calibrated by it. Hence, we are interested in the calibrations ϕ where the dimension of the set $\{\xi : \phi(\xi) = 1\}$ is as large as possible.

Our main goal in this thesis is to understand the case $M = \mathbb{R}^n$ and classify all possible calibrations and corresponding calibrated submanifolds for $n \leq 7$. As we will see one, two, $n-1$, and $n-2$ dimensional calibrations are already determined for arbitrary n. Hence, interesting calibrations start to appear on $\mathbb{R}^6$, which are three-dimensional. This classification problem was solved for $\mathbb{R}^6$ in [8, 9] and for $\mathbb{R}^7$ in [10] which we follow closely. Calibrations are classified equivalently under the action of the symmetry group $SO(n)$ of $\mathbb{R}^n$. By using this action, we can fix a three-dimensional oriented plane in $\mathbb{R}^6$ and consider calibrations that calibrates this plane. There are four different types of calibrations on $\mathbb{R}^6$, namely special Lagrangian, Kähler or $\mathbb{CP}^1$, single point and double point calibrations. Classification on $\mathbb{R}^7$ is more difficult. To do this, first we fix a complex projective plane $\mathbb{CP}^1_0$ and consider the calibrations that calibrate all planes in $\mathbb{CP}^1_0$, there are five types of such calibrations, namely associative, special Lagrangian, $\mathbb{CP}^2$, $\mathbb{CP}^1$, double $\mathbb{CP}^1$. Then by fixing a plane in $\mathbb{CP}^1_0$ and considering calibrations for this plane, we get five new types of calibrations which are single point, double point, S^1, S^2 and S^3.

The organization of this thesis is as follows. In the second chapter we introduce basic definitions and concepts and give a result so called "First Cousin Principle" that will be used frequently in the other chapters. In the third chapter, we study the general theory of calibrations and examples of calibrated submanifolds given in [1]. In the fourth and fifth chapters we classify the calibrations on $\mathbb{R}^6$ and $\mathbb{R}^7$, respectively.

2. PRELIMINARIES

In this chapter, we start with definitions of the exterior algebra and Grassmannian which are essential for this thesis. Then, we explain the concept of a “face” of a Grassmannian. Finally a lemma called “First Cousin Principle” [11] is proved.

2.1. The Exterior Algebra and Grassmannian

Definition 2.1. *The* exterior algebra $\Lambda_k\mathbb{R}^n$ *is a real vector space of dimension* $\binom{n}{k}$ *with a real inner product* $\langle\, ,\rangle$ *induced from the standard inner product on* $\mathbb{R}^n$*. The elements of* $\Lambda_k\mathbb{R}^n$ *are called* k-vectors*. A k-vector* ξ *is* simple *if it can be written as a wedge product of 1-vectors:* $\xi = v_1 \wedge \cdots \wedge v_k$.

For any given orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, the set $\{e_{j_1} \wedge \cdots \wedge e_{j_k} \equiv e_{j_1 \ldots j_k} : j_1 < \cdots < j_k\}$ forms a basis for $\Lambda_k\mathbb{R}^n$. We will also consider the dual space of $\Lambda_k\mathbb{R}^n$, denoted by $\Lambda^k\mathbb{R}^n$ whose basis can be chosen as $\{e^{j_1} \wedge \cdots \wedge e^{j_k} \equiv e^{j_1 \ldots j_k} : j_1 < \cdots < j_k\}$. For more information on the exterior algebra, [12] is a good reference.

Definition 2.2. *The set of all oriented k-dimensional planes through the origin in* $\mathbb{R}^n$ *(or* $\mathbb{C}^n$*) is called the* Grassmannian $G(k, \mathbb{R}^n)$ *(or* $G(k, \mathbb{C}^n)$*).*

We will identify the Grassmannian $G(k, \mathbb{R}^n)$ with the submanifolds of $\Lambda_k\mathbb{R}^n$ which contains unit, simple k-vectors. This can be done by identifying a k-plane $\xi \in G(k, \mathbb{R}^n)$ with the wedge product of some unit vectors $\{v_1, \ldots, v_k\}$ which constitutes an oriented orthonormal basis for ξ. The resulting k-vector is independent of the choice of basis of ξ. Indeed, if we have an oriented basis $\{v_1, \ldots, v_k\}$ for $\xi \in G(k, \mathbb{R}^n)$ such that $\xi = v_1 \wedge \cdots \wedge v_k$ and another oriented orthonormal basis $\{w_1, \ldots, w_k\}$, then we have

$$\xi = v_1 \wedge \cdots \wedge v_k = \left|\frac{\partial(v_1, \ldots, v_k)}{\partial(w_1, \ldots, w_k)}\right| w_1 \wedge \cdots \wedge w_k = w_1 \wedge \cdots \wedge w_k \tag{2.1}$$

Last equality in (2.1) holds since ξ is oriented, so $\left|\dfrac{\partial(v_1,\ldots,v_k)}{\partial(w_1,\ldots,w_k)}\right| = 1$.

Next lemma gives a canonical form of a simple $2k$-vector $\xi \in \Lambda_{2k}\mathbb{R}^{2n}$.

Lemma 2.3. *(Lemma 6.13 of [13]) Given a unit simple $2k$-vector ξ in $\mathbb{C}^n$ with $2k \leq n$, there exists a unitary basis $e_1, Je_1, \ldots, e_n, Je_n$ for $\mathbb{C}^n$ over $\mathbb{R}$ with complex structure J on $\mathbb{C}^n$ and angles*

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{k-1} \leq \pi/2, \quad \theta_{k-1} \leq \theta_k \leq \pi,$$

such that

$$\begin{aligned} \xi \equiv e_1 \wedge (Je_1 \cos\theta_1 + e_2 \sin\theta_1) \wedge e_3 \wedge (Je_3 \cos\theta_2 + e_4 \sin\theta_2) \\ \wedge \cdots \wedge e_{2k-1} \wedge (Je_{2k-1} \cos\theta_k + e_{2k} \sin\theta_k). \end{aligned} \tag{2.2}$$

Proof. We will construct ξ as defined (2.2) step by step. First we choose two unit orthonormal vectors u, v in ξ to maximize $\langle Ju, v\rangle$, and set $cos\theta_1 \equiv \langle Ju, v\rangle$ equal to the maximum value with $0 \leq \theta_1 \leq \pi/2$. Now consider the function

$$f_w(\vartheta) = \langle Ju, (\cos\vartheta)v + (\sin\vartheta)w\rangle \tag{2.3}$$

where w is a unit vector in ξ such that $w \in span\,\{u, v\}^\perp$. This function has a maximum at $\vartheta = 0$, so we have $f'_w(0) = \langle Ju, w\rangle = 0$ for all $w \in span\,\{u, v\}^\perp$. This implies that $w \perp Ju$. Similarly, we can conclude that $w \perp Jv$.

Now define $e_1 \equiv u$. Then $v = \cos\theta_1 Je_1 + \sin\theta_1 z$ where z is a unit vector orthogonal to both e_1 and Je_1, and set $e_2 \equiv z$.

If $w \in span\,\xi$ and $w \perp \{u, v\}$, then $w \perp \{Ju, Jv\}$ as we shown above. This implies that $w \perp span\,\{e_1, Je_1, e_2, Je_2\}$. Consequently, $\xi = e_1 \wedge (\cos\theta_1 Je_1 + \sin\theta_1 e_2) \wedge \eta$ with $\eta \in G(2k-2, V)$ where $V = span\,\{e_1, Je_1, e_2, Je_2\}^\perp \cong \mathbb{R}^{2n-4}$.

For $\theta_1 = 0$, $\xi = e_1 \wedge Je_1 \wedge \eta$ with $\eta \in G(2k-2, V)$ where $V = span\,\{e_1, Je_1\}^\perp \cong \mathbb{R}^{2n-2}$.

The above procedure done for ξ is applied for the $(k-2)$-plane η and by proceeding in this manner we get an oriented basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ for $\mathbb{C}^n$ and for $2k \leq n$ we have ξ as in (2.2). $\square$

2.2. Comass and the Faces of the Grassmannian

Definition 2.4. *The operator norm of a k-form $\phi \in \Lambda^k\mathbb{R}^n$ restricted to the Grassmannian $G(k, \mathbb{R}^n)$ is called the* comass norm *which is given as:*

$$\text{comass}\,\phi \equiv \|\phi\|^* = \max\{\phi(\xi) : \xi \in G(k, \mathbb{R}^n)\}. \tag{2.4}$$

Let

$$G(\phi) = \{\xi \in G(k, \mathbb{R}^n) : \phi(\xi) = \|\phi\|^*\} \tag{2.5}$$

be the set of elements of the Grassmannian $G(k, \mathbb{R}^n)$ at which the maximum of ϕ occurs. $G(\phi)$ is called the *face* of the Grassmannian $G(k, \mathbb{R}^n)$ exposed by ϕ.

Example 2.5. *Consider the most trivial case: the Grassmannian $G(1, \mathbb{R}^2) \subset \mathbb{R}^{\binom{2}{1}}$ consists of oriented lines through the origin in $\mathbb{R}^2$. Such lines can be described by the angle θ that it makes with the x-axis, so $G(1, \mathbb{R}^2)$ is the unit circle in $\mathbb{R}^2$. Hence, the faces of $G(1, \mathbb{R}^2)$ are just single points.*

2.3. First Cousin Principle

In this section we will prove a very useful lemma which will be used to reduce the number of 3-dimensional calibrations ϕ while classifying them on $\mathbb{R}^6$ and $\mathbb{R}^7$.

Definition 2.6. *Let $\xi_0 \in G(k, \mathbb{R}^n)$. An element ξ_1 in $G(k, \mathbb{R}^n)$ is called a* first cousin *of ξ_0 if it satisfies $dim(\xi_0 \cap \xi_1) = k - 1$ and $\langle \xi_0, \xi_1 \rangle = 0$.*

Remark 2.7. If $\xi_0 = e_{1\ldots k}$ for some orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, then first cousins of ξ_0 are of the form $\xi_1 = e_{1\ldots(k-1)(k+1)}$.

Lemma 2.8. *(First cousin principle) If a k-form ϕ attains its maximum at ξ_0, then $\phi(\xi_1) = 0$ for all first cousins ξ_1 of ξ_0.*

Proof. Assume there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$ such that $\xi_0 = e_{1\ldots k}$ and $\xi_1 = e_{1\ldots(k-1)(k+1)}$.. Consider the function

$$f(\theta) = \phi(e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_k \cos\theta + e_{k+1} \sin\theta))$$

which attains its maximum when $\theta = 0$, hence we have

$$\begin{aligned} 0 = \frac{d}{d\theta} f\Big|_{\theta=0} &= \phi(e_1 \wedge \cdots \wedge e_{k-1} \wedge (-e_k \sin\theta + e_{k+1} \cos\theta)\Big|_{\theta=0} \\ &= \phi(e_{1\ldots(k-1)(k+1)}) = \phi(\xi_1). \end{aligned}$$

□

Example 2.9. *If $\phi \in \Lambda^2(\mathbb{R}^4)$ which attains its maximum at e_{12}, $\phi(e_{12}) = \|\phi\|^*$, then $\phi(e_{13}) = \phi(e_{14}) = \phi(e_{23}) = \phi(e_{24}) = 0$ and ϕ can be written as $\phi = ae^{12} + be^{34}$ for some coefficients a,b with $a \geq |b|$. Here $e^{ij} \equiv e^i \wedge e^j$ denotes the differential two-form dual to the two-vector e_{ij}, and the set $\{e^{ij} : i < j\}$ constitutes a basis for $\Lambda^2\mathbb{R}^4$.*

3. CALIBRATED GEOMETRY

In this chapter, we will mainly follow the article written by Harvey and Lawson [1] which introduced this subject in 1982. First, we will give definitions of a calibration and calibrated submanifolds. Then in Sections 3.2, 3.3 and 3.4 we will study basic examples of calibrations, namely complex, special Lagrangian and associative calibrations.

3.1. Introduction to Calibrations

Definition 3.1. *A differential k-form ϕ on a Riemannian manifold (M,g) is called a* calibration *if it is closed and $\phi(e_1 \wedge \cdots \wedge e_k) \leq 1$ for all tangent unit vectors $e_1, \ldots, e_k$ in T_pM at any point $p \in M$.*

Definition 3.2. *An oriented k-dimensional submanifold N of (M^n, g) is* calibrated by a k-form ϕ *if $\phi|_N = vol_N$, where vol is the volume form on N.*

It means that for all $p \in N$, ϕ satisfies $\phi(e_1 \wedge \cdots \wedge e_k) = 1$ for an oriented orthonormal basis $e_1, \ldots, e_k$ of T_pN. This implies that all tangent spaces T_pN lie in the same face $G(\phi)$ of the Grassmannian $G(k, \mathbb{R}^n)$.

Definition 3.3. *Let N be a submanifold of a manifold M. N is said to be* volume-minimizing *if $vol(N) \leq vol(N')$ for every submanifold N' of M such that boundaries of N and N' are the same, that is $\partial N = \partial N'$.*

Theorem 3.4. *[1] Let N be a k-dimensional compact submanifold of M which is calibrated by ϕ. Then N is volume-minimizing.*

Proof. Assume N is calibrated by ϕ. Let N' be another submanifold of M which is homologous to N, in other words $N - N' = \partial X$ for some $(k+1)$-dimensional submanifold X of M. Then,

$$vol(N) = \int_N \phi = \int_{N'} \phi \leq Vol(N')$$

Second equality follows from Stokes' Theorem as

$$\int_{N-N'} \phi = \int_{\partial X} \phi = \int_X d\phi = 0$$

since ϕ is closed. The inequality follows from the definition of calibration since $\phi|_{N'} \leq vol_{N'}$. □

Remark 3.5. Homology class of N contains small variations of it with compact support, in other words N' in the proof of Theorem 3.4. Hence, we have by above theorem volume of N is stationary under small variations. Then as we mentioned in Introduction that calibrated submanifolds are minimal.

Example 3.6. *Theorem 3.4 says that a submanifold N of M is volume-minimizing if all of its tangent planes lie in the same face of the Grassmannian exposed by a closed form ϕ. Straight lines in $\mathbb{R}^n$ are one-dimensional volume-minimizing submanifolds since as in Example 2.5 the faces of Grassmannian are just single points, in other words all of its tangent planes are the same. If we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p\mathbb{R}^n \cong \mathbb{R}^n$ where e_1 is the direction of the tangent lines at any points on a straight line L, then the volume form on L is just $\phi = e^1$ which satisfy $\phi(e_1) = 1$ and $\phi(e_i) = 0$ for all $i = 2, \ldots, n$. Therefore, L is calibrated by $\phi = e^1$.*

In the following sections we discuss some interesting examples of calibrations that are given in [1].

3.2. Complex Submanifolds

On $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, we have the complex structure J and the Kähler 2-form ω

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j. \tag{3.1}$$

Theorem 3.7. *(Wirtinger's Inequality) [14]*

$$\frac{1}{k!}\omega^k(\xi) \leq 1 \quad \textit{for all} \quad \xi \in G(2k, \mathbb{R}^{2n}) \tag{3.2}$$

with equality if and only if $\xi \in G(k, \mathbb{C}^n)$ *where* $\omega^k \equiv \overbrace{\omega \wedge \cdots \wedge \omega}^{k-times}$.

