ADMISSION AND TERMINATION CONTROL OF A TWO CLASS LOSS SYSTEM

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ABSTRACT

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In this thesis, we consider admission and termination control policies in a Markovian loss system with two classes of jobs. A class is characterized by the arrival and service rates, in addition to a fixed reward and termination cost. There are three possible decisions upon an arrival: admitting or rejecting the arriving job, or admitting him/her by terminating a job which is already in the system. The aim is to maximize total expected discounted profit over a finite or infinite horizon. We build a Markov decision model to analyze the structure of optimal policies.

We prove that when there is an idle server in the system, it is never optimal to terminate a job. In addition, we prove that there exists an optimal threshold policy for admission and termination. The threshold levels depend on the jobs of both classes already being served in the system. Furthermore, under certain conditions, we can ensure that a job class is "preferred" or "strongly-preferred". Preferred jobs are always admitted to the system if there are free servers. On the other hand, a strongly-preferred job is always admitted to the system even when the system is full, so that a job of the other class is terminated by incurring the termination cost. We show that both job types cannot be strongly preferred, although it is possible that one of them is strongly-preferred, and the other one is preferred.

ÖZET

İKİ SINIFLI YİTİM SİSTEMLERİNİN KABUL İZİNLERİ VE İHRAÇ KARARLARIYLA DENETİMİ

Bu tezde, iki sınıflı bir Markov yitim sisteminin kabul izinleri ve ihraç kararlarıyla denetimi için eniyi politikalar incelenmiştir. Bir sınıf ona ait olan varış hızı, işgörü hızı, sisteme getirisi ve ihraç maliyeti ile tanımlanmaktadır. Müşterinin varış anında verilebilecek üç tip karar vardır: gelen müşterinin kabul edilmesi, reddedilmesi veya kabul edilip sistemde mevcut olan diğer bir müşterinin ihraç edilmesi. Amaç ise sonlu veya sonsuz ufukta toplam indirilmiş beklenen karın eniyilenmesidir. Bu amaç doğrultusunda optimal politikaları analiz edebilmek için bir Markov karar modeli kurulmuştur.

Sistemde boş işgören var iken ihraç kararının eniyi olmadığı gösterilmiştir. Buna ek olarak, kabul ve ihraç kararları için eniyi bir eşik politikasının var olduğu gösterilmiş tir. Optimal eşik değerleri, sistemde mevcut olan iki müşteri sınıfının sayısına da bağımlıdır. Ayrıca belli koşullar altında "tercih edilen" veya "kuvvetle tercih edilen" müşteri sınıfının var olabileceği gösterilmiştir. Tercih edilen sınıftaki müşteriler, sistemde boş işgören olduğu sürece sisteme kabul edilen müşterilerdir. Kuvvetle tercih edilen sınıftaki müşteriler ise, sistem dolu olsa bile mevcut bir müşterinin ihraç edilmesi ile sisteme kabul edilen müşterilerdir. Aynı anda iki kuvvetle tercih edilen sınıfın olamayacağı, fakat bir sınıfın kuvvetle tercih edilen olup diğerinin ise tercih edilen olabileceği gösterilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$a_n(x;j)$	Optimal action in state $(x;j)$ when there are n stages
$a_n(x;j;s)$	Optimal action in state $(x;j;s)$ when there are n stages
$D_n(ij)(x)$	Difference function
d_1	Lower bound on difference function $D_n(10)(x)$
d_2	Lower bound on difference function $D_n(20)(x)$
d_{12}	Lower bound on difference function $D_n(12)(x)$
e_i	Unit vector
i, j, k	Index of class of jobs
$l_n^i(x_i)$	Threshold values of class-i jobs
n	Number of stages
r_i	Reward brought by class-i job
R_i	Immediate reward of class-i job
s	Number of servers
S	Maximum number of servers
$\mathcal{S}$	Set of states
sc	Fixed cost of increasing number of servers
tc_i	Termination cost of class-i job
$u_n(x)$	Value function in state x when there are n stages
$u_n(x;s)$	Value function in state $(x;s)$ when there are n stages
$u_n^A(x)$	Value function of system A in state x when there are n stages
$v_n(x;j)$	Value function in state $(x;j)$ when there are n stages
$v_n(x;j;s)$	Value function in state $(x;j;s)$ when there are n stages
x	System state vector for the first model
$x;s$	System state vector for the second model
$x^{'}$	System state vector with some specific properties
x_i	Number of class-i customers
$(x;i)$	System state vector at arrival of class-i job for the first model
$(x;i;s)$	System state vector at arrival of class-i job for the second model

β	Discount rate
λ_i	Arrival rate of class-i job
μ_i	Service rate of class-i job
MDP	Markov Decision Process
FCFS	First Come First Served
LCFS	Last Come First Served
$M/E_N/1$	Queueing system with 1 server, Poisson arrival, Erlang-N distributed service time
$M/M/m$	Queueing system with m servers, Poisson arrival, Exponential distributed service time
$M/M/2$	Queueing system with 2 servers, Poisson arrival, Exponential distributed service time

1. INTRODUCTION

In this thesis, we consider the problem of controlling the workload in a production or service system, via dynamic policies of accepting/rejecting incoming jobs and terminating existing jobs. Dynamic admission control is of increasing importance for revenue management in service and manufacturing systems. In telecommunications and in particular in telephone service and support applications, such strategies are commonly used in order to increase flexibility in the allocation of resources among job types.

A static admission rule specifies a priori whether each job class is admitted for service, based on the revenue that jobs of this class generate and their expected service requirements. This specification is made independently of the system state and is equivalent to determining whether one job class is preferred over the other. On the other hand, a dynamic admission policy offers increased flexibility because it makes the decision of admitting an arriving job contingent on the current level of congestion upon arrival, in addition to the job's class.

To the best of our knowledge there are few models in the literature that consider terminating an existing job. That is, once a job is admitted to the system it is kept in the system until the end of its processing time. However, in some practical applications,it may be a good choice to terminate an existing job altogether and start another job.

First example can be given as managing a crime investigation bureau. Jobs arrive randomly in a criminal investigation bureau also there is a limited capacity of criminal experts in this bureau. Furthermore, the nature and the extent of cases may differ greatly. This means that new work is directly associated with a specific type of case. As a result of these uncertainties, bureau has to give some decisions such as accepting or rejecting an incoming case or termination of a case so as to increase the quality of service or profit. Second example is a portfolio manager wishes to invest a certain amount of funds upon arrival of investment opportunities. She is able to remove an investment in her portfolio as well as to accept or reject an arriving of investment opportunity. The last example can be given as management of an intensive care unit. Obviously, patients with different circumstances arrive randomly and they have different service requirements. Termination of the service of an existing patient with certain properties can be very critical as well as accepting and rejecting, so as to maximize total life expectations, total quality of life or profit.

In this thesis, we consider admission and termination control policies in a Markovian loss system with two classes of jobs. The system we study has s identical servers, no waiting room and two classes of jobs. Class-i jobs arrive according to a Poisson Process with arrival rate λ_i and have different exponential service times with mean $1/\mu_i$. The fixed reward r_i is gained after service completion. There are three possible decisions upon an arrival: admitting or rejecting the arriving job, or admitting him/her by terminating a job who is already in the system by paying termination cost tc_i. Our objective is to find dynamic admission and termination control policies that maximize the total expected discounted revenue over a finite or infinite horizon as well as the long-run average revenue.

In this study we make three major contributions: (1) we prove that a termination decision is optimal only if the system is full, (2) we show that the optimal admission and termination policy is characterized by state dependent thresholds, and (3) we present conditions for the existence of preferred and strongly preferred classes. The term “preferred” (“strongly-preferred”) is used to denote a class whose jobs are always admitted to the system when there is at least one free server (even when all servers are occupied through terminating another job). We also demonstrate the effect of various system parameters (such as the termination cost or the reward for completing a job) on a job being preferred or strongly preferred, through examples. In addition to our main model we formulate another MDP model for a system where we can control the number of servers. On the other hand, we do not deeply analyze the structural properties of this model in this thesis. We show that both capacity increase and termination decisions are not optimal if the system is not full.

The rest of the thesis is organized as follows. Chapter 2 provides an overlook for the studies related to our problem. We build a discrete-time Markov decision process (MDP) model in Chapter 3. Our major structural results are given in Chapter 4. In particular, we show that a termination decision is not optimal when the system is not full and also prove that a threshold policy optimally characterizes admission and termination decisions. Furthermore, in Chapter 5, we provide sufficient conditions for both classes to be preferred or strongly preferred. We formulate the new model with capacity decision in Chapter 6. The last chapter is conclusion for the thesis and provides some further research topics.

2. LITERATURE REVIEW

Dynamic admission control policies have been extensively studied in the literature which is also named dynamic stochastic knapsack problem. An extensive review on the admission control problems can be found in Chapter 4 of Ross (1995). The studies included in Ross (1995) analyze the optimal policies, which belong to certain classes of policies, namely coordinate convex, complete partitioning and trunk reservation policies.

There have been earlier studies, which investigate the structural properties of optimal admission policies for certain stochastic knapsacks: Miller (1971) considers a system with s parallel identical servers, no waiting room and k job classes. All jobs demand an exponential service with the same rate, whereas they offer fixed rewards depending on their class. Thus, the optimal policy in Miller's work is a threshold type policy with a preferred class. Lippman and Ross (1971) analyze the optimal admission rule for a system with one server and no waiting room, which receives offers from jobs according to a joint service time, and reward probability distribution.

Harrison (1975) shows that the $r\mu$ rule is optimal for a scheduling problem in a single server queue with two classes of jobs characterized by Poisson arrivals with rates λ_i, exponential service times with different rates μ_i, and lump rewards, r_i, collected at the end of service. So, the optimal admission policy is to admit jobs who bring higher average reward, *i.e.*, $r_i\mu_i$, have to be scheduled first on the single server. The term "preferred" is used to denote a class whose jobs are always admitted to the system when there is at least one free server. We also note that in the studies of Miller (1971), Lewis et al. (1999) and Lippman and Ross (1971), there exists a preferred class, which is determined by the "$r\mu$ rule".

There have been a number of studies that analyze admission control policies in a loss system with two class types. Altman et al. (2001), and Koole (1998) prove the existence of an optimal admission policy that is characterized by acceptance thresholds for both classes in the system, whereas Örmeci et al. (2001) establish the monotonicity of thresholds under certain conditions. Our problem is an extension of Örmeci et al. (2001), in which we have an option to admit the arriving customer by terminating the service of an existing customer. We show that an optimal threshold policy still exists. Furthermore, Gans and Savin (2007) show the existence of threshold policy as well as demonstrate that optimal policy parameters are monotone with respect to system parameters. In our study, we also observe the effects of termination costs on the threshold values, however we do not prove the monotonicity of our threshold values with respect to termination costs. Moreover, Örmeci et al. (2001), Savin et al. (2005) and Gans and Savin (2007) analyze the issue of preferred classes. We propose a new definition, "strongly preferred" class and develop a set of sufficient conditions which ensure that a job class is "preferred" and "strongly preferred".

