SYMMETRIES GLOBAL, LOCAL AND ASYMPTOTIC, APPLICATIONS OF NOETHER AND COVARIANT PHASE SPACE METHODS

by

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B.S., Physics, Boğaziçi University, 2015

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics
Boğaziçi University
2017

ACKNOWLEDGEMENTS

I would like to thank Dieter Van Den Bleeken who kindly accepted to be as my advisor. I am grateful for his guidance and help through the difficulties I encountered during the research process.

In addition to Van Den Bleeken, I am indebted to Suna Ertuğrul who took me by storm with her lectures and inspired me a lot to focus on science. I also want to thank my colleagues, Cihan Pazarbaşı and Emine Şeyma Kutluk who supported this dissertation with nice discussion and advice.

The last but not least, I am grateful to all Abdullah Kuran Library workers who have provided me with the best place to study for seven years.

ABSTRACT

SYMMETRIES GLOBAL, LOCAL AND ASYMPTOTIC, APPLICATIONS OF NOETHER AND COVARIANT PHASE SPACE METHODS

Primarily, global, local and asymptotic symmetries will be described and their correspondence in the conservation laws will be shown in this thesis. The latter will be managed by Noether' s theorems and Covariant Phase Space formalism. Conservation laws related to local gauge transformations will be focused in electrodynamics. A couple of approaches that have already been performed will be examined in detail and charge definitions on different 3D hypersurfaces will be compared.

ÖZET

GLOBAL, LOKAL VE ASİMTOTİK SİMETRİLER, NOETHER VE KOVARYANT FAZ UZAYI METHOTLARI

Bu tezde, öncelikle global, lokal ve asimptotik simetriler betimlenecek ve bu simetrilere tekabül eden doğa kanunları ortaya çıkartılacaktır. Simetrilerle korunum yasaları arasındaki ilişki Noether teoremleri ve kovaryant faz uzayıyla kurulacaktır. Ardından ağırlıklı olarak elektrodinamik sistemlerde lokal ayar dönüşümleriyle alakalı korunum yasalarına odaklanılacaktır. Bu fiziksel sistemde daha evvel çalışılmış olan birkaç yaklaşım ayrıntılı bir şekilde incelenecek ve 3 boyutlu aşırıyüzeylerde tanımlı yükler karşılaştırılacaktır.

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LIST OF SYMBOLS

A^μ	4-vector electromagnetic potential
D_μ	Gauge covariant derivative
E_r	Radius component of electric field in 3D
$F^{\mu\nu}$	Electromagnetic tensor
$\mathcal{H}$	Hamiltonian density
i^0	Spatial infinity
j	Noether's current
$\mathcal{J}^+$	Future null infinity
$\mathcal{J}^+_-$	Past of the future infinity
$\mathcal{J}^-$	Past null infinity
$\mathcal{J}^-_+$	Future of the past infinity
L	Lagrangian or Lie derivative operator
$\mathcal{L}$	Lagrangian density
$\hat{n}$	Normal vector
q	Space-time coordinate
Q	Charge
S	Action functional
S^2	2 dimensional sphere
t	Time parameter
$\vec{\beta}_m$	The speed of an object relative to the speed of light
γ_m	Lorentz factor
Γ	Lateral surface
η	Induced metric on τ constant surface
Σ	Hypersurface
π	Conjugate momentum
$\cong$	Equality when the equation of motion is satisfied

LIST OF ACRONYMS/ABBREVIATIONS

QED	Quantum Electrodynamics
BMS	Bondi-Metzner-Sachs
2D	Two Dimensional
3D	Three Dimensional
4D	Four Dimensional
c.c	Complex conjugate
det	Determinant of

1. INTRODUCTION

In physics, symmetry is a property of the system which leaves some features of laws of physics invariant under some transformations.

> *Symmetry and invariance considerations, and even conservation laws, played undoubtedly an important role in the thinking of the early physicists, such as Galileo and Newton, and probably even before then.* [1]
>
> E.P. Wigner

As Wigner pointed out in 1964, the historical concept of symmetry and related conservations have their origins in Greek antiquity *"summetria"*. Instead of tracing historical methods and thoughts on symmetry and its role in physics back from ancient times to today, the beginning of 20th century will be the reference point of this thesis to start appreciating previous works. A couple of developments, which are significantly important for this dissertation, will be mentioned to form a solid ground for my endeavor.

In 1918, Emmy Noether published the paper "Invariante Variationsprobleme" consisting of two famous theorems [2]. The first one reveals the link between global symmetries and conservation laws, e.g., the connection between translational symmetry and conservation of momentum, rotational symmetry and conservation of angular momentum etc. On the other hand, the second theorem states that one can find a number of constraints on the equations of motion and conservation laws due to local symmetries. A few examples of local symmetries in physics are the local symmetry of diffeomorphism in General Relativity and the local gauge transformations in Electromagnetic theory and Yang-Mills.

A fundemental problem, which in the local symmetries are utilized, is the information problem of a black hole. Taking a brief look at the information problem, in seventies S. W. Hawking argued that a black hole loses mass by the thermal radiation, and eventually evaporates [3–5]. All the information about an object, which falls into the black-hole, becomes inaccessible for an observer outside of the black hole. Considering the object as a quantum state, the state transforms into thermal radiation that is described only by its temperature. Hence, the information related to the original state disappears. This is the information loss problem of a black hole that is still being investigated by researches.

In 2016, S. W. Hawking, M. J. Perry and A. Strominger argued that a black hole carries a number of soft hairs (i.e low-frequency quantum excitations) due to BMS supertranslational symmetries [1] . They release the information after the black hole evaporates.

Physicists working on the information loss problem tend to use Maxwell electrodynamics or QED as an analogy to get more physical intuition for BMS supertranslational symmetry [8–11]. In this dissertation, mainly Maxwell electrodynamics will be considered as the physical system to apply symmetry transformations. Because, understanding soft quantities in Maxwell Electrodynamics will itself be related to a novel conservation laws for the theory which we may not be familiar with at that point. The primary aim of this work is to manifest similarities and compare previous researches.

The outline of this thesis is as follows: In chapter 2, some mathematical tools will be discussed to establish the rest of the thesis. Noether's first and second theorems will be stated with alternative approaches and corroborated by examples.

In Chapter 3, Hamiltonian formalism and distinguished role of time in this formalism will be underlined. Covariant formalism that is a convenient way to make the laws of physics transformable for all inertial observers leads to the need of covariant phase space instead of the Hamiltonian phase space. For this reason, the chapter will end with the construction of the covariant phase space.

[1]BMS supertranslational symmetry refers to the asymptotic symmetry group of asymptotically flat Minkowski space-time [6, 7]

Next, on top of these structures, different subclasses of the gauge transformations and infinitely many charges they produce will be studied. Then, the equivalence of charges defined on different surfaces will be shown. The consistency between two of methods at spatial infinity will be questioned in the last chapter.

2. NOETHER'S THEOREMS

Especially for differential equations, a symmetry is desired because it is the catalyzer solving differential equations. In physics, symmetries have also remarkable and deep meaning. To discover their reflection on physics, Noether's theorems will be used as theoretical tools. In this chapter, theorems will be briefly stated following [12] and they will be digested with a few examples.

2.1. Noether's First Theorem

Noether' s first theorem is based on the particular infinitesimal transformation, denoted by $\hat{\delta}\phi$, that leaves the action invariant up to a possible boundary term. That boundary term will provide a contribution to related conserved quantities. Such transformations are called symmetry transformations. For the action

$$S = \int d^d x \sqrt{g} L(\phi, \nabla_\mu \phi) \tag{2.1}$$

where the Lagrangian density is

$$\mathcal{L}(\phi, \nabla_\mu \phi, g) = \sqrt{g} L(\phi, \nabla_\mu \phi) \tag{2.2}$$

variation of action with respect to the symmetry transformation

$$\hat{\delta} S = S[\phi + \hat{\delta}\phi] - S[\phi] = \int d^d x \sqrt{g} \nabla_\mu K^\mu \tag{2.3}$$

Therefore, K^μ depends on symmetry transformation. On the other hand, a general variation of the action functional is

$$\delta S = \int d^d x \sqrt{g} [-E(\phi)\delta\phi + \nabla_\mu \theta^\mu(\phi; \delta\phi)] \tag{2.4}$$

$$E(\phi) = \nabla_\mu \left(\frac{\delta L}{\delta \nabla_\mu \phi} - \frac{\delta L}{\delta \phi} \right) \qquad \theta^\mu = \left(\frac{\delta L}{\delta \nabla_\mu \phi} \delta \phi \right)$$

where $E(\phi) = 0$ is the equation of motion and $\nabla_\mu \theta^\mu(\phi; \delta\phi)$ is boundary term coming from integration by parts. The symmetry transformation $\hat{\delta}\phi$ on variation of action reveals the current

$$j^\mu = \theta^\mu(\phi; \hat{\delta}\phi) - K^\mu \tag{2.5}$$

which is conserved when equations of motion are imposed.

$$\nabla_\mu j^\mu = E(\phi)\hat{\delta}\phi \cong 0 \tag{2.6}$$

2.2. An Alternative Approach to Noether's First Theorem

This approach is based on deformation of symmetry transformation with an arbitrary function, $\sigma(x)$,

$$\delta_\sigma \phi(x) \quad = \quad \sigma(x)\hat{\delta}\phi(x) \tag{2.7}$$

Thus, variation of the action with respect to $\delta_\sigma \phi(x)$

$$\delta_\sigma S \quad = \quad \int d^d x \sqrt{g} \left(\sigma \nabla_\mu K^\mu + \theta^\mu \nabla_\mu \sigma \right) \tag{2.8}$$

$$\delta_\sigma S \quad = \quad \int d^d x \sqrt{g} \left(\nabla_\mu (\sigma K^\mu) - \nabla_\mu \sigma K^\mu + \theta^\mu \nabla_\mu \sigma \right) \tag{2.9}$$

If $\sigma(x)$ has compact support [2] the action is invariant under (2.7), i.e $K^\mu = 0$

$$\delta_\sigma S = -\int d^d x \sqrt{g} (\sigma(x) \nabla_\mu j^\mu) = 0 \tag{2.10}$$

[2]Compact support means that the function is zero everywhere except some finite region.

2.3. Noether's Second Theorem

In the case that the local symmetry transformation is a set parametrized by at least one (arbitrary) function $\lambda(x)$, Noether's second theorem brings out some identities which restrict the equations of motion and current j^μ. In general, the theorem suggests that infinitesimal transformations parameterized by arbitrary functions and their derivatives give rise to strong identities.