Proof. For an orthonormal basis $e_1, Je_1, \ldots, e_n, Je_n$ for $\mathbb{R}^{2n}$, consider the dual basis $e^1, Je^1, \ldots, e^n, Je^n$. The Kähler form can be written with respect to this basis as

$$\omega = e^1 \wedge Je^1 + \cdots + e^n \wedge Je^n \tag{3.3}$$

Then

$$\omega^k = k! \sum_{i_1 < \cdots < i_k} e^{i_1} \wedge Je^{i_1} \wedge \cdots \wedge e^{i_k} \wedge Je^{i_k} \tag{3.4}$$

Hence by Equation 2.2 for ξ together with (3.4), we have

$$\frac{1}{k!}\omega^k(\xi) = \cos\theta_1 \ldots \cos\theta_k \leq 1, \qquad 2k \leq n. \tag{3.5}$$

Equality holds if and only if $\theta_j = 0$ for all $j = 1, \ldots, k$. In other words by Equation 2.2, we have $\xi = e_1 \wedge Je_1 \wedge \cdots \wedge e_k \wedge Je_k \in G(k, \mathbb{C}^n)$ which is a complex plane with complex structure J. □

Theorem 3.8. *The 2k-form* $\phi \equiv \frac{1}{k!}\omega^k$ *on* $\mathbb{R}^{2n} \cong \mathbb{C}^n$ *is a calibration. The face exposed by* ϕ *is,*

$$G(\phi) = G(k, \mathbb{C}^n) \subset G(2k, \mathbb{R}^{2n})$$

Proof. Note that $d\phi = 0$ since ω is closed. This theorem directly follows from the above theorem. □

Example 3.9. *The complex one-dimensional curve* $N = \{(z, \frac{1}{z}) \in \mathbb{C}^2 : z \in \mathbb{C} - \{0\}\}$ *in* $\mathbb{C}^2$ *is calibrated by* $\phi = \frac{1}{1!}\omega^1 = \omega$ *and* $G(\phi) = G(1, \mathbb{C}^2)$ *by Theorem 3.8.*

3.3. Special Lagrangian Submanifolds

Next, we will consider another example of calibration given by Harvey-Lawson in [1], which is called special Lagrangian calibration. The books [14, 15] are also very good references for this topic. Throughout this section we will denote the volume form on $\mathbb{C}^n$ by dz which can be written as $dz = dz_1 \wedge \cdots \wedge dz_n$ where $\{z_1, \ldots, z_n\}$ are standard complex coordinates for $\mathbb{C}^n$. Let $(\cdot, \cdot) = \sum_j dz_j \, d\bar{z}_j$ denote the Hermitian inner product and $\langle \cdot, \cdot \rangle = \sum_j dx_j^2 + dy_j^2$ denote the standard Euclidean inner product where $z_j = x_j + iy_j$ for $i = 1, \ldots, n$ and ω is the Kähler form as in (3.1). Then we have

$$(u, v) = \langle u, v \rangle - i\omega(u, v) \tag{3.6}$$

for any $u, v \in \mathbb{C}^n$.

Definition 3.10. *An n-plane* $\xi \subset \mathbb{C}^n$ *is called a* Lagrangian plane, *if* $Ju \perp \xi$ *for each vector* $u \in \xi$ *with complex structure* J *on* $\mathbb{C}^n$*. That is,* $\langle Ju, v \rangle = 0$ *for all* $v \in \xi$*. We will denote the set of all Lagrangian n-planes in* $\mathbb{C}^n$ *as LAG.*

Remark 3.11. Definition of a Lagrangian plane is equivalent to $\omega|_\xi = \omega(u, v) = 0$ for all vectors u, v in ξ. Indeed, by Equation 3.6 we have

$$\langle Ju, v \rangle = Re\,(Ju, v) = Re\, i(u, v) = -Im(u, v) = \omega(u, v)$$

Proposition 3.12. *The set of all Lagrangian n-planes in* $\mathbb{C}^n$*, LAG is invariant under the unitary group* $U(n)$*. Furthermore,* $U(n)$ *acts on LAG transitively.*

Proof. For an arbitrary matrix $A \in U(n)$, we have

$$\begin{aligned}\langle JA\epsilon_i, A\epsilon_j\rangle &= Re\,(JA\epsilon_i, A\epsilon_j) = Re\,[i(A\epsilon_i, A\epsilon_j)] \\ &= Re\,[i(\epsilon_i, \bar{A}^T A\epsilon_j)] = Re\,[i(\epsilon_i, \epsilon_j)] = \langle J\epsilon_i, \epsilon_j\rangle = 0\end{aligned}$$

where $\{\epsilon_1, \ldots, \epsilon_n\}$ is an orthonormal basis of ξ. Then by the definition of a Lagrangian plane, we get $A \cdot \xi \in LAG$.

Next, let $\xi = \epsilon_1 \wedge \cdots \wedge \epsilon_n$ and $\xi' = \epsilon'_1 \wedge \cdots \wedge \epsilon'_n$ be two planes in LAG. Then $\{\epsilon_1, \cdots, \epsilon_n, J\epsilon_1, \cdots, J\epsilon_n\}$ and $\{\epsilon'_1, \cdots, \epsilon'_n, J\epsilon'_1, \cdots, J\epsilon'_n\}$ are real bases for $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We can relate these two bases (by using the Einstein summation convention, repeated indices are summed over) as

$$\begin{aligned}\epsilon'_j &= a_{jk}\epsilon_k + b_{jl}J\epsilon_l \\ J\epsilon'_j &= -b_{jl}\epsilon_l + a_{jk}J\epsilon_k.\end{aligned}$$

Set $a = (a_{jk})_{n\times n}$ and $b = (b_{jk})_{n\times n}$, then consider the real matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}_{2n\times 2n}$ which is an orthogonal (change of basis) matrix and consider the complex matrix $(A^{\mathbb{C}})_{ij} = (a + ib)_{ij}$. The matrix $A^{\mathbb{C}}$ is a unitary matrix and satisfies $A \cdot \xi = \xi'$.Hence, $U(n)$ acts on LAG transitively. $\square$

Remark 3.13. $\mathbb{R}^n$ is a Lagrangian plane in $\mathbb{C}^n$ so for every $\xi \in LAG$, we have $\xi = A \cdot \mathbb{R}^n$ for some $A \in U(n)$. Hence, by the group action property we have $LAG \cong U(n)/SO(n)$ since $SO(n)$ is the isotropy subgroup of U(n) at $\xi_0 = \mathbb{R}^n$.

Definition 3.14. *An n-plane $\xi \subset \mathbb{C}^n$ is called a* special Lagrangian plane,*denoted by SLAG if*

(i) ξ is a Lagrangian plane,

(ii) $\xi = A \ldots \xi_0$, where $A \in SU(n)$ and $\xi_0 = \mathbb{R}^n$

Note that $SLAG \cong SU(n)/SO(n)$.

Let us introduce $\alpha \equiv Re\, dz$ and $\beta \equiv Im\, dz$ for the following results.

Theorem 3.15. *For each* $\xi \in G(n, \mathbb{C}^n)$

$$|dz(\xi)|^2 = \alpha(\xi)^2 + \beta(\xi)^2 = |\xi \wedge J\xi| \tag{3.7}$$

Proof. Choose a basis $\{\epsilon_1, \ldots, \epsilon_n\}$ of ξ such that $\lambda\xi = \epsilon_1 \wedge \cdots \wedge \epsilon_n$ for some $\lambda > 0$. Consider the standard Hermitian basis $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ of $\mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n$, and consider a linear transformation A which satisfies $A(e_j) = \epsilon_j$ and $A(Je_j) = J\epsilon_j$. A commutes with J, so A is also a complex linear transformation. In fact, A and $A^{\mathbb{C}}_{n\times n} = a + ib$ are defined as in the proof of Proposition 3.12 with respect to the basis $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$. Then we have

$$dz(A^{\mathbb{C}}(e_1 \wedge \cdots \wedge e_n)) = det A^{\mathbb{C}}. \tag{3.8}$$

On the other hand, by the action of $A^{\mathbb{C}}$, $A^{\mathbb{C}}e_k = a_{kj}e_j + ib_{kj}e_j = \epsilon_k$ we have

$$dz(A^{\mathbb{C}}(e_1 \wedge \cdots \wedge e_n)) = dz(\epsilon_1 \wedge \cdots \wedge \epsilon_n) = dz(\lambda\xi) = \alpha(\lambda\xi) + i\beta(\lambda\xi) \tag{3.9}$$

Then,

$$\begin{aligned}
\lambda^2|dz(\xi)|^2 &= |dz(\lambda\xi)|^2 \\
&= \alpha^2(\lambda\xi) + \beta^2(\lambda\xi) \\
&= Re^2(det A^{\mathbb{C}}) + Im^2(det A^{\mathbb{C}}) = |det A^{\mathbb{C}}|^2 = det A \\
&= |A(e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n)| \\
&= \lambda^2|\xi \wedge J\xi|
\end{aligned} \tag{3.10}$$

Hence, theorem follows. □

Theorem 3.16. *The inequality* $|\xi \wedge J\xi| \leq |\xi|^2$ *is true for all* $\xi \in G(n, \mathbb{C}^n)$*. Moreover, the equality holds if and only if* ξ *is a Lagrangian plane.*

Proof. As above, $\xi = \lambda\epsilon_1 \wedge \cdots \wedge \epsilon_n$ for some orthonormal basis $\{\epsilon_1, \dots, \epsilon_n\}$ of ξ. Then

$$\begin{aligned} |\xi \wedge J\xi| &= \lambda^2 |\epsilon_1 \wedge \cdots \wedge \epsilon_n \wedge J\epsilon_1 \wedge \cdots \wedge J\epsilon_n| \\ &\leq |\epsilon_1| \cdots |\epsilon_n||J\epsilon_1| \cdots |Je_n| \end{aligned} \tag{3.11}$$

Equality holds when $\{\epsilon_1, \dots \epsilon_n, J\epsilon_1, \dots Je_n\}$ is orthonormal vectors in $\mathbb{C}^n$, so for all $u, v \in \xi$, u and Jv is orthogonal, which implies that ξ is a Lagrangian plane. □

Remark 3.17. Theorems 3.15 and 3.16 together imply that we have $dz(\xi) \leq |\xi|$.

Theorem 3.18. *The n-form $\alpha = Re(dz)$ has comass one, namely $\alpha(\xi) \leq 1$ for each $\xi \in G(n, \mathbb{C}^n)$. Moreover, the equality $\alpha(\xi) = 1$ holds if and only if ξ is a SLAG plane.*

Proof. Note that

$$\alpha^2(\xi) \leq \alpha^2(\xi) + \beta^2(\xi) = detA = |\xi \wedge J\xi|^2 \leq |\xi|^2 = 1 \tag{3.12}$$

is true for all $\xi \in G(n, \mathbb{C}^n)$ by Theorem 3.16. Let $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ be the standard basis for $\mathbb{R}^n \oplus \mathbb{R}^n$ and $\{\epsilon_1, \dots, \epsilon_n\}$ be the orthonormal basis for ξ. As in the proof of Theorem 3.14 there exists a complex linear transformation $A^{\mathbb{C}}$ that sends $\{e_1, \dots, e_n\}$ to $\{\epsilon_1, \dots, \epsilon_n\}$.

Assume $\alpha(\xi) = 1$, then (3.12) implies that $|\xi \wedge J\xi|^2 = |\xi|^2$. By Theorem 3.15, ξ is a Lagrangian plane. Moreover, we have $\alpha(\xi) = |\xi| = |\epsilon_1| \dots |\epsilon_n| = 1$, then $detA^{\mathbb{C}} = \alpha(\xi) + i\beta(\xi) = 1$. This gives that $A^{\mathbb{C}} \in SU(n)$, in other words ξ is SLAG plane.

Conversely, if ξ is a SLAG plane then first inequality turns to be equality since $detA^{\mathbb{C}} = 1$, that is $\beta(\xi) = 0$, and second inequality becomes equality by Theorem 3.16 since ξ is a Lagrangian plane. □

Corollary 3.19. *Let $\xi \in G(n, \mathbb{C}^n)$. ξ or $-\xi$ is a SLAG plane if and only if ξ is a Lagrangian plane and $\beta(\xi) = 0$.*

Definition 3.20. *The form $\alpha = Re\, dz$ is said to be a* SLAG calibration *on $\mathbb{C}^n$.*

Definition 3.21. *Let M be an oriented submanifold of $\mathbb{C}^n$. If for each $p \in M$, T_pM is a (special) Lagrangian plane, then M is called a* (special) Lagrangian submanifold.

3.4. Associative Submanifolds

Another example given in [1] is the associative calibrations which exist on $\mathbb{R}^7$. The following definition is the generalization of the cross product on $\mathbb{R}^3$ to an arbitrary inner product space V.

Definition 3.22. *A skew-symmetric bilinear map $V \times V \longrightarrow V$ satisfying:*

$$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0 \tag{3.13}$$

$$|u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2 \tag{3.14}$$

for all $u, v \in V$ is called a cross product *on V.*

Remark 3.23. A cross product also satisfies

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle \tag{3.15}$$

for all $u, v, w \in V$. Indeed, we have

$$\begin{aligned} 0 &= \langle v \times (u + w), u + w \rangle \\ &= \langle v \times w, u \rangle + \langle v \times u, w \rangle \\ &= \langle u, v \times w \rangle - \langle u \times v, w \rangle \end{aligned}$$

Now we will define the associative calibration of V.

Definition 3.24. *The map $\phi : V \times V \times V \to \mathbb{R}$ defined by*

$$\phi(u, v, w) \equiv \langle u \times v, w \rangle \tag{3.16}$$

is called an associative calibration *of V. Note that ϕ is an alternating form, which follows from Equation 3.15.*

Let $\mathbb{O}$ denote the algebra of *octonions* which is an eight dimensional normed division algebra over the real numbers. The multiplication of octonions is nonassociative and noncommutative but alternative, so it defines a cross product satisfying (3.13) and (3.14). Let $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the set of unit octonions. Multiplication table and properties of octonions can be found in [16, 17] extensively. The imaginary part $Im\mathbb{O}$ of octonions is spanned by $\{e_i\}$'s for $i = 1, \ldots, 7$. For two vectors $u, v \in V = \mathbb{R}^7 \equiv Im\mathbb{O}$, the cross product is given by

$$u \times v \equiv \begin{pmatrix} u_2v_3 - u_3v_2 - u_4v_5 + u_5v_4 - u_6v_7 + u_7v_6 \\ u_3v_1 - u_1v_3 - u_4v_6 + u_6v_4 - u_7v_5 + u_5v_7 \\ u_1v_2 - u_2v_1 - u_4v_7 + u_7v_4 - u_5v_6 + u_6v_5 \\ u_1v_5 - u_5v_1 - u_2v_6 - u_6v_2 + u_3v_7 - u_7v_3 \\ -u_1v_4 + u_4v_1 - u_2v_7 + u_7v_2 + u_3v_6 - u_6v_3 \\ u_1v_7 - u_7v_1 - u_2v_4 + u_4v_2 - u_3v_5 + u_5v_3 \\ -u_1v_6 + u_6v_1 + u_2v_5 - u_5v_2 - u_3v_4 + u_4v_3 \end{pmatrix}$$

where u_i's and v_i's are components of u and v. Then we get $\phi = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356}$ which is called *standard associative 3-form* on $\mathbb{R}^7$ where $e^{ijk} \equiv e^i \wedge e^j \wedge e^k$.

Definition 3.25. *A three-dimensional plane $\xi \subset \mathbb{R}^7 \equiv Im\,\mathbb{O}$ is called* associative plane *if $[u, v, w] \equiv (u \times v) \times w + \langle v, w\rangle y - \langle u, w\rangle v = 0$ for all $u, v, w \in \xi$. The set of all associative planes is denoted by ASSOC.*

The next theorem is stated for future references.

Theorem 3.26. *[1] The associative calibration ϕ calibrates all associative planes. In other words, we have $G(\phi) \equiv ASSOC$.*

3.5. General Remarks about Calibrations on $\mathbb{R}^n$

In this section, we will first state a theorem that classifies one, two, $n - 2$ and $n - 1$ dimensional calibrations on $\mathbb{R}^n$ and the faces exposed by them. Second, in the light of this theorem we will study the calibrations on $\mathbb{R}^4$ and the faces calibrated by them explicitly. Then we will prove a very important result called the "Reduction Principle" which will be used later. The following proposition establishes a result for the hodge dual $*\phi$ of a calibration ϕ.

