OPTIMUM VIBRATION ABSORBERS

FOR

CONTINUOUS AND DISCRETE MECHANICAL SYSTEMS

by

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ABSTRACT

OPTIMUM VIBRATION ABSORBERS FOR CONTINUOUS AND DISCRETE MECHANICAL SYSTEMS

In this thesis, the optimum parameters of passive vibration absorbers for continuous and discrete mechanical systems are investigated. Two approaches are undertaken to obtain the optimum absorber parameters. The first approach is built on the continuous system analysis for minimization of vibration amplitudes and the related methodology, motivated by Lagrange's equation and the Lyapunov equation, is derived. The second approach, based on the receptance concept, uses a new methodology which employs the Sherman-Morrison formula to minimize the vibration amplitudes in discrete systems.

A method is developed for calculating the optimum parameters of n absorbers attached to a uniform beam or rectangular plate, the vibrations of which are characterized by the first M modes where n and M can be any positive integers. For the most general case, dissipation due to damping, kinetic and potential energies, and the effects of external forces are all taken into account.

Lagrange's equation is used to generate a state space representation of the system, wherein the Lyapunov equation is used to obtain the H_2 norm of the transfer function. This norm is used to construct the objective function for optimization. The optimal response is compared to cases without an absorber and with randomly selected absorber parameters.

Focusing on the minimization of the vibration amplitudes using the concept of receptance, a method for calculating the receptance of a generic translational mass-spring-damper system with m masses and n absorbers is developed. The dynamic stiffness of the entire system is derived both directly from the equations of motion and through a linear graph representation of the system. The receptance of the combined system, in terms of the parameters of the main and absorber systems is obtained by applying Sherman-Morrison

formula sequentially. The optimal absorber parameters of the system are easily obtained once the receptance is known. The complexity of the formulation is simplified with the proposed approach.

Two basic models, both the continuous and discrete system modeling for suppressing vibration, are built in this thesis. It is shown that continuous and discrete systems are interrelated and the corresponding developed methods can be applied to each type of system mutually.

ÖZET

SÜREKLİ VE AYRIK MEKANİK SİSTEMLER İÇİN OPTİMUM TİTREŞİM SÖNÜMLEYİCİLERİ

Bu tezde, sürekli ve ayrık mekanik sistemler için titreşim sönümleyicilerinin optimum parametreleri incelenmiştir. Optimum sönümleyici parametrelerini elde etmek için iki temel yaklaşım oluşturulmuştur. İlk yaklaşım sürekli sistemlerdeki titreşim genliklerinin en aza indirilmesi üzerine kurulmuş ve Lagrange denklemi ve Lyapunov denkleminden yola çıkarak ilgili yöntem geliştirilmiştir. Dinamik esneklik kavramı üzerine kurulan ikinci yaklaşımda, titreşim genliklerini en aza indirmek için, Sherman-Morrison formulasyonundan türetilen yeni bir yöntem kullanılmıştır.

Pasif sönümleyici yaklaşımı aracılığıyla, M mod biçimine sahip eşbiçimli bir kirişe veya plakaya bağlı olan n sönümleyicinin (n ve M pozitif tamsayıdır) optimum parametrelerini hesaplamak için bir yöntem geliştirilmiştir. En genel durum olarak, sönümleme enerjisine bağlı olarak dağılma, kinetik ve potansiyel enerjileri ve dış kuvvetlerin etkileri sunulmuştur.

Sistemin durum uzayı gösterimini elde etmek için Lagrange denklemi kullanılmıştır. M mod biçimine sahip n sönümleyicili bir sistemin durum uzayı gösterimi oluşturulmuştur. Durum uzayı gösterimi ve Lyapunov denklemi temelinde, sistemin transfer fonksiyonunun H_2 normu optimizasyonda kullanılmıştır. Bu normdan optimizasyondaki hedef fonksiyonu oluşturmak için faydalanılmıştır. Sistem çıktıları minimuma indirilmiş ve sönümleyici bulunmayan ve rastgele seçilen sönümleyici parametreleri bulunan durumlarla karşılaştırma yapılarak gösterilmiştir. M mod biçimine sahip eşbiçimli bir kirişe veya plakaya bağlı olan N sönümleyicinin fiziksel sınırlamalar dahilinde en uygun sönümleyici parametreleri geliştirilen yöntem ile belirlenmiştir.

Dinamik esneklik kavramı aracılığıyla titreşim genliklerinin minimuma indirilmesine odaklanarak, m kütle ve n sönümleyiciden oluşan bir düzlemsel kütle-yay-sönümleyici sisteminin dinamik esnekliğini hesaplamak için yöntem geliştirilmiştir.

Sistemin dinamik esnekliğinin tersi olan dinamik katılık hareket denklemleri ve doğrusal bir grafik temsili kullanılarak ortaya çıkarılmıştır. Birleşik sistemin dinamik esneklik değeri, Sherman-Morrison formulasyonunun ardarda kullanımı ile ana ve sönümleyici sistemlerin parametrelerine bağlı olarak elde edilmiştir. Dinamik esneklik bulunduktan sonra M sönümleyiciden ve n kütleden oluşan bir düzlemsel kütle-yay-sönümleyici sisteminin en uygun sönümleyici parametreleri elde edilmiştir. Önerilen yöntem ile formulasyonların karmaşıklığı en aza indirgenmiştir.

Bu tezde, titreşim sönümleme için iki temel model, sürekli ve ayrık sistemlerin modellemesi olarak, geliştirilmiştir. Sürekli ve ayrık sistemlerin birbirleriyle ilişkili olduğu ve geliştirilen ilgili modellerin her iki sisteme karşılıklı olarak uygulanabildiği gösterilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

a	Location of a driving force on the beam
C_{add}	Damping matrix that arises from adding absorbers to the main system
c_{a1}, c_{a2}, c_{a3}, ...	Optimum damping values of the absorbers
C_{main}	Damping matrix of the main system without absorbers
C_{sec}	Damping matrix that exists in the equation of motion of the absorbers
c_1, c_2, c_3, ...	Damping values of the main system
E	Elastic modulus of the beam / plate
F	External forces applied to the beam / plate
$F(s)$	Laplace transform of $f(t)$
f(x,t)	Transverse force density
$G_{sys}(s)$	Transfer function of the main system
$\|G\|_2$	H_2 norm of the transfer function
$\|G\|_\infty$	H_∞ norm of the transfer function
$\|G_{tot}\|_2^2$	H_2 norm of the total system transfer function (squared)
h	Height of the beam / plate
I	Moment of inertia of the beam / plate
J	Performance index to be minimized
K_{add}	Stiffness matrix that arises from adding absorbers to the main system
k_{a1}, k_{a2}, k_{a3}, ...	Optimum stiffness values of the absorbers
K_{main}	Stiffness matrix of the main system without absorbers
K_{sec}	Stiffness matrix that exists in the equation of motion of the absorbers
k_1, k_2, k_3, ...	Stiffness values of the main system
L	Function that satisfies the Lyapunov equation
l_{beam}	Length of the beam
l_1, l_2, l_3	Optimum location of the absorbers
M	A positive integer indicating the number of initial modes used to model vibration
m	Number of the main masses in translational mass-spring-damper system

m_{a1}, m_{a2}, m_{a3},..	Masses of the absorbers
m_{beam}	Total mass of the beam
M_{main}	Mass matrix of the main system
m_u	Mass of the beam per unit length
M(x,t)	Bending moment
m_1, m_2, m_3,…	Main masses in translational mass-spring-damper system
n	A positive integer indicating the number of absorbers
P	Controllability Gramian matrix
Q	Observability Gramian matrix
$q_{m1}(t)$, $q_{m2}(t)$,..	Displacements of the absorbers with mass m_1, m_2, etc.
$Q_{nc1}(t)$	Generalized non-conservative dissipative force due to damping
$Q_{nc2}(t)$	Generalized non-conservative external force
q(t)	Time-dependent generalized coordinates
Q(x,t)	Shear force acting on a beam element
T(t)	Kinetic energy of beam / plate
$u_i(x)$	Assumed vibration mode of a uniform beam
$u_{ij}(x,y)$	Assumed vibration mode of a rectangular plate
V(t)	Potential energy of beam / plate
$W_f(s)$	Weighting function for the input forces
$\omega(x,t)$	Transverse displacement of the uniform beam
$\omega(x,y,t)$	Transverse displacement of the rectangular plate
$W_y(s)$	Weighting function for the output displacements
$X_a(s)$	Laplace transform of $x_a(t)$
x_{a1}, x_{a2}, x_{a3},…	Displacements of the absorbers
x_1, x_2, x_3, ...	Displacements of the main masses
y	Bending displacement of the beam
y(x,t)	Bending deflection
Z_{tot}	Dynamic stiffness of the entire system
Z_{X_aF}	Dynamic stiffness between $X_a(s)$ and $F(s)$
Z_1, Z_2, Z_3…	Dynamic stiffnesses of branches in a linear graph of the system
α	Receptance of the system

ν	Poisson's ratio
ξ	Damping coefficient
ρ	Density of beam / plate
$\sigma_i(A)$	The i^{th} singular value of matrix A

1. INTRODUCTION

The study of motion in physical systems resulting from internal and external forces is referred to as dynamics. One common type of dynamic behavior is vibratory motion. The vibrational properties of mechanical systems are usually limiting factors in their design, affecting performance requirements and functionality. Engineers have developed countless methods of suppressing unwanted vibrations in industries as diverse as aerospace, manufacturing, defense, and automobiles. If unwanted vibrations can be suppressed, the dynamic response of the mechanical system will improve [1]. Vibration absorbers are an efficient and attractive tool for reducing vibration and noise. This dissertation shows how to optimize a commonly used device: the *dynamic vibration absorber*.

1.1. General Overview of Vibration Absorbers

A vibration absorber mainly consists of three elements: a mass, a damper and a spring (Figure 1.1). Its role is to protect the primary system from a steady-state harmonic disturbance. The absorber needs to be carefully tuned to the driving frequency of the primary system by choosing its mass m_a, stiffness k_a and damping constant c_a. The problem at hand is to find a combination of these parameters that minimizes the motion of the original mass.

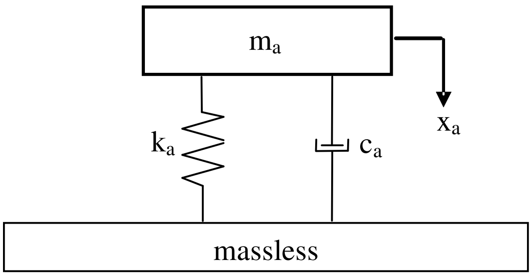

Figure 1.1. Vibration absorber

The theory of dynamic vibration absorbers was first presented by Frahm in 1909. In this model, the absorber consists of a mass and spring attached to the primary system, which itself is a mass-spring system (Figure 1.2). When the undamped absorber is tuned so that its natural frequency coincides with the frequency of a sinusoidal driving force acting on the primary mass, the steady state vibration amplitude of the main device becomes zero [2]. This theoretical system, however, works only for a single frequency.

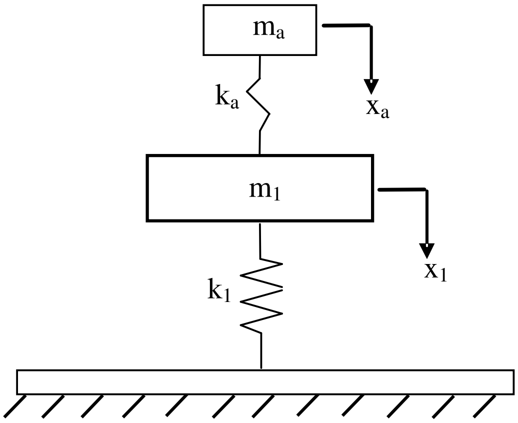

Figure 1.2. Undamped vibration absorber on a mass-spring system

Broadband attenuation can be achieved by using a damped vibration absorber (Figure 1.3), first proposed by Ormondroyd and Den Hartog [3, 4] in 1928. Ormondroyd and Den Hartog derived the optimum absorber parameters that minimize the displacement response of the main structure. Since then, dynamic absorbers have been applied in a wide variety of situations.

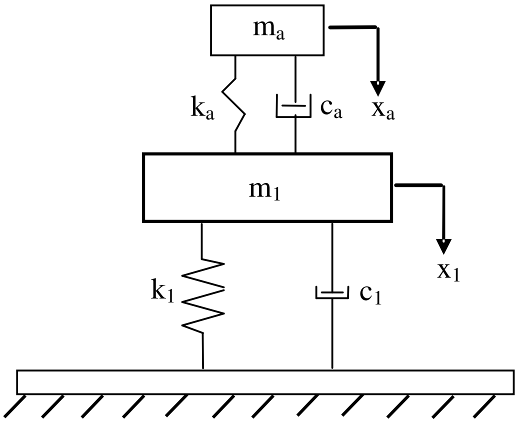

Figure 1.3. Damped vibration absorber

The effect of damping on a vibration absorber is very important. With the addition of damping, the vibration amplitude of the main mass cannot be reduced to zero at the driving frequency. On the other hand, the sensitivity of the system to variations in the driving frequency decreases. The vibration amplitude of the absorber itself also decreases considerably with damping.

Vibration absorbers are commonly used to minimize vibration amplitudes within a specified frequency range. Vibration absorbers are particularly useful for eliminating high-amplitude vibrations at a system's natural frequencies. A vibration absorber thus serves as a controller, applying a counter-effect to eliminate undesirable disturbances.

Vibration absorbers are attached to main systems such as beams, plates, shell structures and frames, etc. Researchers have already spent considerable effort on this approach to minimize vibration amplitudes. Even alternative approaches are validated by comparing their effects to idealized mass-spring-damper systems. In general, however, the vibration absorber approach is preferred. The devices are cheap, robust, and flexible enough to adapt to nearly any mechanical system.

The term "vibration absorber" is reserved in this thesis for passive devices (mass-spring-damper systems) attached to a vibrating structure. A comparison between passive and active absorbers is given in Table 1.1.

Table 1.1. Comparison between passive and active absorbers

	Passive Absorbers	**Active Absorbers**
Composed of	Mass, spring, damper	Sensor, actuator, controller
Advantages	Cheap, robust, simple	Better performance, more flexible
Disadvantages	Less performance	Expensive

1.2. Motivation and Research Objective

The main goal of this thesis is to develop a method for calculating the optimal parameters of a vibration absorber for continuous and discrete systems. This research extends currently available techniques to a more general context and it is applicable to systems with various input characteristics. The study is motivated by the following facts:

- For continuous systems, most of the currently available methods are restricted to specific system characteristics, where constrained number of absorbers (one or two) and main system modes (three or four) are considered. The present work extends a widespread method to the general case of n absorbers and considering M initial modes.

- Physical constraints should also be taken into account by the method, so that specific design criteria can be met when optimizing the absorber parameters.

- For discrete systems, the minimization of vibration amplitudes is usually discussed via the concept of receptance (displacement per unit force). As with continuous systems, a generalized method for calculating the receptance of a translational mass-spring-damper system with n absorbers and m masses is needed.

- The receptance of the combined discrete system (including all main masses and absorbers) must be derived separately, in terms of the parameters of the main and absorber systems, via an analytical approach.

- The analytical expressions derived for the receptance of a discrete system should be validated by a supporting methodology. This work uses two methods to calculate the dynamic stiffness: direct analysis of the equations of motion, and a linear graph representation of the system.

- The problems of continuous and discrete systems are interrelated with each other. Thus, the corresponding methods should be applied to each type of system mutually to achieve comparable results.

- The formulations and results obtained by the new method should be compared to scenarios already published in the literature, for both continuous and discrete systems.

1.3. Literature Review

Dynamic vibration absorbers are *passive* mechanical systems, normally consisting of a mass coupled to the vibrating primary structure by a spring and damping element. Dynamic vibration absorbers can be applied to a wide range of discrete and continuous systems. Note that large machines and structures can not normally be treated as lumped parameter systems; they must be modeled as a combination of individual beams, plates, etc. For this reason, the application of damped vibration absorbers to continuous systems has drawn considerable attention in the literature. In order to avoid dangerous resonance conditions, engineers must also be able to predict the natural frequencies of complex mechanical systems. Thus, some of the research described below focuses on this problem.

Jacquot [5, 6] has analyzed the elimination of excessive vibration in sinusoidally forced Bernoulli–Euler beams, defining a general equation for all possible sets of ordinary boundary conditions and absorber attachment points. The method is demonstrated on a point-forced cantilever beam with only one dynamic absorber (a mass-spring-damper system). The frequency response function is obtained by the Lagrange's equation. The expression found is applicable to driving frequencies upto four times higher than the first resonance of the beam.

