CLASSIFICATION OF NONSYMMETRIC RIEMANNIAN MANIFOLDS USING HOLONOMY GROUPS

by

Bora Ferlengez

B.S., Mathematics, Boğaziçi University, 2005

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics

Boğaziçi University

2008

Anneanneme.

ACKNOWLEDGEMENTS

I am deeply indepted to the following people for this thesis. Without them, I would have studied, learned, produced and more importantly enjoyed mathematics and life much less than I do now.

I would like to express my sincere gratitude to my thesis supervisor Sadık Değer for his endless patience and understanding. He set an example to me not just for mathematics and how to research, but how to be a scientist.

I am thankful to Ferit Öztürk for introducing me to the world of geometry and for being that supportive and encouraging all that years.

I am grateful to Ali Kaya for participating in my thesis committee and sharing his valuable comments and questions.

I am deeply thankful to my officemates, especially to Ercan Atam, Fatih Demirkale, Osman Can İçöz, Deniz Ali Kaptan, Yasemin Kara, Ulaş Karakoç and Birkan Yılmaz. It was an honour for me to work with them. Recep Duygu Ant, Elçim Elgün Kırımlı, Semih Özlem, İhsan Ata Topaloğlu and Murat Sağlam deserve an accolade for their support in this thesis and for the best years of my life.

And my family. Special thanks owed to my life partner Zeynep Şeref for being what she is, and extra special thanks for loving and enduring me all those years. I am mostly indepted to my parents Ahmet Kemal and Şenay Ferlengez and to my sister Burcu Ferlengez for being such a magnificent family. I don't know how to repay, a simple appreciation here would be necessary but certainly not sufficient.

ABSTRACT

CLASSIFICATION OF NONSYMMETRIC RIEMANNIAN MANIFOLDS USING HOLONOMY GROUPS

In this thesis, Simons' proof of Berger's classification of nonsymmetric irreducible Riemannian manifolds with respect to their holonomy groups is studied and Berger's classification is discussed.

The main tools will be principal fibre bundles and vector bundles. Using them, the Ambrose-Singer theorem is investigated, which relates the geometric meaning of curvature to holonomy groups and forms the basis of Simons' proof.

ÖZET

SİMETRİK OLMAYAN RIEMANN MANİFOLDLARININ HOLONOMİ GRUPLARI KULLANILARAK SINIFLANDIRILMASI

Bu tezde Berger'in simetrik olmayan indirgenemez Riemann manifoldlarını holonomi gruplarına göre sınıflandırmasının Simons tarafından yapılan ispatı incelenmiş ve Berger'in sınıflandırması sonucu ortaya çıkan sınıflar yorumlanmıştır.

Asal lif demetleri ve vektör demetleri temel araç olarak kullanılmıştır. Bu araçlar kullanılarak eğriliğin geometrik anlamı ile holonomy gruplarını ilişkilendiren ve Simons'un ispatının temelini oluşturan Ambrose-Singer Teoremi incelenmiştir.

TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

$:=$	definition
$\approx$	isomorphic or diffeomorphic depending on the context
$\square$	end of proof
$\otimes$	tensor product
$\wedge$	wedge product
$\oplus$	direct sum
$\times$	cartesian product
$[\ ,\ ]$	Lie bracket
$\circ$	composition
$\hat{}$	omit the corresponding term

$A^{\#}$	fundamental vector field
A^T	transpose of a matrix A
$\bar{A}$	conjugate of a matrix A
$[a]$	equivalence class of a
ad	adjoint representation
D	exterior covariant derivative on a principal fibre bundle
d	exterior derivative
dim	dimension of a manifold
E	total space of a vector bundle
$E(M, F, G, P)$	associated bundle to the principal bundle $P(M, G)$ with standard fibre F, structure group G
e	identity element of a Lie group
e_i	basis of a vector space
F^E	frame bundle associated to a vector bundle E

F^M	frame bundle of a manifold M
G, H	Lie groups
G_p	vertical subspace at p of a principal fibre bundle
$\mathfrak{g}$,$\mathfrak{h}$	Lie algebras of the Lie groups G and H, respectively
$\mathfrak{g}^*$	space of Maurer-Cartan forms on a Lie group G
$GL(n, \mathbb{R})$	general linear group
$\mathscr{H}(p)$	holonomy group of a principal fibre bundle P at $p \in P$
I_n	n-by-n identity matrix
L_a	left translation by a
M	(Riemannian) manifold
m	point in (base) manifold M
$O(n)$	orthogonal group over $\mathbb{R}$
P	total space in a principal fibre bundle
$P(M, G)$	principal fibre bundle over M with structure group G
R	curvature tensor
R_a	right translation by a
S^n	n-sphere in $\mathbb{R}^{n+1}$
$SL(n, \mathbb{R})$	special linear group over $\mathbb{R}$
$SO(n)$	special orthogonal group over $\mathbb{R}$
$Sp(n)$	symplectic group
$SU(n)$	special unitary group
TM	tangent bundle of the manifold M
T_mM	tangent space of the manifold M at the point $m \in M$
$U(n)$	unitary group
X_m	the valuation of a vector field X at the point p of a manifold
X, Y	vector fields
V, W	vector spaces
$[V, R, G]$	holonomy system

α_*	push-forward of the map α
α^*	pull-back of the map α
$\gamma_{p\to q}$	parallel transport on M induced by the curve γ
γ_t	1-parameter group of transformations
φ	cross section on a vector bundle
$\nabla_X\varphi$	covariant derivative in direction of the vector field X of the cross section φ on a vector bundle
Ω	curvature 2-form on a principal fibre bundle
$\Omega^r(M)$	r-form on the manifold M
ω	(connection) 1-form
ω_α	connection 1-form in the local trivialization
π	projection (of a principal fibre bundle)
ψ_α	local trivialization of a principal fibre bundle
$\psi_{\alpha\beta}$	transition function
ρ	representation of a Lie group G on a vector space V
θ	canonical 1-form on P

ODE	ordinary differential equation

1. INTRODUCTION

Groups are among the most fruitful structures in mathematics. It is natural to try to capture some part of Riemannian geometry applying group theory. This can be done, for example, by holonomy groups using parallel transport. Given two points p and q in a Riemannian manifold M, and a curve γ connecting p to q in M, the parallel transport yields an isometric linear isomorphism $\gamma_{p\to q} : T_pM \to T_qM$ between the tangent spaces of p and q, respectively. Considering $p = q$, a curve gives us an isometric linear isomorphism $\gamma_{p\to p}$ of T_pM onto itself, namely an element of $O(n)$, i.e. the orthogonal group of linear isomorphisms.

Using linear isomorphisms induced by parallel transports, one can define a group, called the holonomy group, at a point p of a Riemannian manifold M, where the composition of two loops corresponds to a product of two Euclidean isomorphisms on T_pM.

Definition 1.1. The subgroup of $O(n)$ consisting of linear isomorpisms in $\gamma_{p\to p} : T_pM \to T_pM$ for a point p in a Riemannian manifold M which are induced by the path γ as γ varies over all possible loops in M based at p is called the *holonomy group* $\mathscr{H}(p)$ at $p \in M$.

Using definition 1.1, we can construct a group structure at each point of a Riemannian manifold. Since we expect the group structure to reflect the Riemannian structure on the manifold strongly and yield some classification of the Riemannian manifolds, we need to extend that group structure defined pointwise above to the whole manifold. This is done by observing that any path η connecting $p, q \in M$ yields an isomorphism $\eta\gamma\eta^{-1} : T_qM \to T_qM$ using the elements of the holonomy group at p. So if one can connect two points $p, q \in M$, then the holonomy groups $\mathscr{H}(p)$ and $\mathscr{H}(q)$ are conjugate to each other. Hence we can speak of *the* holonomy group $\mathscr{H}$ of the manifold M.

The holonomy group $\mathscr{H}$ turns out to be a strong tool. It yields some classifications of Riemannian manifolds with certain properties. For example, such a classification was made by Cartan [1], [2], where he classified symmetric spaces using holonomy groups. We will comment on this classification in section 4.1, for more details see 10.69 and 10.70 of Besse [3]. In 1950's holonomy groups were studied extensively by Berger [4], Borel and Lichnerowicz [5].

After some time gap, in 1990s, the study of holonomy became active in the hands of Bryant, Chi, Joyce, Merkulov, Salamon and Schwachhofer, motivated by its applications in the string theory and mathematical physics. For example, Joyce found the first compact examples of Riemannian manifolds with holonomy group G_2 and $Spin(7)$ in [6] and [7]. On the other hand, Bryant, Chi, Merkulov and Schwachhofer managed to redefine and complete Berger's less known classification of irreducibly acting holonomy groups of torsion-free connections that are not locally symmetric, which are called *exotic holonomies*, see for example [8], [9] and [10]. For more on history of holonomy groups, see Besse [3] or Bryant [11].

We will use holonomy groups to state and prove Berger's classification theorem [4] on nonsymmetric Riemannian manifolds, published in 1955. Since Cartan classified the symmetric spaces in [1] and [2], Berger focused on the nonsymmetric spaces and classifies them. Moreover, although the notion of a holonomy group can be defined for a more general setting than a Riemannian manifold (for example in a vector bundle, on which a linear connection is defined), in such a setting there exists no classification theorem. The reason is the result of Hano and Ozeki [12], which states that any connected Lie subgroup of $GL(n, \mathbb{R})$ can be realized as the holonomy group of a linear connection.

Berger's original proof consists of long calculations. Some years later, in 1962, Simons [13] managed to give a shorter and more geometric proof of that classification. After noting that the groups arising in Berger's theorem are as those in the list of Lie groups acting transitively on the unit sphere [14], Simons managed to prove that for nonsymmetric spaces, the holonomy group $\mathscr{H}$ is not transitive on the unit sphere, hence he reached Berger's theorem in a shorter way.

The aim of this thesis is to understand the classification of nonsymmetric Riemannian manifolds. To reach there, we will work on holonomy groups and prove Ambrose-Singer Theorem, on which Simons' proof relies heavily.

In the second chapter, we will give some preliminaries for the theory of holonomy groups, which consists basically of some results on Lie groups and fibre bundles, and the definitions of some notions in the theory of connections in the principal bundle setting.

In the third chapter, we will first describe holonomy groups in the principal bundle setting and prove the Ambrose-Singer Theorem. The rest of this chapter consists of the translation of the statement of Ambrose-Singer Theorem in the principal bundle setting to the version of it in the vector bundle setting, in where Riemannian manifolds fit naturally and which is used in Berger's classification theorem.

In the fourth and last chapter, we will state Berger's classification theorem and after some simplifications, we will pass to Simons' proof. Rest of the chapter consists of Simons' proof, which is basically two simultaneous inductions relying on an algebraic structure Simons constructs inspired by the Ambrose-Singer Theorem. Finally, we will make some comments on the classes appearing in Berger's list.

Some good references for the theory of holonomy groups are Salamon [15], Joyce [16], Besse [4] and Kobayashi and Nomizu [17].

2. PRELIMINARIES

In this chapter, we will give some background material to fix notations, conventions and definitions which will be useful in understanding the rest of the text. This consists of two main parts. In the first part (sections 2.1 and 2.2), an introduction to the theories of Lie groups and (principal) fibre bundles will be given.

In the second part (sections 2.3,2.4 and 2.5), we will define a connection on a principal fibre bundle, prove its existence, define horizontal lifts of vector fields and curves on the base manifold of a principal fibre bundle and parallel displacement along a curve in the base manifold. Later on, we will introduce the notions of holonomy groups and curvature form on a principal fibre bundle.

We assume that the reader is familiar with the basic properties of topological spaces and Riemannian geometry. We will use Einstein summation convention, that is repeated indices are summed over. We will take a *differentiable (or smooth) manifold* M to be an ordered pair consisting of a Hausdorff, second countable, locally Euclidean topological space and a maximal smooth atlas on this topological space. For introduction to the theory of differentiable manifolds, Boothby [18] and Conlon [19] are good references.

2.1. Lie groups

Before beginning to summarize some results about the Lie theory, we will make some remarks about transformations and local 1-parameter groups of local transformations on a manifold. A *transformation* is simply a diffeomorphism from a manifold M onto itself. A *1-parameter group of transformations* on M is a mapping $\varphi : (t, p) \in \mathbb{R} \times M \mapsto \varphi_t(p) \in M$ such that for each t, $\varphi_t : M \to M$ is a transformation on M and for all $s, t \in \mathbb{R}$, φ_t at $p \in M$ satisfies $\varphi_{s+t}(p) = \varphi_t(\varphi_s(p))$. Each 1-parameter group of transformations defines a vector field X as follows: Given $p \in M$, let X_p be the tangent vector to $\varphi_t(p)$ at $\varphi_0(p) = p$.

A *local 1-parameter group of local transformations* is defined the same way: Given a point $p \in M$, it is a mapping $\varphi : (-\epsilon, +\epsilon) \times U \rightarrow M$ for some positive real number ϵ and an open neighborhood U of p, where

(a) For each $t \in (-\epsilon, +\epsilon)$, $\varphi_t : U \rightarrow \varphi_t(U)$ is a diffeomorphism,
(b) For $s, t, s+t \in (-\epsilon, +\epsilon)$, $\varphi_{s+t}(p) = \varphi_t(\varphi_s(p))$.

A local 1-parameter group of local transformations induces a local vector field as above. The converse is true, as well.

Proposition 2.1. *Given a vector field X on M, for each point $p_0 \in M$, there is a neighborhood U of p_0, a positive real number ϵ and a local 1-parameter group of transformations $\varphi_t : U \rightarrow M$ for $t \in (-\epsilon, +\epsilon)$, which induces the given X.*

Proof. This proposition follows from the existence and uniqueness theorem of ODE's, see page 189 of Nakahara [20]. □

Proposition 2.2. *Let φ be a transformation of M. If a vector field X generates a local 1-parameter group of local transformations φ_t, then the vector field $\varphi_* X$ generates the local 1-parameter group of local transformations $\varphi \circ \varphi_t \circ \varphi^{-1}$, where φ_* is the push-forward of the smooth map φ.*

Proof. Note that $\varphi \circ \varphi_t \circ \varphi^{-1}$ is a local 1-parameter group of local transformations. Let $p \in M$, $q = \varphi^{-1}(p)$. Since φ_t induces X, the vector $X_q \in T_q M$ is tangent to the curve $x(t) = \varphi_t(q)$ at $q = x(0)$. Then the vector $(\varphi_* X)_p = \varphi_*(X_q) \in T_p M$ is tangent to the curve $y(t) = \varphi \circ \varphi_t(q) = \varphi \circ \varphi_t \circ \varphi^{-1}(p)$. □

Now we can start giving some results from the theory of Lie groups. Along the section, we will generally make use of Warner [21] and Helgason [22].

Definition 2.3. A *Lie group* G is a differentiable manifold which is also endowed with a group structure such that the map $(a, b) \in G \times G \to ab^{-1} \in G$ is smooth.

In the following, we will use G and H to denote the Lie groups and e to denote the identity element of a Lie group.

We define the left translation L of G by an element $a \in G$ by $L_a g = ag$ (respectively right translation R by $R_a g = ga$). A vector field X on G is called *left* (resp. right) *invariant* if it is left invariant by all left (resp. right) translations L_a (resp. R_a), i.e. $(L_a)_* X_g = X_{ag}$ (resp. $(R_a)_* X_g = X_{ga}$) for all $a, g \in G$.

Definition 2.4. A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear bracket operator, $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying anti-commutativity and Jacobi identity, namely for $X, Y, Z \in \mathfrak{g}$:

(a) $[X, Y] = -[Y, X]$
(b) $\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0$

On a manifold, the *Lie bracket* of two vector fields X, Y is defined as

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}[Y - (\varphi_t)_* Y]$$

where φ_t is the local 1-parameter group of local transformations generated by the vector field X.

The bracket operation above satisfies the properties of 2.4, so the vector fields on a manifold M, denoted by $\mathfrak{X}(M)$ form a Lie algebra. The left invariant vector fields form a

Lie algebra (namely a Lie subalgebra of $\mathfrak{X}(G)$), as well, by the following proposition.

Proposition 2.5. *Let G be a Lie group and $\mathfrak{g}$ the set of left invariant vector fields of G. Then*

(a) *$\mathfrak{g}$ is a vector space and the map $\alpha : X \in \mathfrak{g} \mapsto X_e \in T_eG$ is an isomorphism from $\mathfrak{g}$ to the tangent space T_eG of the identity element $e \in G$. Therefore* $\dim \mathfrak{g} = \dim T_eG = \dim G$.

(b) *Left invariant vector fields are smooth.*

(c) *The Lie bracket of two left invariant vector fields is itself a left invariant vector field, i.e. $[X, Y] \in \mathfrak{g}$ if $X, Y \in \mathfrak{g}$.*

(d) *$\mathfrak{g}$ forms a Lie algebra under the Lie bracket operation on vector fields.*

Proof. The proof follows from the very definitions, see page 85 of Warner [21]. □

Let $A \in \mathfrak{g}$ be a left invariant vector field on G. As any other vector field, it defines a local 1-parameter group of local transformations φ_t through $e \in G$. In fact this 1-parameter group of local transformations is global. Choose $\epsilon \in \mathbb{R}$ such that $\varphi_t e$ is defined for $|t| < \epsilon$. Then $\varphi_t a$ is defined for any $a \in G$ and $\varphi_t a = L_a(\varphi_t e)$, since being induced by a left invariant vector field, φ_t commutes with L_a for any a. Since $\varphi_t a$ is defined for all $|t| < \epsilon$ and for all $a \in G$, pasting 2ϵ-arcs yields that $a_t := \varphi_t a$ is defined for $|t| < \infty$. Note that $a_{t+s} = a_t a_s$ for all $r, s \in \mathbb{R}$. So, we call a_t the *1-parameter subgroup of G generated by $A \in \mathfrak{g}$*. Since a_t is defined for all t, in particular, $a_1 = \varphi_1 e$ is also defined. We denote a_1 by $\exp A$, the mapping $A \in \mathfrak{g} \mapsto \exp A \in G$ is called the *exponential mapping*. It follows that $\exp tA = a_t$.

Definition 2.6. H is a *Lie subgroup* of a Lie group G if there is a map $\varphi : H \to G$ such that (H, φ) satisfies the following conditions:

(a) H itself is a Lie group,

(b) H is a submanifold of G by φ, i.e. $\varphi : H \to G$ is a smooth 1-1 immersion,

(c) $\varphi : H \to G$ is a group homomorphism.

In other words, H is a subgroup of G which is at the same time a submanifold of G such that H itself is a Lie group with respect to the differentiable structure of G.

Knowing left invariant vector fields motivates us to define left invariant 1-forms. A differential form ω on G is called left invariant, if

$$(L_a)^*\omega = \omega$$

for any $a \in G$, where $(L_a)^*$ denotes the pull-back of the map L_a. They are automatically smooth by the virtue of left invariance as in the case of proposition 2.5,(b). They are known as *Maurer-Cartan forms.* If $A \in \mathfrak{g}$ and ω is a Maurer-Cartan form, then the function $\omega(A)$ is constant on G, so Maurer-Cartan forms form the dual space of $\mathfrak{g}$, denoted as $\mathfrak{g}^*$. Since exterior derivative commutes with pull-back of a smooth map, see page 200 of Nakahara [20], $\mathrm{d}\omega$ is a left invariant form, d denoting the exterior derivative of a form. Moreover, since for an arbitrary r-form

$$\begin{aligned}(d\omega)(X_0, X_1, \ldots, X_r) \quad = \quad & \frac{1}{r+1}\sum_{i=0}^{r}(-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r)) \\ & + \frac{1}{r+1}\sum_{0 \le i < j \le r}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r)\end{aligned} \tag{2.1}$$

holds by proposition 3.11 of Kobayashi and Nomizu [17], where ˆ means that the corresponding term is omitted, we derive the Maurer-Cartan equation:

$$\mathrm{d}\omega(A, B) = -\frac{1}{2}\omega([A, B]) \tag{2.2}$$

for a 1-form $\omega \in \mathfrak{g}^*$ and $A, B \in \mathfrak{g}$.

