DIRAC-DELTA POTENTIALS ON A LATTICE

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ABSTRACT

DIRAC-DELTA POTENTIALS ON A LATTICE

In this thesis, the Dirac delta potential on a one dimensional infinite lattice structure [2] and its band gaps [3] are reviewed. Then, Green's function and regularization & renormalization process for the bound states in two dimensions are mentioned [6–8]. The relation between the Green's function and the renormalization of the interaction cofactor are shown. After that, the work of Albeverio on two dimensional lattice which shows that there is a solution of the Hamiltonian is reviewed [2]. In this model, the positions of Dirac delta potentials are in all lattice points in the one-to-one manner [10, 11].Renormalization process is needed. Semi-relativisitic approach is used to look Dirac-delta potentials on infinite two dimensional lattice. Green's function is found for the classic and semi-relativistic cases. Lastly, a new formula of interaction cofactor is found for the arbitrary shape of Dirac-delta potential in two dimensional lattice in the aspects of Quantum Mechanics. And, the Green's function is written for this model. The formula shows us the existence of a solution of the Hamiltonian. Also, the three dimensional version of the formula is mentioned briefly. This formula is applied on three different patterns being two of them are line and the other pattern is circular. None of these examples do not need renormalization.

ÖZET

ÖRGÜDE DIRAC-DELTA POTANSİYELLERİ

Bu tezde, bir boyuttaki sonsuz örgü için Dirac-delta potansiyeline Schrodinder denklemi uygulandı ve enerji bantları bulundu. Burada, Bloch teoreminin en basit uygulamalarından birini gördük.Sonrasında, 2 boyuttaki tek Dirac-delta potansiyeli için Green fonksiyonuna baktık ve onun etkileşim katsayısının renormalizasyonuyla olan ilişkisini inceledik. Bir sonraki bölümde, Dirac-delta potansiyellerini 2 boyuttaki sonsuz örgüye yerleştirdik ve onun kuvantum mekanik ve kuvantum alanlar teorisi kullanarak açıklamasını yapıp, Green fonksiyonlarını yazdık. En sonunda, noktasal Dirac-delta potansiyeli yerine rastgele eğri kullandık. Bu şekildeki potansiyelleri 2 boyuttaki sonsuz örgüye yerleştirip, etkileşim katsayısı için çözüm denklemini ürettik.3 tane örnek seçip, bunları bulduğumuz formüle uyguladık.

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LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
λ	Interaction coefficient
λ_R	Renormalized interaction coefficient
Λ	Bravais lattice
$\widehat{\Gamma}$	Wigner-Seitz cell
Γ	Dual lattice
$\widehat{\Lambda}$	Primitive cell of dual lattice

LIST OF ACRONYMS/ABBREVIATIONS

1D	One Dimensional
2D	Two Dimensional
3D	Three Dimensional
Ei	Exponential integral
J_ν	Bessel function of the first kind
J_0	Bessel function of the first kind with order zero
Sinc	Sinc function
Γ	Gamma function

1. INTRODUCTION

The Dirac delta function has been a very important tool to physicists and applied on many problems, for example to demonstrate the Poisson's equation for a point charge, since the first deep studies of Kirchhoff (1882, 1891), Heaviside (1893, 1899) and Dirac (1926, 1930) [1]. Many remarkable physicists worked with this tool since it is essential to understand first the concept of the Dirac delta function to explain some critical topics on Maxwell Equations, Quantum mechanics and Quantum Field Theory. For instance, the Dirac-delta potential which we will be interested in is one of the most important subjects for scattering problems in Quantum Theory.

One of the fundamental studies on the applications of the Dirac delta potentials is where the delta function is placed on a lattice which is composed of repeating cells. They can be defined as smallest spatial portion of the lattice. These lattice structures can have arbitrary dimensions. Also, the Dirac delta potentials can be shaped as any patterns instead of being a point. We are free to choose these models to be finite or infinite. They have a difference only in the boundary conditions. Under these circumstances, what happens to the physical variables of a particle can be observed clearly on a view of Quantum Mechanics and that of Quantum Field Theory.

In this thesis, we will review the Dirac delta potential on a one dimensional infinite lattice structure [2] and its band gaps [3]. We will have a chance to see one of the simplest applications of Bloch Theorem [4] for scattering states. Then, we have to mention Green's function and regularization & renormalization process for the bound states in two dimensions [6], [7], [8]. We will work on the single Dirac delta function and the relation between the Green's funtion and the renormalization of the interaction cofactor will be shown. After that, we will review the work of Albeverio on two dimensional lattice which shows that there is a solution of the Hamiltonian [2]. In this model, there are Dirac delta potentials in all lattice points in the one-to-one manner [10], [11]. There is only one band gap. This will be seen by the union of decomposed Hamiltonians corresponding to the one point in Brillouin zone. We will need renormalization process to this model. We can look again Albevrio's work into to use semi-relativisitic approach. Albeverio worked in the world of Quantum Mechanics, but we will use Quantum Field Theory. Green's function will be found for the classic and semi-relativistic cases. Lastly, we will find a new formula of interaction cofactor for the arbitrary shape of Dirac-delta potential in two dimensional lattice in the aspects of Quantum Mechanics. We will write the Green's function for this model. The formula shows us the existence of a solution of the Hamiltonian. Also, we will mention briefly the three dimensional version of the formula. We will apply this formula on three different patterns being two of them are line and the other pattern is circular. We show that none of these examples need renormalization.

2. ONE DIMENSIONAL LATTICE AND DIRAC COMB

In this part, we review of some properties the model where Dirac delta potentials placed in a lattice structure in one dimension and its band gaps. One dimensional lattice [2] can be defined as the Bravais Lattice, Λ, which reads as

$$\Lambda = (na|n\epsilon Z), \quad a > 0 \tag{2.1}$$

The Wigner-Seitz cell is given by

$$\widehat{\Gamma} = [-\frac{a}{2}, \frac{a}{2}], \tag{2.2}$$

We need the notion of the dual lattice:

$$\Gamma = (nb|b\epsilon Z), \quad b = \frac{2\pi}{a} \tag{2.3}$$

Hence, the Brillouin zone equals

$$\widehat{\Lambda} = [-\frac{b}{2}, \frac{b}{2}] \tag{2.4}$$

The potential repeats itself along the base of the Bravais Lattice. In other words, it is periodic.

$$V(x) = V(x+a) \tag{2.5}$$

Let us write time independent Schrodinger equation:

$$\Big(-\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)\Big)\Psi = E\Psi \tag{2.6}$$

From Bloch's theorem [4], the solution to time independent Schrodinger equation have to be satisfied.

$$\Psi(x+a) = e^{iKa}\Psi(x) \tag{2.7}$$

for some constant $K.K$ could depend on E, but is generally independent of x. Let us define an operator $\widehat{D}$ which moves arbitrary eigenfunction along the chain with a base element of Bravais lattice [5].

$$\widehat{D}f(x) = f(x+a) \tag{2.8}$$

Now, we have to find commutation of our translation operator and Hamiltonian.

$$[\widehat{D}, \widehat{H}]f(x) = \widehat{D}\widehat{H}f(x) - \widehat{H}\widehat{D}f(x) \tag{2.9}$$

$$= E\widehat{D}f(x) - \widehat{H}f(x+a) \tag{2.10}$$

$$= E\widehat{D}f(x) - \Big(- \frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)\Big)e^{iKa} \tag{2.11}$$

Hamiltonian operator doesn't act on e^{iKa}, because K is independent of x. And then, we can exchange position of Hamiltonian and that of e^{iKa}:

$$[\widehat{D}, \widehat{H}]f(x) = \widehat{D}\widehat{H}f(x) - \widehat{H}\widehat{D}f(x) \tag{2.12}$$

$$= E\widehat{D} - e^{iKa}\widehat{H}]f(x) \tag{2.13}$$

$$= Ee^{iKa}f(x) - e^{iKa}Ef(x) \tag{2.14}$$

$$= 0 \tag{2.15}$$

For a periodic potential, $\widehat{D}$ commutes with $\widehat{H}$

$$[\widehat{D}, \widehat{H}] = 0 \tag{2.16}$$

For this reason, we can adopt eigenfunctions of $\widehat{H}$ which are simultaneously eigenfunction of $\widehat{D}$.

$$\widehat{D}\Psi(x) = \lambda\Psi(x) \tag{2.17}$$

$$\begin{aligned} \widehat{D}\Psi(x) =& \Psi(x+a) && (2.18) \\ =& \lambda\Psi(x) && (2.19) \end{aligned}$$

Therefore,we can write

$$\lambda = e^{iKa} \tag{2.20}$$

For infinite one dimensional lattice, we cannot impose any other boundary condition.

We assume that $V(x)$ consist of a long string of delta function spikes called the Dirac comb. This is represented algebraically as [3]:

$$V(x) = \lambda \sum_{j=0}^{\infty} \delta(x - ja) \tag{2.21}$$

Firstly, we need to solve the time independent Schrodinger equation within a cell, such that for $0 < x < a$: there is no potential.
Then our Schrodinger equation is

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) =& E\Psi(x) && (2.22) \\ \frac{d^2}{dx^2}\Psi(x) =& -k^2\Psi(x) \qquad where \;\; k = \left(\frac{2mE}{\hbar^2}\right) && (2.23) \end{aligned}$$

The general solution is

$$\Psi(x) = Asin(kx) + Bcos(kx) \;\; where \;\; 0 < x < a \tag{2.24}$$

For $-a < x < 0$: in this region we can use inverse of translation defined above, because it is in the cell immediately to the left of the origin.

Inverse translation operator can be defined as:

$$\widehat{D}^{-1}\Psi(x) = e^{-iKa}\Psi(x) \tag{2.25}$$

When we apply $\widehat{D}^{-1}$ to (2.24), we obtain the solution in this region.

$$\Psi(x) = e^{-iKa}\Big(Asin(kx) + Bcos(kx)\Big) \quad where \quad -a < x < 0 \tag{2.26}$$

At $x = 0$, $\Psi(x)$ must be continuous.

$$\Psi(x)|_{x=-\xi} = \Psi(x)|_{x=\xi} \tag{2.27}$$

Let us write explicitly this feature:

$$B = e^{-iKa}\Big(Asin(ka) + Bcos(ka)\Big) \tag{2.28}$$

After rearranging, we obtain

$$Asin(ka) = B\Big(e^{-iKa} - cos(ka)\Big) \tag{2.29}$$

In the center of the Wigner Seitz cells, the derivative of $\Psi(x)$ exhibits a discontinuity being proportional to λ, the amplitude of the Dirac delta function.

Let us integrate the time independent Schrodinger equation from $-\xi$ to ξ around zero and take the limit as $\xi \to 0$.

$$-\frac{\hbar^2}{2m}\int_{-\xi}^{\xi}\frac{d^2\Psi(x)}{dx^2}dx + \int_{-\xi}^{\xi}V(x)\Psi(x)dx = E\int_{-\xi}^{\xi}\Psi(x)dx \tag{2.30}$$

E and $\Psi(x)$ are finite quantities, and the value of ξ goes to zero. Therefore, we can write

$$E\int_{-\xi}^{\xi}\Psi(x)dx \to 0 \tag{2.31}$$

which yields

$$-\frac{\hbar^2}{2m}\frac{d\Psi(x)}{dx}\Big|_{-\xi}^{\xi} + \int_{-\xi}^{\xi} V(x)\Psi(x)dx = 0 \tag{2.32}$$

Usually we expect that $\left(\frac{d\Psi(x)}{dx}\right)$ is continuous. On the other hand, when $V(x)$ is a Dirac delta function, this argument fails. In this case, where $V(x) = \lambda\delta(x)$, we obtain

$$\begin{aligned}\Delta\Big(\frac{d\Psi(x)}{dx}\Big) =& \Big(\frac{2m\lambda}{\hbar^2}\Big)\Psi(0)\\ =& \Big(\frac{2m\lambda}{\hbar^2}\Big)B\end{aligned} \tag{2.33}$$

Now, we evaluate $\left(\frac{d\Psi(x)}{dx}\right)\Big|_{\xi}$ and $\left(\frac{d\Psi(x)}{dx}\right)\Big|_{-\xi}$, then take their difference at $x = 0$ to obtain

$$Ak - e^{-iKa}k\big[A\cos(ka) - B\sin(ka)\big]\Big(\frac{2m\lambda}{\hbar^2}\Big) \tag{2.34}$$

We have to solve (2.29) for A.

$$A = \frac{B\Big(e^{-iKa} - cos(ka)\Big)}{sin(ka)} \tag{2.35}$$

We can substitute A into (2.34).

$$\frac{Bk\Big(e^{-iKa} - cos(ka)\Big)}{sin(ka)}\big[1 - e^{-iKa}\cos(ka)\big] + Bk\sin(ka) = \Big(\frac{2m\lambda}{\hbar^2}\Big)B \tag{2.36}$$

Cancelling B yields

$$\left[e^{-iKa} - cos(ka)\right]\left[1 - e^{-iKa}\cos(ka)\right] + e^{-iKa}\sin^2(ka) = \left(\frac{2m\lambda}{\hbar^2}\right)\sin(ka) \tag{2.37}$$

After some algebraic works, we simplify the equation to

$$\cos(Ka) = \cos(ka) + \left(\frac{m\lambda}{\hbar^2 k}\right)\sin(ka) \tag{2.38}$$

This equation determines the possible values of k and therefore the permitted energies, because the value of $\cos(Ka)$ is limited within $[-1, 1]$. Let us change variables to look more carefully:

$$z \equiv ka \tag{2.39}$$

$$\eta \equiv \left(\frac{m\lambda a}{\hbar^2}\right) \tag{2.40}$$

then we can write a new function [3]:

$$f(z) = \cos(z) + \eta\frac{\sin(z)}{z} \tag{2.41}$$

This function is separated into two parts:

A. $\cos(ka)$: This part oscillates with z=ka.
B. $\eta\frac{\sin(z)}{z}$: Sinc function scaled by η. This term is localized around z=0 and oscillates, by decaying to zero as $z \to \infty$.

In equation (2.38), $\cos(Ka)$ is bounded above +1 and below -1; in other words $-1 \leq \cos(Ka) \leq 1$. There is not any solutions outside these limits since $|\cos(Ka)|$ cannot be bigger than 1. These regions, that arise from sinc term, correspond to "gaps" and are forbidden energies. Within a band any energy is allowed. This is a nice model to understand appearance of band gaps typical of solids with a lattice structure.

In the next sections, we work out generalizations of this simple model.

3. GREEN'S FUNCTION AND RENORMALIZATION

Let us introduce the Green's function $\mathbf{G}(E)$ [7], [8] associated with some Hamiltonian $\mathbf{H}$.

$$\begin{aligned}\mathbf{G}(E) =& \frac{1}{E - \mathbf{H}} \\ =& \big(E - \mathbf{H}\big)^{-1}\end{aligned} \tag{3.1}$$

Our Hamiltonian $\mathbf{H}$ is the sum of free Hamiltonian and the interaction term.

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_I \tag{3.2}$$

Let us choose the interaction term as Dirac delta function.

$$\mathbf{H}_I|x\rangle = \lambda\delta(x)|x\rangle \tag{3.3}$$

where,we allow for $\lambda < 0$ as well.

The Green's function associated with free Hamiltonian is

$$\mathbf{G}_0(z) = \big(z - \mathbf{H}_0\big)^{-1} \tag{3.4}$$

Let us expand our full Green's function given in terms of $\mathbf{H}$ in another form.

$$\begin{aligned}\mathbf{G}(E) =& \big(E - \mathbf{H}\big)^{-1} \\ =& \big(E - \mathbf{H}_0 - \mathbf{H}_I\big)^{-1}\end{aligned} \tag{3.5}$$

When we pull the Green function essential to the free Hamiltonian $\mathbf{H}_0$, we have to write

$$\mathbf{G}(E) = \Big[\big(E - \mathbf{H}\big)\big(1 - (E - \mathbf{H}_0)^{-1}\mathbf{H}_I\big)\Big]^{-1} \tag{3.6}$$

Inverse of product of two operators is multiplication of inverse of each operator in reverse order [8].

$$\big(\mathbf{AB}\big)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{3.7}$$

Now, Green's function, $\mathbf{G}$(E), is inverse of product of operators, then we can write

$$\mathbf{G}(E) = \Big[1 - \big(E - \mathbf{H}_0\big)^{-1}\mathbf{H}_I\Big]^{-1}\big(E - \mathbf{H}_0\big)^{-1} \tag{3.8}$$

We represent $\big(E - \mathbf{H}_0\big)^{-1}$ as $\mathbf{G}_0(E)$.

$$\mathbf{G}(E) = [1 - \mathbf{G}_0(E)\mathbf{H}_I]^{-1}\,\mathbf{G}_0(E) \tag{3.9}$$

Let us multiply both of sides of the equation $[1 - \mathbf{G}_0(E)\mathbf{H}_I]$.

$$[1 - \mathbf{G}_0(E)\mathbf{H}_I]\,\mathbf{G}(E) = \mathbf{G}_0(E) \tag{3.10}$$

When we recover the left side:

$$\mathbf{G}(E) - \mathbf{G}_0(E)\mathbf{H}_I\mathbf{G}(E) = \mathbf{G}_0(E) \tag{3.11}$$

We may add $\mathbf{G}_0(E)\mathbf{H}_I\mathbf{G}(E)$ to both sides of the equation for a simpler calculation.

$$\mathbf{G}(E) = \mathbf{G}_0(E) + \mathbf{G}_0(E)\mathbf{H}_I\mathbf{G}(E) \tag{3.12}$$

Let us look at some identities relevant to kets. They will be useful for our calculations

$|n\rangle$: complete eigenket of the Hamiltonian.