Örmeci et al. (2002) concentrate on the issues of optimal thresholds and preferred jobs when the rewards are random. Carrizosa et al. (1998) analyze an optimal static control policy for a stochastic knapsack with k classes of jobs, where the service times follow a general distribution. Likewise, Örmeci and van der Wal (2006) consider the problem with two classes of jobs and establish the existence of optimal acceptance thresholds for both job classes and show that under certain conditions there exists a preferred class for a two class loss system with general interarrival times.

Besides single arrival, Örmeci and Burnetas (2004), Örmeci and Burnetas (2005), Ku and Jordan (2002) and Ku and Jordan (2003) investigate the problems with batch arrivals in Markovian loss systems. Örmeci and Burnetas (2005) consider the problem of a Markovian loss system with two classes in which, jobs arrive at the system in batches, where each admitted job requires different service rates and brings different revenues depending on its class. They derive sufficient conditions for each system to have a preferred class and establish a monotonicity property of the optimal value functions. Ku and Jordan (2002) prove that the policy that maximizes total discounted revenue consists of a set of monotonically decreasing thresholds in a system with parallel multiserver loss queues. Moreover, Çil et al. (2007) present the structural properties of the optimal batch acceptance policy in a queueing problem with several customer classes. Çil et al. (2007) show the monotonicity properties of the optimal policy in a single server and constant batch size problem and give counter examples to show that optimal batch acceptance policies do not have any structural properties when the constant batch size assumption is relaxed. There are some other types of queueing control problems studied in the literature, in addition to admission control problems in loss systems, such as Ghoneim and Stidham (1985), Stidham (1985), Blanc et al. (1992). Stidham (2002) provides an extensive literature review on queueing control problems and their applications.

The very first idea to remove jobs from the system is given by Xu and Shanthikumar (1993) by using expulsion control, in which jobs are removed from the end of a queue. They obtain an optimal admission policy for a First Come First Served (FCFS) $M/M/m$ ordered entry queueing system with non-identical servers. A dual preemptive Last Come First Served (LCFS) $M/M/m$ ordered-entry system with expulsion control is constructed and they show that the two systems have the same probabilistic behavior. Xu (1994) determines the optimal admission and scheduling control policy in a FCFS $M/M/2$ queueing system with non-identical servers by using the dual approach. Righter (2000) extends the results to an $M/M/2$ queueing system with non-identical servers and multiple classes of jobs, where preemption is allowed. These studies use expulsion control not as a policy but as a tool whereas we use termination as a policy to control the original system. The main difference between termination control and expulsion control is as follows: Expulsion control removes a job from the end of a queue. In particular, a job whose service has started is never removed from the system. In termination control, on the other hand, a job which is currently in service can be immediately removed from the system.

Termination control is not heavily studied in the literature. The concept of termination control is first introduced by Brouns and van der Wal (2003). They consider an $M/E_N/1$ system with one type of job. The service of a job can be ended at any point in time, and it can be removed from the queue as well. Under certain regularity conditions, they show that there exist optimal threshold policies for the decision to accept or reject a new job and the decision to continue or terminate the service of a job.

In addition, Brouns and van der Wal (2006) investigate the system with two classes of jobs with different arrival rates and show that there exist optimal threshold policies for these two types of decisions. In the model of Brouns and van der Wal (2006), jobs have different arrival rates, but same service rates. Conversely, we investigate an optimal threshold policy in a multi-server system with different arrival rates and different service rates. Table 2.1 and Table 2.2 present some of the studies in the literature along with their positioning with respect to various system characteristics. In these tables we also position our study as compared to others.

Table 2.1. Some key studies in the literature

		Loss System		Queueing System	
		General Service	Exponential Service	General Service	Exponential Service
Termination	Single Arrival		Our Study	Brouns and van der Wal (2003)	Brouns and van der Wal (2006)
	Batch Arrival				
No Termination	Single Arrival	Carrizosa et al. (1998)	Altman et al. (2001) Örmeci et al. (2001)	Several studies, Stidham (2002)	
	Batch Arrival		Örmeci and Burnetas (2005) Ku and Jordan (2002)		

Table 2.2. Comparison of admission control related papers

	Loss	Queueing	Single Arr.	Batch Arr.	Termin.	No Termin.	Exp. Serv.	Gen. Serv.	Ident. Serv.	Non-Ident. Serv.
Our Study	√		√		√		√		√	
Altman et al. (2001)	√		√			√	√		√	
Brouns and van der Wal (2003)		√	√		√			√	√	
Brouns and van der Wal (2006)		√	√		√		√		√	
Carrizosa et al. (1998)	√		√			√		√	√	
Gans and Savin (2007)	√		√			√	√		√	
Koole (1998)	√		√			√	√		√	
Ku and Jordan (2002)	√			√		√	√		√	
Örmeci et al. (2001)	√		√			√	√		√	
Örmeci and Burnetas (2005)	√			√		√	√		√	
Righter (2000)		√	√		√		√			√
Xu and Shanthikumar (1993)		√	√		√		√			√

3. DESCRIPTION OF THE MATHEMATICAL MODEL

3.1. Markov Decision Model for Finite Horizon

In this section, we formulate a discrete-time, finite-horizon Markov decision process (MDP) model for the system described earlier. Our objective is maximizing total expected discounted return over a finite time horizon with β discount rate. We describe our MDP model as follows:

States: We have two types of state descriptions, upon an arrival of a job and upon a service completion: If there is a potential service completion, we denote the system state by $x = (x_1, x_2)$, where x_i is the number of class-i jobs and let $\mathcal{S}$ be the set of states, *i.e.*, $\mathcal{S} = \{x : x_1 + x_2 \leq s\}$. If there is a class-$j$ arrival, then the state is $(x; j) = (x_1, x_2; j)$ so that there are x_i class-i jobs in the system and a class-j job has just arrived.

Uniformization: The original process of the system evolves in continuous time, but we can apply uniformization technique to build the discrete time equivalent of the system (Lippman and Ross (1971)). We assume, without loss of generality, $\mu_1 \leq \mu_2$. Then, the maximum possible rate out of any state is $\lambda_1 + \lambda_2 + s\mu_2 + \beta$. If we set $\lambda_1 + \lambda_2 + s\mu_2 + \beta = 1$, so the system will be observed at exponentially distributed intervals with mean 1. Then, with probability λ_i a class-i job arrives, with probability $x_i\mu_i$ service completion due to a class-i job occurs, and with probability $s\mu_2 - x_1\mu_1 - x_2\mu_2$ a fictitious service completion occurs.

Value Functions: We define a value function for each type of states: $u_n(x)$ is the maximal expected β-discounted reward for the system starting in state x and and $v_n(x; j)$ is the maximal expected β-discounted reward for the system starting in state $(x; j)$, when n observation points remain in the horizon.

*Actions:*The decisions are taken at arrival epochs. We define $a_n(x; j)$ as the optimal action in state $(x; j)$ when there are n more transitions: $a_n(x; j)$ is 1 if it is optimal to accept the arriving job of class j, 0 if it is optimal to reject the arriving job of class-j and 2 if it is optimal to accept the arriving job of class-j by terminating one of the class-k ($k \neq j$) jobs which is already in the system. We choose to accept arriving job if it is optimal to accept and reject or terminate, and we choose to reject if it is both optimal to reject and terminate. We also define e_1 and e_2 as the vectors of size 2, which have a 1 at the 1^{st} and 2^{nd} coordinate, respectively, and 0 elsewhere.

Now we present the optimality equations:

For $j = 1, 2$ and $n \geq 0$:

$$\begin{aligned}
v_n(x;j) &= \max\left\{u_n(x+e_j), u_n(x), u_n(x+e_j-e_k) - tc_k\right\}, \text{ for } x_1+x_2 < s, \\
v_n(x;j) &= \max\left\{u_n(x), u_n(x+e_j-e_k) - tc_k\right\}, \text{ for } x_1+x_2 = s, \text{ and, } x_j \neq s, \\
v_n(x;j) &= u_n(x), \text{ for } x_j = s. \qquad (3.1)
\end{aligned}$$

For $x_1 + x_2 \leq s$:

$$\begin{aligned}
u_{n+1}(x) = {} & x_1\mu_1 r_1 + x_2\mu_2 r_2 + \lambda_1 v_n(x;1) + \lambda_2 v_n(x;2) + x_1\mu_1 u_n(x-e_1) \\
& + x_2\mu_2 u_n(x-e_2) + (s\mu_2 - x_1\mu_1 - x_2\mu_2)u_n(x). \qquad (3.2)
\end{aligned}$$

Further, we define $u_n(-1, x_2) = u_n(0, x_2)$ and $u_n(x_1, -1) = u_n(x_1, 0)$. Note that the definition of $v_n(x;j)$ is similar to the definition of optimality operator in standard admission control models (see, for example, Örmeci et al. (2001)) with the exception of the termination possibility and the resulting termination cost.

3.2. Infinite Horizon Model

We use induction technique to prove all our results. All the results for discounted finite horizon problem *i.e.*, for finite n, are also valid for discounted infinite horizon problem *i.e.*, limit $n \to \infty$ (see Puterman (1994)). Furthermore for $\beta = 0$, the same results are valid for maximizing the long-run average reward, since the state and action spaces are finite (see Puterman (1994)).

3.3. Coupling

In most of the proofs, we use the coupling method, which is a widely used method in Markov decision models. The idea is to compare two systems by coupling all the random variables for the two systems. Both systems will have the same arrival pattern. The service times of jobs in the two systems are coupled as follows: If the coupled jobs are of the same class, they depart at the same time. If they are different, we use the assumption $\mu_1 \leq \mu_2$, which implies that class-1 jobs are "slow" jobs. With probability μ_1, both jobs leaves the system, and with probability $\mu_2 - \mu_1$, the class-2 job departs the system while the coupled class-1 job stays in the system. If a class-1 job leaves the system, the coupled class-2 job also leaves the system. Hence, a coupled class-1 job can never leave the system while the coupled class-2 job is still there.

3.4. Effect of an Additional Job

The basis of our analysis stems from the differences between two systems. For instance, $u_n(x + e_j) - u_n(x)$ can be interpreted as the net benefit of the system due to an additional class-j job in state x, when there are n more transitions, and $u_n(x + e_1) - u_n(x + e_2)$ is the benefit of changing a class-2 job to a class-1 job in state $x + e_2$, when there are n more transitions. Therefore, we define $D_n(ij)(x)$ as the difference in the total expected β discounted rewards between system A and system B, if system A starts in state $x + e_i$ and system B starts in state $x + e_j$ ($x + e_0 = x$ with $e_0 = (0, 0)$), when there are n more steps to go. Hence, we are interested in four $D_n(ij)$ functions, which are:

$$D_n(10)(x) = u_n(x + e_1) - u_n(x), \tag{3.3}$$

$$D_n(20)(x) = u_n(x + e_2) - u_n(x), \tag{3.4}$$

$$D_n(12)(x) = -D_n(21)(x) = u_n(x + e_1) - u_n(x + e_2). \tag{3.5}$$

3.5. Rewards

We assume that the rewards are collected at the end of service completion. The present value of the reward brought by a class-i job is $(\frac{r_i\mu_i}{\mu_i+\beta})$ due to discounting. We call this quantity immediate reward of a class-i job and denote it by R_i. Furthermore, termination costs are incurred at the instants of termination decisions given. Since the termination cost tc_i is incurred at the instance of arrival, the immediate reward, R_i, is compared to tc_i to identify optimal actions. We note that because rewards are not collected until service completion, we do not repay any reward to the terminated job.