Lie groups arise in many natural ways as transformation groups of differentiable manifolds.

Definition 2.7. A Lie group G is a *Lie transformation group on a manifold* M, or G acts (differentiably) on M (on the right), if there is a mapping $R : G \times M \to M$,

(a) R is a differentiable mapping,
(b) For any $a \in G$, $R(a, \cdot) : M \to M$ is a transformation, namely a diffeomorphism from M to M, denoted by R_a,
(c) $R_{ab}(p) = R_b \circ R_a(p)$, for $a, b \in G$, $p \in M$,
(d) R_e is the identity transformation on M.

We shortly write, $pa = R_a(p)$. The left Lie group action is defined similarly, with the difference that the condition (c) of definition 2.7 is replaced by $L_{ab} = L_a \circ L_b$.

One kind of group action plays a fundamental role in many branches of mathematics: Representation of a Lie group G on a vector space, and in particular the adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$. A *Lie group homomorphism* is a smooth map between Lie groups which is at the same time a group homomorphism. A *Lie algebra homomorphism* σ is defined similarly, being a linear map from a Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{h}$, which preserves Lie brackets: $\sigma[X, Y] = [\sigma(X), \sigma(Y)]$ for X, $Y \in \mathfrak{g}$. We will shortly say a Lie homomorphism instead of a Lie group homomorphism or a Lie algebra homomorphism. For a finite dimensional real or complex vector space V, let $GL(V)$ denote the set of invertible linear transformations from V to itself. $GL(V)$ is a Lie group since any basis of V yields an isomorphism from $GL(V)$ to $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ depending on the dimension n of V and the field of scalars of V. If G is a Lie group, a *(finite dimensional) representation of* G is a Lie homomorphism $\rho : G \to GL(V)$. Any representation yields a smooth left action of G on V defined by $L_g v := \rho(g)v$ for $g \in G$, $v \in V$.

Consider the natural Lie group action of G on itself on the left by the *conjugation map*:

$$\begin{aligned} C_g : G &\rightarrow G \\ h &\mapsto ghg^{-1} \end{aligned} \tag{2.3}$$

Then C_g is a Lie homomorphism, since multiplication is smooth and it is a group homomorphism by simple calculation. Let $\mathrm{ad}(g) := (C_g)_* : \mathfrak{g} \rightarrow \mathfrak{g}$ denote its induced Lie algebra homomorphism. Then $\mathrm{ad} : G \rightarrow GL(\mathfrak{g})$ is a representation of G on the vector space $\mathfrak{g}$, see page 211 of Lee [23], called the *adjoint representation of G*.

Some Lie group actions deserve to have a special name:

Definition 2.8. (1) G acts *effectively* on M, if $R_a p = p$ for all $p \in M$ implies that $a = e$.

(2) G acts *freely* on M, if $R_a p = p$ for some $p \in M$ implies that $a = e$.

(3) G acts *transitively* on M, if for every $p \in M$ and $q \in M$, there exists an element $a \in G$ such that $R_a p = q$.

If G acts on M on the right, we assign to each element $A \in \mathfrak{g}$ a natural vector field, denoted by $A^\#$ on M which is obtained by the action of 1-parameter subgroup $a_t = \exp tA$ on M.

Proposition 2.9. *Let a Lie group G act on a manifold M on the right. The assignment defined above $\sigma : A \in \mathfrak{g} \mapsto A^\# \in \mathfrak{X}(M)$ is a Lie algebra homomorphism. If G acts effectively on M, then σ is an injection of $\mathfrak{g}$ into $\mathfrak{X}(M)$. If G acts freely on M, then $\sigma(A)$ never vanishes on M, for all non-zero $A \in \mathfrak{g}$.*

Proof. Define a map $\sigma_p : a \in G \mapsto pa \in M$ for each $p \in M$. Then $(\sigma_p)_*(A_e) = \frac{\mathrm{d}}{\mathrm{d}t}(p \exp tA)\Big|_{t=0} = (\sigma A)_p$. So σ is a linear mapping. Let $A, B \in \mathfrak{g}$, $A^\# = \sigma A$, $B^\# = \sigma B$ and $a_t = \exp tA$. Then by the definition of Lie bracket for vector fields on a manifold,

$$[A^\#, B^\#] = \lim_{t \to 0} \frac{1}{t}[B^\# - R_{a_t} B^\#]$$

Since $R_{a_t} \circ \sigma_{pa_t^{-1}}(g) = xa_t^{-1} g a_t$, for $g \in G$,

$$((R_{a_t})_* B^\#)\big|_p = (R_{a_t})_* \circ (\sigma_{pa_t^{-1}})_* B_e = (\sigma_p)_*(\mathrm{ad}(a_t^{-1})B_e)$$

Then

$$[A^\#, B^\#] = (\sigma_p)_* \Big(\lim_{t \to 0} \frac{1}{t}[B_e - \mathrm{ad}(a_t^{-1})B_e] \Big) = (\sigma_p)_*([A, B]_e) = (\sigma[A, B])_p$$

So σ commutes with the bracket, and hence σ is a Lie homomorphism.

Now, assume that $\sigma A = 0$ everywhere. Then R_{a_t} becomes the identity transformation of M for every t. If G acts effectively on M, then $a_t = e$ for each t and thus $A = 0$, σ is an injection.

Lastly, assume that σA vanishes at some point $p \in M$. Then $R_{a_t} p = p$ for all t. If G acts freely on M, this means that $a_t = e$ for each t and therefore $A = 0$. □

2.2. Fibre Bundles

In this section, we will introduce principal fibre bundles. Note that, as we will show vector bundles can be constructed from principal bundles. We first define a principal fibre bundle:

Definition 2.10. Let M be a differentiable manifold, and G be a Lie group. A *principal fibre bundle* $P(M, G)$ *over* M *with fibre* G is a manifold P equipped with a differentiable projection $\pi : P \to M$, and a right action of G on P, denoted as $(u, g) \in P \times G \mapsto u \cdot g = R_g u \in P$ such that

(a) G acts on P freely and differentiably,

(b) M is the quotient space of P by the equivalence relation induced by G,

(c) P is locally trivial, i.e. every point $m \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G$ by a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$, where φ is a mapping of $\pi^{-1}(U)$ to G satisfying that $\varphi(ug) = (\varphi(u))g$ for $u \in \pi^{-1}(U)$, $g \in G$.

The projection π is sometimes also called a *fibration*. Note that each *fibre* $\pi^{-1}(m)$, which is the orbit of G-action, is a copy of G. P is called the *total space* or *bundle space*, M is called the *base space*, and G is called the *structure group*.

Now we define fundamental vector fields on a principal fibre bundle, which we make use of in the theory of connections in section 2.3.

Definition 2.11. Let $P(M, G)$ be a principal fibre bundle. The action of G on P induces a homomorphism $\sigma : \mathfrak{g} \to \mathfrak{X}(P)$ by proposition 2.9. The vector field $A^{\#} = \sigma(A)$ for $A \in \mathfrak{g}$ is called a *fundamental vector field*.

Since G acts fibrewise on P, $A^{\#}_u$ is tangent to the fibre $G_m = \pi^{-1}(m)$ for $m \in M$, $u \in \pi^{-1}(m) \in P$. Moreover since G acts freely on P, by proposition 2.9, $A^{\#}$ never vanishes on P for nonzero $A \in \mathfrak{g}$. Thus, since $\dim G_m = \dim \mathfrak{g}$, σ is a linear isomorphism of $\mathfrak{g}$ onto the tangent subspace at u through the fibre G_m of u.

Proposition 2.12. *Let $A^{\#}$ be the fundamental vector field corresponding to $A \in \mathfrak{g}$. For $a \in G$, $(R_a)_* A^{\#}$ is the fundamental vector field corresponding to* $(\mathrm{ad}(a^{-1}))A \in \mathfrak{g}$.

Proof. Remember that $A^{\#}$ is induced by the 1-parameter group of transformations R_{a_t}, where $a_t = \exp tA$. Then by proposition 2.9, $(R_a)_* A^{\#}$ is induced by the 1-parameter group of transformations $R_a R_{a_t} R_{a^{-1}} = R_{a^{-1}a_t a}$. Since $a^{-1}a_t a$ is the 1-parameter group generated by $(\mathrm{ad}(a^{-1}))A \in \mathfrak{g}$, the proposition follows. □

Given a principal fibre bundle, we automatically get a family of transition functions. Let $P(M,G)$ be a principal fibre bundle, $\{U_\alpha, \psi_\alpha\}$ be a family of local trivializations and let $u \in P$ such that $\pi(u) \in U_\alpha \cap U_\beta$ for some U_α and U_β. We define the transition functions as $\psi_{\beta\alpha} : \pi(u) \in U_\alpha \cap U_\beta \mapsto \varphi_\beta(u)(\varphi_\alpha(u))^{-1} \in G$, where φ_α (resp. φ_β) is the second coordinate of the isomorphism $\psi_\alpha(u) = (\pi(u), \varphi_\alpha(u)) \in U_\alpha \times G$ (resp. ψ_β). Since $\varphi_\alpha(ua) = \varphi_\alpha(u)a$, for all φ_α and $a \in G$, $\psi_{\beta\alpha}$ depends only on $\pi(u)$ but not on $u \in P$ and the transition functions are well-defined. Moreover, they satisfy the identity

$$\psi_{\alpha\beta}(m) = \psi_{\alpha\gamma}(m) \cdot \psi_{\gamma\beta}(m) \tag{2.4}$$

for $m \in U_\alpha \cap U_\beta \cap U_\gamma$.

The converse holds, too. In other words,

Proposition 2.13. *Assume M is a differentiable manifold provided by an open cover $\{U_\alpha\}$ and G is a Lie group. If mappings $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ are given for each nonempty $U_\alpha \cap U_\beta$ satisfying equation 2.4, then there exists a principal fibre bundle $P(M,G)$ with transition functions $\psi_{\beta\alpha}$.*

For the construction see page 64 of Husemoller [24].

It will be important for us to study bundle homomorphisms and reduced bundles for the proof of Ambrose-Singer Theorem 3.9.

Definition 2.14. A *homomorphism f of a principal bundle $P'(M', G')$ into another principal bundle $P(M, G)$* consists of a mapping $f_1 : P' \to P$ and a Lie group homomorphism $f_2 : G' \to G$ such that it respects the principal bundle structure, i.e. $f_1(u'a') = f_1(u')f_2(a')$ for $u \in P$ and $a \in G$.

In the following, we will denote f_1 and f_2 also by the same letter f. Since f sends fibres of $P'(M', G')$ to fibres of $P(M, G)$, it induces a mapping M' into M, denoted also by the letter f.

A homomorphism f is called an *imbedding*, if the induced mapping $f : M' \to M$ is an imbedding and $f : G' \to G$ is a monomorphism. If f is an imbedding, we call image of $P'(M', G')$ under f a *subbundle* of $P(M, G)$. If the induced map $f : M' \to M$ is the identity mapping on M, then $P'(M, G')$ is called a *reduced bundle* and f a *reduction.* The structure group G of a principal fibre bundle $P(M, G)$ is said to be *reducible* to a Lie subgroup G' of G, if $P(M, G')$ is a reduced bundle of $P(M, G)$.

Vector bundles and principal fibre bundles are closely related, as mentioned before. Now, we will study this relationship. Firstly, we define a vector bundle:

Definition 2.15. Let M be a differentiable manifold, and G be a Lie group. A *vector bundle E over M with fibre G* is a manifold E equipped with a differentiable projection $\pi : E \to M$ such that

(a) Each fibre $E_m = \pi^{-1}(m)$ has a vector space structure, i.e. isomorphic to a vector space V, called the fibre of E, for $m \in M$,

(b) E is locally trivial, i.e. every point $m \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times V$.

Let us pass from a vector bundle to a fibre bundle, called the *frame bundle*:

Let M be a manifold, $E(M,\mathbb{R}^n)$ a vector bundle with fibre $\mathbb{R}^n$. Define a manifold

$$F^E := \big\{(m,e_1,e_2,\dots,e_k)\,|\,m\in M\,,\,(e_1,\dots,e_k)\text{ is a basis for }E_m\big\}$$

where points $m\in M$ are associated with all linear frames at m, where a *linear frame u at $m\in M$* is an ordered basis $\{X_1,\dots,X_n\}$ of the tangent space T_mM. and a differentiable projection mapping $\tilde{\pi}:(m,e_1,e_2,\dots,e_k)\in F^E\mapsto m\in M$. Define a left action of $GL(k,\mathbb{R})$ on F^E by

$$A\cdot(m,e_1,e_2,\dots,e_k)=(m,e_1',e_2',\dots,e_k')=(m,A_{1j}e_j,A_{2j}e_j,\dots,A_{kj}e_k)$$

We can introduce a differentiable structure on F^E by taking the coordinates (x^i) of M and X_i^j's, where

$$(e_1,e_2,\dots,e_k)=(X_1^j\frac{\partial}{\partial x^j},X_2^j\frac{\partial}{\partial x^j},\dots,X_k^j\frac{\partial}{\partial x^j})$$

as a local coordinate system in $\pi^{-1}(U)$. Then $\pi^{-1}(U)\approx U\times GL(k,\mathbb{R})$ and F^E is a principal fibre bundle over M with fibre $GL(k,\mathbb{R})$. We call F^E as the *bundle of linear frames of E*, or shortly *frame bundle of E*. If $E=TM$, then we denote F^{TM} by F^M and call it the *frame bundle of M*. Here, TM denotes the *tangent bundle* of the manifold M, the vector bundle with basis M, standard fibre $\mathbb{R}^n$ and the structure group $GL(n,\mathbb{R})$.

Now we pass from a principal fibre bundle to a vector bundle, called the *associated bundle*. Let $P(M,G)$ be a principal fibre bundle, V be a vector space on which G has a representation ρ, namely there is a homomorphism ρ from G into $GL(V)$, the set of invertible transformations of V onto V. This representation induces a left action of G on V: $(a,\xi)\in G\times V\mapsto\rho(a)v\in V$. Take the product manifold $P\times V$. Then G acts on this product space on the right as $(u,\xi)\in P\times V\overset{a}{\mapsto}(ua,\rho(a)^{-1}\xi)\in P\times V$. Consider the quotient space of $P \times V$ by this action, denote it by $E = P \times_G V = \{[u, \xi] \mid u \in P, \xi \in V\}$, where $[u, \xi]$ is the equivalence class of (u, ξ): $\{(v, \eta) \mid (v, \eta) = (ua, \rho(a)^{-1}\xi)$ for some $a \in G\}$. Define the projection mapping $\pi_E : E \to M$ as $\pi_E([u, \xi]) := \pi(u)$. Since for every $m \in M$ there is a locally trivializable neighborhood U of m satisfying $\pi^{-1}(U) \approx U \times G$, the action of G on $\pi^{-1}(U) \times V$ is given as

$$(m, a, \xi) \xrightarrow{b} (m, ab, \rho(b)^{-1}\xi)$$

for $(m, a, \xi) \in U \times G \times V$, $b \in G$, after identifying $\pi^{-1}(U)$ with $U \times G$. Then the isomorphism $\pi^{-1}(U) \approx U \times G$ yields another isomorphism $\pi_E^{-1}(U) \approx U \times V$, where $\rho(G)$ acts freely on V. We introduce a differentiable structure on E by requiring that $\pi_E^{-1}(U)$ is an open submanifold diffeomorphic to $U \times V$ under the isomorpism $\pi_E^{-1}(U) \approx U \times V$. i.e. requiring that U is a local trivializable with respect to the projection map π_E. Then π_E becomes a differentiable mapping of E onto M and E becomes a vector bundle with fibre E.

$E(M, V, G, P)$ is said to be *the fibre bundle associated with the principal fibre bundle P, over the base M with standard fibre V and structure group G.*

Remark 2.16. The construction above can be generalized. Instead of taking a vector space as a fibre, one can build a fibre bundle with an arbitrary manifold F on which the structure group G acts on the left. Then we obtain $E(M, F, G, P)$, a fibre bundle with standard fibre F which is associated with $P(M, G)$.

Remark 2.17. The two constructions above give rise to a one-to-one correspondence between vector bundles over a manifold M with standard fibre $\mathbb{R}^k$ and principal bundles over M with fibre $GL(k, \mathbb{R})$. Since there are other Lie groups than $GL(k, \mathbb{R})$ which may arise as fibres of principal bundles over M, principal bundles are more general than vector bundles.

Remark 2.18. The tangent bundle TM over M with structure group $GL(n, \mathbb{R})$ is the bundle associated F^M with standard fibre $\mathbb{R}^n$. Then the fibre of TM over $m \in M$ turns out to be $T_m M$.

Next, we will state an important proposition from Kobayashi and Nomizu [17], which enables us to pass from the principal bundle setting to vector bundle setting in the theory of connections.

Proposition 2.19. *Let $P(M,G)$ be a principal fibre bundle and F a manifold on which G acts on the left. Let $E(M,F,G,P)$ be the fibre bundle associated with P. For each $u \in P$ and each $\xi \in F$, denote by $u\xi$ the image of $(u,\xi) \in P \times F$ by the natural projection $P \times F \to E = P \times_G F$ given above, namely the equivalence class of $(u,\xi) \in P \times F$ in E. Then each u is a mapping of F onto F_m, where $F_m = \pi_E^{-1}(m)$ and $m = \pi(u)$. Moreover*

$$(ua)\xi = u(a\xi)$$

for $u \in P$, $a \in G$, $\xi \in F$.

Due to the above proposition, we consider elements $u \in P$ as (invertible) mappings from the standard fibre F to the fibre $F_m = \pi_E^{-1}(m)$, where $m = \pi(u)$. By an isomorphism of a fibre $F_{m_1} = \pi_E^{-1}(m_1)$ onto another fibre $F_{m_2} = \pi_E^{-1}(m_2)$ with $m_1, m_2 \in M$, we mean a diffeomorphism represented by $v \circ u^{-1}$, where $u \in \pi^{-1}(m_1)$ and $v \in \pi^{-1}(m_2)$. In particular, if $m_1 = m_2$, then $v = ua$ for some $a \in G$. Therefore any automorphism of F_m can be represented in the form $u \circ a \circ u^{-1}$, where u is an arbitrary fixed point in $\pi^{-1}(m)$. So, the automorphism group of F_m is isomorphic to the structure group G.

2.3. Connections in a principal fibre bundle

A fundamental notion for a differentiable manifold is to be able to relate tangent spaces of distinct points in the manifold. Using the structure induced by a connection, one can specify how vectors, or more generally how tensors, are transported along a curve connecting two points in a manifold.

In the context of principal fibre bundles, transporting tensors along a curve γ on the base manifold will be done by using the horizontal lift $\hat{\gamma}$ of γ. Horizontal lifts are defined again using a connection, which this time leads to a correspondence between two fibres along γ.