Completeness can be defined mathematically as $\mathbf{1} = \sum_n |n\rangle\langle n|$,where $\mathbf{1}$ is called identity operator.

When Green operator is applied to its eigenket, we obtain

$$\frac{1}{E-\mathbf{H}}|n\rangle = \frac{1}{E-E_n}|n\rangle \tag{3.13}$$

where, $\mathbf{H}|n\rangle = E_n|n\rangle$, and E_n is an eigenvalue of $\mathbf{H}$. Let us apply position ket from left and right to Green's function. After these applications, we can write new form of this:

$$\langle x|\mathbf{G}(E)|y\rangle = G(E;\overrightarrow{x},\overrightarrow{y}) \tag{3.14}$$

We can plug identity operators to places which we choose.

$$G(E;\overrightarrow{x},\overrightarrow{y}) = \sum_n\sum_m \langle x|m\rangle\langle m|\mathbf{G}(E)|n\rangle\langle n|y\rangle \tag{3.15}$$

where, $|n\rangle$ is an orthonormal basis which can be defined as $\langle m|n\rangle = \delta_{m,n}$

$$G(E;\overrightarrow{x},\overrightarrow{y}) = \sum_n\sum_m \frac{\langle x|m\rangle\delta_{m,n}\langle n|y\rangle}{E-E_n} \tag{3.16}$$

Therefore, after applying $\sum_m$, we obtain:

$$G(E;\overrightarrow{x},\overrightarrow{y}) = \sum_n \frac{\Psi_n(x)\Psi_n^*(y)}{E-E_n} \tag{3.17}$$

where,$\langle x|n\rangle = \Psi_n(x)$ which is called by eigenfunction.

Let us write (3.12) in the position basis:

$$G(E;\overrightarrow{x},\overrightarrow{y}) = \langle x|\mathbf{G}_0(E)|y\rangle + \langle x|\mathbf{G}_0(E)\mathbf{H}_I\mathbf{G}(E)|y\rangle \tag{3.18}$$

We define $\langle x|\mathbf{G}_0(E)|y\rangle = G_0(E;\overrightarrow{x},\overrightarrow{y})$, and then plug identity operator, which is $\mathbf{1} = \int d^2z|z\rangle\langle z|$, in position form between $\mathbf{H}_I$ and $\mathbf{G}(E)$.

$$G(E;\overrightarrow{x},\overrightarrow{y}) = G_0(E;\overrightarrow{x},\overrightarrow{y}) + \int d^2z\langle x|\mathbf{G}_0(E)\mathbf{H}_I|z\rangle\langle z|\mathbf{G}(E)|y\rangle \tag{3.19}$$

After using (3.13) and (3.14), we obtain

$$G(E;\overrightarrow{x},\overrightarrow{y}) = G_0(E;\overrightarrow{x},\overrightarrow{y}) + \int d^2zG_0(E;\overrightarrow{x},\overrightarrow{y})\lambda\delta(\overrightarrow{z})G(E;\overrightarrow{z},\overrightarrow{y}) \tag{3.20}$$

Let us integrate second term of right side of the equation.

$$G(E;\overrightarrow{x},\overrightarrow{y}) = G_0(E;\overrightarrow{x},\overrightarrow{y}) + \lambda G_0(E;\overrightarrow{x},0)G(E;0,\overrightarrow{y}) \tag{3.21}$$

Now, we put $\overrightarrow{x} = 0$ in the expression above.

$$G(E;0,\overrightarrow{y}) = G_0(E;0,\overrightarrow{y}) + \lambda G_0(E,0,0)G(E;0,\overrightarrow{y}) \tag{3.22}$$

We can solve this for $G(E;0,\overrightarrow{y})$.

$$\left[1 - \lambda G_0(E,0,0)\right]G(E;0,\overrightarrow{y}) = G_0(E;0,\overrightarrow{y})$$

$$G(E;0,\overrightarrow{y}) = \frac{G_0(E;0,\overrightarrow{y})}{\left[1 - \lambda G_0(E,0,0)\right]} \tag{3.23}$$

If we insert the result in (3.20), we obtain an explicit expression for Green's function associated with the Hamiltonian $\mathbf{H}$: [6]

$$G(E;\overrightarrow{x},\overrightarrow{y}) = G_0(E;\overrightarrow{x},\overrightarrow{y}) + \frac{G_0(E;\overrightarrow{x},0)G_0(E;0,\overrightarrow{y})}{\frac{1}{\lambda} - G_0(E,0,0)} \tag{3.24}$$

Let us investigate the bound states of Hamiltonian (3.2), with $\mathbf{H}_0$ the Hamiltonian of a free particle in 2 dimensions:

$$\begin{aligned}\mathbf{H}_0 &= -\frac{\hbar^2}{2m}\nabla^2 \\ &= -\frac{\hbar^2}{2m}\sum_{j=1}^{2}\frac{\partial^2}{\partial x_j^2}\end{aligned} \tag{3.25}$$

The energy levels of bound states are the real poles of the Green's function. Because there are no bound states for the free particle problem, such poles can only appear as zeros of the denominator of second term of the right hand side of (3.24).

Let us write Green's function for free particle and plug the identity operator between bra and ket.

$$\begin{aligned}G_0(E; \overrightarrow{x}, \overrightarrow{y}) &= \int d^2k \langle x| \frac{1}{E - \mathbf{H}_0} |k\rangle\langle k|y\rangle \\ &= \int d^2k \frac{\langle x|k\rangle\langle k|y\rangle}{E - \frac{\hbar^2 k^2}{2m}}\end{aligned} \tag{3.26}$$

where, $\mathbf{H}_0|k\rangle = \frac{k^2}{2m}|k\rangle$ such that $|k\rangle$ is an eigenket of $\mathbf{H}_0$.

We will use the postulate of Quantum mechanics which is defined mathematically:

$$\langle x|k\rangle = (2\pi)^{-2} e^{i\overrightarrow{k}\,\overrightarrow{x}} \tag{3.27}$$

For the sake of simplicity, we choose units such that $\hbar = 2m = 1$. Thus, we find

$$G_0(E; \overrightarrow{x}, \overrightarrow{y}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\overrightarrow{k}(\overrightarrow{x} - \overrightarrow{y})}}{E - k^2} \tag{3.28}$$

Therefore, in order to find energy of the bound states we have to solve equation (3.24) $(-\nu^2 = E)$

$$\frac{1}{\lambda} + \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \nu^2} = 0 \tag{3.29}$$

There is a problem which is described as that $G_0(E, 0, 0)$ is divergent. We must introduce a cut-off in the integral in order to deal with this problem. In Quantum field theory this procedure is known as regularization [9].

$$\begin{aligned} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2+\nu^2} =& \frac{1}{2\pi} \int_0^\Lambda \frac{kdk}{k^2+\nu^2} && (3.30) \\ =& \frac{1}{4\pi} \ln\left(\frac{\Lambda^2+\nu^2}{\nu^2}\right) && (3.31) \end{aligned}$$

The next step called renormalization is to absorb the divergent part which is dependent on the cut-off in a redefinition of coupling constant.

$$\frac{1}{\lambda_R} \equiv \frac{1}{\lambda} + \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{\eta^2}\right) \tag{3.32}$$

The parameter η is arbitrary. It keeps the argument of the logarithm dimensionless. Let us take the limit $\Lambda \to \infty$, varying the bare coupling constant λ in such a way that the renormalized coupling constant λ_R remains finite. This process is called renormalization [6]; then,

$$\frac{1}{\lambda_R} - \frac{1}{4\pi} \ln\left(\frac{\nu^2}{\eta^2}\right) = 0 \tag{3.33}$$

We have to solve this equation for ν^2 to find the energy of the bound state.

$$\begin{aligned} E_B =& -\nu^2 && (3.34) \\ =& -\eta^2 \exp\left(\frac{4\pi}{\lambda_R}\right) && (3.35) \end{aligned}$$

In spite of the fact that the Hamiltonian contains only one parameter, λ, we have obtained an energy, E_B, depending on two parameters (λ_R and η). However, it is possible to show that Green's function depends on a single parameter in addition to E, $\vec{x}$ and $\vec{y}$. To make it clear, we have to write the denominator of the second term on the right hand side on (3.24) in regularized form.

$$\frac{1}{\lambda} - G_0(E,0,0) = \frac{1}{\lambda} + \frac{1}{2\pi}\int_0^{\Lambda} \frac{kdk}{k^2+E} \tag{3.36}$$

$$= \frac{1}{\lambda} + \frac{1}{4\pi}\ln\Big(\frac{\Lambda^2 - E}{-E}\Big) \tag{3.37}$$

Let us look at (3.32)-(3.33). Plug the definition of $\frac{1}{\lambda_R}$ in (3.33).

$$\frac{1}{\lambda} + \frac{1}{4\pi}\ln\Big(\frac{\Lambda^2}{\eta^2}\Big) - \frac{1}{4\pi}\ln\Big(\frac{\nu^2}{\eta^2}\Big) = 0 \tag{3.38}$$

We can find

$$\frac{1}{\lambda} = -\frac{1}{4\pi}\ln\Big(-\frac{\Lambda^2}{\nu^2}\Big) \tag{3.39}$$

Then, from (3.34)

$$\frac{1}{\lambda} = -\frac{1}{4\pi}\ln\Big(-\frac{\Lambda^2}{E_B}\Big) \tag{3.40}$$

Therefore, we can substitute this in the expression which is (3.37):

$$\frac{1}{\lambda} - G_0(E;\overrightarrow{x},\overrightarrow{y}) = -\frac{1}{4\pi}\ln\Big(-\frac{\Lambda^2}{E_B}\Big) + \frac{1}{4\pi}\ln\Big(\frac{\Lambda^2 - E}{-E}\Big) \tag{3.41}$$

$$= \frac{1}{4\pi}\ln\Big(\frac{E_B(\Lambda^2 - E)}{E\Lambda^2}\Big) \tag{3.42}$$

After taking the limit $\lambda \to \infty$, we obtain:

$$\frac{1}{\lambda} - G_0(E,0,0) = -\frac{1}{4\pi}\ln\Big(-\frac{E}{E_B}\Big) \tag{3.43}$$

This is called the dimensional transmutation, since λ had no dimensions previously, renormalization introduced a dimensionful parameter, namely E_B itself.

Thus (3.24) becomes a well-defined Green's function:

$$G(E; \overrightarrow{x}, \overrightarrow{y}) = G_0(E; \overrightarrow{x}, \overrightarrow{y}) + G_0(E; \overrightarrow{x}, 0) \left[\frac{1}{-\frac{1}{4\pi} \ln\Big(- \frac{E}{E_B} \Big)} \right] . \Big(G_0(E; 0, \overrightarrow{y}), . \Big) \tag{3.44}$$

4. TWO DIMENSIONAL LATTICE WITH DIRAC-DELTA POTENTIALS

4.1. Review of Quantum Mechanical Description

Now,it is time to analyze the two dimensional lattice using Quantum mechanics. We can define an arbitrary position in it.

$$\overrightarrow{x} = \overrightarrow{R} + \overrightarrow{S} \tag{4.1}$$

$\overrightarrow{R}$ is an element of Λ, $\overrightarrow{R}\epsilon\Lambda$ where Λ is called Bravais Lattice.

$$\Lambda = (n_1\overrightarrow{a_1} + n_2\overrightarrow{a_2})\,\epsilon\ \mathrm{R}^2|\,(n_1, n_2)\,\epsilon Z^2 \tag{4.2}$$

$\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are two linearly independent vectors in $\mathbf{R}^2$.
$\overrightarrow{S}$ is an element of $\widehat{\Gamma}$, $\overrightarrow{S}\epsilon\widehat{\Gamma}$, where $\widehat{\Gamma}$ can be identified with Wigner-Seitz cell of Bravais Lattice

$$\widehat{\Gamma} = (s_1\overrightarrow{a_1} + s_2\overrightarrow{a_2})\,\epsilon\ \mathrm{R}^2|s_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)\,, j = 1, 2 \tag{4.3}$$

Any point of the momentum space of lattice can be written as:

$$\overrightarrow{p} = \overrightarrow{k} + \overrightarrow{\theta} \tag{4.4}$$

Linear combinations of Γ make up $\overrightarrow{k}$-vector. In other words, $\overrightarrow{k}\,\epsilon\Gamma$. Γ is named the dual lattice (or orthogonal lattice or reciprocal lattice) corresponding to Λ

$$\Gamma = \left(n_1\overrightarrow{b_1} + n_2\overrightarrow{b_2}\right)\,\epsilon\ \mathrm{R}^2|\,(n_1, n_2)\,\epsilon Z^2 \tag{4.5}$$

where the dual basis $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ satisfy [10], [11]

$$\overrightarrow{a}_j \overrightarrow{b}_{j'} = 2\pi\delta_{jj'} \qquad j, j' = 1, 2 \tag{4.6}$$

in analogy with relation between Λ and $\widehat{\Gamma}$, Γ is corresponding to $\widehat{\Lambda}$ defined by

$$\widehat{\Lambda} = \left(z_1\overrightarrow{b_1} + z_2\overrightarrow{b_2}\right) \epsilon \ \mathrm{R}^2 | z_j \in \left[-\frac{1}{2}, \frac{1}{2}\right), j = 1, 2 \tag{4.7}$$

the absolute value of $\widehat{\Gamma}$ is

$$|\widehat{\Gamma}| = 1 \tag{4.8}$$

where, $\widehat{\Gamma}$ can be interpreted as area of the cell. Let us decompose the Hilbert space $L^2(\mathbf{R}^2)$ according to

$$U : L^2(\mathbf{R}^2 \to L^2\big(\widehat{\Lambda}, \ell^2(\Gamma)\big) = \int_{\widehat{\Lambda}}^{\oplus} d^2\theta \ell^2(\Gamma) \tag{4.9}$$

$$(U\widehat{f})(\gamma, \theta) = \widehat{f}(\gamma + \theta) \qquad \theta\epsilon\widehat{\Lambda}, \gamma\epsilon\Gamma, \widehat{f}\epsilon L^2\big(\widehat{\Lambda}, \ell^2(\Gamma)\big) \tag{4.10}$$

the decomposition corresponds that of momentum vector.

$$\overrightarrow{p} = \overrightarrow{\theta} + \overrightarrow{\gamma} \qquad \theta\epsilon\widehat{\Lambda}, \gamma\epsilon\Gamma \tag{4.11}$$

This decomposition of $L^2(\mathbf{R}^2)$ will also decompose the Hamiltonian operator $\widehat{\mathbf{H}}$ [2].

$$U\widehat{\mathbf{H}}U^{-1} = \int_{\widehat{\Lambda}}^{\oplus} d^2 H(\theta) \tag{4.12}$$

$H(\theta)$ corresponds to a point in Brillouin zone and cover all points which are elements of the reciprocal lattice, called by "Λ".

We can separate decomposed Hamiltonian, $H(\theta)$, as kinetic and potential term like ordinary Hamiltonian.

$$H(\theta) = H_0(\theta) + V(\theta) \tag{4.13}$$

$H(\theta)$ can be applied on an eigenket associated with an element of orthogonal lattice, $|\gamma\rangle$.

$$H_0(\theta)|\gamma\rangle = \frac{(\overrightarrow{\gamma} + \overrightarrow{\theta})^2}{2M}|\gamma\rangle \tag{4.14}$$

$$= E(\theta)|\gamma\rangle \tag{4.15}$$

Our lattice have a symmetry. This can be described as

$$V(x + \lambda) = V(x) \quad \lambda \epsilon \Lambda \tag{4.16}$$

Therefore, potential can be expressed such that:

$$V(x) = \sum_{\gamma \epsilon \Gamma} V_\gamma \exp(i \overrightarrow{\gamma} \overrightarrow{x}) \tag{4.17}$$

Potential in our model is

$$V(x) = -\sum_{\sigma \epsilon \Lambda} \lambda \delta(x - \sigma) \tag{4.18}$$

We have chosen Wigner-Seitz cell as a union of the two half ordinary cells. Then, Wigner Seitz cell can be used to find the Fourier transform of the Dirac delta potentials which are the midpoints of cells and V_γ.

$$V_\gamma = \sum_{\sigma \epsilon \Lambda} \int_{\widehat{\Gamma}} d^2 \upsilon V(\upsilon) e^{-i\gamma\upsilon} \tag{4.19}$$

$$= -\sum_{\sigma \epsilon \Lambda} \int_{\widehat{\Gamma}} d^2 \upsilon \lambda \delta(\upsilon - \sigma) e^{-i\gamma\upsilon} \tag{4.20}$$

where $\sigma\epsilon\Lambda$.