Proposition 3.27. *If a k-form ϕ is a calibration on an n-dimensional manifold M and $*\phi$ is closed, then $*\phi$ is a calibration which is a $(n-k)$-form on M. Moreover, if ϕ calibrates the plane $\xi = e_1 \wedge \cdots \wedge e_k$ for some oriented basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of T_pM at a point $p \in M$, then the plane $\zeta = e_{k+1} \wedge \cdots \wedge e_n$ is calibrated by $*\phi$.*

Proof. We know that $\phi \wedge *\phi$ is equal to the volume form of M. Then for vectors $e_1, \ldots, e_{n-k}$ in T_pM, we have $*\phi(e_1 \wedge \cdots \wedge e_{n-k}) \leq 1$, hence it is a calibration. The other statement is straight forward to prove. □

Theorem 3.28. *[18] Let $(\mathbb{R}^n, \phi)$ be a Riemannian manifold with calibration ϕ, which is a constant coefficient k-form. Then there are orthonormal coordinates $\{e_1, \ldots, e_n\}$ on $\mathbb{R}^n$ such that the following holds:*

(i) *For $k = 1$, a calibration can be written as $\phi = e^1$. Then ϕ-calibrated submanifolds are the real lines parallel to e_1.*

(ii) *For $k = 2$, a calibration can be written as $\phi = \phi_m = \sum_{j=1}^{m} e^{2j-1} \wedge e^{2j}$, for some $1 \leq m \leq n/2$. Then $G(\phi) \cong \mathbb{CP}^{m-1}$ and the manifolds of the form $\Sigma \times \{v\}$ where Σ is a holomorphic curve in $\mathbb{R}^{2m} \equiv \mathbb{C}^m$ with complex coordinates $z_j = e_{2j-1} + ie_{2j}$ for $j = 1, \ldots, m$, and $v \in \mathbb{R}^{n-2m}$ with coordinates $(e_{2m+1}, \ldots, e_n)$ are calibrated by ϕ_m.*

(iii) *For $k = n-2$, a calibration can be written as $*\phi$ where ϕ is given as in ii. Then $G(*\phi) \cong \mathbb{CP}^{m-1}$ and the manifolds of the form $\Sigma \times \mathbb{R}^{n-2m}$ where Σ is a hypersurface in $\mathbb{R}^{2m} \equiv \mathbb{C}^m$ with complex coordinates $z_j = e_{2j-1} + ie_{2j}$ for $j = 1, \ldots, m$, and $v \in \mathbb{R}^{n-2m}$ with coordinates $(e_{2m+1}, \ldots, e_n)$ are calibrated by ϕ_m.*

(iv) *For $k = n-1$, $\phi = e^2 \wedge \cdots \wedge e^n$. Then $G(\phi) = \{e_2 \wedge \cdots \wedge e_n\}$ and ϕ-calibrated submanifolds in $\mathbb{R}^n$ are real hyperplanes $e_1 = (c, 0, \ldots, 0)$ for $c \in \mathbb{R}$.*

Proof. The cases (i) and (iv) are just simple observations. To prove (ii), assume a 2-form ϕ on $\mathbb{R}^n$ is a calibration. Then for all unit vectors $u, v \in \mathbb{R}^n$, we have $\phi(u \wedge v) \leq 1$. Take all unit vectors u, v such that $\phi(u, v) = 1$. Observe that we can not have $\phi(u, v) = \phi(u, w) = 1$ by the First Cousin Principle, Lemma 2.8. Hence we can say that $2m$ many such unit vectors arise. Then ϕ can be written as in (ii) and by extending the number of unit vectors orthonormally to get a basis for $\mathbb{R}^n$. The planes $\{e_{2j-1} \wedge e_{2j}\}$ are calibrated by ϕ for $1 \leq j \leq m$, so the face of the Grassmannian exposed by ϕ is equal $\mathbb{CP}^{m-1}$. The submanifolds of $\mathbb{R}^n$ defined as above have the tangent planes lying in the face of ϕ. By Proposition 3.27, we get the case (iii). □

By this theorem, we are able to classify all calibrations on $\mathbb{R}^n$ for $n \leq 5$. To illustrate the results in Theorem 3.28ii and iii, we consider the faces of the Grassmannian $G(2, \mathbb{R}^4)$ which has a simple description, as in the article [11]. First we will generalize hodge dual operator, which is usually defined for differential forms, to two-vectors.

Definition 3.29. *Given an orientation of $\mathbb{R}^4$ by a 4-vector $E \in \Lambda_4\mathbb{R}^4$, there is a symmetric map $*$ from $\Lambda_2\mathbb{R}^4$ to itself, which is called* Hodge dual operator *for two-vectors satisfying*

$$\xi \wedge \eta = \langle \xi, *\eta \rangle E.$$

This operation satisfies $*^2 = I$, hence $*$ is an orthogonal map with eigenvalues ± 1. By this result, we get $\Lambda_2\mathbb{R}^4 \cong P_1 \oplus P_{-1}$ where $P_{\pm 1}$ are eigenspaces corresponding to ± 1 with dimensions 3.

Remark 3.30. The classification of calibrations on $\mathbb{R}^5$ is also given by Theorem 3.28, but unlike $\mathbb{R}^4$ it can not be done by decomposing $\Lambda_2\mathbb{R}^5$ into eigenspaces of Hodge dual operator since it maps $\Lambda_2\mathbb{R}^5$ into $\Lambda_3\mathbb{R}^5$.

Now, for an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$, $P_{\pm 1}$ have orthonormal bases

$\{E_1, E_2, E_3\}$ and $\{E_4, E_5, E_6\}$ (respectively) given by

$$E_1 = \frac{e_{12} + e_{34}}{\sqrt{2}}, \quad E_2 = \frac{e_{13} + e_{24}}{\sqrt{2}}, \quad E_3 = \frac{e_{14} + e_{23}}{\sqrt{2}}$$

$$E_4 = \frac{e_{12} - e_{34}}{\sqrt{2}}, \quad E_5 = \frac{e_{13} - e_{24}}{\sqrt{2}}, \quad E_6 = \frac{e_{14} - e_{23}}{\sqrt{2}}$$

Proposition 3.31. *Let $\xi \in \Lambda_2(\mathbb{R}^4)$. The following statements are equivalent:*

(i) ξ *is simple,*
(ii) $\xi \wedge \xi = 0$,
(iii) $\langle \xi_1, \xi_1 \rangle = \langle \xi_{-1}, \xi_{-1} \rangle$,
(iv) If $\xi = \sum_{j=1}^{6} b_j E_j$ *then* $b_1^2 + b_2^2 + b_3^2 = b_4^2 + b_5^2 + b_6^2$.

Proof. Choose an orthonormal basis $\{e'_1, e'_2, e'_3, e'_4\}$ for $\mathbb{R}^4$ such that $\langle \xi, e'_{12} \rangle$ is maximum. As in the proof of the First Cousin Principle, we have $\langle \xi, e'_{13} \rangle = \langle \xi, e'_{14} \rangle = \langle \xi, e'_{23} \rangle = \langle \xi, e'_{24} \rangle = 0$. Then ξ is of the form $\xi = ae'_{12} + be'_{34}$, with $a \geq |b|$. Then we have $\xi \wedge \xi = 2abe'_{1234}$. Now ξ is simple if and only if $b = 0$ if and only if $\xi \wedge \xi = 0$. This proves that (i) if and only (ii).
Now write ξ in terms of the basis $\{E_j\}$ as $\xi = \sum b_j E_j$. Then

$$\begin{aligned} \xi \wedge \xi &= (\sum b_j E_j) \wedge (\sum b_j E_j) \\ &= (b_1^2 + b_2^2 + b_3^2 - b_4^2 - b_5^2 - b_6^2) e_{1234} \end{aligned}$$

This computation shows that $\xi \wedge \xi = 0$ if and only if $b_1^2 + b_2^2 + b_3^2 = b_4^2 + b_5^2 + b_6^2$.
In this notation $\xi = \xi_1 + \xi_{-1}$, we have, $\xi_1 = b_1 E_1 + b_2 E_2 + b_3 E_3$ and $\xi_{-1} = b_4 E_4 + b_5 E_5 + b_6 E_6$. Then $\langle \xi_1, \xi_1 \rangle = b_1^2 + b_2^2 + b_3^2$ and $\langle \xi_{-1} \wedge \xi_{-1} \rangle = b_4^2 + b_5^2 + b_6^2$. Hence, proof is completed. $\square$

Corollary 3.32.

$$G(2, \mathbb{R}^4) = S^2_{1/\sqrt{2}} \times S^2_{1/\sqrt{2}} \subset P_1 \times P_{-1} \cong \Lambda_2 \mathbb{R}^4 \tag{3.17}$$

where $S^2_{1/\sqrt{2}}$ *is a two-dimensional sphere with radius* $1/\sqrt{2}$.

Proof. Any unit simple 2-vector $\xi \in G(2,\mathbb{R}^4)$ can be written as $\xi = \xi_1 + \xi_{-1} \in P_1 \oplus P_{-1}$ since $\Lambda_2\mathbb{R}^4 \cong\in P_1 \oplus P_{-1}$. By Proposition 3.31, we have $|\xi_1| = |\xi_{-1}|$. This is true for ξ if and only if $|\xi_1| = |\xi_{-1}| = 1/\sqrt{2}$, that is $\xi \in \boldsymbol{S}^2_{1/\sqrt{2}} \times \boldsymbol{S}^2_{1/\sqrt{2}}$ □

Corollary 3.33. *Let* $\zeta = \zeta_1 + \zeta_{-1}$ *be a nonzero element of* $\Lambda_2\mathbb{R}^4 = P_1 \oplus P_{-1}$. *The face of* ζ^*, *that contains elements of the Grassmannian* $\xi \in G(2,\mathbb{R}^4)$ *where* $\zeta^*(\xi)$ *is maximum. Then*

$$G(\zeta^*) = \boldsymbol{S}^2_{1/\sqrt{2}} \oplus \left\{\frac{\zeta_{-1}}{\sqrt{2}|\zeta_{-1}|}\right\} \quad \text{if} \quad \zeta_1 = 0, \tag{3.18}$$

$$G(\zeta^*) = \left\{\frac{\zeta_1}{\sqrt{2}|\zeta_1|}\right\} \oplus \boldsymbol{S}^2_{1/\sqrt{2}} \quad \text{if} \quad \zeta_{-1} = 0, \tag{3.19}$$

$$G(\zeta^*) = \left\{\frac{\zeta_1}{\sqrt{2}|\zeta_1|}\right\} \oplus \left\{\frac{\zeta_{-1}}{\sqrt{2}|\zeta_{-1}|}\right\} \quad \text{otherwise.} \tag{3.20}$$

In each case, we have

$$\|\zeta^*\|^* \equiv max\{\langle \xi, \zeta\rangle : \xi \in G(2,\mathbb{R}^4)\} = \frac{|\zeta_1| + |\zeta_{-1}|}{\sqrt{2}}. \tag{3.21}$$

Proof. Let $\xi = \xi_1 + \xi_{-1} \in G(2,\mathbb{R}^4)$. Since $|\xi_1| = |\xi_{-1}| = 1/\sqrt{2}$, we have

$$\zeta^*(\xi) = \langle \xi_1, \zeta_1\rangle + \langle \xi_{-1}, \zeta_{-1}\rangle \leq \frac{|\zeta_1| + |\zeta_{-1}|}{\sqrt{2}} \tag{3.22}$$

Equality holds if and only if ζ_j is a nonnegative multiple of ξ_j. Therefore $G(\zeta^*)$ and $\|\zeta^*\|^*$ are as claimed. □

Proposition 3.34. *(Reduction Principle) Let* $\phi \in \Lambda^k\mathbb{R}^n$ *and* L *be a (k+1)-dimensional linear subspace of* $\mathbb{R}^n$. *For* $x \in \mathbb{R}^n$, $\phi[x]$ *denotes the restriction of* ϕ *onto* $(span\, x)^\perp$. *We have that* $\|\phi\|^* \leq 1$ *if and only if* $\|\phi[x]\|^* \leq 1$ *for all* $x \in L$. *Moreover, if* $\|\phi\|^* = 1$, *then*

$$G(\phi) = \bigcup \{G(\phi[x]) : \|\phi[x]\|^* = 1, \quad x \in L\}$$

Proof. For $\xi \in G(k, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$|\langle \xi, \phi[x] \rangle| \leq |\langle \xi, \phi \rangle|$$

since for the left hand side, the value of ϕ is computed on $G(k, \mathbb{R}^n - \{x\})$. Equality holds if and only if $\xi \in (span\, x)^\perp$. If $\|\phi\|^* \leq 1$, then clearly $\|\phi[x]\|^* \leq 1$ for all x in L. Conversely, assume $\|\phi[x]\|^* \leq 1$ for all x in L. Since dimension of L is $k+1$, we have $L \cap (span\, \xi)^\perp \neq 0$ for all $\xi \in G(k, \mathbb{R}^n)$. Then for all $\xi \in G(k, \mathbb{R}^n)$, there exists $x \in L \cap (span\, \xi)^\perp$ for which $\langle \xi, \phi \rangle = \langle \xi, \phi[x] \rangle \leq 1$. $\square$

4. CLASSIFICATION OF 3-FORMS ON $\mathbb{R}^6$

In this chapter, we discuss the first nontrivial case which is not covered by Theorem 3.28 for $\mathbb{R}^n$ that is to classify all possible calibrated geometries arising from a 3-form ϕ in $\mathbb{R}^6$. This characterization was begun in the article [8] by Harvey and Dadok and completed in [9] by Harvey and Morgan, so we will follow the articles [8,9] throughout this chapter. We begin with definitions and observations about the Grassmannian $G(3, \mathbb{R}^6)$.

Definition 4.1. *The smallest convex closed set containing Grassmannian $G(3, \mathbb{R}^6) \subset \Lambda_3\mathbb{R}^6$ is called the* mass ball K*. The* comass ball K^* *is the dual convex body of the mass ball in $\Lambda^3\mathbb{R}^6$. The* dual face $F^*(\xi)$*, of a point $\xi \in G(3, \mathbb{R}^6)$, is defined by*

$$F^*(\xi) \equiv \{\phi \in K^* : \phi(\xi) = 1\} \tag{4.1}$$

If $\xi \in \partial K$, boundary of K, then $\xi = a\xi_1 + (1-a)\xi_2$ is a convex linear combination of $\xi_1, \xi_2 \in G(3, \mathbb{R}^6)$. For $\phi \in F^*(\xi)$, we have $\phi(\xi) = \phi(a\xi_1 + (1-a)\xi_2) = a\phi(\xi_1) + (1-a)\phi(\xi_2)$ since ϕ is linear. This implies that $1 = \phi(\xi) = a\phi(\xi_1) + (1-a)\phi(\xi_2) \leq a + (1-a) = 1$, then $\phi \in F^*(\xi_1) \cap F^*(\xi_2)$. Conversely, we have $F^*(\xi_1) \cap F^*(\xi_2) \subset F^*(\xi)$. We can conclude that the only maximal faces of K^* (not contained in other faces) are of the form $F^*(\xi)$, $\xi \in G(3, \mathbb{R}^6)$.

$SO(6)$ acts on $G(3, \mathbb{R}^6)$ transitively. The points in $G(3, \mathbb{R}^6)$ describe three dimensional oriented planes in $\mathbb{R}^6$, so the symmetry group of $\mathbb{R}^6$ which preserves orientation consists of rotations, that is $SO(6)$. Therefore, for any two elements ξ, ζ in $G(3, \mathbb{R}^6)$, there is a matrix $A \in SO(6)$ such that $\xi = A \cdot \zeta$. By vector space duality, we also have equivalence of $F^*(\xi)$ and $F^*(\zeta)$ under the action of $SO(6)$.