4. STRUCTURE OF THE OPTIMAL POLICY

In this chapter, we establish several structural properties of the MDP model formulated in Chapter 3. In section 4.1, we show that termination decision is not optimal if there are free servers in the system, which is the most important result of our study. In section 4.2, we provide a lower bound on the difference function $D_n(i0)(x)$, which we use to show our results. In section 4.3, we prove that a threshold policy optimally characterizes admission and termination decisions. We also give some examples to illustrate the effect of termination cost on threshold values. In addition, in section 4.4, we discuss a threshold policy on a full server system, unfortunately we cannot prove it analytically.

4.1. Condition for Non-optimality of Termination

We show that termination is never an optimal decision if there are free servers in the system. The following theorem states our main result regarding the termination decision, which is proven by coupling.

Theorem 4.1. *The termination decision is not optimal if all servers are not busy, i.e.,* $a_n(x;j) \neq 2$, *if* $x_1 + x_2 < s$.

Proof. We prove the statement by a sample path argument. Assume that systems A and B are both in state $(x;j)$ with $x_1 + x_2 < s$. System A accepts an arriving class-j job by terminating a class-k job with a cost of tc_k, where $j \neq k$. On the other hand, system B accepts the arriving job without termination. Now, system A is in state $(x + e_j - e_k)$, and system B is in state $(x + e_j)$. We couple the two systems in such a way that except for the additional job in system B, all service and interarrival times in both systems are the same. Moreover, we let system A to follow the optimal policy from now on and system B imitates the decisions of system A, which preserves the difference between two systems except three cases:

(i.) The process of the additional class-k job in system B can be finished with an additional reward r_k, then the two systems couple. System B has more profit than system A.

(ii.) If a class-k job arrives and system A accepts it, system B rejects this job to couple with system A. Thus, system B has tc_k more profit than system A.

(iii.) When the system B is full and system A accepts an arriving class-j job, system B also accepts it by terminating one of the class-k jobs with a termination cost of tc_k, so that the two systems couple. We know that system A already paid tc_k in period $n+1$, so again system B has more profit than system A, since paying tc_k in later stages is more profitable for $\beta > 0$. Even if $\beta = 0$, difference between two systems is 0.

In all cases above, system B performs better (or the same) than system A. This concludes that system A cannot follow the optimal policy. *i.e.*, $u_n(x + e_j - e_k) - tc_k \leq u_n(x + e_j)$, if $x_1 + x_2 < s$. Thus, terminating a job is never optimal when the system is not full. □

4.2. A Lower Bound on Difference of Value Functions

We show that the value of having one more class-j job is always greater than or equal to the negative of termination cost of class-j job.

Lemma 4.1. *For all x with $x_1 + x_2 + 1 \leq s$:*

$$u_n(x + e_j) - u_n(x) \geq -tc_j, \ \ \forall n \geq 0. \tag{4.1}$$

Proof. Define two systems A and B in states $x + e_j$ and x, respectively, and let system A has an extra reward of tc_j. We couple the two systems in such a way that except for the additional job in A, all service and interarrival times in both systems are the same. Moreover, we let system B to follow the optimal policy and system A imitates the decisions of system B, which preserves the difference between two systems except for the following three cases:

(i.) The process of the additional class-j job in system A can be finished with an additional reward r_j, then the two systems couple. System A has an extra profit of $tc_j + r_j$.

(ii.) If a class-j job arrives and system B accepts it, system A rejects the arriving job and couples with system B. Thus system A has tc_j more profit than system B.

(iii.) When system A is full and a class-k job arrives, where $k \neq j$, and if system B accepts arriving class-k job, system A also accepts it and terminates one of the class-j jobs by paying termination cost tc_j, again the two systems couple. Since system A has already extra tc_j, system A still has more profit than system B.

In all the above cases, the inequality $u_n(x + e_j) + tc_j \geq u_n(x)$ holds. □

Lemma 4.1 has a direct consequence on the structure of optimal policy as summarized in the following proposition:

Proposition 4.1. *a. If it is optimal to reject a class-k job in a state x, then it can never be optimal to accept a class-k job by terminating a class-j customer in state* $x + e_j$.

b. If it optimal to accept a class-j job by terminating a class-k job in a state x, then it is always optimal to accept a class-k job in state $x - e_j$.

Proof. *a.* Assume that it is optimal to reject a class-k job in state x. Then we have:

$$u_n(x + e_k) < u_n(x) \leq u_n(x + e_j) + tc_j,$$

where the first inequality is by our assumption and the second by Lemma 4.1.

b. Assume that it is optimal to accept a class-k job by terminating a class-j job in state x. Then we have:

$$u_n(x - e_j) < u_n(x) + tc_j \leq u_n(x - e_j + e_k),$$

where the first inequality is due to Lemma 4.1, and the second by our assumption. □

4.3. Optimal Threshold Policy

In this section we show that there exists an optimal policy that can be determined by optimal thresholds. We expect that it should be less profitable to accept jobs when there are more jobs already in the system, and so the benefit of additional jobs to the system should decrease in the number of jobs in the system. The following lemma shows that the benefit due to an additional class-j job is decreasing in the number of class-i jobs.

Lemma 4.2. *For all x with $x_1 + x_2 + 2 \leq s$:*

$$u_n(x + e_1 + e_2) - u_n(x + e_1) \leq u_n(x + e_2) - u_n(x), \;\; \forall n \geq 0. \tag{4.2}$$

Proof. When $x_1 + x_2 + 2 < s$, Lemma 4.2 is the same with Lemma 4 of Örmeci et al. (2001), since the system is concerned only with admission decisions due to Theorem 4.1. So, proving the inequality for the case of $x_1 + x_2 + 2 = s$ is sufficient. The value function at step n is:

$$\begin{aligned} u_n(x) &= x_1\mu_1(r_1 + u_{n-1}(x - e_1)) + x_2\mu_2(r_2 + u_{n-1}(x - e_2)) + (s\mu_2 - x_1\mu_1 - x_2\mu_2)u_{n-1}(x) \\ &\quad + \lambda_1[\max\{u_{n-1}(x + e_1), u_{n-1}(x), u_{n-1}(x + e_1 - e_2) - tc_2)\}] \\ &\quad + \lambda_2[\max\{u_{n-1}(x + e_2), u_{n-1}(x), u_{n-1}(x - e_1 + e_2) - tc_1)\}]. \end{aligned} \tag{4.3}$$

The idea of the proof is as follows. We assume that the inequality (4.4) holds for $n-1$ and show that it also holds for n:

$$u_n(x_1 + 1, x_2 + 1) - u_n(x_1, x_2 + 1) \leq u_n(x_1 + 1, x_2) - u_n(x_1, x_2). \tag{4.4}$$

We show that the inequality (4.4) holds term by term by using equality (4.3). For the first three terms which do not involve a maximization, the inequality (4.4) holds obviously. We consider the fourth term in (4.3), since the inequality (4.4) can be shown similarly.

We define four systems A, B, C, and D in states $(x_1+1, x_2+1), (x_1, x_2+1), (x_1+1, x_2), (x_1, x_2)$, respectively, with $x_1 + x_2 + 2 = s$. We let system A and D follow the optimal policy. Upon arrival of a class-1 job, system A cannot admit the arriving job, instead it can either reject it or accept it by terminating a class-2 job. On the other hand, the other systems can admit or reject the arriving job, while they cannot give a termination decision by Theorem 4.1. We have four possible cases depending on the decisions of systems A and D. Note that if the inequality (4.4) holds for some feasible actions of B and C, it also holds for optimal actions of B and C due to the maximization.

(i.) Assume that it is optimal to reject the incoming class-1 job in both systems A and D. Let system B and C also reject the arriving class-1 job. The inequality (4.4) becomes:

$$u_{n-1}(x_1+1, x_2+1) - u_{n-1}(x_1, x_2+1) \leq u_{n-1}(x_1+1, x_2) - u_{n-1}(x_1, x_2),$$

which is true by the induction hypothesis.

(ii.) Assume that the optimal action for system A is to reject and for system D to accept the arriving class-1 job. Let system B accept the arriving job and system C reject it. So, the inequality (4.4) becomes:

$$u_{n-1}(x_1+1, x_2+1) - u_{n-1}(x_1+1, x_2+1) \leq u_{n-1}(x_1+1, x_2) - u_{n-1}(x_1+1, x_2),$$

which is trivially true.

(iii.) Assume that the optimal action for system A is to accept arriving class-1 job by terminating a class-2 job and for system D is to accept it. Let system B and C both accept the arriving job. So, the inequality (4.4) turns out to be as below:

$$
\begin{aligned}
&u_{n-1}(x_1+2,x_2)-tc_2-u_{n-1}(x_1+1,x_2+1)\le u_{n-1}(x_1+2,x_2)-u_{n-1}(x_1+1,x_2)\\
\iff\ &u_{n-1}(x_1+1,x_2)-u_{n-1}(x_1+1,x_2+1)\le tc_2,
\end{aligned}
$$

which holds by Theorem 4.1.

(iv.) Assume that the optimal action for system A is to accept the arriving class-1 job by terminating a class-2 job and system D is rejecting the arriving class-1 job. Let system B reject and system C accept the arriving job. So the inequality is:

$$
\begin{aligned}
&u_{n-1}(x_1+2,x_2)-tc_2-u_{n-1}(x_1,x_2+1)\le u_{n-1}(x_1+2,x_2)-u_{n-1}(x_1,x_2)\\
\iff\ &-tc_2-u_{n-1}(x_1,x_2+1)\le -u_{n-1}(x_1,x_2)\\
\iff\ &u_{n-1}(x_1,x_2+1)\ge u_{n-1}(x_1,x_2)-tc_2,
\end{aligned}
$$

which is true by Lemma 4.1.

In all possible cases, the inequality (4.4) holds for n whenever it is true for $n-1$. Hence, we conclude, by induction, that the inequality (4.4) holds for all $n \ge 0$. □

Lemma 4.2 implies that an additional class-j job brings more benefit in state x than it does in state $x+e_i$, for $i \neq j$. This guarantees an optimal threshold policy:

Theorem 4.2. *There exists an optimal policy of threshold type, which is characterized as follows: Let* $l_n^i(x_i) = \min\{x_j : a_n(x;i)=0\}$*, where we set* $l_n^i(x_i) = s - x_i + 1$ *if there is no such* x*. Then if* $l_n^i(x_i) \le s - x_i$*:*

$$
a_n(x;i) = \begin{cases} 0, & if\ \ x_j \ge l_n^i(x_i) \\ 1, & otherwise, \end{cases}
$$

and if $l_n^i(x_i) = s - x_i + 1$*:*

$$a_n(x;i) = \begin{cases} 2, & if \;\; x_j = s - x_i \\ 1, & otherwise. \end{cases}$$

We would also expect it to be more difficult to accept class-i jobs when there are more class-i jobs, which corresponds to the concavity of the value functions:

$$u_n(x + 2e_i) - u_n(x + e_i) \leq u_n(x + e_i) - u_n(x), \;\; \forall n \geq 1.$$

However, this property has not been shown in full generality even for the simpler systems with no termination (Örmeci et al., 2001).