Here, transformations of fields will be restricted to depend only on $\lambda(x)$ and its first derivative because this particular way of transformation has physically interesting examples and relatively simplify calculations.

For this case, the form of transformation is[3]

$$\delta_\lambda \phi = f(\phi)\lambda(x) + f^\mu(\phi)\partial_\mu \lambda(x)$$

For $\lambda(x)$ that has compact support, the boundary term is zero.

At that point, how the variation of the action will be applied under local transformations may bring a question whether variation of the action will act on coordinates, as well. The action S is a map from fields to reals S: $\mathcal{F} \to R$. Therefore, the values of an action does not depend on coordinates because covariant form inside the integral of action implies that coordinates are dummy variables. For this reason, instead of transformation of coordinates, we will focus on transformations of fields at the same point in space-time.

Variation of the action is already given in (2.4) with δ_λ

$$\delta S_\lambda = \int d^d x \sqrt{g}[-E(\phi)\delta_\lambda \phi + \nabla_\mu \theta^\mu(\phi; \delta_\lambda \phi)]$$

for transformations parametrized by $\lambda(x)$.

[3] $\delta_\lambda \phi$ is used instead of $\delta\phi$ to specify the transformation parameter.

The contribution of θ^μ should not contribute again since it is evaluated on the boundary where transformation $\lambda(x)$ vanishes. Therefore, we have

$$\int d^d x \sqrt{g} E(\phi) \delta_\lambda \phi = 0$$

The key point is that variation δ_λ may still depend on $\lambda(x)$. This dependence constraints the equations of motion and current j$^\mu$ as follows

$$E(\phi)[f(\phi)\lambda(x) + f^\mu(\phi)\partial_\mu \lambda(x)] = 0 \tag{2.11}$$

Variation of this equation in terms of $\lambda(x)$ gives that

$$\Delta E(\phi) \equiv E(\phi) f(\phi) - \nabla_\mu (E(\phi) f^\mu(\phi)) = 0 \tag{2.12}$$

using integration by parts. This equation imposes the strong identity which puts constraints on the equation of motion. Hence, the equations of motion are not all independent under local transformations. Now,

$$\begin{aligned} E(\phi)\delta_\lambda \phi &= \Delta E(\phi)\lambda + E(\phi) f^\mu(\phi) \partial_\mu \lambda(x) + \nabla_\mu (E(\phi) f^\mu(\phi)) \lambda(x) && (2.13) \\ &= \Delta E(\phi)\lambda + \nabla_\mu (E(\phi) f^\mu(\phi) \lambda(x)) && (2.14) \\ &= \Delta E(\phi)\lambda + \nabla_\mu S^\mu && (2.15) \end{aligned}$$

When the equations of motion are satisfied,

$$\nabla_\mu (E(\phi) f^\mu(\phi) \lambda(x)) = 0 \tag{2.16}$$

(2.16) satisfies the same equation as the Noether current j^μ, derived from the first theorem. However, the term inside of the total derivative, called S^μ, is not necessarily equal to j^μ.

Although j^μ and S^μ satisfy the same equation, they may differ by a divergence-free vector field κ^μ

$$j^\mu(\lambda) = S^\mu(\lambda) + \kappa^\mu \tag{2.17}$$

where

$$\nabla_\mu j^\mu(\lambda) = \nabla_\mu S^\mu(\lambda) + \nabla_\mu \kappa^\mu \tag{2.18}$$

when equations of motion are imposed, $\nabla_\mu j^\mu(\lambda)$ and $S^\mu(\lambda)$ are zero. Then,

$$\nabla_\mu j^\mu(\lambda) \cong \nabla_\mu \kappa^\mu \tag{2.19}$$

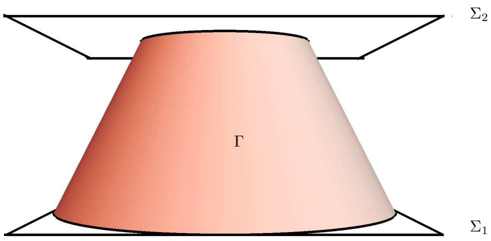

Figure 2.1: Lateral surface Γ, between two hypersurface Σ_2 and Σ_1

$$\int_\vee d^d x \sqrt{g} \nabla_\nu j^\nu = 0 \tag{2.20}$$

Then, by Stokes' theorem

$$\int_\vee d^d x \sqrt{g} \nabla_\nu j^\nu = \int_{\Sigma_2} d^{d-1} x \sqrt{g_\Sigma} \hat{n}_{2\nu} j^\nu + \int_{\Sigma_1} d^{d-1} x \sqrt{g_\Sigma} \hat{n}_{1\nu} j^\nu + \int_\Gamma d^{d-1} x \sqrt{g_\Gamma} \hat{n}^\Gamma_\nu j^\nu$$

Γ is chosen to be the lateral area between Σ_1 and Σ_2, $\hat{n}$ is normal vector on the surface that the integral is evaluated [4] , $\sqrt{g_\Sigma}$ is the absolute value of induced metric on surface Σ and the subscript of g refers to the induced metric on that surface.

$$\hat{n}_{2\nu} = \hat{n}_\nu \tag{2.21}$$

$$\hat{n}_{1\nu} = -\hat{n}_\nu \tag{2.22}$$

since $\hat{n}_\nu$ is outward to the closed surface Σ.

If the last term in (2.20) vanishes,

$$\int_{\Sigma_1} d^{d-1}x\sqrt{g_\Sigma}\hat{n}_\nu j^\nu = \int_{\Sigma_2} d^{d-1}x\sqrt{g_\Sigma}\hat{n}_\nu j^\nu \tag{2.23}$$

This means that for a hypersurface Σ on which the integral is evaluated, the expression $\int_\Sigma d^{d-1}x\sqrt{g_\Sigma}j^\nu$ is conserved along its normal vector.

However, without any assumption, Q is conserved up to contribution coming form the lateral surface Γ

$$Q_2 - Q_1 = -\int_\Gamma d^{d-1}x\sqrt{g_\Sigma}\hat{n}^\Gamma_\nu j^\nu \tag{2.24}$$

Q_1 and Q_2 are evaluated on Σ_1 and Σ_2 respectively. The non-zero charges will come from those κ^μ for which λ is non-zero on $\partial\Sigma$ which is a $d-2$ dimensional surface. This statement will be more clear with examples below.

2.4. Examples

Noether's first and second theorem are shown above with the modern approach.

[4] If Σ is null surface, timelike or spacelike, the convention for normal vectors, is expressed in the Appendix A.3

Examples of a free particle under Galilean transformation, Maxwell fields and dynamics of a relativistic particle will be considered in this framework.

2.4.1. A Free Particle Under Galilean Boost

Galilean coordinate transformations are used to relate coordinates of two frames moving at a constant velocity ε relative to each other. One interesting example is the free particle which gives non-zero θ^μ and boundary term K^μ

$$S = \int \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 dt \tag{2.25}$$

The infinitesimal Galilean boost is described as;

$$\hat{\delta} x = \epsilon t \tag{2.26}$$

$$\hat{\delta} t = 0 \tag{2.27}$$

where ϵ is infinitesimally small and constant. As mentioned before, the symmetry transformations are specific transformations that leave the action invariant up to boundary term.

$$\delta S = \int dt \frac{d(mx\epsilon)}{dt} \qquad K = mx\epsilon \tag{2.28}$$

neglecting higher orders in ϵ.

When the equation of motion is imposed, the variation of the action under Galilean boost, $\hat{\delta}x{=}\epsilon t$, is

$$\hat{\delta} S = \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \hat{\delta} x\right) = \int dt \frac{d}{dt}\left(m\dot{x}t\epsilon\right) \tag{2.29}$$

Thus,

$$\theta = m\dot{x}t\epsilon \tag{2.30}$$

Since the parameter of the lagrangian inside (2.25) is only time, conservation of the current is expressed as $\partial_t j^t = 0$. The time component of the current corresponds to charge, j^t called Q_G. Combining (2.28) and (2.29)

$$j^t = m\dot{x}t - mx = pt - mx$$

where the conjugate momentum, p, defined as $\frac{\partial L}{\partial \dot{q}}$. Analogously, space translational symmetry corresponds to conservation of momentum,

$$Q_p = m\dot{x} = p \tag{2.31}$$

The poisson bracket of two conserved charges gives another charge.

$$\{Q_G, Q_p\} = m \tag{2.32}$$

Here, the mass appears as a central charge because it commutes with all other conserved charges.

2.4.2. Maxwell's Electromagnetic Theory

The system with the combination of electromagnetic and matter fields will be considered for this example. Both global and local symmetries will be applied to them.

$$\mathcal{L} = \sqrt{g}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\delta_{AB}}{2}D_\mu\phi^A D^\mu\phi^{*B}\right\} \tag{2.33}$$

D is the gauge covariant derivative of ϕ[5] which ensures invariance under gauge transformation. Let's begin with the following global transformations

$$\delta\phi^A = \epsilon^A{}_B\phi^B$$
$$\delta\phi^{*A} = \epsilon^A{}_B\phi^{*B}$$

[5] $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$ and $D_\mu\phi^* = \partial_\mu\phi^* - ieA_\mu\phi^*$

$$\delta A_\mu = 0$$

As discussed before the Lagrangian should be invariant up to a boundary term for the transformation to be a symmetry. This implies

$$\epsilon^A{}_B - \epsilon^B{}_A = 0.$$

Now, evaluate the equations of motion.

$$E(\phi)\delta\phi^A = \frac{\delta_{AB}}{2} D_\mu D^\mu \phi^{*B} \delta\phi^A \tag{2.34}$$

$$E(\phi^*)\delta\phi^{*A} = \frac{\delta_{AB}}{2} D_\mu D^\mu \phi^B \delta\phi^{*A} \tag{2.35}$$

This Lagrangian has K^μ=0 for these symmetries. The non-zero contribution to the current comes from θ^μ

$$j^\mu = \frac{\delta_{AB}}{2} \epsilon^A{}_C \left((D^\mu \phi^{*B})\phi^C + (D^\mu \phi^B)\phi^{*C} \right) \tag{2.36}$$

$$Q = \int_{\Sigma_\mu} d^{d-1}x \hat{n}_\mu \sqrt{g_{ind}} \left\{ \frac{\delta_{AB}}{2} \epsilon^A{}_C \left((D^\mu \phi^{*B})\phi^C + (D^\mu \phi^B)\phi^{*C} \right) \right\} \tag{2.37}$$

The charge associated with a global symmetry is given by $d-1$ dimensional volume integral.