In another work of Jacquot [7], two variations of the cantilever beam are investigated. The first is modified by a simple tip damper, the second by a damped vibration absorber attached at the tip. A method is proposed to predict the response of both structures. In an additional example using a clamped-pinned beam, it is shown that increasing the damping eliminates resonances near the cantilever's natural frequencies, but induces new resonances near the original natural frequencies.

Transverse vibrations of a cantilever beam carrying a concentrated mass (with no damping effect) have been studied by many researchers. Generally, these studies are related with the situation where the mass is rigidly attached to the beam tip. In many cases the mass is elastically attached to the structural element, due to the fact that an engine is mounted on an elastic foundation or simply because the connection of the mass to the beam possesses elastic properties [8]. Both cases have been analyzed by Rossit and Laura [9], who also report the natural frequencies for a wide range of intervening physical parameters.

Nagaya et al. [10] have discussed a method of vibration control using undamped vibration absorbers. A single undamped vibration absorber can be tuned to suppress the principal mode of a fixed–fixed beam. In order to indicate the validity of the proposed approach, experimental tests are performed.

Wu and Whittaker [11] have studied the natural frequencies and the corresponding vibration modes of a uniform beam carrying multiple spring-mass systems (each with two degrees of freedom).

Wu [12] has characterized the free vibrations of a beam carrying multiple two degrees of freedom spring-damper-mass systems (each system consists of two springs and two dampers attached to a single lumped mass). The study concentrates on replacing their effects with an equivalent set of dampers in the theoretical model. The natural frequencies of such a beam could be approximated more easily by considering the equivalent dampers.

Qiao et al. [13] have presented a new and powerful method for analyzing the free vibrations of non-uniform Euler-Bernoulli beams. Their approach works for an arbitrary number of spring-mass systems, with one or two degrees of freedom, in any configuration. The spring-mass systems are replaced with massless equivalent springs. Exact analytical solutions are derived for various cases. To simplify analysis, the fundamental solutions are based on the derived exact solutions.

Thompson [14] has analyzed the effect of a damped mass-spring absorber system attached continuously along a beam. Because it is not confined to a single point, this absorber is effective for driving forces applied at any location along the beam. The method is somewhat similar to constrained layer damping, but remains useful in many situations where the latter is impractical (particularly for stiff beams and low frequencies). The parameters controlling its behavior are investigated and simple formulae are developed to optimize its performance.

Gurgoze et al. [15] have investigated the natural-vibrations of a primary system consisting of two clamped-free, laterally vibrating Bernoulli-Euler beams carrying tip masses. A secondary system with two springs and one mass is attached. First, the equations of motion for portions of each beam are derived. Using the method of separation of variables in the corresponding boundary and continuity equations, the frequency equation of the combined system is formulated. The effects of the variation of certain parameters on the natural frequencies are investigated through numerical examples. Inceoglu and Gurgoze [16] have extended this work, deriving the natural frequencies of coupled beams with several double spring mass systems attached.

Manikanahally and Crocker [17] have analyzed the case of a mass-loaded beam with variable cross-section, subject to an arbitrarily distributed simple harmonic force. The studied process provides some flexibility in choosing the number of absorbers, depending on the number of significant modes to be suppressed. The design parameters are the mass, the stiffness and the damping coefficients of each absorber. The interactions between absorbers are also demonstrated. The authors derive a procedure for choosing the vibration absorbers that minimize a given resonance, and provide a simple example: a mass loaded free-free beam driven by a simple harmonic force at a single point. First the equations are obtained for the general case with n absorbers, and then a specific case with two absorbers is analyzed. Next, the question of whether to optimize all of the absorber parameters or not is investigated. In this case, an upper and a lower bound for each parameter are set so that fully optimizing the system is feasible. The stiffness and the damping coefficients are found to lie within these limits, but the mass is driven to the upper bound. It is concluded that heavier absorber masses reduce the dynamic response. Finally, the mass is held constant while the stiffness and damping coefficients of the absorbers are optimized.

In addition to the problem of absorbers applied to beams, there has been research on plates, cylindrical shells, shallow shells, and other forms. A few of these works will now be summarized.

Jacquot [18] has considered the effect of damped spring-mass absorbers on simply supported rectangular plates, which have several real-world applications (bridges, building structures, naval structures, and pressure vessels for example). The design parameters are the absorber locations, masses, and stiffness/damping coefficients. The plates are assumed to be thin, uniform, elastic, and supported only along the edges. The frequency response is found, and plotted for various damping ratios and mass ratios. Finally, the paper shows how to minimize the mean squared motion by choosing suitable absorber parameters.

Huang and Fuller [19] have examined multiple vibration absorbers on a closed cylindrical shell. The results are discussed for two different types of external forces: point forces and a uniformly distributed pressure. The effects of absorber parameters on the shell's interior acoustic sound pressure are also investigated.

Aida et al. [20] have analyzed the vibration absorber technique for a shallow shell structure. The authors' aim is to suppress several modes of bending vibration of a main system. A dynamic shell, with the same boundary condition and the shape as the primary system, is tuned to absorb the vibration. It therefore acts like a plate-type dynamic vibration absorber. Springs and dampers are uniformly distributed in the vertical direction between shells. An approximate tuning method is also developed in the study.

The literature also contains investigations on different types and configurations of absorbers, as described below.

A hybrid dynamic vibration absorber has been proposed by Burdisso and Heilmann [21]. The absorber consists of two masses attached to the main structure in parallel through elastic elements. Its performance is evaluated by controlling the response of a cantilever beam. The proposed dual-mass dynamic vibration absorber achieved the same level of vibration reduction as a single mass dynamic absorber, but required only half the control effort.

Wong et al. [22] have presented a new type of dynamic vibration absorber combining a translational type and a rotational type of absorber for isolation of beam vibration under point or distributed harmonic excitation. Its efficiency is evaluated by simulating the response of a finite element beam model. It is proved that the vibration of the forced beam, under point or distributed harmonic excitation, could be isolated in one part of the beam.

Nader D. Ebrahimi [23] has analyzed the effects of vibration reduction on the simplest possible primary system (a mass-spring-damper system). With the help of a secondary vibration absorber with the same characteristics, the vibration of the main mass can be minimized for any given frequency. The locations of absorbers are very important. The optimum absorber should minimize the vibration amplitude for the specified degree of freedom with the least control effort. During the design stage, the frequency range, within which the disturbances are effective, is considered.

Eskinat and Ozturk [24] have considered the design of active and passive vibration absorbers and an approach was presented to solve the absorber design problem (i.e., choosing their locations, stiffness coefficients and damping values) by a technique that is similar to loop shaping technique used in control system design. If multiple sinusoidal forces with different frequencies are acting on a structure, multiple absorbers are needed. The approach is also extended to the design of multiple absorbers. First, an absorber is designed for one driving force, ignoring all others. Other absorbers are designed and added to the structure sequentially. It is shown that the optimum absorber location is the point where the transfer function between the absorber force and the displacement of the main mass whose motion is to be minimized, is at its maximum, over the frequency range of interest. A few iterations are usually necessary for the absorber parameters converge. An example is given to illustrate the results.

Pennestri [25] has applied Chebyshev's criterion to determine the optimum parameters of damped vibration absorbers. In this case, the primary goal is to minimize the maximum displacement of the primary mass, rather than effectively absorbing energy over a range of frequencies. The algorithm searches for the maximum displacement over a frequency range centered on the resonance to be suppressed. The min-max problem is numerically solved; usually by finding the maximum of the two peaks first (due to the absorber and original beam resonances) occurring in the frequency range of concern by using a global maximization algorithm, and then determining the tuning and damping ratios to minimize this maximum value.

Rana and Soong [26] have presented a simplified parametric procedure for tuning mass dampers. In addition, a study is performed to investigate the possibility of controlling multiple structural modes using multi-tuned mass dampers. A mass damper tuned to the first mode is found to also reduce the second and third modes.

Some scientists focus on receptance theory to minimize the vibration amplitudes of discrete systems. The main complexity of this method is it relies on inverting stiffness matrices. Since the expression is becoming complex, the receptance expression is hard to solve from the inverse of the dynamic stiffness matrix. Karakas and Gurgoze [27] have analyzed the problem of solving the receptance matrix, aiming to show that it is possible to obtain the receptance matrix directly without using iterations. An exact method to evaluate the receptances of non-proportionally damped dynamic systems has been developed by Yang [28]. This is an iterative method for the calculation of the receptance matrix when the damping matrix is expressed as a sum of the dyadic products. In this method, there is no need to apply an inverse matrix operation. The iteration process starts by calculating the receptance matrix of the undamped system. Receptance matrix of the damped system is acquired after several iterations, one for each dyadic product in the damping matrix.

In contrast with Yang's iterative method, Gurgoze [29] has focused on obtaining the receptance matrix directly. The Sherman–Morrison formula can be used to invert the sum of a regular matrix and one dyadic product (i.e., another matrix of rank one). The more general Woodbury formula gives the inverse of the sum of a regular matrix and a matrix product, whose rank can be greater than one. Gurgoze [29] has derived the receptance matrix using the latter formula and a dyadic product formulation to decompose the matrices. Variations of the damping mechanism and the eigen characteristics of the system are studied.

Lin and Lim [30] have found a method of calculating the receptance, eigenvalues and eigenvector sensitivities using vibration test data. The relationship between the receptance sensitivities, eigenvalues and eigenvectors is established. A method to determine the sensitivities of the eigenvalues and eigenvectors can then be developed.

Ozer and Royston [31] have proposed a semi-analytic method to determine the optimal parameters of a damped vibration absorber attached to a damped system with multiple degrees of freedom. The Sherman–Morrison formula is used to obtain the receptance matrix. Due to the advantage of using an absorber attached to the main system, the Sherman–Morrison formula is only used once. Both the main system's equation of motion and the absorber's equation of motion are derived, thereby, clearly defining what matrices should be added to the main system.

There are many other works based on the Sherman–Morrison formula, for example that of Riedel [32]. Instead of using a second term consisting of a combination of two products, three matrices defined in the study are analyzed and decomposed as a second term.

Another investigation carried out by Gurgoze [33] to combine the receptance approach with constrained equations. Gurgoze aims to establish the receptance matrix of the constrained system in terms of the receptance matrix of the unconstrained system and the coefficient vectors of the constraint equations. The coordinates of the system are assumed to be subject to several linear constraint equations. The receptance matrix of the unconstrained system is displayed and combined with the generalized coordinates by using the Lagrange multipliers.

Shankar and Keane [34] have developed an effective new method based on receptance theory. Substructures of the main system are analyzed separately using finite element models. The vibrational energy levels of the substructures are calculated by evaluating the net energy between input and dissipated energies and the energy transferred to other substructures through coupling nodes. It is beneficial to carry out this energy into statistical energy analysis. The formulation developed uses a finite elements analysis program and the response of the total structure is predicted by using substructure modal information with the help of the study.

In order to reach desired dynamic behaviour, Kyprianou et al. [35] also vary the mass, stiffness and damping properties of their absorbers. This can be achieved by adding or removing mass, spring and damper elements. By adding mass and spring into the system (two cases; modification by a simple spring mass oscillator and modification by a mass connected by two springs), the effect of such changes on the receptance function and response function are investigated.

Motivated by the efforts of finding the optimal absorber parameters for suppressing vibration in mechanical systems in literature, it is aimed in this thesis to develop a general method for calculating the optimal absorber parameters for continuous and discrete systems. This research extends the literature by deriving a more general approach to optimization that works for a wide range of structures and is applicable for various different input characteristics.

1.4. Vibrating Systems

A complex structure consisting of several elements with similar or dissimilar properties is called a system. The vibrating systems are basically divided into two main branches: Continuous and Discrete Systems. Systems with an infinite number of degrees of freedom are called continuous systems. Those with a finite number of degrees of freedom are called discrete systems. Continuous systems, especially those that include deformable elements, have an infinite number of degrees of freedom. For example, to fully describe the vibration of an elastic beam, an infinite number of coordinates are needed. Discrete systems are reducible to the lumped characteristics of their element and can be modeled as if they had a finite number of degrees of freedom. [36].

Although it is not immediately obvious, there is a very close relationship between continuous and discrete systems. In fact, they generally represent two distinct mathematical models of the same physical system. Despite the fact that discrete systems are described by ordinary differential equations and continuous systems by partial differential equations, their behaviors are quite similar and the same results can be obtained using either model [37].

1.4.1. Continuous Systems

Most physical systems are continuous in character, and in that their structural parameters are distributed. Indeed, it is almost impossible to find systems that can truly be described with a finite number of degrees of freedom. A beam, for example, has its mass and elasticity distributed along its length. Its dynamic behavior is therefore described using partial differential equations. Due to the difficulties inherent in solving partial differential equations numerically, an approach is developed where the distributed parameters can be replaced by discrete ones by suitable lumping of the continuous systems. This should be carried out whenever possible, since lumped parameter systems are described by ordinary differential equations, which are far easier to solve than the partial differential equations.

1.4.2. Discrete Systems

Discrete models are formed in order to simplify the structural complexity of the vibrating systems for dynamic analysis purposes [38].

The starting point for construction of a discrete system is to specify the number of degrees of freedom, and the number of independent coordinates necessary to define the system dynamics.

A level of accuracy suitable for engineering purposes can often be reached by describing continuous systems with a finite number of coordinates. For example, an elastic beam can be divided into a number of segments along its long axis. It is obvious that using more coordinates yields a better approximation. The form of the continuous elastic line between approximation points is then assumed to be parabolic (or some other convenient curve).

Every system, discrete or continuous, can be adequately described by a finite number of independent coordinates or degrees of freedom. A lumped system is a particular case of discrete system that consists of discrete elements, that is, elements which relate forces to displacements, velocities, and accelerations.

1.5. Research Contribution

The contributions of this thesis can be summarized as follows:

- A method for calculating the optimal absorber parameters for uniform beams and thin rectangular plates is developed for the most general case, n absorbers and considering M initial modes. Physical constraints on the parameters are taken into consideration in order to be applicable to specified design criteria in the study of continuous systems. A desired output with weighting functions of input and output can be defined. Weighting functions can also be used to generate a more complex objective function.

- In the analysis of discrete systems, this research focuses on the concept of receptance. A method calculating the receptance of a generic translational mass-spring-damper system in a generalized manner with m masses and n absorbers is developed for providing the optimal absorber parameters.

- It is shown that continuous and discrete systems are interrelated with each other. The corresponding methods are applied to each type of system mutually.

The study enlarges two researches ([39] and [31]) published in the vibration literature, which focus on optimizing vibration absorbers in continuous and discrete systems.

Ozguven and Candir [39] have developed a method for determining the optimum parameters of two dynamic vibration absorbers tuned to the first two resonances of a beam. Their method is applied to a cantilever beam and the optimum parameters of absorbers tuned to the first and second resonances are calculated. However, the study of Ozguven and Candir [39] and the specific investigations on this subject in the vibration absorber literature are constrained to limited number of modes and absorbers. The approach in this thesis does not limit the number of modes or the number of absorbers. With reference to these studies, in this thesis, a new method is developed for providing the optimal absorber parameters of a continuous system where the number of modes and absorbers (attached to the system) can be any positive integers. The method is presented in Section 2.1 for uniform beams and the solution technique is then extended to thin rectangular plates in Section 2.2.

In the study of discrete systems via vibration absorbers, Ozer and Royston [31] have introduced a method based on the Sherman–Morrison matrix inversion formula for determining the optimum parameters of a single–degree of freedom damped vibration absorber attached to a multi-degree of freedom damped main system. Ozer and Royston's [31] studies and the other researches followed are constrained to limited number of main masses and absorbers. The method presented in Section 4.2 of this thesis provides the receptance of a generic discrete system where any combination of main masses and absorbers is allowed. Another contribution is that a general formulation is derived for finding the total system receptance. The total system receptance is obtained in terms of the receptance of the system with one less absorber by applying the Sherman–Morrison formula sequentially. The complexity of the formulation is simplified with the proposed approach. Then, the displacement expression is found through the total system receptance. Using the receptance method, the optimal parameters of the absorbers of a discrete system (translational mass-spring-damper) are obtained subject to design criteria and physical constraints.

1.6. Thesis Outline

The introduction has provided a literature survey including a general overview and historical progress of vibration absorbers. It has also explained the difference between the discrete and continuous systems, and described relevant investigations on how vibration absorbers can be optimized for each type.