From equation (4.20), we obtain

$$V_\gamma = -\lambda \tag{4.21}$$

where, we have chosen zero point of Wigner Seitz cell at the position of Dirac delta potential. Let us look at Schrodinger equation after we use identities.

$$\widehat{V}(p) = (2\pi)\sum_{\gamma\epsilon\Gamma} V_\gamma \delta(p-\gamma) \tag{4.22}$$

our Schrodinger equation becomes:

$$Hf(p) = \frac{p^2}{2M}f(p) + \sum_{\gamma\epsilon\Gamma} V_\gamma f(p-\gamma) \tag{4.23}$$

$$= \frac{p^2}{2M}f(p) - \lambda\sum_{\gamma\epsilon\Gamma} f(p-\gamma) \tag{4.24}$$

After applying the decomposition process to our Schrodinger equation, we can write

$$\widehat{\mathbf{H}}(\theta)g(\gamma) = \frac{|\gamma+\theta|^2}{2M}g(\gamma) - \lambda\sum_{\gamma'\epsilon\Gamma} g(\gamma-\gamma') \tag{4.25}$$

We are working on infinite lattice. It has a special feature which is

$$\sum_{\gamma} \equiv \sum_{\gamma\pm\gamma'} \tag{4.26}$$

We can use this as:

$$\widehat{\mathbf{H}}(\theta)g(\gamma) = \frac{|\gamma+\theta|^2}{2M}g(\gamma) - \lambda\sum_{\gamma-\gamma'\epsilon\Gamma} g(\gamma-\gamma') \tag{4.27}$$

$$= \frac{|\gamma+\theta|^2}{2M}g(\gamma) - \lambda\sum_{\gamma'\epsilon\Gamma} g(\gamma') \tag{4.28}$$

We have chosen that $g(\gamma)$ is an eigenfunction of $\widehat{\mathbf{H}}$:

$$\widehat{\mathbf{H}}(\theta)g(\gamma) = E_\theta g(\gamma) \tag{4.29}$$

$$= \frac{|\gamma+\theta|^2}{2M} g(\gamma) - \lambda \sum_{\gamma' \epsilon \Gamma} g(\gamma') \tag{4.30}$$

If we change the positions of the energy term and interaction term, we obtain

$$\lambda \sum_{\gamma' \epsilon \Gamma} g(\gamma') = \Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big) g(\gamma) \tag{4.31}$$

One can divide both sides of the equation above with $\Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big)$.

$$\frac{\lambda \sum_{\gamma' \epsilon \Gamma} g(\gamma')}{\Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big)} = g(\gamma) \tag{4.32}$$

After summing over γ both sides of the equation, we obtain

$$\sum_{\gamma \epsilon \Gamma} \frac{\lambda \sum_{\gamma' \epsilon \Gamma} g(\gamma')}{\Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big)} = \sum_{\gamma \epsilon \Gamma} g(\gamma) \tag{4.33}$$

It is easy to see that summations of eigenfunctions can be omitted.

$$\sum_{\gamma \epsilon \Gamma} \frac{\lambda}{\Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big)} = 1 \tag{4.34}$$

And then, one could obtain the equation on the form which we want, after pulling λ into the left hand side:

$$\frac{1}{\lambda} = \sum_{\gamma \epsilon \Gamma} \frac{1}{\Big(- E_\theta + \frac{|\gamma+\theta|^2}{2M} \Big)} \tag{4.35}$$

We work in the bound state such that energy is negative.

$$E_\theta = -\vartheta^2 \tag{4.36}$$

Therefore, we can write

$$\frac{1}{\lambda} = \sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\vartheta^2 + \frac{|\overrightarrow{\gamma}+\overrightarrow{\theta}|^2}{2M}\right)} \tag{4.37}$$

This term, $\frac{1}{\lambda}$, diverges. And then, we have to renormalize it. The divergent part can be absorbed by redefinition of the coupling constant.

$$\frac{1}{\lambda} = \frac{1}{\lambda_R} + \sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\frac{|\overrightarrow{\gamma}|^2}{2M} + \mu^2\right)} \tag{4.38}$$

We can write the resolvent in a separable Hilbert space as:

$$\begin{aligned} G_\theta(E) =& \frac{1}{H_\theta - E} - \frac{1}{H_\theta - E}\left[\sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\vartheta^2 + \frac{|\overrightarrow{\gamma}+\overrightarrow{\theta}|^2}{2M}\right)} - \sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\frac{|\overrightarrow{\gamma}|^2}{2M} + \mu^2\right)} - \frac{1}{\lambda_R}\right]^{-1} \\ & \times \left(\frac{1}{H_\theta - E}, \cdot\right) \end{aligned} \tag{4.39}$$

Let us show that the difference is finite:

$$\begin{aligned} & \sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\vartheta^2 + \frac{|\overrightarrow{\gamma}+\overrightarrow{\theta}|^2}{2M}\right)} - \sum_{\gamma\epsilon\Gamma} \frac{1}{\left(\frac{|\overrightarrow{\gamma}|^2}{2M} + \mu^2\right)} = \\ & \sum_{\gamma\epsilon\Gamma} \frac{\left(\frac{|\overrightarrow{\gamma}|^2}{2M} + \mu^2\right) - \left(\vartheta^2 + \frac{|\overrightarrow{\gamma}+\overrightarrow{\theta}|^2}{2M}\right)}{\left(\frac{|\overrightarrow{\gamma}|^2}{2M} + \mu^2\right)\left(\vartheta^2 + \frac{|\overrightarrow{\gamma}+\overrightarrow{\theta}|^2}{2M}\right)} = \\ & \sum_{\gamma\epsilon\Gamma} \frac{-2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1} + |\overrightarrow{\theta}|^2(2M)^{-1} + \mu^2 - \vartheta^2}{|\overrightarrow{\gamma} + \overrightarrow{\theta}|^2|\overrightarrow{\gamma}|^2(2M)^{-2} + |\overrightarrow{\gamma} + \overrightarrow{\theta}|^2\mu^2(2M)^{-1} + |\overrightarrow{\gamma}|^2\vartheta^2(2M)^{-1} + \vartheta^2\mu^2} \end{aligned} \tag{4.40}$$

where, α is the angle between $\overrightarrow{\gamma}$ and $\overrightarrow{\theta}$ which is chosen as arbitrarily. We are interested in infinite lattice, then it can be written as

$$\sum_{\gamma\epsilon\Gamma} \frac{-2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1} + |\overrightarrow{\theta}|^2(2M)^{-1} + \mu^2 - \vartheta^2}{|\overrightarrow{\gamma} + \overrightarrow{\theta}|^2|\overrightarrow{\gamma}|^2(2M)^{-2} + |\overrightarrow{\gamma} + \overrightarrow{\theta}|^2\mu^2(2M)^{-1} + |\overrightarrow{\gamma}|^2\vartheta^2(2M)^{-1} + \vartheta^2\mu^2} =$$
$$2\sum_{\gamma\epsilon\Gamma,\gamma\geq 0} \frac{-2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1} + |\overrightarrow{\theta}|^2(2M)^{-1} + \mu^2 - \vartheta^2}{|\overrightarrow{\gamma} + \overrightarrow{\theta}|^2|\overrightarrow{\gamma}|^2(2M)^{-2} + |\overrightarrow{\gamma} + \overrightarrow{\theta}|^2\mu^2(2M)^{-1} + |\overrightarrow{\gamma}|^2\vartheta^2(2M)^{-1} + \vartheta^2\mu^2}$$
$$- \frac{|\overrightarrow{\theta}|^2(2M)^{-1} + \mu^2 - \vartheta^2}{|\overrightarrow{\theta}|^2\mu^2(2M)^{-1} + \vartheta^2\mu^2} \tag{4.41}$$

It is clear that the last term of the right hand side of (4.41) is a constant, then we can omit it.

Let us look at dual lattice and its summation rigorously:

$$|\overrightarrow{\gamma}| = \sqrt{m_1^2 + m_2^2} \qquad m_1, m_2\epsilon\mathbb{Z} \tag{4.42}$$

$$\sum_{\gamma\epsilon\Gamma,\gamma\geq 0}^{\infty} f(|\overrightarrow{\gamma}|) < f(0) + 4\pi\int_0^{\infty} f(\rho)\rho d\rho \tag{4.43}$$

If applying (4.43) to first term of the right hand side of (4.41), we can write

$$\sum_{\gamma\epsilon\Gamma,\gamma\geq 0} \frac{-2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1}+|\overrightarrow{\theta}|^2(2M)^{-1}+\mu^2-\vartheta^2}{|\overrightarrow{\gamma}|^4(2M)^{-2}+2|\overrightarrow{\gamma}|^3(2M)^{-2}|\overrightarrow{\theta}|+|\overrightarrow{\gamma}|^2\left(\frac{|\overrightarrow{\theta}|^2}{(2M)^2}+\frac{|\overrightarrow{\theta}|^2+\mu^2+\vartheta^2}{2M}\right)+\frac{2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\mu^2}{2M}+c} =$$

$$\sum_{m_1}\sum_{m_2}\left(-2\sqrt{m_1^2+m_2^2}|\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1}+|\overrightarrow{\theta}|^2(2M)^{-1}+\mu^2-\vartheta^2\right)\times$$
$$\Bigg((\sqrt{m_1^2+m_2^2})^4(2M)^{-2}+2(\sqrt{m_1^2+m_2^2})^3(2M)^{-2}|\overrightarrow{\theta}|$$
$$+(\sqrt{m_1^2+m_2^2})^2\left(\frac{|\overrightarrow{\theta}|^2}{(2M)^2}+\frac{|\overrightarrow{\theta}|^2+\mu^2+\vartheta^2}{2M}\right)+\frac{2|\overrightarrow{\gamma}||\overrightarrow{\theta}|\mu^2}{2M}+c\Bigg)^{-1} <$$
$$\frac{|\overrightarrow{\theta}|^2(2M)^{-1}+\mu^2-\vartheta^2}{c}+$$
$$4\pi\int_0^\infty \frac{-2\rho|\overrightarrow{\theta}|\cos(\alpha)(2M)^{-1}+|\overrightarrow{\theta}|^2(2M)^{-1}+\mu^2-\vartheta^2}{\rho^4(2M)^{-2}+2\rho^3(2M)^{-2}|\overrightarrow{\theta}|+\rho^2\left(\frac{|\overrightarrow{\theta}|^2}{(2M)^2}+\frac{|\overrightarrow{\theta}|^2+\mu^2+\vartheta^2}{2M}\right)+\rho\frac{2|\overrightarrow{\theta}|\mu^2}{2M}+c}\rho d\rho$$
$$<\infty \tag{4.44}$$

where $c=\vartheta^2\mu^2+(2M)^{-1}\mu^2|\overrightarrow{\theta}|^2$

Therefore, $\frac{1}{\lambda}$ converges.

4.2. Semi-Relativistic Approach

We have mentioned the Dirac delta potential points on a lattice in the usual Quantum Mechanics setting. We extend this problem to relativistic particles ignoring pair creation. We use Quantum field theory to describe our system. Let us start with some concepts [12], [13] which we need.

In lattice structure, arbitrary momentum can be written as

$$\overrightarrow{p}=\overrightarrow{k}+\overrightarrow{\theta} \tag{4.45}$$

And the momentum integral can be separated in terms of orthogonal lattice and Wigner Seitz cell of dual lattice [10], [11].

$$\int d^2p = \sum_{\overrightarrow{k}\,\epsilon\Gamma} \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d^2\theta \tag{4.46}$$

Free scalar field in Schrodinger picture can be described as

$$\Phi(x) = \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\overrightarrow{k}\,\epsilon\Gamma} \left(\frac{\widehat{a}_{\overrightarrow{p}}}{\sqrt{2\omega_{\overrightarrow{p}}}} e^{i\overrightarrow{p}\,\overrightarrow{x}} + \frac{\widehat{a}^{\dagger}_{\overrightarrow{p}}}{\sqrt{2\omega_{\overrightarrow{p}}}} e^{-i\overrightarrow{p}\,\overrightarrow{x}} \right) \tag{4.47}$$

and conjugate momentum to this field is written as

$$\Pi(x) = \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\overrightarrow{k}\,\epsilon\Gamma} (-i)\sqrt{\frac{\omega_{\overrightarrow{p}}}{2}} \left(\widehat{a}_{\overrightarrow{p}} e^{i\overrightarrow{p}\,\overrightarrow{x}} - \widehat{a}^{\dagger}_{\overrightarrow{p}} e^{-i\overrightarrow{p}\,\overrightarrow{x}} \right) \tag{4.48}$$

where $\omega_{\overrightarrow{p}}$ is frequency which is defined as

$$\omega_{\overrightarrow{p}} = \sqrt{(\overrightarrow{p})^2 + m^2} \tag{4.49}$$

We have already employed proper formalism of dual lattice in own construction. We have written $\Phi(x)$ and $\Pi(x)$ as a linear sum of an infinite number of creation an annihilation operators $\widehat{a}^{\dagger}_{\overrightarrow{p}}$ and $\widehat{a}_{\overrightarrow{p}}$ indexed by 3-momentum $\overrightarrow{p}$.

The commutation relations for $\Phi(x)$ and $\Pi(x)$ are equivalent to the following commutation relations for $\widehat{a}^{\dagger}_{\overrightarrow{p}}$ and $\widehat{a}_{\overrightarrow{p}}$.

$$[\Phi(x), \Phi(y)] = [\Pi(x), \Pi(y)] = 0 \qquad [\widehat{a}_{\overrightarrow{p}}, \widehat{a}_{\overrightarrow{q}}] = [\widehat{a}^{\dagger}_{\overrightarrow{p}}, \widehat{a}^{\dagger}_{\overrightarrow{q}}] = 0 \tag{4.50}$$

$$[\Phi(x), \Pi(y)] = \delta^2(\overrightarrow{x} - \overrightarrow{y}) \qquad [\widehat{a}_{\overrightarrow{p}}, \widehat{a}^{\dagger}_{\overrightarrow{q}}] = (2\pi)^2\delta^2(\overrightarrow{p} - \overrightarrow{q}) \tag{4.51}$$

$|0\rangle$ is a vacuum state.

A particle with momentum, $\overrightarrow{p}$, can be defined as

$$|p\rangle = \sqrt{2\omega_{\overrightarrow{p}}}\,\widehat{a}_{\overrightarrow{p}}|0\rangle \tag{4.52}$$

And the momentum integral can be separated in terms of orthogonal lattice and Wigner Seitz cell of dual lattice [10], [11].

$$\int d^2p = \sum_{\vec{k}\,\epsilon\Gamma} \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d^2\theta \tag{4.53}$$

Let us apply this momentum expansion to our field, and its conjugate momentum and frequency:

$$\Phi(x) = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} \left(\frac{\widehat{a}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} + \frac{\widehat{a}^{\dagger}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}+\vec{\theta})\vec{x}} \right) \tag{4.54}$$

$$\Pi(x) = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} (-i) \sqrt{\frac{\omega_{\vec{k}+\vec{\theta}}}{2}} \left(\widehat{a}_{\vec{k}+\vec{\theta}} e^{i(\vec{k}+\vec{\theta})\vec{x}} - \widehat{a}^{\dagger}_{\vec{k}+\vec{\theta}} e^{-i(\vec{k}+\vec{\theta})\vec{x}} \right) \tag{4.55}$$

$$\omega_{\vec{k}+\vec{\theta}} = \sqrt{(\vec{k} + \vec{\theta})^2 + m^2} \tag{4.56}$$

Quantum field can be divided to two parts. The first has positive frequency with creation operator. The other has an annihilation operator via negative frequency.

$$\Phi^-(x) = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} \frac{\widehat{a}^{\dagger}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}+\vec{\theta})\vec{x}} \tag{4.57}$$

$$\Phi^+(x) = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} \frac{\widehat{a}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} \tag{4.58}$$

Let us apply this field to vacuum state

$$\Phi(x)|0\rangle = \Phi^-(x)|0\rangle + \Phi^+(x)|0\rangle \tag{4.59}$$

When we look at the first term in right hand side of (4.59), the relation between annihilation operator and vacuum state which can be written as $\widehat{a}_{\vec{k}+\vec{\theta}}|0\rangle = 0$ have to be used. And then, it is easy to see:

$$\Phi^{+}(x)|0\rangle = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} \frac{1}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} \widehat{a}_{\vec{k}+\vec{\theta}}|0\rangle = 0 \tag{4.60}$$

Therefore, we can write

$$\Phi(x)|0\rangle = \Phi^{-}(x)|0\rangle = \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} \frac{1}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} |\vec{k} + \vec{\theta}\rangle \tag{4.61}$$

This is a linear superposition of single particle states that have well defined momentum constrained by lattice structure.

In addition to these, we introduce a normal ordering prescription

$$: \widehat{a}_{\vec{p}}\widehat{a}^{+}_{\vec{q}} := \widehat{a}^{+}_{\vec{p}}\widehat{a}_{\vec{q}} \qquad : \widehat{a}^{+}_{\vec{p}}\widehat{a}_{\vec{q}} := \widehat{a}^{+}_{\vec{q}}\widehat{a}_{\vec{p}} \tag{4.62}$$

Let us write Hamiltonian of this semi-relativistic approach

$$H = \int d^2x \left[: \Phi(x)\left(-\nabla^2 + m^2\right)\Phi(x) + \Pi^2(x) :\right] - \lambda \sum_{R\epsilon\Lambda} \Phi^{-}(R)\Phi^{+}(R) \tag{4.63}$$

where $\lambda > 0$, because we are searching for bound states, nevertheless this is not essential. It is good to recall lattice structure [2]:

$$\vec{x} = \vec{R} + \vec{\eta} \qquad \vec{R}\epsilon\Lambda \quad and \quad \vec{\eta}\,\epsilon\widehat{\Gamma} \tag{4.64}$$

$$\vec{R} = \sum_{i=1}^{2} n_i \vec{a_i} \qquad \vec{a_i} : basis \;\; at \;\; \Lambda \tag{4.65}$$

We have used the adjective, semi-relativistic, because the interaction term is composed of $\Phi^{-}(x)$ and $\Phi^{+}(x)$ instead of the full field, $-\lambda \sum_{R\epsilon\Lambda} \Phi(R)\Phi(R)$, our semi relativistic approach truncates this interaction into $-\lambda \sum_{R\epsilon\Lambda} \Phi^{-}(R)\Phi^{+}(R)$.