Before defining a connection, we first define what a distribution on a manifold is. Given a manifold M, a *distribution* Δ of dimension r is an assignment which maps to each $m \in M$ an r-dimensional vector subspace Δ_m of the tangent space $T_m M$. A distribution Δ is called a *differentiable distribution*, if every point $m \in M$ has a neighborhood U and r differentiable vector fields $X_1, X_2, \ldots, X_r$ which form a basis for Δ_{m_1} for each point $m_1 \in U$.

Definition 2.20. Let $P(M, G)$ be a principal fibre bundle over a manifold M with projection π and group G. Let $p \in P$, $T_p P$ be its tangent space and G_p be the vectors that are tangent to the fibre through p. A *connection* Γ is a distribution $\Gamma : p \mapsto H_p$ such that

(a) $T_p P = G_p \oplus H_p P$,

(b) $H_{R_g p} = (R_g)_* H_p P$, i.e. Γ is invariant by the right action of G on P,

(c) H_p depends differentiably on p, i.e. Γ is a differentiable distribution.

Given a tangent vector $X \in T_p P$, by (a), $X = X_V + X_H$, where $X_V \in G_p P$ is called the *vertical part* of X and $X_H \in H_p P$ the *horizontal part.* Moreover, we call G_p the *vertical subspace* of $T_p P$ and $H_p P$ the *horizontal subspace.*

Remark 2.21. The conditions that we expect the connection Γ to meet are quite natural. It preserves the differentiable structure (by (c)) and, the fibre structure (by (b)) of the principal bundle. Condition (a) will give us the infinitesimal horizontal counterpart of the horizontal lift $\hat{\gamma}$ which we will define later as the integral curve through $p \in P$ obtained by these horizontal vectors.

Since G acts effectively on P by R_g, there is a natural vector isomorphism $A \in \mathfrak{g} \mapsto (A^{\#})_p \in G_p$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G and $A^{\#}$ is the fundamental vector field in P induced by A, defined as in proposition 2.9. Given a connection Γ, we have a well-defined mapping $\omega \in \mathfrak{g} \otimes T_p^*P$ for each $p \in P$, which maps a vector $X \in T_pP$ to the unique element $A \in \mathfrak{g}$, by contraction, such that $A^{\#}$ equals the vertical component X_V of X. The mapping ω is called the *connection 1-form* of the given connection Γ.

Note that for a horizontal vector X, $\omega(X) = 0$, since it has no vertical component and for $A \in \mathfrak{g}$, $\omega(A^{\#}) = A$ by the very definition of ω.

If (ϵ_α) is a basis for $\mathfrak{g}$ and (θ^i) is a basis for T_p^*P, then ω can be written as

$$\omega = \omega^\alpha \otimes \epsilon_\alpha = \omega_i^\alpha \theta^i \otimes \epsilon_\alpha \tag{2.5}$$

where ω^α's are 1-forms on P and ω_i^α's are functions on P.

Since connection is a differentiable distribution, ω^α's are differentiable $\mathfrak{g}$-valued 1-forms on P. Thus ω is a differentiable $\mathfrak{g}$-valued 1-form on P.

If $X \in T_pP$ is a horizontal vector, then so is $(R_g)_*X \in T_{R_gp}P$. Thus both $\omega((R_g)_*X)$ and $\mathrm{ad}(g^{-1})\omega(X)$ vanish. In the case that $X \in T_pP$ is a vertical vector, we can take X as the vector of a fundamental vector field $A^{\#}$ at p. Then proposition 2.12 yields that $(R_g)_*X$ is the fundamental vector field corresponding to $\mathrm{ad}(g^{-1})X$. So,

$$\omega_{R_gp}((R_g)_*X) = \mathrm{ad}(g^{-1})A = \mathrm{ad}(g^{-1})(\omega_p(X)) \tag{2.6}$$

Since each vector X in T_pP is decomposed as $X = X_V + X_H$, the above calculations yield

$$(R_g)^*\omega = \mathrm{ad}(g^{-1})\omega$$

Summarizing the above properties, we give the formal definition of a connection 1-form:

Definition 2.22. A *connection 1-form* $\omega \in \mathfrak{g} \otimes T^*P$ is a projection of T_pP onto the vertical component $G_p \cong \mathfrak{g}$ that satisfies

(a) $\omega(A^\#) = A$ for $A \in \mathfrak{g}$

(b) $R_g^*\omega = \mathrm{ad}(g^{-1})\omega$, i.e. for $X \in T_pP$, $R_g^*\omega_{R_gp}(X) = \omega_{R_gp}(R_{g*}X) = g^{-1}\omega_p(X)g$

Actually, defining connection 1-form ω is equivalent to define a connection Γ by the following proposition, if we define the horizontal subspaces H_pP by the kernel of ω:

$$H_pP := \{X \in T_pP \,|\, \omega(X) = 0\} \tag{2.7}$$

Proposition 2.23. *The distribution defined as in equation 2.7 satisfies*

$$R_{g*}H_pP = H_{R_gp}P$$

Proof. Let $p \in P$ and define H_pP as in equation 2.7. Take $X \in H_pP$ and consider $R_{g*}X \in T_{R_gp}P$. Then

$$\omega(R_{g*}X) = \mathrm{ad}(g^{-1})\omega(X) = 0$$

by condition (b) of definition 2.22 and since $\omega(X) = 0$. Therefore, $R_{g*}X \in H_{R_gp}P$. Moreover, R_{g*} is an invertible linear map, where it is induced by the group action of G on P. Hence, for any vector $Y \in H_{R_gp}P$, $Y = R_{g*}X$ for some $X \in H_pP$. □

The differentiability of the distribution of H_pP follows from the differentiability of ω. Since ω is a linear isomorphism from G_p onto $\mathfrak{g}$, T_pP is decomposed as $G_p \oplus H_pP$ as in the definition of connection on a principal fibre bundle, i.e. the definition of connection Γ is equivalent to the definition of the $\mathfrak{g}$-valued connection form ω, known as the *Ehresmann connection* in the literature.

Now, we are ready to prove the existence of a connection on a principal fibre bundle $P(M, G)$. Let A be a subset of M. A connection is said to be defined over A, if there is a distribution H_p for each $p \in \pi^{-1}(m)$, $m \in A$, satisfying (a) and (b) of definition 2.20 and for each $x \in A$, there is an open neighborhood $U \subset M$ and a connection on $P|_U = \pi^{-1}(U)$ such that the horizontal space of the connection on $\pi^{-1}(U)$ coincides with H_p given by the distribution.

Before proving the existence of a connection 1-form we define what a cross section on a fibre bundle is:

Definition 2.24. Let $P(M, G)$ (or $E(M, G)$) be a principal fibre bundle (or a vector bundle). A *cross section of the bundle $P(M, G)$ (or $E(M, G)$)* is a mapping $\sigma : M \to P$ (or $\sigma : M \to P$) such that $\pi \circ \sigma = Id_M$, identity map on M (or $\pi_E \circ \sigma = Id_M$), where π and π_E denote the natural projections from P and E to M, respectively.

Theorem 2.25. *Let $P(M, G)$ be a principal fibre bundle, A be a closed (possibly empty) subset of M. If M is paracompact, every connection defined over A can be extended to a connection on P. In particular, taking $A = \emptyset$, P admits a connection, if M is paracompact.*

To prove the theorem, we will make use of two lemmas.

Lemma 2.26. *A differentiable function defined on a closed subset of $\mathbb{R}^n$ can be extended to a differentiable function on $\mathbb{R}^n$.*

This lemma is proved using bump functions in the lemma 2.27 (extension lemma) of

Lee [23]. We omit the proof here.

Lemma 2.27. *Each point in M has a neighborhood U such that every connection defined on a closed subset contained in U can be extended to a connection defined on U.*

Proof. Let $m \in M$, and U be a coordinate neighborhood of m such that $\pi^{-1}(U) \approx U \times G$. Consider the natural cross section $\sigma : m \in U \mapsto (m, e) \in U \times G \approx \pi^{-1}(U)$. First note that, if we know how a connection ω behaves at $U \times \{e\}$, then we know how it behaves on $U \times G$, where $R_g^*(\omega) = \mathrm{ad}(g^{-1})\omega$.

Let $X \in T_{\sigma(m)}(U \times G)$. Then $X = Y + Z$ for Y being a vector tangent to $U \times \{e\}$, i.e. $Y = \sigma_*(\pi_* u)$ and Z being a vertical vector. Then

$$\omega(X) = \omega(Y) + \omega(Z) = \omega(\sigma_*(\pi_* X)) + \omega(Z) = (\sigma^*\omega)(\pi_* X) + A$$

where $A \in \mathfrak{g}$ such that $A^{\#}_{\sigma(m)} = Z$, $A^{\#}$ denoting the fundamental vector field corresponding to A. Since $\omega(A^{\#}_{\sigma(m)}) = A$ is true for every connection ω, A does not depend on ω, but only on X. Therefore ω is completely determined by $\sigma^*\omega$. The converse is true, as well, namely every $\mathfrak{g}$-valued 1-form on U determines a unique connection 1-form ω on $U \times G$, by the same equation above.

Thus, finding the extension of a connection over a closed set in U to a connection on U is reduced to the extension of a $\mathfrak{g}$-valued 1-form over a closed set in U to a $\mathfrak{g}$-valued 1-form over U. Since U is a coordinate neighborhood, every $\mathfrak{g}$-valued 1-form on U, can be written as $\omega = \omega_i^\alpha \theta^i \otimes \epsilon_\alpha$ as in equation 2.5. Given a connection form on a closed set in U, we can extend the coefficient functions ω_i^α by lemma 2.26. So lemma 2.27 is proved. □

Proof of the theorem 2.25. We are going to define an order relation on some special subsets of M and use Zorn's lemma to prove the theorem.

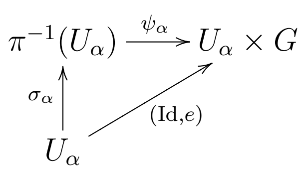
Let $\{U_i\}_{i\in I}$ be a locally finite open covering of M such that each $\{U_i\}$ satisfies lemma 2.27. Since M is paracompact, there is an open refinement $\{V_i\}$ such that $\bar{V}_i \subset U_i$ for all $i \in I$. Put $S_J := \bigcup_{i\in J} \bar{V}_i$ for each $J \subset I$. Define

$$T := \{(\tau, J) \,|\, \tau \text{ is a connection defined on } S_J,\, \tau = \text{the given connection on } A \cap S_J\}$$

Now, put an order relation $\prec$ on T, by defining $(\tau', J') \prec (\tau'', J'')$ if $J' \subset J''$ and $\tau' = \tau''$ on S'_J. By Zorn's lemma, let (τ, J) be a maximal element of T. By the way of contradiction, assume $J \neq I$. Pick $i \in I \setminus J$. On the closed set $(A \cup S_J) \cap \bar{V}_i \subset U_i$, we have a well-defined connection τ_i such that τ_i coincides with the given connection on $A \cap \bar{V}_i \subset A$ and coincides with τ on $S_J \cap \bar{V}_i$. By lemma 2.27, we can extend τ_i to a connection on $\bar{V}_i$. Then letting $J' := J \cup \{i\}$ and τ' the extended connection on $S_{J'}$, defined by $\tau' = \tau$ on S_J and $\tau' = \tau_i$ on $\bar{V}_i$, implies that $(\tau, J) \prec (\tau', J')$ contradicting the maximality of (τ, J). Therefore $I = J$ and the connection on A is extended to whole of M. □

Next, we will investigate local connection 1-forms on M.

Let a connection 1-form ω on P and an open covering $\{U_\alpha\}$ of M with a family of isomorphisms (local trivializations) $\psi_\alpha : \pi^{-1}(U) \to U_\alpha \times G$ and the corresponding transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be given. Take the natural cross sections for each α, namely $\sigma_\alpha : m \in U_\alpha \to \psi_\alpha^{-1}(m, e) \in P$.



Define for each non-empty $U_\alpha \cap U_\beta$,

$$\theta_{\alpha\beta} := \psi^*_{\alpha\beta}\theta$$

and for each α,

$$\omega_\alpha := \sigma^*_\alpha \omega$$

where θ is the left-invariant canonical 1-form on G, namely the Maurer-Cartan form that is defined in section 2.1, and ω is the connection 1-form on P.

Definition 2.28. $\omega_\alpha := \sigma^*_\alpha \omega$ on U_α defined above is called the *connection 1-form in the local trivialization* ψ_α.

These connection 1-forms in the local trivialization turn out to be very crucial in Riemannian geometry and physics. The following proposition yields the transformation formula of Christoffel symbols in Riemannian manifolds, see page 366 of Bruhat [25]. We will make use of it in the proof of the existence of a horizontal lift of a curve γ in the base manifold M in proposition 2.34.

Proposition 2.29. *The forms* $\theta_{\alpha\beta}$ *and* ω_α *satisfy the equation*

$$\omega_\beta = \mathrm{ad}(\psi^{-1}_{\alpha\beta})\omega_\alpha + \theta_{\alpha\beta} \tag{2.8}$$

Proof. Note that ${\psi^{-1}}^*_\alpha \omega : TU_\alpha \times TG \to \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on $U_\alpha \times G$. Let $(X, Y) \in T_m U_\alpha \times T_g G$, then

$$({\psi^{-1}}^*_\alpha \omega)(X, Y) = (\psi^*_\alpha \omega)(X, 0_g) + (\psi^*_\alpha \omega)(0_m, Y)$$

where 0_g and 0_m are zero vectors at $g \in G$ and $m \in U_\alpha$, respectively.

$$\begin{aligned}
(\psi^{-1*}_{\ \alpha}\omega)(X, 0_g) &= (\psi^{-1*}_{\ \alpha})(X, R_{g*}0_e) \\
&= \mathrm{ad}(g^{-1})(\psi^{-1*}_{\ \alpha})(X, 0_e) \\
&= \mathrm{ad}(g^{-1})(\psi^{-1} \circ (Id, e))^*\omega(X) \\
&= \mathrm{ad}(g^{-1})\sigma^*_\alpha\omega(X)
\end{aligned}$$

On the other hand,

$$(\psi^{-1*}_{\ \alpha}\omega)(0_m, Y) = \theta(Y)$$

since $(0_m, Y)$ is a vertical vector. Hence

$$(\psi^{-1*}_{\ \alpha}\omega)(X, Y) = \mathrm{ad}(g^{-1})\sigma^*_\alpha\omega(X) + \theta(Y) \tag{2.9}$$

To obtain ω_α in terms of ω_β,

$$\begin{aligned}
((\psi^{-1}_\beta \circ \psi_\beta \circ \psi^{-1}_\alpha)^*\omega)(X, Y) &= (\psi^{-1*}_\beta\omega)((\psi_\beta \circ \psi^{-1}_\alpha)_*|_{(m,g)}(X, Y)) \\
&= (\psi^{-1*}_\beta\omega)((Id_M, L_{\psi_{\beta\alpha}(m)})_*|_{(m,g)}(X, Y)) \\
&= \mathrm{ad}(g^{-1})(\mathrm{ad}\psi_{\beta\alpha}(m))^{-1}\omega_\beta(X) \\
&\quad +\theta(\psi_{\beta\alpha_*}|_m(X)) + \theta(Y) \qquad (2.10)
\end{aligned}$$

Equation 2.8 follows then from equation 2.9 and 2.10. □

2.4. Parallelism in a principal fibre bundle

Next, we define the horizontal lift of a vector field on the base manifold using the given connection Γ on the principal fibre bundle $P(M,G)$.

Definition 2.30. The *horizontal lift* of a vector field X on M is the unique vector field X^* on P which is horizontal and which projects onto X, namely $\pi(X^*_p) = X_{\pi(p)}$ for all $p \in P$.

Proposition 2.31. *Given a connection on P and a vector field X on M, there exists a unique horizontal lift X^* of X. X^* is invariant under the action of G on P. Moreover, the converse holds as well, that is, every horizontal vector field X^* on P invariant by G-action is the lift of a vector field on M.*

Proof. Let U be a neigborhood of $m \in M$ which is locally trivializable, i.e. $\pi^{-1}(U) \approx U \times G$. Let $Y := \pi^*(X) \in T\pi^{-1}(U)$. Let $X^* := Y_H$, be the horizontal component of Y. Then X^* is a differentiable horizontal vector field on $\pi^{-1}(U)$. Since ker $\pi_* = G_p$, π_* is a linear isomorphism $H_pP \to T_{\pi(p)}M$ for every $p \in P$, thus the existence and uniqueness follows.

For the converse, let X^* be a horizontal vector field on P which is G-invariant. For $m \in M$, pick $p \in \pi^{-1}(m)$ and define $X_m := \pi_*(X^*_p)$. X_m is well-defined, since if $p = R_g p'$ for some $p' \in P$, then $\pi_*(X^*_{p'}) = \pi_*(R_{g*}X^*_p) = \pi_*(X^*_p)$, since X^* is G-invariant. Then clearly X^* is the horizontal lift of X by the construction. $\square$

Remark 2.32. Clearly, if X^* and Y^* are horizontal lifts of X and Y, respectively, then $X^* + Y^*$ is the horizontal lift of $X + Y$. Moreover,

$$\pi_*([X^*, Y^*]_H) = \pi_*([X^*, Y^*]_H + [X^*, Y^*]_V) = \pi_*([X^*, Y^*]) = [\pi_* X^*, \pi_* Y^*] = [X, Y] \tag{2.11}$$

In other words, the horizontal component of $[X^*, Y^*]$ is the horizontal lift of $[X, Y]$.

After defining horizontal lifts of vector fields defined on the base manifold M, we simply take the integral curve through p of the horizontal vectors projecting on the tangent vectors to the curve γ, namely the integral curve of the tangent vectors to it, in M to obtain the horizontal lift $\hat{\gamma}$ of γ.

Definition 2.33. Let $\gamma : I \subset \mathbb{R} \to M$ be a piecewise differentiable curve in M. A *horizontal lift* of γ is a horizontal curve $\hat{\gamma} : I \to P$ such that $\pi(\hat{\gamma}(t)) = \gamma(t)$ for $t \in I$. By a *horizontal curve*, we mean a piecewise differentiable curve in P having tangent vectors which are all horizontal.

Let $\hat{X}$ be a tangent vector to $\hat{\gamma}$, horizontal lift of a curve $\gamma \subset M$. Then clearly, $\omega(\hat{X}) = 0$, since it is horizontal. This yields to an ordinary differential equation and the fundamental theorem of ODE's gives us the local existence and uniqueness of the horizontal lift of γ.

Proposition 2.34. *Let $\gamma : I \to M$ be a curve in M, $p_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\hat{\gamma}(t)$ in P such that $\hat{\gamma}(0) = p_0$.*

Proof. Without loss of generality, let U_α be a coordinate neighborhood which contains $\gamma(I)$. Take a cross section σ_α over U_α. If there exists a horizontal lift $\hat{\gamma}$, then $\hat{\gamma} = R_{(g_\alpha(t))}\sigma_\alpha$ for some curve $g_\alpha(t)$ in G. Choose the cross section such that $\sigma_\alpha(\gamma(0)) = \hat{\gamma}(0) = p_0$, i.e. $g_\alpha(0) = e \in G$. Let $X = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(0)$. Then $\hat{X} = \hat{\gamma}_* v$ is tangent to $\hat{\gamma}$ at $p_0 = \hat{\gamma}(0)$. $\hat{X}$ is horizontal, so $\omega(\hat{X}) = 0$.