The thresholds can be computed only as a result of finding the optimal policy, i.e., there is no closed form solution for the optimal thresholds. We can analyze their behavior as the termination costs vary. In one extreme when $tc_j = \infty$, the problem coincides with the admission control problem, whereas in the other extreme with $tc_j = 0$, all jobs are always admitted although some will be terminated later on upon an arrival of "better" class. In between these extreme cases, the threshold values of class-i job decrease in tc_i. It is easier to terminate a class-i job when tc_i is low, so it is also easier to accept it. As tc_i increases, the system becomes more conservative to accept class-i jobs since it will be expensive to terminate them. The following example demonstrates this:

Example 1: We take $s = 8$ and other parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 2$, $r_1 = 1$, $r_2 = 50$, $\beta = 0.01$, $tc_2 = 2$. Class-2 jobs are always admitted to the system when there are free servers, also a class-1 job is terminated to accept a class-2 job for $tc_1 = 10$ and $tc_1 = 1$. Hence, the thresholds for class-2 are given by $l_n^2(x_2) = s - x_2$ for $tc_1 = 100$, $l_n^2(x_2) = s - x_2 + 1$ for $tc_1 = 10$ and $tc_1 = 1$. The threshold values of class 1 for three different termination cost values are given in Table 4.1, while Figure 4.1 provides a visual depiction of these thresholds. We observe that the thresholds decrease in the termination cost.

Table 4.1. Thresholds in Example 1

tc_1	$l_n^1(0)$	$l_n^1(1)$	$l_n^1(2)$	$l_n^1(3)$	$l_n^1(4)$	$l_n^1(5)$	$l_n^1(6)$	$l_n^1(7)$
100	6	5	4	3	2	1	0	0
10	7	6	5	4	3	2	1	0
1	8	7	6	5	4	3	2	1

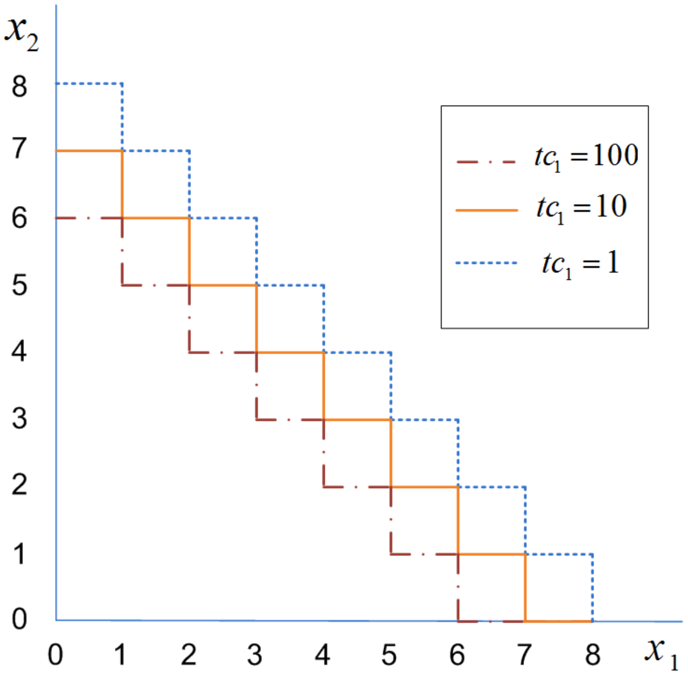

Figure 4.1. Thresholds in Example 1

4.4. A Discussion on an Optimal Threshold Policy When the System is Full

We start this section by an example to illustrate how a threshold policy can be defined on the diagonal represented by the line $x_1 + x_2 = s$, where the system is full.

Example 2: We take $s = 8$ and other parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 2$, $r_1 = 1$, $r_2 = 2$, $\beta = 0.01$, $tc_1 = 1.05$, and $tc_2 = 1.05$. The optimal policy upon arrival of a class-2 job is in Figure 4.2. Let $x_1 + x_2 = 8$. Then for $x_2 \leq 5$, it is optimal to terminate a class-1 job to admit an arriving class-2 job, otherwise, i.e., when $x_2 \geq 6$, it is optimal to reject the incoming class-2 job. Hence, the threshold on x_2 to reject class-2 jobs on the line $x_1 + x_2 = 8$ is specified as 6.

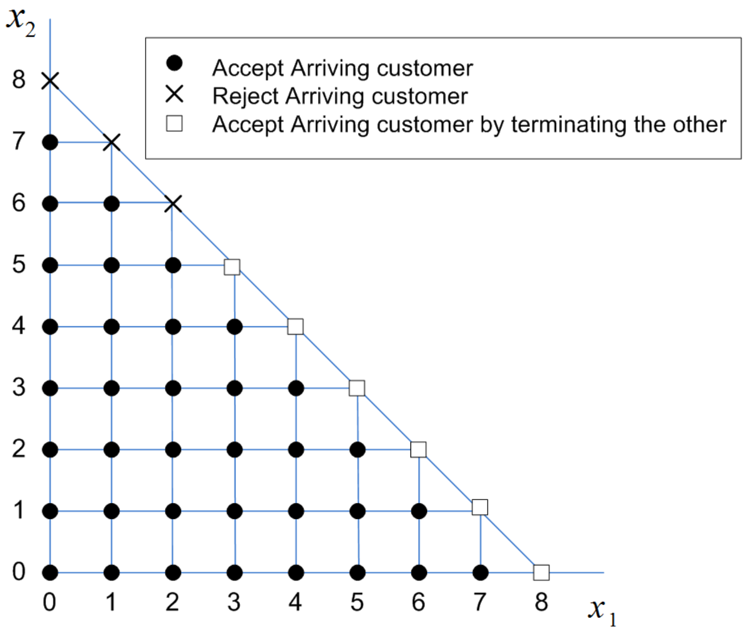

Figure 4.2. The optimal policy for the system in Example 2

In all the examples we have constructed, we observe such an optimal threshold policy. It is straightforward to show that such a threshold policy is direct consequence of the following inequality:

$$u_n(x + e_1 - e_2) - u_n(x) \le u_n(x) - u_n(x - e_1 + e_2), \ \ \forall n \ge 1. \tag{4.5}$$

We refer to this property of the value functions as monotonicity on the diagonal. We first observe inequality is symmetric in x_1 and x_2. In this form, the inequality compares the net benefit of changing a class-2 job to class 1 in states x and $x - e_1 + e_2$, and it concludes that this benefit is increasing in x_2 with $x_1 + x_2 = s$. Unfortunately, we are not able to show this inequality, although it holds in all the examples we have constructed.

5. EXISTENCE OF PREFERRED AND STRONGLY PREFERRED CLASSES

In this chapter, we develop sufficient conditions for each class type to be preferred and strongly preferred. In our problem context, jobs of a preferred class are always admitted to the system whenever there is an idle server and jobs of a strongly preferred class are always admitted to the system, even when the system is full, by terminating one job of the other class (except for the state that there is no other class of job to terminate). Note that by definition, if a job is strongly preferred, it is also preferred.

For class i to be preferred, having one more class-i job must be profitable for all x and n, when there is an idle server, *i.e.*, $D_n(i0)(x) \geq 0$ for all $x \in \mathcal{S}$, and for all $n \geq 0$. On the other hand, for class i to be strongly preferred, the profit of having one more class-i job and having one less class-j job must be greater than the termination cost of a class-j job, *i.e.*, $D_n(ij)(x) > tc_j$, for all $x \in \mathcal{S}$, and for all $n \geq 0$. The following theorem provides sufficient conditions for class 2 to be preferred and strongly preferred:

Theorem 5.1. *(a) Class 2 is a strongly preferred class for all* $n \geq 0$, *if*

$$R_2 > tc_1 + R_1.$$

(b) Class 2 is a preferred class for all $n \geq 0$, *if*

$$R_2 \geq \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} R_1.$$

The proof of Theorem 5.1 is based on the upper and lower bounds of the differences $D_n(21)(x)$ and $D_n(20)(x)$, and it can be found in Appendix. As stated earlier, if class 2 is strongly preferred, it is also preferred. This is also verified by Theorem 5.1, since the bound for class 2 to be strongly preferred is tighter than the bound for class 2 to be preferred. Theorem 5.1 shows that if the difference between the immediate rewards of class 2 and class 1 is greater than termination cost of a class-1 job, class 2 is a strongly preferred class. We note that the condition to be preferred is the same with that in a pure admission control system.

The following theorem summarizes the results for class 1 to be preferred and strongly preferred:

Theorem 5.2. *(a) Class 1 is a strongly preferred class for all* $n \geq 0$*, if*

$$\frac{R_1(\mu_1+\beta)}{\lambda_1+\mu_1+\beta} > \frac{(R_2+tc_2)(\mu_2+\beta)}{\lambda_1+\mu_2+\beta}.$$

(b) Class 1 is a preferred class for all $n \geq 0$*, if*

$$\frac{R_1(\mu_1+\beta)}{\lambda_2} \geq \frac{R_2(\mu_2+\beta)}{\lambda_2+\mu_2+\beta}.$$

The proof of Theorem 5.2 can be found in Appendix. Similarly to class 2, it can be shown that the bound for class 1 to be strongly preferred is tighter than the bound for class 1 to be preferred. Also, the condition to be preferred is the same with that in a pure admission control system. Unfortunately, it is very difficult to give an intuitive interpretation of these bounds. The following example demonstrates, termination cost is an important factor for a class to be strongly preferred:

Example 3: We have a system with $s = 8$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\mu_1 = 1$, $\mu_2 = 1.1$, $\beta = 0.01$, and let r_1 and r_2 change between 0 and 10. In Figure 5.1, the area under each line represents the region where class-1 jobs are strongly preferred. We observe how this region becomes narrower as tc_2 increases.

To clarify the meanings of preferred and strongly preferred classes, we will present two corollaries and several examples.