4.1. The Face of e_{123}

By the discussion above, we can conclude that it is enough to compute $F^*(e_{123})$ to classify three form calibrations on $\mathbb{R}^6$. It follows from the First Cousin Principle, Lemma 2.8 that $\phi \in F^*(e_{123})$ is of the form

$$\begin{aligned}\phi(A,\mu) = e^{123} &+ a_{11}e^{156} + a_{12}e^{416} + a_{13}e^{451} \\ &+ a_{21}e^{256} + a_{22}e^{426} + a_{23}e^{452} \\ &+ a_{31}e^{356} + a_{32}e^{436} + a_{33}e^{453} + \mu e^{456}. \end{aligned} \tag{4.2}$$

since ϕ attains its maximum at e_{123} where $A_{3\times 3} = (a_{ij})$. The next theorem shows that some coefficients in (4.2) can be chosen to be zero.

Theorem 4.2. *$H = SO(3) \times SO(3) \subset SO(6)$ is the isotropy subgroup at e^{123}. The face $F^*(e_{123})$ is stable under H, and*

$$F^*(e_{123}) = H \cdot F \tag{4.3}$$

where $F \equiv F^*(e_{123}) \cap span\{e^{123}, e^{156}, e^{426}, e^{453}, e^{456}\} \equiv F^*(e_{123}) \cap V^5$.
Moreover,

$$\partial F^*(e_{123}) = H \cdot \partial F \tag{4.4}$$

Proof. Let us decompose orthogonally $\mathbb{R}^6 = \mathbb{R}^3_1 \oplus \mathbb{R}^3_2$ where $\mathbb{R}^3_1 = span\{e_1, e_2, e_3\}$ and $\mathbb{R}^3_2 = span\{e_4, e_5, e_6\}$. Then its dual is $(\mathbb{R}^6)^* = (\mathbb{R}^3_1)^* \oplus (\mathbb{R}^3_2)^*$. For an arbitrary element ϕ in $F^*(e_{123})$, by the First Cousin Principle, Lemma 2.8 we have $\langle \phi, \xi \rangle = 0$ for all $\xi \in \Lambda_2\mathbb{R}^3_1 \otimes \Lambda_1\mathbb{R}^3_2$. Then $\phi \in span\{e^{123}, e^{456}\} \oplus (\Lambda^1\mathbb{R}^3_1) \otimes (\Lambda^2\mathbb{R}^3_2) \equiv V^{11}$. The isotropy subgroup $H \subset SO(6)$ at e^{123} fixes e^{123} and e^{456}. We claim that H acts on $(\Lambda^1\mathbb{R}^3_1) \otimes (\Lambda^2\mathbb{R}^3_2)$ transitively. Let us choose a basis for $(\Lambda^1\mathbb{R}^3_1) \otimes (\Lambda^2\mathbb{R}^3_2)$ as $\{e^1, e^2, e^3\} \otimes \{e^{56}, e^{46}, e^{45}\}$. We can denote an element in $(\Lambda^1\mathbb{R}^3_1) \otimes (\Lambda^2\mathbb{R}^3_2)$ by a matrix $\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^T \cdot \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$ where a_i's are coefficients of e^i's for $i = 1, 2, 3$ and b_1 is the coefficient of e^{56}, b_2 is the coefficient of e^{46} and b_3 is the coefficient of e^{45}. Then

H acts on this 3×3 matrix equivalently as the action of $SO(3) \times SO(3)$ on the set of 3×3 real matrices A by $(g,h) \cdot A = gAh^{-1}$ for some $(g,h) \in SO(3) \times SO(3)$. By the singular value decomposition of real matrices, see for example [19], for every $A \in H$, we have the decomposition $(g,h) \cdot A = gAh^{-1} = D$ where D is the diagonal matrix. This means that every H-orbit in $(\Lambda^1\mathbb{R}^3_1) \otimes (\Lambda^2\mathbb{R}^3_2)$ intersects nontrivially with the set $\{e^{156}, e^{426}, e^{453}\}$ which corresponds the diagonal matrices. Therefore, $F^*(e_{123}) = H \cdot F$. We have $H \cdot \partial F \subset \partial F^*(e_{123})$. Now assume, for a contradiction, there is an element ξ in the boundary $\partial F^*(e_{123})$ and in the interior $\mathring{F}$ of F. There exists a sequence $\xi_n \in V^{11}$ outside $F^*(e_{123})$ that converges to ξ and $\langle e_{123}, \xi_n \rangle = 1$. Since $\xi_n \in V^{11}$, by the first part of the proof, we can write $\xi_n = h_n u_n \in H \cdot V^5$. It can be assumed, by compactness of H and V^5, h_n converges to h in H and u_n converges to u in V^5. Then we have $\xi = hu$. Now since ξ and u are in V^5 which lie in the same H-orbit in V^5, there exists $h^{'} \in H$ such that $h^{'} \cdot u = \xi$. Then we get $h^{'} \cdot u_n$ converges to ξ in $\mathring{F}$, $h^{'}u_n \in V^5$, $\langle e_{123}, h^{'}u_n \rangle = 1$ and $\{h^{'}u_n\} \cap F = \emptyset$, which gives a contradiction with $\xi \in \mathring{F}$. Hence, we get $\partial F^*(e_{123}) = H \cdot \partial F$. $\square$

Remark 4.3. In the proof of Theorem 4.2, we have shown that each orbit of the action of $SO(6)$ on $\Lambda^3(\mathbb{R}^6)$ contains an element from the five-dimensional subspace

$$\begin{aligned} V^5 &= span\,\{e^{123}, e^{156}, e^{426}, e^{453}, e^{456}\} \qquad (4.5) \\ &= \{\phi(\lambda,\mu) \equiv \lambda_0 e^{123} + \lambda_1 e^{156} + \lambda_2 e^{426} + \lambda_3 e^{453} + \mu e^{456} : \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4, \mu \in \mathbb{R}\} \end{aligned}$$

We want to classify the 3-forms in $F^*(e_{123})$, so we are interested in forms $\phi(\lambda,\mu)$ with $\lambda_0 = 1$

$$\phi(\lambda,\mu) = e^{123} + \lambda_1 e^{156} + \lambda_2 e^{426} + \lambda_3 e^{453} + \mu e^{456} \qquad (4.6)$$

and $\phi(e_{123}) = 1$. The next theorem describes F, defined in Theorem 4.2, with respect to the coefficients λ and μ for which we need the following two definitions.

Definition 4.4. *The open set*

$$T \equiv \{\phi(\lambda, 0) : |\lambda_j| \leq 1 \quad j = 1, 2, 3 \quad \textit{and} \quad \mathcal{H}(\lambda) = 1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 > 0\} \tag{4.7}$$

is called as the inflated tetrahedron.

The double point tetrahedron D *is defined as*

$$D \equiv \{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (0, \pi) \times (0, \pi) \times (0, \pi) : \alpha_i < \alpha_j + \alpha_k \quad \{i, j, k\} = \{1, 2, 3\}$$
$$\alpha_1 + \alpha_2 + \alpha_3 < 2\pi\} \tag{4.8}$$

Definition 4.5. *Let* $T^3 = S^1 \times S^1 \times S^1$ *denote the three dimensional torus in* $\mathbb{R}^6$. *On* T^3, *we have* 3-torus vectors *in* $G(3, \mathbb{R}^6)$, *which are defined by*

$$\eta(\alpha) = (c_1e_1 + s_1e_4) \wedge (c_2e_2 + s_2e_5) \wedge (c_3e_3 + s_3e_6) \equiv \exp(i\alpha_1)e_1 \wedge \exp(i\alpha_2)e_2 \wedge \exp(i\alpha_3)e_3 \tag{4.9}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ *and* $s_i \equiv sin\alpha_i$, $c_i \equiv cos\alpha_i$.

Lemma 4.6. *[20] Consider* $\mathbb{R}^{2n} \cong \mathbb{C}^n$ *with an orthonormal basis* $\{e_1, .., e_n, ie_1, .., ie_n\}$. *Let* $\phi \in \Lambda^n(\mathbb{R}^{2n})$ *satisfy*

$$(e_j \wedge ie_j) \lrcorner \phi = 0 \quad 1 \leq j \leq n. \tag{4.10}$$

Then ϕ *attains its maximum at a unit simple n-vector of the form*

$$\eta(\alpha) = \exp(i\alpha_1)e_1 \wedge \cdots \wedge \exp(i\alpha_n)e_n. \tag{4.11}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$

Lemma 4.6 says that $\phi(\lambda, \mu)$, given in (4.6) also attains its maximum at a 3-torus vector $\eta(\alpha)$ since it satisfies the condition (4.10). Now we will examine $\eta(\alpha)$'s on T^3 such that $\phi(\eta(\alpha)) = 1$ and $\dfrac{d}{d\alpha}\phi(\eta(\alpha)) = 0$. By computing these explicitly we get the so called *critical equations*:

$$1 = c_1c_2c_3 + \lambda_1c_1s_2s_3 + \lambda_2s_1c_2s_3 + \lambda_3s_1s_2c_3 + \mu s_1s_2s_3 \tag{4.12}$$

$$0 = -s_1c_2c_3 - \lambda_1s_1s_2s_3 + \lambda_2c_1c_2s_3 + \lambda_3c_1s_2c_3 + \mu c_1s_2s_3 \tag{4.13}$$

$$0 = -c_1s_2c_3 + \lambda_1c_1c_2s_3 - \lambda_2s_1s_2s_3 + \lambda_3s_1c_2c_3 + \mu s_1c_2s_3 \tag{4.14}$$

$$0 = -c_1c_2s_3 + \lambda_1c_1s_2c_3 + \lambda_2s_1c_2c_3 - \lambda_3s_1s_2s_3 + \mu s_1s_2c_3 \tag{4.15}$$

We will determine the solutions (λ, μ, α) of these equations and define the solutions set $\Sigma \equiv \{(\lambda, \mu) :$ the critical equations have a solution (λ, μ, α) and $\eta(\alpha) \neq e_{123}\}$.

Remark 4.7. This set may not be closed but its limit points also satisfy the critical equations since cosine and sine functions are continuous and these equations depend on λ and μ linearly. This means that we want to find the closure of the solutions set, $\overline{\Sigma}$ that is called as *solution shadow*.

Consider the space containing $(\lambda, \mu, c_1, c_2, c_3, s_1, s_2, s_3)$. The solution shadow is a subspace of it and contains elements satisfying $c_i^2 + s_i^2 = 1$. If $s_i = 0$ for all $i = 1, 2, 3$ and $c_i = \pm 1$ such that $c_1c_2c_3 = 1$, then $\eta(\alpha) = e_{123}$ which does not give any information about λ and μ. Hence we are not interested in such (c_i, s_i). Therefore, we will assume some $s_i \neq 0$.

Lemma 4.8. *The solution shadow* $\overline{\Sigma}$ *contains the locus* $(1 - \lambda_1^2)(1 - \lambda_2^2)(1 - \lambda_3^2) = 0$.

Proof. Assume, without loss of generality, $\lambda_1 = 1$ and μ is arbitrary. Now put $c_1 = 1$ and $c_2 = c_3 = c, s_2 = s_3 = s$. Then critical equations (4.12-4.15) are satisfied automatically except (4.13). It is also satisfied if and only if $(\lambda_2 + \lambda_3)sc + \mu s^2 = 0$ and $c^2 + s^2 = 1$. If we choose λ_2, λ_3 satisfying $\lambda_2 + \lambda_3 \neq 0$, then c, s can be set as $\frac{c}{s} = -\frac{\mu}{\lambda_2 + \lambda_3}$, $c^2 + s^2 = 1$ and $s \neq 0$. Hence this proves that $(\lambda, \mu) \in \overline{\Sigma}$. □

Lemma 4.9. *Suppose* $(\lambda, \mu, \alpha) \in \Sigma$. *Then* $s_i = 0$ *implies that* $\lambda_i^2 = 1$ *and* $s_js_k \neq 0$ *where* $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Assume $s_i = 0$. Then by critical equations (4.12-4.15) we get $s_j c_k = \lambda_i c_j s_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Hence $s_j c_k(1 - \lambda_i^2) = 0$. The case $1 - \lambda_i^2 \neq 0$ implies either $s_j = s_k = 0$ or $c_j = c_k = 0$. First implication can not hold since $\eta(\alpha) \neq e_{123}$. Second implication and Equation 4.12 give that $\lambda_i = 1$, and $s_j s_k = \pm 1$. Hence, for all cases we have $\lambda_i = 1$ and $s_j s_k \neq 0$. □

Lemma 4.10. *Suppose* $(\lambda, \mu, \alpha) \in \Sigma$. *Then for* $\{i, j, k\} = \{1, 2, 3\}$

(i) $s_j s_k \lambda_i = c_i - c_j c_k$
(ii) $s_i s_j s_k \mu = 1 - \sum c_j^2 + 2c_1 c_2 c_3$
(iii) $s_j^2 s_k^2 (1 - \lambda_i^2) = 1 - \sum c_j^2 + 2c_1 c_2 c_3$
(iv) $s_i^2 s_j s_k (\lambda_i + \lambda_j \lambda_k) = c_i(1 - \sum c_j^2 + 2c_1 c_2 c_3)$
(v) $s_i s_j s_k (\mathcal{H}(\lambda) - \mu^2) = 0$

Proof. To show (i) for $i = 1$, $j = 2$, $k = 3$, multiply Equation 4.12 by c_1 and subtract s_1 times the Equation 4.13. This gives that $c_1 = c_2 c_3 + \lambda_1 s_2 s_3$. The other cases are treated similarly. If we substitute $\lambda_i s_j s_k$ in Equation 4.13, we will get (ii). To find (iii), square (i) which will give

$$s_j^2 s_k^2 \lambda_i^2 = c_i^2 - 2c_1 c_2 c_3 + c_j^2 c_k^2 = \sum c_l^2 - 2c_1 c_2 c_3 - 1 + s_j^2 s_k^2.$$

From (i) we have $\lambda_i = \dfrac{c_i - c_j c_k}{s_j s_k}$, plug this into the left hand side of equation (iv), this proves (iv). For $i = 1$, $j = 2$, $k = 3$, by adding c_1 times square of (ii) and square of (iv), we get $s_1^4 s_2^2 s_3^2 (\mu^2 + (\lambda_1 + \lambda_2 \lambda_3)^2) = 1 - \sum c_j^2 + 2c_1 c_2 c_3$. Then by (iii), we have $s_1^4 s_2^2 s_3^2 (\mu^2 + (\lambda_1 + \lambda_2 \lambda_3)^2) = s_1^4 s_2^2 s_3^2 (1 - \lambda_2^2)(1 - \lambda_3^2)$. Also by using the identity $\mathcal{H}(\lambda) + (\lambda_1 + \lambda_2 \lambda_3)^2 = (1 - \lambda_2^2)(1 - \lambda_3^2)$, we get (v). □

Corollary 4.11. *The solution shadow* $\overline{\Sigma}$ *is the set of zeros of*

$$(1 - \lambda_1^2)(1 - \lambda_2^2)(1 - \lambda_3^2)(1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1 \lambda_2 \lambda_3 - \mu^2). \tag{4.16}$$

Proof. This result follows from the Lemmas 4.8, 4.9 and 4.10v. □

Lemma 4.12. *F is contained in $\{\phi(\lambda,\mu) : \lambda \in \overline{T}\}$ where $T = \{\lambda : |\lambda_j| \leq 1 \quad and$ $1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 > 0\}$ is the inflated tetrahedron.*

Proof. Let $\phi(\lambda,\mu) \in F$. Consider the function $f(\alpha) = \langle \phi(\lambda,\mu), \eta(\alpha)\rangle$. This function attains its maximum value at $\alpha = 0$, so the Hessian of f must be negative semidefinite at $\alpha = 0$.

$$Hessian(f)|_{\alpha=0} = \begin{vmatrix} -1 & \lambda_3 & \lambda_2 \\ \lambda_3 & -1 & \lambda_1 \\ \lambda_2 & \lambda_1 & -1 \end{vmatrix} = -1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2\lambda_3 \leq 0$$

so $\mathcal{H}(\lambda) \geq 0$ and $|\lambda_j| \leq 1 \quad j = 1,2,3$. In other words, $\lambda \in \overline{T}$, and no condition appears for μ. Moreover, since a convex combination of the nonnegative matrices remains nonnegative, the above set is convex. $\square$

Lemma 4.13. *The boundary of F satisfies :*

(i) $\partial T \subset \partial F$
(ii) the graph of $\mu = \pm\sqrt{\mathcal{H}(\lambda)}$ over $\lambda \in T$ is also a subset of ∂F.