Chapter 2 begins by modeling a uniform beam with n passive absorbers. A uniform beam, to which n vibration absorbers are attached, is studied and a new method for calculating the optimum absorber parameters is proposed. First, for the most general case, dissipation due to damping, kinetic and potential energies of the beam, and the effects of external forces are all taken into account. Lagrange's equation is used to derive the state space representation of the system for the general case of n absorber and considering M initial modes. Using the state space equations of motion and the Lyapunov equation, the H_2 norm of the transfer function of the system is obtained and selected as the basis for the optimization. The method is then extended to a thin rectangular plate.

In Chapter 3, numerical applications and a variety of specific cases are carried out for beams and rectangular plates. The results are compared to those of related literature.

Chapter 4 suggests a concept of receptance method to suppress vibration amplitudes in a configuration of discrete system, a translational mass-spring-damper system. A new method based on using the Sherman–Morrison matrix inversion formula to calculate the optimal parameters of the absorbers on a translational mass-spring-damper system with n absorbers and m masses is studied. The system receptance is developed by applying the Sherman-Morrison formula sequentially: the receptance of any system can be expressed in terms of the same system with one less absorber. Finally, the optimal parameters of the absorbers are calculated subject to specified design criteria and physical constraints.

Chapter 5 numerically evaluates some specific cases and compares them to results from the literature.

The purpose of Chapter 6 is to show that discrete and continuous systems are interrelated and in fact that the two viewpoints are physically equivalent. As such, it revisits the translational mass-spring-damper system of the previous chapter and calculates its optimal absorber parameters using the method developed in Chapter 2 for continuous systems. Similarly, the method described in Chapter 4 for discrete systems is applied to a uniform beam.

In Chapter 7, conclusions and the main contributions of this thesis are outlined.

2. VIBRATION SUPPRESSION OF CONTINUOUS SYSTEMS USING PASSIVE ABSORBERS

2.1. Vibration Suppression of Uniform Beams

In the present study, a new method for calculating the optimal parameters of passive vibration absorbers placed on a uniform beam, allowing for design constraints on the absorber parameters is developed [40]. The method delivers the optimum parameters (i.e. l, the location of each absorber, k, the stiffness of each absorber, and c, the damping of each absorber) for the most general case of n absorbers and considering M initial modes of the beam.

2.1.1. Overview of Beam Theories

A fundamental principle of all deformable continuous systems is that their dynamic behavior is described by partial differential equations [41].

Several mathematical models of continuous beams exist, based on various theories of their dynamic behavior [42]. All the models under consideration take into account only transverse displacements of the beam. Examples of beam theories are Euler-Bernoulli, Rayleigh, Timoshenko, Bress, Volterra, and Ambartsunyan. These theories are distinguished from each other by different driving forces and assumptions.

As a comparison, the choice between two models, Euler-Bernoulli and Timoshenko, is largely dependent on the beam geometry and which modes are of greatest interest. For a steel beam 12 m long, 15 cm wide, and 0.6 m deep, shows a difference in only 0.4% between the first natural frequency of the Euler-Bernoulli model and that of the Timoshenko model. Their predictions of the fifth natural frequency, however, deviate by 10%. If only the first mode is of interest, the Euler-Bernoulli model would be good enough. On the other hand, if the fifth mode is of interest, the Timoshenko model might worth its extra complexity [1].

In literature, different types of beam models are compared to each other. The study of Ruge and Birk [43], for example, compares the translational and rotational dynamic stiffness coefficients of both Timoshenko and Euler-Bernoulli beams on a Winkler foundation.

Vibrating Timoshenko beams have recently been studied by a few authors, dealing with problems such as free vibrations with intermediate flexible constraints, free and forced vibrations of non-homogeneous beams with varying cross-section, and beams carrying a concentrated mass. When modal analysis for damped continuous systems is applied to the particular case of the Timoshenko beam model, the eigenvalue problem is solved by combining a state-space representation with a transfer matrix. This technique yields closed-form expressions for the eigenfunctions [44].

Elastic bodies are systems having an infinite number of particles between which elastic forces act. Therefore, there is infinite number of degrees of freedom for continuous systems.

The beam model under consideration in this study is the Euler-Bernoulli beam model, which is the elementary beam model and it is the most commonly used one because of its simplicity and basic structure.

The Euler-Bernoulli beam model includes the strain energy of bending and kinetic energy of transverse displacements. In this theory, the inertia due to transverse translation is considered and the inertia forces due to shear deflection and rotation are neglected. It is assumed that the rotation of a differential element is small compared to its translation, and that any angular distortion due to shear can be safely neglected. Each cross-section is assumed to remain planar and orthogonal to the mid-plane of the beam during a deformation.

To derive the equation of motion, Newton's second law is used. Considering the beam of length l in Figure 2.1 and the beam element in Figure 2.2**,** the net force on an element in the vertical direction is given by:

$$\left(Q(x,t) + \frac{\partial M(x,t)}{\partial x} dx \right) - Q(x,t) + f(x,t)dx = m_u(x)dx \frac{\partial^2 y(x,t)}{dt^2}, \quad 0 < x < l \quad (2.1)$$

The equilibrium equation for moments of the beam element in bending – assuming the moment of inertia of the element and its angular acceleration are negligible – is given by:

$$\left(M(x,t) + \frac{\partial M(x,t)}{\partial x} dx \right) - M(x,t) + \left(Q(x,t) + \frac{\partial Q(x,t)}{\partial x} dx \right) dx + f(x,t)dx \frac{dx}{2} = 0, \quad 0 < x < l \quad (2.2)$$

In both equations $M(x,t)$ denotes the bending moment, $Q(x,t)$ is the shear force acting on the beam element, $f(x,t)$ is the transverse external force per unit length, $m_u(x)$ is the mass per unit length, $I(x)$ is the moment of inertia along the length of the beam, and E is the modulus of elasticity.

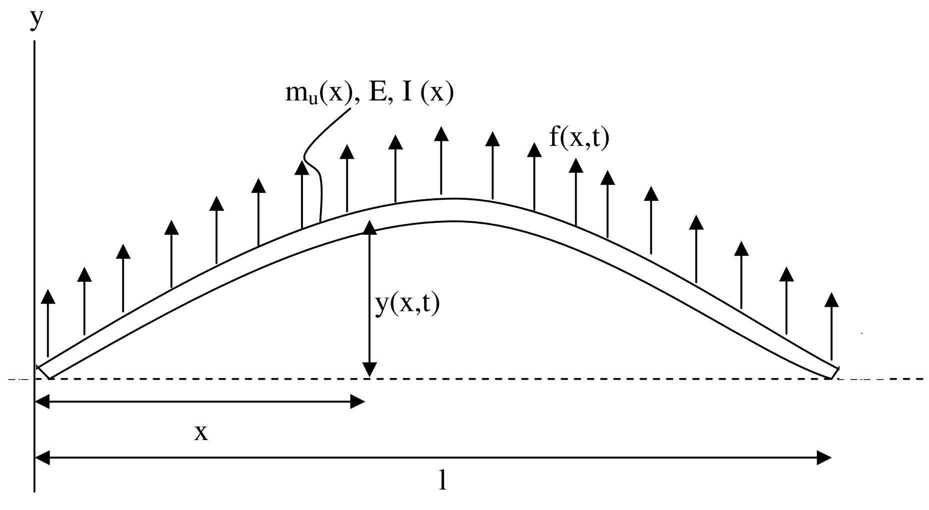

Figure 2.1. Beam in bending [41]

By ignoring second-order dx terms in Eq. (2.2), and inserting it into Eq. (2.1), and then dividing Eq. (2.1) by dx, the following equation is obtained:

$$-\frac{\partial^2 M(x,t)}{\partial x^2} + f(x,t) = m_u(x) \frac{\partial^2 y(x,t)}{\partial t^2}, \qquad 0 < x < l \qquad (2.3)$$

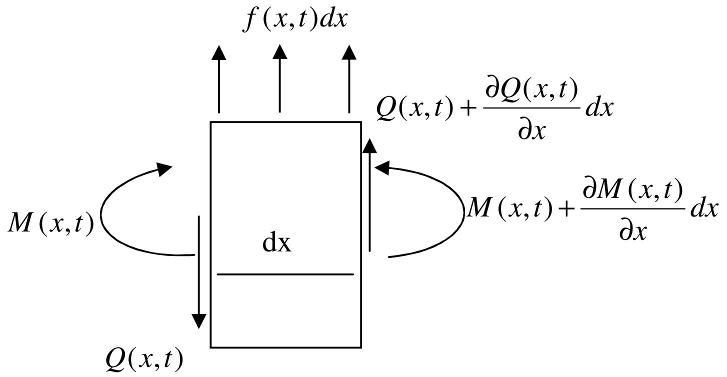

Figure 2.2. Free-body diagram for a bending beam element [41]

Eq. (2.3) relates the bending moment $M(x,t)$ and external force $f(x,t)$ to the transverse bending deflection $y(x,t)$. However, the bending moment can also be defined as follows:

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \tag{2.4}$$

Inserting Eq. (2.4) into Eq. (2.3) yields the relationship between the transverse displacement and the external force density:

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) + f(x,t) = m_u(x)\frac{\partial^2 y(x,t)}{\partial t^2}, \qquad 0 < x < l \tag{2.5}$$

The equations of motion with boundary conditions form a boundary value problem which can be solved by the method of separation of variables. To complete the boundary value problem, two boundary conditions for each end of the beam should be specified. For different types of attachment, the regarding boundary conditions are used for solving the above partial differential equation [1].

2.1.2. Vibration Absorbers Attached to a Beam

This analysis focuses on a uniform beam to which n vibration absorbers are attached. The formulation is developed from the most general case, displayed in Figure 2.3.

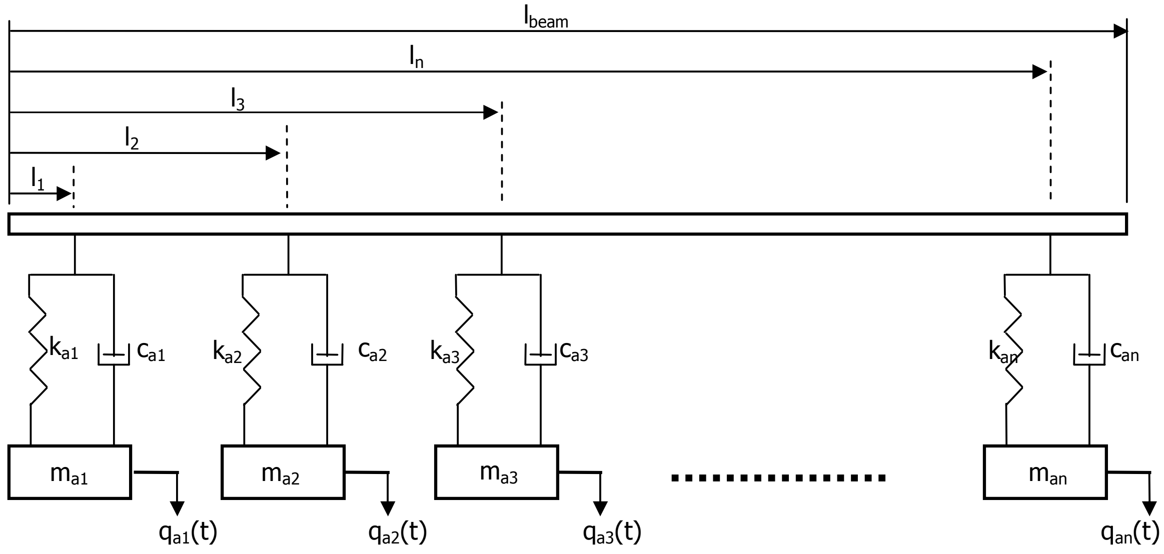

Figure 2.3. Vibration absorbers attached to a beam

The "Assumed Modes Method" consists of assuming a solution of the free vibration in the form of a series composed of a linear combination of admissible functions, $u_i(x)$, which are functions of the spatial coordinates, multiplied by time-dependent generalized coordinates, $q_i(t)$. The method improves the understanding of discretization by means of a series solution [37, 45].

Thus, the transverse displacement $\omega(x,t)$ of the beam is assumed to be of the form

$$\omega(x,t) = u_1(x)q_1(t) + u_2(x)q_2(t) + u_3(x)q_3(t) + ... + u_M(x)q_M(t) \tag{2.6}$$

where $q_1(t)$, $q_2(t)$ ... $q_M(t)$ are time dependent generalized coordinates.

The velocity of the beam is formulated as

$$\dot{\omega}(x,t) = u^T(x)\dot{q}(t)\,, \tag{2.7}$$

where

$$u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ \vdots \\ \vdots \\ u_M(x) \end{bmatrix},\ q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \\ \vdots \\ q_M(t) \end{bmatrix}. \tag{2.8}$$

$$\dot{\omega}(x,t) = u^T(x)\dot{q}(t) = \begin{bmatrix} u_1(x) & u_2(x) & u_3(x) & \cdots & \cdots & u_M(x) \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \vdots \\ \vdots \\ \dot{q}_M(t) \end{bmatrix} \tag{2.9}$$

$$= u_1(x)\dot{q}_1(t) + u_2(x)\dot{q}_2(t) + u_3(x)\dot{q}_3(t) + \ldots\ldots\ldots + u_M(x)\dot{q}_M(t)$$

The squared velocity is

$$\dot{\omega}^2(x,t) = \dot{\omega}^T\dot{\omega} = (u^T(x)\dot{q}(t))^T(u^T(x)\dot{q}(t))\text{, or} \tag{2.10}$$

$$\dot{\omega}^2(x,t) = \dot{q}^T(t)u(x)u^T(x)\dot{q}(t) = \dot{q}^T(t)U(x)\dot{q}(t) \tag{2.11}$$

where the matrix $U(x)$ is defined below:

$$U(x) = u(x)u^T(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ \vdots \\ \vdots \\ u_M(x) \end{bmatrix} \begin{bmatrix} u_1(x) & u_2(x) & u_3(x) & \cdots & \cdots & u_M(x) \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2(x) & \cdots & \cdots & u_1(x)u_M(x) \\ u_2(x)u_1(x) & u_2^2(x) & & \vdots \\ u_3(x)u_1(x) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ u_M(x)u_1(x) & \cdots & \cdots & u_M^2(x) \end{bmatrix}. \tag{2.12}$$

The expressions formulated above will be used in Lagrange's equations.

2.1.3. Lagrangian Equations of Motion

The general form of Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = Q_r \qquad r = 1.........M\ . \tag{2.13}$$

The kinetic and potential energies of a dynamical system can always be expressed in terms of generalized coordinates and their time derivatives. Thus, the equations of motion can also be written in terms of generalized coordinates using Lagrange's equation.