Manipulating and taking ω of both sides of the equation 2.8, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t} g_\alpha(t) = -\omega(\sigma_{\alpha *} v) g_\alpha(t) \tag{2.12}$$

which is an ordinary linear differential equation. The existence and uniqueness theorem of ODE's gives us the solution, namely the horizontal lift. □

Remark 2.35. Although the existence theorem of ODE's yields the existence of the horizontal lift $\hat{\gamma}$ only in a neighborhood of p_0, it is possible to obtain the horizontal lift of the whole curve by covering γ by a finite number of arcs. A change of parametrization of γ changes the parametrization of $\hat{\gamma}$, as well, however the geometric curve $\hat{\gamma}$ is independent of the parametrization of γ.

The uniqueness of a horizontal lift allows us to define parallel displacement of fibres

Definition 2.36. Let $\gamma(t)$ be a curve in M connecting two points m_0 and m_1 in M. Let p_0 be an arbitrary point of P such that $\pi(p_0) = m_0$. The unique horizontal lift $gam\hat{m}a(t)$ of $\gamma(t)$ has the end point $p_1 \in \pi^{-1}(m_1)$. By varying p_0 in the fibre $\pi^{-1}(m_0)$, we obtain a mapping of the fibre $\pi^{-1}(m_0)$ on the the fibre $\pi^{-1}(m_1)$ which maps p_0 into p_1. This map, denoted as τ, is called the *parallel displacement*(or *parallel transport*) along the curve γ.

The fact that $\tau : \pi^{-1}(m_0) \to \pi^{-1}(m_1)$ is actually an isomorphism comes from the following remark:

Remark 2.37. The parallel displacement along any curve γ commutes with the action of G on P, i.e. $\tau \circ R_g = R_g \circ \tau$ for every $g \in G$, since every horizontal curve is mapped into a horizontal curve by R_g and the inverse τ^{-1} of the mapping τ is obtained by parallel displacement along the same geometric curve with the opposite orientation.

2.5. Curvature form in a principal fibre bundle

First, we define the notion of exterior covariant derivative on a principal bundle.

Consider $\phi \in \Omega^r(P) \otimes V$, a vector-valued r-form,

$$\phi : \underbrace{TP \wedge TP \wedge ... \wedge TP}_{r \text{ times}} \to V$$

where $\dim V = k$.

Then $\phi = \sum_{\alpha=1}^{k} \phi^\alpha \otimes e_\alpha$, $\{e_\alpha\}$ being a basis for V and $\phi^\alpha \in \Omega^r(P)$. The *exterior derivative* $\mathrm{d}_P\phi$ is defined as

$$\mathrm{d}_P\phi := \sum_{\alpha=1}^{k} \mathrm{d}_P\phi^\alpha \otimes e_\alpha$$

Definition 2.38. Let $\phi \in \Omega^r(P) \otimes V$, $X_1, ..., X_{r+1} \in T_U P$. The exterior covariant derivative of ϕ is defined by

$$D\phi(X_1, ..., X_{r+1}) := d_P\phi(X_{1H}, ..., X_{(r+1)_H}) \tag{2.13}$$

where $X_i \in T_U P$ is decomposed as $X_i = X_{iH} + X_{iV}$ using a connection 1-form ω on P.

Definition 2.39. The curvature Ω is a 2-form on P which is the exterior covariant derivative of the connection 1-form $\omega \in T^*P \otimes \mathfrak{g}$

$$\Omega := D\omega \in \Omega^2(P) \otimes \mathfrak{g} \tag{2.14}$$

Proposition 2.40. *The curvature 2-form satisfies*

$$R_g^*\Omega = \mathrm{ad}(g^{-1})\Omega \quad g \in G \tag{2.15}$$

Proof. First note that

$$(R_{g_*}X)_H + (R_{g_*}X)_V = R_{g_*}X = R_{g*}(X_H + X_V) = R_{g*}X_H + R_{g*}X_V$$

Hence $(R_{g*}X)_H = R_{g*}(X_H)$. where ω decomposes T_pP uniquely into $H_pP \oplus V_pP$. Moreover also note that $\mathrm{d}_P R_g^* = R_g^* \mathrm{d}_P$. Then

$$\begin{aligned}
R_g^*\Omega(X,Y) &= \Omega(R_{g*}X, R_{g*}Y) = \mathrm{d}_P\omega((R_{g*}X)_H, (R_{g*}Y)_H) \\
&= \mathrm{d}_P\omega(R_{g*}X_H, R_{g*}Y_H) = R_g^*\mathrm{d}_P\omega(X_H, Y_H) \\
&= \mathrm{d}_P R_g^*\omega(X_H, Y_H) = \mathrm{d}_P(\mathrm{ad}(g^{-1})\omega)(X_H, Y_H) \\
&= \mathrm{ad}(g^{-1})\mathrm{d}_P\omega(X_H, Y_H) \\
&= \mathrm{ad}(g^{-1})\Omega(X,Y)
\end{aligned}$$

□

For a $\mathfrak{g}$-valued p-form $\zeta = \zeta^\alpha \otimes e_\alpha$ and a $\mathfrak{g}$-valued q-form $\eta = \eta^\alpha \otimes e_\alpha$ the commutator of ζ and η is defined by

$$\begin{aligned}
[\zeta, \eta] &:= \zeta \wedge \eta - (-1)^{pq}\eta \wedge \zeta \\
&= [e_\alpha, e_\beta] \otimes \zeta^\alpha \wedge \eta^\beta = c_{\alpha\beta}{}^\gamma e_\gamma \otimes \zeta^\alpha \wedge \eta^\beta \qquad (2.16)
\end{aligned}$$

where $\{e_\alpha\}$ is a basis of $\mathfrak{g}$ and $c_{\alpha\beta}{}^\gamma$ are the *structure constants* of the Lie algebra $\mathfrak{g}$ with respect to that basis.

If $\zeta = \eta$ and $p = q$ are odd, then $[\zeta, \zeta] = 2\zeta \wedge \zeta = c_{\alpha\beta}{}^\gamma e_\gamma \otimes \zeta^\alpha \wedge \zeta^\beta$

Lemma 2.41. *If $X \in H_pP$ and $Y \in G_p$, then $[X, Y] \in H_pP$.*

Proof.

$$[X, Y] = -[Y, X] = -\lim_{t\to 0} t^{-1}(R_{g(t)*}X - X)$$

where $g(t) = \exp(tw)$ generates the vector field w. Hence $R_{g(t)*}X$ horizontal, so is the difference $R_{g(t)*}v - v$. □

Now, we are ready to derive the Cartan structure equations. Using them, we will be able to comment on the geometric meaning of the curvature tensor.

Theorem 2.42. *Let $X, Y \in T_pP$. Then Ω and ω satisfy Cartan's structure equation*

$$\Omega(X, Y) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)] \tag{2.17a}$$

or also written as

$$\Omega = \mathrm{d}_P\omega + \omega \wedge \omega. \tag{2.17b}$$

Proof. We will investigate three cases:

Case 1: $X, Y \in H_pP$. Then $\Omega(X, Y) = \mathrm{d}_P\omega(X_H, Y_H) = \mathrm{d}_P\omega(X, Y)$

Case 2: $X \in H_pP$, $Y \in G_p$. Since Y is vertical, $Y_H = 0$, hence $\Omega(X, Y) = 0$. On the other hand, $X \in H_pP$, so $\omega(X) = 0$. By equation 2.1,

$$\mathrm{d}_P\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y])$$

Since $Y \in G_p$, there is a $A \in \mathfrak{g}$ such that $w = A^{\#}$. Thus $X\omega(Y) = XA = 0$, $\omega(Y) = A$ being a constant. By lemma 2.41, $[X, Y] \in H_pP$, so $\omega([X, Y]) = 0$. Hence $\mathrm{d}_P\omega(X, Y) = 0$.

Case 3: $X, Y \in G_p$. Then $\Omega(X, Y) = 0$ and therefore

$$\mathrm{d}_P\omega(X, Y) = -\omega([X, Y]) \tag{2.18}$$

by the same argument as in case 2. Since $X, Y \in G_p$, $[X, Y] \in G_p$. Thus there exists an $A \in \mathfrak{g}$ such that $\omega([X, Y]) = A$ and $A^\# = [X, Y]$. Letting $B^\# := X, C^\# := Y$ implies that $[\omega(X), \omega(Y)] = [B, C]$. Since $[B, C]^\# = [B^\#, C^\#] = [X, Y] = A^\#$,

$$[B, C] = A \tag{2.19}$$

By equations 2.18 and 2.19,

$$0 = \mathrm{d}_P\omega(X, Y) + \omega([X, Y]) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)]$$

Ω is bilinear and skew-symmetric, so the theorem holds for any $X, Y \in T_pP$. □

It is known that the Riemannian curvature tensor expresses the non-commutativity of the parallel transport of vectors on a Riemannian manifold, see page 89 of do Carmo [26]. There is a similar interpretation of a curvature two-form in a principal fibre bundle. If $X, Y \in H_uP$ are horizontal vectors, then equation 2.1 implies that

$$\begin{aligned} \Omega(X, Y) &= \mathrm{D}\omega(X, Y) = \mathrm{d}_P\omega(X_H, Y_H) \\ &= \mathrm{d}_P\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]) \end{aligned} \tag{2.20}$$

In other words, Ω measures the vertical component of the Lie bracket $[X, Y]$ of horizontal vectors X, Y. Take an infinitesimal parallelogram γ in a coordinate system $\{x_\mu\}$ on a chart $U \subset M$ whose corners are $O = (0, 0, \ldots, 0)$, $P = (\epsilon, 0, \ldots, 0)$, $Q = (\epsilon, \delta, 0, \ldots, 0)$ and $R = (0, \delta, 0, \ldots, 0)$. Consider the horizontal lift $\hat{\gamma}$ of γ. Let $X, Y \in H_uP$ such that $\pi_*(X) = \epsilon\frac{\partial}{\partial x^1}$ and $\pi_*(Y) = \delta\frac{\partial}{\partial x^2}$. Then

$$\pi_*([X, Y]_H) = \epsilon\delta\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0 \tag{2.21}$$

Namely, $[X, Y]$ is vertical. By the previous observation in equation 2.20, we conclude that the curvature measures the failure of the initial and final point of the horizontal lift of a closed curve, where that failure is proportional to the vertical vector $[X, Y]$ and is equal to an element $A = \Omega(X, Y) = -\omega([X, Y]) \in \mathfrak{g}$ such that $[X, Y] = A^{\#}$.

3. HOLONOMY GROUPS AND AMBROSE-SINGER THEOREM

In this chapter, we will define holonomy groups and prove the Ambrose-Singer theorem first in the principal bundle setting. In that setting, it is easy to visualize that the holonomy group turns out to be a subgroup of the structure group. Also, the holonomy group is closely related to the curvature defined in section 2.5. The relationship is given by the Ambrose-Singer theorem. After proving the Ambrose-Singer theorem for principal fiber bundles, we will translate it to its version on the vector bundles, which happens to be the basis of Simons' proof of Berger's classification theorem.

3.1. Ambrose-Singer Theorem: Principal fibre bundle version

Let $\gamma(t)$ be a closed curve in M, starting and ending at $m \in M$. The parallel displacement $\hat{\gamma}$ along γ of the elements p of the fibre $G_m = \pi^{-1}(m)$ defines a mapping $G_m \to G_m$, which is denoted by τ.

By remark 2.37, the mapping $\tau : G_m \to G_m$ defined by the parallel transport along the loop γ is a diffeomorphism which commutes with R_g. Holonomy groups are defined via parallel displacement defined in section 2.4.

Definition 3.1. The set Θ_m of mappings τ corresponding to the loops at m forms a group (the product of $\tau_1 \cdot \tau_2$ is defined to be the parallel transport first along γ_1, then along γ_2), which is called the *holonomy group* of the connection ω with the reference point m.

Definition 3.2. The subgroup of the holonomy group consisting of the parallel displacement along loops which are contractible in M is called the *restricted holonomy group*.

Remark 3.3. The relation of the the holonomy definition above and the definition given in the introduction chapter will be clear in section 3.2 from the study of parallel transport in the tangent bundle TM which corresponds to parallel transport of vectors M.

Remark 3.4. The holonomy group at $m \in M$ can be identified with a subgroup of the structure group G. Let $p \in P$ such that $\pi(p) = m \in M$. By parallel transport along a loop γ at m, we get a point $\tau(p) \in \pi^{-1}(m)$, thus there exists an element $a \in G$ with $\tau(p) = R_a p$. If a_1, a_2 correspond to loops γ_1 and γ_2, respectively, then $a_1 a_2 \in G$ corresponds to the loop $\gamma_1\gamma_2$, since $R_{a_1}R_{a_2} = R_{a_1a_2}$.

Hence we have defined for each $p \in P$, an isomorphism between the holonomy group at m (or the restricted holonomy group at m) and a subgroup $\mathscr{H}(p)$ ($\mathscr{H}_0(p)$ for the restricted holonomy group) with reference point $p \in P$. If we change the reference point $p \in G_m$ to $p' = R_g p$, then we change $\mathscr{H}(p)$ to the isomorphic subgroup $\mathscr{H}(p') = g\mathscr{H}(p)g^{-1}$ ($\mathscr{H}_0(p') = g\mathscr{H}_0(p)g^{-1}$, respectively for the restricted holonomy groups).

Let $p_0 \in P$ be chosen. Define $P(p_0)$ as the set of points of P which can be obtained by parallel transport from p_0. $P(p_0)$ is called the *holonomy bundle* at p_0. If we replace p_0 by a point in $P(p_0)$, then clearly we obtain by parallel transport the same holonomy bundle $P(p_0)$. This gives us an equivalence relation on P, and implies that P is the disjoint union of holonomy bundles of points which cannot be joined by a horizontal curve in P.

Proposition 3.5. *The holonomy group $\mathscr{H}(p)$ is the same subgroup of G at each point of a holonomy bundle $P(p_0)$.*

Proof. If $a \in \mathscr{H}(p_0)$, there is a loop γ at m_0 such that the parallel transport of p_0 along γ results $\tau(p_0) = R_a p_0$. If p is obtained by parallel transport from p_0 along some curve φ from m_0 to m, the same is true of $R_a p$ from $R_a p_0$, since parallel transport commutes with the right action of G on P: Thus $R_a p$ is obtained by parallel transport from p along the loop $\varphi \circ \gamma \circ \varphi^{-1}$, thus $a \in \mathscr{H}(p)$. Conversely, if $a \in \mathscr{H}(p)$, then $a \in \mathscr{H}(p_0)$. Therefore $\mathscr{H}(p_0) = \mathscr{H}(p)$ (the same is true for contractible curves, i.e. $\mathscr{H}_0(p_0) = \mathscr{H}_0(p)$). $\square$

The following theorem relates the topological structure of a principal fibre bundle and its holonomy bundle relative to some connection defined on it:

Theorem 3.6. *Let $P(M,G)$ be a principal fibre bundle with a connection ω and M be connected. Then each holonomy bundle $P(p_0)$ is a reduced bundle of P, with structure group $\mathscr{H}(p_0)$.*

Proof. First note that the projection $\pi : P \to M$ maps also $P(p_0)$ onto M if M is connected, since $p_0 \in \pi^{-1}(m_0)$ can be parallel transported to some element $p \in \pi^{-1}(m)$ above any point $m \in M$.

If $p \in P(p_0)$ and $a \in \mathscr{H}(p_0)$, then $R_a p \in P(p_0)$. Conversely if $p, p' \in P(p_0)$, $\pi(p) = \pi(p')$, then there exists an $a \in \mathscr{H}(p_0)$ such that $p' = R_a p$. Therefore if $\psi : \pi^{-1}(U) \to U \times G$ is a local trivialization of P, the restriction $\psi|_{P(p_0)}$ defines a bijection $\phi : P(p_0) \cap \pi^{-1}(U) \to U \times \mathscr{H}(p_0)$

But then $P(p_0)(M, \mathscr{H}(p_0))$ is a principal fibre bundle which is a reduction of $P(M,G)$. □

Definition 3.7. Let $f : P_1(M, G_1) \to P(M,G)$ be a reduction of a principal fibre bundle. A connection ω on P is said to be a *reducible connection* by f if there is a connection ω_1 on P_1 such that ω is its image, namely $f(\omega_1)$.

Note that, if such a connection ω_1 exists, then the horizontal spaces of ω_1 are mapped into horizontal spaces of ω by f. Moreover, again it is clear that if a connection ω on P is reducible by f if the 1-form $f^*\omega$ on P_1 has its values in $\mathfrak{g}_1$, the Lie algebra of G_1, identified with a Lie subalgebra of G.

Proposition 3.8. *A connection ω on a principal fibre bundle is reducible to a connection in any of its holonomy bundles.*

Proof. The bundle $P(M, G)$ is reducible to the holonomy bundle $P(p)(M, \mathscr{H}(p))$. We can define a connection in $P(p)$ by setting horizontal subspaces equal to the horizontal subspaces of the original connection of P at the points of $P(p)$, since these spaces are tangent to $P(p)$ by the very definition. It is easy to show that the assigned horizontal spaces satisfy the required properties to define a connection ω_1, reduction of ω. □

As we have observed in equation 2.21, the curvature 2-form measures the discrepancy between the initial and final points of the horizontal lift of a closed curve. But since that discrepancy corresponds simply to the holonomy, we expect that the holonomy group is expressed in terms of the curvature and that brings us to the Ambrose-Singer Theorem [27]:

Theorem 3.9 (Ambrose-Singer Theorem on a principal fibre bundle)**.** *Let $P(M, G)$ be a principal fibre bundle with a connection 1-form ω, M be connected. The Lie algebra of an holonomy group $\mathscr{H}(p_0)$ is equal to the subspace of the Lie algebra $\mathfrak{g}$ of G which is spanned by all elements of the form $\Omega_p(X, Y)$, where Ω_p is the curvature 2-form of ω at an arbitrary point $p \in P(p_0)$ and X, Y are arbitrary horizontal vectors at p.*

To prove this theorem, we need the Frobenius theorem which produces us a submanifold when a suitable distribution is given.

Given a distribution on Δ on a manifold, Δ is called *involutive*, if $[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k$ for $1 \leq i, j \leq r$ and $c_{ij}^k \in \mathbb{R}^n$. In other words, Δ is an involutive distribution, if the bracket $[X, Y]$ of the vector fields X, Y belongs to Δ, whenever X, Y belong to Δ, where a vector field X is said to belong to a distribution Δ, if $X_m \in \Delta_m$ for any $m \in M$.

A connected submanifold N of M is called an *integral submanifold of the distribution* Δ, if $\varphi_*(T_m N) = \Delta_m$ for all $m \in N$, where φ is the imbedding of the submanifold N to M, namely N admits Δ as its tangent bundle TN. If there is no other integral submanifold of Δ containing N, then N is called a *maximal integral submanifold of* Δ. Section IV.8 of Boothby [18] is a good reference for the Frobenius theorem and its corollaries.

Theorem 3.10 (Frobenius Theorem)**.** *Let Δ be an involutive distribution on a manifold M. Through every point $p \in M$, there passes a unique maximal integral submanifold $N(p)$ of Δ. Any integral manifold of Δ through p is an open submanifold of $N(p)$.*

After citing Frobenius theorem, we can prove Ambrose-Singer theorem.

Proof of Ambrose-Singer theorem. By our construction on the previous page, we may assume $P = P(p_0)$, $G = \mathscr{H}(p_0)$. Denote by V the Lie algebra, subspace of $\mathfrak{g}$, spanned by all elements of the form $\Omega_p(X, Y)$, $p \in P(p_0)$, where X, Y are horizontal vectors at p.