For the gauge transformations,

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda(x)$$

$$\delta_\Lambda \phi = -ie\Lambda$$

Noether's first theorem gives that

$$j^\mu = -F^{\mu\nu}\partial_\nu \Lambda(x) - ie\Lambda \left(\phi D^\mu \phi^* + \phi^* D^\mu \phi \right) \tag{2.38}$$

In terms of the formalism of Noether's Second Theorem

$$f(\phi) = -ie \qquad f^{\mu}(\phi) = 0 \tag{2.39}$$

$$f(A_{\nu}) = 0 \qquad f^{\mu}(A_{\nu}) = \delta^{\mu}_{\nu} \tag{2.40}$$

The term S^{μ} which satisfies the same equation as j^{μ} is contributed only by vector potential since $f^{\mu}(\phi)$=0.

$$S^{\mu} = E(A_{\mu})\Lambda = \{(\nabla_{\nu} F^{\mu\nu}) + (-ie\phi D^{\mu}\phi^{*} + ie\phi^{*} D^{\mu}\phi)\}\,\Lambda \tag{2.41}$$

By using (2.38) and (2.41)

$$j^{\mu} - S^{\mu} = \kappa^{\mu} = -\nabla_{\nu}(F^{\mu\nu}\Lambda). \tag{2.42}$$

Hence,

$$Q = \int_{\Sigma} d^{d-1}x\sqrt{g_{\Sigma}}\hat{n}_{\mu}\nabla_{\nu}(F^{\nu\mu}\Lambda) = \int_{\partial\Sigma} d^{d-2}x\sqrt{g_{\partial\Sigma}}F^{\nu\mu}\Lambda \tag{2.43}$$

Now, the conserved quantity is evaluated on $d-2$ dimensional surface.

2.4.3. A Relativistic Particle with A Modification

The action of a relativistic particle is

$$S = \int d\sigma \sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}}$$

The change of the parameter from σ to $\tilde{\sigma}$;

$$\frac{dx^{\mu}}{d\tilde{\sigma}} = \frac{dx^{\mu}}{d\sigma}\frac{d\tilde{\sigma}}{d\sigma} \tag{2.44}$$

leaves the action invariant. The conserved current related to reparametrization will be evaluated by Noether's second theorem. However, for 1D parameter space the Noether's current is zero. By extending parameter space, a non-zero current will be investigated. It is provided by replacing σ by ξ^a.

The action for the relaticistic particle that its dimension of parameter space increased is obtained in [13] called the Nambu-Goto action

$$\tilde{S} = C \int d\xi^2 \sqrt{-detG_{ab}} \tag{2.45}$$

where

$$G_{ab} \equiv g_{\mu\nu} \frac{dX^\mu}{d\xi^a} \frac{dX^\nu}{d\xi^b}$$

under local transformations

$$\delta\xi^\mu = \alpha^\mu \tag{2.46}$$

$$\delta_\xi X^\mu(\xi) = -\alpha^\nu \partial_\nu X^\mu(\xi) \tag{2.47}$$

By using the same notation used in the section 2.3,

$$f(X) = -\partial_\nu X^\mu \qquad f^\mu(X) = 0 \tag{2.48}$$

The equations of motion for this particular Lagrangian

$$E_\delta(X) = g_{\delta\tau}\sqrt{G}G^{ab} \left\{ \delta^\tau_\lambda - g_{\rho\lambda}G^{nm} \frac{dX^\tau}{d\xi^m} \frac{dX^\rho}{d\xi^n} \right\} \left\{ \frac{d^2X^\lambda}{d\xi^a d\xi^b} + \Gamma^\lambda_{\mu\nu} \frac{dX^\mu}{d\xi^a} \frac{dX^\nu}{d\xi^b} \right\} \tag{2.49}$$

$$(\Delta E(X))_\nu = -E_\lambda(X)\partial_\nu X^\lambda = 0 \tag{2.50}$$

The operator P^τ_λ is defined as

$$P^\tau_\lambda \equiv \left\{ \delta^\tau_\lambda - g_{\rho\lambda} G^{nm} \frac{dX^\tau}{d\xi^m} \frac{dX^\rho}{d\xi^n} \right\}$$

It is orthogonal to the velocity of the particle

$$\left\{ \delta^\tau_\lambda - g_{\rho\lambda} G^{nm} \frac{dX^\tau}{d\xi^m} \frac{dX^\rho}{d\xi^n} \right\} \frac{dX^\lambda}{d\xi^\nu} = 0$$

Therefore, all solutions of equations of motion are not allowed but should obey this identity which is predicted by Noether's second theorem.

Moreover, κ, K and θ are evaluated as described before

$$\begin{aligned} \delta_\xi L &= \nabla_a K^a = \nabla_a(\epsilon^a \sqrt{G}), \\ \theta^a &= \epsilon^a \sqrt{G} \end{aligned}$$

Then, j is obviously zero and S is zero because $f^\mu(X)$ is zero for this particular transformation . Then, $j - S = \kappa$ implies that it has no non-zero divergence free vector field κ.

3. HAMILTONIAN DYNAMICS AND COVARIANT PHASE SPACE

Equations of motion in the Hamiltonian formalism are first order in the time derivative. It is usually prefered over the Lagrangian formalism that yields second order differential equations. Moreover, Hamiltonian formalism presents an opportunity to see how the dynamical systems evolve in time by defining a phase space. Therefore, this framework splits space-time into spacial slices that evolve in time. A symplectic form defined on the phase space is not covariant under Lorentz coordinate transformations. Covariant formalism also ensures manifestly that the laws of physics do not depend on choice of coordinate. To get a conservation law which can be transformed from one frame to another, the phase space should be generalized which will be called covariant phase space. In this chapter, the Hamiltonian Dynamics and Covariant formalism related to it will be discussed by following [14], [15]

3.1. The Hamiltonian Dynamics and The Symplectic Structure

The Hamiltonian of a dynamical system is the Legendre transformation of the Lagrangian.

$$S = \int dt L(q, \dot{q}) \tag{3.1}$$

The equation of motion obtained by variation of action is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.2}$$

Under the Legendre transformation [16],

$$p = \frac{\partial L}{\partial \dot{q}} \tag{3.3}$$

$$H = p\dot{q} - L(q, \dot{q}) \tag{3.4}$$

The equation of motion in the Hamiltonian formalism is

$$\frac{\partial H}{\partial p} = \dot{q} \qquad\qquad \frac{\partial H}{\partial q} = -\dot{p} \tag{3.5}$$

Solutions of these yield trajectories in terms of position and momentum as a function of time.

In classical mechanics, the phase space is the space of all possible states of a physical system characterized by not only its position but also its momentum. Therefore, the phase space is given by q and p. The geometry of the phase space is symplectic manifold as will be described now.

Let a symplectic manifold ($\mathcal{M}$, Ω) be called the phase space. Any point on $\mathcal{M}$ is specified by (p, q). Ω is a symplectic form that is admitted by the symplectic manifold.

$$\Omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j \tag{3.6}$$

A symplectic form must satisfy the following properties,

(i) Ω is a closed form $d\Omega = 0$,

(ii) Ω is non-degenerate, namely $\omega(v, u) = 0$ for all $v \neq 0$ if and only if u=0.

This description is for n dimensional configuration space and $2n$ dimensional symplectic manifold. For one particle case, the symplectic manifold is 2 dimensional (p, q), rewriting (3.6)

$$\Omega = \frac{1}{2}\Omega_{qp} dq \wedge dp + \frac{1}{2}\Omega_{pq} dp \wedge dq \tag{3.7}$$

Consider a curve on $\mathcal{M}$ which is a solution of (3.5) . Its tangent vector is

$$V = \frac{d}{dt} = \frac{dq}{dt}\frac{\partial}{\partial q} + \frac{dp}{dt}\frac{\partial}{\partial p} \tag{3.8}$$

Along the tangent vector on the manifold, the symplectic form is left invariant. This property can be shown by

$$\mathcal{L}_V \Omega = d[\Omega(V)] - d\Omega(V) \tag{3.9}$$

which is Cartan's formula that is the definition of the Lie derivative on the space of differential forms. However, due to closedness of the symplectic form, the second term vanishes. Then, with proper symplectic matrix Ω_{qp}

$$\Omega = dq \wedge dp \tag{3.10}$$

which obeys $\mathcal{L}_V \Omega = 0$. The proof is following

$$\begin{aligned} \Omega(V) &= \langle dq, V\rangle dp - \langle dp, V\rangle dq && (3.11)\\ &= \dot{q}dp - \dot{p}dq && (3.12) \end{aligned}$$

By using Hamilton's equation of motion

$$\Omega(V) = \frac{\partial H}{\partial p}dp - \frac{\partial H}{\partial q}dq = dH \tag{3.13}$$

$d\Omega(V)$ automatically vanishes by

$$d^2 = 0 \tag{3.14}$$

which states that any exact form, i.e $\beta = d\alpha$ is closed.

Then, the vector field V, satisfies $\Omega(V) = dH$ symplectic manifold $(\mathcal{M}, \omega)$, is called a Hamiltonian vector field.

3.2. Covariant Phase Space

The covariant phase space consists of all solutions of equations of motion and in contrast to the Hamiltonian phase space, does not restrict the states to have boundary conditions at the initial and final times. In this section, covariant phase space will be constructed by following [15]. As pointed out before, the purpose is to find the conservation law which is related to symplectic form in a covariant way.

The fields are described as a map from $4D$ space-time M into M'.

$$\phi : M \to M' \tag{3.15}$$

$\mathcal{F}$ is a manifold which involves the collection of allowed field configurations. Therefore, the action S can be thought as a map from configuration space to real numbers.

$$S : \mathcal{F} \to R \tag{3.16}$$

$$S(\phi) = \int_M \mathcal{L} \tag{3.17}$$

ϕ can be represented locally as a collection of ϕ^a. Let $\phi(\sigma)$ be one-parameter field, then

$$\delta\phi^a(x) \equiv \frac{\partial \phi^a(\sigma, x)}{\partial \sigma}|_{\sigma=0}. \tag{3.18}$$

Variation of L where $\mathcal{L} = \sqrt{g}L(\phi, \nabla_\mu \phi)$ is

$$\delta_1 L = E_a \delta_1 \phi^a + \nabla_\mu \theta^\mu(\phi, \delta_1 \phi) \tag{3.19}$$

where E_a are equations of motion for each variable. δ_2 is another transformation,

$$(\delta_2\delta_1 - \delta_1\delta_2)L = (\delta_1 E_a)(\delta_2\phi^a) - (\delta_2 E_a)(\delta_1 \phi^a) + \nabla_\mu \omega^\mu \tag{3.20}$$

if δ_1, δ_2 commute with each other,

$$0 = (\delta_1 E_a)(\delta_2 \phi^a) - (\delta_2 E_a)(\delta_1 \phi^a) + \nabla_\mu \omega^\mu \tag{3.21}$$

where $\omega^\mu(\phi, \delta_1\phi, \delta_2\phi) {=} \delta_1\theta^\mu(\phi, \delta_2\phi) - \delta_2\theta^\mu(\phi, \delta_1\phi)$.