The kinetic energy of a beam with n absorbers can be expressed as follows:

$$T(t) = \frac{1}{2} m_u \int_0^{l_{beam}} \left(\frac{\partial \omega}{\partial t}\right)^2 dx + \frac{1}{2} \sum_{i=1}^{n} m_{ai} \dot{q}_{ai}{}^2(t)\,, \tag{2.14}$$

$$T(t) = \underbrace{\frac{1}{2} m_u \int_0^{l_{beam}} \left(\frac{\partial \omega}{\partial t}\right)^2 dx}_{\mathrm{A}} + \underbrace{\frac{1}{2} m_{a1} \dot{q}_{a1}{}^2(t)}_{\mathrm{B_1}} + \underbrace{\frac{1}{2} m_{a2} \dot{q}_{a2}{}^2(t)}_{\mathrm{B_2}} + \underbrace{\frac{1}{2} m_{a3} \dot{q}_{a3}{}^2(t)}_{\mathrm{B_3}} + ... + \underbrace{\frac{1}{2} m_{an} \dot{q}_{an}{}^2(t)}_{\mathrm{B_n}} \tag{2.15}$$

$$T(t) = A + B_1 + B_2 + B_3 + ... + B_n \tag{2.16}$$

Note that $q_{a1}(t)$, $q_{a2}(t)$... $q_{an}(t)$ are the displacements of the absorbing masses. The kinetic energies of both the beam and the absorber's masses of the total system are derived below:

$$A = \frac{1}{2} m_u \int_0^{l_{beam}} \begin{bmatrix} \dot{q}_1(t) & \dot{q}_2(t) & \dot{q}_3(t) & \cdots & \cdots & \dot{q}_M(t) \end{bmatrix} \begin{bmatrix} u_1^2(x) & \cdots & \cdots & u_1(x)u_M(x) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ u_M(x)u_1(x) & \cdots & \cdots & u_M^2(x) \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \vdots \\ \vdots \\ \dot{q}_M(t) \end{bmatrix} dx \tag{2.17}$$

$$B_1 = \frac{1}{2} m_{a1} \dot{q}_{a1}{}^2(t) \tag{2.18}$$

$$B_2 = \frac{1}{2} m_{a2} \dot{q}_{a2}{}^2(t) \tag{2.19}$$

$$\vdots$$

$$B_n = \frac{1}{2} m_{an} \dot{q}_{an}{}^2(t) \tag{2.20}$$

The potential energy of a beam with n absorbers can be expressed as follows:

$$V(t) = \frac{1}{2} EI \int_0^{l_{beam}} \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 dx + \frac{1}{2} \sum_{i=1}^{n} k_{ai} \left(\omega(l_i, t) - q_{ai}(t) \right)^2 \tag{2.21}$$

$$V(t) = \underbrace{\frac{1}{2} EI \int_0^{l_{beam}} \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 dx}_{\mathrm{C}} + \underbrace{\frac{1}{2} k_{a1} \left(\omega(l_1, t) - q_{a1}(t) \right)^2}_{\mathrm{D_1}} + \underbrace{\frac{1}{2} k_{a2} \left(\omega(l_2, t) - q_{a2}(t) \right)^2}_{\mathrm{D_2}} + \underbrace{\frac{1}{2} k_{a3} \left(\omega(l_3, t) - q_{a3}(t) \right)^2}_{\mathrm{D_3}} + \ldots$$

$$+ \underbrace{\frac{1}{2} k_{an} \left(\omega(l_n, t) - q_{an}(t) \right)^2}_{\mathrm{D_n}} \tag{2.22}$$

$$V(t) = C + D_1 + D_2 + D_3 + \ldots\ldots\ldots + D_n \tag{2.23}$$

The beam potential energy expands to

$$C=\frac{1}{2}EI\int_{0}^{l_{beam}}\left(\frac{d^2u_1(x)}{dx^2}q_1(t)+\frac{d^2u_2(x)}{dx^2}q_2(t)+\frac{d^2u_3(x)}{dx^2}q_3(t)+.........+\frac{d^2u_M(x)}{dx^2}q_M(t)\right)^2dx$$

(2.24)

$$C=\frac{1}{2}EI\left[q_1(t)\quad q_2(t)\quad q_3(t)\quad\cdots\quad\cdots\quad q_M(t)\right]$$
$$\int_{0}^{l_{beam}}\begin{bmatrix}\left(\frac{d^2u_1(x)}{dx^2}\right)^2 & \cdots\ \cdots & \left(\frac{d^2u_1(x)}{dx^2}\right)\left(\frac{d^2u_M(x)}{dx^2}\right)\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ \left(\frac{d^2u_M(x)}{dx^2}\right)\left(\frac{d^2u_1(x)}{dx^2}\right) & \cdots\ \cdots & \left(\frac{d^2u_M(x)}{dx^2}\right)^2\end{bmatrix}dx\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\\ \vdots\\ \vdots\\ q_M(t)\end{bmatrix} \tag{2.25}$$

The absorber potential energies expand to

$$D_1=\frac{1}{2}k_{a1}(u_1(l_1)q_1(t)+u_2(l_1)q_2(t)+u_3(l_1)q_3(t)+.........+u_M(l_1)q_M(t)-q_{a1}(t))^2 \tag{2.26}$$

$$D_2=\frac{1}{2}k_{a2}(u_1(l_2)q_1(t)+u_2(l_2)q_2(t)+u_3(l_2)q_3(t)+.........+u_M(l_2)q_M(t)-q_{a2}(t))^2 \tag{2.27}$$

$$\vdots$$

$$D_n=\frac{1}{2}k_{an}(u_1(l_n)q_1(t)+u_2(l_n)q_2(t)+u_3(l_n)q_3(t)+.........+u_M(l_n)q_M(t)-q_{an}(t))^2 \tag{2.28}$$

The generalized non-conservative forces effect (Q_r) can be distinguished in Lagrange's equation:

$Q_{nc1}(t)$**:** Dissipative forces due to damping.

$Q_{nc2}(t)$**:** External forces.

$$Q_r=-\frac{\partial Q_{nc1}}{\partial\dot{q}_k}-\frac{\partial Q_{nc2}}{\partial\dot{q}_k} \tag{2.29}$$

$$Q_{nc1}=\underbrace{\frac{1}{2}c_{a1}\left(\dot{\omega}^2(l_1,t)-\dot{q}_{a1}(t)\right)^2}_{E_1}+\underbrace{\frac{1}{2}c_{a2}\left(\dot{\omega}^2(l_2,t)-\dot{q}_{a2}(t)\right)^2}_{E_2}+\underbrace{\frac{1}{2}c_{a3}\left(\dot{\omega}^2(l_3,t)-\dot{q}_{a3}(t)\right)^2}_{E_3}+.........$$
$$+\underbrace{\frac{1}{2}c_{an}\left(\dot{\omega}^2(l_n,t)-\dot{q}_{an}(t)\right)^2}_{E_n} \tag{2.30}$$

where

$$E_1 = \frac{1}{2} c_{a1} (u_1(l_1)\dot{q}_1(t) + u_2(l_1)\dot{q}_2(t) + u_3(l_1)\dot{q}_3(t) + ... + u_M(l_1)\dot{q}_M(t) - \dot{q}_{a1}(t))^2 \tag{2.31}$$

$$E_2 = \frac{1}{2} c_{a2} (u_1(l_2)\dot{q}_1(t) + u_2(l_2)\dot{q}_2(t) + u_3(l_2)\dot{q}_3(t) + ... + u_M(l_2)\dot{q}_M(t) - \dot{q}_{a2}(t))^2 \tag{2.32}$$

$$\vdots$$

$$E_n = \frac{1}{2} c_{an} (u_1(l_n)\dot{q}_1(t) + u_2(l_n)\dot{q}_2(t) + u_3(l_n)\dot{q}_3(t) + ... + u_M(l_n)\dot{q}_M(t) - \dot{q}_{an}(t))^2 \tag{2.33}$$

The partial derivative of a generalized non-conservative external force with respect to the first derivative of the generalized coordinate is defined below:

For the external forces (Q_{nc2}),

$$\delta W = \int_0^{l_{beam}} f(x,t)\delta W(x,t) = \int_0^{l_{beam}} f(x,t)\sum_{i=1}^{M} u_i(x)\delta q_i dx = -\sum_{i=1}^{M} Q_{nc2}(t)\delta q_i(t) \tag{2.34}$$

and

$$\frac{\partial Q_{nc2}}{\partial \dot{q}_k} = -\int_0^{l_{beam}} f(x,t)u_i(x)dx\,. \tag{2.35}$$

Substituting the energies and the generalized forces into Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = Q_r\,, \tag{2.36}$$

yields the following matrix equation:

$$
\left[\begin{array}{cccc|cccc}
m_u \int_0^{l_{beam}} u_1(x)u_1(x)dx & m_u \int_0^{l_{beam}} u_1(x)u_2(x)dx & \cdots & m_u \int_0^{l_{beam}} u_1(x)u_M(x)dx & 0 & 0 & 0 & 0 \\
m_u \int_0^{l_{beam}} u_2(x)u_1(x)dx & m_u \int_0^{l_{beam}} u_2(x)u_2(x)dx & \cdots & m_u \int_0^{l_{beam}} u_2(x)u_M(x)dx & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots & 0 \\
m_u \int_0^{l_{beam}} u_M(x)u_1(x)dx & \cdots & \cdots & m_u \int_0^{l_{beam}} u_M(x)u_M(x)dx & 0 & 0 & 0 & 0 \\
\hline
0 & \cdots & \cdots & 0 & m_{a1} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & m_{a2} & 0 & 0 \\
0 & \cdots & \cdots & 0 & \vdots & 0 & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & m_{an}
\end{array}\right]
\left[\begin{array}{c}
\ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \vdots \\ \ddot{q}_M \\ \hline \ddot{q}_{a1} \\ \ddot{q}_{a2} \\ \vdots \\ \ddot{q}_{an}
\end{array}\right]
$$

$$
+\left[\begin{array}{cccc|cccc}
\sum_{i=1}^{n} c_{ai}u_1(l_i)u_1(l_i) & \sum_{i=1}^{n} c_{ai}u_2(l_i)u_1(l_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_M(l_i)u_1(l_i) & -c_{a1}u_1(l_1) & -c_{a2}u_1(l_2) & \cdots & -c_{an}u_1(l_n) \\
\sum_{i=1}^{n} c_{ai}u_1(l_i)u_2(l_i) & \sum_{i=1}^{n} c_{ai}u_2(l_i)u_2(l_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_M(l_i)u_2(l_i) & -c_{a1}u_2(l_1) & -c_{a2}u_2(l_2) & \cdots & -c_{an}u_2(l_n) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{n} c_{ai}u_1(l_i)u_M(l_i) & \sum_{i=1}^{n} c_{ai}u_2(l_i)u_M(l_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_M(l_i)u_M(l_i) & -c_{a1}u_M(l_1) & -c_{a2}u_M(l_2) & \cdots & -c_{an}u_M(l_n) \\
\hline
-c_{a1}u_1(l_1) & -c_{a1}u_2(l_1) & \cdots & -c_{a1}u_M(l_1) & c_{a1} & 0 & \cdots & 0 \\
-c_{a2}u_1(l_2) & -c_{a2}u_2(l_2) & \cdots & -c_{a2}u_M(l_2) & 0 & c_{a2} & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & 0 \\
-c_{an}u_1(l_n) & -c_{an}u_2(l_n) & \cdots & -c_{an}u_M(l_n) & 0 & 0 & \cdots & c_{an}
\end{array}\right]
\left[\begin{array}{c}
\dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \vdots \\ \dot{q}_M \\ \hline \dot{q}_{a1} \\ \dot{q}_{a2} \\ \vdots \\ \dot{q}_{an}
\end{array}\right]
$$

$$
+\left[\begin{array}{cccc|cccc}
EI\int_0^{l_{beam}}\frac{d^2u_1}{dx^2}\frac{d^2u_1}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_1(l_i)u_1(l_i) & \cdots & \cdots & EI\int_0^{l_{beam}}\frac{d^2u_M}{dx^2}\frac{d^2u_1}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_1(l_i)u_M(l_i) & -k_{a1}u_1(l_1) & -k_{a2}u_1(l_2) & \cdots & -k_{an}u_1(l_n) \\
EI\int_0^{l_{beam}}\frac{d^2u_1}{dx^2}\frac{d^2u_2}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_1(l_i)u_2(l_i) & \cdots & \cdots & EI\int_0^{l_{beam}}\frac{d^2u_M}{dx^2}\frac{d^2u_2}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_2(l_i)u_M(l_i) & -k_{a1}u_2(l_1) & -k_{a2}u_2(l_2) & \cdots & -k_{an}u_2(l_n) \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
EI\int_0^{l_{beam}}\frac{d^2u_1}{dx^2}\frac{d^2u_M}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_1(l_i)u_M(l_i) & \cdots & \cdots & EI\int_0^{l_{beam}}\frac{d^2u_M}{dx^2}\frac{d^2u_M}{dx^2}dx+\sum_{i=1}^{n}k_{ai}u_M(l_i)u_M(l_i) & -k_{a1}u_M(l_1) & -k_{a2}u_M(l_2) & \cdots & k_{an}u_M(l_n) \\
\hline
-k_{a1}u_1(l_1) & -k_{a1}u_2(l_1) & \cdots & -k_{a1}u_M(l_1) & k_{a1} & 0 & \cdots & 0 \\
-k_{a2}u_1(l_2) & -k_{a2}u_2(l_2) & \cdots & -k_{a2}u_M(l_2) & 0 & k_{a2} & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & 0 \\
-k_{an}u_1(l_n) & -k_{an}u_2(l_n) & \cdots & -k_{an}u_M(l_n) & 0 & 0 & 0 & k_{an}
\end{array}\right]
\left[\begin{array}{c}
q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_M \\ \hline q_{a1} \\ q_{a2} \\ \vdots \\ q_{an}
\end{array}\right]
$$

$$
=\left[\begin{array}{c}
\int_0^{l_{beam}} f(x,t)u_1(x)dx \\
\int_0^{l_{beam}} f(x,t)u_2(x)dx \\
\vdots \\
\int_0^{l_{beam}} f(x,t)u_M(x)dx \\
\hline
0 \\ \vdots \\ \vdots \\ 0
\end{array}\right]
\tag{2.37}
$$

2.2. Vibration Suppression of Rectangular Plates

In the present study, a method that can be effectively used for the vibration analysis of a rectangular plate is investigated. The framework described for uniform beams in the previous section is extended to thin rectangular plates. The method developed provides the optimum absorber parameters (i.e. their locations $x_{1...n}$ and $y_{1...n}$, k, the stiffness, and c, the damping of the absorbers) for the most general case of n absorbers and considering M initial modes of a rectangular plate.

2.2.1. Overview of Rectangular Plates Approaches

The dynamic behavior of all deformable continuous systems can be described by partial differential equations. Hence, the analysis of vibration in continuous systems requires the determination of dynamic characteristics of the system.

Vera et al. [46] have analyzed the natural frequencies and vibration modes of a plate attached to a spring-mass system with two degrees of freedom. Both simply supported and cantilever plates are considered. The vibrations are characterized using Lagrange multipliers, and validated by the finite element method.

Kwak et al. [47] have investigated several methods of free vibration analysis for the case of a rectangular plate with a rectangular or a circular hole. The kinetic and potential energies of the plate are calculated by subtracting the hole domain from the integrals. The "missing" energies due to the hole can also be expressed in closed form. A methodology called Independent Coordinate Coupling is developed, in which energies corresponding to the rectangular plate domain and hole domain are derived independently. The two coordinates are then coupled by imposing kinematic relations.

An effective method for analyzing the free vibrations of a simply supported rectangular plate with thickness variation (with linearly varying and with quadratic thickness) was studied by Kang and Kim [48]. In their analysis, the plate is divided into a number of small regions over which the thickness is assumed to be constant. A closed form for the frequency function is derived by requiring continuity in the displacement and slope across regions. Their new method yields accurate eigenvalues and modes, and is compared to other approaches including the finite elements method.

Wong [49] has analyzed plate vibrations with a distributed mass load instead of point masses. In particular, the free bending vibration of a simply supported rectangular plate is analyzed by the Rayleigh-Ritz method. The frequencies and vibration modes for various loading distributions are determined numerically.

Jacquot [50] studied a simply supported, thin rectangular plate under the influence of two external driving forces. The first is distributed (a force density expression), and the second is transmitted at a single point near the attached dynamic absorber. The location, mass, stiffness and damping parameters are chosen to minimize the mean squared motion of the plate at the absorber. It is also possible to choose absorber parameters that minimize the mean squared motion at some other point on the plate, but due to the lengthy computations involved, this application is not considered.

2.2.2. Vibration Absorbers Attached to a Rectangular Plate

A rectangular plate to which n vibration absorbers are attached will now be analyzed. The formulation is developed for the most general case, displayed in Figure 2.4.