Now, it is time to calculate $\nabla^2\Phi$,$\Phi^2(x)$, and $\Pi^2(x)$ simpler as possible to write the Hamiltonian more explicitly.

Firstly, the Laplacian of our field is written as

$$\nabla^2\Phi = \int_{\vec{\theta}\,\epsilon\hat{\Lambda}} d\theta \sum_{\vec{k}\,\epsilon\Gamma} (\vec{k} + \vec{\theta})^2 \left(\frac{\hat{a}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} + \frac{\hat{a}^{\dagger}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}+\vec{\theta})\vec{x}} \right) \tag{4.66}$$

Secondly, square of our field have to be examined.

$$\Phi^2(x) = \int_{\vec{\theta}\,\epsilon\hat{\Lambda}} \int_{\vec{\theta'}\,\epsilon\hat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\,\epsilon\Gamma} \left(\frac{\hat{a}_{\vec{k'}+\vec{\theta'}}}{\sqrt{2\omega_{\vec{k'}+\vec{\theta'}}}} e^{i(\vec{k'}+\vec{\theta'})\vec{x'}} + \frac{\hat{a}^{\dagger}_{\vec{k'}+\vec{\theta'}}}{\sqrt{2\omega_{\vec{k'}+\vec{\theta'}}}} e^{-i(\vec{k'}+\vec{\theta'})\vec{x}} \right)$$
$$\times \left(\frac{\hat{a}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{\theta})\vec{x}} + \frac{\hat{a}^{\dagger}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}+\vec{\theta})\vec{x}} \right) d\theta d\theta' \tag{4.67}$$

If we expand parentheses, we obtain

$$\Phi^2(x) = \int_{\vec{\theta}\,\epsilon\hat{\Lambda}} \int_{\vec{\theta'}\,\epsilon\hat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\,\epsilon\Gamma} \frac{\hat{a}_{\vec{k'}+\vec{\theta'}}\hat{a}_{\vec{k}+\vec{\theta}}}{2\sqrt{\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}}$$
$$+ \frac{\hat{a}_{\vec{k'}+\vec{\theta'}}\widehat{a^{\dagger}}_{\vec{k}+\vec{\theta}}}{2\sqrt{\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}-\vec{k'}+\vec{\theta}-\vec{\theta'})\vec{x}} + \frac{\widehat{a^{\dagger}}_{\vec{k'}+\vec{\theta'}}\hat{a}_{\vec{k}+\vec{\theta}}}{2\sqrt{\omega_{\vec{k}+\vec{\theta}}}} e^{i(\vec{k}-\vec{k'}+\vec{\theta}-\vec{\theta'})\vec{x}}$$
$$+ \frac{\widehat{a^{\dagger}}_{\vec{k'}+\vec{\theta'}}\widehat{a^{\dagger}}_{\vec{k}+\vec{\theta}}}{2\sqrt{\omega_{\vec{k}+\vec{\theta}}}} e^{-i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}} d\theta d\theta' \tag{4.68}$$

Finally, we have to look at square of conjugate momentum.

$$\Pi^2(x) = -\int_{\vec{\theta}\,\epsilon\hat{\Lambda}} \int_{\vec{\theta'}\,\epsilon\hat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\,\epsilon\Gamma} \frac{\sqrt{\omega_{\vec{k}+\vec{\theta}}.\omega_{\vec{k'}+\vec{\theta'}}}}{2}$$
$$\times \left[\left(\hat{a}_{\vec{k'}+\vec{\theta'}} e^{i(\vec{k'}+\vec{\theta'})\vec{x}} - \widehat{a^{\dagger}}_{\vec{k}+\vec{\theta}} e^{i(\vec{k'}+\vec{\theta'})\vec{x}} \right) \right.$$
$$\left. \times \left(\hat{a}_{\vec{k}+\vec{\theta}} e^{i(\vec{k}+\vec{\theta})\vec{x}} - \widehat{a^{\dagger}}_{\vec{k'}+\vec{\theta'}} e^{i(\vec{k'}+\vec{\theta'})\vec{x}} - \right) \right] d\theta d\theta' \tag{4.69}$$

we can write it more clear after some easy algebra have been used:

$$\Pi^2(x) = -\int_{\overrightarrow{\theta}\epsilon\widehat{\Lambda}}\int_{\overrightarrow{\theta'}\epsilon\widehat{\Lambda}}\sum_{\overrightarrow{k}\epsilon\Gamma}\sum_{\overrightarrow{k'}\epsilon\Gamma}\frac{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}.\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}}{2}$$
$$\times\Big[\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}} - \widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(-\overrightarrow{k}+\overrightarrow{k'}-\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}}$$
$$-\widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(\overrightarrow{k}-\overrightarrow{k'}+\overrightarrow{\theta}-\overrightarrow{\theta'})\overrightarrow{x}} + \widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{-i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}}\Big]d\theta d\theta' \tag{4.70}$$

Let us plug these into free term in Hamiltonian and apply normal ordering process.

$$:\Phi(x)\left(-\nabla^2+m^2\right)\Phi(x)+\Pi^2(x):=\int_{\overrightarrow{\theta}\epsilon\widehat{\Lambda}}\int_{\overrightarrow{\theta'}\epsilon\widehat{\Lambda}}\sum_{\overrightarrow{k}\epsilon\Gamma}\sum_{\overrightarrow{k'}\epsilon\Gamma}\left[(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2\right]\times$$
$$\times\Big[\frac{\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}e^{i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}} + \frac{\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}e^{-i(\overrightarrow{k}-\overrightarrow{k'}+\overrightarrow{\theta}-\overrightarrow{\theta'})\overrightarrow{x}}$$
$$+\frac{\widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}e^{i(\overrightarrow{k}-\overrightarrow{k'}+\overrightarrow{\theta}-\overrightarrow{\theta'})\overrightarrow{x}} + \frac{\widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}e^{-i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}}\Big]$$
$$-\int_{\overrightarrow{\theta}\epsilon\widehat{\Lambda}}\int_{\overrightarrow{\theta'}\epsilon\widehat{\Lambda}}\sum_{\overrightarrow{k}\epsilon\Gamma}\sum_{\overrightarrow{k'}\epsilon\Gamma}\frac{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\times\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}}{2}$$
$$\times\Big[\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}} - \widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(-\overrightarrow{k}+\overrightarrow{k'}-\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}}$$
$$-\widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{i(\overrightarrow{k}-\overrightarrow{k'}+\overrightarrow{\theta}-\overrightarrow{\theta'})\overrightarrow{x}} + \widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}e^{-i(\overrightarrow{k}+\overrightarrow{k'}+\overrightarrow{\theta}+\overrightarrow{\theta'})\overrightarrow{x}}\Big]d\theta d\theta' \tag{4.71}$$

We can write full Hamiltonian in more compact form.

$$
\begin{aligned}
H = & \int_{\vec{\theta}\epsilon\hat{\Lambda}} \int_{\vec{\theta'}\epsilon\hat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\epsilon\Gamma} \Bigg[\frac{\omega^2_{\vec{k}+\vec{\theta}}}{2\sqrt{\omega_{\vec{k}+\vec{\theta}}\omega_{\vec{k'}+\vec{\theta'}}}} \\
& \Bigg(\widehat{a}_{\vec{k'}+\vec{\theta'}} \widehat{a}_{\vec{k}+\vec{\theta}} \int d^2x e^{i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}} + \widehat{a^\dagger}_{\vec{k}+\vec{\theta}} \widehat{a}_{\vec{k'}+\vec{\theta'}} \int d^2x e^{i(-\vec{k}+\vec{k'}-\vec{\theta}+\vec{\theta'})\vec{x}} \\
& + \widehat{a^\dagger}_{\vec{k'}+\vec{\theta'}} \widehat{a}_{\vec{k}+\vec{\theta}} \int d^2x e^{i(\vec{k}-\vec{k'}+\vec{\theta}-\vec{\theta'})\vec{x}} + \widehat{a^\dagger}_{\vec{k'}+\vec{\theta'}} \widehat{a^\dagger}_{\vec{k}+\vec{\theta}} e^{-i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}} \Bigg) \\
& - \Bigg(\frac{\sqrt{\omega_{\vec{k}+\vec{\theta}} \times \omega_{\vec{k'}+\vec{\theta'}}}}{2} \\
& \times \big[\widehat{a}_{\vec{k'}+\vec{\theta'}} \widehat{a}_{\vec{k}+\vec{\theta}} \int d^2x \; e^{i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}} - \widehat{a^\dagger}_{\vec{k}+\vec{\theta}} \widehat{a}_{\vec{k'}+\vec{\theta'}} \int d^2x \; e^{i(-\vec{k}+\vec{k'}-\vec{\theta}+\vec{\theta'})\vec{x}} \\
& - \widehat{a^\dagger}_{\vec{k'}+\vec{\theta'}} \widehat{a}_{\vec{k}+\vec{\theta}} \int d^2x \; e^{i(\vec{k}-\vec{k'}+\vec{\theta}-\vec{\theta'})\vec{x}} + \widehat{a^\dagger}_{\vec{k'}+\vec{\theta'}} \widehat{a^\dagger}_{\vec{k}+\vec{\theta}} \int d^2x \; e^{-i(\vec{k}+\vec{k'}+\vec{\theta}+\vec{\theta'})\vec{x}} \big) \Bigg) \\
& - \lambda \frac{\widehat{a^\dagger}_{\vec{k'}+\vec{\theta'}} \widehat{a}_{\vec{k}+\vec{\theta}}}{\sqrt{2\omega_{\vec{k}+\vec{\theta}}\omega_{\vec{k'}+\vec{\theta'}}}} \sum_{R\epsilon\Lambda} e^{i(\vec{k}-\vec{k'})R} e^{i(\vec{\theta}-\vec{\theta'})R} \Bigg] d\theta d\theta''
\end{aligned}
\tag{4.72}
$$

Now, we have to recall the identity:

$$
\int d^2x e^{[i(\vec{k}+\vec{\theta})\vec{x}]} = 4\pi^2 \delta(\vec{k} + \vec{\theta}) \tag{4.73}
$$

In addition to this, we have the equation [10], [11] related to lattice which goes to infinity in all directions.

$$
\sum_{\vec{R}\epsilon\Lambda} e^{(i\vec{\theta}\vec{R})} = \delta(\vec{\theta}) \tag{4.74}
$$

After we apply (4.73), and (4.74) to our Hamiltonian, it is obtained in clearer expression.

$$
\begin{aligned}
H = & \int_{\overrightarrow{\theta}\epsilon\widehat{\Lambda}} \int_{\overrightarrow{\theta'}\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\epsilon\Gamma} \sum_{\overrightarrow{k'}\epsilon\Gamma} \\
& \Bigg[\Bigg(\frac{\omega^2_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}} \big(\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} + \overrightarrow{k'} + \overrightarrow{\theta} + \overrightarrow{\theta'}) \\
& + \widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}} \widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}} 4\pi^2 \delta(-\overrightarrow{k} + \overrightarrow{k'} - \overrightarrow{\theta} + \overrightarrow{\theta'}) \\
& + \widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} - \overrightarrow{k'} + \overrightarrow{\theta} - \overrightarrow{\theta'}) \\
& + \widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} + \overrightarrow{k'} + \overrightarrow{\theta} + \overrightarrow{\theta'}) \big) \Bigg) \\
& - \Bigg(\frac{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}} \times \omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}}{2} \big(\widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} + \overrightarrow{k'} + \overrightarrow{\theta} + \overrightarrow{\theta'}) \\
& - \widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}} \widehat{a}_{\overrightarrow{k'}+\overrightarrow{\theta'}} 4\pi^2 \delta(-\overrightarrow{k} + \overrightarrow{k'} - \overrightarrow{\theta} + \overrightarrow{\theta'}) \\
& - \widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} - \overrightarrow{k'} + \overrightarrow{\theta} - \overrightarrow{\theta'}) \\
& + \widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}} 4\pi^2 \delta(\overrightarrow{k} + \overrightarrow{k'} + \overrightarrow{\theta} + \overrightarrow{\theta'}) \big) \Bigg) \Bigg] d\theta d\theta'' \\
& - \lambda \int_{\overrightarrow{\theta}\epsilon\widehat{\Lambda}} \int_{\overrightarrow{\theta'}\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\epsilon\Gamma} \sum_{\overrightarrow{k'}\epsilon\Gamma} \frac{\widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}} \widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{\sqrt{2}\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}} \delta(\theta - \theta') d\theta d\theta'' \qquad (4.75)
\end{aligned}
$$

For the lattice, it is easy to see that

$$
\delta(\overrightarrow{k} - \overrightarrow{k'} + \overrightarrow{\theta} - \overrightarrow{\theta'}) = \delta_{\overrightarrow{k},\overrightarrow{k'}} \delta(\theta - \theta') \qquad (4.76)
$$

Let us use equation (4.76) & sum over $\overrightarrow{k'}$ and integrate over $\overrightarrow{\theta'}$ in equation (4.75).

$$H = \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\overrightarrow{k}\,\epsilon\Gamma} 2\pi^2 \omega_{\overrightarrow{k}+\overrightarrow{\theta}}$$
$$\Bigg[\Bigg(\widehat{a}_{-\overrightarrow{k}-\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} + 2\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} + \widehat{a}^{\dagger}_{-\overrightarrow{k}-\overrightarrow{\theta}}\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\Bigg)$$
$$-\Bigg(\widehat{a}_{-\overrightarrow{k}-\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} - 2\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}} + \widehat{a}^{\dagger}_{-\overrightarrow{k}-\overrightarrow{\theta}}\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\Bigg)\Bigg]$$
$$-\lambda \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k'}\,\epsilon\Gamma} \frac{\widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}} \tag{4.77}$$

There are same terms with opposite signs in equation (4.77). After equation (4.77) is simplified, we obtain final form:

$$H = \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\overrightarrow{k}\,\epsilon\Gamma} 8\pi^2 \omega_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}$$
$$-\lambda \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\theta \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k'}\,\epsilon\Gamma} \frac{\widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}} \tag{4.78}$$

Let us define an arbitrary single particle eigenket for the lattice:

$$|\Psi\rangle = \sum_{\overrightarrow{k''}\epsilon\Gamma} \int_{\overrightarrow{\theta''}\epsilon\widehat{\Lambda}} d\theta'' \frac{\Psi_{\theta''}(k'')}{\sqrt{\omega_{\overrightarrow{k''}+\overrightarrow{\theta''}}}}\widehat{a}^{\dagger}_{\overrightarrow{k''}+\overrightarrow{\theta''}}|0\rangle \tag{4.79}$$

Now, we can apply Hamiltonian to equation (4.79).

$$H|\Psi\rangle = \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \int_{\overrightarrow{\theta''}\epsilon\widehat{\Lambda}} d\theta d\theta'' \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k''}\epsilon\Gamma} 8\pi^2 \omega_{\overrightarrow{k}+\overrightarrow{\theta}}$$
$$\frac{\Psi_{\theta''}(k'')}{\sqrt{\omega_{\overrightarrow{k''}+\overrightarrow{\theta''}}}}\widehat{a}^{\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}\widehat{a}^{\dagger}_{\overrightarrow{k''}+\overrightarrow{\theta''}}|0\rangle$$
$$-\lambda \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \int_{\overrightarrow{\theta''}\epsilon\widehat{\Lambda}} d\theta d\theta'' \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k'}\,\epsilon\Gamma}$$
$$\frac{\Psi_{\theta''}(k'')}{\sqrt{\omega_{\overrightarrow{k''}+\overrightarrow{\theta''}}}} \frac{\widehat{a}^{\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta'}}\widehat{a}_{\overrightarrow{k}+\overrightarrow{\theta}}}{2\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\omega_{\overrightarrow{k'}+\overrightarrow{\theta'}}}}\widehat{a}^{\dagger}_{\overrightarrow{k''}+\overrightarrow{\theta''}}|0\rangle \tag{4.80}$$

When we use equation (4.50) and (4.51), we obtain

$$\begin{aligned} H|\Psi\rangle = & \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \int_{\vec{\theta''}\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k''}\epsilon\Gamma} 16\pi^2 \frac{\omega_{\vec{k}+\vec{\theta}}}{\sqrt{\omega_{\vec{k''}+\vec{\theta''}}}} \Psi_{\theta''}(k'') \widehat{a}^{\dagger}{}_{\vec{k}+\vec{\theta}} \\ & \times \delta(\vec{k} - \vec{k''} + \vec{\theta} - \vec{\theta''})|0\rangle \\ & - \lambda \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \int_{\vec{\theta''}\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\epsilon\Gamma} \sum_{\vec{k''}\epsilon\Gamma} 2\pi^2 \frac{\widehat{a}^{\dagger}{}_{\vec{k'}+\vec{\theta}}}{\sqrt{\omega_{\vec{k}+\vec{\theta}}\omega_{\vec{k'}+\vec{\theta}}}} \\ & \times \delta(\vec{k} - \vec{k''} + \vec{\theta} - \vec{\theta''}) \frac{\Psi_{\theta''}(k'')}{\sqrt{\omega_{\vec{k''}+\vec{\theta''}}}}|0\rangle \end{aligned} \tag{4.81}$$

After applying some integration and summation process, we can write

$$\begin{aligned} H|\Psi\rangle = & \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} 16\pi^4 \sqrt{\omega_{\vec{k}+\vec{\theta}}}\, \Psi_{\theta}(k) \widehat{a}^{\dagger}{}_{\vec{k}+\vec{\theta}}|0\rangle \\ & - \lambda \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\epsilon\Gamma} 2\pi^2 \frac{\Psi_{\theta}(k)}{\omega_{\vec{k}+\vec{\theta}} \sqrt{\omega_{\vec{k'}+\vec{\theta}}}} \widehat{a}^{\dagger}{}_{\vec{k'}+\vec{\theta}}|0\rangle \end{aligned} \tag{4.82}$$

We can shift $k \mapsto k'$ or $k' \mapsto k$ indices freely in interaction term.

$$\begin{aligned} H|\Psi\rangle = & \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} 16\pi^4 \sqrt{\omega_{\vec{k}+\vec{\theta}}}\, \Psi_{\theta}(k) \widehat{a}^{\dagger}{}_{\vec{k}+\vec{\theta}}|0\rangle \\ & - \lambda \int_{\vec{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\vec{k}\,\epsilon\Gamma} \sum_{\vec{k'}\epsilon\Gamma} 2\pi^2 \frac{\Psi_{\theta}(k')}{\omega_{\vec{k'}+\vec{\theta}} \sqrt{\omega_{\vec{k}+\vec{\theta}}}} \widehat{a}^{\dagger}{}_{\vec{k}+\vec{\theta}}|0\rangle \end{aligned} \tag{4.83}$$

We know that $|\Psi\rangle$ is an eigenket of H with eigenvalue E:

$$H|\Psi\rangle = E\rangle \tag{4.84}$$

Let us choose E negative such that $E = -\nu_{\theta}^2$, since the delta potential is attractive. We look for bound states.