At each point $p \in P$ we consider the subspace S_p of the tangent space T_pP spanned by the horizontal space H_p and the vertical subspace corresponding to V. The subspace $S_p \subset T_pP$ has dimension $n+r$ with $r = \dim\ V$, $n = \dim\ M$. The distribution S_p depends differentiably on p and satisfies Frobenius condition for complete integrability, using the facts that V is a Lie subalgebra of $\mathfrak{g}$ and the bracket of a vertical vector field with a horizontal field is horizontal and that the bracket of two horizontal vectors has a vertical component which belongs to V, by the structure equation 2.17a, $\omega([X, Y]) = -\Omega(X, Y)$ if X and Y are horizontal vector fields.

The family of hyperplanes S_p admits by the Frobenius theorem an integral manifold S of dimension $n+r$ passing through p_0. It contains all the horizontal curves starting from p_0, thus coincide with P. Since $\dim\ P = n + \dim\ \mathfrak{g}$, we have $\dim\ \mathfrak{g} = r$, i.e. $V = \mathfrak{g}$. $\square$

3.2. Ambrose-Singer Theorem: Vector bundle version

In this section, we will translate most of what we have done so far to the vector bundle setting. We will work with vector bundles induced from principal bundles, in particular with tangent and frame bundles. We will assume that the field of scalars of the vector bundle is $\mathbb{R}$, namely vector bundles have standard fibre $F = \mathbb{R}^n$. Interchanging $\mathbb{C}^n$ with $\mathbb{R}^n$ leaves the definitions and propositions below unchanged.

We will modify our definitions of horizontal lift, curvature 2-form and holonomy group that we gave in chapter 2 for an associated vector bundle $E(M, F, G, P)$. Naturally they are closely related, as the vector bundle $E(M, F, G, P)$ is constructed from the principal bundle $P(M, G)$. Also note that, since by remark 2.17, any vector bundle can be seen as an associated vector bundle to a principal fibre bundle, the definitions and results in this section can be generalized to any vector bundle. In the following, we will make use of proposition 2.19 frequently, which actually relates the associated vector bundle structure to the principal bundle structure.

Let $P(M, G)$ be a principal fibre bundle, $E(M, F, G, P)$ be the associated vector bundle with standard fibre $F = \mathbb{R}^n$ on which G acts through the representation ρ. As in proposition 2.19, for $m \in M$ and $u \in \pi^{-1}(m)$, π denoting the projection mapping of $P(M, G)$, u can be considered as a linear isomorphism of F onto $F_m = \pi_E^{-1}(m)$, where π_E^{-1} denotes the natural projection mapping $E \to M$.

Let S be the set of cross sections $\varphi : M \to E$. S forms a vector space over $\mathbb{R}$ and a module over the algebra of $\mathbb{R}$-valued functions on M with the addition and multiplication operations defined as below:

$$\begin{aligned}(\varphi+\psi)(m) &:= \varphi(m)+\psi(m)\\(\lambda\varphi(m)) &:= \lambda(m)\varphi(m)\end{aligned}$$

for $\varphi, \psi \in S$, $m \in M$ and λ an $\mathbb{R}$-valued function on M.

Let Γ be a connection in P. The parallel displacement of fibres of $E(M, F, G, P)$ with standard fibre F is defined via the definition of a horizontal lift in E as follows:

Let $w \in E$. Then w is the equivalence class of some $(u, \xi) \in P \times F$, namely $w = [(u, \xi)]$ for some $u \in P$, $\xi \in F$. Fix this $\xi \in F$ and consider the mapping $\nu : v \in P \mapsto [(v, \xi)] \in E$. The *horizontal space at* w, denoted by H_w is defined to be the image of the horizontal space $H_u \in T_uP$ under the pushforward of ν, i.e. $H_w = \nu_*(H_u)$.

Two easy conclusions are that the definition of H_w does not depend on the representative $(u, \xi) \in P \times \mathbb{R}^n$, and T_wE can be decomposed into $T_wE = F_w \oplus H_w$, where F_w denotes the tangent subspace of T_wE along the fibre $\pi_E^{-1}(m)$ with $\pi_E(w) = m$.

Definition 3.11. A curve in E is said to be *horizontal*, if its tangent vectors are horizontal at each point. Consider a curve $\gamma(t)$ in M. The *horizontal lift of* γ *in* E is a horizontal curve $\tilde{\gamma}$ in E that satisfies $\pi_E(\tilde{\gamma}) = \gamma$.

Proposition 3.12. *Given a curve* $\gamma(t)$ *in* M, $0 \le t \le 1$ *and a point* $w_0 \in E(M, F, G, P)$, *there exists a unique lift* $\tilde{\gamma}(t)$ *starting from* w_0.

Proof. Choose a point $(u_0, \xi) \in P \times F$ such that $[(u_0, \xi)] = w_0$. Let $u_t := \hat{\gamma}(t)$ be the unique horizontal lift of $\gamma(t)$ starting from u_0, whose existence and uniqueness are given in proposition 2.34. Then $w_t := u_t\xi$ is a horizontal lift of $\gamma(t)$ starting from w_0. The uniqueness of w_t follows from the uniqueness of u_t and the fundamental theorem of ODE's.

□

Then parallel displacement of fibres of $E(M, F, G, P)$ is defined similar to definition 2.36:

Definition 3.13. Let $\gamma(t)$ be a curve in M connecting two points m_0 and m_1 in M. Let w_0 be an arbitrary point of E such that $\pi_E(w_0) = m_0$. The unique horizontal lift $\tilde{\gamma}(t)$ of $\gamma(t)$ has the end point $w_1 \in \pi_E^{-1}(m_1)$. The map $\tau : w_0 \mapsto w_1$ is called the *parallel displacement*(or *parallel transport*) along the curve γ.

Remark 3.14. Note that, beside the similarities between the definitions of a horizontal lift of a curve in M to a principal bundle (definition 2.33) and horizontal lift to vector bundle definition (3.11), definition 3.11 is induced by the definition as observed in the proof of proposition 3.12.

Let φ be a section of E over a given curve $\gamma(t)$ so that $\pi_E \circ \varphi(\gamma(t)) = \gamma(t)$. The existence of such a section is guaranteed by proposition 12.2 in Steenrod [28]. Let $\gamma'(t)$ denote the vector tangent to $\gamma(t)$ at $t \in [0, 1]$.

Definition 3.15. The *covariant derivative* $\nabla_{\gamma'(t)}\varphi$ *of* φ *in the direction of* $\gamma'(t)$ is defined as

$$\nabla_{\gamma'(t)}\varphi := \lim_{h \to 0} \frac{1}{h} \left[\tilde{\gamma}_t^{t+h}(\varphi(\gamma(t+h))) - \varphi(\gamma(t)) \right] \tag{3.1}$$

where $\tilde{\gamma}_t^{t+h}$ denotes the parallel displacement of the fibre $\pi_E^{-1}(\gamma(t+h))$ to the fibre $\pi_E^{-1}(\gamma(t))$.

By definition 3.15, $\nabla_{\gamma'(t)}\varphi \in \pi_E^{-1}(\gamma(t))$ for all t. So, $\nabla_{\gamma'(t)}\varphi$ defines a cross section in E over $\gamma(t) \subset M$. Moreover, the cross section φ is parallel, namely $\varphi(\gamma(t))$ is a horizontal curve in E if and only if $\nabla_{\gamma'(t)}\varphi = 0$ for all t.

Remark 3.16. If $X \in T_mM$ is a vector at $m \in M$, and φ is defined in a neighborhood of m, then $\nabla_X\varphi$ is defined naturally as follows: Take a curve $\gamma(t)$ for $-\varepsilon \leq 0 \leq \varepsilon$ such that $\gamma(0) = m$ and $\gamma'(0) = X$. Define $\nabla_X\varphi := \nabla_{\gamma'(0)}\varphi$. It is clear this definition depends only on γ' but not on $\gamma(t)$.

Remark 3.17. If φ is a cross section in E defined on M and X is a vector field on M, then *the covariant derivative of* φ *in the direction of* X is defined as

$$(\nabla_X\varphi)(m) = \nabla_{X_m}\varphi$$

One can define covariant derivative in an equivalent alternative way:

Let U be an open neighborhood in M, on where a cross section φ is defined. Using the proposition 2.19, we can associate with φ an $\mathbb{R}^n$-valued function f on $\pi^{-1}(U) \subset P$ as follows:

$$f : v \in \pi^{-1}(u) \mapsto v^{-1}(\varphi(\pi(v))) \in \mathbb{R}^n \tag{3.2}$$

where v^{-1} is considered as a linear isomorphism from $F_m = \pi_E^{-1}(m) \subset E$ to the standard fibre $F = \mathbb{R}^n$.

Proposition 3.18. *For $X \in T_mM$, $u \in \pi^{-1}(m)$, the cross section φ defined on a neighborhood at $m \in M$ and f defined as above, the following equality holds:*

$$\nabla_X\varphi = u(X^*f) \tag{3.3}$$

where X^* is the horizontal lift of X to T_uP.

Proof. First, observe that $u(X * f)$ is an element of $\pi_E^{-1}(m)$: Since $X^* \in T_uP$ and f is an

$\mathbb{R}^n$-valued function on $\pi^{-1}(U)$, X^*f takes values in $\mathbb{R}^n$. Then $u(X^*f)$ belongs to $\pi_E^{-1}(m)$ as in proposition 2.19.

Now, let $\gamma(t)$, $-\varepsilon \leq t \leq \varepsilon$ be a curve in M such that $X = \gamma'(0)$. Take the horizontal lift $u_t = \hat{\gamma}(t)$ of γ to P such that $u_0 = u$. Then $\frac{\mathrm{d}}{\mathrm{d}t}u_0 = X^*$ and

$$X^*f = \lim_{h\to 0}\frac{1}{h}\big[f(u_h) - f(u_0)\big] = \lim_{h\to 0}\frac{1}{h}\big[u_h^{-1}(\varphi(\gamma(h))) - u^{-1}(\varphi(\gamma(0)))\big]$$

Applying u to both sides,

$$u(X^*f) = \lim_{h\to 0}\frac{1}{h}\big[uu_h^{-1}(\varphi(\gamma(h))) - (\varphi(\gamma(0)))\big]$$

Set $\xi := u_h^{-1}(\varphi(\gamma(h)))$. Then, by very definition, $u_t\xi$ is a horizontal curve in E through u. But $u_h\xi = \varphi(\gamma(h))$ implies that $\varphi(\gamma(h))$ is an element of E obtained by the parallel displacement of $u_0\xi = uu_h^{-1}(\varphi(\gamma(h)))$ along γ from $\gamma(0)$ to $\gamma(h)$. Then by definition of $\tilde{\gamma}_0^h$, $\tilde{\gamma}_0^h(\varphi(\gamma(h))) = uu_h^{-1}(\varphi(\gamma(h)))$. So,

$$\tilde{\gamma}_0^h(\varphi(\gamma(h))) = uu_h^{-1}(\varphi(\gamma(h)))$$

and the proposition follows. $\square$

Using proposition 3.18, the fact that $X^* + Y^*$ is the horizontal lift of $X + Y$ in P and the definition 3.15, it is easy to prove the following consequences:

Proposition 3.19. *Let X and Y be vector fields on M, φ and ψ be cross sections of E over M and λ is an $\mathbb{R}$-valued function on M. Then*

(1) $\nabla_{(X+Y)}\varphi = \nabla_X\varphi + \nabla_Y\varphi$
(2) $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$

(3) $\nabla_{(\lambda X)}\varphi = \lambda\nabla_X\varphi$

(4) $\nabla_X(\lambda\varphi) = \lambda\nabla_X\varphi + (X\lambda)\nabla_X\varphi$

Before defining linear connections, we investigate the bundle of linear frames:

Let M be a manifold of dimension n. If F^M is the frame bundle as defined in section 2.2, then the projection map π sends a linear frame $u \in F^M$ to m.

$GL(n, \mathbb{R})$ acts on F^M on the right: If $a = (a_{ij}) \in GL(n, \mathbb{R})$ and $u = (X_1, \ldots, X_n)_m$, then $ua := (a_{j1}X_1, \ldots, a_{jn}X_n)$. This action is free.

As in proposition 2.19, a linear frame u maps $\xi \in \mathbb{R}^n$ to $[(u, \xi)] \in T_mM$ such that $ue_i = X_i \in T_mM$, where e_i is a basis element of $\mathbb{R}^n$ and X_i is a vector in T_mM. Then the action of $GL(n, \mathbb{R})$ on F^M can be interpreted as the composition of the maps

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_mM \tag{3.4}$$

Let M be a manifold, F^M be its frame bundle and $GL(n, \mathbb{R})$ be the Lie group acting on the frame bundle.

Definition 3.20. A connection in the frame bundle F^M of M is called a *linear connection*.

Before we define the curvature tensor field of a manifold M, we first define the canonical 1-form θ on F^M:

Definition 3.21. Let $X \in T_uF^M$, for some $u \in F^M$. The $\mathbb{R}^n$-valued 1-form θ on F^M defined as

$$\theta(X) := u^{-1}(\pi_*(X))$$

is called the *canonical 1-form on F^M*.

Proposition 3.22. *The canonical 1-form θ satisfies:*

(1) θ is horizontal, i.e. $\theta(X) = 0$ for vertical vectors $X \in T_u F^M$,

(2) $R_a^* \theta = a^{-1}\theta$, for $a \in GL(n\mathbb{R})$.

Proof. (1) If X is a vertical vector, then $\pi_*(X) = 0$, hence $\theta(X) = 0$.

(2) Let $X \in T_u F^M$, $a \in GL(n\mathbb{R})$. Then $R_{a*}X$ is a vector at $ua \in F^M$. Then

$$R_a^*\theta(X) = \theta((R_a)_*X) = (ua)^{-1}(\pi_*(R_{a*})X) = a^{-1}u^{-1}(\pi_*(X)) = a^{-1}(\theta(X))$$

□

We are ready to define the curvature tensor field R on a manifold.

Definition 3.23. Let $u \in F^M$ such that $\pi(u) = m \in M$ and X^*, Y^* are vectors in $T_u F^M$ such that $\pi_*(X^*) = X$ and $\pi_*(Y^*) = Y$, where $X, Y \in T_m M$. The *curvature tensor* R is defined as

$$R(X,Y)Z := u(2\Omega(X^*, Y^*))(u^{-1}Z)$$

where u is considered as a non-singular map from $\mathbb{R}^n$ to $T_m M$, as above and Ω is the curvature form of F^M as in definition 2.39.

Note that $R(X,Y)Z$ depends only on X, Y and Z, but not on u, X^* and Y^*, applying equation 2.15 to the curvature 2-form Ω. In the definition 3.23, $(2\Omega(X^*, Y^*))(u^{-1}Z)$ denotes the image of $u^{-1}Z \in \mathbb{R}^n$ by the linear endomorphism $2\Omega(X^*, Y^*) \in \mathfrak{gl}(n, \mathbb{R})$ of

$\mathbb{R}^n$. Thus $R(X, Y)$ is an endomorphism of T_mM and is called the *curvature transformation* of T_mM determined by X and Y. It follows R is a tensor of type $(1, 3)$ such that $R(X, Y) = -R(Y, X)$.

There is an equivalent definition of curvature tensor in terms of covariant derivative:

Theorem 3.24. *In terms of covariant differentiation, the curvature R can be expressed as*

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

where X, Y and Z are vector fields on M.

Proof. Let X^*, Y^* and Z^* be the horizontal lifts of X, Y and Z, respectively. By proposition 3.18, $(\nabla_X Y)_m = u(X^*_u f)$, where f is an $\mathbb{R}^n$-valued function defined by $f(u) = u^{-1}(Y_m)$. Hence $f(u) = \theta(Y^*_u)$ for $u \in F^M$. Therefore we obtain

$$(\nabla_X Y)_m = u(X^*_u(\theta(Y^*))) \tag{3.5}$$

where $\pi(u) = m$.

Set $f = \theta(Z^*)$ so that f is an $\mathbb{R}^n$-valued function on F^M that satisfies the conditions in proposition 3.22. We have then

$$\begin{aligned} ([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)_m &= u(X^*_u(Y^* f) - Y^*_u(X^* f) - ([X^*, Y^*]_u)_H f) \\ &= u(([X^*, Y^*]_u)_V f) \end{aligned}$$

where $\cdot_H$, and $\cdot_V$ denote the horizontal and vertical components of the corresponding vectors, respectively. Let A be an element of $\mathfrak{gl}(n, \mathbb{R})$ such that $A^\#_u = ([X^*, Y^*]_u)_V$, where $A^\#$ is the fundamental vector field corresponding to A, as in the proposition 2.9. Then by equation 2.17a,

$$2\Omega(X_u^*, Y_u^*) = \omega([X^*, Y^*]_u) = -A$$

where ω is the linear connection of M.

On the other hand, if $a_t = \exp tA$ is the 1-parameter subgroup of $GL(n, \mathbb{R})$ generated by A, then

$$\begin{aligned} A_u^{\#} f &= \lim_{t \to 0} \frac{1}{t} \left[f(ua_t) - f(u) \right] \\ &= \lim_{t \to 0} \frac{1}{t} \left[a_t^{-1} f(u) - f(u) \right] \\ &= -A(f(u)) \end{aligned}$$

where $A(f(u))$ denotes the result of the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ applied to $f(u) \in \mathbb{R}^n$. Therefore, we have

$$\begin{aligned} ([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)_m &= u(([X^*, Y^*]_u)_V f) \\ &= u(-A(f(u))) \\ &= u(2\Omega(X_u^*, Y_u^*)(f(u))) \\ &= u(2\Omega(X_u^*, Y_u^*)(u^{-1}Z)) \\ &= R(X, Y)Z \end{aligned} \tag{3.6}$$

□

Remember that we have identified $\mathscr{H}(u)$ with $u \in F^M$ with the holonomy group Θ_m at $m \in M$ by remark 3.4. Moreover, since curvature tensor on the manifold M is related closely to the curvature 2-form on the principal bundle F^M, and horizontal lifts on TM is defined by horizontal lifts in F^M, we expect Ambrose-Singer theorem to have a corresponding version for the tangent bundle TM of a manifold M. Now, we are ready to state and prove the Ambrose-Singer theorem using curvature tensor.

Theorem 3.25 (Ambrose-Singer Theorem)**.** *The Lie algebra $\mathfrak{h}(m)$ of the holonomy group Θ_m is equal to the subspace of linear endomorphisms of T_mM spanned by all elements of the form $(g_\gamma R)(X,Y) = g_\gamma^{-1} \circ R(g_\gamma X, g_\gamma Y) \circ g_\gamma$, where $X, Y \in T_mM$ and $g_\gamma \in G$ corresponds to the parallel displacement along an arbitrary piecewise differentiable curve γ starting from m.*

Proof. Let Γ be a linear connection on a manifold M. Let m be a point in the manifold M and X, Y be vectors in T_mM. Choose $u \in F^M$ such that $\pi(u) = m$.

From theorem 3.9, we know that the Lie algebra $\mathfrak{h}(u)$ of the holonomy group $\mathscr{H}(u)$ equals the span of elements of the form $\Omega(X^*, Y^*)$, where X^*, Y^* vary among the horizontal vectors at $u \in F^M$.

By definition of curvature tensor R on M,

$$R(X,Y) := u(2\Omega(X^*, Y^*))(u^{-1}) \tag{3.7}$$

where X^*, Y^* are horizontal vectors at u.