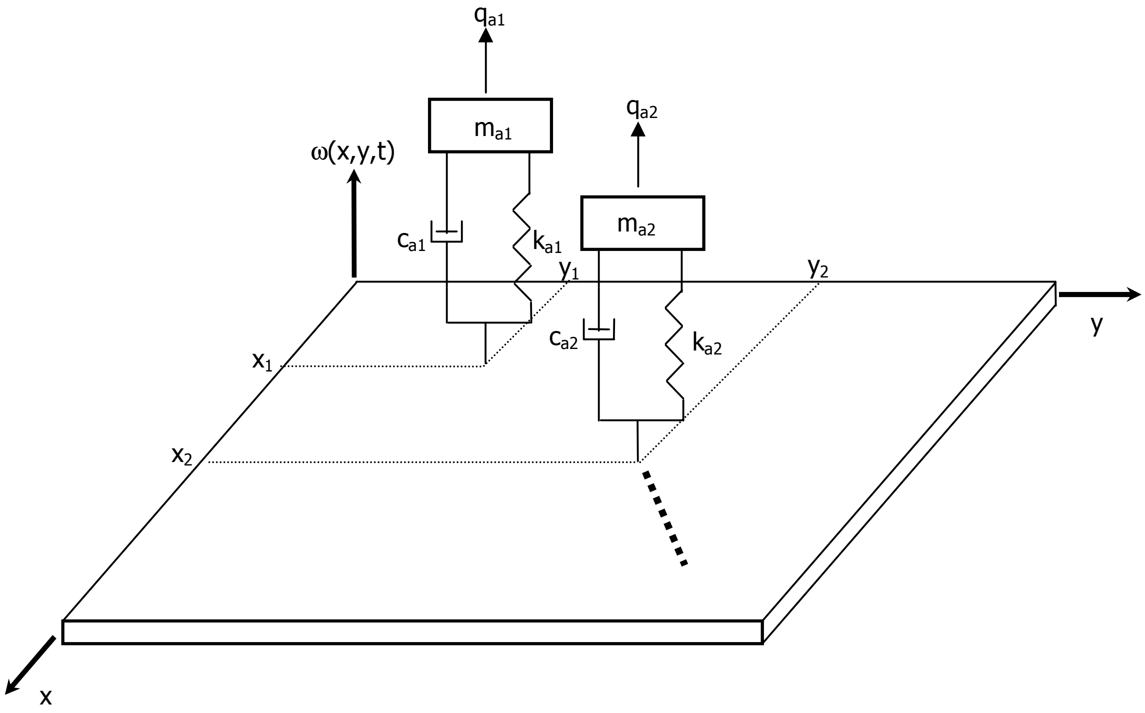

Figure 2.4. Vibration absorbers attached to a rectangular plate

The free vibration of the plate is again assumed to a linear combination of separable functions $u_{ij}(x, y)$, which are functions of the spatial coordinates, multiplied by the time-dependent generalized coordinates, $q_{ij}(t)$.

The transverse displacement $\omega(x, y, t)$ of the rectangular plate is thus of the form

$$\omega(x, y, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_{ij}(x, y) q_{ij}(t), \tag{2.38}$$

where the $q_{ij}(t)$ is the time-dependent generalized coordinates.

2.2.3. Lagrangian Equations of Motion

The general form of Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}+\frac{\partial V}{\partial q_r}=Q_r \qquad r=1........M \tag{2.39}$$

The kinetic energy of a rectangular plate with n absorbers can be expressed as:

$$T(t)=\frac{1}{2}\rho h\int_0^a\int_0^b\left(\frac{\partial \omega(x,y,t)}{\partial t}\right)^2 dxdy+\frac{1}{2}\sum_{i=1}^{n} m_{ai}\dot{q}_{ai}{}^2(t) \tag{2.40}$$

$$T(t)=\frac{1}{2}\rho\, h\int_0^a\int_0^b \dot{q}^2(t)u^2(x,y)dxdy+\frac{1}{2}m_{a1}\dot{q}_{a1}^2(t)+\frac{1}{2}m_{a2}\dot{q}_{a2}^2(t)+..........+\frac{1}{2}m_{an}\dot{q}_{an}^2(t) \tag{2.41}$$

The potential energy of a rectangular plate with n absorbers can be expressed as:

$$V(t)=\frac{1}{2}\frac{Eh^3}{12(1-\nu^2)}\int_0^a\int_0^b\left[\begin{array}{l}\left(\frac{\partial^2\omega(x,y,t)}{\partial x^2}\right)^2+\left(\frac{\partial^2\omega(x,y,t)}{\partial y^2}\right)^2+2\nu\left(\frac{\partial^2\omega(x,y,t)}{\partial x^2}\frac{\partial^2\omega(x,y,t)}{\partial y^2}\right)\\ +2(1-\nu)\left(\frac{\partial^2\omega(x,y,t)}{\partial x\partial y}\right)^2\end{array}\right]dxdy$$
$$+\frac{1}{2}\sum_{i=1}^{n}k_{ai}\left(\omega(x_i,y_i,t)-q_{ai}(t)\right)^2 \tag{2.42}$$

$$V(t)=\frac{1}{2}\frac{Eh^3}{12(1-\nu^2)}\int_0^a\int_0^b\left[\begin{array}{l}\left(q(t)\frac{\partial^2u(x,y)}{\partial x^2}\right)^2+\left(q(t)\frac{\partial^2u(x,y)}{\partial y^2}\right)^2+2\nu\left(q(t)\frac{\partial^2u(x,y)}{\partial x^2}q(t)\frac{\partial^2u(x,y)}{\partial y^2}\right)\\ +2(1-\nu)\left(q(t)\frac{\partial^2u(x,y)}{\partial x\partial y}\right)^2\end{array}\right]dxdy$$
$$+\frac{1}{2}\sum_{i=1}^{n}k_{ai}\left(\omega(x_i,y_i,t)-q_{ai}(t)\right)^2 \tag{2.43}$$

$$Q_r = -\frac{\partial Q_{nc1}}{\partial \dot{q}_k} - \frac{\partial Q_{nc2}}{\partial \dot{q}_k} \tag{2.44}$$

$$\begin{aligned} Q_{nc1} &= \frac{1}{2}c_{a1}\left(\dot{\omega}^2(x_1, y_1, t) - \dot{q}_{a1}(t)\right)^2 + \frac{1}{2}c_{a2}\left(\dot{\omega}^2(x_2, y_2, t) - \dot{q}_{a2}(t)\right)^2 + \dots\dots\dots \\ &+ \frac{1}{2}c_{an}\left(\dot{\omega}^2(x_n, y_n, t) - \dot{q}_{an}(t)\right)^2 \end{aligned} \tag{2.45}$$

The partial derivative of a generalized non-conservative external force with respect to the first derivative of the generalized coordinate is defined as follows:
For the external force (Q_{nc2}),

$$\delta W = \int_0^a\int_0^b f(x,y,t)\delta W(x,y,t) = \int_0^a\int_0^b f(x,y,t)\sum_{i=1}^{M} u_i(x,y)\delta q_i(t)dxdy = -\sum_{i=1}^{M} Q_{nc2}(t)\delta q_i(t) \tag{2.46}$$

$$\frac{\partial Q_{nc2}}{\partial \dot{q}} = -\int_0^a\int_0^b f(x,y,t)u_i(x,y)dxdy \tag{2.47}$$

Substituting the energies and generalized forces into Lagrange's equation yields an equation for each vibration mode and absorber displacement:

$\underline{q_r = q_1(t)}$

$$\begin{aligned} &\rho h\int_0^a\int_0^b\left(\ddot{q}_1(t)u_1^2(x,y) + \ddot{q}_2(t)u_1(x,y)u_2(x,y) + \dots\dots + \ddot{q}_M(t)u_1(x,y)u_M(x,y)\right)dxdy \\ &+ \sum_{j=1}^{n} k_{aj}\left[\left(\sum_{i=1}^{M} q_i(t)u_1(x_j,y_j)u_i(x_j,y_j)\right) - u_1(x_j,y_j)q_{aj}\right] \\ &+ \frac{Eh^3}{12(1-\nu^2)}\left[\begin{aligned} &\int_0^a\int_0^b\sum_{i=0}^{M} q_i(t)\left(\frac{\partial^2 u_1}{\partial x^2}\right)\left(\frac{\partial^2 u_i}{\partial x^2}\right)dxdy + \int_0^a\int_0^b\sum_{i=1}^{M} q_i(t)\left(\frac{\partial^2 u_1}{\partial y^2}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right)dxdy \\ &+ 2\nu\int_0^a\int_0^b\sum_{i=1}^{M} q_i(t)\left(\frac{\partial^2 u_1}{\partial x^2}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right)dxdy + 2(1-\nu)\int_0^a\int_0^b\sum_{i=1}^{M} q_i(t)\left(\frac{\partial^2 u_1}{\partial x\partial y}\right)\left(\frac{\partial^2 u_i}{\partial x\partial y}\right)dxdy \end{aligned}\right] \\ &= \sum_{j=1}^{n} c_{aj}\left[\left(\sum_{i=1}^{M} \dot{q}_i(t)u_1(x_j,y_j)u_i(x_j,y_j)\right) - u_1(x_j,y_j)\dot{q}_{aj}\right] + \int_0^a\int_0^b f(x,t)u_1(x,y)dxdy \end{aligned}$$

(2.48)

$\underline{q_r = q_2(t)}$

$$\rho h \int_0^a \int_0^b \left(\ddot{q}_1(t) u_2(x,y) u_1(x,y) + \ddot{q}_2(t) u_2(x,y) u_2(x,y) + ... + \ddot{q}_M(t) u_2(x,y) u_M(x,y) \right) dxdy$$
$$+ \sum_{j=1}^{n} k_{aj} \left[\left(\sum_{i=1}^{M} q_i(t) u_2(x_j, y_j) u_i(x_j, y_j) \right) - u_2(x_j, y_j) q_{aj} \right]$$
$$+ \frac{Eh^3}{12(1-\nu^2)} \left[\begin{array}{l} \int_0^a \int_0^b \sum_{i=0}^{M} q_i(t) \left(\frac{\partial^2 u_2}{\partial x^2} \right) \left(\frac{\partial^2 u_i}{\partial x^2} \right) dxdy + \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_2}{\partial y^2} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dxdy \\ + 2\nu \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_2}{\partial x^2} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dxdy + 2(1-\nu) \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_2}{\partial x \partial y} \right) \left(\frac{\partial^2 u_i}{\partial x \partial y} \right) dxdy \end{array} \right]$$
$$= \sum_{j=1}^{n} c_{aj} \left[\left(\sum_{i=1}^{M} \dot{q}_i(t) u_2(x_j, y_j) u_i(x_j, y_j) \right) - u_2(x_j, y_j) \dot{q}_{aj} \right] + \int_0^a \int_0^b f(x,t) u_2(x,y) dxdy \tag{2.49}$$

$\vdots$

$\underline{q_r = q_M(t)}$

$$\rho h \int_0^a \int_0^b \left(\ddot{q}_1(t) u_M(x,y) u_1(x,y) + \ddot{q}_2(t) u_M(x,y) u_2(x,y) + ... + \ddot{q}_M(t) u_M(x,y) u_M(x,y) \right) dxdy$$
$$+ \sum_{j=1}^{n} k_{aj} \left[\left(\sum_{i=1}^{M} q_i(t) u_M(x_j, y_j) u_i(x_j, y_j) \right) - u_M(x_j, y_j) q_{aj} \right]$$
$$+ \frac{Eh^3}{12(1-\nu^2)} \left[\begin{array}{l} \int_0^a \int_0^b \sum_{i=0}^{M} q_i(t) \left(\frac{\partial^2 u_M}{\partial x^2} \right) \left(\frac{\partial^2 u_i}{\partial x^2} \right) dxdy + \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_M}{\partial y^2} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dxdy \\ + 2\nu \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_M}{\partial x^2} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dxdy + 2(1-\nu) \int_0^a \int_0^b \sum_{i=1}^{M} q_i(t) \left(\frac{\partial^2 u_M}{\partial x \partial y} \right) \left(\frac{\partial^2 u_i}{\partial x \partial y} \right) dxdy \end{array} \right]$$
$$= \sum_{j=1}^{n} c_{aj} \left[\left(\sum_{i=1}^{M} \dot{q}_i(t) u_M(x_j, y_j) u_i(x_j, y_j) \right) - u_M(x_j, y_j) \dot{q}_{aj} \right] + \int_0^a \int_0^b f(x,t) u_M(x,y) dxdy \tag{2.50}$$

$\underline{q_r = q_{a1}(t)}$

$$\begin{aligned} & m_{a1} \ddot{z}_1 - k_{a1} \left[u_1(x_1, y_1) q_1(t) + u_2(x_1, y_1) q_2(t) + ... + u_M(x_1, y_1) q_M(t) - q_{a1}(t) \right] \\ & = c_{a1} \left[u_1(x_1, y_1) \dot{q}_1(t) + u_2(x_1, y_1) \dot{q}_2(t) + ... + u_M(x_1, y_1) \dot{q}_M(t) - \dot{q}_{a1}(t) \right] \end{aligned} \tag{2.51}$$

$\underline{q_r = q_{a2}(t)}$

$$\begin{aligned} &m_{a2}\ddot{z}_2 - k_{a2}\left[u_1(x_2, y_2)q_1(t) + u_2(x_2, y_2)q_2(t) + ... + u_M(x_2, y_2)q_M(t) - q_{a2}(t)\right] \\ &= c_{a2}\left[u_1(x_2, y_2)\dot{q}_1(t) + u_2(x_2, y_2)\dot{q}_2(t) + ... + u_M(x_2, y_2)\dot{q}_M(t) - \dot{q}_{a2}(t)\right] \end{aligned} \tag{2.52}$$

$\vdots$

$\underline{q_r = q_{an}(t)}$

$$\begin{aligned} &m_{an}\ddot{z}_n - k_{an}\left[u_1(x_n, y_n)q_1(t) + u_2(x_n, y_n)q_2(t) + ... + u_M(x_n, y_n)q_M(t) - q_{an}(t)\right] \\ &= c_{an}\left[u_1(x_n, y_n)\dot{q}_1(t) + u_2(x_n, y_n)\dot{q}_2(t) + ... + u_M(x_n, y_n)\dot{q}_M(t) - \dot{q}_{an}(t)\right] \end{aligned} \tag{2.53}$$

When these expressions are substituted into Lagrange's equation, the matrix equation below Eq. (2.54) is obtained.

$$
\left[\begin{array}{cccc:cccc}
\rho h\int_0^a\int_0^b u_1(x,y)u_1(x,y)dxdy & \rho h\int_0^a\int_0^b u_1(x,y)u_2(x,y)dxdy & \cdots & \rho h\int_0^a\int_0^b u_1(x,y)u_M(x,y)dxdy & 0 & 0 & 0 & 0 \\
\rho h\int_0^a\int_0^b u_2(x,y)u_1(x,y)dxdy & \rho h\int_0^a\int_0^b u_2(x,y)u_2(x,y)dxdy & \cdots & \rho h\int_0^a\int_0^b u_2(x,y)u_M(x,y)dxdy & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots & 0 \\
\rho h\int_0^a\int_0^b u_M(x,y)u_1(x,y)dxdy & \cdots & \cdots & \rho h\int_0^a\int_0^b u_M(x,y)u_M(x,y)dxdy & 0 & 0 & 0 & 0 \\
\hdashline
0 & \cdots & \cdots & 0 & m_{a1} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & m_{a2} & 0 & 0 \\
0 & \cdots & \cdots & 0 & \vdots & 0 & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & m_{an}
\end{array}\right]
\left[\begin{array}{c}
\ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \vdots \\ \ddot{q}_M \\ \hdashline \ddot{q}_{a1} \\ \ddot{q}_{a2} \\ \vdots \\ \ddot{q}_{an}
\end{array}\right]
$$

$$
+\left[\begin{array}{cccc:cccc}
\sum_{i=1}^{n} c_{ai}u_1(x_i,y_i)u_1(x_i,y_i) & \sum_{i=1}^{n} c_{ai}u_1(x_i,y_i)u_2(x_i,y_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_1(x_i,y_i)u_M(x_i,y_i) & -c_{a1}u_1(x_1,y_1) & -c_{a2}u_1(x_2,y_2) & \cdots & -c_{an}u_1(x_n,y_n) \\
\sum_{i=1}^{n} c_{ai}u_2(x_i,y_i)u_1(x_i,y_i) & \sum_{i=1}^{n} c_{ai}u_2(x_i,y_i)u_2(x_i,y_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_2(x_i,y_i)u_M(x_i,y_i) & -c_{a1}u_2(x_1,y_1) & -c_{a2}u_2(x_2,y_2) & \cdots & -c_{an}u_2(x_n,y_n) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{n} c_{ai}u_M(x_i,y_i)u_1(x_i,y_i) & \sum_{i=1}^{n} c_{ai}u_M(x_i,y_i)u_2(x_i,y_i) & \cdots & \sum_{i=1}^{n} c_{ai}u_M(x_i,y_i)u_M(x_i,y_i) & -c_{a1}u_M(x_1,y_1) & -c_{a2}u_M(x_2,y_2) & \cdots & -c_{an}u_M(x_n,y_n) \\
\hdashline
-c_{a1}u_1(x_1,y_1) & -c_{a1}u_2(x_1,y_1) & \cdots & -c_{a1}u_M(x_1,y_1) & c_{a1} & 0 & \cdots & 0 \\
-c_{a2}u_1(x_2,y_2) & -c_{a2}u_2(x_2,y_2) & \cdots & -c_{a2}u_M(x_2,y_2) & 0 & c_{a2} & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & 0 \\
-c_{an}u_1(x_n,y_n) & -c_{an}u_2(x_n,y_n) & \cdots & -c_{an}u_M(x_n,y_n) & 0 & \cdots & 0 & c_{an}
\end{array}\right]
\left[\begin{array}{c}
\dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \vdots \\ \dot{q}_M \\ \hdashline \dot{q}_{a1} \\ \dot{q}_{a2} \\ \vdots \\ \dot{q}_{an}
\end{array}\right]
$$