Proof. For an arbitrary element $\lambda \in \partial T$, we will consider the ray $t \to (t\lambda, 0)$, $t \geq 0$. The initial point of the ray $\phi(0,0) = e^{123}$ lies in F. Assume for some t_0, $\phi(t_0\lambda, 0)$ is on the boundary of F. Now there is two cases: the face of $\phi(t_0\lambda, 0)$ also contains a torus three vector $\eta(\alpha)$ or not. In the first case we have $G(\phi(t_0\lambda,0)) \supset \{e_{123}, \eta(\alpha)\}$, so $(t_0\lambda, 0)$ must be in the solution shadow $\overline{\Sigma}$, hence by the Corollary 4.11, we get $t_0 = 1$ (since λ satisfies $\mathcal{H}(\lambda) = 0$). In the second case, we have $G(\phi(t_0\lambda,0)) = e_{123}$. Then Hessian at $\alpha = 0$ of $\alpha \to \langle\phi(t_0\lambda,0), \eta(\alpha)\rangle$ is not negative definite. By Lemma 4.12 , we have $\mathcal{H}(t_0\lambda, 0) = 0$. Hence, we get $t_0 = 1$. In each case $\lambda \in \partial F$.

In order to prove (ii), consider the line $t \to (\lambda, t)$ for fixed $\lambda \in T$, $t \in \mathbb{R}$. As we have seen in the proof of Lemma 4.12, the Hessian of $\alpha \to \langle\phi(\lambda,t), \eta(\alpha)\rangle$ does not depend on t at $\alpha = 0$. This line lie in F if and only if $\mathcal{H}(\lambda) \geq \mu^2$. At $t = \mu$, we must have $G(\phi, (\lambda,\mu)) \supset \{e_{123}, \eta(\alpha)\}$, $\eta(\alpha) \neq e_{123}$. Again by Corollary 4.11, we must have

$\mu = \pm\sqrt{\mathcal{H}(\lambda)}$. $\square$

Theorem 4.14. *F is determined as:*

$F = \{\phi(\lambda,\mu) : |\lambda_j| \leq 1 \quad j = 1,2,3 \quad \textit{and} \quad 1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 - \mu^2 \geq 0\}$

and ∂F is given as:

$\partial F = \{\phi(\lambda,\mu) : |\lambda_j| \leq 1 \quad j = 1,2,3 \quad \textit{and} \quad 1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 - \mu^2 = 0\}$

Proof. We have by Lemma 4.12 and Lemma 4.13

$$S \equiv \{\phi(\lambda,\mu) : \mu^2 = \mathcal{H}(\lambda) \quad \text{and} \quad |\lambda_j| \leq 1 \quad j = 1,2,3\} \subset \partial F.$$

The set S is closed since it is the graph of $\mu = \pm\sqrt{\mathcal{H}(\lambda)}$. The complement of S in $\mathbb{R}^3 \times \mathbb{R}$ is open and has two components. Since F is convex, it is connected hence the complement of ∂F has two components. Therefore, we get $S = \partial F$ and $F = \{\phi(\lambda,\mu) : \mu^2 \leq \mathcal{H}(\lambda) \quad \text{and} \quad |\lambda_j| \leq 1 \quad j = 1,2,3\}$ $\square$

Remark 4.15. Theorem 4.14 describes the convex set F. For the form $\phi(\lambda,\mu)$, the conditions $|\lambda_j| \leq 1$ and $\mathcal{H}(\lambda) - \mu^2 \geq 0$ can be seen as: $\|D\| \leq 1$ and $1 - |D|^2 - 2\det(D) - \mu^2 \geq 0$ where D is the diagonal matrix $D = (d_{jj}) = \lambda_j$ and $\|D\| = \max|Dx|$ for unit vectors $x \in \mathbb{R}^3$ and $|D|$ is the L^2-norm of D. These conditions also holds for the general form $\phi(A,\mu) \in F^*(e_{123})$, as in (4.2) as follows:

$$\|A\| \leq 1 \tag{4.17}$$

$$1 - |A|^2 - 2\det(A) - \mu^2 \geq 0 \tag{4.18}$$

since A is equivalent to a diagonal matrix D by the singular value decomposition, $GAH = D$ for some special orthogonal matrices G, H and the above norms and determinant function are preserved by such matrices.

Theorem 4.16. *The boundary of F contains the following four distinct cases:*

(i) The singular points of ∂F. There are four of them given as $|\lambda_j| = \pm 1$ and $\mathcal{H}(\lambda) = 0$.

(ii) The open line segments obtained by joining any pair of SLAG calibrations.

(iii) The points $\phi(\lambda, 0)$ where $|\lambda_j| < 1$ for all $j \in \{1, 2, 3\}$, which are interior points of ∂T.

(iv) For $\mu \neq 0$, the graph of $\mu = \pm\sqrt{\mathcal{H}}$ over $\lambda \in T$.

Proof. $\phi(\lambda, \mu) \in \partial F$ can be described by the variables λ and μ with the relation $\mathcal{H}(\lambda) - \mu^2 \geq 0$. The gradient of $\mathcal{H}(\lambda) - \mu^2$ is equal to

$$(-2\lambda_1 - 2\lambda_2\lambda_3, -2\lambda_2 - 2\lambda_1\lambda_3, -2\lambda_3 - 2\lambda_1\lambda_2)$$

This vanishes when $\lambda_1 + \lambda_2\lambda_3 = \lambda_2 + \lambda_1\lambda_3 = \lambda_3 + \lambda_1\lambda_2 = 0$. Hence, the singular points of ∂F are attained when $\lambda_j = \pm 1$ with the odd number of negative signs. Moreover, by the equality $(1 - \lambda_2^2)(1 - \lambda_3^2) = \mathcal{H}(\lambda) + (\lambda_1 + \lambda_2\lambda_3)^2$, we can say that the sign of $1 - \lambda_i$ and $1 - \lambda_j$ must be the same on $\mathcal{H}(\lambda) - \mu^2 = 0$. Therefore, if we exclude the four vertices (which result in special Lagrangian forms), ∂F is the submanifold, $\mathcal{H}(\lambda) - \mu^2 = 0$, of the open set $\lambda_j^2 < 1$ some j. This proves (i).
Without loss of generality, assume $\lambda_1 = \pm 1$ then we have $-(\lambda_2 \pm \lambda_3)^2 = \mu^2$. Then $\lambda_2 \pm \lambda_3 = \mu = 0$. This gives the second type of calibrations on ∂F.
For $\mu = 0$, there is one remaining case which stems from the condition $|\lambda_j| < 1$, $j = 1, 2, 3$. This produces the third type of calibrations.
For $\mu \neq 0$, we have the fourth type which are considered as the graph of $\mu = \pm\sqrt{\mathcal{H}(\lambda)}$. $\square$

Remark 4.17. This theorem classifies the calibrations in ∂F, from which we can get the calibrations in F since F is convex. Moreover we have $F^*(e_{123}) = H \cdot F$ where $H = SO(3) \times SO(3)$ as in Theorem 4.2. Therefore we have the classification of calibrations in $F^*(e_{123})$.

4.2. The Classification of Faces of $G(3, \mathbb{R}^6)$

Next we will study the calibrated planes that are calibrated by the four distinct types of calibrations given in Theorem 4.16.

Theorem 4.18. *(Special Lagrangian ϕ) Suppose $\phi \in \partial F$ is a singular point. Then ϕ is one of the four possible SLAG calibrations :*

$$\phi = e^{123} \pm e^{156} \pm e^{426} \pm e^{453} \quad \textit{(for odd number of negative signs)}$$

All of these four are $SO(6)$ equivalent to $Re\, dz$. Then the face of ϕ is equivalent to special Lagrangian planes:

$$G(Re\, dz) \cong SU(3)/SO(3) \equiv SLAG$$

Proof. Under the complex structure J determined by $Je_1 = e_4$, $Je_2 = e_5$, $Je_3 = e_6$, we have

$$\begin{aligned}
Re\, dz_1 \wedge dz_2 \wedge dz_3 &= \phi(-1,-1,-1,0) = e^{123} - e^{156} - e^{426} - e^{453} \\
Re\, dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 &= \phi(-1,1,1,0) = e^{123} - e^{156} + e^{426} + e^{453} \\
Re d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 &= \phi(1,-1,1,0) = e^{123} + e^{156} - e^{426} + e^{453} \\
Re d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 &= \phi(1,1,-1,0) = e^{123} + e^{156} + e^{426} - e^{453}
\end{aligned}$$

These four special Lagrangian calibrations are $SO(6)$ equivalent. By the discussion in the Section 3.3, $G(Re\, dz)$ is equal to the five-dimensional manifold of special Lagrangian planes. In other words, $G(Re\, dz) = SU(3)/SO(3) \equiv SLAG$. □

Theorem 4.19. *(Kähler ϕ) Suppose $\phi \in \partial F$ is a second type calibrations as in Theorem 4.16(ii). Then ϕ can be written as:*

$$\phi \equiv t\psi_1 + (1-t)\psi_2, \quad t \in (0,1)$$

where ψ_1, ψ_2 are some SLAG calibrations. Then the face of ϕ, $G(\phi) = G(\psi_1) \cap G(\psi_2)$ is independent of t.

By taking $t = \frac{1}{2}$ and using action of $SO(6)$ we may assume

$$\phi = e_1 \wedge \omega = e_1 \wedge (e_2 \wedge e_3 - e_5 \wedge e_6)$$

where ω is the standard Kähler form with $Je_2 = e_3$, $Je_5 = -e_6$. Then $G(\phi) \cong SU(2)/SU(1) \cong \mathbb{C}P^1 \equiv G(1, \mathbb{C})$.

Theorem 4.20. *(Single point calibration ϕ) The points $\phi(\lambda, 0)$ where $|\lambda_j| < 1$ for all $j \in \{1, 2, 3\}$ calibrates only the plane e_{123}. In other words, $F(\phi) = \{e_{123}\}$. For instance, $\phi = e^{123} + \frac{1}{2}e^{156} + \frac{1}{2}e^{426} + \frac{1}{2}e^{453}$.*

Proof. Assume $\mu = 0$ and $1 - \lambda_j^2 > 0$ for all $j = 1, 2, 3$. Take an element $\eta(\alpha) \in G(\phi(\lambda, 0))$. Then we have by Lemma 4.10ii and iii $s_j^2 s_k^2(1 - \lambda_i^2) = s_1 s_2 s_3 \mu = 0$ for $j = 1, 2, 3$. Hence, we have $s_1 s_2 s_3 = 0$ that implies $\eta(\alpha) = e_{123}$ by Lemma 4.9. This proves that $G(\phi) \cap T^3 = \{e_{123}\}$. Then, $G(\phi)$ contains only the plane e_{123}. Therefore $F(\phi) = \{e_{123}\}$. $\square$

Theorem 4.21. *(Double point calibration ϕ) Suppose $\phi(\lambda, \mu) \in \partial F$ and $\mu \neq 0$. Then the face of ϕ contains two elements e_{123} and $\eta(\alpha)$ for some uniquely chosen α in double point tetrahedron D (respectively -D) if $\mu > 0$ (respectively $\mu < 0$) satisfying following:*

$$\cot \alpha_j = \frac{\lambda_i + \lambda_j \lambda_k}{\mu} \quad \{i, j, k\} = \{1, 2, 3\}$$

Conversely, there is a unique $\phi \in \partial K^*$ such that $G(\phi) = \{e_{123}, \eta(\alpha)\}$ for any $\alpha \in \pm D$. Moreover, this $\phi = \phi(\lambda, \mu) \in \partial F$, $\mu \neq 0$ is determined by

$$\lambda_i = \frac{c_i - c_j c_k}{s_j s_k} \qquad \{i, j, k\} = \{1, 2, 3\}$$
$$\mu = \frac{1 - \sum c_j^2 + 2c_i c_j c_k}{s_1 s_2 s_3}$$

Proof. Assume $\phi(\lambda, \mu) \in \partial F$ and $\mu \neq 0$. We have $G(\phi(\lambda, \mu)) \supset \{e_{123}, \eta(\alpha)\}$, $\eta(\alpha) \neq e_{123}$ since $s_j^2 s_k^2(1 - \lambda_i^2) = s_1 s_2 s_3 \mu \neq 0$ and $\lambda \in T$ (i.e. it satisfies $|\lambda_j| < 1$). Also we have $s_1s_2s_3 \neq 0$. Hence, we can define

$$\frac{\cos\alpha_j}{\sin\alpha_j} = \cot\alpha_j = \frac{\lambda_i + \lambda_j\lambda_k}{\mu} \qquad \{i,j,k\} = \{1,2,3\} \tag{4.19}$$

Now, we need to show that $\alpha \in D$ if $\mu > 0$ and $-\alpha \in D$ if $\mu < 0$. On the three-torus T^3, adding π to any pair of the angles $\alpha_1, \alpha_2, \alpha_3$ does not change the point $\eta(\alpha)$, we can assume that either $\alpha_j \in (0,\pi)$ or $-\alpha_j \in (0,\pi)$ for $j = 1,2,3$. Since we have $s_j^2 s_k^2(1-\lambda_i^2) = s_1s_2s_3\mu > 0$ by Lemma 4.10 and our assumptions, it can be concluded that sign of $s_1s_2s_3$ and μ are the same. Hence Equation 4.19 uniquely determines α as a function of λ and μ.

First assume $\mu > 0$, then $s_j > 0$ for all j. Then by Lemma 4.10i, we have the following equivalent formulas.

$$-1 < \lambda_i = \frac{c_i - c_jc_k}{s_js_k} \tag{4.20}$$

$$\cos(\alpha_j + \alpha_k) \equiv c_jc_k - s_js_k < c_i \tag{4.21}$$

$$0 < \alpha_i < \alpha_j + \alpha_k < 2\pi - \alpha_i \tag{4.22}$$

These show that $\alpha \in D$. Similarly, it follows that for $\mu < 0$, $-\alpha \in D$. This proves one direction of the theorem.

Conversely, take an arbitrary $\pm\alpha \in D$. We will show that there exists a unique $\phi \in K^*$ such that $G(\phi) \cap T^3 = \{e_{123}, \eta(\alpha)\}$. Define

$$\lambda_i \equiv \frac{c_i - c_jc_k}{s_js_k} \qquad \{i,j,k\} = \{1,2,3\} \tag{4.23}$$

$$\mu \equiv \frac{1 - \sum c_j^2 + 2c_1c_2c_3}{s_1s_2s_3} \tag{4.24}$$

By Lemma 4.10, with the above λ and μ, $\phi = \phi(\lambda,\mu)$ satisfy the critical equations and we have $\mathcal{H}(\alpha) = \mu^2$ since $\max|\lambda_j| < 1$. Lemma 4.10i says that $\phi(\lambda,\mu) \in F$. By the first part of the proof, we can conclude that $G(\phi(\lambda,\mu)) \cap T^3 = \{e_{123}, \eta(\alpha)\}$. □

The following lemma gives equivalent conditions to Equation 4.18 and then a proposition about results obtained in this chapter is given for later applications.