Since T_mM is isomorphic to the horizontal space H_u, as X, Y vary in T_mM, X^*, Y^* vary in H_u and therefore

$$\text{span } \{R(X,Y)\} \approx \text{span } \{\Omega(X^*, Y^*)\}$$

by equation 3.7.

Let γ be a loop based at m. Denote the element of the holonomy group $\mathscr{H}(u)$ by g_γ. Then the $u \in F^M$ is translated to $v = ug_\gamma \in F^M$ by the horizontal lift of γ. So, the vectors X^* and Y^* are parallel translated to $(R_{g_\gamma})_* X^*$ and $(R_{g_\gamma})_* Y^*$, respectively. Then

$$\Omega((R_{g_\gamma})_* X^*, (R_{g_\gamma})_* Y^*) = g_\gamma \Omega(X^*, Y^*) g_\gamma^{-1} \tag{3.8}$$

by equation 2.15. So, span $\{\Omega(X^*, Y^*)\} \approx g_\gamma \Omega(X^*, Y^*) g_\gamma^{-1}$, since parallel translation g_γ induces an isomorphism between the horizontal space H_u and H_v.

But this implies that

$$\text{span } \{R(g_\gamma X, g_\gamma Y)\} \approx \text{span } g_\gamma \Omega(X^*, Y^*) g_\gamma^{-1}$$

and the second version of the Ambrose-Singer Theorem follows from the following isomorphism and theorem 3.9

$$\text{span } \{g_\gamma^{-1} R(g_\gamma X, g_\gamma Y) g_\gamma\} \approx \text{span } \Omega(X^*, Y^*)$$

□

Remark 3.26. As it can be observed from its proof, Ambrose-Singer theorem looks redundant at the first sight, where one has to use all elements of the holonomy group to generate the Lie algebra of the holonomy group, so it is not too effective for computing the holonomy group. On the other hand, it has a theoretical importance, where it relates the curvature form (or tensor) to the holonomy group and we will use the theorem heavily in the next chapter to prove Berger's classification theorem.

4. CLASSIFICATION OF RIEMANNIAN MANIFOLDS WITH RESPECT TO THEIR HOLONOMY GROUPS

4.1. Reducible and Symmetric Spaces

In this section we will state some important theorems on the reducible and symmetric Riemannian spaces without going into details of their proofs which we will make use of in the next section.

A Riemannian manifold (M,g) is said to be *reducible*, if it is isometric to a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$, i.e. there is an isometry from (M, g) to the product space $M_1 \times M_2$ with the metric $g_1 \times g_2$

Proposition 4.1. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy $\mathscr{H}(g_1 \times g_2) = \mathscr{H}(g_1) \times \mathscr{H}(g_2)$*

This proposition follows from the *de Rham decomposition theorem.* For its proof, see section IV.6 of Kobayashi and Nomizu [17].

Definition 4.2. A Riemannian manifold (M, g) is said to be a *symmetric space* if for every point $p \in M$, there is an isometry $s_p : M \to M$ such that s_p^2 is the identity transformation on M and p is an isolated fixed point of s_p. (M, g) is said to be *locally symmetric* if the isometry s_p above is defined locally.

Definition 4.3. A *homogeneous space* is a manifold M on which a Lie group G acts transitively.

Proposition 4.4. *Let (M, g) be a connected, simply connected Riemannian symmetric space. Then g is complete, namely every geodesic can be extended infinitely. Let G be the group of isometries of (M, g) generated by elements of the form $s_p \circ s_q$ for $p, q \in M$. Then*

G is a connected Lie group acting transitively on M. Choose $p \in M$, let H be the subgroup of G fixing p. Then H is a closed, connected Lie subgroup of G and M is the homogeneous space G/H.

Sketch of the proof. In the proof, one starts with a geodesic γ passing through a point $p \in M$, then it is extended to the whole of M using the fact that $s_p(\exp_p(X)) = exp_p(-X)$ for s_p being an isometry as in definition 4.2 and composing two such isometries at points lying closely on the geodesic. Using geodesic segments, whose existence are proved, one can show that G acts transitively on M and the generating elements $s_q \circ s_p$ of G can be joined to the identity isometry by smooth paths, yielding G to be connected. The claim $M \approx G/H$ is proved using the fact that the isometry group of (M, g) is a Lie group acting smoothly on M by the Myers-Steenrod Theorem, see page 39 of [4] and Yamabe Theorem, see [29], which states that an arcwise connected subgroup (in this proposition G) of a Lie group (the isometry group of M) itself is a Lie group. For the details of the proof, see page 48 of Joyce [16]. □

Definition 4.5. Given a tensor field K of type (r, s), the *covariant derivative of K in the direction of a vector field X* is defined to be a tensor field ∇K of type $(r, s+1)$:

$$(\nabla K)(X_1, \ldots, X_s, X) := \nabla_X(K(X_1, \ldots, X_s)) - \sum_{i=1}^{s} K(X_1, \ldots, \nabla_X X_i, \ldots, X_s)$$

Proposition 4.6. *Let M, g, G, p, H be as above. Then the holonomy group of M is H and the Riemannian curvature satisfies $\nabla R = 0$.*

Sketch of the proof. To show that H is the holonomy group of M, one decomposes $\mathfrak{g}$ into direct sum $\mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{g}$, $\mathfrak{h}$ denote the Lie algebras of G and H, respectively, $\mathfrak{m}$ is the eigenspace of s_p with eigenvalue -1 and $\mathfrak{h}$ is corresponds to the eigenspace with eigenvalue $+1$. Using this splittin, one constructs a linear connection ∇, which satisfies $\nabla g = 0$, hence coincides with the Levi-Civita connection. Then, one uses the Ambrose-Singer Theorem to show that H is the holonomy group of M.

Since M is symmetric, at $p \in M$, any tensor field of type (r, s) where $r + s$ is odd which is invariant by the local symmetry at p, is 0 at $p \in M$, see page 303 of Kobayashi and Nomizu [17]. Hence $\nabla R = 0$.

For the details of the proof, see page 50 of Joyce [16]. □

These propositions enable us to classify simply-connected Riemannian symmetric spaces by looking at Lie groups. Cartan has classified all these groups in 1926-1927, see [1], [2]. For Cartan's proof, see Helgason [22] and for the tables of all possibilities see Besse [4].

4.2. Berger's List

After the discussions above, we are confronted with the problem of classification of Riemannian manifolds having the following properties:

(a) We assume that M is simply-connected so that the problem of classification reduces to a local one. Then it is sufficient for us to work with the restricted holonomy group $\mathscr{H}_0(g)$.

(b) We suppose that the Riemannian metric g is irreducible. We know that if g is reducible then by the proposition 4.1, the holonomy group of g is the product of holonomy groups in lower dimensions.

(c) If g is locally symmetric, then $\mathscr{H}_0(g)$ lies on the list of holonomy groups of Riemannian symmetric spaces.

So, we will restrict ourselves to simply-connected, irreducible and nonsymmetric Riemannian manifolds. Marcel Berger [4] proved that:

Theorem 4.7 (Berger's List)**.** *If M is a simply-connected manifold of dimension n and g is a Riemannian metric on M which is irreducible and nonsymmetric, then exacly one of the following cases holds:*

(i) $\mathscr{H}_0(g) = SO(n)$

(ii) $n = 2m$ *and* $\mathscr{H}_0(g) = U(m) \subset SO(2m)$ *for* $m \geq 2$

(iii) $n = 2m$ *and* $\mathscr{H}_0(g) = SU(m) \subset SO(2m)$ *for* $m \geq 2$

(iv) $n = 4m$ *and* $\mathscr{H}_0(g) = Sp(1) \cdot Sp(m) \subset SO(4m)$ *for* $m \geq 2$

(v) $n = 4m$ *and* $\mathscr{H}_0(g) = Sp(m) \subset SO(4m)$ *for* $m \geq 2$

(vi) $n = 16$ *and* $\mathscr{H}_0(g) = Spin(9) \subset SO(16)$

(vii) $n = 8$ *and* $\mathscr{H}_0(g) = Spin(7) \subset SO(8)$

(viii) $n = 7$ *and* $\mathscr{H}_0(g) = G_2 \subset SO(7)$

Remark 4.8. The definitions of the above groups are given in section 4.4.

Remark 4.9. In cases (ii)-(v), we require $m \geq 2$ to avoid repetitions in the list. For $m = 1$, the above cases yield:

(ii) $U(1) = SO(2)$

(iii) $SU(1) = \{1\}$, which acts reducibly and can be regarded as $SO(1) \times SO(1)$ acting on $\mathbb{R} \oplus \mathbb{R}$

(iv) $Sp(1) = SU(2)$

(v) $Sp(1)Sp(1) = SO(4)$

The original proof of Berger was based on Elie Cartan's classification of irreducible linear real representations (in this case orthogonal ones) of real Lie groups, see [1] and [2]. As M is simply-connected, $\mathscr{H}(g)$ is a closed subgroup of $SO(n)$, since g is irreducible the representation of $\mathscr{H}(g)$ on $\mathbb{R}^n$ is irreducible. Berger took all closed, connected subgroups of $SO(n)$ with irreducible representations on $\mathbb{R}^n$ from the above two lists of Cartan. He then put these groups into two tests containing Jacobi identities as in definition 2.4 and $\nabla R = 0$ condition. The list turns out to be exactly the same as the list of Lie groups acting transitively on spheres except one case, case (vi). The list of Lie groups acting transitively on spheres was found by Montgomery and Samelson [14], [30] and Borel [31], [32]:

Theorem 4.10 (Classification of Lie groups acting transitively on S^{n-1})**.** *Assume the Lie group G acts transitively on S^{n-1}. Then G belongs to one of the following cases*

(i) $G = SO(n)$

(ii) $n = 2m$ *and* $G = U(m)$ *for* $m \geq 2$

(iii) $n = 2m$ *and* $G = SU(m)$ *for* $m \geq 2$

(iv) $n = 4m$ *and* $G = Sp(1) \cdot Sp(m)$ *for* $m \geq 2$

(iv)' $n = 4m$ *and* $G = T \cdot Sp(m)$ *for* $m \geq 2$

(v) $n = 4m$ *and* $G = Sp(m)$ *for* $m \geq 2$

(vi) $n = 16$ *and* $G = Spin(9)$

(vii) $n = 8$ *and* $G = Spin(7)$

(viii) $n = 7$ *and* $G = G_2$

Remark 4.11. In this list, T denotes the matrix Lie group which consists complex numbers modulus 1, which is called *torus*, see Helgason [22].

The nonequivalent cases are (vi) in the first list and $(iv)'$ in the second. By 10.66 of Besse [4], if the holonomy representation of some Riemannian manifold is contained in $T \cdot SP(m)$, then it is necessarily contained in $Sp(m)$. Alekseevskii [33] showed that any Riemannian metric with holonomy group $Spin(9)$ is symmetric. After excluding them the remaning cases are the same. Using that observation Simons managed to prove Berger's theorem:

Theorem 4.12. *Let M be an irreducible Riemannian manifold. If $\mathscr{H}_p$ is not transitive on the unit sphere S^{n-1} in T_pM, then M is a symmetric space of dimension ≥ 2.*

4.3. Simons' Proof

In this section, we will try to follow Simons' proof of Berger's classification theorem from his article *On The Transitivity of Holonomy Systems* [13], by pointing out his crucial arguments and motivations behind them. Good discussions of [13] can also be found in Salamon [15] and Besse [4].

We can divide the proof into three main steps:

Part 1: Simons constructs an algebraic structure which he calls a *holonomy system* $[V, R, G]$ using an Euclidean vector space $V \cong \mathbb{R}^n$, a special tensor R of type $(1, 3)$ and a compact group of isometries G of V. He first proves that a holonomy system which is irreducible and not transitive on the unit sphere is symmetric.

Part 2: He shows that the Ambrose-Singer Theorem (theorem 3.25) provides a holonomy system $[T_pM, R(p), \mathscr{H}_0(p)]$ at every point $p \in M$. Then assuming irreducibility and nontransitiveness at each point, $[T_pM, R(p), \mathscr{H}_0(p)]$ becomes symmetric.

Part 3: The last part is to organize these systems and extend the pointwise proven theorem to M by showing that curvature tensors at various points of M are proportional using an unpublished theorem of Konstant.

Simons' proof consists of five sections, which we will follow in the remaining:

(i) *Definitions*: Simons defines a holonomy group, a holonomy system, a curvature tensor and discusses the symmetry and reducibility of holonomy systems. These definitions are usually analogous to those for Riemannian manifolds but there are some differences which we will comment on.

(ii) *Flat subspaces and transitivity*: Simons defines flat spaces and gives a characterization of transitive action of holonomy group on a unit sphere.

(iii) *Main theorem*: Simons defines totally geodesic vector subspaces of V and proves the main theorem which tells that an irreducible holonomy system is symmetric if the holonomy group is not transitive on S^{n-1}. He uses a complicated induction on totally geodesic vector subspaces and the characterization he found in the second section.

(iv) *A partial classification*: In this section of Simons' article [13], Simons gives a corollary of his algebraic construction which enables to construct a symmetric space of rank 1 given a nonzero curvature and an irreducible holonomy system $S = [V, R, G]$. Since this part is not used in the proof of Berger's classification theorem, we will not discuss it.

(v) *Applications*: Using Konstant's result, Simons extends the theorem to M.

4.3.1. Definitions

For the algebraic construction the most important guide will be the Ambrose-Singer Theorem 3.25. We will mimic what happens in theorem 3.25 to construct our algebraic setup. Let V be an Euclidean vector space of dimension n like T_pM. Let $\mathcal{P}$ to denote $(1,3)$-type tensors on V. For $P \in \mathcal{P}$, this means that $P \in V \otimes V^* \otimes V^* \otimes V^*$, V^* denoting the dual space of V. Let $O(n)$ denote the group of isometries of V. Define a *representation of $O(n)$ on $\mathcal{P}$* as

$$g(P)_{x,y} := g \circ P_{g^{-1}(x) \cdot g^{-1}(y)} \circ g^{-1} \tag{4.1}$$

Here $P_{x,y}$ denotes the tensor P contracted with vectors $x, y \in V$ and can be considered as a linear map from V to V. This action of $O(n)$ is a reproduction of the action of the holonomy group on the curvature tensor as in 3.25.

Denote the skew-symmetric operators on V as $\mathcal{A}$. Since the skew-symmetric operators form the Lie algebra of $O(n)$, every $A \in \mathcal{A}$ can be considered as a tangent vector of a curve $g(t) \in O(n)$ with $g(0) = e$. The corresponding representation of the Lie algebra $\mathcal{A}$ on $\mathcal{P}$ is given as:

$$A(P)_{x,y} = -P_{A(x),y} - P_{x,A(y)} - [P_{x,y}, A] \tag{4.2}$$

for $A \in \mathcal{A}$, $x, y \in V$ and $P \in \mathcal{P}$. Since $\mathcal{A}$ form the Lie algebra of $O(n)$, the exponential map is defined $\exp : \mathcal{A} \to O(n)$ and equations 4.1 and 4.2 are related as

$$\lim_{t \to 0} \frac{\exp tA(P) - P}{t} = A(P) \tag{4.3}$$

Definition 4.13. Let $R \in \mathcal{P}$. R is called a *curvature* tensor if the following hold:

$$\begin{aligned} R_{x,y} &= -R_{y,x} && (4.4) \\ R_{x,y}z + R_{z,x}y + R_{y,z}x &= 0 && (4.5) \\ < R_{x,y}z, w > &= - < R_{x,y}w, z > && (4.6) \\ < R_{x,y}z, w > &= < R_{z,w}x, y > && (4.7) \end{aligned}$$

Remark 4.14. Note that a curvature tensor on a Riemannian manifold satisfies these conditions at any point.

It is not hard to show that given a curvature tensor R on V, $g(R)$ and $A(R)$ satisfy the equations 4.4-4.7. The linear combination of curvature tensors are again curvature tensors. Denote $G(R)$ the linear span of all $g(R)$ as $g \in G$. It contains all the curvature tensors induced by R by the G representation on R and it is a vector space.

Definition 4.15. Let R be a curvature tensor on V, G a compact group of isometries of V (a subset of $O(n)$) with Lie algebra $\mathfrak{g}$ (a subset of $\mathcal{A}$). Then G is called a *holonomy group* of R if $R_{x,y} \in \mathfrak{g}$ for all $x, y \in V$.

Remark 4.16. G being a compact group of isometries, it is a subgroup of $O(n)$. Also note that the restricted holonomy group of a Riemannian manifold M is a closed subgroup of $SO(n)$, where dim$M = n$, see page 186 of [17].

Moreover, if $R_{x,y} \in \mathfrak{g}$ for all $x, y \in V$, then $g(R)_{x,y} \in \mathfrak{g}$ using the adjoint representation of G on $\mathfrak{g}$ and $A(R)_{x,y} \in \mathfrak{g}$ by equation 4.2. Thus if G is a holonomy group for R, it is a holonomy group for $g(R)$ and $A(R)$ for any $g \in G$, $A \in \mathcal{A}$.

Definition 4.17. A triple $[V, R, G]$ is called a *holonomy system*, if G is a connected holonomy group for the curvature tensor R on the Euclidean space V.

For a holonomy system, if $Q \in G(R)$, then $g(Q) \in G(R)$, $A(Q) \in G(R)$ and $Q_{x,y} \in \mathfrak{g}$ where $g \in G$, $A \in \mathfrak{g}$. Let $\mathfrak{g}^R$ denote the subspace of $\mathfrak{g}$ spanned by $Q_{x,y}$, $Q \in G(R)$. $\mathfrak{g}$ is a ring with respect to the bracket operation and real numbers as scalars. Then $\mathfrak{g}^R$ is an ideal in $\mathfrak{g}$, where it is a subring and bracket of elements from $\mathfrak{g}^R$ with other elements of $\mathfrak{g}$ belong to $\mathfrak{g}^R$.

By the assumption that G being compact and theorem 5.8 of Sepanski [34] $\mathfrak{g}^R$ has a complimentary ideal $\mathfrak{g}_R$, $\mathfrak{g} = \mathfrak{g}^R \oplus \mathfrak{g}_R$. Using 4.2 and 4.7, it is easy to prove:

Proposition 4.18. *If* $A \in \mathfrak{g}_R$, *then* $A(Q) = 0$ *for any* $Q \in G(R)$.

Keeping the $\nabla R = 0$ characterization of a locally symmetric space in mind, see proposition 4.4 of do Carmo [26], we define a holonomy system to be symmetric as follows:

Definition 4.19. A holonomy system $S = [V, R, G]$ is said to be *symmetric* if $g(R) = R$ for all $g \in G$, or equivalently by equation 4.3 $A(R) = 0$ for all $A \in \mathfrak{g}$.

Definition 4.20. A holonomy system $S = [V, R, G]$ is called *reducible*, if G acts reducibly on V, i.e. G leaves a non-trivial subspace of V invariant.

The rest of this section is divided to explain two phenomena:

(1) For a symmetric holonomy system, there is a corresponding simply connected Riemannian symmetric space M whose tangent space, curvature and holonomy algebra can be identified with V, R and $\mathfrak{g}^R$ of the holonomy system, respectively, see proposition 3.6 of Helgason [22] for details.