$$
+\left[\begin{array}{cccc|cccc}
\frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_1(x_i,y_i)u_1(x_i,y_i) & \cdots & \cdots & \frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_M(x_i,y_i)u_1(x_i,y_i) & -k_{a1}u_1(x_1,y_1) & \cdots & \cdots & -k_{an}u_1(x_n,y_n) \\
\frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_1(x_i,y_i)u_2(x_i,y_i) & \cdots & \cdots & \frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_M(x_i,y_i)u_2(x_i,y_i) & -k_{a1}u_2(x_1,y_1) & \cdots & \cdots & -k_{an}u_2(x_n,y_n) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_1(x_i,y_i)u_M(x_i,y_i) & \cdots & \cdots & \frac{Eh^3}{12(1-\nu^2)}RR_{ij}(x,y)+\sum_{i=1}^{n}k_{ai}u_M(x_i,y_i)u_M(x_i,y_i) & -k_{a1}u_M(x_1,y_1) & \cdots & \cdots & -k_{an}u_M(x_n,y_n) \\
\hline
-k_{a1}u_1(x_1,y_1) & \cdots & \cdots & -k_{a1}u_M(x_1,y_1) & k_{a1} & 0 & \cdots & 0 \\
-k_{a2}u_1(x_2,y_2) & \vdots & \cdots & -k_{a2}u_M(x_2,y_2) & 0 & k_{a2} & 0 & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
-k_{an}u_1(x_n,y_n) & \cdots & \cdots & -k_{an}u_M(x_n,y_n) & 0 & \cdots & 0 & k_{an}
\end{array}\right]
\left[\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_M \\ \hline q_{a1} \\ q_{a2} \\ \vdots \\ q_{an} \end{array}\right]
$$

$$
=\left[\begin{array}{c}
\int_0^a\int_0^b f(x,y,t)u_1(x,y)dxdy \\
\int_0^a\int_0^b f(x,y,t)u_2(x,y)dxdy \\
\vdots \\
\int_0^a\int_0^b f(x,y,t)u_M(x,y)dxdy \\
\hline
0 \\ \vdots \\ \vdots \\ 0
\end{array}\right]
\quad \text{where } RR_{ij}(x,y)=\int_0^a\int_0^b\left[\frac{\partial^2u_i(x,y)}{\partial x^2}\frac{\partial^2u_i(x,y)}{\partial x^2}+\frac{\partial^2u_i(x,y)}{\partial y^2}\frac{\partial^2u_i(x,y)}{\partial y^2}+2\nu\frac{\partial^2u_i(x,y)}{\partial x^2}\frac{\partial^2u_i(x,y)}{\partial y^2}+2(1-\nu)\frac{\partial^2u_i(x,y)}{\partial x\partial y}\frac{\partial^2u_i(x,y)}{\partial x\partial y}\right]dxdy
$$

i, j are the respective row and column locations. (2.54)

2.3. State Space Representation of the Equations

The general equation of motion for beams and plates can be written as below:

$$M\ddot{q}_r + C\dot{q}_r + Kq_r = Q_r \text{, or} \tag{2.55}$$

$$\ddot{q}_r = -M^{-1}C\dot{q}_r - M^{-1}Kq_r + M^{-1}Q_r . \tag{2.56}$$

By defining the state variables

$$x = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \\ q_{a1} \\ q_{a2} \\ \vdots \\ q_{an} \\ \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \\ \dot{q}_{a1} \\ \dot{q}_{a2} \\ \vdots \\ \dot{q}_{an} \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \\ \dot{q}_{a1} \\ \dot{q}_{a2} \\ \vdots \\ \dot{q}_{an} \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_M \\ \ddot{q}_{a1} \\ \ddot{q}_{a2} \\ \vdots \\ \ddot{q}_{an} \end{bmatrix} \begin{array}{l} = x_{M+n+1} \\ = x_{M+n+2} \\ \\ = x_{2M+n} \\ \\ \\ \\ = x_{2(M+n)} \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \ddot{q}_r = -M^{-1}C\dot{q}_r - M^{-1}Kq_r + M^{-1}Q_r \end{array} \tag{2.57}$$

The state space representation of the equations of the system is as follows:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \\ \vdots \\ \dot{q}_{an} \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \dot{q}_M \\ \vdots \\ \ddot{q}_{an} \end{bmatrix} = \left[\begin{array}{cccccc|cccccc} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & \vdots & 0 & 1 & 0 & & & 0 \\ \vdots & & \ddots & & & \vdots & \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots & \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & -M^{-1}K & & & & & & -M^{-1}C & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \end{array}\right] \left[\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_M \\ \vdots \\ q_{an} \\ \hline \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \\ \vdots \\ \dot{q}_{an} \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \hline \\ \\ M^{-1}Q_r \\ \\ \\ \end{array}\right] F \quad (2.58)$$

Output of the system is $y(t) = Cx$.

If the aim is to minimize the vibration amplitudes for a uniform beam at $x = l_{beam}/2$, the output function is as follows:

$$\omega(l_{beam}/2,t) = u_1(l_{beam}/2)q_1(t) + u_2(l_{beam}/2)q_2(t) + u_3(l_{beam}/2)q_3(t) + \ldots\ldots\ldots\ldots + u_M(l_{beam}/2)q_M(t) \quad (2.59)$$

$$y = \left[\begin{array}{cccccccccc|ccc} u_1(l_{beam}/2) & u_2(l_{beam}/2) & u_3(l_{beam}/2) & \cdots & \cdots & u_M(l_{beam}/2) & 0 & \cdots & 0 & & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_M \\ \vdots \\ q_{an} \\ \hline \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \\ \vdots \\ \dot{q}_{an} \end{array}\right] \quad (2.60)$$

2.4. Optimization of Absorber Parameters in Continuous Mechanical Systems

This section develops a method to find the optimal absorber parameters. The goal is to minimize the amplitude of vibration over a specified frequency range, a specific resonance, or at specific locations. A mixed goal can also be defined, for example with different weights given to displacements at different frequencies or locations on the beam and plate. A general framework for the optimization process is displayed in Figure 2.5. $\mathbf{G}_{\mathbf{sys}}(s)$ is the main system transfer function, while $\mathbf{W}_{\mathbf{f}}(s)$ and $\mathbf{W}_{\mathbf{y}}(s)$ are weighting functions for the input (force) and output (displacement), respectively.

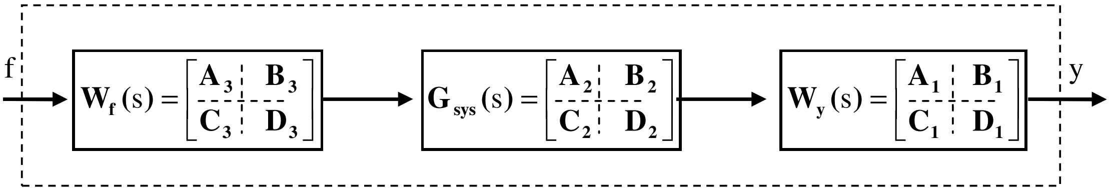

Figure 2.5. Transfer function diagram

For single-input single-output (SISO) systems, the corresponding state space representation of Figure 2.5 (constituting $W_f(s), G_{sys}(s), W_y(s)$) is shown below respectively:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1C_2 & B_1D_2C_3 \\ 0 & A_2 & B_2C_3 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_1D_2D_3 \\ B_2D_3 \\ B_3 \end{bmatrix} f \tag{2.61}$$

$$y = \begin{bmatrix} C_1 & D_1C_2 & D_1D_2C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} D_1D_2D_3 \end{bmatrix} f \tag{2.62}$$

Equations (2.61) and (2.62) can be combined as follows:

$$G_{tot}(s) = \left[\begin{array}{ccc:c} A_1 & B_1C_2 & B_1D_2C_3 & B_1D_2D_3 \\ 0 & A_2 & B_2C_3 & B_2D_3 \\ 0 & 0 & A_3 & B_3 \\ \hdashline C_1 & D_1C_2 & D_1D_2C_3 & D_1D_2D_3 \end{array}\right] \text{(one-input one-output)} \tag{2.63}$$

If $W_y(s)$ does not exist, the state space representation is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1.D_2 \\ B_2 \end{bmatrix} f \tag{2.64}$$

$$y = \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1D_2 \end{bmatrix} f\ . \tag{2.65}$$

In this case, multi-input multi-output (MIMO) system is considered. For a two-input two-output system, $G_{tot}(s)$ is constructed as follows. Taking the transfer functions $G_i(s)$ between each input-output pair,

$$G_i(s) = \left[\begin{array}{c:c} A_i & B_i \\ \hdashline C_i & D_i \end{array}\right] \text{ where } i = 1,...,4\ , \tag{2.66}$$

$G_{tot}(s)$ is found as

$$G_{tot}(s) = \left[\begin{array}{cccc:cc} A_1 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 & 0 \\ 0 & 0 & 0 & A_4 & 0 & B_4 \\ \hdashline C_1 & C_2 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & C_3 & C_4 & D_3 & D_4 \end{array}\right] \text{(two inputs-two outputs).} \tag{2.67}$$

$G_{tot}(s)$ can be obtained in the same way for larger numbers of inputs and outputs.

The objective of a control system is to achieve certain performance specifications while providing internal stability. One way to describe the performance of a control system is by measuring certain signals of interest, or more often errors and variations in the signals. For example, the performance of a tracking system could be measured by the size of the tracking error. The choice of the signal norm is therefore quite important; the most appropriate formula depends on the situation at hand.

The H_2 and H_∞ norms [51, 52, 53] are presented below, and the choice for continuous systems is described. The discussion of these norms relies on the following notation:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right] \text{(a state space representation of the total system)} \tag{2.68}$$

$$G_{tot}(s) = C(sI - A)^{-1}B + D = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \tag{2.69}$$

System norms serve as a measure of system size and are commonly used in model reduction and procedures for actuator/sensor/absorber placement. The system norms: H_2 and H_∞ are analyzed. It is shown that for flexible structures the H_2 norm has an additive property : it is a root-mean-square sum of the norms of individual modes. The H_∞ norm is determined from the corresponding modal norms, by choosing the largest one. All the norms of a structure with multiple inputs (or outputs) can be decomposed into the root-mean-square sum of norms of a structure with a single input (or output).

2.4.1. H_2 Norm

Consider a *strictly proper* system with transfer matrix $G(s)$, meaning that $D = 0$ in a state space realization. $G(s)$ is a specific case of Eq. (2.69). The $\|G\|_2$ norm is defined as

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} trace\{G^*(j\omega)G(j\omega)\}d\omega} \,. \tag{2.70}$$

Consider the transfer matrix

$$G_{tot}(s) = C(sI - A)^{-1} B = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \tag{2.71}$$

where A is stable. Then

$$\|G\|_2^2 = trace(B^* Q B) = trace(C^* P C), \tag{2.72}$$

where Q and P are the "observability" and "controllability" Gramians respectively. They can be obtained by solving the following Lyapunov equations:

$$AP + PA^T + BB^T = 0 \tag{2.73}$$

$$A^T Q + QA + C^T C = 0 \tag{2.74}$$

The controllability Gramian of (A, B) and the observability gramian of (C, A) can be represented as

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt \ , \ P = \int_0^\infty e^{At} BB^T e^{A^T t} dt \ . \qquad (2.75),(2.76)$$

2.4.2. H_∞ Norm

Consider a proper, linearly stable system $G(s)$ (i.e., $D \neq \mathbf{0}$ is allowed). The $\|G\|_\infty$ norm is defined as

$$\|G\|_\infty = \sup \overline{\sigma}(G(j\omega)). \tag{2.77}$$

The H_∞ norm of a single-input single-output system is the peak of the transfer function magnitude (in terms of its singular values). The H_∞ norm can be interpreted as the magnitude of some closed loop transfer function relative to a specified upper bound.

2.4.3. Differences between the H_2 and H_∞ Norms

H_∞ refers to the set of transfer functions with a bounded ∞ norm (the symbol ∞ comes from the maximum magnitude over frequency), which is simply the set of stable and proper transfer functions. Similarly, the symbol H_2 stands for the Hardy space of transfer functions with a bounded 2-norm, which is the set of stable and strictly proper transfer functions.

The H_∞ norm is convenient for representing unstructured model uncertainty, because it satisfies the multiplicative property.

$$\|AB\|_\infty \leq \|A\|_\infty . \|B\|_\infty \tag{2.78}$$

The H_2 norm has a number of useful mathematical and numerical properties, and its minimization has important engineering implications. However, the H_2 norm does not satisfy the multiplicative property above. This implies that evaluating the H_2 norm of individual components can not give an idea about the behavior of their series interconnection. The 2-norm of the transfer function can also be expressed in terms of singular values:

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2 (G(j\omega))d\omega} . \tag{2.79}$$

From the above equation, it can be concluded that minimizing the H_∞ norm corresponds to minimizing the peak of the largest singular value (worst direction, worst frequency), whereas minimizing the H_2 norm corresponds to minimizing the sum of squares of all the singular values over all frequencies (average direction, average frequency).

In summary, H_∞ optimization will push down the largest singular value, while H_2 optimization pushes down the whole frequency range.

In this study, the optimization is performed using the squared H_2 norm ($\|G_{tot}\|_2^2$) of the system:

$$\|G_{tot}\|_2^2 = CLC^T . \tag{2.80}$$

The matrix L satisfies the Lyapunov equation,

$$AL + LA^T + BB^T = 0 . \tag{2.81}$$

Thus, by minimizing the H_2 norm, the output power of the generalized system is minimized [54].

To begin the optimization process, an initial estimate of the absorber parameters is needed. From this starting point, the goal is to find a constrained minimum of the H_2 norm over the entire system's transfer function at a specific location or combination of locations.

The best location for a vibration absorber attached to a beam is the point which has the maximum amplitude of vibration at the resonance to which the absorber is tuned. As far as cantilever beams are concerned, a vibration absorber tuned to the first, second or a higher resonance is most effective when it is attached to the free end. These results will be verified by numerical examples studied in the following chapter.

This method described in this chapter is executed in Matlab. It is also possible to analyze systems with more than one absorber. The required inputs are as follows:

- Elastic modulus, moment of inertia, mass and dimensions of the beam or plate;
- Structure of the beam or plate;
- Magnitudes and locations of external forces applied to the beam or plate;
- Number and structure of the absorbers;
- Masses of the absorbers.

The outputs are as follows:

- The optimal absorber parameters (locations, stiffness and damping coefficients) meeting specified performance criteria (constrained minimization, goal function with weighing, etc.).

The following tasks can be fulfilled:

- Analyze a system with n absorbers and M initial modes, where n and M are any positive integers.
- Display plots of the displacement versus the frequency (including all displacements).
- Display plots comparing the "without absorber", "with randomly selected absorber" and "with optimum absorber" cases.

3. CASE STUDIES FOR VIBRATION SUPPRESSION OF UNIFORM BEAMS AND RECTANGULAR PLATES

Numerical studies on physical systems are performed to verify the developed method in the previous chapter.

3.1. Case Study 1

A case where one absorber is attached to a pinned–pinned beam modeled with initial five mode shapes is studied (see Figure 3.1).