In the case that equations of motion are satisfied, $\nabla_\mu \omega^\mu = 0$. By choosing a Cauchy surface Σ [6]

$$\Omega[\phi, \delta_1\phi^a, \delta_2\phi^a] = \int_\Sigma d^{d-1}x \sqrt{g_\Sigma}\hat{n}_\mu \omega^\mu[\phi, \delta_1\phi^a, \delta_2\phi^a] \tag{3.22}$$

What stated before is that under symmetry transformation $\hat{\delta}$ the Noether' s current can be expressed as

$$\nabla_\mu j^\mu(\phi, \hat{\delta}\phi) = \nabla_\mu \theta^\mu(\phi, \hat{\delta}\phi) - \nabla_\mu K^\mu(\phi, \hat{\delta}\phi) \tag{3.23}$$

By inserting $\hat{\delta}L = \nabla_\mu K^\mu(\phi, \hat{\delta}\phi)$

$$= (\hat{\delta}L - E_a\hat{\delta}\phi^a) - \hat{\delta}L \tag{3.24}$$

$$= -E_a\hat{\delta}\phi^a \tag{3.25}$$

Lemma. If ϕ is a solution of the equation of motion, $E_a = 0$. $\hat{\delta}\phi^a$ be an infinitesimal local symmetry at ϕ. Then, the variation of E_a vanishes under corresponding symmetry transformation.[7]

Let δ_2 be symmetry transformation where δ_1 is arbitrary transformation. Then, the second term in (3.21) vanishes for local symmetry transformations and first term can be written as

[6]The Cauchy surface is a hypersurface such that it is possible to move from one to another uniquely along a timelike curve

[7]for proof, see [15]

$$\delta_1(-E_a\hat{\delta}\phi^a)=\nabla_\mu\omega^\mu(\phi,\delta_1\phi,\hat{\delta}\phi)$$

And by lemma 1

$$\nabla_\mu(-\delta_1 j^\mu(\phi,\hat{\delta}\phi)-\omega^\mu(\phi,\delta_1\phi,\hat{\delta}\phi))=0 \tag{3.26}$$

By applying Stokes' s theorem, Noether's charge comes from first theorem is zero as proved in [15]. Then, it is possible to define a charge Q that is conserved up to the lateral surface Γ as shown in the section 2.3.

$$\delta_1 Q(\phi,\hat{\delta}\phi)=\int_\Sigma d^{d-1}x\sqrt{g_\Sigma}\hat{n}_\mu\omega^\mu(\phi,\delta_1\phi,\hat{\delta}\phi)\equiv\Omega(\delta_1\phi,\hat{\delta}\phi) \tag{3.27}$$

The point to take into consideration is that j^μ may differ from ω^μ by $\tilde{\omega}^\nu$ that is divergence-free vector field. This brings a freedom to determine symplectic current that will be used in the section 4.3.

Ω is a symplectic form on the covariant phase space

$$\Omega(\delta_1\phi,\delta_2\phi)=\Omega_{AB}(\delta_1\phi)^A(\delta_2\phi)^B \tag{3.28}$$

where $(\delta_i\phi)^A$ is tangent vector on the covariant phase space. Because of the contribution comes from lateral surface, Ω is not necessarily conserved. However, there are a couple of ways to reach conserved symplectic form on the covariant phase space. One will be achieved in 4.3. Another way is redefinition of Ω on Σ_1 as suggested in [17],

$$\Omega_{\Sigma_1}(\delta_1\phi,\delta_2\phi)=\int_{\Sigma_1} d^3x\sqrt{g_\Sigma}\hat{n}_\mu\omega^\mu(\phi,\delta_1\phi,\delta_2\phi)+\int_{S_1} d^2x\sqrt{g_S}\zeta(\phi,\delta_1\phi,\delta_2\phi) \tag{3.29}$$

where $\zeta(\phi,\delta_1\phi,\delta_2\phi)$ is a function that satisfies

$$\int_\Gamma d^3x\sqrt{g_\Gamma}\hat{n}_\mu\omega^\mu(\phi,\delta_1\phi,\hat{\delta}\phi)=\oint_{S_2} d^2x\sqrt{g_S}\zeta(\phi,\delta_1\phi,\delta_2\phi)-\oint_{S_1} d^2x\sqrt{g_S}\zeta(\phi,\delta_1\phi,\delta_2\phi) \tag{3.30}$$

where S_1 and S_2 intersects with Σ_1 and Σ_2 respectively.

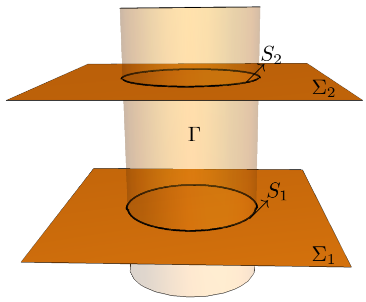

Figure 3.1: S's represent 2 spheres, one dimension is supressed.

3.3. Examples

Hamiltonian formalism and covariant phase space is discussed above. The purpose of the first example is to shed light on space/time spliting of Hamiltonian formalism and the constraint on equation of motion. The second is applicaiton of finding conserved quantity by using covariant phase space formalism which is compatible with Noether's second theorem.

3.3.1. Hamiltonian Formalism for Maxwell Theory

The Lagrangian density for electromagnetic fields, [8]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.31}$$

When $F^{\mu\nu}$ is expanded in $\dot{A}$ and $\partial_i A_0$ etc.[9] ,

$$\mathcal{L} = \frac{1}{2}\dot{A}_i\dot{A}^i - \dot{A}_i\partial^i A_0 + \frac{1}{2}\partial_i A_0 \partial^i A_0 - \frac{1}{4}F_{ij}F^{ij} \tag{3.32}$$

[8]Lagrangian density is called because $\sqrt{g} = 1$ for this particular example
[9]Here, latin indices are used for spatial components

The conjugate momenta, $\pi_\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu}$.

$$\pi_i = \dot{A}_i - \partial_i A_0 \qquad \pi_0 = 0 \tag{3.33}$$

Then,

$$\mathcal{H} = \pi_i \dot{A}^i - \mathcal{L} \tag{3.34}$$
$$\mathcal{H} = \frac{1}{2}(\pi^2 + \vec{B}^2) + \pi^i \partial_i A_0 \tag{3.35}$$

The second term in the right hand side is rewritten by using total derivative.

$$\pi^i \partial_i A_0 = \partial_i(\pi^i A_0) - \partial_i \pi^i A_0 \tag{3.36}$$

Note that Hamiltonian is in integral over t, with usual assumptions of electromagnetic fields such that $A_0 \to 0$ at spatial infinity, the Hamiltonian density can be written as that

$$\mathcal{H} = \frac{1}{2}\pi_i\pi^i + \frac{1}{4}F_{ij}F^{ij} - A_0\partial_i\pi^i$$

Equations of motion are

$$\begin{aligned} \dot{\pi}_0 &= -\frac{\partial H}{\partial A^0} = \partial_i \pi^i \\ \dot{\pi}_i &= -\frac{\partial H}{\partial A^i} = 0 \qquad \dot{A}_i = \frac{\partial H}{\partial \pi^i} = \pi_i \end{aligned}$$

By using (3.33)

$$\begin{aligned} \dot{\pi}_0 = \partial_i \pi^i &= 0 \\ \partial_i \pi^i = \nabla . \vec{E} &= 0 \end{aligned}$$

The constraint $\nabla . \vec{E} = 0$, called Gauss law, is not a dynamic equation. For this case, the phase space is specified by A_i and π_i with the constraint.

3.3.2. Conservation Law in Electrodynamics

Electromagnetic field in terms of differential forms can be written as

$$\mathcal{L} = -\tfrac{1}{4}(F \wedge *F)$$

where F is electrogmagnetic 2-form, * stands for hodge dual and $F \equiv dA$.

Let δ_2 be local gauge transformation with gauge parameter Λ. Then,

$$\omega[A, \delta A, \delta_\Lambda A] = -[\delta(*F)\delta_\Lambda A + \delta_\Lambda(*F)\delta A]$$

Since F is gauge invariant, the second term vanishes.

$$\int_\Sigma \delta j(A, \delta_\Lambda A) = -\int_\Sigma \delta(*F) d\Lambda \tag{3.37}$$

Assuming that gauge transformation is field independent

$$\int_\Sigma j = \int_\Sigma d(*F)\Lambda - \int_{\partial\Sigma} *F\Lambda \tag{3.38}$$

The first term is zero when the equations of motion is satisfied, non-zero charge is obtained if Λ does not vanish on boundary $\partial\Sigma$. As a result, the charge related to gauge transformation Λ

$$Q_\Lambda = \int_\Sigma j = -\int_{\partial\Sigma} *F\Lambda \tag{3.39}$$

4. GAUGE TRANSFORMATIONS AND CHARGES

Physical transformations such as translation or rotation map the value of physical observable to the another. For example, consider a system involving a classical free particle. its momentum is conserved but the particle does not sit at the same point no longer after spatial translation. So, at least one physical quantity associated with the free particle changed. On the other hand, gauge transformations correspond to a particular symmetry that leaves physics invariant but changes non-observable quantities. Here, the attention will be restricted to gauge transformations and related charges within the scope of Maxwell's electrodynamics.

However, gauge transformations deserve special consideration because of redundant degrees of freedom in fields A_μ. Considering quantum electrodynamics, photons are quanta of electromagnetic fields that describes electromagnetic interaction between particles. Photons are massless and spin-1 particles which have two degrees of freedom [18]. Fields in covariant formalism are 4-vector objects. Embedding two degrees of freedom into 4-vector brings two unphysical degrees of freedom. Those should be fixed by the gauge condition to obtain unique solutions of fields. Redundancy in gauge transformation would be also killed by this process. However, the way of killing extra degrees of freedom is not unique. Accordingly, gauge transformations may be eliminated by imposing different gauge conditions. Different gauge conditions reduce the gauge transformations into subclass of them.

Previous chapters set the stage for the rest of dissertation. Methods enabling us to evaluate conserved quantities are already developed above. In the following chapter, conserved quantities resulted from subclass of gauge transformations, namely residual and asymptotic gauge transformations [8, 11] and few attitudes about them will be discussed, respectively.