When we equate (4.83) and (4.84), we obtain

$$\int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\,\epsilon\Gamma} 16\pi^4 \sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}\Psi_\theta(k)\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}|0\rangle$$
$$-\lambda \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k'}\,\epsilon\Gamma} 2\pi^2 \frac{\Psi_\theta(k')}{\omega_{\overrightarrow{k'}+\overrightarrow{\theta}}\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}|0\rangle =$$
$$-\sum_{\overrightarrow{k''}\epsilon\Gamma} \int_{\overrightarrow{\theta''}\epsilon\widehat{\Lambda}} d\theta'' \frac{\Psi_{\theta''}(k'')}{\sqrt{\omega_{\overrightarrow{k''}+\overrightarrow{\theta''}}}}\widehat{a^\dagger}_{\overrightarrow{k''}+\overrightarrow{\theta''}}\nu^2_{\theta''}|0\rangle \tag{4.85}$$

If we change the positions of interaction term and energy term, we can write

$$\int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\,\epsilon\Gamma} \left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu^2_\theta\right] \frac{\Psi_\theta(k)}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\sqrt{\omega_{\overrightarrow{k'}+\overrightarrow{\theta}}}}\widehat{a^\dagger}_{\overrightarrow{k'}+\overrightarrow{\theta}}|0\rangle =$$
$$\lambda \int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} \sum_{\overrightarrow{k}\,\epsilon\Gamma} \sum_{\overrightarrow{k'}\,\epsilon\Gamma} 2\pi^2 \frac{\Psi_\theta(k')}{\omega_{\overrightarrow{k'}+\overrightarrow{\theta}}\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}}\widehat{a^\dagger}_{\overrightarrow{k}+\overrightarrow{\theta}}|0\rangle \tag{4.86}$$

$\int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\overrightarrow{\theta} \sum_{\overrightarrow{k}\,\epsilon\Gamma}$ is a complete set such that $\int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\overrightarrow{\theta} \sum_{\overrightarrow{k}\,\epsilon\Gamma} = \int d^2p$. Therefore we can omit $\int_{\overrightarrow{\theta}\,\epsilon\widehat{\Lambda}} d\overrightarrow{\theta} \sum_{\overrightarrow{k}\,\epsilon\Gamma}$ at two sides of (4.86).

$$\left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu^2_\theta\right] \frac{\Psi_\theta(k)}{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}} = \frac{2\pi^2\lambda}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}} \sum_{\overrightarrow{k'}\,\epsilon\Gamma} \frac{\Psi_\theta(k')}{\sqrt{\omega_{\overrightarrow{k'}+\overrightarrow{\theta}}}} \tag{4.87}$$

We can define $\mho(\theta)$ as

$$\mho(\theta) = \sum_{\overrightarrow{k'}\,\epsilon\Gamma} \frac{\Psi_\theta(k')}{\sqrt{\omega_{\overrightarrow{k'}+\overrightarrow{\theta}}}} \tag{4.88}$$

If we rearrange (4.87) using new our new variable, $\mho(\theta)$, we can write

$$\frac{\Psi_\theta(k)}{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}} = \frac{2\pi^2\lambda}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu^2_\theta]}\mho(\theta) \tag{4.89}$$

Let us sum two sides of the equation over $\overrightarrow{k}$:

$$\sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{\Psi_\theta(k)}{\sqrt{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}}} = \sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{2\pi^2\lambda}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu_\theta^2\right]} \mho(\theta) \tag{4.90}$$

Now, you can see left hand side of (4.90) equals to $\mho(\theta)$.

$$\mho(\theta) = \sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{2\pi^2\lambda}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu_\theta^2\right]} \mho(\theta) \tag{4.91}$$

These are the same term on both of sides in (4.91). And then, it can be omitted.

$$1 = \sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{2\pi^2\lambda}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu_\theta^2\right]} \tag{4.92}$$

Let us pull λ^{-1} from right side

$$.\frac{1}{\lambda} = \sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{2\pi^2}{\omega_{\overrightarrow{k}+\overrightarrow{\theta}}\left[16\pi^4\omega_{\overrightarrow{k}+\overrightarrow{\theta}} + \nu_\theta^2\right]} \tag{4.93}$$

Finally, we have found λ^{-1} whose situation of convergence determines existence of solution.

We can use new notation for the sake of simplicity.

$\eta = \frac{E(\theta)}{16\pi^4} = -\frac{\nu_\theta^2}{16\pi^4}$

After plugging definition of $\omega_{\overrightarrow{k}+\overrightarrow{\theta}}$ in $\frac{1}{\lambda}$ formula, we obtain

$$\frac{1}{\lambda} = \frac{1}{8\pi^2} \sum_{\overrightarrow{k}\,\epsilon\Gamma} \frac{1}{\left(\sqrt{(\overrightarrow{k} + \overrightarrow{\theta})^2 + m^2} - \eta\right)\left(\sqrt{(\overrightarrow{k} + \overrightarrow{\theta})^2 + m^2}\right)} \tag{4.94}$$

We have a formula which can be described as

$$\frac{1}{a(a-b)} = \int_0^1 \frac{du}{(a-ub)^2} \tag{4.95}$$

We can apply this formula to 4.94:

$$\frac{1}{\left(\sqrt{(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2}-\eta\right)\left(\sqrt{(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2}\right)}=$$
$$\int_0^1 \frac{du}{\left[\sqrt{(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2}-u\eta\right]^2}=$$
$$\int_0^\infty\int_0^1 dsdu\exp\left(u\eta s\right)\exp\left(-s\sqrt{(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2}\right) \tag{4.96}$$

Let us take integrals:

$$\int_0^1 \exp(su\eta)du=\frac{e^{s\eta}-1}{\eta s} \tag{4.97}$$

and

$$\exp\left(-s\sqrt{(\overrightarrow{k}+\overrightarrow{\theta})^2+m^2}\right)=\frac{s}{2\sqrt{\pi}}\int_0^\infty \frac{\exp\left(-\frac{s^2}{4t}-m^2t-(\overrightarrow{k}+\overrightarrow{\theta})^2t\right)}{t^{\frac{3}{2}}}dt \tag{4.98}$$

After some arrangement, we obtain

$$\frac{1}{\lambda}=\frac{1}{16\pi^{\frac{5}{2}}}\sum_{\overrightarrow{k}\epsilon\Gamma}\int_0^\infty\frac{e^{s\eta}-1}{\eta}\int_0^\infty\frac{\exp\left(-\frac{s^2}{4t}-m^2t-(\overrightarrow{k}+\overrightarrow{\theta})^2t\right)}{t^{\frac{3}{2}}}dtds \tag{4.99}$$

We set,

$\eta=\frac{E(\theta)}{16\pi^4}$, $\mu=\frac{E(0)}{16\pi^4}$

One can find the value of $\frac{1}{\lambda}$ at $\theta=0$:

$$\frac{1}{\lambda}\Big|_{\theta=0}=\frac{1}{16\pi^{\frac{5}{2}}}\sum_{\overrightarrow{k}\epsilon\Gamma}\frac{1}{\left(\sqrt{(\overrightarrow{k}^2+m^2}-\mu\right)\left(\sqrt{(\overrightarrow{k}^2+m^2}\right)} \tag{4.100}$$

or

$$\frac{1}{\lambda}\Big|_{\theta=0} = \frac{1}{16\pi^{\frac{5}{2}}} \sum_{\vec{k}\,\epsilon\Gamma} \int_0^\infty \frac{e^{s\mu}-1}{\mu} \int_0^\infty \frac{\exp\Big(-\frac{s^2}{4t} - m^2 t - (\vec{k})^2 t\Big)}{t^{\frac{3}{2}}} dtds \tag{4.101}$$

We require λ to be independent of θ.

Note that if this condition is satisfied, the solution exists. However there is a divergence. We have to apply renormalization process which does not change observables to make the solution convergent.

$$\frac{1}{16\pi^{\frac{5}{2}}} \sum_{\vec{k}\,\epsilon\Gamma} \int_0^\infty ds \int_0^\infty dt \Bigg[\frac{\exp\Big(-\frac{s^2}{4t} - m^2 t - (\vec{k} + \vec{\theta})^2 t\Big)}{t^{\frac{3}{2}}} \times \frac{e^{s\eta}-1}{\eta} - \frac{\exp\Big(-\frac{s^2}{4t} - m^2 t - (\vec{k})^2 t\Big)}{t^{\frac{3}{2}}} \times \frac{e^{s\mu}-1}{\mu}\Bigg] = 0 \tag{4.102}$$

gives the solution for the bound state.

$$\frac{1}{16\pi^{\frac{5}{2}}} \int_0^\infty ds \int_0^\infty dt \frac{\exp\Big(-\frac{s^2}{4t} - m^2 t\Big)}{t^{\frac{3}{2}}} \Bigg[\frac{(e^{s\eta}-1)}{\eta} \sum_{\vec{k}\,\epsilon\Gamma} \exp\big(-(\vec{k} + \vec{\theta})^2\big) - \frac{(e^{s\mu}-1)}{\mu} \sum_{\vec{k}\,\epsilon\Gamma} \exp\big(-(\vec{k})^2\big)\Bigg] = 0 \tag{4.103}$$

In a similar way, we can find the Green function:

$$G(k+\theta) = G_0(k+\theta) + G_0(k+\theta)\Bigg[\frac{1}{16\pi^{\frac{5}{2}}} \int_0^\infty ds \int_0^\infty dt \frac{\exp\Big(-\frac{s^2}{4t} - m^2 t\Big)}{t^{\frac{3}{2}}} \times \Big[\frac{(e^{s\eta}-1)}{\eta} \sum_{\vec{k}\,\epsilon\Gamma} \exp\big(-(\vec{k} + \vec{\theta})^2\big) - \frac{(e^{s\mu}-1)}{\mu} \sum_{\vec{k}\,\epsilon\Gamma} \exp\big(-(\vec{k})^2\big)\Big]\Bigg]^{-1} . \Big(G_0(k+\theta), .\Big) \tag{4.104}$$

5. PATTERNS OF DIRAC-DELTA POTENTIAL

5.1. General Concepts

In this model, Dirac delta potentials have arbitrary shape instead of being a point. Therefore, interaction term of Hamiltonian is defined as:

$$-\lambda \sum_{\chi} \delta\,(\overrightarrow{\chi} - \overrightarrow{x})\,\Psi(\overrightarrow{\chi}) \tag{5.1}$$

where, $|\chi\rangle$ which is an image determines the shape of Dirac delta potential in a cell.

Let us introduce a map, γ, such that

$$\gamma : \mathbb{R} \longrightarrow \widehat{\Gamma} \tag{5.2}$$

$$\gamma : s \longrightarrow d\overrightarrow{\chi} \tag{5.3}$$

Ordinary real number corresponds to differential of the shape in the Wigner-Seitz cell.

To find the ket which represents the pattern choosing, we have to integrate over a variable.

$$|\chi\rangle = \int_{\gamma} ds\,|\gamma(s)\rangle \tag{5.4}$$

where χ is called a curve representation.

Dirac delta function over the pattern is

$$\delta\,(\overrightarrow{\chi} - \overrightarrow{x}) = \int_{\gamma} ds\,\langle\gamma(s)|\Psi\rangle \tag{5.5}$$

The wave function can be defined as:

$$\Psi\left(\chi\right) = \int_{\gamma} ds \left\langle \gamma(s) | x \right\rangle \tag{5.6}$$

Our full Hamiltonian is

$$H = H_0 - \lambda \sum_{\chi} \left|\chi\right\rangle \left\langle\chi\right| \tag{5.7}$$

When free Hamiltonian, H_0, acts upon its momentum eigenstate:

$$H_0 \left|p\right\rangle = \frac{p^2}{2m} \left|p\right\rangle \tag{5.8}$$

In lattice representation, our potential is repeated in same form.

$$\sum_{\chi} \left|\chi\right\rangle \left\langle\chi\right| = \sum_{n_1} \sum_{n_2} \left|\overrightarrow{\chi} + n_1 \overrightarrow{a_1} + n_2 \overrightarrow{a_2}\right\rangle \left\langle \overrightarrow{\chi} + n_1 \overrightarrow{a_1} + n_2 \overrightarrow{a_2}\right| \tag{5.9}$$

where

$$\Lambda = \left(n_1 \overrightarrow{a_1} + n_2 \overrightarrow{a_2}\right) \epsilon \ \mathrm{R}^2 | \left(n_1, n_2\right) \epsilon Z^2 \tag{5.10}$$

Let us act Hamiltonian to abstract ket, and plug identity operator in momentum form:

$$H \left|\Psi\right\rangle = \int H \left|p\right\rangle \left\langle p | \Psi \right\rangle d^2 p \tag{5.11}$$

$$= = \int \left|p\right\rangle E\left(\theta\right) \Psi\left(p\right) d^2 p \tag{5.12}$$

If we want to write it more explicitly:

$$H \left|\Psi\right\rangle = \left(H_0 - \lambda \sum_{n_1} \sum_{n_2} \left|\overrightarrow{\chi} + n_1 \overrightarrow{a_1} + n_2 \overrightarrow{a_2}\right\rangle \left\langle \overrightarrow{\chi} + n_1 \overrightarrow{a_1} + n_2 \overrightarrow{a_2}\right| \right) \left|\Psi\right\rangle \tag{5.13}$$

We can plug three identity operator between kets and bras for simpler calculation in function form.

$$
\begin{aligned}
H\left|\Psi\right\rangle = &\int H_0 \left|p\right\rangle \left\langle p|\Psi\right\rangle d^2p \\
&-\lambda\sum_{n_1}\sum_{n_2}\int d^2pd^2p' \left|p\right\rangle \left\langle p|\overrightarrow{\chi}+n_1\overrightarrow{a_1}+n_2\overrightarrow{a_2}\right\rangle \left\langle\overrightarrow{\chi}+n_1\overrightarrow{a_1}+n_2\overrightarrow{a_2}|p'\right\rangle \left\langle p'|\Psi\right\rangle
\end{aligned} \tag{5.14}
$$

If (5.8) is used, we can obtain

$$
\begin{aligned}
H\left|\Psi\right\rangle = &\int \left|p\right\rangle \left\langle p|\Psi\right\rangle \frac{p^2}{2m} d^2p \\
&-\lambda\sum_{n_1}\sum_{n_2}\int d^2pd^2p' \left|p\right\rangle \left\langle p|\overrightarrow{\chi}+n_1\overrightarrow{a_1}+n_2\overrightarrow{a_2}\right\rangle \left\langle\overrightarrow{\chi}+n_1\overrightarrow{a_1}+n_2\overrightarrow{a_2}|p'\right\rangle \left\langle p'|\Psi\right\rangle
\end{aligned} \tag{5.15}
$$

When applying identity operator, we can write as

$$
\begin{aligned}
\int d^2pH|p\rangle\langle p|\left|\Psi\right\rangle = &\int d^2p \left|p\right\rangle \frac{p^2}{2m}\Psi\left(p\right) \\
&-\lambda\sum_{n_1}\sum_{n_2}\int d^2p|p\rangle d^2p' \exp\left[-i\left(\overrightarrow{p}\,\overrightarrow{\chi}+n_1\overrightarrow{p}\,\overrightarrow{a_1}+n_2\overrightarrow{p}\,\overrightarrow{a_2}\right)\right] \\
&.\exp\left[i\left(\overrightarrow{p'}\,\overrightarrow{\chi}+n_1\overrightarrow{p'}\,\overrightarrow{a_1}+n_2\overrightarrow{p'}\,\overrightarrow{a_2}\right)\right]\Psi\left(p'\right).\frac{1}{2\pi}
\end{aligned} \tag{5.16}
$$