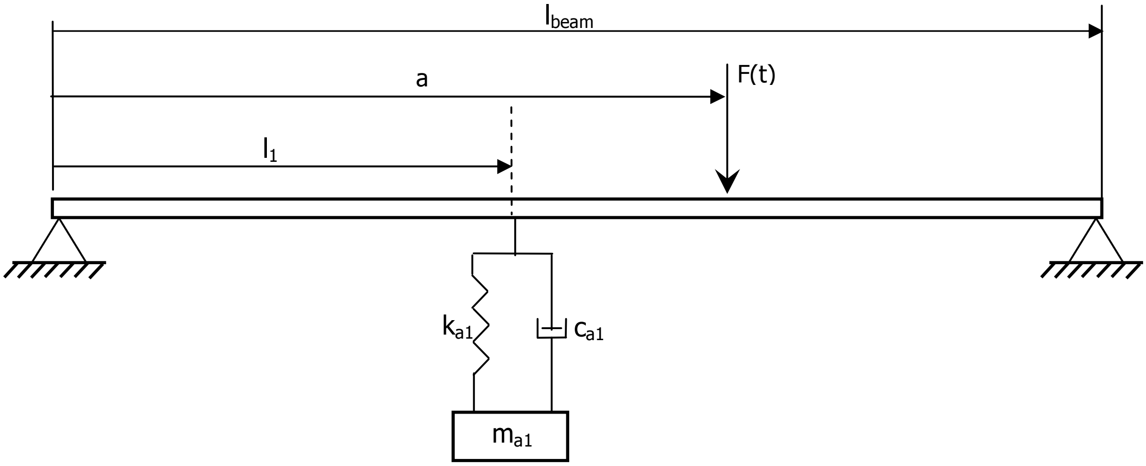

Figure 3.1. One absorber attached to a pinned–pinned beam

The goal of optimization is to minimize the vibration amplitude at $x = \mathrm{l}_{\mathrm{beam}}/2$. The beam material is selected as aluminum alloy 6061-T6 [55], with a square cross section of 0.004 m × 0.004 m. An external force (F) is applied at the point $x = 3\mathrm{l}_{\mathrm{beam}}/4$. The input weighting function is

$$\mathrm{W}_{\mathrm{f}}(s) = \left| \frac{\omega_f^{\ 2}}{s^2 + 2\xi\,\omega_f\, s + \omega_f^{\ 2}} \right|. \tag{3.1}$$

The numerical values of the physical parameters are given as: $\xi = 0.18$, $\omega_f = 500\,\text{rad/s}$, $E = 70*10^9\ \text{Pa}$, $I = 2.1333*10^{-11}\ \text{m}^4$, $m_{beam} = 0.0434\,\text{kg}$, $l_{beam} = 1\,\text{m}$, and $m_{a1} = 0.0087\,\text{kg}$. The mode shapes are displayed below:

$$MS = \sin\left(\frac{n\pi x}{l_{beam}}\right) \quad n = 1,2,3,4,5. \tag{3.2}$$

Focusing on the first resonance, Figure 3.2 compares the beam responses with optimal absorber parameters and randomly selected absorber parameters. The randomly selected parameters are $l_1 = 0.65\,\text{m}$, $k_{a1} = 300\,\text{N/m}$, and $c_{a1} = 2\,\text{N s/m}$. The tolerances are so narrow that regardless of the initial parameter estimates, the output values converge to the same absorber configuration.

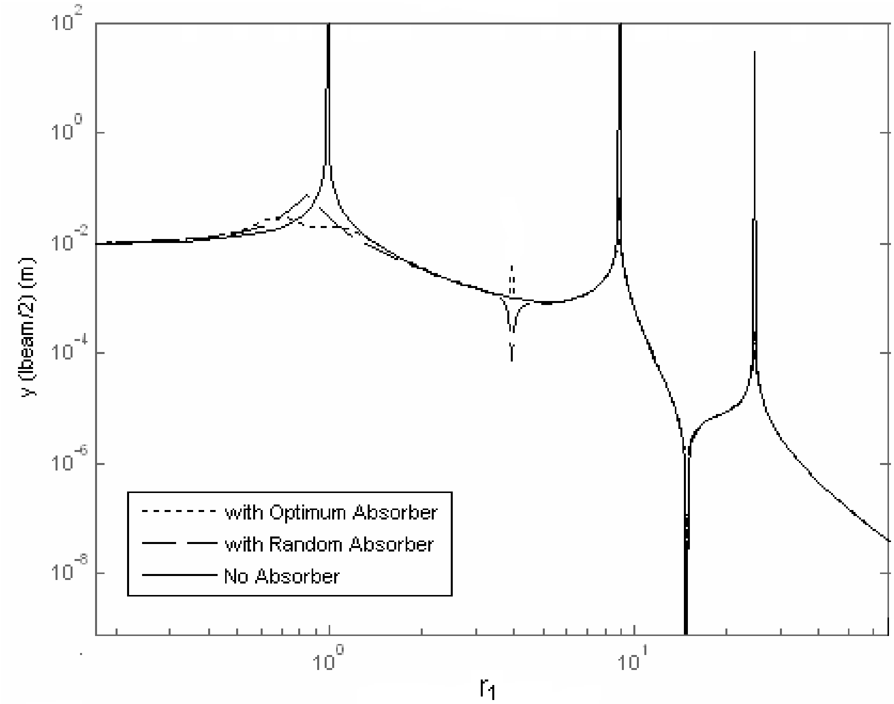

Figure 3.2. Displacement of the beam at x=l_{beam}/2 – Case Study 1

When a vibration absorber is tuned to a specific resonance, the original peak is replaced by two smaller peaks around the primary system's resonant frequency, one is due to the absorber and the other is due to the original resonance. The optimum parameters are found as: $\mathrm{l_1 = 0.4969\,m}$, $\mathrm{k_{a1} = 17.9425\,N/m}$, and $\mathrm{c_{a1} = 0.2517\,N\,s/m}$.

For non-dimensional representation of the solution, the following non-dimensional parameters are introduced.

$$\mu = \frac{m_{a1}}{m_{beam}} \text{ (Mass ratio)} \tag{3.3}$$

$$Y_1 = \frac{c_{a1}}{2m_{a1}\omega_1} \text{ (Damping ratio)} \tag{3.4}$$

$$\beta_1^{\ 2} = \frac{k_{a1}}{m_{a1}\omega_1^2} \text{ (Tuning ratio)} \tag{3.5}$$

$$r_i = \frac{\omega_i}{\omega_1} \text{ (Frequency ratio)} \tag{3.6}$$

where ω_1 is the first resonance frequency of the beam.

The tuning ratio (β) and damping ratio (Y) are known as the absorber parameters, because for given mass ratios, the spring stiffnesses and viscous damping coefficients of the absorbers can be found. It is possible to vary the spring and damping elements such that a specific resonance in the response of the vibrating structure will be reduced to a minimum value. The process of adjusting absorber parameters to get the best possible response curve around a specified resonance is called "tuning".

The effectiveness of the vibration absorber is largely dependent on the mass ratio (μ) and attachment point of the absorber. The optimum absorber parameters, k_1, c_1, β_1,

Y_1 are plotted as a function of μ in Figures 3.3, 3.4, 3.5 and 3.6, respectively. The square of H_2 norm of the system with respect to μ is plotted in Figure 3.7.

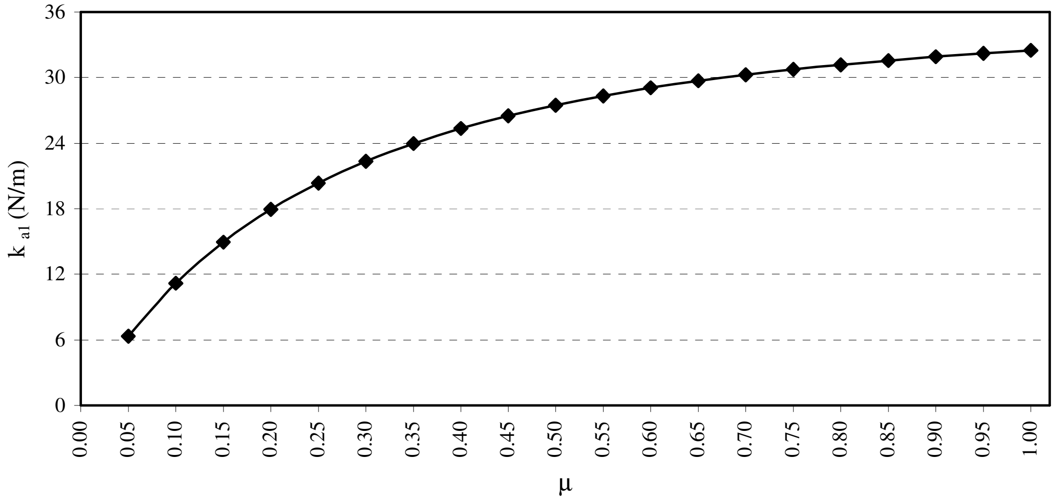

Figure 3.3. Optimum value of absorber stiffness (k_{a1}) vs. mass ratio (μ)

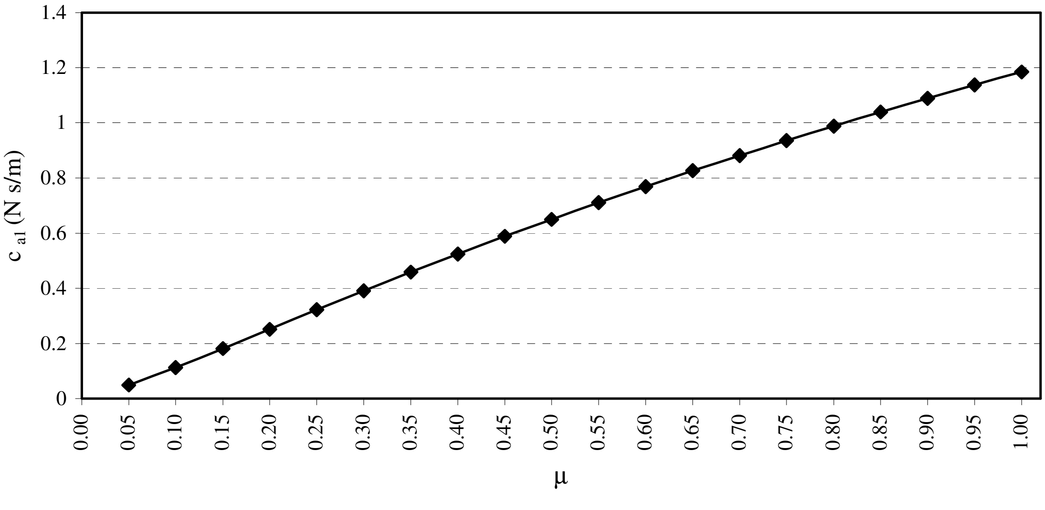

Figure 3.4. Optimum value of absorber damping (c_{a1}) vs. mass ratio (μ)

As mass ratio (μ) increases, the optimum value of spring constant (k_{a1}) and the optimum value of damping constant (c_{a1}) increase. The increase amount of stiffness value is considerably sensed in the lower range of mass ratio values. But, for the high values of mass ratio, the increase rate of stiffness value decreases.

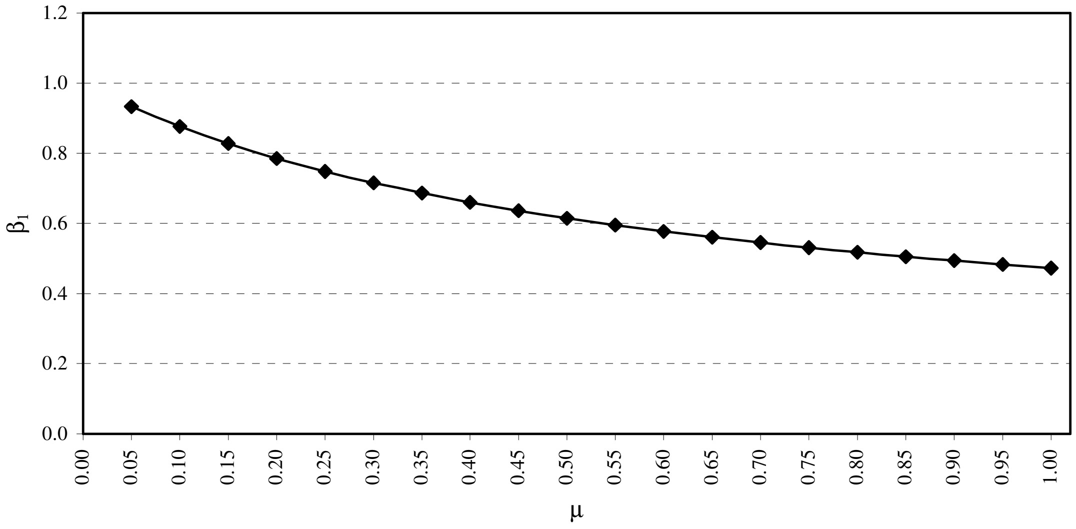

Figure 3.5. Optimum value of tuning ratio (β_1) vs. mass ratio (μ)

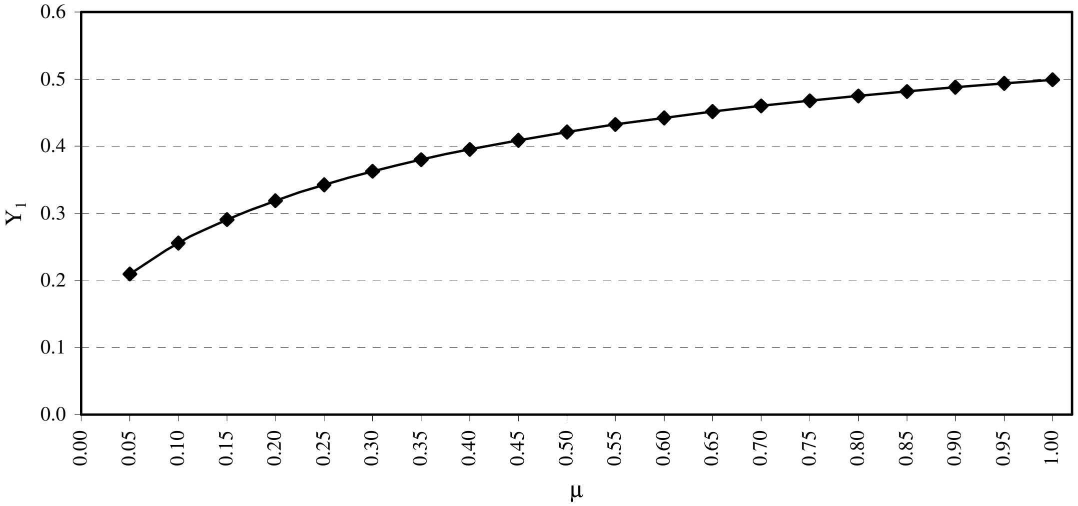

Figure 3.6. Optimum value of damping ratio (Y_1) vs. mass ratio (μ)

As mass ratio (μ) increases, the optimum value of tuning ratio (β_1) decreases and the optimum value of damping ratio (Y_1) increases. The main purpose of increasing μ is to reduce the value of H_2 norm of the system.

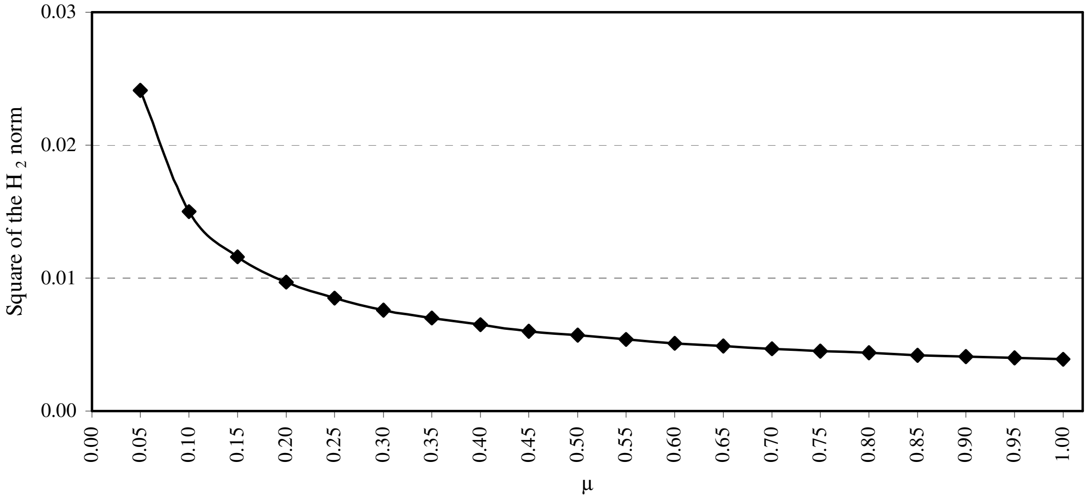

Figure 3.7. Minimized square of the H_2 norm ($\left\|G_{tot}\right\|_2^2$) vs. mass ratio ($\mu$)

If a vibration absorber is attached to a position different from the mid-point of the pinned-pinned beam (instead of concentrating on the first mode shape), the H_2 norm of the system increases, i.e. the performance which could be obtained with a specific mass ratio decreases. With the increasing mass ratio, the H_2 norm of the system decrease rapidly (displayed in Figure 3.7) and the rate of decrease drops after a certain value of the mass ratio. Therefore, in practice, it can be said that values of mass ratio (μ) higher than 0.4 do not provide much benefit.