Corollary 5.1. *It is possible to have class* i *to be strongly preferred, and class* j *to be preferred.*

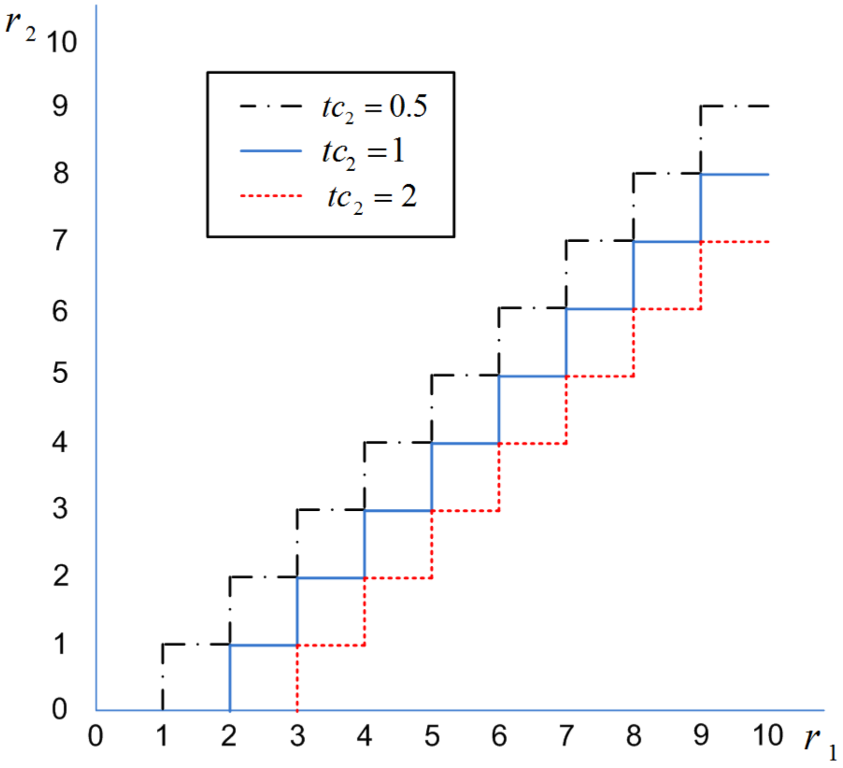

Figure 5.1. Strongly Preferred Region of Class-1 job

This corollary states that it is possible to have the following situation: all arriving jobs of class-j are always admitted to the system if there is an idle server, whereas if the system is full, an arriving job of strongly preferred class-i is admitted by terminating a class-j job. The proof follows from the above theorems by combining the appropriate conditions to show that there exists feasible regions in which these conditions are satisfied (see Appendix for the details). The following example presents such a system.

Example 4: We take $s = 8$ and other parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 2$, $r_1 = 1.8$, $r_2 = 2$, $\beta = 0.01$, $tc_1 = 0.2$, and $tc_2 = 1.5$. Class 2 is strongly preferred, whereas class 1 is preferred. Figure 5.2 shows the optimal policy of this system.

Corollary 5.2. *Both of the jobs cannot be strongly preferred at the same time.*

Proof. If both jobs are strongly preferred, when the system is full, it is always optimal to admit an arriving class-1 job by terminating a class-2 job regardless of the state, in period n, also it is always optimal admit an arriving class-2 job by terminating a

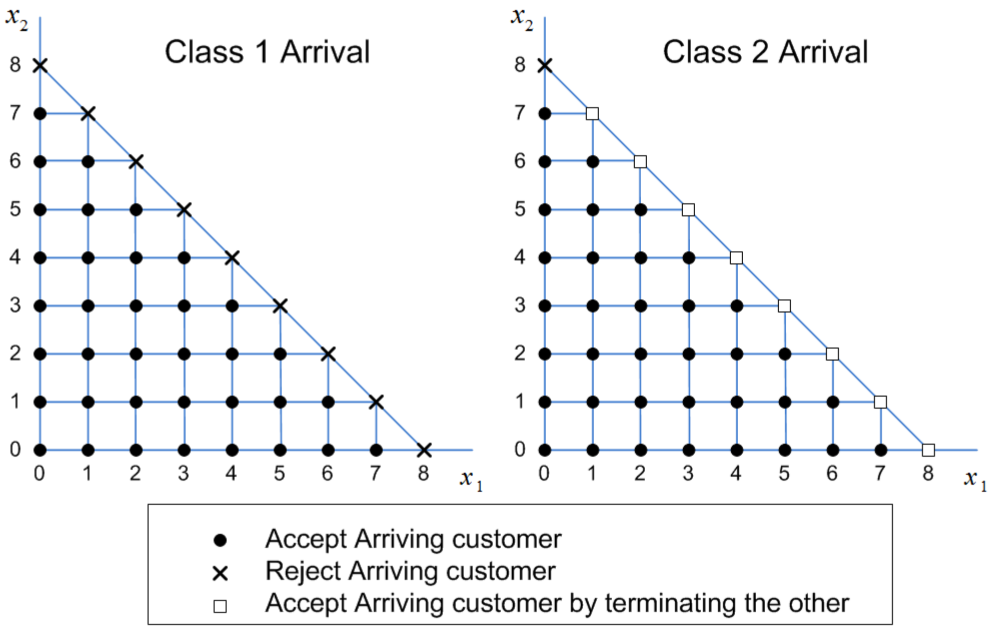

Figure 5.2. The optimal policy for the system in Example 4

class-1 job regardless of the state, in period n. This means both $D_n(21)(x) > tc_1$ and $D_n(12)(x) > tc_2$ conditions hold, for all x, such that $x_1 + x_2 = s$. On the other hand, we know that $D_n(12)(x) = -D_n(21)(x)$, hence if $D_n(12)(x)$ is positive, then $D_n(21)(x)$ is negative, and vice versa. In addition, tc_1 and tc_2 are non-negative values, so that both conditions cannot hold at the same time, which is a contradiction. Thus, both jobs cannot be strongly preferred. $\square$

This corollary states that it is not possible to have two strongly preferred classes. However, it is still possible to terminate both of the classes in the same system. There can be a state x where it is optimal to terminate a class-1 job, and a state $x^{'}$ where it is optimal to terminate a class-2 job. An example of such a system is presented below:

Example 5: We take $s = 8$ and other parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 2$, $r_1 = 2.1$, $r_2 = 2$, $\beta = 0.01$, $tc_1 = 0$, and $tc_2 = 0$. Figure 6.4 presents the optimal policy for this system.

In this example, we can also observe the thresholds on the diagonal. The threshold on x_1 to reject class-1 jobs is 4, since class-1 jobs are admitted to the system when

$x_1 \leq 3$, and rejected otherwise, when the system is full. Similarly, the threshold on x_2 to reject class-2 jobs is 4.

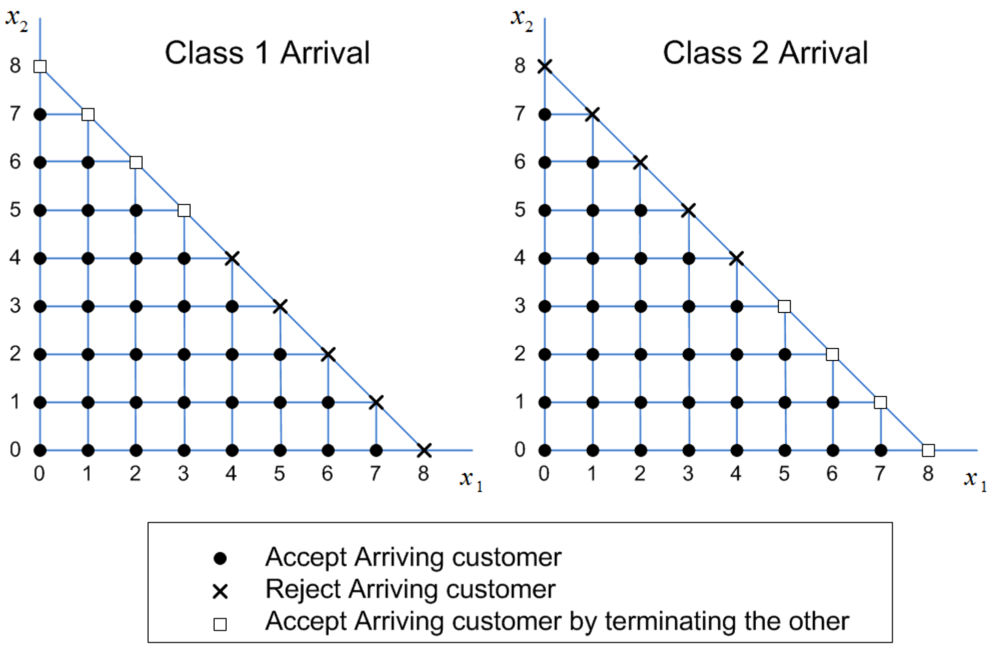

Figure 5.3. The optimal policy for the system in Example 5

6. A MODEL WITH SERVER CAPACITY INCREASE DECISION

6.1. Markov Decision Model for Finite Horizon

In the second model, we consider admission, termination and capacity increase decisions in a two class loss system. Again, we have s identical servers, no waiting room. Class-i jobs arrive according to a Poisson Process with arrival rate λ_i and have different exponential service times with mean $1/\mu_i$. The fixed reward r_i is gained after service completion. There are four possible decisions upon an arrival: admitting or rejecting the arriving job, admitting him/her by terminating a job who is already in the system by paying termination cost tc_i or admitting him/her by increasing the number of servers by one. The fixed cost of increasing number of servers by one is sc. The discount rate is β. Our objective is to find dynamic policies that maximize the total expected discounted revenue over a finite or infinite horizon as well as the long-run average revenue. Our MDP model is as follows:

States: As in the first model, we have two types of state descriptions, upon an arrival of a job and upon a service completion: If there is a potential service completion, we denote the system state by $(x;s) = (x_1, x_2, s)$, where x_i is the number of class-i jobs and s is the current number of servers. If there is a class-j arrival, then the state is $(x;j;s) = (x_1, x_2; j; s)$ so that there are x_i class-i jobs in the system, a class-j job has just arrived and the number of servers is s.

Uniformization: To apply uniformization to this model, we assume that there is a limit to the number of servers, so S is the maximum number of allowable servers. We assume, without loss of generality, $\mu_1 \leq \mu_2$. Then, the maximum possible rate out of any state is $\lambda_1 + \lambda_2 + S\mu_2 + \beta$. If we set $\lambda_1 + \lambda_2 + S\mu_2 + \beta = 1$, so the system will be observed at exponentially distributed intervals with mean 1. Then, with probability λ_i a class-i job arrives, with probability $x_i\mu_i$ service completion due to a class-i job occurs, and with probability $S\mu_2 - x_1\mu_1 - x_2\mu_2$ a fictitious service completion occurs.

Value Functions: We define a value function for each type of states: $u_n(x;s)$ is the maximal expected β-discounted reward for the system starting in state $(x;s)$ and and $v_n(x;j;s)$ is the maximal expected β-discounted reward for the system starting in state $(x;j;s)$, when n observation points remain in the horizon.