When we omit $\int d^2p|p\rangle$, we obtain

$$
\begin{aligned}
E(\theta)\Psi\left(p\right) = &\frac{p^2}{2m}\Psi\left(p\right) \\
&-\lambda\sum_{n_1}\sum_{n_2}\int d^2p' \exp\left[-i\left(\overrightarrow{p}\,\overrightarrow{\chi}+n_1\overrightarrow{p}\,\overrightarrow{a_1}+n_2\overrightarrow{p}\,\overrightarrow{a_2}\right)\right] \\
&\times\exp\left[i\left(\overrightarrow{p'}\,\overrightarrow{\chi}+n_1\overrightarrow{p'}\,\overrightarrow{a_1}+n_2\overrightarrow{p'}\,\overrightarrow{a_2}\right)\right]\Psi\left(p'\right)\times\frac{1}{2\pi}
\end{aligned} \tag{5.17}
$$

The basic periodic or primitive cell $\widehat{\Gamma}$ can be identified with Wigner-Seitz cell [2].

$$
\widehat{\Gamma} = \left(s_1\overrightarrow{a_1}+s_2\overrightarrow{a_2}\right)\epsilon\ \mathrm{R}^2|s_j\in\left[-\frac{1}{2},\frac{1}{2}\right), j=1,2 \tag{5.18}
$$

The shape of Dirac delta potential is an element of the Wigner-Seitz cell.

$$\chi \,\epsilon\, \widehat{\Gamma}$$

Arbitrary momentum can be decomposed as an element of dual lattice and that of Brillouin zone.

$$\overrightarrow{p} = \overrightarrow{k} + \overrightarrow{\theta} \quad \overrightarrow{k}\,\epsilon\Gamma,\ \ \overrightarrow{\theta}\,\epsilon\widehat{\Lambda}$$
$$\int d^2p = \sum_k \int d^2\theta$$

$\overrightarrow{k}$ can be expanded in their basis.

$$\overrightarrow{k} = m_1\overrightarrow{b_1} + m_2\overrightarrow{b_2}$$
$$\sum_k = \sum_{m_1}\sum_{m_2} \qquad m_1, m_2 \in \mathbb{Z}$$

$\overrightarrow{\theta}$ is also decomposed in its basis.

$$\overrightarrow{\theta} = z_1\overrightarrow{b_1} + z_2\overrightarrow{b_2} \quad z_1, z_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Let us make Schrodinger equation simpler by using the relation between basis of Bravais lattice and that of reciprocal lattice. We can choose that $(\overrightarrow{a_1}, \overrightarrow{a_2})$ and $(\overrightarrow{b_1}, \overrightarrow{b_2})$ are orthonormal bases such that: [10], [11]

$$\overrightarrow{a_i}.\overrightarrow{a_j} = \delta_{ij}$$
$$\overrightarrow{b_i}.\overrightarrow{b_j} = \delta_{ij} \quad i, j = 1, 2$$

We define A and B which are parts of Schrodinger equation.

$$A = exp\Big[-i\big((\overrightarrow{k}+\overrightarrow{\theta})\overrightarrow{\chi} + n_1m_1\underbrace{\overrightarrow{a_1}\overrightarrow{b_1}}_{2\pi} + n_1m_2\underbrace{\overrightarrow{a_1}\overrightarrow{b_2}}_{0} + n_2m_1\underbrace{\overrightarrow{a_2}\overrightarrow{b_1}}_{0} + n_2m_2\underbrace{\overrightarrow{a_2}\overrightarrow{b_2}}_{2\pi} + n_1\overrightarrow{\theta}\overrightarrow{a_1} + n_2\overrightarrow{\theta}\overrightarrow{a_2}\big)\Big] \quad (5.19)$$

Values of some terms were shown by underbraces above.

After applying the simple identity such that $e^{2\pi \mathrm{i}} = 1, \quad t\epsilon\mathbb{Z}$, we can write

$$A = exp\left[-i\left(\overrightarrow{k}+\overrightarrow{\theta}\right)\overrightarrow{\chi}\right] exp\left[-i\left(\overrightarrow{\theta}\,\overrightarrow{a}\right)\right] \quad (5.20)$$

where, $\overrightarrow{a}$ is defined by

$$\overrightarrow{a} = n_1\overrightarrow{a_1} + n_2\overrightarrow{a_2}$$
$$\sum_{n_1}\sum_{n_2} = \sum_{\overrightarrow{a}}$$

In the same manner, we can write

$$B = exp\Big[i\big((\overrightarrow{k'}+\overrightarrow{\theta'})\overrightarrow{\chi} + n_1m'_1\underbrace{\overrightarrow{a_1}\overrightarrow{b_1}}_{2\pi} + n_1m'_2\underbrace{\overrightarrow{a_1}\overrightarrow{b_2}}_{0} + n_2m'_1\underbrace{\overrightarrow{a_2}\overrightarrow{b_1}}_{0} + n_2m'_2\underbrace{\overrightarrow{a_2}\overrightarrow{b_2}}_{2\pi} + n_1\overrightarrow{\theta'}\overrightarrow{a_1} + n_2\overrightarrow{\theta'}\overrightarrow{a_2}\big)\Big] \quad (5.21)$$

We can use the same procedure.

$$B = exp\left[i\left(\overrightarrow{\overrightarrow{k'}}+\overrightarrow{\theta'}\right)\overrightarrow{\chi}\right] exp\left[i\left(\overrightarrow{\theta'}\,\overrightarrow{a}\right)\right] \quad (5.22)$$

Now, we put kinetic term of the equation from right to left with changing its sign:

$$\Big(E(\theta) - \frac{(\overrightarrow{k} + \overrightarrow{\theta})^2}{2m}\Big)\Psi(\overrightarrow{k} + \overrightarrow{\theta})) = -\lambda \sum_{\overrightarrow{k'}\epsilon\Gamma} \int d\theta' exp\Big(-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}\Big). \\ exp\Big(i(\overrightarrow{k'} + \overrightarrow{\theta'})\overrightarrow{\chi}\Big) \sum_{\overrightarrow{a}} exp\Big(i\overrightarrow{a}(\overrightarrow{\theta} - \overrightarrow{\theta'})\Big). \\ \Psi(\overrightarrow{k'} + \overrightarrow{\theta'}) \quad (5.23)$$

There is an identity for lattice structure.

$$\sum_{\overrightarrow{a}} exp\left(i\overrightarrow{a}(\overrightarrow{\theta} - \overrightarrow{\theta'})\right) = \delta(\overrightarrow{\theta} - \overrightarrow{\theta'}) \quad (5.24)$$

We have to use this identity for (5.23)

$$\left(E(\theta) - \frac{(\overrightarrow{k} + \overrightarrow{\theta})^2}{2m}\right)\Psi(\overrightarrow{k} + \overrightarrow{\theta}) = -\lambda \int ds exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right] \\ \sum_{\overrightarrow{k'}\epsilon\Gamma} \int ds' exp\left[i(\overrightarrow{k'} + \overrightarrow{\theta})\overrightarrow{\chi}(s')\right] \\ \Psi(\overrightarrow{k'} + \overrightarrow{\theta}) \quad (5.25)$$

Let us look at a property which a shape of potential has

$$f(\overrightarrow{\chi})g(\overrightarrow{\chi}) = \int ds \int ds' f(\overrightarrow{\chi}(s))g(\overrightarrow{\chi}(s')) \quad (5.26)$$

where, s is a variable whose trajectory is $\chi(s)$, and f and g are arbitrary functionals.

At (5.26), all points of one trajectory have to interact with that of other one. We can apply (5.26) to (5.25)

$$\left(E(\theta) - \frac{(\overrightarrow{k} + \overrightarrow{\theta})^2}{2m}\right)\Psi(\overrightarrow{k} + \overrightarrow{\theta}) = -\lambda \int ds exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right] \sum_{\overrightarrow{k'}\epsilon\Gamma} \int ds' exp\left[i(\overrightarrow{k'} + \overrightarrow{\theta})\overrightarrow{\chi}(s')\right] \Psi(\overrightarrow{k'} + \overrightarrow{\theta}) \quad (5.27)$$

Now, we define

$$\sum_{\overrightarrow{k'}\epsilon\Gamma} \int ds' exp\left[i(\overrightarrow{k'} + \overrightarrow{\theta})\overrightarrow{\chi}(s')\right] \Psi(\overrightarrow{k'} + \overrightarrow{\theta}) = \Xi \quad (5.28)$$

Therefore, (5.27) can be redefined.

$$\Psi(\overrightarrow{k} + \overrightarrow{\theta}) = \frac{-1}{\left(E(\theta) - \frac{(\overrightarrow{k} + \overrightarrow{\theta})^2}{2m}\right)}\lambda \int ds exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right] \Xi \quad (5.29)$$

Let us multiply both sides of eqn(5.29) by $\int ds'' exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s'')\right]$ and sum over $\overrightarrow{k}$. After that, left hand side of the equation is equal to Ξ:

$$(5.30)$$

$$(5.31)$$

$$\Xi = \sum_{\overrightarrow{k}\epsilon\Gamma} \int ds'' exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s'')\right] \Psi(\overrightarrow{k} + \overrightarrow{\theta}) \sum_{\overrightarrow{k}\epsilon\Gamma} \int ds'' exp\left[i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s'')\right] \frac{-1}{\left(E(\theta) - \frac{(\overrightarrow{k} + \overrightarrow{\theta})^2}{2m}\right)}\lambda \int ds exp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right] \Xi \quad (5.32)$$

Ξ can be omitted from both sides of the equation. After dividing both sides by λ, we obtain

$$\frac{1}{\lambda} = \sum_{\overrightarrow{k}\epsilon\Gamma} \int ds'' exp\left[i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s'')\right] \int dsexp\left[-i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right] \frac{-1}{\left(E(\theta) - \frac{(\overrightarrow{k}+\overrightarrow{\theta})^2}{2m}\right)} \tag{5.33}$$

After rearranging 5.33, we obtain:

$$\frac{1}{\lambda} = \sum_{\overrightarrow{k}\epsilon\Gamma} \left| \int ds\, exp\left[i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right]\right|^2 \frac{1}{-E(\theta) + \frac{(\overrightarrow{k}+\overrightarrow{\theta})^2}{2m}} \tag{5.34}$$

We are working for bound states, then the notation, $E(\theta)$, can be changed

$$E(\theta) = -\nu^2(\theta) \tag{5.35}$$

Finally, this can be written as

$$\frac{1}{\lambda} = \sum_{\overrightarrow{k}\epsilon\Gamma} \left| \int ds\, exp\left[i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right]\right|^2 \frac{1}{\nu^2(\theta) + \frac{(\overrightarrow{k}+\overrightarrow{\theta})^2}{2m}} \tag{5.36}$$

We have found $\frac{1}{\lambda}$ for arbitrary shape. It determines that solution exists or not. We can separate the value of $\frac{1}{\lambda}$.

- $\frac{1}{\lambda} \to \infty$: There is no solution for the shape we are working. $\lambda \to 0$ means that there is no interaction term or potential
- $\frac{1}{\lambda}$ is convergent. This means that any solutions are possible.

Similarly we find the Green's function in the dual lattice vector representation.

$$
\begin{aligned}
G_\theta(E) =& \frac{1}{H_\theta - E} + \frac{1}{H_\theta - E} \\
& \times \left[\frac{1}{\lambda} - \sum_{\vec{k}\epsilon\Gamma} \left| \int ds\, exp\left[i(\vec{k} + \vec{\theta})\vec{\chi}(s) \right] \right|^2 \frac{1}{-E + \frac{(\vec{k}+\vec{\theta})^2}{2m}} \right] \\
& \times \frac{1}{H_\theta - E}
\end{aligned} \tag{5.37}
$$

Same can be repeated for 3-dim lattice. Then we face a similar divergence, we need to introduce renormalization process. We choose

$$
\frac{1}{\lambda} = \sum_{\vec{k}\epsilon\Gamma} \left| \int ds\, exp\left[i\vec{k}\vec{\chi}(s) \right] \right|^2 \frac{1}{\mu^2 + \frac{\vec{k}^2}{2m}} \tag{5.38}
$$

where

$$
-\mu^2 = E_{\theta=0} \tag{5.39}
$$

We can write one of the most important term to find resolvent.

$$
\Phi_\theta^{-1} = \frac{1}{\sum_{\vec{k}\epsilon\Gamma} \left| \int ds\, exp\left[i\vec{k}\vec{\chi}(s) \right] \right|^2 \left[\frac{1}{\mu^2 + \frac{\vec{k}^2}{2m}} - \frac{1}{-E + \frac{(\vec{k}+\vec{\theta})^2}{2m}} \right]} \tag{5.40}
$$

5.2. Applications

5.2.1. Line Pattern1

We have found the formula for coefficient of interaction for arbitrary shape of Dirac delta potentials in lattice structure. Now, we choose a line which is parallel to one of the orthogonal bases as the shape of the pattern. Let us write this mathematically

$$
\chi(s) = s\vec{a_1} \qquad \vec{a_i}\vec{a_j} = \delta_{ij} \tag{5.41}
$$

I want to remind our formula to apply the shape we choose

$$\frac{1}{\lambda} = \sum_{\vec{k}\,\epsilon\Gamma} \left| \int ds\, exp \left[i(\vec{k} + \vec{\theta})\vec{\chi}(s) \right] \right|^2 \frac{1}{\nu^2(\theta) + \frac{(\vec{k}+\vec{\theta})}{2m}} \tag{5.42}$$

First of all, we have to write some terms in (5.42) more explicitly to find the value of the cofactor.

Arbitrary point in momentum space is

$$\vec{k} + \vec{\theta} = (m_1 + z_1)\vec{b_1} + (m_2 + z_2)\vec{b_2} \tag{5.43}$$

where, [2]

$$m_i \epsilon \mathbb{Z} \quad and \quad z_i \epsilon \left[-\frac{1}{2}, \frac{1}{2} \right]$$

The dot product of the vector which explain the shape and arbitrary momentum can be written as

$$(\vec{k} + \vec{\theta})\vec{\chi}(s) = \left((m_1 + z_1)\vec{b_1} + (m_2 + z_2)\vec{b_2} \right) (s\vec{a_1}) \tag{5.44}$$

After using $\vec{a_i}\vec{b_j} = 2\pi\delta_{ij}$ [10], [11], we obtain

$$(\vec{k} + \vec{\theta})\vec{\chi}(s) = 2\pi(m_1 + z_1)s \tag{5.45}$$

In addition to these, we have to write square of momentum clearly.

$$(\vec{k} + \vec{\theta})^2 = (m_1 + z_1)^2 + (m_2 + z_2)^2 \tag{5.46}$$

Finally, we need summation equivalence which is

$$\sum_{\overrightarrow{k}\,\epsilon\Gamma} \rightarrow \sum_{m_1} \sum_{m_2}$$

If we plug all term in (5.42), we can write

$$\frac{1}{\lambda} = \sum_{m_1} \sum_{m_2} \frac{\left| \int \; exp\left[2\pi i(m_1 + z_1)s\right] ds \right|^2}{\frac{(m_1+z_1)^2+(m_2+z_2)^2}{2M} + \vartheta_\theta^2} \tag{5.47}$$

We can change summation for the sake of simplicity.

$$m_1 + z_1 = m_1' \qquad\qquad m_2 + z_2 = m_2'$$

$$\sum_{m_1} \sum_{m_2} \rightarrow \sum_{m_1'} \sum_{m_2'}$$

m_i start at zero but m_i' begin at z_i.

And we choose the border of integration in the exponential from $-\frac{L}{2}$ to $\frac{L}{2}$ to make the integral definite.

Therefore, we can write

$$\frac{1}{\lambda} = \sum_{m_1'} \sum_{m_2'} \frac{\left| \int_{-\frac{L}{2}}^{\frac{L}{2}} \; exp\left[2\pi i m_1' s\right] \; ds \right|^2}{\frac{(m_1')^2+(m_2')^2}{2M} + \vartheta_\theta^2} \tag{5.48}$$

Let us integrate the term in exponential.

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[2\pi i m_1' s\right] ds = \frac{exp\left[m_1'\, 2\pi i s\right]}{2\pi i m_1'}\bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$
$$= \frac{\left[exp\left(m_1'\, 2\pi i \frac{L}{2}\right)\right] - \left[exp\left(-m_1'\, 2\pi i \frac{L}{2}\right)\right]}{2\pi i m_1'} \tag{5.49}$$

And then, we expand these exponential terms:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[2\pi i m_1' s\right] ds = \frac{\cos\left(\pi m_1' L\right) + i\sin\left(\pi m_1' L\right) - \cos\left(\pi m_1' L\right) + i\sin\left(\pi m_1' L\right)}{2\pi i m_1'}$$
$$= \frac{1}{\pi m_1'} \sin\left(\pi m_1' L\right) \tag{5.50}$$

Let us plug this simple expression in (5.48).

$$\frac{1}{\lambda} = \sum_{m_1'} \sum_{m_2'} \frac{1}{\pi^2} \sin^2\left(\pi m_1' L\right) \times \frac{1}{m_1'^2 \times \left(\frac{(m_1')^2 + (m_2')^2)}{2M} + \vartheta_\theta^2\right)} \tag{5.51}$$

Now, we have to look situation of convergence (or divergence) of $\frac{1}{\lambda}$ to determine existence of solution.