4.1. Residual Gauge Transformations

The characteristics of the residual gauge transformation are based on removing redundant gauge transformations by Lorenz gauge fixing and putting boundary condition on gauge invariant fields. It is worth noting that the latter does not restrict the gauge transformations.

$$\nabla_\mu A^\mu = 0 \tag{4.1}$$

$$E_r \propto \frac{1}{r^2} \tag{4.2}$$

The equation 4.2 is a typical boundary condition.

4.1.1. Noether's Charge by Residual Gauge Transformations

The description of Q in the (3.27) based on the assumption that fields vanish on the boundary. Non-zero fields on the boundary imply conservation up to boundary term as stated before on page 9. The corresponding charge is

$$Q_\lambda = \int_\Sigma d\Sigma_\nu \nabla_\mu (F^{\nu\mu}\lambda) \tag{4.3}$$

$$Q_\lambda = \oint_{\partial\Sigma} d\Sigma_{\nu\mu} F^{\nu\mu}\lambda \tag{4.4}$$

$$= \int_\Sigma d\Sigma_\nu \nabla_\mu F^{\nu\mu}\lambda + \int_\Sigma d\Sigma_\nu F^{\nu\mu}\nabla_\mu\lambda \tag{4.5}$$

where λ is local gauge parameter. First term of the right hand side is called hard part associated with current j^μ. When λ is constant, it gives nothing else but the electric charge. The second one is the soft part which is related to electromagnetic field.

The characteristic of this work is to choose finite t,r surfaces, then sending them to infinity. To simplify the calculation, 4D space-time is flat everywhere

$$ds^2 = -dt^2 + dr^2 + r^2 d\Xi^2 \tag{4.6}$$

$$Q_\lambda = \oint_{\partial\Sigma} d\Xi r^2 F^{tr} \lambda \quad (4.7)$$

$$Q_\lambda = \int_\Sigma r^2 d\Xi \nabla_\mu F^{t\mu} \lambda + \int_\Sigma r^2 d\Xi F^{t\mu} \nabla_\mu \lambda \quad (4.8)$$

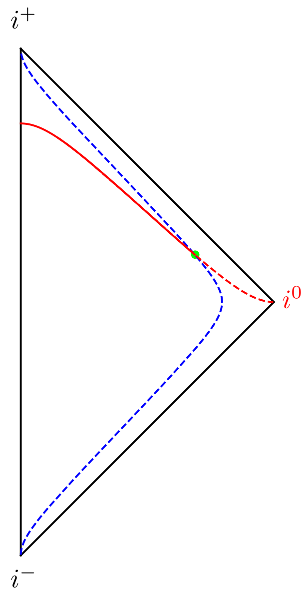

Figure 4.1: Penrose Diagram : The red curve is t constant and the blue is r constant curves. The green point represents S^2 the integral is taken over

By the Lorenz gauge, the Maxwell's equations become inhomogeneous wave equation.

$$\Box A^\mu = j^\mu$$

The gauge transformation, which corresponds to non-trivial charge,

$$\nabla_\mu A^\mu = 0 \longmapsto \Box \lambda = 0 \quad (4.9)$$

By separation of variables, time dependence of the gauge parameter can be expressed as

$$\lambda(t, x^i) = e^{-ikt} \lambda_k(x^i)$$

The wave equation becomes Helmholtz equation with two different sets of solutions. The solutions are the spherical Hankel functions of the first and second kind. Since electromagnetic fields vanish faster than increase in the gauge parameter, they do not reveal finite charge. However, if k is zero, the gauge parameter obeys the Laplace equation

$$\nabla^2 \lambda = 0 \tag{4.10}$$

$$\lambda(x^i) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\alpha^+_{l,m} r^l + \alpha^-_{l,m} r^{-(l+1)}) Y^*_{l,m}(\theta, \phi)$$

The minus sign is just a convention. The zero frequency modes, which do not vanish on the boundary, are candidate to rise non-zero charges. As an application of this approch, the conservation of multipole moments is showed in [11] for electrostatic case

$$Q_{l,m} = \frac{l+1}{2l+1} q_{l,m} \tag{4.11}$$

$$Q^h_{l,m} = q_{l,m} \qquad Q^s_{l,m} = -\frac{l}{l+1} q_{l,m} \tag{4.12}$$

The total charge (4.11) is proportional to electric multipole moments. Since spherical harmonics are not only orthogonal but also complete set, for each (l, m) conservation of multipole charges is satisfied.

4.2. Asymptotic Gauge Transformations

The asymptotic gauge transformations arise out of a gauge fixing and asymptotic boundary condition. In this case, the asymptotic boundary condition restricts gauge transformations. They correspond to infinitely many conserved charges as well if the boundary conditions are weak enough to allow all physically reasonable solutions. This approach has an ambiguity in the sense that there is no proof on having all physically reasonable solutions yet.

Based on previous researches [8, 9, 19] to end up with conserved quantity there are two approaches, one is based on Noether's Second theorem that is already studied in detail, the other is based on imposing antipodal matching. In this chapter, charges related to asymptotic gauge transformations will be investigated in the leading order of fields. Then, how antipodal matching conditions on fields and gauge transformations impose conserved quantity will be stated.

4.2.1. Noether's Charge by Asymptotic Gauge Transformations

Asymptotically flat Minkowski metric in the retarded coordinates (u,r,z,$\bar{z}$) near the future null infinity $\mathcal{I}^+$, figure 4.2

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$

where $z = e^{i\phi}tan\frac{\theta}{2}$ and $\bar{z} = e^{-i\phi}tan\frac{\theta}{2}$.
A particular gauge condition is imposed

$$A_u = 0 \tag{4.13}$$

The general form of gauge parameter $\epsilon(u, r, z, \bar{z})$ restricted by (4.13).

$$A_u = \partial_u\epsilon(u, r, z, \bar{z}) = 0 \tag{4.14}$$

Hence, ϵ is independent of u. The asymptotic boundary conditions at $\mathcal{J}^+$

$$A_r \sim \mathcal{O}(\frac{1}{r^2}) \quad A_z \sim \mathcal{O}(1) \quad A_{\bar{z}} \sim \mathcal{O}(1) \tag{4.15}$$

They are suggested to have finite charges. Asymptotic expansions of the gauge fields which are compatible with boundary conditions near $\mathcal{J}^+$

$$\begin{aligned}A_r(r,u,z,\bar{z}) &= \sum_{n=0}^{\infty}\frac{A_r^{(n)}(u,z,\bar{z})}{r^{n+2}}\\ A_z(r,u,z,\bar{z}) &= \sum_{n=0}^{\infty}\frac{A_z^{(n)}(u,z,\bar{z})}{r^{n}}\\ A_{\bar{z}}(r,u,z,\bar{z}) &= \sum_{n=0}^{\infty}\frac{A_{\bar{z}}^{(n)}(u,z,\bar{z})}{r^{n}}\end{aligned}$$

Since $A_r|_{\mathcal{J}^+}=0$, the gauge transformation is radius independent in the leading order near future null infinity $\mathcal{J}^+$ as well. Hence, the gauge parameter λ is an arbitrary function of $z,\bar{z}$ at $\mathcal{J}^+$. [10] As stated in [8], the remnant gauge transformation is partially fixed at $\mathcal{J}^-$ by imposing,

$$r^2\gamma_{z\bar{z}}\nabla^{\mu}A_{\mu}(u\to\infty,r,z,\bar{z}) = 0 \tag{4.16}$$

Then, the solutions of the Maxwell equations near $\mathcal{J}^+$ shows that with two initial data, one can reach the unique solution [8]. However, it restricts the residual gauge transformation as well

$$\begin{aligned}r^2\gamma_{z\bar{z}}\nabla^{\mu}A_{\mu}(u\to\infty,r,z,\bar{z}) &= r^2\gamma_{z\bar{z}}\frac{1}{\sqrt{g}}\partial_{\mu}(\sqrt{g}A^{\mu})|_{\mathcal{J}^-}\\ &= [\partial_z A_{\bar{z}}+\partial_{\bar{z}}A_z+(\partial_r-\partial_u)(\gamma_{z\bar{z}}r^2A_r)]|_{\mathcal{J}^-}\end{aligned}$$

Then, replacing gauge fields with gauge transformations

$$2\partial_z\partial_{\bar{z}}\epsilon+\gamma_{z\bar{z}}\partial_r(r^2\partial_r\epsilon)=0$$

[10] $\epsilon(z,\bar{z})$ is used for gauge parameter of asymptotic symmetries along this thesis

Locally,

$$2\partial_z\partial_{\bar{z}}\epsilon(z,\bar{z}) = 0$$
$$\epsilon(z,\bar{z}) = \epsilon_z(z) + \epsilon_{\bar{z}}(\bar{z})$$

The Noether's charge as described in (4.3)

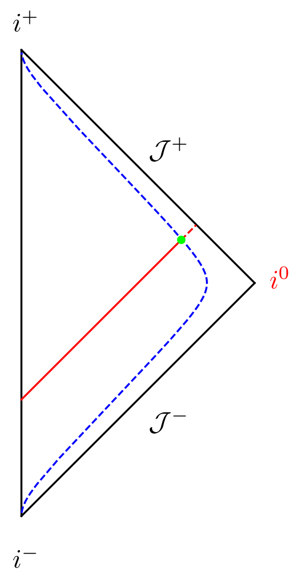

Figure 4.2: Penrose Diagram: Red line is u constant, blue is r constant surfaces. The green dot is intersection of these two surfaces.

$$\begin{aligned} Q_\epsilon &= \int_{\partial\Sigma} d\Sigma_{ur} F^{ur}\epsilon(z,\bar{z}) \\ &= \int_{\Sigma} d\Sigma_u \nabla_\mu F^{u\mu}\epsilon(z,\bar{z}) + \int_{\Sigma} d\Sigma_u F^{u\mu}\nabla_\mu\epsilon(z,\bar{z}) \end{aligned}$$

Taking the asymptotic boundary conditions and asymptotic expansion of gauge fields into account

$$\begin{aligned} Q_\epsilon &= \int_{\partial\Sigma} d\Sigma_{ur} \frac{F^{(2)}_{ru}}{r^2}\epsilon(z,\bar{z}) \\ &= \int_{\Sigma} drdzd\bar{z}\gamma_{z\bar{z}} g^{u\nu} g^{\mu\gamma}\nabla_\mu F^{(2)}_{\nu\gamma}\epsilon(z,\bar{z}) + \int_{\Sigma} drdzd\bar{z}\gamma_{z\bar{z}} g^{u\nu} g^{\mu\gamma} F^{(2)}_{\nu\gamma}\nabla_\mu\epsilon(z,\bar{z}) \end{aligned}$$

This result introduces a conserved charge for each $\epsilon(z,\bar{z})$. For constant gauge parameter, the expression gives the total electric charge again.