Actions: We define $a_n(x;j;s)$ as the optimal action in state $(x;j;s)$ when there are n more transitions: $a_n(x;j;s)$ is 1 if it is optimal to accept the arriving job of class j, 0 if it is optimal to reject the arriving job of class-j, 2 if it is optimal to accept the arriving job of class-j by terminating one of the class-k $(k \neq j)$ jobs which is already in the system and 3 if it is optimal to accept the arriving job of class-j by increasing the number of servers by one. The optimality equations are as follows:

For $j = 1,2$ and $n \geq 0$:

For $x_1 + x_2 < s$

$$\begin{aligned} v_n(x;j;s) = \max\{&u_n(x+e_j;s), u_n(x;s), u_n(x+e_j-e_k;s) - tc_k, \\ &u_n(x+e_j;s+1) - sc\}. \end{aligned} \tag{6.1}$$

For $x_1 + x_2 = s < S$

$$\begin{aligned} v_n(x;j;s) &= \max\{u_n(x;s), u_n(x+e_j-e_k;s) - tc_k, u_n(x+e_j;s+1) - sc\}, \ x_j \neq s, \\ v_n(x;j;s) &= \max\{u_n(x;s), u_n(x+e_j;s+1) - sc\}, \ x_j = s. \end{aligned} \tag{6.2}$$

For $x_1 + x_2 = s = S$

$$\begin{aligned} v_n(x;j;s) &= \max\{u_n(x;s), u_n(x+e_j-e_k;s) - tc_k\}, \ x_j \neq s, \\ v_n(x;j;s) &= u_n(x;s), \ x_j = s. \end{aligned} \tag{6.3}$$

For $x_1 + x_2 \leq s$:

$$\begin{aligned} u_{n+1}(x;s) = &x_1\mu_1 r_1 + x_2\mu_2 r_2 + \lambda_1 v_n(x;1;s) + \lambda_2 v_n(x;2;s) + x_1\mu_1 u_n(x-e_1;s) \\ &+ x_2\mu_2 u_n(x-e_2;s) + (S\mu_2 - x_1\mu_1 - x_2\mu_2) u_n(x;s). \end{aligned} \tag{6.4}$$

6.2. Structural Properties

In this section, we show that both termination and server increase decision is not an optimal decision unless we have a full system. On the other hand, further structural properties of the optimal policy are not analyzed. The next two theorems summarize the results.

Theorem 6.1. *Server increase decision is not an optimal decision if all servers are not busy, i.e.,* $a_n(x, s; j) \neq 3$, *if* $x_1 + x_2 < s$.

Proof. Assume that systems A and B are both in state $(x, s; j)$ which means a class-j job has just arrived in period $n+1$ and $x_1 + x_2 < s$. System A accepted the class-j job and increased the number of servers one with a cost of sc. On the other hand, system B accepted the arriving job without increasing the server. Now, system A is in state $(x + e_j, s + 1)$, system B is in state $(x + e_j, s)$. We let system A to follow the optimal policy from now on. Both systems are said to be coupled in the sense that system A and B would observe the same service times and interarrival times. The only difference for system A and B being an additional server in system A. The difference between two systems is preserved except one case. If a class-k job arrives and system A accepts it, system B couples with system A by accepting it and increasing server by one. But system B pays the service cost on a later stage. Paying the service cost on a later stage is always more profitable, since we are looking the present values. So, system A has always less profit than system B. This concludes that system A cannot be the one that follows the optimal policy. So, giving server increase decision when the system is not full is not an optimal decision. □

Theorem 6.2. *Termination decision is not an optimal decision if all servers are not busy, i.e.,* $a_n(x, s; j) \neq 2$, *if* $x_1 + x_2 < s$.

We skip the proof of the theorem since the same idea can be repeated as in the proof of Theorem 4.1. The following example illustrates the optimal policy.

Example 6: We take $S = 10$ and other parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 2$,

$r_1 = 1$, $r_2 = 2$, $\beta = 0.1$, $tc_1 = 0.2$, $tc_2 = 0.2$ and $s_c = 2$. We present the optimal policy for $s = 3$, $s = 4$, $s = 5$ and $s = 8$. From the figures below one can easily see that server increase decisions are given when we have less number of servers. For instance if we have 3 servers, upon an arrival of both class-1 job and class-2 job server increase decision is optimal, whereas if we have 5 servers, server increase decision is optimal only in state (0,5). Furthermore, if we have 8 servers, server increase decision is not optimal at all. Thus, we expect that marginal benefit of having a server decreases in the number of servers s.

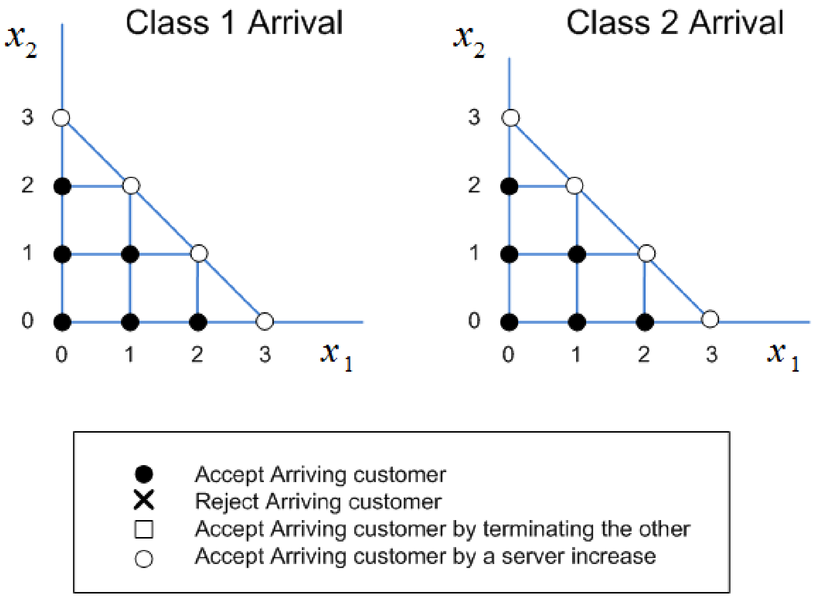

Figure 6.1. The optimal policy when $s = 3$ for the system in Example 6

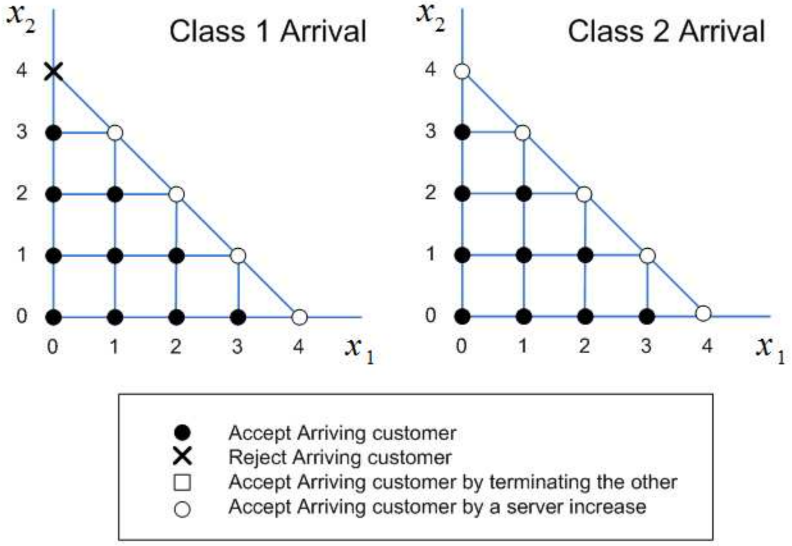

Figure 6.2. The optimal policy when $s = 4$ for the system in Example 6

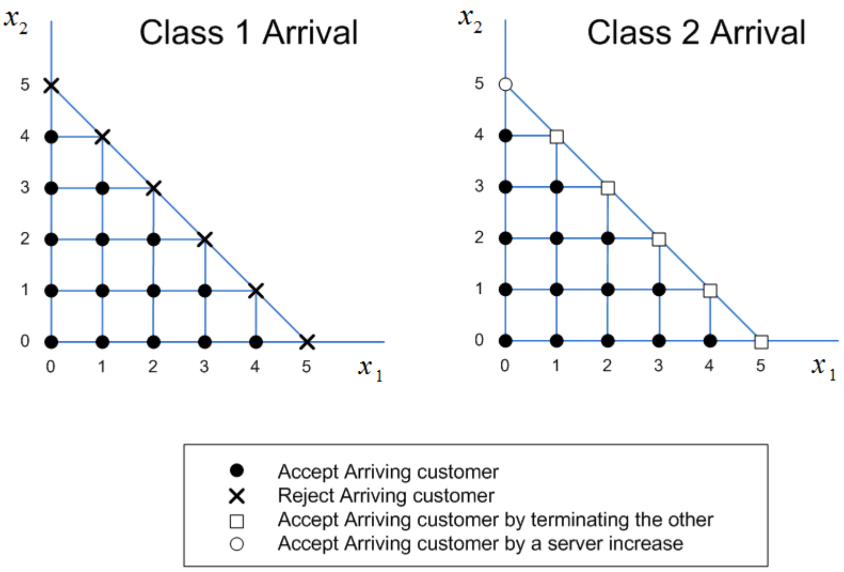

Figure 6.3. The optimal policy when $s = 5$ for the system in Example 6

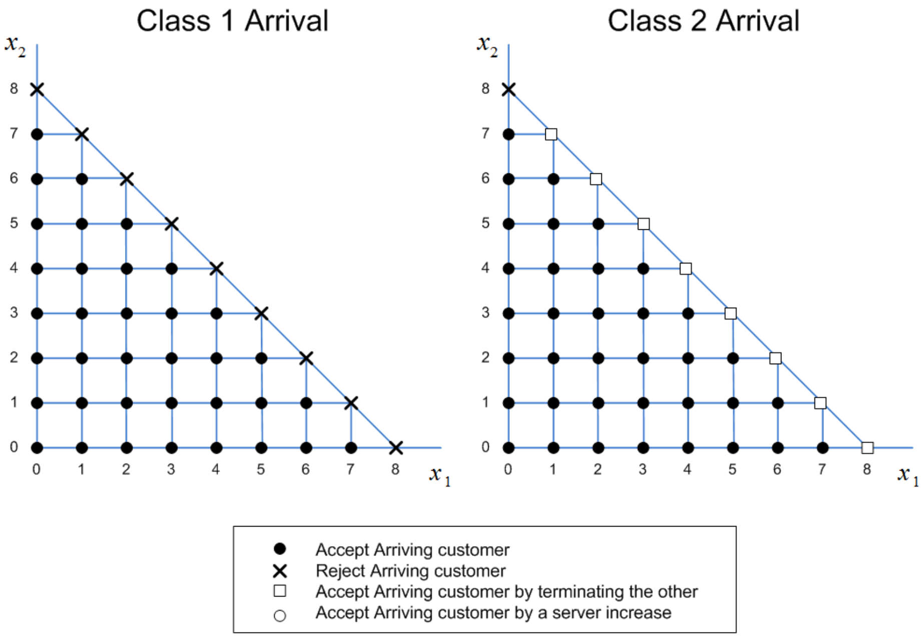

Figure 6.4. The optimal policy when $s = 8$ for the system in Example 6

7. CONCLUSIONS

In this thesis, we considered admission and termination control policies in a Markovian loss system with two classes of jobs and with an objective to maximize total expected discounted profit over a finite or infinite horizon. Although dynamic admission control policies have been extensively studied in the literature, there is limited research on systems with a possibility of terminating a job before its service completion. We formulated a Markov decision model to analyze the structure of optimal policies and presented several properties of the optimal policy. In particular, we proved that when there is an idle server in the system it is never optimal to terminate a job. We showed that optimal admission and termination policy is characterized as a state dependent threshold policy. Furthermore, we provided sufficient conditions for a job class to be preferred or strongly-preferred. We showed that both job types cannot be strongly preferred, although it is possible that one of them is strongly-preferred, and the other one is preferred. We also demonstrated the effect of various system parameters on a job being preferred or strongly preferred through examples.