Lemma 4.22. *(Lemma 3.3 of [10]) For a* 3×3 *matrix* $A = (a_{ij})$ *satisfying the condition* $1 - |A|^2 - 2\det(A) \geq 0$*, the following statements are equivalent:*

$$\|A\| \leq 1 \tag{4.25}$$

$$\sum_j a_{ij}^2 \leq 1 \quad \textit{for all } i, \tag{4.26}$$

$$|\det A| \leq 1 \tag{4.27}$$

$$|A|^2 \leq 3. \tag{4.28}$$

where $|A|^2 = \sum_{i,j} a_{ij}^2$ *and* $\|A\| = \max|Ax|$ *where* x *is a unit vector in* $\mathbb{R}^3$*. Furthermore if* A *is triangular,*

$$\max|a_{jj}| \leq 1 \tag{4.29}$$

is also equivalent to the above conditions.

Proof. We may assume A is a diagonal since norms and determinant function that appear in (4.25-4.48) are not changed under the singular value decomposition of A. Then the proof is immediate. If A is triangular, (4.25) implies (4.29) and (4.29)implies (4.27). □

Proposition 4.23. *(Proposition* 3.4 *of [10]) Suppose* $\phi \in \Lambda^3(\mathbb{R}^6)$ *is of the form* $\phi(A,\mu)$*. Let* φ *denote the projection of* ϕ *into* $\Lambda^1(\mathrm{span}\, e_{123}) \otimes \Lambda^2(\mathrm{span}\, e_{456})$*. Denote* $P_0 = |\varphi|^2$*,* $E = e_{456}$*, and define* D_0 *by*

$$D_0 E \equiv [(e_1 \lrcorner \varphi)\lrcorner E] \wedge [(e_2 \lrcorner \varphi)\lrcorner E] \wedge [(e_3 \lrcorner \varphi)\lrcorner E]$$

where $\lrcorner$ *denotes the interior multiplication [14]. Then* $\|\phi\|^* \leq 1$ *if and only if*

$$1 - P_0 - 2D_0 - \mu^2 \quad \geq 0 \tag{4.30}$$

$$3 - P_0 \quad \geq 0. \tag{4.31}$$

The (4.31) is equivalent to

$$|(e_i \lrcorner \varphi) \lrcorner E| \leq 1 \quad \textit{for} \quad i = 1, 2, 3. \tag{4.32}$$

If $\|\phi\|^* = 1$ *and* $G(\phi) \neq \{e_{123}\}$, *then* $1 - P_0 - 2D_0 = \mu^2$.
If $\|\phi\|^* = 1$, *then* $G(\phi)$ *is a double point face if and only if* $1 - P_0 - 2D_0 = \mu^2 > 0$.

Proof. Note that $P_0 = |A|^2$ since $\varphi = a_{11}e^{156} + a_{12}e^{416} + a_{13}e^{451} + a_{21}e^{256} + a_{22}e^{426} + a_{23}e^{452} + a_{31}e^{356} + a_{32}e^{436} + a_{33}e^{453}$ and $D_0 = \det A$ since we compute

$$(e_i \lrcorner \varphi) \lrcorner E = (a_{i1}e^{56} - a_{i2}e^{46} + a_{i3}e^{45}) \lrcorner E = a_{i1}e_4 + a_{i2}e_5 + a_{i3}e_6 \quad \text{for i} = 1, 2, 3.$$

Then we get

$$\begin{aligned} D_0 E &= (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})E \\ &= (\det A)E \end{aligned}$$

Hence, (4.30) is equivalent to $1 - |A|^2 - 2\det\ A - \mu^2 \geq 0$. (4.31) and (4.32) are equivalent to (4.28) in Lemma 4.21. The other statements follow from Theorem 4.16. $\square$

5. CLASSIFICATION OF 3-FORMS ON $\mathbb{R}^7$

In this chapter, we will classify the faces of the Grassmannian $G(3, \mathbb{R}^7)$ by closely following the article [10]. We will separate calibrations ϕ with the faces $G(\phi)$ into two cases with respect to whether they contain a $\mathbb{CP}^1$ or not. In Section 5.3, we will give the complete classification of the faces of the Grassmannian $G(3, \mathbb{R}^7)$.

Note that for the Grassmannian $G(3, \mathbb{R}^6)$, we can say that the special Lagrangian faces contain a $\mathbb{CP}^1$ by Theorems 4.17 and 4.18; but the double-point faces do not by Theorem 4.20.

Throughout this chapter we will denote calibrations on $\mathbb{R}^6$ by $\phi(\cdot)$ with round brackets as in the Chapter 4 and on $\mathbb{R}^7$ by $\phi[\cdot]$ with square brackets.

5.1. The Face of $\mathbb{CP}^1$

In this section, we will describe the calibrations $\phi \in \Lambda^3(\mathbb{R}^7)$ such that the face $G(\phi)$ of $G(3, \mathbb{R}^7)$ contains a $\mathbb{CP}^1$. Since the face $G(\phi)$ contains three real dimensional planes, we have to clarify what we mean by a $\mathbb{CP}^1$. Here by a $\mathbb{CP}^1$ we mean the Cartesian product of a fixed real line with the set of complex lines in an orthogonal $\mathbb{C}^2 \cong \mathbb{R}^4$. By the action of $O(7)$ on $\mathbb{R}^7$, it can be assumed $G(\phi)$ contains the "standard" $\mathbb{CP}^1_0$ where

$$\mathbb{CP}^1_0 \equiv e_1 \wedge G(1, span\, e_{2365})$$

with an orthonormal basis $\{e_1, \ldots, e_7\}$ of $\mathbb{R}^7 \equiv \mathbb{R}^1 \times \mathbb{R}^4 \times \mathbb{R}^2$ where $\mathbb{R}^1 = span\,\{e_1\}$, $\mathbb{C}^2 \cong \mathbb{R}^4 = span\,\{e_2, e_3, e_6, e_5\}$ with the complex structure J satisfying $e_3 = Je_2$ and $e_5 = Je_6$, and $\mathbb{R}^2 = span\,\{e_4, e_7\}$. The calibrations ϕ satisfying $G(\phi) \supset \mathbb{CP}^1_0$ lies in the dual face

$$F^*(\mathbb{CP}^1_0) = \{\phi \in \Lambda^3(\mathbb{R}^7) : \phi(\xi) = \|\phi\|^* = 1 \text{ for all } \xi \in \mathbb{CP}^1_0\}.$$

As in the case for $F^*(e_{123})$ on $\mathbb{R}^6$, we want to reduce the number of basis 3-forms to describe the calibrations that are in $F^*(\mathbb{CP}^1_0)$. Note that $\mathbb{CP}^1_0$ and so $F^*(\mathbb{CP}^1_0)$ are invariant under the group action of $id \times U(2) \times O(2)$ on $\mathbb{R}^1 \times \mathbb{R}^4 \times \mathbb{R}^2$.

Proposition 5.1. *Every $\phi \in F^*(\mathbb{CP}^1_0)$ can be written as*

$$\begin{aligned}\phi[a, B, c] = e^1 \wedge (e^{23} - e^{56}) + ae^{147} \\ + [b_{11}(e^{25} + e^{36}) + b_{12}(e^{26} + e^{53})] \wedge e^7 \\ + [b_{21}(e^{25} + e^{36}) + b_{22}(e^{26} + e^{53})] \wedge e^*_4 \\ + c \wedge e^{47}\end{aligned} \tag{5.1}$$

where $B = (b_{ij})$ is a 2×2 matrix and

$$c = c_2e^2 + c_3e^3 + c_6e^6 + c_5e^5 \in \Lambda^1(\mathbb{R}^4) \cong \mathbb{R}^4.$$

Proof. There are $\binom{7}{3} = 35$ simple 3-vectors in the basis of $\Lambda^3\mathbb{R}^7$, namely $\{e^{ijk} \equiv e^i \wedge e^j \wedge e^k : i < j < k\}$ forms a basis for $\Lambda^3\mathbb{R}^7$. Hence every $\phi \in \Lambda^3\mathbb{R}^7$ can be written as

$$\phi = \sum_{i<j<k} \phi_{ijk} e^{ijk}$$

Since $\phi \in F^*(\mathbb{CP}^1_0)$, it attains its maximum at e_{123} and $-e_{156}$. Then we have $\phi_{123} = 1 = -\phi_{156}$, and by Lemma 2.8, that is the First Cousin Principle, we have $\phi_{ijk} = 0$ for $card(\{i, j, k\} \cap \{1, 2, 3\}) = 2$ or $card(\{i, j, k\} \cap \{1, 5, 6\}) = 2$. Moreover, we have the equalities:

$$\phi_{267} = -\phi_{357} \tag{5.2}$$

$$\phi_{257} = \phi_{367} \tag{5.3}$$

$$\phi_{245} = \phi_{346} \tag{5.4}$$

$$\phi_{246} = -\phi_{345} \tag{5.5}$$

which follow from $\phi(\xi) = 1$ where

$$\xi \in \{e_1 \wedge \frac{e_2 + e_5}{\sqrt{2}} \wedge \frac{e_3 - e_6}{\sqrt{2}}, e_1 \wedge \frac{e_2 + e_6}{\sqrt{2}} \wedge \frac{e_3 + e_5}{\sqrt{2}}\} \subset \mathbb{CP}^1_0.$$

For instance, consider the function $f(\theta) = \phi((e_1 \cos\theta + e_7 \sin\theta) \wedge \frac{e_2 + e_5}{\sqrt{2}} \wedge \frac{e_3 - e_6}{\sqrt{2}})$ which attains its maximum at $\theta = 0$, so we have

$$\begin{aligned} 0 = f'(0) &= \frac{1}{2}\phi(e_{723} - e_{726} + e_{753} - e_{756}) \\ &= \frac{1}{2}(0 - \phi_{267} - \phi_{357} - 0) \end{aligned}$$

that gives (5.2). The others follow similarly. On the remaining coefficients there are no more conditions, thus we get the form $\phi = \phi[a, B, c]$ as in (5.1). □

Remark 5.2. If we put a complex structure J on $span\, e_{2365} \cong \mathbb{C}^2$ satisfying $e_3 = Je_2$, $e_5 = Je_6$, then the standard Kähler form ω will be equal to $e^{23} + e^{65}$ and the volume form dz will be equal to $(e^{26} + e^{53}) + i(e^{25} + e^{36})$. Thus $\phi[a, B, c]$ in (5.1) can be written in terms of ω and $Re\, dz$ as

$$\begin{aligned} \phi[a, B, c] = e^1 \wedge \omega + ae^{147} & \\ + [b_{11}(Im\, dz) + b_{12}(Re\, dz)] \wedge e^7 & \\ + [b_{21}(Im\, dz) + b_{22}(Re\, dz)] \wedge e^4 & \\ + c \wedge e^{47} & \end{aligned} \tag{5.6}$$

with $c \in \Lambda^1(span\, e_{2365})$.

Proposition 5.3. *Under the group action $id \times U(2) \times O(2)$, every $\phi \in F^*(\mathbb{CP}^1_0)$ can* *be written as*

$$\begin{aligned}\phi[a,\mu] &= e^{123} - e^{156} + a_1 e^{147} \\ &+ a_2(e^{25}+e^{36})\wedge e^7 \\ &+ a_3(e^{26}+e^{53})\wedge e^4 \\ &+ \mu e^{247}\end{aligned} \tag{5.7}$$

where $a = (a_1, a_2, a_3)$. *Each* $\phi \in \Lambda^3(\mathbb{R}^7)$ *of the form* $\phi[a,\mu]$ *or* $\phi[a,B,c]$ *satisfies* $\phi(\xi) = 1$ *for all* $\xi \in \mathbb{CP}^1_0$.

Proof. Let $\phi \in F^*(\mathbb{CP}^1_0)$ be of the form $\phi[a,B,c]$ as in (5.6). To get the reduced form $\phi[a,\mu]$ of ϕ, we can act on $\phi[a,B,c]$ first by an element in $U(1)$ on *span* e_{23}, then by an element in $SO(2)$ on *span* e_{47} and finally by an element in $SU(2)$ on *span* e_{2365}. The action $U(1)$ on *span* e_{23} leaves ω fixed and multiplies dz by a complex number with modulus 1, and rotates c. Then with the action of $SO(2)$ on *span* e_{47}, we can diagonalize B. Moreover, the action of $SU(2)$ on *span* e_{2365} leaves ω, dz fixed, and acts transitively on *span* e_{2365}, and maps c to e^2. Thus, we get $\phi = \phi[a,\mu]$, as the required form.
Next, let $\xi \in \mathbb{CP}^1_0$. By the action of $id \times U(2)$ on $\mathbb{R}\times\mathbb{R}^4$, it can be assumed $\xi = e_{123}$. Then, we have $\phi(\xi) = 1$ for $\phi[a,B,c]$ and $\phi[a,\mu]$. □

Theorem 5.4. *Let* $\phi \in \Lambda^3(\mathbb{R}^7)$ *be as in (5.7). Then* $\phi \in F^*(\mathbb{CP}^1_0)$ *if and only if*

$$\mathcal{H}(a_1,a_2,a_3) \equiv 1 - a_1^2 - a_2^2 - a_3^2 - 2a_1a_2a_3 \geq \mu^2 \quad \textit{and} \quad \max|a_j| \leq 1. \tag{5.8}$$

Proof. Let $\phi \in \Lambda^3(\mathbb{R}^7)$ be of the form $\phi[a,\mu]$. Proposition 5.3 says that $\phi(\xi) = 1$ for all $\xi \in \mathbb{CP}^1_0$. Then we have $\phi \in F^*(\mathbb{CP}^1_0)$ if and only if $\|\phi\|^* \leq 1$. By the Reduction principle, Proposition 3.34 with $L = span\, e_{2365}$, we have the equivalent condition $\|\phi[x]\|^* \leq 1$ for all unit vectors $x \in L$.