To show that the same charges emerge with different gauge fixings a and boundary conditions, was expected at the beginning of this project by me. At least, in Quantum Electrodynamic, Faddev Popov trick shows that different gauge fixings do not change the physical conditions. However, boundary conditions play a crucial role here because for asymptotic case, they restrict gauge transformations. In short, showing their equivalence was one of my goals for this dissertation. I have fallen short of my aim. In time, I realized that physically same boundary conditions, $E_r \propto \frac{1}{r^2}$ and $A_r \propto \frac{1}{r^2}$ may correspond to different conserved quantities. I still have not fully comprehended both their effect on physical conservation laws and justification of putting boundary condition on gauge fields.

4.2.2. Conserved Charge by Antipodal Matching

This approach starts with

$$Q_\epsilon^+ \quad = \quad \int_{\mathcal{J}_-^+} dz d\bar{z} \gamma_{z\bar{z}} F^{ur} \epsilon \tag{4.17}$$

as a candidate to be conserved, [9]. Here, it is not necessarily the Noether's charge. If antipodal matching conditions and particular boundary conditions near spatial infinity are imposed, then the quantity becomes a conserved quantity at i^0.

The motivation comes from the Lienart-Wiechert solution for radial electric field

$$F_{rt}(\vec{x},t) = \frac{e^2}{4\pi} \sum_{m=1}^{n} \frac{Q_m \gamma_m (r - t\hat{x}.\vec{\beta}_m)}{|\gamma_m^2 (t - r\hat{x}\vec{\beta}_m)^2 - t^2 + r^2|^{3/2}} \tag{4.18}$$

First, this expression of F_{rt} is rewritten in retarded and advanced coordinates. Then taking limits of the expression in retarded coordinates $r \to \infty$ and $u \to -\infty$ in order and taking limits of the expression in advanced coordinates $r \to \infty$ and $v \to \infty$ the spatial infinity i^0 is reached on the Penrose Diagram. To get continuous fields, Lienart-Wiechert solutions must obey

$$\lim_{r\to\infty} F^{(2)}_{ru}(\hat{x})|_{\mathcal{J}^+_-} = \lim_{r\to\infty} F^{(2)}_{rv}(-\hat{x})|_{\mathcal{J}^-_+} \tag{4.19}$$

is imposed. The right hand side of the first equation is in advanced coordinates (v,r,z,$\bar{z}$) where $v = t + r$,

In addition, ϵ which was an arbitrary function in $z, \bar{z}$ should satisfy (4.20) to provide conservation of (4.17).

$$\epsilon(z, \bar{z})|_{\mathcal{J}^+_-} = \epsilon(z, \bar{z})|_{\mathcal{J}^-_+} \tag{4.20}$$

Under asymptotic boundary conditions (4.15), only the leading order of (4.17) gives non-zero charge using asymptotic expansion of fields. Hence,

$$Q^+_\epsilon = \int_{\mathcal{J}^+_-} dz d\bar{z}\gamma_{z\bar{z}} F^{(2)}_{ru}\epsilon(z, \bar{z}) \tag{4.21}$$

$$Q^-_\epsilon = \int_{\mathcal{J}^-_+} dz d\bar{z}\gamma_{z\bar{z}} F^{(2)}_{rv}\epsilon(z, \bar{z}) \tag{4.22}$$

$$Q^+_\epsilon = Q^-_\epsilon \tag{4.23}$$

Below, the paper [10] will discuss the point of the charge definition further. Though, this work shows that antipodal matching can be read as a consequence of Maxwell equations near spatial infinity too.

4.3. Conserved Charge at Spatial Infinity

Although physical meaning of antipodal matching has not been explained thoroughly, the antipodal condition (4.19) is patially lightened in [10]. A careful examination on the Maxwell equations reveals the equality at spatial infinity i^0 figure:(4.1) with the assumption that the Maxwell equations are homogeneous because massive charged particles cannot reach a point at spatial infinity.

However, the purpose to make this work one of the current issues here is to compare definitions of charges and surfaces they described instead.

In the $r > |t|$ region, the Minkowski metric and coordinates ρ,τ as a function of t,r are given as

$$ds^2 = d\rho^2 + \rho^2 \left[-\frac{d\tau^2}{1+\tau^2} + (1+\tau^2) q_{AB} dx^A dx^B \right] \tag{4.24}$$

$$= d\rho^2 + \rho^2 \left[-\frac{d\tau^2}{1+\tau^2} + (1+\tau^2) d\Xi^2 \right] \tag{4.25}$$

$$\tau = \frac{t}{(r^2 - t^2)^{1/2}} \qquad \rho = (r^2 - t^2)^{1/2} \tag{4.26}$$

The asymptotic boundary conditions in (τ, ρ, x^A, x^B)

$$A_\rho \sim \mathcal{O}(\rho^{-1}) \quad A_\tau \sim \mathcal{O}(\rho^0) \quad A_A \sim \mathcal{O}(\rho^0)$$

imposed. The symplectic current on τ- constant surface

$$\omega^a_{bulk} = \delta F^{ab} \wedge \delta A_b + (\delta D^a \phi)^* \wedge \delta\phi + c.c \tag{4.27}$$

The leakage of the current between two τ - constant surface near the spatial infinity

$$\lim_{\rho\to\infty} \int_{\tau_1}^{\tau_2} d\tau \int d\Xi \omega^\rho_{bulk}$$

$$\omega^\rho_{bulk} = (\delta F^{\rho b} \wedge \delta A_b + (\delta D^\rho \phi)^* \wedge \delta\phi + c.c) \tag{4.28}$$

The matter field part vanishes under gauge transformation, then

$$\omega^\rho_{bulk} = (\delta F^{\rho b} \wedge \delta A_b) \tag{4.29}$$

The results are obtained in the sections 2.3 and 3.2 are based on the gauge invariance of the electromagnetic tensor, namely

$$\delta_\lambda F^{ab} = 0 \tag{4.30}$$

Then, the term $\delta_\lambda F^{ab}\delta A_b$ in (4.29) vanishes, the other term $\delta F^{ab}\delta_\lambda A_b$ survive. Alternatively, in the leading order (4.29) becomes

$$\begin{aligned} h^{\alpha\beta}\delta F^{(1)}_{\rho b} \wedge \delta A^{(0)}_b &= -\delta D^\alpha A^{(1)}_\rho \wedge \delta A^{(0)}_\alpha & (4.31)\\ &= -D^\alpha(\delta A^{(1)}_\rho \wedge \delta A^{(0)}_\alpha) + \delta A^{(1)}_\rho \wedge \delta D^\alpha A^{(0)}_\alpha & (4.32)\end{aligned}$$

The first term in (4.32) is a total derivative can be cancelled by adding to ω^α_{bulk}

$$\omega^\alpha_{leak} = \nabla_b(\delta A^a \wedge \delta A^b) \tag{4.33}$$

The last term in (4.32) vanishes by demanding the Lorenz gauge condition

$$A^{(1)} \sim D^\alpha A^{(0)}_\alpha \tag{4.34}$$

Actually, it is the example of the freedom to determine symplectic current which is mentioned before. Then,

$$\begin{aligned} \omega^a_{total} &= \omega^a_{bulk} + \omega^a_{leak} = -\nabla_b(\lambda F^{ab}) + \nabla_b(\nabla^a\lambda A^b - A^a\nabla^b\lambda)\\ &= -\nabla_b(\lambda F^{ab}) + \nabla_b(\nabla^a\lambda A^b - A^a\nabla^b\lambda)\end{aligned}$$

The charge at ρ_0,

$$\begin{aligned} Q_\lambda &= \int_{\Sigma_\rho} d\Sigma_\rho\left[-\nabla_b(\lambda F^{\rho b}) + \nabla_b(\nabla^\rho\lambda A^b - A^\rho\nabla^b\lambda)\right] & (4.35)\\ &= \int_{\Sigma_\rho} d\Sigma_\rho\left[\nabla_b(\lambda F^{b\rho}) + \nabla_b(\nabla^\rho\lambda A^b - A^\rho\nabla^b\lambda)\right] & (4.36)\end{aligned}$$

where $n_\rho = (0, 1, 0, 0)$, Σ_ρ is a hypersurface of ρ_0 and $d\Sigma_\rho = \rho_0^2(1 + \tau^2) d\tau\, d\Xi$
Appliyng Stoke's theorem, the charge Q_λ on τ constant surface on Σ

$$Q_\lambda[\tau] = \int_{\Sigma_{\rho\tau}} d\Xi \rho_0^2(1 + \tau_0^2) n_\tau \left[(\lambda F^{\tau\rho}) + (\nabla^\rho \lambda A^\tau - A^\rho \nabla^\tau \lambda)\right] \tag{4.37}$$

where $n_\tau = \left(-\frac{\sqrt{1+\tau_0^2}}{\rho}, 0, 0, 0\right)$. By sending ρ to ∞ and taking the leading order, the expression becomes

$$Q_\lambda[\tau] \;=\; \int_{\Sigma_{\rho\tau}} d\Xi (1 + \tau_0^2)^{3/2} [\lambda F^{(1)}_{\tau\rho} + \nabla_\tau \lambda A^{(1)}_\rho] \tag{4.38}$$

Then, limits of τ match with the charges described in section 4.2.2.

$$\lim_{\tau \to \pm\infty} Q_\lambda[\tau] \;=\; Q^\pm_\epsilon \tag{4.39}$$

On the right hand side, $Q^\pm_\epsilon$ is the charge obtained by gauge fixing $A_u = 0$ and asymptotic boundary condition $A|^+_{\mathcal{J}}$.

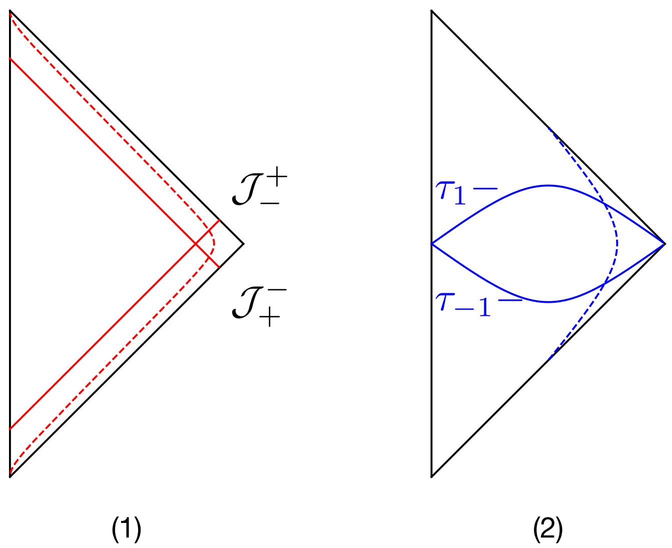

Figure 4.3: (1): The dashed line is large r constant, the red line which touches $\mathcal{J}^+$ is u constant, the other red line which touches $\mathcal{J}^-$ is v constant curve. (2):The dashed blue line is ρ constant.