Let us separate $\frac{1}{\lambda}$.

$$\frac{1}{\lambda} = \frac{1}{\pi^2} \frac{\sin^2\left(\pi z_1 L\right)}{(z_1)^2 \left(\frac{(z_1)^2 + (z_2)^2)}{2M} + \vartheta_\theta^2\right)}$$
$$+ \frac{1}{\pi^2} \sum_{m_1' - (z_1)} \frac{\sin^2\left(\pi m_1' L\right)}{(m_1')^2 \left(\frac{(m_1')^2 + (z_2)^2)}{2M} + \vartheta_\theta^2\right)}$$
$$+ \frac{1}{\pi^2} \sum_{m_2' - (z_2)} \frac{\sin^2\left(\pi z_1 L\right)}{(z_1)^2 \left(\frac{(z_1)^2 + (m_2')^2)}{2M} + \vartheta_\theta^2\right)}$$
$$+ \frac{1}{\pi^2} \sum_{m_1' - (z_1)} \sum_{m_2' - (z_2)} \frac{\sin^2\left(\pi m_1' L\right)}{(m_1')^2 \left(\frac{(m_1')^2 + (m_2')^2)}{2M} + \vartheta_\theta^2\right)} \tag{5.52}$$

If we settle the boundaries of the summations, we obtain

$$\begin{aligned}
\frac{1}{\lambda} =&\frac{1}{\pi^2}\frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{1}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\frac{\sin^2(\pi m_1' L)}{(m_1')^2\left(\frac{(m_1')^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{1}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2\left(\frac{(m_1')^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{1}{\pi^2}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{1}{\pi^2}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{1}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\sum_{m_2'=-\infty}^{z_2-1}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}+\sum_{m_2'=1+z_2}^{\infty}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}\right) \\
&+\frac{1}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\sum_{m_2'=-\infty}^{z_2+1}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)}+\sum_{m_2'=1+z_2}^{\infty}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}\right)
\end{aligned} \tag{5.53}$$

There are squares in our equation, then we can write

$$\sum_{m_{1,2}'=1+(z_{1,2})}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2\left(\frac{(m_1')^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \approx \sum_{m_{1,2}'=-\infty}^{z_{1,2}-1}\frac{\sin^2(\pi m_1' L)}{(m_1')^2\left(\frac{(m_1')^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \tag{5.54}$$

Therefore, we obtain

$$
\begin{aligned}
\frac{1}{\lambda} \approx & \frac{1}{\pi^2} \frac{\sin^2(\pi z_1 L)}{(z_1)^2 \left(\frac{(z_1)^2+(z_2)^2}{2M} + \vartheta_\theta^2 \right)} \\
& + \frac{2}{\pi^2} \sum_{m_1'=1+z_1}^{\infty} \frac{\sin^2(\pi m_1' L)}{(m_1')^2 \left(\frac{(m_1')^2+(z_2)^2}{2M} + \vartheta_\theta^2 \right)} \\
& + \frac{2}{\pi^2} \sum_{m_2'=1+z_2}^{\infty} \frac{\sin^2(\pi z_1 L)}{(z_1)^2 \left(\frac{(z_1)^2+(m_2')^2}{2M} + \vartheta_\theta^2 \right)} \\
& + \frac{4}{\pi^2} \sum_{m_1'=1+z_1}^{\infty} \frac{\sin^2(\pi m_1' L)}{(m_1')^2} \left(\sum_{m_2'=1+z_2}^{\infty} \frac{1}{\left(\frac{(m_1')^2+(m_2')^2}{2M} + \vartheta_\theta^2 \right)} \right)
\end{aligned}
\tag{5.55}
$$

Now, there is an special case in the first term of (5.55).

Consider $z_1 \to 0$ and $z_2 \to 0$ such that $z_1 + z_2 \to 0$. In that case, $\frac{1}{\lambda} \to \infty$ that may be seen as a problem. Actually this is not.

We have to use L'Hospital's rule [18] which can be defined as below:
Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then,

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{5.56}
$$

We can apply this rule to the first term of (5.55):

$$
\frac{1}{\pi^2} \frac{\sin^2(\pi z_1 L)}{(z_1)^2 \left(\frac{(z_1)^2+(z_2)^2}{2M} + \vartheta_\theta^2 \right)} \approx \frac{L^2}{\vartheta_\theta} \tag{5.57}
$$

Therefore, the first term of (5.55) does not cause divergence. We have to define integral test [15], [16] to show that other terms converges.

Let us consider an integer N and a non-negative, continuous function f defined on the unbounded interval [N,∞), on which it is monotone decreasing. Therefore, the infinite series, $\sum_{n=N}^{\infty} f(n)$, converges to a real number if and only if the improper integral $\int_N^{\infty} f(x)dx$ is finite. In other words, if the integral diverges, then the series diverges as well.

If the improper integral is finite, then the proof also gives the lower and upper bounds.

$$\int_N^\infty f(x)dx \le \sum_{n=N}^{\infty} f(n) \le f(N) + \int_N^\infty f(x)dx \tag{5.58}$$

for the infinite series.

Firstly, we can apply integral test to the second term.

$$\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty} \frac{\sin^2(\pi m_1' L)}{(m_1')^2\left(\frac{(m_1')^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \le \frac{2}{\pi^2}\frac{\sin^2(\pi(z_1+1)L)}{(z_1+1)^2\left(\frac{(z_1+1)^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} + \frac{2}{\pi^2}\int_{z_1+1}^{\infty} dx \frac{\sin^2(\pi x L)}{x^2\left(\frac{x^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \tag{5.59}$$

We can say that the second term converges.

Secondly,

$$\frac{2}{\pi^2}\sum_{m_2'=z_2+1}^{\infty} \frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \le \frac{2}{\pi^2}\frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+(z_2+1)^2}{2M}+\vartheta_\theta^2\right)} + \frac{2}{\pi^2}\int_{z_2+1}^{\infty} dx \frac{\sin^2(\pi z_1 L)}{(z_1)^2\left(\frac{(z_1)^2+x^2}{2M}+\vartheta_\theta^2\right)} \tag{5.60}$$

It is clear that the third term converges.

Finally, we have to look at the last term rigorously. We can apply integral test inside the parenthesis.

$$\frac{4}{\pi^2}\sum_{m_1'=1+z_1}^{\infty} \frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\sum_{m_2'=1+z_2}^{\infty}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}\right) \le \frac{4}{\pi^2}\sum_{m_1'=1+z_1}^{\infty} \frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\frac{1}{\left(\frac{(m_1')^2+(z_2+1)^2}{2M}+\vartheta_\theta^2\right)} + \int_{z_2+1}^{\infty}\frac{dx}{\left(\frac{(m_1')^2+x^2}{2M}+\vartheta_\theta^2\right)}\right) \tag{5.61}$$

We know an identity related to the integral.

$$\int \frac{1}{x^2+a}dx = \frac{\arctan(\frac{x}{\sqrt{a}})}{\sqrt{a}} + cst \tag{5.62}$$

After integrating this, we can write

$$\begin{aligned}
&\frac{4}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\sum_{m_2'=1+z_2}^{\infty}\frac{1}{\left(\frac{(m_1')^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)}\right) \leq \\
&\frac{4}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2}\left(\frac{1}{\left(\frac{(m_1')^2+(z_2+1)^2}{2M}+\vartheta_\theta^2\right)}\right. \\
&\left. + \frac{\arctan(\infty)-\arctan\left(\frac{z_2}{\sqrt{(m_1')^2+2M\vartheta_\theta^2}}\right)}{\sqrt{((m_1')^2+2M\vartheta_\theta^2)}}\times 2M\right)
\end{aligned} \tag{5.63}$$

where,

$\arctan(\infty) = \frac{\pi}{2}$.

The first term of the inequality above is similar to the second term of (5.55). Then, it converges.

Let us see the second term of the inequality above.
Because it does not affect convergence of the problem, the term, $\arctan(\infty) - \arctan\left(\frac{z_2}{\sqrt{(m_1')^2+2M\vartheta_\theta^2}}\right)$, can be omitted.

$$\begin{aligned}
\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2(\pi m_1' L)}{(m_1')^2\sqrt{((m_1')^2+2M\vartheta_\theta^2)}} \leq &\frac{\sin^2(\pi(z_1+1)L)}{(1+z_1)^2\sqrt{((1+z_1)^2+2M\vartheta_\theta^2)}} \\
&\int_{z_1+1}^{\infty} dx\frac{\sin^2(\pi x L)}{x^2\sqrt{(x^2+2M\vartheta_\theta^2)}}
\end{aligned} \tag{5.64}$$

We can see that it does not cause any problems.

Therefore, $\frac{1}{\lambda}$ converges,there is no need for renormalization. Moreover, we now have an explicit formula for the resolvent.

5.2.2. Line Pattern 2

Now, we have come to the second example in which the formula for coefficient of interaction for arbitrary shape of Dirac delta potentials in lattice structure is applied to the line pattern whose direction is not parallel to both of bases. For the sake of simplicity, the angles between pattern line and basis can be chosen as "$\frac{\pi}{4}$". Let us write the equation which describes the pattern structure.

$$\overrightarrow{\chi}(s) = \frac{s}{\sqrt{2}}\overrightarrow{a_1} + \frac{s}{\sqrt{2}}\overrightarrow{a_2} \tag{5.65}$$

where we have chosen orthonormal bases such that

$$\overrightarrow{a_1}.\overrightarrow{a_1} = \delta_{i,j}$$

We have to recall our formula to apply the shape which we are working.

$$\frac{1}{\lambda} = \sum_{\overrightarrow{k}\epsilon\Gamma} \left| \int ds\; exp\left[i(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s)\right]\right|^2 \frac{1}{\nu^2(\theta) + \frac{(\overrightarrow{k}+\overrightarrow{\theta})}{2m}} \tag{5.66}$$

Firstly, some terms in (5.66) have to be written in more explicit manner to discuss the cofactor. We write arbitrary momentum in lattice structure in the first example.

$$\overrightarrow{k} + \overrightarrow{\theta} = (m_1 + z_1)\overrightarrow{b_1} + (m_2 + z_2)\overrightarrow{b_2} \tag{5.67}$$

Therefore, we can write dot product of the vector which explains our shape and arbitrary momentum as

$$\begin{aligned}(\overrightarrow{k} + \overrightarrow{\theta})\overrightarrow{\chi}(s) &= \left((m_1 + z_1)\overrightarrow{b_1} + (m_2 + z_2)\overrightarrow{b_2}\right) . \left(\frac{s}{\sqrt{2}}\overrightarrow{a_1} + \frac{s}{\sqrt{2}}\overrightarrow{a_2}\right) \\ &= \sqrt{2}\pi\left[(m_1 + z_1)s + (m_2 + z_2)s\right] \end{aligned} \tag{5.68}$$

where, the identity that is "$\overrightarrow{a_j}\overrightarrow{b_{j'}} = 2\pi\delta_{jj'}$ " have been used

We need the expression that is the square of momentum.

$$(\overrightarrow{k} + \overrightarrow{\theta})^2 = (m_1 + z_1)^2 + (m_2 + z_2)^2 \tag{5.69}$$

Additionally, the summation over $\overrightarrow{k}$ can be re expressed as

$$\sum_{\overrightarrow{k} \epsilon \Gamma} \longrightarrow \sum_{m_1} \sum_{m_2} \tag{5.70}$$

Let us plug all terms in the equation:

$$\frac{1}{\lambda} = \sum_{m_1} \sum_{m_2} \frac{\left| \int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[(m_1 + z_1 + (m_2 + z_2)\sqrt{2}\pi i s \right] ds \right|^2}{\frac{(m_1+z_1)^2+(m_2+z_2)^2}{2M} + \vartheta_\theta^2} \tag{5.71}$$

We can change boundaries of summations for simpler calculation.

$$m_1 + z_1 = m_1' \qquad m_2 + z_2 = m_2'$$

$$\sum_{m'_1} \sum_{m'_2} \longrightarrow \sum_{m_1} \sum_{m_2} \tag{5.72}$$

m_i goes through, but m'_i start at $z_i$and goes through $m_i + z_i$. And the border of integration in the exponential can be chosen from $-\frac{L}{2}$ to $\frac{L}{2}$ in order to make the integral definite.

$-\frac{L}{2}$ and $\frac{L}{2}$ have to be in the Wigner-Seitz cell.

$$\Big(-\frac{L}{2}\overrightarrow{n}, \frac{L}{2}\overrightarrow{n} \Big) \epsilon \widehat{\Gamma}$$

, where

$$\overrightarrow{n} = \frac{\overrightarrow{a_1}}{\sqrt{2}} + \frac{\overrightarrow{a_2}}{\sqrt{2}}$$

If we apply these conditions to formula, we obtain

$$\frac{1}{\lambda} = \sum_{m_1'} \sum_{m_2'} \frac{\left| \int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[(m_1' + m_2') \sqrt{2} \pi i s \right] \, ds \right|^2}{\frac{(m_1')^2 + m_2')^2}{2M} + \vartheta_\theta^2} \tag{5.73}$$

Let us integrate the term inside absolute-square in the numerator:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[(m_1' + m_2') \sqrt{2} \pi i s \right] \, ds = \left. \frac{exp\left[(m_1' + m_2') \sqrt{2} \pi i s \right]}{\sqrt{2} \pi i (m_1' + m_2')} \right|_{-\frac{L}{2}}^{\frac{L}{2}} \tag{5.74}$$

After the boundary value of the integral, we obtain

$$\begin{aligned} \int_{-\frac{L}{2}}^{\frac{L}{2}} exp\left[(m_1' + m_2') \sqrt{2} \pi i s \right] \, ds = & \frac{\cos\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right) + i \sin\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right)}{\sqrt{2} \pi i (m_1' + m_2')} \\ & + \frac{i \sin\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right) \cos\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right)}{\sqrt{2} \pi i (m_1' + m_2')} \\ = & \frac{\sqrt{2}}{\pi (m_1' + m_2')} \sin\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right) \end{aligned} \tag{5.75}$$

Now, we can plug this simple expression in the equation:

$$\frac{1}{\lambda} = \sum_{m_1'} \sum_{m_2'} \frac{2}{\pi^2} \sin^2\left(\pi (m_1' + m_2') \frac{L}{\sqrt{2}} \right) \times \frac{1}{(m_1' + m_2')^2 \times \left(\frac{(m_1')^2 + (m_2')^2)}{2M} + \vartheta_\theta^2 \right)} \tag{5.76}$$

Let us separate $\frac{1}{\lambda}$:

$$\begin{aligned}\frac{1}{\lambda} =& \frac{2}{\pi^2}\frac{\sin^2\left(\pi(z_1+z_2)\frac{L}{\sqrt{2}}\right)}{(z_1+z_2)^2.\left(\frac{(z_1)^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)} \\ &+\frac{2}{\pi^2}\sum_{m_1'-(z_1)}\frac{\sin^2\left(\pi(m_1'+z_2)\frac{L}{\sqrt{2}}\right)}{(m_1'+z_2)^2.\left(\frac{(m_1')^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)} \\ &+\frac{2}{\pi^2}\sum_{m_2'-(z_2)}\frac{\sin^2\left(\pi(z_1+m_2')\frac{L}{\sqrt{2}}\right)}{(z_1+m_2')^2.\left(\frac{(z_1)^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \\ &+\frac{2}{\pi^2}\sum_{m_1'-(z_1)}\sum_{m_2'-(z_2)}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}\end{aligned} \tag{5.77}$$

If we settle the boundaries of the summations, we obtain

$$
\begin{aligned}
\frac{1}{\lambda} =& \frac{2}{\pi^2}\frac{\sin^2\left(\pi(z_1+z_2)\frac{L}{\sqrt{2}}\right)}{(z_1+z_2)^2.\left(\frac{(z_1)^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\frac{\sin^2\left(\pi(m_1'+z_2)\frac{L}{\sqrt{2}}\right)}{(m_1'+z_2)^2.\left(\frac{(m_1')^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2\left(\pi(m_1'+z_2)\frac{L}{\sqrt{2}}\right)}{(m_1'+z_2)^2.\left(\frac{(m_1')^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(z_1+m_2')\frac{L}{\sqrt{2}}\right)}{(z_1+m_2')^2.\left(\frac{(z_1)^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(z_1+m_2')\frac{L}{\sqrt{2}}\right)}{(z_1+m_2')^2.\left(\frac{(z_1)^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \\
&+\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}
\end{aligned} \tag{5.78}
$$

We can estimate:

$$\frac{1}{\lambda} \approx \frac{2}{\pi^2}\frac{\sin^2\left(\pi(z_1+z_2)\frac{L}{\sqrt{2}}\right)}{(z_1+z_2)^2.\left(\frac{(z_1)^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)}$$
$$+\frac{6}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2\left(\pi(m_1'+z_2)\frac{L}{\sqrt{2}}\right)}{(m_1'+z_2)^2.\left(\frac{(m_1')^2+(z_2)^2)}{2M}+\vartheta_\theta^2\right)}$$
$$+\frac{4}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}$$
$$+\frac{2}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}$$
$$+\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \tag{5.79}$$

Let us look situation of convergence(or divergence) of $\frac{1}{\lambda}$ in order to determine existence of solution. Now, there is an special case in the first term of (5.79). Consider $z_1 \to 0$ and $z_2 \to 0$ such that $z_1 + z_2 \to 0$. In that case, $\frac{1}{\lambda'} \to \infty$ may be seen as a problem. Actually it is not.

$$\lim_{(z_1+z_2)\to 0}\frac{\sin^2\left(\pi(z_1+z_2)\frac{L}{\sqrt{2}}\right)}{(z_1+z_2)^2} \approx \frac{\pi^2 L^2}{2} \tag{5.80}$$

Therefore, the first term of Eqn (5.79) does not cause divergence.