(2) Using the following proposition

Proposition 4.21. *Let $S = [V, R, G]$ be a holonomy system. Let $G^R = \exp \mathfrak{g}^R$. Then G^R is compact and so $S^R := [V, R, G^R]$ is a holonomy system.*

which is proved by decomposing V to subspaces V_i on which G^R acts irreducibly, restricting G^R to V_i's, then decomposing $\mathfrak{g}$ to subideals $\mathfrak{g}_i$ spanned by elements of the form Q_{x_i,y_i} with $Q \in G(R)$ and $x_i, y_i \in V$, defining $G_i^R := \exp \mathfrak{g}_i$, showing that $S_i = \{V_i, R_i, G_i^R\}$'s are holonomy systems and finally decomposing $S^R \cong S_1 \times \cdots \times S_p$ as a product of holonomy systems, we decompose a holonomy system to a product of smaller irreducible holonomy systems. Note that, this proposition corresponds to the decomposition of holonomy groups as in proposition 4.1.

4.3.2. Flat subspaces and transitivity

The reason that we define flat subspaces is that they provide a good characterization of transitive action of G^R on the unit sphere.

Definition 4.22. A subspace $W \subset V$ is called *flat* if $Q_{w,v} = 0$ for any $w, v \in W$ and for all $Q \in G(R)$.

Remark 4.23. For a flat Riemannian manifold the curvature vanishes. Thus extending the pointwise definition of flatness to a submanifold N of M, N would be a flat submanifold of M.

There is a characterization of flatness of a subspace W of M:

Lemma 4.24. *W is flat if and only if $A(W) \subset W^{\perp}$ for all $A \in \mathfrak{g}^R$.*

Proof. The proof is easy and relies on equation 4.7: W is flat if and only if

$$< Q_{w,v}x, y >= 0$$

for $Q \in G(R)$, $w, v \in W$ and $x, y \in V$. By 4.7

$$< Q_{w,v}x, y >=< Q_{x,y}w, v >$$

But $Q_{x.y}w$ is an arbitrary element of $\mathfrak{g}^R(W)$. Hence $\mathfrak{g}^R(W) \subset W$. □

Every 1-dimensional subspace of V is flat by equation 4.4. Thus a flat W is called *nontrivial* if dim $W \geq 2$.

The definition of flatness brings us to the first important theorem Simons' proof:

Theorem 4.25. *G^R is transitive on the unit sphere, $S^{n-1} \subset V$ if and only if there exists no nontrivial flat subspace of V.*

Proof. Restrict the action of G^R to S^{n-1}. Let $p \in S^{n-1}$, let T_pS^{n-1} be the tangent space at p. Moreover take an arbitrary $h \in G^R$. The group action of G^R on S^{n-1} induces a map for fixed $p \in S^{n-1}$:

$$\phi_p : g \in G^R \mapsto g(p) \in S^{n-1}$$

Decomposing $\mathrm{d}\phi_p = L_h \circ \phi_p \circ L_h^{-1}$, where L_h stands for left translation, we see that

$$\mathrm{d}\phi_p(G_h^R) = \mathrm{d}L_h \circ \mathrm{d}\phi_p \circ \mathrm{d}L_h^{-1}(G_h^R) = \mathrm{d}L_h \circ \mathrm{d}\phi_p(G_e^R) \tag{4.8}$$

Since L_h is smooth and invertible, it is of full rank, hence $\mathrm{d}\phi_p(G_h^R) \approx \mathrm{d}\phi_p(G_e^R)$. If $\mathrm{d}\phi_p$ is not of maximal rank everywhere, it is of maximal rank nowhere and this holds for any $p \in S^{n-1}$. Since G^R acts transitively on S^{n-1}, ϕ_p maps G^R onto S^{n-1} and $\mathrm{d}\phi_p$ is of maximal rank everywhere. To prove the theorem, we make some identifications:

$$\begin{aligned} G_e^R &\approx \mathfrak{g}^R \\ S_P^{n-1} &\approx p^\perp \end{aligned}$$

Since $A = \frac{\mathrm{d}}{\mathrm{d}t} g(t)$ for some curve $g(t) \subset G^R$ with $g(0) = e$, it follows that

$$\mathrm{d}\phi_p(A) = A(p) \tag{4.9}$$

and therefore $\mathrm{d}\phi_p(G_e^R) = S_p^{n-1}$, i.e. $\mathfrak{g}^R(p) = p^\perp$ Assuming that p lies in a nontrivial flat subspace W of V, lemma 4.24 implies that $W^\perp \supset \mathfrak{g}^R(W) \supset p^\perp$. This is a contradiction, since $\mathrm{dim} W^\perp \leq n-2$ and $\mathrm{dim} p^\perp = n-1$.

Conversely, assuming that G^R is not transitive on S^{n-1} implies by equation 4.9 and theorem 7.15 of Lee [23] that $\mathrm{d}\phi_p$ is of maximal rank nowhere. Therefore $\mathfrak{g}^R(p) = \mathrm{d}\phi_p(\mathfrak{g}^R) \subsetneq p^\perp$. Choosing w such that $w \in p^\perp$ and $w \in [\mathfrak{g}^R(p)]^\perp$ and defining $W := \mathrm{span}\{p, w\}$ yields the nontrivial flat subspace W of V. $\square$

4.3.3. The main theorem

In this section, we are going to finish the first and most important part of the proof. To do this we need to define totally geodesic subspaces of V.

Definition 4.26. A subspace $M \subset V$ is called *totally geodesic* if

$$Q_{l,m}n \in M$$

for all $Q \in G(R)$ and $l, m, n \in M$. In other words $Q|_M$ is a curvature tensor on M.

Remark 4.27. To capture the motivation behind this definition, remember that if N is a totally geodesic submanifold on M, then its second fundamental form vanishes, see page 132 of do Carmo [26]. Then by Gauss' theorem on do Carmo [26] page 130, the sectional curvatures of M and N turns out to be equal on N. Then again by do Carmo [26], lemma 3.3 on page 94, the curvature tensor restricted to N equals to the curvature tensor on M, as in definition 4.26.

Remark 4.28. By proposition 7.1 on page 224 of Helgason [22], if N is a totally geodesic submanifold of M, then M being locally symmetric implies N to be totally geodesic. This motivates us to do an induction over totally geodesic submanifolds of M.

Theorem 4.29. *Let M be totally geodesic. Let $\mathfrak{I} = \{A \in \mathfrak{g} \,|\, A(Q)|_M = 0$ for any $Q \in G(R)\}$ Then $\mathfrak{I}$ is an ideal in $\mathfrak{g}$. Moreover,*

(a) $\mathfrak{g}_R \subset \mathfrak{I}$,

(b) $Q_{m,x} \in \mathfrak{I}$ *for* $m \in M$, $x \in M^{\perp}$ *and* $Q \in G(R)$.

Proof. The restriction map $Q \in G(R) \mapsto Q_M$ is a linear homomorphism. Consider the kernel of this homomorphism:

$$K := \{Q \in G(R) \,|\, Q_{l,m}n = 0 \text{ for } l, m, n \in M\}$$

Using equation 4.5, it is easily shown that for $A \in \mathfrak{g}$ and $Q \in K$, $A(Q) \in K$, so K is a subspace of $G(R)$ invariant under the representation of $\mathfrak{g}$ on $G(R)$. Thus, $A \in \mathfrak{I}$ if and only if $A(G(R)) \subset K$. For such an $A \in \mathfrak{g}$, for any $B \in \mathfrak{g}$ and $Q \in G(R)$,

$$[B, A](Q) = B(A(Q)) - A(B(Q)) \in K$$

Therefore, $\mathfrak{I}$ is an ideal in $\mathfrak{g}$.

(a) follows from proposition 4.18. To prove (b), note that if for $A \in \mathfrak{g}$, $A(M) \subset M^{\perp}$, then $A \in \mathfrak{I}$, by skew-symmetry of A and properties of curvature. For $Q_{m,x}$ such that $Q \in G(R)$, $m \in M$, $x \in M^{\perp}$, $Q_{m,x}(M) \subset M^{\perp}$ and part (b) follows. □

In the remaining part of this section our aim is to prove 4.33, which plays a very important role in Simon's proof. First, we give some results on which the proof of 4.33 depends. For the rest of this section, let $[S, R, G]$ be an irreducible holonomy system, and let G^R be nontransitive on S^{n-1}.

Since G^R is assumed to be nontransitive on S^{n-1}, we can choose a nontrivial flat subspace of V by 4.25. For $Q \in G(R)$, we define the bilinear mapping:

$$T_Q : (v, w) \in W \times W \mapsto Q_{v,x}w$$

where $Q_{v,x}w$ is a symmetric bilinear transformation by identity 4.5 and the fact that W is flat. Moreover,

$$[T_Q(v, w), T_Q(s, t)] = 0 \tag{4.10}$$

for $v, w, s, t \in W$ by the fact that $Q_{A(w),v} + Q_{w,A(v)} = 0$ for $Q \in G(R)$, $A \in \mathfrak{g}$ and $w, v \in W$ and symmetry of T_Q.

By equation 4.10, we obtain a commutative family of symmetric transformations $\{T_Q(v,w)\}_{v,w \in W}$ of V, for fixed $Q \in G(R)$. $\{T_Q(v,w)\}_{v,w \in W}$ can be diagonalized, since they are symmetric. Moreover they have the same set of orthonormal eigenvectors $\{X_k\}$, where they commute, so the diagonalization can be made simultaneously with respect to $\{X_k\}$. But then we obtain symmetric bilinear forms k that satisfy

$$T_Q(v,w)(X_k) = k(v,w)X_k$$

We will call the symmetric bilinear forms k *eigenvalues* of T_Q, and they depend on $(v,w) \in W \times W$. If X_k is a nontrivial eigenvector of $T_Q(v,w)$ for fixed Q, then k is a nonzero symmetric bilinear form on W. Hence there exists $x \in W$ such that $k(x,x) \neq 0$ and

$$X_k = \frac{1}{k(x,x)} Q_{x,X_k} x \tag{4.11}$$

Defining

$$U_k := \{u \in W \mid k(u,x) = 0\}$$

U_k is a nonempty subset of W such that $W = U_k \oplus \operatorname{span}\{x\}$, and using 4.11 and the fact that $Q_{A(w),v} + Q_{w,A(v)} = 0$ for $Q \in G(R)$, we prove a useful lemma:

Lemma 4.30. *For any $P \in G(R)$ and $u \in U_k$,*

$$P_{u,X_k} = 0$$

In particular $Q_{u,X_k} = 0$ and U_k is the kernel of k, where $k(w,u) = 0$ for any $w \in W$, $u \in U_k$.

Define a totally geodesic subspace of V depending on U_k by

$$M_k := \{m \in V \,|\, P_{u,m} = 0 \text{ for all } u \in U_k, P \in G(R)\} \tag{4.12}$$

The fact that M_k is a totally geodesic subspace follows from the definition of M_k and and the equations 4.2 and identity 4.5.

Remark 4.31. Summarizing what is done above: For fixed $Q \in G(R)$, we have obtained subspaces of U_k of W with codimension 1 corresponding to non-trivial eigenvectors (if they exist). Then U_k gives rise to M_k, which are totally geodesic and which contain W for each k.

Lemma 4.32. *Let X_k be one of the nontrivial eigenvectors of $T_Q(v,w)$. Let M_l be a totally geodesic subspace determined by $T_P(v,w)$ for $P \in G(R)$. Then $X_k \in M_l$ or $X_k \in M_l^{\perp}$.*

Proof. Assume $X_k \notin M_l$. Then since $X_k \in M_k$ by 4.30, $U_k \neq U_l$. Pick $w \in U_l \setminus U_k$. Since $U_k = \ker k$ and $\dim W \setminus U_k = 1$, $k(w,w) \neq 0$. So $X_k = \frac{1}{k(w,w)} Q_{w,X_k} w$. Let $m \in M_l$, Then

$$< X_k, m >= \frac{1}{k(w,w)} < Q_{w,m} w, X_k >= 0$$

where $w \in U_l$, $m \in M_l$ and therefore $Q_{w,m} w = 0$ □

Now we are about to define a subspace $Z(W)$ which is totally geodesic and contains W. The strategy after this definition is to show that this subspace $Z(W)$ is built in such a way that $Z(W) \supset V$ will imply the symmetry of $S = [V, R, G]$.

Let W be a flat subspace of V. Define

$$Z(W) = \{z \in V \,|\, T_Q(w,v)(z) = 0 \text{ for all } v, w \in W \text{ and } Q \in G(R)\}$$

By definition, $Z(W)$ is the intersection of the nullspaces of all the $T_Q(v,w)$. If we set $X := \text{span}\{X_k \in V \,|\, X_k$ is a nontrivial eigenvector for $Q \in G(R)\}$, then clearly,

$$V = Z(W) \oplus X \tag{4.13}$$

Since W is spanned by eigenvectors corresponding to 0 for every Q, $W \subset Z(W)$. The fact that $Z(W)$ is totally geodesic is not hard to prove and follows from its very definition and identity 4.5.

We are ready to state and prove the main theorem:

Theorem 4.33. *Let $S = [V, R, G]$ be an irreducible holonomy system. If G^R is not transitive on S^{n-1}, then S is symmetric.*

Proof. The theorem is true for $\dim V = 1$ or 2, where $S^0 = \{-1, +1\}$, and S^1 is 1-dimensional. We will prove the theorem by a strong induction on $\dim V$. Assume the theorem is true for $\dim V \leq n$. Let $S = [V, R, G]$ be an irreducible holonomy system with $\dim V = n + 1$. Let W be a maximal nontrivial flat subspace of V. Since $g(W)$ is flat for any $g \in G$, every $v \in V$ lies in some maximal flat W. We can choose W such that $Z(W) \neq V$, since if for every maximal W, $Z(W) = V$, then for all $x, y \in V$, $R_{x,y}x = 0$. This would imply that $< R_{x,y}x, y >= 0$ for any $x, y \in V$. Then by lemmas 3.3 and 3.4 of do Carmo [26] $R \equiv 0$, so S would be a symmetric holonomy system.

Now fix W to be a maximal flat subspace of V such that $Z(W) \neq V$. Put H' to be a subgroup of G under whose action $Z(W)$ remains invariant. Then H' is a compact subgroup of G. If $y, z \in Z(W)$, $Q \in G(R)$, then $Q_{y,z} \in \mathfrak{h}'$, the Lie algebra of H', since $Z(W)$ is totally geodesic.

Let $H = H'|_{Z(W)}$, $\mathfrak{h}$ be the Lie algebra of H. Denote $\hat{Q} := Q|_{Z(W)}$. By the previous observation, for any $Q \in G(R)$, $Z_Q = [Z(W), \hat{Q}, H]$ is a holonomy system.

Since $\dim Z(W) < \dim V$, the induction hypothesis holds for Z_Q. Assume Z_Q is irreducible. We will first show that for $w \in W$, $y, z \in Z(W)$ and $Q \in G(R)$,

$$Q_{w,y} z = 0 \tag{4.14}$$

There are two cases. First, let Z_Q be symmetric. $\hat{Q}_{y,z} = 0$ if and only if $< \hat{Q}_{y,z} y, z >= 0$, by lemmas 3.3 and 3.4 of do Carmo [26]. Since $w \in W$, for any $y \in Z(W)$, $\hat{Q}_{w,y} w = 0$, thus $< \hat{Q}_{w,y} w, y = 0 >$, So, $Q_{w,y} z = 0$ for $w \in W$ and $y, z \in Z(W)$.

By induction hypothesis, if Z_Q is not symmetric, then H is transitive on the unit sphere of $Z(W)$. Let $h' \in H' \subset G$. Then for $y \in Z(W)$ and $w \in W$,

$$h'(\hat{Q})_{w,y} y = 0$$

by and equations 4.5 and 4.4. Set $h = h'|_{Z(W)}$, then $h(\hat{Q})_{w,y} w = 0$. By equation 4.1 and arbitrariness of y,

$$\hat{Q}_{h(w),y} h(w) = 0$$

Since H is transitive in $Z(W)$ and $W \subset Z(W)$,

$$\hat{Q}_{z,y} z = 0$$

for all $y, z \in Z(W)$. Then $< \hat{Q}_{z,y}z, y >= 0$ implies that $\hat{Q} \equiv 0$, in particular $\hat{Q}_{w,y} = 0$, so again $Q_{w,y}z = 0$ for $w \in W$ and $y, z \in Z(W)$.

If Z_Q is reducible, then by proposition 4.21, one can decompose Z_Q into smaller irreducible systems and then prove the equality $Q_{w,y}z = 0$ in those irreducible systems.

Let $y \in [\sum_k M_k]^\perp$, where the sum is over nonzero eigenvalues k of $T_Q(v, w)$ and M_k is defined as in 4.12. Since for $w \in W$, $Q \in G(R)$, $Q_{w,y}(M_k) \subset M_k$ and $Q_{w,y}(M_k) \subset (M_k)^\perp$ by the proof of theorem 4.29. This forces $Q_{y,w}(M_k)$ to be zero. On the other hand, $Q_{w,y}(Z(W)) = 0$. But since $V = Z(W) \oplus X$, $Q_{w,y} = 0$ for any $Q \in G(R)$. Since W is maximal, $y \in W$, contradicting the assumption $y \in [\sum_k M_k]^\perp$. Therefore $[\sum_k M_k]^\perp = \emptyset$ and

$$V = \sum_{k \text{ nonzero eigenvalue of } T_Q(v,w)} M_k \tag{4.15}$$

Now, we will prove that $\bigcap M_k = W$: If there is only one M_k, then by lemma 4.15, $V = M_k$. Thus for $u \in U_k$, $x \in V$, $Q \in G(R)$, $Q_{u,x} = 0$. Defining an invariant subspace

$$J := \{v \in V \mid Q_{v,x} = 0 \text{for } x \in V, Q \in G(R)\}$$

under G by 4.1, we reach the conclusion $J = V$, since G is irreducible. Thus $R = 0$ and S is symmetric.

If there are several M_k's, take M_k and M_l such that $U_k \neq U_l$. Since each U_k has codimension 1 in W, $W = U_k + U_l$. So, if $x \in M_k \cap M_l$, then $Q_{w,x} = 0$ for all $w \in W$ and $Q \in G(R)$. Since W is a maximal flat subspace, $x \in W$. So, $\bigcap M_k = W$ and we reach the conclusion

$$\sum_k M_k^{\perp} = W^{\perp} \tag{4.16}$$

Remark 4.34. Let $w \in W$ and $x \in M_k^{\perp}$ for some M_k. Define A to be the set $A := P_{w,x}$ for $P \in G(R)$. Take a nontrivial eigenvector $X_k \in M_k$. Then letting $W' := \mathit{span}\{W, X_k\}$ implies that W' is flat in the holonomy system $S^A = [V, A(R), G]$, by theorem 4.29, lemma 4.32 and equation 4.15.

Let A be defined as in remark 4.34. Since for $Q \in G(R)$, $A \in \mathfrak{g}$, $A(Q) \in G(R)$, $\mathfrak{g}^{A(R)} \subset \mathfrak{g}^R$. Moreover by the same reason, $\mathfrak{g}^{A(R)}$ is an ideal in $\mathfrak{g}^R$. Let W be a maximal subspace of V such that

$$\mathfrak{g}^R(W) \subset W^{\perp}$$

Since $W' = \mathrm{span}\{W, X_k\}$, $W \subset W'$ is a proper subset of W and

$$\mathfrak{g}^R(W') \subset W'^{\perp}$$

$\mathfrak{g}^{A(R)}$ is a proper ideal of $\mathfrak{g}^R$.