Let $x = x_2e_2 + x_3e_3 + x_6e_6 + x_5e_5$ be a unit vector in L, and consider

$$g = \begin{bmatrix} x_3 & x_2 & -x_5 & x_6 \\ -x_2 & x_3 & -x_6 & -x_5 \\ x_5 & x_6 & x_3 & -x_2 \\ -x_6 & x_5 & x_2 & x_3 \end{bmatrix}$$

Note that $ge_3 = x$ and $g^{-1} = g^T$. Then $\phi[x] = \phi[ge_3] = (g^{-1}\phi)[e_3]$. Observe that $\phi - \mu e^{247}$ and e^{47} are invariant under the action of g, so we can compute $(g^{-1}\phi)[e_3]$ by dividing it into two parts as:

$$\begin{aligned} \varphi[x] \equiv (g^{-1}\phi)[e_3] &= (\phi - \mu e^*_{247})[e_3] + \mu((g^{-1}e^2) \wedge e^{47}[e_3]) \\ &= -e^{156} + a_1e^{147} + 0e^{217} + 0e^{241} \\ &+ \mu x_6 e^{547} + a_2e^{257} + 0e^{245} \\ &- \mu x_5 e^{647} - 0e^{267} - a_3e^{246} \\ &+ \mu x_3 e^{247}. \end{aligned}$$

$\varphi[x]$ is a differential form in six variables, of the form $\phi(A, \bar{\mu})$ as in (4.2) by mapping $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ to $\{e_1, e_5, -e_6, e_2, e_4, e_7\}$, where

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ \mu x_6 & a_2 & 0 \\ \mu x_5 & 0 & a_3 \end{bmatrix},$$

and $\bar{\mu} \equiv \mu x_3$. Thus we have

$$\begin{aligned} &\|\phi[x]\|^* = \|\varphi[x]\|^* \leq 1 \\ &\quad \Leftrightarrow 1 - |A|^2 - 2\det A \geq \mu^2 x_3^2 \quad \text{and} \quad \|A\| \leq 1 \\ &\quad \Leftrightarrow \mathcal{H}(a) = 1 - a_1^2 - a_2^2 - a_3^2 - 2a_1a_2a_3 \geq \mu_2(x_3^2 + x_6^2 + x_5^2) \\ &\quad 3 - a_1^2 - a_2^2 - a_3^2 \geq \mu_2(x_3^2 + x_6^2 + x_5^2) \end{aligned}$$

by Lemma 4.21. Hence

$$\begin{aligned}\|\phi\|^* \leq 1 &\Leftrightarrow \|\phi[x]\|^* \leq 1 \quad \text{for all unit } x \text{ in } L\\ &\Leftrightarrow \mathcal{H}(a) \geq \mu^2 \quad \text{and} \quad 3 - a_1^2 - a_2^2 - a_3^2 \geq \mu_2\\ &\Leftrightarrow \mathcal{H}(a) \geq \mu^2 \quad \text{and} \quad \max|a_j| \leq 1\end{aligned}$$

by Lemma 4.21 which proves the theorem. □

Remark 5.5. Note that for $\phi \in F^*(\mathbb{CP}^1_0)$, if $\mathcal{H}(a) > \mu^2$ then $1 - |A|^2 - 2\det A > \mu^2 x_3^2 = \bar{\mu}^2$ for all $x \in L$, which implies that $G(\phi[x])$ is a single point face by Theorem 4.16iii. Moreover if $\mathcal{H}(a) = \mu^2$ and $x_2 \neq 0$ then $G(\phi[x])$ is also a single point face.

Theorem 5.6. *$\phi = \phi[a,\mu] \in F^*(\mathbb{CP}^1_0)$ and the corresponding faces $G(\phi)$ can be classified in terms of the parameter a and μ satisfying*

$$\mathcal{H}(a) \equiv 1 - a_1^2 - a_2^2 - a_3^2 - 2a_1a_2a_3 \geq \mu^2$$

and $\max|a_j| \leq 1$.

(i) If $a = (\pm1, \pm1, \pm1)$ (with odd number of negative signs), then $\mu = 0$, ϕ is an associative form, and $G(\phi)$ is a manifold of associative planes with dimension eight.

(ii) If a lies on one of the four edges of the tetrahedron where $|a_1| < 1$, then $\mu = 0$ and $G(\phi)$ is the corresponding five-dimensional manifold of special Lagrangian planes in $(span\, e_4)^\perp$ or in $(span\, e_7)^\perp$. For the associative forms ϕ_0, ϕ_1 at vertices of such an edge , $\phi_t = (1-t)\phi_0 + t\phi_1$ denotes the forms along the edge. For example, $\phi_{1/2}$ is one of the following special Lagrangian calibrations:

$$e^{123} - e^{156} \pm (e^{426} + e^{453}) \in \Lambda^3(span\, e_7)^\perp,$$
$$e^{123} - e^{156} \pm (e^{257} + e^{367}) \in \Lambda^3(span\, e_4)^\perp.$$

Then for all $t \in (0,1)$, $G(\phi_t) = G(\phi_{1/2}) = G(\phi_0) \cup G(\phi_1)$.

(iii) If a lies on one of the two edges where $a_1 = \pm1$, then $\mu = 0$ and $G(\phi) = \mathbb{CP}^2$: the

product of e_1 *with complex two-planes in* $(\mathrm{span}\, e_1)^{\perp}$ *with the complex structure* $e_3 = Je_2, e_5 = Je_6, e_7 = \pm Je_4$. *For the associative forms* ϕ_0, ϕ_1 *at vertices of such an edge ,* $\phi_t = (1-t)\phi_0 + t\phi_1$ *denotes the forms along the edge, then* $\phi_{1/2} = e^1 \wedge (e^{23} - e^{56} \pm e^{47})$ *and for all* $t \in (0,1)$, $G(\phi_t) = G(\phi_{1/2}) = G(\phi_0) \cup G(\phi_1)$.

(iv) If a *lies elsewhere in the inflated tetrahedron and* $\mu = 0$*, or if* $\mu^2 < \mathcal{H}(a)$*, then* $G(\phi) = \mathbb{CP}^1_0$.

(v) If $\mu^2 = \mathcal{H}(a) \neq 0$*, then* a *lies in the interior of* T *and* $G(\phi)$ *is the union of two* $\mathbb{CP}^1$*'s, namely* $\mathbb{CP}^1_0$ *and another* $\mathbb{CP}^1$ *which denote by* $\boldsymbol{C}$*, which intersect only at* e_{123}.

Proof. Let a be one of the four vertices of the tetrahedron T,

$$(-1,-1,-1), (-1,1,1), (1,-1,1), (1,1,-1).$$

By Theorem 4.16, in such a case $\mu = 0$ and $\phi[a, 0]$ has the forms:

$$\begin{aligned}
\phi[(-1,-1,-1),0] &= e^{123} - e^{156} - e^{147} - e^{257} - e^{367} - e^{426} - e^{453},\\
\phi[(-1,1,1),0] &= e^{123} - e^{156} - e^{147} + e^{257} + e^{367} + e^{426} + e^{453},\\
\phi[(1,-1,1),0] &= e^{123} - e^{156} + e^{147} - e^{257} - e^{367} + e^{426} + e^{453},\\
\phi[(1,1,-1),0] &= e^{123} - e^{156} + e^{147} + e^{257} + e^{367} - e^{426} - e^{453}.
\end{aligned}$$

The first one is the standard associative form given in [1] and the others results follows from changing the signs of e_4 and e_7. This proves (i).

Let a lies on an edge joining any pair of the four vertices of the tetrahedron which are associative calibrations. There are six edges. Four of them with $|a_1| \leq 1$ give special Lagrangian calibrations: two of them are the special Lagrangian form and its image under the change of sign of e_4, the others arise from changing (e_2, e_3, e_4) with $(e_3, -e_2, e_7)$. Two of edges represents the product of e^1 with the Kähler forms on

$(span\, e_1)^\perp$. Furthermore, for $0 < t < 1$, $\xi \in G(3, \mathbb{R}^7)$ satisfies for each case,

$$\phi_t(\xi) = (1-t)\phi_0(\xi) + t\phi_1(\xi) \leq (1-t) + t = 1.$$

Equality holds if and only if $\phi_0(\xi) = \phi_1(\xi) = 1$. Thus, $G(\phi_t) = G(\phi_0) \cap G(\phi_1)$ for $t \in (0,1)$.

Next for $\mu^2 < \mathcal{H}(a)$, we will show $G(\phi) = \mathbb{CP}^1_0$. Assume, for a contradiction, there exists $\xi \in G(\phi) - \mathbb{CP}^1_0$, and write $\phi = e^{123} - e^{156} + \varphi$. Then $\phi(\xi) = 1$ implies that $(e^{123} - e^{156})(\xi) < 1$ and $\varphi(\xi) > 0$. The condition $\mu^2 < \mathcal{H}(a)$ gives also $\max|a_j| < 1$ since otherwise we will have $\mathcal{H}(a) = \mu^2 = 0$. The condition $\max|a_j| < 1$ implies by Theorem 5.4 that $\|\phi'\|^* \leq 1$ where $\phi' = e^{123} - e^{156} + \delta\varphi$ for some $\delta > 1$. But then we have $\phi'(\xi) = \phi(\xi) + (\delta - 1) > 1$, which contradicts with $\|\phi'\|^* \leq 1$. Hence, we have $G(\phi) = \mathbb{CP}^1_0$. Next we will show $G(\phi) = \mathbb{CP}^1_0$ for a lies elsewhere in the inflated tetrahedron and $\mu = 0$. In this case, ϕ is invariant under the action of the group $SU(2)$ on $span\, e_{2365}$. Then the restriction of ϕ on $(span\, e_3)^\perp$,

$$\phi[e_3] = -e^{156} + a_1 e^{147} + a_2 e^{257} + a_3 e^{426}$$

By Theorem 4.16iii , $\phi[e_3]$ is a single point calibration. Then we have $G(\phi[e_3]) = \{-e_{156}\} \subset \mathbb{CP}^1_0$. For all $x \in span\, e_{2365}$ we have $G(\phi[x]) \subset \mathbb{CP}^1_0$ since ϕ and $\mathbb{CP}^1_0$ are invariant under the action of $SU(2)$ and $SU(2)$ acts transitively on $span\, e_{2365}$. Therefore, by the Reduction Principle, Proposition 3.34, $G(\phi) \subset \mathbb{CP}^1_0$ as desired.

Now assume $\mathcal{H}(a) = \mu^2 \neq 0$. Consider the restriction of $\phi(a, \mu)$ on $span\,(e_3)^\perp$ with $\mu \neq 0$,

$$\phi[e_3] = -e^{156} + a_1 e^{147} + a_2 e^{257} + a_3 e^{426} + \mu e^{427}$$

By mapping $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ to $\{e_1, e_5, -e_6, e_2, e_4, e_7, \}$, $\phi[e_3]$ is a double point calibration which calibrates $-e_{156}$ and $\xi = (c_1 e_1 + s_1 e_2) \wedge (c_2 e_5 + s_2 e_4) \wedge (-c_3 e_6 + s_3 e_7)$ where $c_i \equiv \cos\alpha_i$ and $s_i \equiv \sin\alpha_i$ with $\alpha \in (sign\, \mu)D$, D is the double point tetrahedron as in (4.8). By the Reduction Principle, Proposition 3.34 we have $\{-e_{156}, \xi\} \in G(\phi)$. Next we will show that $G(\phi)$ can not contain any elements of the form $v\wedge\eta$ where unit vector $v \in span\, e_{123}$ and unit 2-vector $\eta \in span\, e_{4567}$, except $-e_{156}$ and ξ. Assume $\xi' \neq -e_{156}$ is such an element in $G(\phi)$. If we choose a unit vector $x \in span\, e_{23}$ perpendicular to ξ' then we have $\{-e_{156}, \xi'\} \in G(\phi[x])$. By Remark 5.5, $x = e_3$ since $G(\phi[x])$ is not a single point face so $x_2 = 0$. Therefore $\{-e_{156}, \xi'\} \in G(\phi[e_3] = \{-e_{156}, \xi\})$ implies that $\xi' = \xi$.

By the Reduction Principle, Proposition 3.34, $G(\phi) = \bigcup\{G(\phi[x]) : x \in span\, e_{4567}\}$. But this implies $\mu = 0$ for all $x \in span\, e_{4567}$, hence $\phi[x]$ can not be a double point calibration as well as it is not a single point face.

Assume $G(\phi[x])$ is a special Lagrangian face for some $x \in span\, e_{4567}$. By using the action of $SO(3)\times SO(4)$, we can assume $x = e_7$ and $G(\phi[x]) = G(e^{123} - e^{156} - e^{426} - e^{453})$. But then we get three elements in $G(\phi[x])$, namely $-e_{156}, -e_{426}$ and $-e_{453}$ which are of the form $\nu \wedge \eta$, described above. Hence, $G(\phi[x])$ can not be a special Lagrangian face for any $x \in span\, e_{4567}$. Then by Theorem 4.16 $G(\phi[x])$ is a $\mathbb{CP}^1$ face through e_{123} for all $x \in span\, e_{4567}$. Hence, $G(\phi) = \bigcup\{G(\phi[x]) : x \in span\, e_{4567}\}$ is either $\mathbb{CP}^1_0$ or $\mathbb{CP}^1_0 \cup C$. Since $\xi \in G(\phi[e_3]) \subset G(\phi)$ and $\xi \notin \mathbb{CP}^1_0$, we get $G(\phi) = \mathbb{CP}^1_0 \cup C$ with intersection $\mathbb{CP}^1_0 \cap C = e_{123}$. This proves (v). $\square$

The next theorem classifies the faces of the Grassmannian $G(3,\mathbb{R}^7)$ without using the action of the group $id \times U(2) \times O(2)$. In other words, we will consider $\phi[a,B,c] \in F(\mathbb{CP}^1_0)$ as in (5.1).

Theorem 5.7. *Let $\phi \in \Lambda^3\mathbb{R}^7$ as in the Equation 5.1. Then $\phi[a,B,c]$ is in $F(\mathbb{CP}^1_0)$ if and only if*

$$1 - a^2 - |B|^2 - 2a\det B \geq |c|^2 \tag{5.9}$$

$$3 - a^2 - |B|^2 \geq 0. \tag{5.10}$$

where $|B|^2 = b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2$ and $c = c_2e^2 + c_3e^3 + c_6e^6 + c_5e^5 \in \Lambda^1\mathbb{R}^4 \cong \mathbb{R}^4$. The faces $G(\phi)$ can be classified as follows:

(i) If B is orthogonal and $a = -\det B \in \{-1, 1\}$, then $c = 0$ and ϕ is an associative

form.

(ii) If $|B|^2 = (\det B)^2 + 1 < 2$*, then* $c = 0$ *and* $G(\phi)$ *is a five dimensional manifold of special Lagrangian planes in* $(\mathit{span}\, v)^\perp$ *for some* $v \in \mathit{span}\, e_{47}$*.*

(iii) If $a = \pm 1$ *and* $|B|^2 < 2$*, then* $c = 0$ *and* $G(\phi) = \mathbb{CP}^2$

(iv) If $1 - a^2 - |B|^2 - 2a \det B > |c|^2$*, or if* $c = 0$*,* $a \notin \{-1, 1\}$*, and* $|B|^2 \neq \det^2 B + 1$*, then* $G(\phi) = \mathbb{CP}^1_0$*.*

(v) If $1 - a^2 - |B|^2 - 2a \det B = |c|^2 > 0$*, then* $G(\phi)$ *is the union of two* $\mathbb{CP}^1$*'s, namely* $\mathbb{CP}^1_0$ *and* C*, which intersect at a single point.*

Proof. For $\phi[a, \mu] \in F^*(\mathbb{CP}^1_0)$ as in (5.7), we have the conditions
$\mathcal{H}(a) = 1 - a_1^2 - a_2^2 - a_3^2 - 2a_1a_2a_3 \geq \mu^2$ and $\max |a_j| \leq 1$ by Theorem 5.6. Then we have $\max |a_j| \leq 1$ that implies $3 - a_1^2 - a_2^2 - a_3^2 \geq 0$ by Lemma 4.22. These conditions are equivalent to 5.9 and 5.10 for $\phi[a, \mu]$. Since the conditions 5.9 and 5.10 are $id \times U(2) \times SO(2)$ invariant, they hold for the general case $\phi(a, B, c)$.
Since the classification is also invariant under $id \times U(2) \times SO(2)$, we may check the conditions for $\phi[a, \mu] = \phi[a, B, c]$ where

$$a = a_1, \quad B = \begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix}, \quad c = \mu e^2.$$

Assume B is orthogonal matrix and $a = -\det B \in \{-1, 1\}$. Then we have $\{a_2, a_3\} \in \{-1, 1\}$ and $(a_1, a_2, a_3) \in \{(-1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$. Hence, we have associative forms.
Next, we have $|B|^2 = (\det B)^2 + 1 \Leftrightarrow a_2^2 + a_3^2 = a_2^2 a_3^2 + 1 \Leftrightarrow (1 - a_2^2)(1 - a_3^2) = 0 \Leftrightarrow a_2$ or a_3 is $\pm 1 \Leftrightarrow (a_1, a_2, a_3)$ are as in Theorem 5.6ii, which give the special Lagrangian calibrations.
The case $a = \pm 1$ holds if and only if (a_1, a_2, a_3) lies on the edge $a_1 = \pm 1$, which leads the type of calibrations as in Theorem 5.6iii.
First condition in (iv) is equivalent to $\mathcal{H}(a) > \mu^2$. The second conditions in (iv) lead to $\mu = 0$ and a lying inside the inflated tetrahedron. Hence we have $G(\phi) = \mathbb{CP}^1_0$ in each case by Theorem 5.6iv. (v) also directly follows from Theorem 5.6v since the conditions are equivalent. □

5.2. The Face of e_{123}

In this section, we will describe the face of e_{123}, that is $F^*(e_{123})$ which is given by

$$F^*(e_{123}) \equiv \{\phi \in \Lambda^3\mathbb{R}^7 : \phi(e_{123}) = \|\phi\|^* = 1\}.$$

As in the case of $F^*(e_{123}) \subset \Lambda^3\mathbb{R}^6$, a similar procedure will be applied to $F^*(e_{123})$ in $\Lambda^3\mathbb{R}^7$ by reducing the number of basis 3-forms to describe $F^*(e_{123})$. Thanks to the First Cousin Principle, Lemma 2.8, we have the following proposition immediately.