In the figure 4.3, two different paths that reach the spatial infinity is demonstrated. In (1), charges on u constant line reach to $\mathcal{J}^+$ first and approach the spatial infinity from $\mathcal{J}^+_-$, and charges on v constant line reach to $\mathcal{J}^-$ and approach the spatial infinity from $\mathcal{J}^-_+$. In (2), on ρ constant line, τ is send to infinity. The result eq.(4.39) is obtained in [10] shows equivalence of two paths, as well.

5. COMPARISONS

The purpose of this chapter is to present the comparison of surfaces on which charges are evaluated. And, the statement for charges by antipodal matching near spatial infinity will be compared to the general formula we have from Noether's second theorem.

5.1. tr vs ur

Charges related to residual and asymptotic gauge transformations are studied in (t, r, θ, ϕ) and (u, r, θ, ϕ). Now, equivalence of surfaces charges are evaluated will be compared.

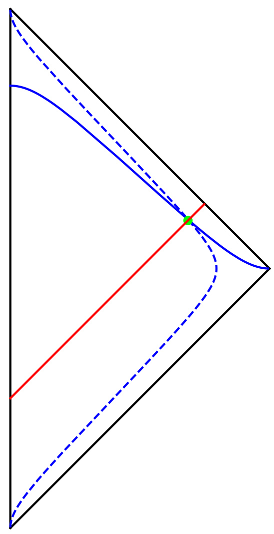

Figure 5.1: Penrose Diagram: Blue line represents u constant and r constant surfaces, red lines are t and r constant surfaces

In the figure 5.1, the charge defined on t_0 constant surface drags on the green point which intersects with r_0 constant surface by appliying Stokes' Theorem. The point represents S^2 surface. On the other hand the charge defined on u_0 constant surface drags on the point which intersects with r_0 constant curve. The charge defined on t constant surface will be compared with the charge on u constant surface.

Recall that 4.7 is charge on t constant surface. To compare this with the study that considers the charge asymptotically, t-constant implies

$$u = t_0 - r \tag{5.1}$$

$$r = r \tag{5.2}$$

The normal vector of t constant surface $\hat{n}_\mu = (-1, -1, 0, 0)$ in (u,r,z,$\bar{z}$) basis. Then,

$$Q_\lambda = \int_{\Sigma_t} r^2 d\Xi dr \left\{ -\nabla_\mu (F^{u\mu}\lambda) - \nabla_\mu (F^{r\mu}\lambda) \right\} \tag{5.3}$$

Applying Stokes' theorem, the second term vanishes because the electromagnetic tensor $F^{\mu\nu}$ is an anti-symmetric tensor

$$Q_\lambda = -\int_{\partial\Sigma} r^2 d\Xi F^{ur}\lambda = \int r^2 dz d\bar{z}\gamma z\bar{z} F_{ru}\lambda \tag{5.4}$$

Then, with (4.17)

$$Q_\lambda = Q_{\epsilon \to \lambda} \tag{5.5}$$

$Q_{\epsilon\to\lambda}$ means that when the charge related to asymptotic gauge transformation is sent to residual gauge parameter, evaluation of integral gives the same charges. This can be achieved by imposing lorenz gauge condition and boundary conditions as stated for residual gauge transformations.

5.2. tr vs $\rho\tau$

In this instance, the charges defined on t constant surface will be rewritten in terms of ρ constant surface and will be compared with the charge definition at spatial infinity. The expression of ρ and τ in terms of t_0 and r is

$$\rho = \sqrt{r^2 - t_0^2} \qquad \tau = \frac{t_0}{\sqrt{r^2 - t_0^2}} \tag{5.6}$$

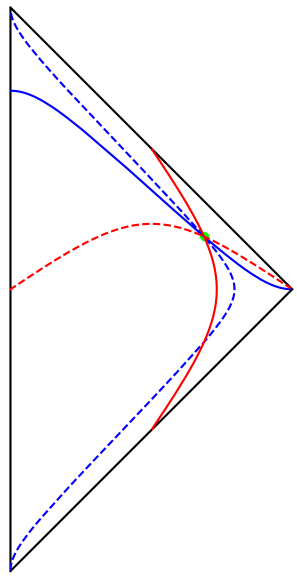

Figure 5.2: Penrose Diagram: The blue line is t and the dashed blue line is r constant curves, the red line is ρ and the dashed red line is τ constant curves

This automatically implies ,

$$dr = \frac{\rho}{-\tau\sqrt{1+\tau^2}} d\tau \tag{5.7}$$

The definition of charge on t_0 hypersurface

$$Q_\lambda = \int_{\Sigma_{t_0}} d\Sigma \hat{n}_\nu \nabla_\mu (F^{\nu\mu}\lambda) = \tag{5.8}$$

$$= \int_{\Sigma_{t_0}} dr d\Xi r^2 [-\rho \nabla_\mu (F^{\tau\mu}\lambda) - \tau \nabla_\mu (F^{\rho\mu}\lambda)] \tag{5.9}$$

where $\hat{n}_\nu = (-\rho, -\tau, 0, 0)$ in $(\tau, \rho, \theta, \phi)$ basis.

$$\int_{\Sigma_{t_0}} d\tau d\Xi \frac{\rho^3\sqrt{1+\tau^2}}{\tau} [\rho \nabla_\mu (F^{\tau\mu}\lambda)] + \int_{\Sigma_{t_0}} d\tau d\Xi \frac{\rho^3\sqrt{1+\tau^2}}{\tau} [\tau \nabla_\mu (F^{\rho\mu}\lambda)]$$

Appliying Stokes' theorem,

$$Q_\lambda = -\int_{\Sigma_{t_0 \tau_0}} d\Xi \rho_0 (1+\tau_0^2)^{3/2} F_{\rho\tau}\lambda \tag{5.10}$$

where τ_0 normal vector $(\rho/\sqrt{1+\tau_0^2}, 0, 0, 0)$ This result shows that definitions of charges on two different surfaces match.

5.3. Conserved Charge by Antipodal Matching vs by Residual Gauge

The charge defined in the section 4.2.1 is conserved up to leakage over lateral surface Γ. In 4.3, the leakage from lateral surface is included partially into ω and the rest vanishes imposing Lorenz gauge fixing. At the end of that section, (4.39) argued that the contribution over the lateral surface at spatial infinity vanishes. The purpose of this section is to show that two results in (2.20) (4.17) are compatible with each other. As a first step, considering massive charged fields,

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\phi^* + m^2\phi^2 \tag{5.11}$$

where the related action in 4D is

$$S = \int d^4x\sqrt{g}L \tag{5.12}$$

For this particular L, one can easily check that for Noether's second theorem, j^μ and S^μ does not change so that κ^μ remains same. On the other hand, in covariant phase space method, ω^μ is also same since θ's do not care ϕ dependence of L and by adding $m^2\phi^2$ the equation of motion of gauge fields A_μ will be same. Therefore, the charge found above will be used here.

Rewriting the equation (2.24) for the charges evaluated on t constant surface

$$\begin{aligned} Q_{t_2} - Q_{t_1} &= \int_\Gamma dt d\Xi r^2 \nabla_\mu(F^{r\mu}\lambda) && (5.13)\\ &= \int_\Gamma dt d\Xi r^2 \nabla_\mu(F^{r\mu})\lambda + \int_\Gamma dt d\Xi r^2 F^{r\mu}\nabla_\mu\lambda && (5.14)\\ &= \int_\Gamma dt d\Xi r^2 j^r\lambda + \int_\Gamma dt d\Xi r^2 F^{r\mu}\partial_\mu\lambda && (5.15)\end{aligned}$$

The first term vanishes because massive charged particles follow timelike geodesics.

Therefore, they cannot touch i^0.The second term in the right hand side of (5.15) is

$$\int_\Gamma dtd\Xi r^2 F^{r\mu}\partial_\mu\lambda = \int_\Gamma dtd\Xi r^2 F^{rt}\partial_t\lambda + \int_\Gamma dtd\Xi r^2 F^{r\theta}\partial_\theta\lambda + \int_\Gamma dtd\Xi r^2 F^{r\phi}\partial_\phi\lambda \quad (5.16)$$

With these asymptotic expansions, the last two terms vanish if (5.16) gives finite charges. Recall that the finite soft charges are evaluated on $d\Sigma_t$

$$\int_{\Sigma_t} drd\Xi r^2 F^{t\mu}\partial_\mu\lambda = \int_{\Sigma_t} drd\Xi r^2 F^{tr}\partial_t\lambda + \int_{\Sigma_t} drd\Xi r^2 F^{t\theta}\partial_\theta\lambda + \int_{\Sigma_t} drd\Xi r^2 F^{t\phi}\partial_\phi\lambda$$

As an order of magnitude, the right hand side of (5.16) goes with $\frac{1}{r}$ of (5.17) except the first term. More precisely, physical asymptotic boundaries on the electromagnetic tensor is suggested as

$$F_{t\theta} = \sum_{l=0}^{n} \frac{F_{t\theta}^{(l+1)}}{r^{l+1}} \qquad F_{t\phi} = \sum_{l=0}^{n} \frac{F_{t\phi}^{(l+1)}}{r^{l+1}} \quad (5.17)$$

$$F_{r\theta} = \sum_{l=0}^{n} \frac{F_{r\theta}^{(l+2)}}{r^{l+2}} \qquad F_{r\phi} = \sum_{l=0}^{n} \frac{F_{r\phi}^{(l+2)}}{r^{l+2}} \quad (5.18)$$

Hence, the last two terms

$$\int_\Gamma dtd\Xi r^2 F^{t\theta}\partial_\theta\lambda + \int_\Gamma dtd\Xi r^2 F^{t\phi}\partial_\phi\lambda \quad (5.19)$$

$$= \int_\Gamma dtd\Xi r^2 \frac{F_{t\theta}^{(l+1)}}{r^{l+3}}\partial_\theta\lambda_l(\theta,\phi) r^l + \int_\Gamma drd\Xi r^2 \frac{F_{t\phi}^{(l+1)}}{r^{l+3}}\partial_\phi\lambda_l(\theta,\phi) r^l \quad (5.20)$$

In the equation (5.16) the first term gives zero charges because solutions of wave equation, obtained by Lorenz gauge fixing, are Hankel functions and they show that charges, which are related to time-dependent, λ vanish asymptotically.

$$\lim_{r\to\infty}\int_\Gamma dtd\Xi r^2 F^{r\mu}\partial_\mu\lambda = \int_{\Gamma_\infty} dtd\Xi r^2 F_{rt}^{(2)}\partial_t\lambda = 0 \quad (5.21)$$

This shows that the leakage from lateral surface Γ at spatial infinity vanishes under the assumption and asymptotic boundary conditions, which are imposed on gauge invariant fields.