Since by 4.16, $\sum_k M_k^{\perp} = W^{\perp}$, we can choose a basis $\{x_i\}$ of $W^{\perp}$ such that $x_i \in M_k^{\perp}$ for different k's. Let $\{y_i\}$ be a basis of W. Then they form together a basis for V. By condition 4.34, $\dim \mathfrak{g}^{Q_{w,x_i}(R)} < \dim \mathfrak{g}^R$. For $\{y_i\} \in W$, $\dim \mathfrak{g}^{Q_{w,y_i}(R)} < \dim \mathfrak{g}^R$ holds trivially, hence for the basis $\{y_i\}$ of V, $\dim \mathfrak{g}^{Q_{w,y_i}(R)} < \dim \mathfrak{g}^R$. $g(W)$ is a flat subspace for any $g \in G$. Moreover it is a maximal flat subspace, since W is a maximal flat subspace. Note that $Z(g(W)) = g(Z(W))$ by the very definition of $Z(W)$. Since $Z(W) \neq V$, $Z(g(W)) \neq V$. Then for the maximal flat subspace $g(W)$ all the constructions above hold.

Taking $z_i := g(y_i)$, $\{z_i\}$ is a basis of V satisfying that

$$\dim \mathfrak{g}^{Q_{g(w),z_i}(R)} < \dim \mathfrak{g}^R$$

Since $\mathfrak{g}$ is irreducible on V, $g(w)$ span V for $g \in G$, $w \in W$. Therefore $\{Q_{g(w),z_i}\}$ span $\mathfrak{g}^R$ and we can choose a basis $\{A_i\}$ among them such that

$$\dim \mathfrak{g}^{A_i(R)} < \dim \mathfrak{g}^R \tag{4.17}$$

Now, we proceed by the second induction on $\dim \mathfrak{g}^R$. Assume $S = [V, R, G]$ is a holonomy system of dimension $n+1$ (Remember that we are proving $\dim V = n+1$ case in the first induction). Let $k = \dim \mathfrak{g}^R$. The theorem 4.33 is true for $k = 0$ trivially. Now, by induction hypothesis, assume the theorem holds for $k \leq p$. Let $S = [V, R, G]$ as above of dimension $n+1$ such that $\dim \mathfrak{g}^R = p+1$. For any $B \in \mathfrak{g}_R$, $B(R) = 0$ by proposition 4.18. Then the condition 4.17 assures the existence of a basis $\{A_i\} \subset \mathfrak{g}$ such that

$$\dim \mathfrak{g}^{A_i(R)} \leq p < \dim \mathfrak{g}^R$$

Let $S^{A_i} := [V, A_i(R), G]$. Then $\dim S^{A_i} = \dim V = n+1$ and S^{A_i} is irreducible. By the second induction hypothesis, the systems S^{A_i} are symmetric.

Let $B \in \mathfrak{g}$. For any A_i, $B(A_i(R)) = 0$ by the definition of symmetry. Since $\{A_i\}$ span $\mathfrak{g}$, $B(A(R)) = 0$ for any $A, B \in \mathfrak{g}$. In particular, taking $A = B$, $B(B(R)) = 0$.

But B is a skew-symmetric operator. Then $B(B(R)) = 0$ implies that $B(R) = 0$. Since B is an arbitrary element of $\mathfrak{g}$, S is symmetric.

Thus both inductions are complete. □

4.3.4. Applications

In this section, we will go from the algebraic statements in the previous sections to a conclusion on a Riemannian manifold, stating that if the holonomy group $\mathscr{H}_p$ is not transitive on the unit sphere in T_pM for $p \in M$, then the Riemannian manifold M is symmetric of rank ≥ 2, given that M is not a reducible Riemannian manifold.

Ambrose-Singer Theorem 3.25 provides us a holonomy system $[V = T_pM, R(p), \mathscr{H}_0(p)]$ at every point in M. This is because the curvature tensor evaluated at a point $p \in M$ yields by the Ambrose-Singer theorem the holonomy group $\mathscr{H}_0(p)$, where the action of the holonomy group on the term $R(X, Y)$ for $X, Y \in T_pM$ at $p \in M$ as in 4.1 corresponds to the Lie algebra of the holonomy group. Assuming irreducibility and non-transitiveness, we know that every $[V = T_pM, R(p), \mathscr{H}_0(p)]$ is symmetric.

We have to organize these systems. Simons does it relying on an unpublished result of Konstant:

Theorem 4.35 (Konstant's theorem)**.** *Let R and R' be two nonzero curvature tensors on V. Let G be a compact group of isometries acting irreducibly on V. Suppose that $S = [V, R, G]$ and $S' = [V, R', G]$ are both symmetric holonomy systems. Then $R = cR'$, where c is a non-zero real number.*

After defining the Ricci tensor:

Definition 4.36. Ricci tensor L_R is defined

$$L_R(x, y) := \mathrm{tr}\varphi(x, y)$$

where $\varphi(x, y) : v \in V \mapsto R_{x,v}y \in V$.

Simons cites a theorem of Yano and Bochner [35]:

Theorem 4.37 (Yano-Bochner). *Let M be a Riemmanian manifold of dimension ≥ 3. Denote by R the curvature tensor field on M, and by R^p the curvature tensor on T_pM for each $p \in M$. Suppose there is a real valued function f on M such that*

$$L_{R^p} = f(p) <, >$$

at each $p \in M$. Then f is a constant.

Now, we are able to extend the main theorem 4.33 to a Riemannian manifold M:

Theorem 4.38. *Let M be an irreducible Riemannian manifold of dimension ≥ 3. At each point $p \in M$ let H_p denote the connected component of the holonomy group on T_pM. By theorem 3.25, $S_p = [T_pM, R_p, H_p]$ is an irreducible holonomy system. If each S_p is symmetric, then M is symmetric.*

Proof. Firstly, Simons shows that if G acts reducibly, then there is a real number β

$$L_{R_p}(,) = \beta <, > \tag{4.18}$$

for an irreducible symmetric holonomy system at each point $p \in M$, by using the very definition of the Ricci curvature L_p at $p \in M$. By the previous theorem 4.37, the above equation 4.18 holds for all $p \in M$.

Let γ be the parallel transport on the Riemannian manifold M from q to p, where p, q are two fixed points in M. Then by Ambrose-Singer Theorem, $S_{\gamma(p)} = [T_pM, \gamma(R_q), H_p]$ is an irreducible holonomy system, where $\gamma(R_q)$ is defined as

$$\gamma(R_q)_{x,y} = \gamma^{-1} \circ R_{\gamma^{-1}(x),\gamma^{-1}(y)} \circ \gamma$$

For $g \in \mathscr{H}_p$, $\gamma^{-1} \circ g \circ \gamma \in H_q$. Therefore, since $g(\gamma(R_q)) = R_q$, $S_{\gamma(p)}$ is an irreducible symmetric holonomy system, too.

So, $g(\gamma(R_q)) = cR_p$ by Konstant's theorem 4.35. Putting all this together, we get

$$cL_{R_p}(\, , \,) = L_{\gamma(R_q)}(\, , \,) = \beta < , >= L_{R_p}(\, , \,)$$

which implies that $c = 1$. Hence

$$\gamma(R_q) = R_p$$

Then $\nabla R = 0$, since R does not change under any parallel transport. Therefore M is symmetric. □

Theorem 4.39. *Let M be an irreducible Riemannian manifold. If H_p is not transitive on the unit sphere in T_pM, then M is a symmetric space of rank ≥ 2.*

Proof. For $\dim M = 2$, note that the only connected group of isometries acting non-transitively on $S^1 \subset T_pM$ for $p \in M$ is the identity. This is so, since connectedness implies path-connectedness for Lie groups. But identity mapping leaves any vector subspace of T_pM invariant trivially, contradicting the irreducibility assumption. For $\dim M \geq 3$, the theorem follows as a corollary of theorems 4.33 and 4.38. □

4.4. Some comments on Berger's List and Developments

In the following, let $(x_1, \ldots, x_n)$ and $(z_1, \ldots, z_n)$ be points in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

4.4.1. $SO(n)$

The *special orthogonal group* $SO(n)$ is the group of matrices $A \in SL(n,\mathbb{R})$ which leave invariant the quadratic form $x_1^2 + \cdots + x_n^2$, namely satisfying $A^T A = I_n$, where $SL(n,\mathbb{R})$ is the group of real matrices of det $= 1$, the *special linear group.*

This is the generic case. The restricted holonomy group of a Riemannian manifold M is a closed subgroup of $SO(n)$. For the proof, see page 186 of Kobayashi and Nomizu [17] or Borel and Lichnerowicz [5].

4.4.2. $U(m)$**,** $n = 2m$**,** $m \geq 2$

The *unitary group* $U(m)$ is the group of complex matrices $A \in GL(m,\mathbb{C})$ which leave invariant the Hermitian form $z_1\bar{z}_1 + \cdots + z_m\bar{z}_m$, namely satisfying $A^T I \bar{A} = I$.

This is the general case of a Kähler manifold which is an important class of complex manifolds. For the theory of Kähler manifolds, see for example Griffiths and Harris [36] and Kobayashi and Nomizu [37].

4.4.3. $SU(m)$**,** $n = 2m$

The *special unitary group* $SU(m)$ is the subgroup of the unitary group consisting if matrices with determinant 1.

In 1960, Calabi gave the first examples of such special Kähler manifolds, but his examples were only local, not complete. In 1970, Calabi got complete examples, see Calabi [38] which were quite difficult to exhibit. The first compact examples were given by Yau, after he proved Calabi's conjecture. On any compact Kähler manifold whose first Chern class is zero there will exists some Kähler metric with $\mathscr{H}_0 \subset SU(m)$, see Yau [39]. To obtain a manifold with $\mathscr{H}_0 = SU(m)$, one can take some algebraic hypersurface of degree $m+2$ in $\mathbb{C}P^{m+1}$ as a Kähler manifold such that its first Chern class vanishes. If g is Kähler, then $\mathscr{H}_0 \subset SU(m)$ if and only if g is Ricci-flat. Thus Calabi-Yau metrics are locally the same as Ricci-flat Kähler metrics. The best known example of a Calabi-Yau manifold is the $K3$ surface with Kummer construction, which admits a family of metrics with holonomy $SU(2)$.

4.4.4. $Sp(m)$, $n = 4m$, $m \geq 2$

The *symplectic group* $Sp(m)$ is the subgroup $GL(2m, \mathbb{R})$ which leave invariant the exterior form $z_1 \wedge z_{n+1} + z_2 \wedge z_{n+2} + \cdots + z_m \wedge z_{2m}$, which is also called the *symplectic form.*

Such metrics are called hyperkähler. As $Sp(m) \subset SU(2m) \subset U(2m)$, hyperkähler metrics are Ricci-flat and Kähler. Calabi managed to give complete, but not compact, hyperkähler metrics in 1979 [38]. Compact examples were only gotten in 1981 for $m = 2$ by Fujiki [40]. Yau's proof of Calabi conjecture is essential in the construction. For higher m, it was achieved in 1982 by Beauville [41].

4.4.5. $Sp(1) \cdot Sp(m)$, $n = 4m$, $m \geq 2$

The group of unit quaternions acting on $\mathbb{H}^m \approx \mathbb{R}^{4m}$ by right multiplication defines a subgroup $Sp(1)$ of $SO(4m)$. It's centralizer in $SO(4m)$ is precisely the symplectic group $Sp(m)$. The intersection of $Sp(1)$ and $Sp(m)$ consists of plus and minus the identity. If $m > 1$, the two groups generate a proper subgroup of $SO(4m)$: $Sp(1) \cdot Sp(m) := Sp(1) \times_{\mathbb{Z}_2} Sp(m)$.

The metrics having $Sp(1) \cdot Sp(m)$ as their holonomy groups are called *quaternionic Kähler metrics.* They are Einstein, but not Ricci-flat. The best known example is the quoternionic projective space $\mathbb{H}P^m$. For more examples, see Galicki and Lawson [42]. For the theory of quaternionic Kähler manifolds, see Salamon [43].

4.4.6. $Spin(7)$ and G_2

G_2 is the subgroup of $GL(7,\mathbb{R})$ that preserves the exterior 3-form $\varphi_0 = \mathrm{d}x_{123}+\mathrm{d}x_{145}+\mathrm{d}x_{167}+\mathrm{d}x_{246}-\mathrm{d}x_{257}-\mathrm{d}x_{347}-\mathrm{d}x_{356}$, where $\mathrm{d}x_{ijk}$ denotes the exterior form $\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k$ on $\mathbb{R}^7$.

$Spin(7)$ is the subgroup of $GL(8,\mathbb{R})$ that preserves the exterior 4-form $\psi_0 = \mathrm{d}x_{1234}+\mathrm{d}x_{1256}+\mathrm{d}x_{1278}+\mathrm{d}x_{1357}-\mathrm{d}x_{1368}-\mathrm{d}x_{1458}-\mathrm{d}x_{1467}-\mathrm{d}x_{2358}-\mathrm{d}x_{2367}-\mathrm{d}x_{2457}-\mathrm{d}x_{2468}+\mathrm{d}x_{3456}+\mathrm{d}x_{3478}+\mathrm{d}x_{5678}$

They are called exceptional holonomy groups, since they don't belong to the set of classical Lie groups. The existence of metrics with holonomy G_2 and $Spin(7)$ was first established by Bryant [44], using the theory of exterior differential systems. Explicit examples of complete metrics with holonomy G_2 and $Spin(7)$ are found by Bryant and Salamon, reference [45]. Metrics with compact manifolds were constructed by Joyce, for the G_2 case [6],[7], for the $Spin(7)$ case [46]. G_2 case provides the only known examples of compact, simply connected Ricci-flat manifolds of odd dimension.

REFERENCES

1. Cartan, E., *Les groupes projectifs qui ne laissent invariante aucune multiplicite plane*, J. Math. pures et appliquees 10, p. 149-186, 1914.

2. Cartan, E., *Les groupes projectifs continus reels qui ne laissent invariante aucune multiplicite plane*, Bull. Soc. Math. Fr 41, p. 53-96, 1913.

3. Besse, A. L., *Einstein Manifolds*, Springer, Berlin, 1987.

4. Berger, M., *Sur les groupes d'holonomie des varietes a connexion affine et des varietes riemmaniennes*, Bull. Soc. Math. Fr. 83, p. 279-330, 1955.

5. Borel, A. and A. Lichnerowicz, *Groupes d'holonomie des varietes riemanniennes*, C. R. Acad. Sci. Paris 234, p.1853-1837, 1952.

6. Joyce, D. D., *Compact Riemannian 7-manifolds with holonomy G_2 I*, Journal of Differential Geometry 43, p.291-328, 1996.

7. Joyce, D. D., *Compact Riemannian 7-manifolds with holonomy G_2 II*, Journal of Differential Geometry 43, p.329-375, 1996.

8. Bryant, R. L., *Two exotic holonomies in dimension four, path geometries and twistor geometry*, AMS Proc. Symp. Pure Math. 53, p.33-88, 1991.

9. Chi, Q. S., S. Merkulov and L. Schwachhofer, *On the existence of infinite series of exotic holonomies*, Inventiones Math. 126, p. 391-411, 1996.

10. Chi, Q. S. and L. Schwachhofer, *Exotic holonomy on moduli spaces of rational curves*, Differential Geom. App. 8, p.105-134, 1998.

11. Bryant, R. L., *Recent advances in the theory of holonomy*, Seminaire N. Bourbaki 861, p. 351-374, 1998.

12. Hano J. and H. Ozeki, *On the holonomy group of linear connections*, Nagoya Math. Journal 10, p.97-100, 1956.

13. Simons, J., *On the transitivity of holonomy Systems*, Annals of Math. 76, p.213-234, 1962.

14. Montgomery, D. and H. Samelson, *Transformation groups of spheres*, Annals of Math. 44, p. 454-470, 1943.

15. Salamon, S. M., *Riemannian Geometry and Holonomy Groups*, volume 201 of Pitman Research Notes in Mathematics, Longman, Harlow, 1989.

16. Joyce, D. D., *Compact Manifolds with Special Holonomy*, Oxford University Press, New York, 2000.

17. Kobayashi S. and K. Nomizu, *Foundations of Differential Geometry*, volume 1, Wiley, New York, 1963.

18. Boothby, W. M., *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, San Diego, 2003.

19. Conlon, L., *Differentiable Manifolds*, Birkhäuser, Boston, 2nd Ed., 2001.

20. Nakahara, M., *Geometry, Topology and Physics*, Institute of Physics Publishing, London, 2nd Ed., 2005.

21. Warner, F. W., *Foundations of Differential Manifolds and Lie Groups*, Scott, Foresman and co., Illinois, 1971.

22. Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces*, American Mathematical Society, Rhode Island, 2001.

23. Lee, J. M., *Introduction to Smooth Manifolds*, Springer Verlag, New York, 2003.

24. Husemoller, D., *Fibre Bundles*, Springer Verlag, New York, 1994.

25. Choquet-Bruhat, Y., C. DeWitt-Morette and M. Dilliard-Bleick, *Analysis, Manifolds and Physics*, volume 1, North-Holland, Amsterdam, 1989.

26. do Carmo, M. P., *Riemannian Geometry*, Birkhäuser, Boston, ed. 2, 1993.

27. Ambrose, W. and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math Soc. 234, p.428-443, 1953.

28. Steenrod, N., *The Topology of Fibre Bundles*, Princeton University Press, New Jersey, 1951.

29. Yamabe, H., *On the arcwise connected subgroup of a Lie group*, Osaka Math Journal 2, p.13-14, 1950.

30. Montgomery, D. and H. Samelson, *Groups transitive on the n-dimensional torus*, Bull. AMS 49, p. 455-456, 1943.

31. Borel, A., *Some remarks about the Lie groups transitive on spheres and tori*, Bull. AMS 55, p. 580-587, 1949.

32. Borel, A., *La plan projectif des octaves et les spheres comme espaces homogenes*, C. R. Acad. Sc. Paris 230, p. 1378-1380, 1950.

33. Alekseevskii, D. V., *Riemannian symmetric spaces with exceptional holonomy*, Func-

tional Analysis and its Applications 2, p. 97-105, 1968.

34. Sepanski, M., *Compact Lie Groups*, Springer, New York, 2007.

35. Yano, K. and S. Bochner, *Curvature and Betti numbers*, Annals of Math. Studies, no 32, Princeton, 1953

36. Griffiths, P. and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

37. Kobayashi S. and K. Nomizu, *Foundations of Differential Geometry*, volume 2, Wiley, New York, 1963.

38. Calabi, E., *Metriques kahleriennes et fibres holomorphes*, Annales scientifiques de l'Ecole Normale Superieure 12, p.269-294, 1979.

39. Yau, S. T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equations*, Communications on pure and applied mathematics 31, p.339-411, 1978.

40. Fujiki, A., *On primetively symplectic Kähler V-manifolds of dimension 4* in Ueno, K. editor, *Classification of algebraic and analytic manifolds*, Progress in Mathematics 39, p.71-250, Birkhäuser, Boston, 1983.

41. Beauville, A., *Varietes Kähleriennes dont la lere classe de Chern est nulle*, Journ. Diff. Geom. 18, 755-782, 1983.

42. Galicki, K. and H. B. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Mathematische Annalen 282, p. 1-21, 1988.

43. Salamon, S. M., *Quaternionic Kaehler manifolds*, Inventiones mathematicae 67, p.143-171, 1982.

44. Bryant, R. L., *Metrics with exceptional holonomy*, Annals of Math. 126, p.525-576, 1987.

45. Bryant, R. L. and S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Mathematical Journal 58, p. 829-850, 1989.

46. Joyce, D. D., *Compact 8-manifolds with holonomy* $Spin(7)$, Inventiones mathematicae 123, p.507-552, 1996.