Proposition 5.8. *Every $\phi \in F^*(e_{123})$ can be written as*

$$\phi = e^{123} + \varphi + \mu \lrcorner e^{4567} \tag{5.11}$$

where $\varphi \in \Lambda^1(span\, e_{123}) \otimes \Lambda^2(span\, e_{4567})$ and $\mu \in span\, e_{4567}$.

Proof. Since $\phi \in F^*(e_{123})$, ϕ attains its maximum at e_{123}. By the First Cousin Principle, Lemma 2.8 , φ can not be in $\Lambda^2(span\, e_{123}) \otimes \Lambda^1(span\, e_{4567})$, which is the set containing the nonzero forms at the first cousins of e_{123}. $\square$

Remark 5.9. $\phi \in F^*(e_{123})$ can be written explicitly as

$$\begin{aligned}
\phi[A, B, \mu] = e^{123} &+ a_{11}e^{156} + a_{12}e^{416} + a_{13}e^{145} \\
&+ a_{21}e^{256} + a_{22}e^{426} + a_{23}e^{245} \\
&+ a_{31}e^{356} + a_{32}e^{436} + a_{33}e^{345} \\
&+ b_{11}e^{147} + b_{12}e^{157} + b_{13}e^{167} \\
&+ b_{21}e^{247} + b_{22}e^{257} + b_{23}e^{267} \\
&+ b_{31}e^{347} + b_{32}e^{357} + b_{33}e^{367} \\
&- \mu_1 e^{567} + \mu_2 e^{467} - \mu_3 e^{457} + \mu_4 e^{456}.
\end{aligned} \tag{5.12}$$

where the coefficients can be seen as matrices $A_{3\times 3} = (a_{ij})$ and $B_{3\times 3} = (b_{ij})$ and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$. We will denote the columns of A, B as $a_k = (a_{ik})$ and $b_k = (b_{jk})$

for $k = 1, 2, 3$.

Theorem 5.10. *There exist quadratic forms Q and R on span e_{4567} whose coefficients are polynomials in A, B and μ where $\phi[A, B, \mu] \in F^*(e_{123})$ described as in (5.11).*

Proof. $\phi \in F^*(e_{123})$ if and only if $\phi[A, B, \mu]$ as in Proposition 5.3 and $\|\phi\|^* \leq 1$. By the Reduction Principle, Lemma 3.34 we have $\|\phi\|^* \leq 1$ if and only if $\|\phi[x]\|^* \leq 1$ for all unit vector $x \in \mathit{span}\, e_{4567}$. Then

$$\phi[x] = e^{123} + \varphi[x] + (\mu \cdot x)(x \lrcorner e^{4567}).$$

Since $\phi[x] \in \Lambda^3(\mathit{span}\, x)^{\perp} = \Lambda^3(\mathit{span}\, e_{123} \wedge E(x)) \cong \Lambda^3(\mathbb{R}^6)$, by Proposition 4.22, we have $\|\phi[x]\|^* \leq 1$ if and only if

$$\begin{aligned} Q(x) &\equiv |x|^2 - P(x) - 2D(x) - (\mu \cdot x)^2, \\ R(x) &\equiv 3|x|^2 - P(x) \end{aligned}$$

are positive semidefinite, where $P(x) = |\varphi[x]|^2$ and $D(x)$ is defined by

$$D(x)E(x) = [(e_1 \lrcorner \varphi[x]) \lrcorner E(x)] \wedge [(e_2 \lrcorner \varphi[x]) \lrcorner E(x)] \wedge [(e_3 \lrcorner \varphi[x]) \lrcorner E(x)]$$

$P(x) = |\varphi \wedge x^*|^2$ where x^* is the dual form of the vector x is clearly quadratic in x and coefficients of R are polynomials in A, B and μ. $\square$

Remark 5.11. Let $\phi \in F^*(e_{123})$. If $\phi[x]$ is not a single point calibration, then we have $Q(x) = 0$ where $x \in \mathit{span}\, e_{2365}$. The face $G(\phi[x])$ is a double point calibration if and only if $Q(x) = 0$ and $\mu \cdot x \neq 0$, by applying Theorem 4.16.

The following corollary easily follows from Remark 5.11 and the Reduction Principle, Proposition 3.34.

Corollary 5.12. *For $\phi \in F^*(e_{123})$,*

$$G(\phi) = \{e_{123}\} \cup \bigcup \{G(\phi[x]) : x \in \mathit{nullspace}\, Q\}$$

5.3. The Classification of Faces of $G(3, \mathbb{R}^7)$

Our final goal is to determine the faces of the Grassmannian $G(3, \mathbb{R}^7)$ corresponding to the faces of $F^*(e_{123})$. There are ten types, first five of them are given in Theorem 5.6 which contain a $\mathbb{CP}^1$ and others are new types that do not contain any $\mathbb{CP}^1$.

Proposition 5.13. *Let $\phi \in F^*(e_{123})$. Then under the action of $SO(3) \times SO(4)$, ϕ can be written in the following form*

$$\begin{aligned}\phi = {} & e^{123} + a_{11}e^{156} + a_{22}e^{426} + a_{33}e^{453} \\ & + b_{11}e^{147} \\ & + b_{21}e^{247} + b_{22}e^{257} \\ & + b_{31}e^{347} + b_{32}e^{357} + b_{33}e^{367} \\ & - \mu_1 e^{567} + \mu_2 e^{467} - \mu_3 e^{457} + \mu_4 e^{456}.\end{aligned} \tag{5.13}$$

by maximizing $-a_{11}$. In this case, $G(\phi)$ contains a $\mathbb{CP}^1$ if and only if $a_{11} = -1$.

Proof. For $\xi \in G(3, \mathbb{R}^7)$ and $\phi \in F^*(\xi)$, we may choose an orthonormal basis $\{e_1, \ldots, e_7\}$ such that $\xi = e_{123}$. In other words, ξ is equivalent orthonormally to e_{123}. First choose $e_1 \in \mathit{span}\, e_{123}$ and $e_{56} \in \Lambda_2(\mathit{span}\, e_{4567})$ to maximize $-\phi(e_1 \wedge e_{56})$. Then we have $e_{23} = e^1 \lrcorner e_{123}$ and $e_{47} = e^{56} \lrcorner e_{4567}$. Next choose $e_4 \in \mathit{span}\, e_{47}$, $e_2 \in \mathit{span}\, e_{23}$ and $e_6 \in \mathit{span}\, e_{56}$ to maximize $\phi(e_4 \wedge e_2 \wedge e_6)$. Then we have $e_3 = e^2 \lrcorner e_{23}$, $e_5 = -e^6 \lrcorner e_{56}$. Under these choices, by the First Cousin Principle, Lemma 2.8, we have

$$\begin{aligned}&\phi(e_{256}) = \phi(e_{356}) = \phi(e_{145}) = \phi(e_{146}) = \phi(e_{157}) = \phi(e_{167}) = 0 \\ &\phi(e_{726}) = \phi(e_{436}) = \phi(e_{425}) = 0\end{aligned}$$

Therefore, ϕ is of the desired form with $-a_{11} = -\phi(e_{156})$ as large as possible. $\square$

Theorem 5.14. *The faces $G(\phi)$ of the Grassmannian $G(3, \mathbb{R}^7)$ exposed by $\phi \in \Lambda^3\mathbb{R}^7$ can be divided into following classes. It can be assumed $\phi \in F^*(e_{123})$ so that $\phi(e_{123}) = \|\phi\|^* = 1$ and can be written in (5.13). Let Q, R be the associated quadratic forms.*

(i) If $a_{11} = -1$ and $(b_{11}, a_{22} = a_{33}, b_{22} = b_{33}) \in \{(-1,-1,-1), (-1,1,1), (1,-1,1), (1,1,-1)\}$, then ϕ is an associative form and the other terms vanish.

(ii) If $a_{11} = -1$ and exactly one of the numbers $a_{22} = a_{33}$, $b_{22} = b_{33}$ belongs to $\{1,-1\}$, then ϕ is a special Lagrangian form.

(iii) If $a_{11} = -1$, $b_{11} = \pm 1$, and $a_{22}^2 + b_{22}^2 < 1$ then $G(\phi) = \mathbb{CP}^2$.

(iv) If $a_{11} = -1$ and

$$1 - b_{11}^2 - a_{22}^2 - b_{22}^2 - 2b_{11}a_{22}b_{22} > b_{21}^2 + b_{21}^2 + \mu_2^2 + \mu_3^2;$$

or if $a_{11} = -1$, $b_{21} = b_{31} = \mu_2 = \mu_3 = 0$, and $\max\{|b_{11}|, |a_{22}|, b_{22}|\} < 1$, then $G(\phi) = \mathbb{CP}^1_0$

(v) If $1 - b_{11}^2 - a_{22}^2 - b_{22}^2 - 2b_{11}a_{22}b_{22} = b_{21}^2 + b_{21}^2 + \mu_2^2 + \mu_3^2 > 0$, then $G(\phi) = C \cup \mathbb{CP}^1_0$ where C is defined as in Theorem 5.5 (v).

(vi) If $\mu \cdot x = 0$ for all $x \in$nullspace Q and $a_{11} \neq -1$, then $G(\phi) = \{e_{123}\}$.

(vii) If nullity $Q = 1$ and $\mu \cdot x \neq 0$ for some $x \in$ nullspace Q, then $G(\phi) = \{e_{123}, \xi\}$.

(viii) If nullity $Q = 4$ and $\mu \cdot x \neq 0$ for some $x \in$ nullspace1, Q, and $a_{11} \neq -1$, then $G(\phi)$ is an embedded, real-analytic, 3-dimensional sphere.

(ix) If nullity $Q = 3$ and $\mu \cdot x \neq 0$ for some $x \in$ nullspace Q, and $a_{11} \neq -1$, then $G(\phi)$ is an embedded, real-analytic, 2-dimensional sphere.

(x) If nullity $Q = 2$ and $\mu \cdot x \neq 0$ for some $x \in$ nullspace Q, then $G(\phi)$ is an embedded, real-analytic, unit circle.

Proof. Let $\phi \in F^*(e_{123})$ as in (5.13). First five cases of theorem follows from Theorem 5.6 as follows: If $a_{11} = -1$, then $G(\phi)$ contains a $\mathbb{CP}^1$ by Proposition 5.13 and ϕ can be written of the form $\phi[a, B, c]$ as in 5.1. Then we have $a_{22} = a_{33}$, $b_{22} = b_{33}$ and $b_{32} = \mu_1 = \mu_4 = 0$.

For the remaining cases, consider the face of ϕ as in Corollary 5.12. By Remark 5.13, $G(\phi[x])$ is a double point face $\{e_{123}, \xi\}$ if and only if $Q(x) = 0$ and $\mu \cdot x \neq 0$.

If $\mu \cdot x = 0$ and $a_{11} \neq -1$, then $G(\phi[x])$ does not contain a $\mathbb{CP}^1$, so it can not be a special Lagrangian face or complex face. Then the only case is $G(\phi[x]) = \{e_{123}\}$, which implies $G(\phi) = \{e_{123}\}$.

Let x be a unit vector in $nullspace\, Q$ and $\mu \cdot x \neq 0$. Hence $G(\phi[x])$ is a double point face, then $G(\phi) = \{e_{123}, \xi\}$.

For the other cases, assume $k = nullity\, Q$ and $\mu \cdot x \neq 0$ where $x \in nullspace\, Q$. Since we have $\phi[x] = \phi[-x]$, it is enough to compute $G(\phi[x])$ for $\mu \cdot x > 0$, then $G(\phi) = \bigcup_{\mu \cdot x > 0} G(\phi[x])$. For all $x \in nullspace\, Q$ with $\mu \cdot x > 0$, we have $G(\phi[x]) = \{e_{123}, \xi(x)\}$. Then we have a map:

$$\begin{aligned} \xi : U &\to G(3, \mathbb{R}^7) \\ x &\mapsto \xi(x) \end{aligned}$$

where $U = \{unit\; x \in nullspace\, Q : \mu \cdot x > 0\}$. Note that $\xi(x)$ is a three-torus vector, $\xi(x) = \eta(\alpha)$ for some $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in the double point tetrahedron D. Hence $\xi(x)$ can be found explicitly by determining the angles α as in Theorem 4.20. This shows that the map ξ is real-analytic. Also we know that e_{123} and $\xi(x)$ intersects transversely, so that $e_{123} \oplus \xi(x)$ must equal to $span(x)^{\perp}$, which means ξ is an embedding. Hence $G(\phi)$ is the image of ξ is an embedded, real-analytic k-dimensional manifold except possibly at e_{123}, which can be described topologically with S^{k-1}. $\square$

6. CONCLUSION

In this thesis we studied the theory of calibrated geometry and mainly focused on classification of calibrations on $\mathbb{R}^n$ for $n \leq 7$. We did the classification of 3-calibrations on $\mathbb{R}^6$ and got four different types of calibrations. Among them, the most interesting one is special Lagrangian calibration since the face corresponding to it is the biggest. More information about such a face can be found in [8]. We also work out main classification of 3-calibrations on $\mathbb{R}^7$, in [10] a more detailed description of the diagonal case is given, in which $\phi[A, B, \mu]$ is as in the Equation 5.12 with the matrices A and B are diagonal and $\mu_i = 0$ except for $i = 4$. For higher dimensional $\mathbb{R}^n$, only $\mathbb{R}^8$ case has been solved explicitly [21]. As the dimension of $\mathbb{R}^n$ gets higher the classification problem becomes more challenging since the dimension of $\Lambda^k\mathbb{R}^n$ gets very large. In [22], a less complicated problem is considered: "What are all calibrations, whose faces contain the face of a given calibration?" While classifying calibrations on $\mathbb{R}^7$, we solved this problem for the given calibration being Kähler (or $\mathbb{CP}^1$) on $\mathbb{R}^7$. In [22], this problem is also solved for the given calibration being special Lagrangian on $\mathbb{R}^8$.

In Chapter 5, we considered $\mathbb{CP}^1$ as a Cartesian product of a fixed real line and an orthogonal complex plane in $\mathbb{C}^2$. We know that a real line is calibrated by the dual of its tangent vector and a complex plane is calibrated by the Kähler form on it. Hence these two planes are volume-minimizing separately. We also found the dual face $F^*(\mathbb{CP}^1)$. In other words, $\mathbb{CP}^1$ as a Cartesian product of two planes is a minimal submanifold of $\mathbb{R}^7$ since $F^*(\mathbb{CP}^1)$ is nontrivial. From this example we may ask the following: "When is the Cartesian product of two volume-minimizing surfaces volume-minimizing?". This is an open question for arbitrary dimensions but some progress is achieved in [11].

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