6. CONCLUSION

The mathematical tools were utilized to reveal the link between symmetry transformations and conserved quantities. They are exemplified for different systems and set ground for the rest. This dissertation is mainly based on gauge transformations and related conserved charges. Few approaches on gauge symmetry that were already published aggregated and derived in detail. The main point of this dissertation is to show the equivalence of charge definitions on different surfaces as in the chapter 5. However, two different subclasses of gauge transformations, asymptotic and residual gauge transformations needs to further work to compare the associated charges and understand more.

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APPENDIX A: HYPERSURFACES ANS STOKES' THEOREM

In this appendix, a few basic calculations used above will be discussed in detail following [22].

A.1. Normal Vectors

In d-dimensional space-time manifold, $d-1$ dimensional submanifold is a hypersurface. Hypersurface Σ is be selected by imposing a constraint on the coordinates, $\phi = 0$, or by putting parametric equations in the form of $x^a = x^a(y^a)$, where y^a are coordinates on the hypersurface. The values of ϕ changes only in the direction of normal

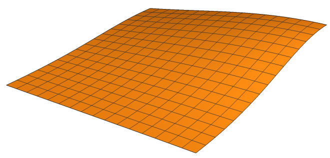

Figure A.1: A hypersurface with $\phi = 0$ in 3D

vector on hypersurface, Σ. If n_α is not null-vector, by normalizing

$$n_\alpha = \frac{\epsilon \partial_\alpha \phi}{|g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi|^{1/2}} \tag{A.1}$$

where ϵ is determined by

$$n^\alpha n_\alpha = \epsilon \tag{A.2}$$

where the normal vector of the surface is not null. If denominator is zero, the hypersurface is null, and the normal vector is not normalizable.
Then, the sign of null vector k is chosen, as $k_\alpha = -\partial_\alpha \phi$.

$$k^\alpha k_\alpha = 0 \tag{A.3}$$

It means null vectors are both orthogonal and tangent to the null hypersurface, Σ.

A.2. Induced Metric

The hypersurface Σ in 4D space-time can be described by a constraint $\phi(x_i) = 0$. The metric of 4D space-time

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{A.4}$$

The induced metric can be thought as coordinate transformation where the constraint imposed on new coordinates.

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^i} dy^i \frac{\partial x^\alpha}{\partial y^j} dy^j \\ &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\alpha}{\partial y^j} dy^i dy^j \end{aligned}$$

Now, the induced metric h can be written as,

$$h_{ij} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\alpha}{\partial y^j} \tag{A.5}$$

A.2.1. Example

$$ds^2 = -du^2 - 2dudr - r^2 \gamma_{z\bar{z}} dz d\bar{z} \tag{A.6}$$

induced metric on u constant surface, Σ_u.

$$\phi = u = 0 \tag{A.7}$$

Coordinate transformation form $(u, r, z,\bar{z})$ to $(r, z,\bar{z})$

$$\begin{aligned} h_{rr} &= g_{ru}\frac{\partial r}{\partial r}\frac{\partial u}{\partial r} + g_{uu}\frac{\partial u}{\partial r}\frac{\partial u}{\partial r} && \text{(A.8)} \\ &= 0 && \text{(A.9)} \end{aligned}$$

where h_{rz}, $h_{r\bar{z}}$ are obviously zero, and $h_{z\bar{z}}$, $h_{\bar{z}z}$ are equal to $g_{z\bar{z}}, g_{\bar{z}z}$ respectively.

$$ds^2_{\Sigma_u} = -r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{A.10}$$

A.3. Integration on Hypersurfaces

For non-null hypersurfaces,

$$d\Sigma \equiv |h|^{(1/2)}d^3y \tag{A.11}$$

In general,

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial y^1}\frac{\partial x^\beta}{\partial y^2}\frac{\partial x^\gamma}{\partial y^3}d^3y \tag{A.12}$$

where $\epsilon_{\mu\alpha\beta\gamma}$ is $\sqrt{g}$ times permutation of $[\mu\alpha\beta\gamma]$ By using (A.11) and (A.12), directed hypersurface becomes

$$d\Sigma_\alpha = \epsilon n_\alpha d\Sigma \tag{A.13}$$

On null hypersurface, $(\nabla_\beta k^\alpha)k^\beta$ is proportional to k^α.

$$\nabla_\beta\nabla_\alpha\phi\nabla^\beta\phi = (\nabla_\alpha\nabla_\beta\phi)\phi^\beta = \frac{1}{2}\nabla_\alpha(\nabla_\beta\phi\nabla^\beta\phi) \tag{A.14}$$

It is zero on Σ because $\nabla_\beta\phi\nabla^\beta\phi = 0$. Its gradient must be parallel to k_α because only change of function can be along the vector k_α. Then,

$$\nabla_\beta\nabla_\alpha\phi\nabla^\beta\phi \;=\; (\partial_\beta\partial_\alpha\phi)\partial^\beta\phi + \Gamma^c_{\beta\alpha}\partial_c\phi\partial^\beta\phi \propto k^\alpha \tag{A.15}$$

where k^α is tangent to null geodesics, $\nabla_\beta\nabla_\alpha\phi\nabla^\beta\phi$. The geodesics, parametrized by σ imposes $dx^\alpha = k^\alpha d\lambda$. Then, $d^3y = d\lambda d^2y$

$$d\Sigma_\alpha = -k_\alpha\sqrt{s}d^2yd\lambda \tag{A.16}$$

where $\sqrt{s}$ is the square root of the absolute value of the metric on the null hypersurface.

A.4. Stokes' Theorem

The general form of Stokes' s theorem is

$$\int_{\mathcal{V}} d\omega = \int_{\partial\mathcal{V}} \omega \tag{A.17}$$

in form language.

The adapted version to calculations above is for any vector field A^μ defined within a finite region $\mathcal{V}$ of 4D space-time manifold

$$\int_{\mathcal{V}} d^4x\sqrt{g}\nabla_\mu A^\mu = \oint_{\partial\mathcal{V}} d\Sigma_\mu A^\mu \tag{A.18}$$

where $\mathcal{V}$ bounded by a closed surface $\partial\mathcal{V}$.

APPENDIX B: PENROSE DIAGRAM

Penrose diagrams help to imagine infinities by bringing them in finite positions. It is called conformal compactification, obtained by a series of coordinate transformations. The Penrose diagram is discussed below relying on [23] and [24]. First,

$$u = t - r \tag{B.1}$$

$$v = t + r \tag{B.2}$$

where $r > 0$, $\infty > t > -\infty$. Hence, $v > u$ for all t,r and $\infty > v > -\infty$, $\infty > u > -\infty$. The behaviour of arctan at infinities makes it a candidate to compactify space-time.

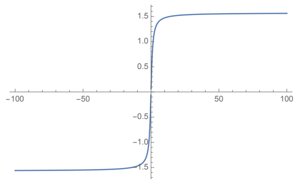

Figure B.1: Plot of arctan(x)

$$p = \arctan(u) \tag{B.3}$$

$$q = \arctan(v) \tag{B.4}$$

where $\frac{\pi}{2} > p > -\frac{\pi}{2}$, $\frac{\pi}{2} > q > -\frac{\pi}{2}$ and $p \geqslant q$

$$T = q + p \tag{B.5}$$

$$R = q - p \tag{B.6}$$

where $\pi \geqslant T \geqslant -\pi$, $\pi \geqslant R \geqslant 0$, $R \geqslant 0$. The metric becomes,

$$ds^2 = [\frac{1}{4}\sec^2(1/2(T+R))\sec^2(1/2(T-R))]\left\{-dT^2 + dR^2 + \sin^2 R(d^2\theta + \sin^2\theta d^2\phi)\right\}$$

$$ds^2 = [\frac{1}{4}\sec^2(1/2(T+R))\sec^2(1/2(T-R))]d^2\bar{s}$$

As a consequence, the initial metric g_{ab} is related to transformed metric $\bar{g}_{ab}$ such that

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{B.7}$$

. $\bar{g}_{ab}$ is conformally related to g_{ab}. In conformally related metrics, null geodesics are the same in the sense that ingoing and outgoing null geodesics are straight lines at $-45°$, $45°$, respectively in the Penrose diagrams as in the lightcone. Suppressing two dimensions (θ, ϕ), a Penrose diagram looks like

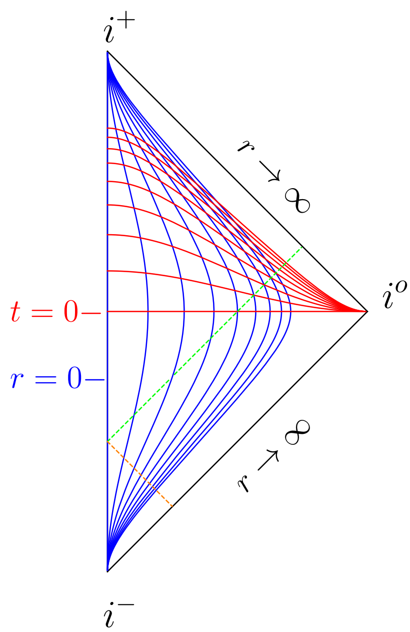

Figure B.2: Penrose diagram: red curves and blue curves are t and r-constant lines, respectively, The dashed green line is u-constant (outgoing radial geodesic), the dashed orange line is v-constant (ingoing radial geodesic)

$$(p = \tfrac{1}{2}\pi,\ q = \tfrac{1}{2}\pi) \text{ is } i^0$$
$$(p = \tfrac{1}{2}\pi,\ q = -\tfrac{1}{2}\pi) \text{ is } i^-$$
$$(p = -\tfrac{1}{2}\pi,\ q = \tfrac{1}{2}\pi) \text{ is } i^+$$

The geodesics of massive particles start from i^- and end up at i^+.