We have to use the integral test to show the second term converges.

$$\frac{6}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\frac{\sin^2\left(\pi(m_1'+z_2)\frac{L}{\sqrt{2}}\right)}{(m_1'+z_2)^2\left(\frac{(m_1')^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \le \frac{6}{\pi^2}\frac{\sin^2\left(\pi(z_1+z_2+1)\frac{L}{\sqrt{2}}\right)}{(z_1+z_2+1)^2\left(\frac{(z_1+1)^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)}+$$
$$\frac{6}{\pi^2}\int_{z_1+1}^{\infty}dx\frac{\sin^2\left(\pi(x+z_2)\frac{L}{\sqrt{2}}\right)}{(x+z_2)^2\left(\frac{x^2+(z_2)^2}{2M}+\vartheta_\theta^2\right)} \tag{5.81}$$

It is easy to see that there is no problem.

When looking at the third term, we can see similar algebraic structure of the final term of $\frac{1}{\lambda}$ in the previous example.

$$\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}$$
$$<\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi m_1' L\right)}{(m_1')^2\left(\frac{(m_1')^2+(m_2')^2)}{2M}+\vartheta_\theta^2\right)} \tag{5.82}$$

We can say that it converges.

Let us look at the last two terms. They have approximately same value.

$$\frac{2}{\pi^2}\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=1+z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}\approx$$
$$\frac{2}{\pi^2}\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \tag{5.83}$$

We can write it with changing the sign of the second summation.

$$\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=-\infty}^{z_2-1}\frac{\sin^2\left(\pi(m_1'+m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'+m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}=$$
$$\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1-z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)} \tag{5.84}$$

$$\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=1-z_2}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}=$$

$$2\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=m_1'+1}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}+\sum_{m_1'=1+z_1}^{\infty}\frac{\pi^2L^2}{2}\frac{1}{\frac{(m_1')^2}{M}+\vartheta_\theta^2} \tag{5.85}$$

It is clear that second term of (5.85) converges. We have to look at the first term rigorously. Then, it can be written in a new form for the sake of easy calculations.

$$\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=m_1'+1}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left(\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2\right)}=$$

$$2M\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=m_1'+1}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left((m_1')^2+(m_2')^2+2M\vartheta_\theta^2\right)}$$

$$2M\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=m_1'+1}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left((m_1'+m_2')^2+(m_1'-m_2')^2+2M\vartheta_\theta^2\right)} \tag{5.86}$$

We can change variables such that

$$m_1'-m_2'=\upsilon \qquad\qquad m_1'+m_2'=\zeta \tag{5.87}$$

as continuous variables.

$$2M\sum_{m_1'=1+z_1}^{\infty}\sum_{m_2'=m_1'+1}^{\infty}\frac{\sin^2\left(\pi(m_1'-m_2')\frac{L}{\sqrt{2}}\right)}{(m_1'-m_2')^2.\left((m_1'+m_2')^2+(m_1'-m_2')^2+2M\vartheta_\theta^2\right)}< \tag{5.88}$$

$$2M\int_0^{\infty}d\upsilon\int_{-\upsilon}^{\upsilon}d\zeta\frac{1}{\upsilon^2+\zeta^2+2M\vartheta_\theta^2}\frac{sin^2(\zeta\frac{\pi L}{\sqrt{2}})}{\zeta^2}d\zeta< \tag{5.89}$$

$$2M\int_0^{\infty}d\upsilon\frac{1}{\upsilon^2+2M\vartheta_\theta^2}\int_{-\upsilon}^{\upsilon}\frac{sin^2(\zeta\frac{\pi L}{\sqrt{2}})}{\zeta^2}d\zeta< \tag{5.90}$$

$$2M\int_0^{\infty}d\upsilon\frac{1}{\upsilon^2+2M\vartheta_\theta^2}\int_{-\infty}^{\infty}\frac{sin^2(\zeta\frac{\pi L}{\sqrt{2}})}{\zeta^2}d\zeta \tag{5.91}$$

We know that

$$\int_{-\infty}^{\infty} \frac{sin^2(\varsigma)}{\varsigma^2} d\varsigma = \frac{\pi}{2} \tag{5.92}$$

It is clar that

$$M \int_0^{\infty} dv \frac{1}{\upsilon^2 + 2M\vartheta_\theta^2} < \infty \tag{5.93}$$

Therefore, $\frac{1}{\lambda}$ converges.

5.2.3. Circular Pattern

Finally, we have reached to the last example of our formula. In this example, we are working with circular shaped Dirac delta potentials in each cell. Let us write the pattern structure of circular shape:

$$\overrightarrow{\chi(s)} = Rcos(\theta)\overrightarrow{a_1} + Rsin(\theta)\overrightarrow{a_2} \tag{5.94}$$

Our bases are orthonormal.

$$\overrightarrow{a_1}.\overrightarrow{a_1} = \delta_{i,j}$$

We have to remind our formula in order to apply circular pattern structure.

$$\frac{1}{\lambda} = \sum_{\vec{k}\epsilon\Gamma} \left| \int ds\, exp\left[i(\vec{k} + \vec{\theta})\vec{\chi}(s) \right] \right|^2 \frac{1}{\nu^2(\theta) + \frac{(\vec{k}+\vec{\theta})^2}{2M}} \tag{5.95}$$

The dot product between circular pattern vector and arbitrary momentum is

$$(\vec{k} + \vec{\theta})\vec{\chi}(s) = 2\pi R(m_1 + z_1)cos(\theta) + 2\pi R(m_2 + z_2)sin(\theta) \tag{5.96}$$

We need some expressions from previous examples such that

$$\overrightarrow{k} + \overrightarrow{\theta} = (m_1 + z_1)\overrightarrow{b_1} + (m_2 + z_2)\overrightarrow{b_2} \tag{5.97}$$

$$\overrightarrow{a_i}\overrightarrow{b_j} = 2\pi\delta_{ij} \tag{5.98}$$

$$(\overrightarrow{k} + \overrightarrow{\theta})^2 = (m_1 + z_1)^2 + (m_2 + z_2)^2 \tag{5.99}$$

$$\sum_{\overrightarrow{k}\epsilon\Gamma} \longrightarrow \sum_{m_1}\sum_{m_2} \tag{5.100}$$

Let us plug all terms in their positions:

$$\frac{1}{\lambda} = \sum_{m_1}\sum_{m_2}\left|\int_0^{2\pi} d\theta\, exp\left[2\pi i\left((m_1+z_1)cos(\theta)R + (m_2+z_2)sin(\theta)\right)R\right]\right|^2 \times \frac{1}{\frac{(m_1+z_1)^2+(m_2+z_2)^2}{2M} + \vartheta_\theta^2} \tag{5.101}$$

Now, we can change variables easily. First of all, components of momentum vectors have to be changed:

$$\big(m_1 + z_1, m_2 + z_2\big) \Longrightarrow \big(\sqrt{(m_1+z_1)^2 + (m_2+z_2)^2}, 0\big) \tag{5.102}$$

Secondly, the angle between non-zero component, $sqrt(m_1 + z_1)^2 + (m_2 + z_2)^2$, and radius R, is arbitrary. It can be chosen as zero with changing angle variable, because the integral in absolute square is from 0 to 2π.

$$\theta \Longrightarrow \theta' \tag{5.103}$$

such that

$$\overrightarrow{p}.\overrightarrow{R} = \left(\sqrt{(m_1+z_1)^2+(m_2+z_2)^2}R\right)\cos(\theta') \tag{5.104}$$

Now, θ is used instead of θ' due to aesthetic reasons. Therefore, we can write the coefficient in simpler form.

$$\frac{1}{\lambda} = \sum_{m_1}\sum_{m_2}\left|\int_0^{2\pi} d\theta\, exp\left[2\pi i\left(\sqrt{(m_1+z_1)^2+(m_2+z_2)^2}R\right)\cos(\theta')\right]\right|^2 \times \frac{1}{\frac{(m_1+z_1)^2+(m_2+z_2)^2}{2M}+\vartheta_\theta^2} \tag{5.105}$$

Let us look inside absolute square. There is a Bessel function of the first kind with zero order [14] which is defined as

$$\mathrm{J}_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp(ia\ cos(\theta))d\theta \tag{5.106}$$

This term in (5.105) is written as

$$\left|\int_0^{2\pi} d\theta\, exp\left[2\pi i\sqrt{(m_1+z_1)^2+(m_2+z_2)^2}cos(\theta)R\right]\right|^2 = \left|2\pi\ \mathrm{J}_0(2\pi R\sqrt{(m_1+z_1)^2+(m_2+z_2)^2})\right|^2 \tag{5.107}$$

The summation term can be changed for simpler expression.

$$m_1+z_1 = m_1' \qquad m_2+z_2 = m_2' \tag{5.108}$$

If we write (5.105) with using new notation, it is written as

$$\frac{1}{\lambda} = \sum_{m_1'}\sum_{m_2'}\left|2\pi\ \mathrm{J}_0(2\pi\sqrt{(m_1')^2+(m_2')^2}R)\right|^2 \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} \tag{5.109}$$

Arbitrary Bessel function of the first kind with zero order can be expanded as

$$J_0(x) = \sum_{m}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \tag{5.110}$$

(5.110) implies that $\mathrm{J}_0(x)$ is real, if x is real. It implies that

$$\left| J_0(2\pi\sqrt{(m_1')^2 + (m_2')^2} R) \right|^2 = \mathrm{J}_0^2(2\pi\sqrt{(m_1')^2 + (m_2')^2} R) \tag{5.111}$$

Let us apply this property to our equation:

$$\frac{1}{\lambda} = \sum_{m_1'} \sum_{m_2'} \Big(2\pi \, \mathrm{J}_0(2\pi\sqrt{(m_1')^2 + (m_2')^2} R) \Big)^2 \times \frac{1}{\frac{(m_1')^2 + (m_2')^2}{2M} + \vartheta_\theta^2} \tag{5.112}$$

A Bessel function of the first kind has a special feature in inequality form [14]. This can be demonstrated as

$$\mathrm{J}_\nu(x) < (\frac{x}{2})^\nu \frac{1}{\Gamma(\nu+1)} \exp\Big(-\frac{x^2}{4(\nu+1)} \Big) \tag{5.113}$$

$$\mathrm{J}_0(x) < \frac{1}{\Gamma(1)} \exp\Big(-\frac{x^2}{4} \Big) \tag{5.114}$$

where, Γ is gamma function and

$$\Gamma(1) = 1 \tag{5.115}$$

[17]

Let us apply Eqn (5.114) to Eqn(5.112):

$$\frac{1}{\lambda} < \sum_{m_1'} \sum_{m_2'} 4\pi^2 exp(-\pi^2 \sqrt{(m_1')^2 + (m_2')^2} R) \times \frac{1}{\frac{(m_1')^2 + (m_2')^2}{2M} + \vartheta_\theta^2} \tag{5.116}$$

We can separate the summation to see more clear.

$$\sum_{m_1'}\sum_{m_2'} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} =$$
$$4\pi^2 exp(-\pi^2\sqrt{(z_1)^2+(z_2)^2}R) \times \frac{1}{\frac{(z_1)^2+(z_2)^2}{2M}+\vartheta_\theta^2} +$$
$$\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=-\infty}^{z_2-1} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} +$$
$$\sum_{m_1'=-\infty}^{z_1-1}\sum_{m_2'=1+z_1}^{\infty} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} +$$
$$\sum_{m_1'=z_1+1}^{\infty}\sum_{m_2'=-\infty}^{z_2-1} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} +$$
$$\sum_{m_1'=z_1+1}^{\infty}\sum_{m_2'=z_2+1}^{\infty} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} + \quad (5.117)$$

The equation above can be approximated as

$$\sum_{m_1'}\sum_{m_2'} 4\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} \approx$$
$$\sum_{m_1'=0}^{\infty}\sum_{m_2'=0}^{\infty} 16\pi^2 exp(-\pi^2\sqrt{(m_1')^2+(m_2')^2}R) \times \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M}+\vartheta_\theta^2} \quad (5.118)$$

We have to change variables ie, m_1') and m_2' to make calculation in more comfortable manner.

Now it is time to define a new variable h:

$$(m_1')^2 + (m_2')^2 = h^2 \quad (5.119)$$

We can focus on the new variable, h^2, to use integral test which have been used in previous examples. The integral over new variable is bigger than the summation over two old variables. Let us write inequality which we need with the help of integral test.

$$\sum_{m_1'=z_1, m_2'=z_2}^{\infty} f((m_1')^2 + (m_2')^2) < f(0) + 4\pi \int_0^{\infty} f(h^2) h dh \tag{5.120}$$

We can write this more explicitly as

$$\begin{aligned} &\sum_{m_1'} \sum_{m_2'} 4\pi^2 exp(-\pi^2 \sqrt{(m_1')^2 + (m_2')^2} R). \frac{1}{\frac{(m_1')^2+(m_2')^2}{2M} + \vartheta_\theta^2} \\ &< 16\pi^2 \frac{1}{\vartheta_\theta^2} + 16\pi^2 \int_0^{\infty} exp(-\pi^2 h R) \frac{h}{\frac{h^2}{2M} + \vartheta_\theta^2} dh \end{aligned} \tag{5.121}$$

We have chosen that energy such that the term, $16\pi^2 \frac{1}{\vartheta_\theta^2}$, converges.
Now we have to look at the second term on the right hand side of the inequality.

Therefore, we can ignore this for now.

$$\begin{aligned} \int_0^{\infty} \frac{exp(-\pi^2 h R) h}{h^2 + 2M\vartheta_\theta^2} dh =& \frac{1}{2} e^{-i\pi R\sqrt{2M\vartheta_\theta^2}} \Bigg(e^{2i\pi R\sqrt{2M\vartheta_\theta^2}} \operatorname{Ei}\big(-\pi R(x + i\sqrt{2M\vartheta_\theta^2})\big) \\ &+ \operatorname{Ei}\big(i\pi R\sqrt{2M\vartheta_\theta^2} - \pi R x\big) \Bigg)\Bigg|_0^{\infty} \\ &= \frac{1}{2} e^{-i\pi R\sqrt{2M\vartheta_\theta^2}} \Bigg(e^{2i\pi R\sqrt{2M\vartheta_\theta^2}} \operatorname{Ei}\big(-\pi R(\infty + i\sqrt{2M\vartheta_\theta^2})\big) \\ &+ \operatorname{Ei}\big(i\pi R\sqrt{2M\vartheta_\theta^2} - \pi R\infty\big) \\ &- e^{2i\pi R\sqrt{2M\vartheta_\theta^2}} \operatorname{Ei}\big(-i\pi R\sqrt{2M\vartheta_\theta^2}\big) \\ &+ \operatorname{Ei}\big(i\pi R\sqrt{2M\vartheta_\theta^2}\big) \Bigg) \end{aligned} \tag{5.122}$$

where Ei represents exponential integral which can be described as

$$\operatorname{Ei}(x) = \int_{-\infty}^{x} \frac{e^{-t} dt}{t} \tag{5.123}$$

We know that

$$\operatorname{Ei}\big(-\pi R(\infty + i\sqrt{2M\vartheta_\theta^2})\big) = \operatorname{Ei}\big(i\pi R\sqrt{2M\vartheta_\theta^2} - \pi R\infty\big) = 0 \tag{5.124}$$

and

$$e^{-i\pi R\sqrt{2M\vartheta_{\theta}^{2}}}\,\mathrm{Ei}\left(i\pi R\sqrt{2M\vartheta_{\theta}^{2}}\right)-e^{i\pi R\sqrt{2M\vartheta_{\theta}^{2}}}\,\mathrm{Ei}\left(-i\pi R\sqrt{2M\vartheta_{\theta}^{2}}\right)\epsilon\mathbb{R} \tag{5.125}$$

Therefore, we can say that $\frac{1}{\lambda}$ converges.

6. CONCLUSION

In conclusion, we have reviewed application of the Dirac-delta potential on a one dimensional lattice structure. Then, we have mentioned Green's function and regularization & renormalization process for the bound states in two dimensions. We have worked on the single Dirac-delta function and the relation between the Green's function and the renormalization of the interaction cofactor have been shown. After that, we have reviewed the work of Albeverio on two dimensional lattice which shows that there was a solution of the Hamiltonian. In this model, there were Dirac delta potentials in all lattice points in the one-to-one manner. There was only one band gap. This was an interesting result. We have applied renormalization process to this model. We have looked again Albevrio's work into to use semi-relativisitic approach. Albeverio worked in the world of Quantum Mechanics, but we have used Quantum Field Theory. Green's function have been found for the classic and semi-relativistic cases. Lastly, we found a new formula of interaction cofactor for the arbitrary shape of Dirac-delta potential in two dimensional lattice in the aspects of Quantum Mechanics. We have written the Green's function for this model. The formula showed us the existence of a solution of the Hamiltonian. Also, we have mentioned briefly the three dimensional version of the formula. We have applied this formula on three different patterns being two of them are line and the other pattern was circular. We have showed that none of these examples need renormalization.

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