In addition to our main model, we also extended our model with capacity increase decision. We showed that both termination and capacity increase decisions are not optimal for a system with free servers. However, we have not deeply analyzed the optimal policies for this model. Obviously, further analysis of this model is an interesting research topic.

This study can be extended in several directions. One further research issue is to prove both the concavity of the value functions and monotonicity on diagonal (inequality (4.5)). A promising research direction is to enrich the cost and reward structure of the model. For instance, random or time dependent rewards and termination costs with time dependent jumps can be considered. Furthermore, the time that a job spends in the system can be generalized to Erlang random variable. Also, a model with batch arrivals and batch termination possibility can be investigated.

APPENDIX A: Proof of Theorem 5.1

We assume that the later rewards of jobs who are still in the system at $n = 0$, are collected at $n = 0$, *i.e.*, we specify the initial value function, u_0 as follows:

$$u_0(x) = x_1 R_1 + x_2 R_2, \ \ \forall x \in S. \tag{A.1}$$

In order to show the first part of the theorem we need to show $D_n(21)(x) > tc_1$, for all x and for all $n \geq 0$, under some conditions. The following lemma provides upper bound on difference and $D_n(12)(x)$. Note that an upper bound for $D_n(12)(x)$ is a lower bound for $D_n(21)$.

Lemma A.1. *For all $x \in \mathcal{S}$ and for all $n \geq 0$:*

$$D_n(12)(x) = -D_n(21) \leq \max\{R_1 - R_2, -tc_1\}, \ \textit{for}\, i = 1, 2.$$

Proof. We can use sample path argument to get this bound. Assume that we have two systems, such that system A is in state $x + e_1$ and system B is in state $x + e_2$. We couple the additional class-1 job, say job j_1 in system A with additional job class-2 job, say job j_2, as well as all other service and interarrival times, *i.e.*, if j_1 leaves the system, j_2 also leaves. Now, let system A follow the optimal policy and B imitate all decisions of A except one case. Say, system A is in state $x' + e_1$ and system B is in $x' + e_2$, and there is no class-1 job in system B. Upon an arrival of a class-2 job, if system A accepts it by terminating a class-1 job, system B cannot imitate the decisions of system A, since there is no class-1 job in system B to give a termination decision. We assume that system B gives a rejection decision in this case. Now, let A is in state $x + e_1$ and B is in state $x + e_2$, in which there is at least one class-1 job in system B, the difference in the expected returns of A and B is:

$$D_n(12)(x) = u_n(x + e_1) - u_n(x + e_2) \leq u_n(x + e_1) - u_n^B(x + e_2) = R_1 - R_2$$

If system A is in $x^{'} + e_1$ and B in $x^{'} + e_2$, in which there is no class-1 job in system B, the difference is:

$$D_n(12)(x^{'}) = u_n(x^{'} + e_1) - u_n(x^{'} + e_2) \leq u_n(x^{'} + e_1) - u_n^B(x^{'} + e_2) = R_1 - R_2.$$

If system A rejects arriving class-2 job,

$$D_n(12)(x^{'}) = u_n(x^{'} + e_1) - u_n(x^{'} + e_2) \leq u_n(x^{'} + e_1) - u_n^B(x^{'} + e_2) = -tc_1.$$

If system A accepts arriving class-2 job and by terminating class-1 job. So the difference in state $x^{'}$:

$$D_n(12)(x^{'}) = u_n(x^{'}+e_1) - u_n(x^{'}+e_2) \leq u_n(x^{'}+e_1) - u_n^B(x^{'}+e_2) = \max\left\{R_1 - R_2, -tc_1\right\}.$$

Since $\max\left\{R_1 - R_2, -tc_1\right\}$ is a looser upper-bound than $R_1 - R_2$, we come up with the following lower bound for all $x \in \mathcal{S}$ and for all $n \geq 0$ is:

$$D_n(12)(x) = u_n(x+e_1) - u_n(x+e_2) \leq u_n(x+e_1) - u_n^B(x+e_2) = \max\left\{R_1 - R_2, -tc_1\right\}.$$

$\square$

As stated earlier, for class-2 to be strongly preferred, $D_n(21)(x) > tc_1$, for all x and for all $n \geq 0$. By Lemma A.1, we have $D_n(21)(x) < \min\left\{R_2 - R_1, tc_1\right\}$, hence we come up with the following condition:

$$R_2 - R_1 > tc_1$$

which concludes the proof of the first part of Theorem 5.1.

For the second part of the theorem we need to show $D_n(20)(x) > 0$, for all $x \in \mathcal{S}$ and for all $n \geq 0$. The following lemma provides lower bounds for $D_n(20)(x)$.

Lemma A.2. *For all $x \in \mathcal{S}$ and for all $n \geq 0$:*

- *i* . *if* $tc_1 < R_2 - R_1$ *and* $tc_1 < \frac{\mu_2 r_2}{\lambda_2+\mu_2+\beta}$, $\frac{\lambda_1 tc_1+\mu_2 r_2}{\lambda_1+\lambda_2+\mu_2+\beta} \leq D_n(20)(x)$,
- *ii* . *if* $tc_1 < R_2 - R_1$ *and* $tc_1 \geq \frac{\mu_2 r_2}{\lambda_2+\mu_2+\beta}$,$\frac{\mu_2 r_2}{\lambda_2+\mu_2+\beta} \leq D_n(20)(x)$,
- *iii* . *if* $tc_1 \geq R_2 - R_1$ *and* $\frac{\lambda_2+\mu_2+\beta}{\lambda_2} < \frac{R_2}{R_1}$, $\frac{\mu_2 r_2}{\lambda_2+\mu_2+\beta} \leq D_n(20)(x)$,
- *iv* . *if* $tc_1 \geq R_2 - R_1$ *and* $\frac{\lambda_1}{\lambda_1+\mu_2+\beta} \leq \frac{R_2}{R_1} \leq \frac{\lambda_2+\mu_2+\beta}{\lambda_2}$,$\frac{(\lambda_1+\mu_2+\beta)R_2-\lambda_1 R_1}{\lambda_1+\lambda_2+\mu_2+\beta} \leq D_n(20)(x)$.

Proof. The proof is by induction, we can show that the bounds are all valid for u_0. Assume that all statements are true for n and consider $n+1$. To build the induction step we use coupling. Assume we have two systems such that system A is in state $x+e_2$ and system B is in state x in period $n+1$. Now, let system B follow the optimal policy whereas system A rejects the next arrival and follows the optimal policy afterwards. Let d_2 be a lower bound on $D_n(20)(x)$ for all $x \in S$ and for some $n \geq 0$, so that $d_2 \leq \min_{\{x \in S\}} \{D_n(20)(x)\}$. Consider an arrival. If system B also rejects the arriving job, both system remain in their current states, which leads to a minimum difference of d_2 in the rewards of the two systems by definition of d_2. If system B accepts an arriving class-1 job, then system B goes to a different state $x + e_1$ whereas system A remains in state $x + e_2$. If system B accepts arriving class-2 job, then the two systems couple with no reward. Termination is not taken into consideration since it is not optimal when there are idle servers in the system and by definition system B, it has at least one idle server. If the additional class-2 job departs the system with a return of r_2, the systems again enter the same state. All other service completions keep the difference between two systems, which means that the difference between two systems is at least d_2. If we let u^A_{n+1} be total expected discounted reward of system A, then:

$$\begin{aligned}
D_{n+1}(20)(x) &= u_{n+1}(x+e_2) - u_{n+1}(x) \geq u^A_{n+1}(x+e_2) - u_{n+1}(x) \\
&\geq \lambda_1 \min\{D_n(21)(x), D_n(20)(x)\} + \lambda_2 \min\{0, D_n(20)(x)\} + \mu_2 r_2 \\
&+ (s-1)\mu_2 \min_{y \in S} \{D_n(20)(y)\} \\
&\geq \lambda_1 \min\{\min\{R_2 - R_1, tc_1\}, d_2\} + \lambda_2 \min\{0, d_2\} + \mu_2 r_2 + (s-1)\mu_2 d_2.
\end{aligned}$$

where the first inequality is by definition of the policy that system A follows, since the policy that system A follows may not necessarily be the optimal policy. The second one is due to the coupling described above, and the last one follows from Lemma A.1, and by definition of d_2.

For the first part of the lemma, if $tc_1 < R_2 - R_1$ and $tc_1 < \frac{\mu_2 r_2}{\lambda_2+\mu_2+\beta}$, and assume $\frac{\lambda_1 tc_1+\mu_2 r_2}{\lambda_1+\lambda_2+\mu_2+\beta} \leq D_n(20)(x)$ for all $x \in \mathcal{S}$ and for some $n \geq 0$, *i.e.*, $d_2 = \frac{\lambda_1 tc_1+\mu_2 r_2}{\lambda_1+\lambda_2+\mu_2+\beta}$. Then:

$$\begin{aligned} D_{n+1}(20)(x) &\geq \lambda_1 tc_1 + \mu_2 r_2 + (s-1)\mu_2 d_2 \\ &= \lambda_1 tc_1 + \mu_2 r_2 + (s-1)\mu_2 \left(\frac{\lambda_1 tc_1 + \mu_2 r_2}{\lambda_1 + \lambda_2 + \mu_2 + \beta} \right) \\ &= \frac{\lambda_1 tc_1 + \mu_2 r_2}{\lambda_1 + \lambda_2 + \mu_2 + \beta} \end{aligned}$$

Note that, we have proven the first part of Lemma A.2 by induction. One can easily show the other parts of the lemma in a similar way. □

It can be easily seen that all bounds are non-negative in Lemma A.2, so we can come up with the following condition by just combining these bounds:

$$R_2 \geq \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} R_1,$$

which concludes the proof of Theorem 5.1.

APPENDIX B: Proof of Theorem 5.2

The construction of the proof is similar to the proof of class-2 job. In order to show the first part and second part of the theorem we need to show $D_n(12)(x) > tc_2$, and $D_n(10)(x) \geq 0$, for all x and for all $n \geq 0$ respectively under some conditions. The following lemma provides lower bounds on the differences $D_n(10)(x)$ and $D_n(12)(x)$.

Lemma B.1. *For all $x \in S$ and for all $n \geq 0$:*

i . If $tc_2 < \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}$ *and* $tc_2 > \frac{r_2\mu_2}{\lambda_1}$,

$$\frac{r_1\mu_1}{\lambda_1+\mu_1+\beta} \leq D_n(10)(x), \text{ and}$$
$$\frac{r_1\mu_1}{\lambda_1+\mu_1+\beta} - \frac{r_2\mu_2}{\lambda_1+\mu_2+\beta} + \frac{\lambda_1 tc_2}{\lambda_1+\mu_2+\beta} \leq D_n(12)(x).$$

ii . If $tc_2 < \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}$ *and* $tc_2 \leq \frac{r_2\mu_2}{\lambda_1}$,

$$\frac{r_1\mu_1(\lambda_1+\lambda_2+\mu_2+\beta) - \lambda_2 r_2\mu_2 + \lambda_1\lambda_2 tc_2}{(\lambda_1+\lambda_2+\mu_2+\beta)(\lambda_1+\mu_1+\beta)} \leq D_n(10)(x), \text{ and}$$
$$\frac{r_1\mu_1(\lambda_1+\lambda_2+\mu_2+\beta)(\lambda_1 tc_2 - r_2\mu_2)(\lambda_1+\lambda_2+\mu_1+\beta)}{(\lambda_1+\lambda_2+\mu_2+\beta)(\lambda_1+\mu_1+\beta)} \leq D_n(12)(x).$$

iii . If $tc_2 \geq \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}$ *and* $\frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_1}{\lambda_1+\mu_1+\beta}$,

$$\frac{r_1\mu_1}{\lambda_1+\mu_1+\beta} \leq D_n(10)(x), \text{ and}$$
$$\frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta} \leq D_n(12)(x).$$

iv . If $tc_2 \geq \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}$ *and* $\frac{\lambda_1}{\lambda_1+\mu_1+\beta} \leq \frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_2+\mu_2+\beta}{\lambda_2}$,

$$\frac{r_1\mu_1(\lambda_2+\mu_2+\beta) - \lambda_2 r_2\mu_2}{(\lambda_1+\lambda_2+\mu_2+\beta)(\mu_1+\beta)+\lambda_1(\mu_2-\mu_1)} \leq D_n(10)(x), \text{ and}$$
$$\frac{r_1\mu_1(\lambda_1+\lambda_2+\mu_2+\beta) - r_2\mu_2(\lambda_1+\lambda_2+\mu_1+\beta)}{(\lambda_1+\lambda_2+\mu_2+\beta)(\mu_1+\beta)+\lambda_1(\mu_2-\mu_1)} \leq D_n(12)(x).$$

Proof. The proof is by induction, we can show that the bounds are all valid for $n = 0$. Next, we assume that these statements are true for some n and try to show they are also valid for $n + 1$. We will consider two pairs of coupled systems, one for $D_n(10)(x)$ and the other for $D_n(12)(x)$.

Now consider the first pair. Assume that system A is in state $x + e_1$ and system B is in x, and couple the two systems in such a way that A rejects all jobs in the next transition and then continues with the optimal policy in all transitions, whereas system B always follows the optimal policy. Let d_1 and d_{12} be lower bounds for $D_n(10)(x)$ and $D_n(12)(x)$ for all $x \in S$ and for some $n \geq 0$, respectively. The coupling arguments are similar to the coupled system in the proof of Lemma A.2. We have:

$$\begin{aligned} D_{n+1}(10)(x) &= u_{n+1}(x + e_1) - u_{n+1}(x) \geq u^A_{n+1}(x + e_1) - u_{n+1}(x) \\ &\geq \lambda_1 \min\{0, D_n(10)(x)\} + \lambda_2 \min\{D_n(12), D_n(10)(x)\} + \mu_1 r_1 \\ &+ (s\mu_2 - \mu_1) \min_{y \in S}\{D_n(10)(y)\} \\ &\geq \lambda_1 \min\{0, d_1\} + \lambda_2 \min\{d_{12}, d_1\} + \mu_1 r_1 + (s\mu_2 - \mu_1) d_1. \end{aligned}$$

The second inequality is due to coupling, and the third follows from d_1 and d_{12} being lower bounds.

Now consider the second pair of systems. Let system A$'$ be in state $x + e_1$ and system B$'$ in $x + e_2$. As before, we couple the additional class-2 job, say j_2, in system B$'$ with the additional class-1 job, say job j_1 in system A$'$, as well as all other service and interarrival times. System B$'$ takes the optimal actions and system A$'$ imitates all the actions of system B$'$ in the next transition and afterwards follows the optimal policy. A$'$ can imitate all decisions of B$'$ except one case. Say x' is a state in which there are no class-2 jobs, A$'$ is in state $x' + e_1$ and B$'$ in $x' + e_2$. If the next transition is arrival of a class-1 and the optimal action of system B$'$ is termination of class-2 job, A$'$ cannot imitate the action since it has no class-2 job to terminate. So, if this case occurs A$'$ rejects arriving class-2 job which makes the two system coupled. If j_1 leaves the system, which happens with probability μ_1, j_2 also leaves. The departure of j_1

leads the system to couple with a reward of $r_1 - r_2$, the departure of j_2 alone , which happens with probability $\mu_2 - \mu_1$, takes the system to two different states, $x + e_1$ and x with reward of $-r_2$. If there is any other transition, both system continue to have their additional jobs. So we have the following inequality:

$$\begin{aligned}
D_{n+1}(12)(x) &\geq \mu_1(r_1 - r_2) + (\mu_2 - \mu_1)(-r_2 + D_n(10)(x)) + \lambda_1 \min\{tc_2, D_n(12)(x)\} \\
&\quad + (\lambda_2 + (s-1)\mu_2) \min_{y \in S}\{D_n(12)(y)\} \\
&\geq r_1\mu_1 - r_2\mu_2 + (\mu_2 - \mu_1)d_1 + (1 - \lambda_1 - \mu_2 - \beta)d_{12} + \lambda_1 \min\{tc_2, d_{12}\}.
\end{aligned}$$

The first inequality is due to coupling and the second follows from d_1 and d_{12} being lower bounds. For the first part of the lemma, if $tc_2 < \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}$ and $tc_2 > \frac{r_2\mu_2}{\lambda_1}$, and assume the following statements are true for all $x \in \mathcal{S}$ and for some $n \geq 0$:

$$\begin{aligned}
&\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta} \leq D_n(10)(x) \\
&\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta} - \frac{r_2\mu_2}{\lambda_1 + \mu_2 + \beta} + \frac{\lambda_1 tc_2}{\lambda_1 + \mu_2 + \beta} \leq D_n(12)(x).
\end{aligned}$$

Then:

$$\begin{aligned}
D_{n+1}(10)(x) &\geq \lambda_1 \min\{0, d_1\} + \lambda_2 \min\{d_{12}, d_1\} + \mu_1 r_1 + (s\mu_2 - \mu_1)d_1 \\
&= \lambda_2\left(\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta}\right) + \mu_1 r_1 + (s\mu_2 - \mu_1)\left(\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta}\right) \\
&= \frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta}
\end{aligned}$$

Also:

$$\begin{aligned}
D_{n+1}(12)(x) &\geq r_1\mu_1 - r_2\mu_2 + (\mu_2 - \mu_1)d_1 + (1 - \lambda_1 - \mu_2 - \beta)d_{12} + \lambda_1 \min\{tc_2, d_{12}\} \\
&= r_1\mu_1 - r_2\mu_2 + (\mu_2 - \mu_1)\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta} \\
&\quad + (1 - \lambda_1 - \mu_2 - \beta)\left(\frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta} - \frac{r_2\mu_2}{\lambda_1 + \mu_2 + \beta} + \frac{\lambda_1 tc_2}{\lambda_1 + \mu_2 + \beta}\right) + \lambda_1 tc_2 \\
&= \frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta} - \frac{r_2\mu_2}{\lambda_1 + \mu_2 + \beta} + \frac{\lambda_1 tc_2}{\lambda_1 + \mu_2 + \beta}
\end{aligned}$$

Note that, we have proven the first part of Lemma B.1 by induction. One can show the other parts of the lemma in a similar way. □

Note that all lower bounds for $D_n(10)(x) \geq 0$, with some algebra we can come up with the preferred class conditions in Theorem 5.2. We can also derive the strongly preferred condition for class-1 by using the bounds for $D_n(12)(x)$.

APPENDIX C: Proof of Corollary 5.1

i . The Condition for class-1 job to be strongly preferred is:

$$tc_2 < \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}.$$

The condition for class-2 job to be just preferred is:

$$tc_1 \geq R_2 - R_1 \text{ and } \frac{r_2\mu_2}{\mu_2+\beta} \geq \frac{\lambda_1}{(\lambda_1+\mu_2+\beta)}\frac{r_1\mu_1}{(\mu_1+\beta)}.$$

By combining the conditions above we get a condition, which is the condition of class-1 being strongly preferred and class-2 being preferred:

$$tc_2 < \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_1\mu_1}{(\mu_1+\beta)}\frac{\lambda_1}{(\lambda_1+\mu_2+\beta)} \text{ and } tc_1 \geq R_2 - R_1. \tag{C.1}$$

We have to verify the region above is a feasible region. Since $tc_2 \geq 0$, the inequality below must be feasible:

$$\begin{aligned}
\frac{\lambda_1(\mu_1 r_1)}{(\lambda_1+\mu_2+\beta)(\mu_1+\beta)} &< \frac{(\mu_1 r_1)(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} \\
\frac{\lambda_1}{(\lambda_1+\mu_2+\beta)(\mu_1+\beta)} &< \frac{(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} \\
\lambda_1 &< \frac{(\lambda_1+\mu_2+\beta)^2(\mu_1+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)}.
\end{aligned}$$

Say $\lambda_1 = \mu_1 + \beta$, then:

$$1 < \frac{(\lambda_1+\mu_2+\beta)^2}{(\lambda_1+\mu_1+\beta)(\mu_2+\beta)}$$

which is true, since $\mu_2 \geq \mu_1$. This means the condition (C.1) is a feasible condition. So, if class-1 is strongly preferred class-2 can be preferred.

ii . The condition for class-1 job to be just preferred is:

$$tc_2 \geq \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta} \text{ and } r_1\mu_1 \geq \frac{\lambda_2}{\lambda_2+\mu_2+\beta}r_2\mu_2.$$

The condition for class-2 job to be strongly preferred is:

$$tc_1 > R_2 - R_1.$$

By combining the conditions above we get a condition as below, which is the condition of class-2 being strongly preferred and class-1 being preferred:

$$\begin{aligned} tc_1 &< \frac{\mu_2 r_2}{\mu_2+\beta} - \frac{\lambda_2\mu_2 r_2}{(\lambda_2+\mu_2+\beta)(\mu_1+\beta)} \text{ and} \\ tc_2 &\geq \frac{r_1\mu_1(\lambda_1+\mu_2+\beta)}{(\mu_2+\beta)(\lambda_1+\mu_1+\beta)} - \frac{r_2\mu_2}{\mu_2+\beta}. \end{aligned} \tag{C.2}$$

We have to verify the region above is a feasible region. Since $tc_1 \geq 0$, the inequality below must be feasible:

$$\begin{aligned} \frac{\lambda_2\mu_2 r_2}{(\lambda_2+\mu_2+\beta)(\mu_1+\beta)} &< \frac{\mu_2 r_2}{\mu_2+\beta} \\ \frac{\lambda_2}{(\lambda_2+\mu_2+\beta)(\mu_1+\beta)} &< \frac{1}{\mu_2+\beta} \\ \frac{\lambda_2}{(\lambda_2+\mu_2+\beta)} &< \frac{\mu_1+\beta}{\mu_2+\beta}. \end{aligned}$$

Say $\lambda_2 = \mu_1 + \beta$, then:

$$\frac{1}{(\lambda_2+\mu_2+\beta)} < \frac{1}{\mu_2+\beta}$$

which is valid. This means the condition (C.2) is a feasible condition. So, if class-2 is strongly preferred job class-1 can be preferred.

This concludes that, if class-i job is strongly preferred, class-j job can be preferred, where $i \neq j$.

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