3.2. Case Study 2 (with Different Locations of the Forcing and Different Forcing Frequencies)

This case study uses the system shown in Figure 3.1, but the location and the frequency of the external force are varied. The goal is to see whether the optimum absorber parameters are sensitive to these changes.

Figure 3.8 shows the optimum location of the absorber with respect to the forcing frequencies and locations. The numerical results are reported in Table 3.1. When the force is applied to the middle of the beam (a = 0.5 m), whatever the forcing frequency is, the optimum location for the absorber is a = 0.5 m. As the applied force moves toward one end of the beam, the absorber location shifts slightly away from the midpoint toward the other end.

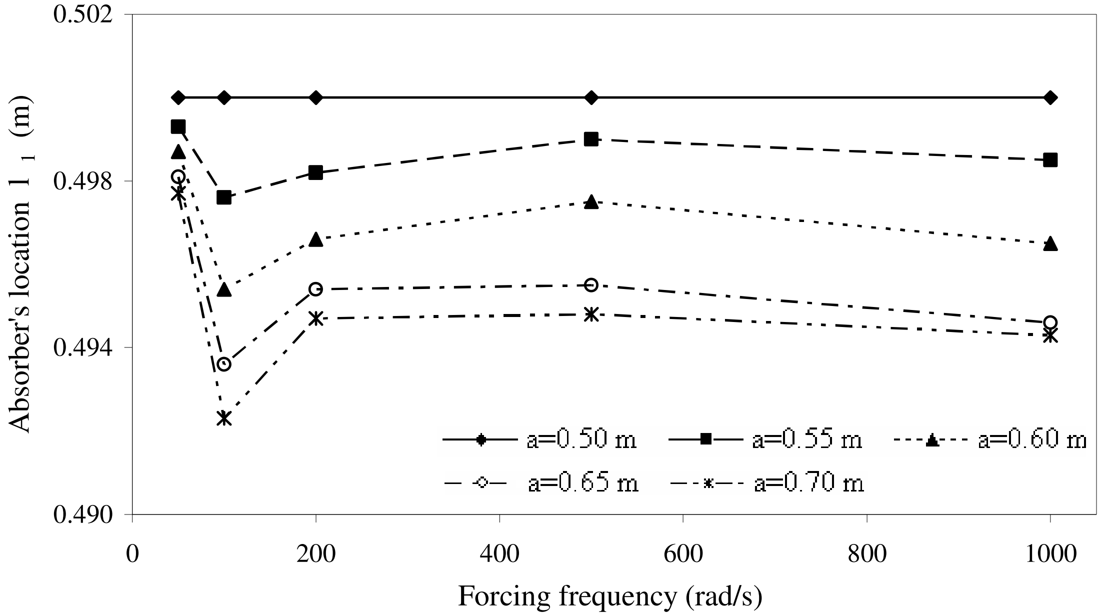

Figure 3.8. Optimum absorber locations vs. forcing frequencies and locations

Table 3.1. Optimum absorber locations vs. forcing frequencies and locations

ω (rad/sec) \ a (m)	0.50	0.55	0.60	0.65	0.70
50	0.5000	0.4993	0.4987	0.4981	0.4977
75	0.5000	0.4973	0.4948	0.4928	0.4913
100	0.5000	0.4976	0.4954	0.4936	0.4923
125	0.5000	0.4980	0.4962	0.4947	0.4936
150	0.5000	0.4981	0.4963	0.4949	0.4939
175	0.5000	0.4981	0.4964	0.4950	0.4941
200	0.5000	0.4982	0.4966	0.4954	0.4947
500	0.5000	0.4990	0.4975	0.4955	0.4948
1000	0.5000	0.4985	0.4965	0.4946	0.4943

The stiffness and damping values of the absorber are displayed in Figures 3.9 and 3.10, respectively, and tabulated in Tables 3.2 and 3.3. The optimum stiffness value increases slightly as the forcing location moves away from the beam’s center. The same tendency can be observed in the damping values. Above $\omega_f = 200\,\text{rad}/s$, however, the damping value trend is reversed

Table 3.2. Optimum absorber stiffness vs. forcing frequencies and locations

a (m) / ω (rad/sec)	0.50	0.55	0.60	0.65	0.70
50	16.4414	16.4463	16.4601	16.4800	16.5023
75	21.9074	21.9204	21.9565	22.0079	22.0647
100	19.5881	19.5984	19.6271	19.6680	19.7134
125	18.7180	18.7264	18.7498	18.7831	18.8198
150	18.3763	18.3846	18.4074	18.4397	18.4753
175	18.1937	18.2020	18.2249	18.2572	18.2926
200	18.0809	18.0891	18.1118	18.1437	18.1783
500	17.8046	17.8084	17.8220	17.8462	17.8929
1000	17.7484	17.7559	17.7782	17.8127	17.8496

Table 3.3. Optimum absorber damping vs. forcing frequencies and locations

a (m) / ω (rad/sec)	0.50	0.55	0.60	0.65	0.70
50	0.1296	0.1297	0.1298	0.1300	0.1303
75	0.2605	0.2607	0.2613	0.2621	0.2630
100	0.2689	0.2693	0.2703	0.2717	0.2734
125	0.2461	0.2464	0.2472	0.2484	0.2498
150	0.2348	0.2351	0.2358	0.2368	0.2379
175	0.2292	0.2295	0.2301	0.2309	0.2319
200	0.2261	0.2263	0.2268	0.2276	0.2285
500	0.2492	0.2440	0.2316	0.2213	0.2256
1000	0.2271	0.2255	0.2220	0.2199	0.2221

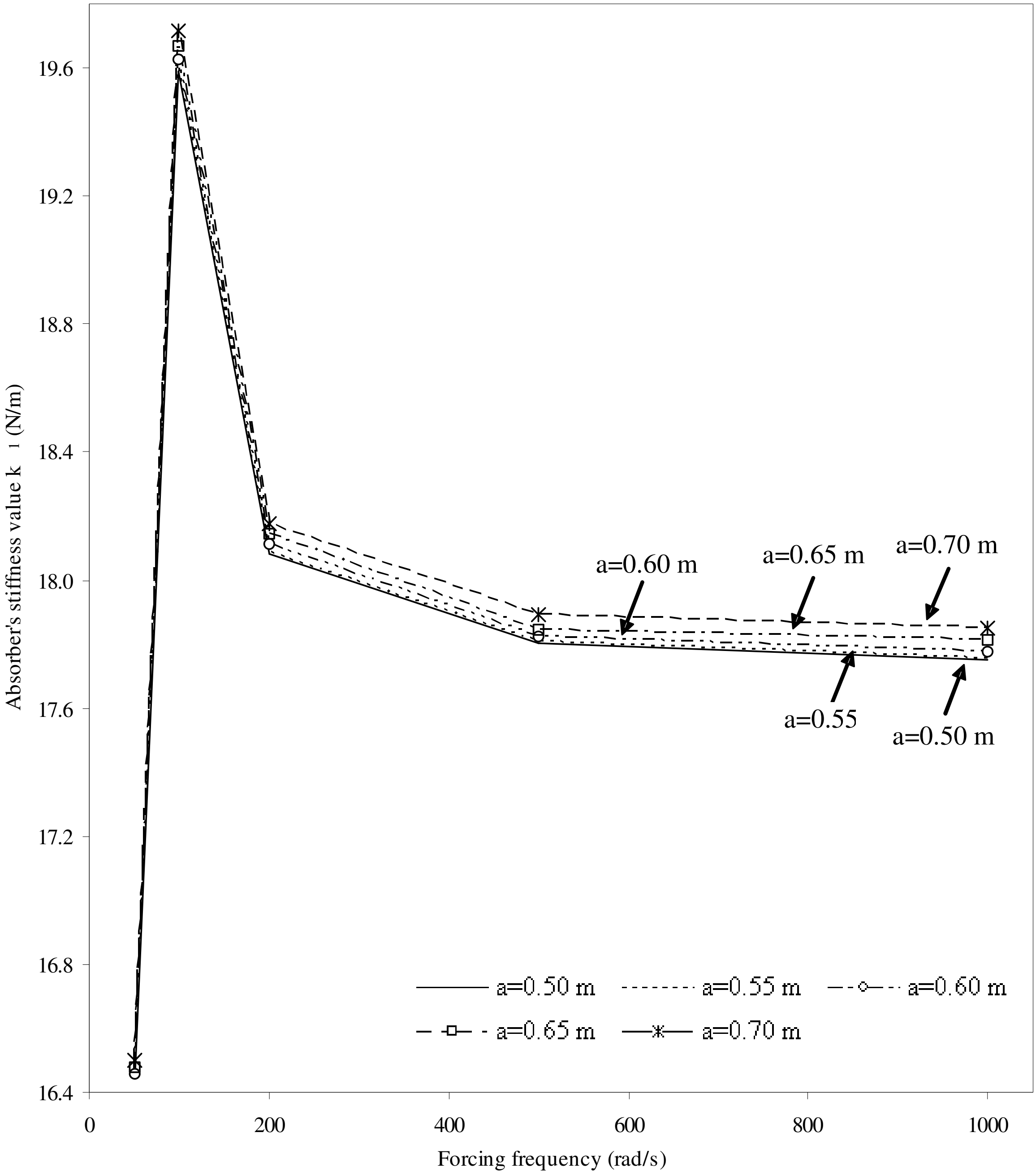

Figure 3.9. Optimum absorber stiffness vs. forcing frequencies and locations

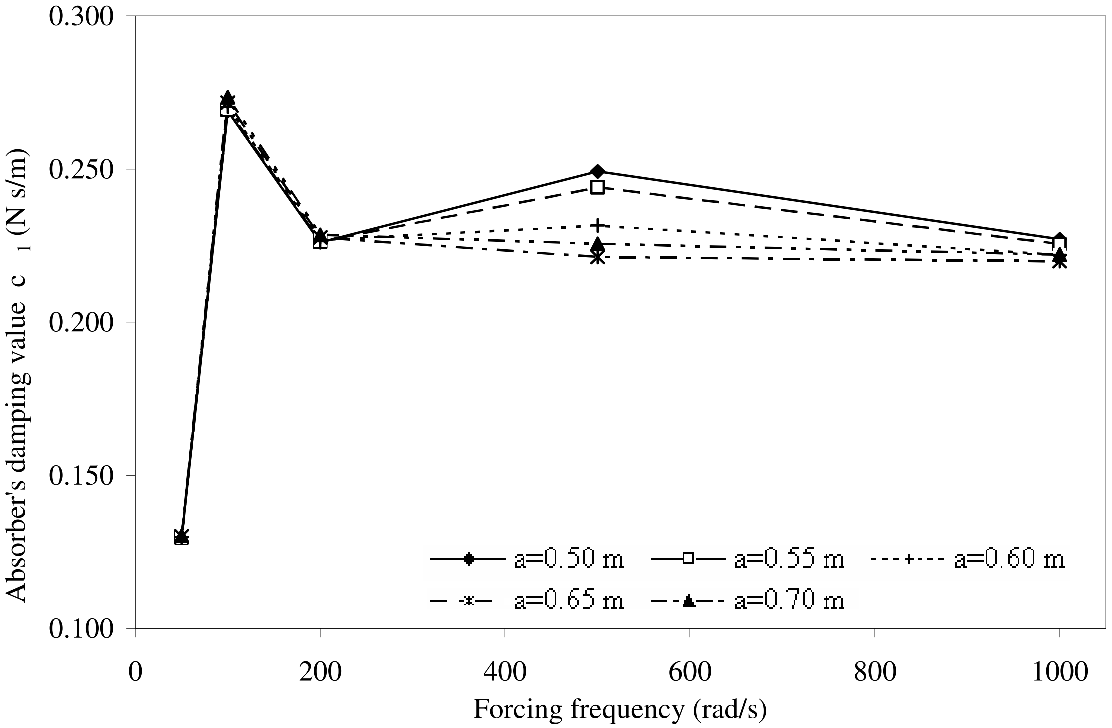

Figure 3.10. Optimum absorber damping vs. forcing frequencies and locations

The stiffness and damping values of the absorber are displayed in Figures 3.9 and 3.10, respectively, and tabulated in Tables 3.2 and 3.3. The optimum stiffness value increases slightly as the forcing location moves away from the beam's center. The same tendency can be observed in the damping values. Above $\omega_f = 200\,\text{rad}/s$, however, the damping value trend is reversed.

Table 3.4. Minimized H_2 norm vs. forcing frequencies and locations

ω (rad/sec) \ a (m)	0.50	0.55	0.60	0.65	0.70
50	0.0452	0.0440	0.0407	0.0356	0.0292
75	0.0471	0.0459	0.0426	0.0373	0.0308
100	0.0328	0.0320	0.0297	0.0261	0.0216
125	0.0257	0.0251	0.0233	0.0205	0.0170
150	0.0224	0.0219	0.0203	0.0179	0.0148
175	0.0206	0.0201	0.0187	0.0164	0.0136
200	0.0196	0.0191	0.0177	0.0156	0.0129
500	0.0194	0.0185	0.0163	0.0136	0.0115
1000	0.0175	0.0170	0.0155	0.0134	0.0111

Figure 3.11 displays the square of the H_2 norm of the total system transfer function, ($\|G_{tot}\|_2^2$), which was defined as the minimization goal to determine the optimal absorber coefficients. The values of this norm are presented in Table 3.4. As the forcing location approaches the endpoint of the beam, the disturbance due to forcing and also the minimum value of the H_2 norm decrease. In order to suppress the vibration amplitudes to a high level, the H_2 norm is targeted to be as small as possible.

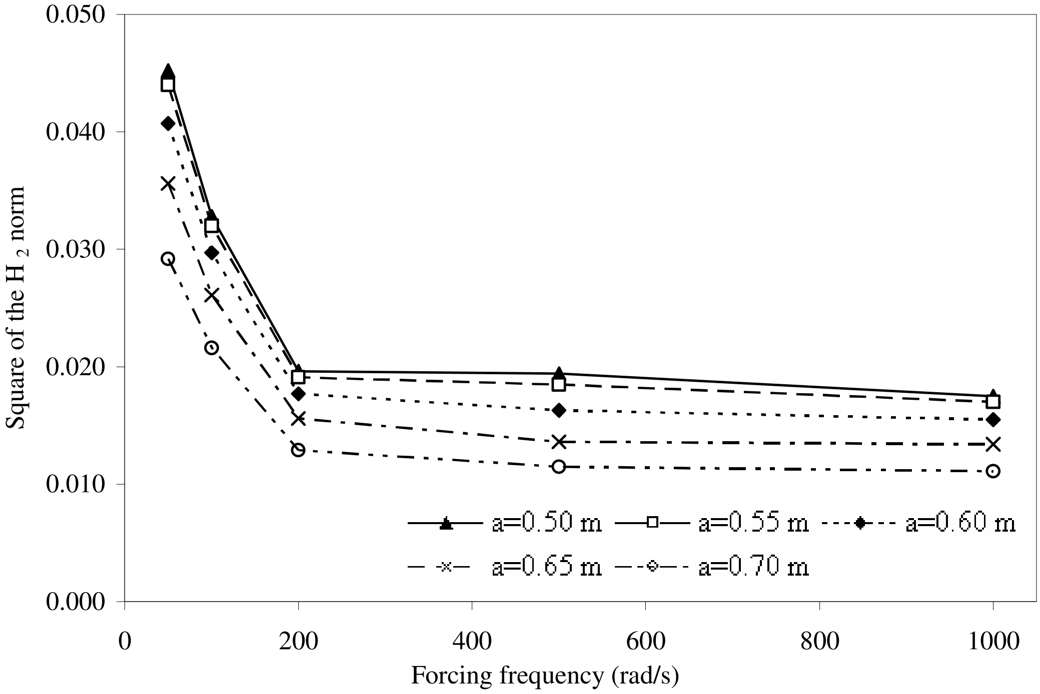

Figure 3.11. Minimized square of the H_2 norm vs. forcing frequencies and locations

3.3. Case Study 3 (with Five Absorbers)

This case uses the same beam material and structure as in Section 3.1 (Case Study 1). An external force (F) is applied at $x = 0.65\,l_{\text{beam}}$. Five absorbers are attached to a pinned-pinned beam which is modeled with six modes as displayed in Figure 3.12. The aim is to minimize the vibration amplitudes at $x = l_{\text{beam}}/2$. Fifteen different parameters must be considered in the optimization package, three for each absorber. A comparison between cases with no absorber, randomly selected absorbers, and the optimal absorbers is shown in Figure 3.13. The optimal values for the absorbers are as follows: l_1 = 0.5101 m, l_2 = 0.5048 m, l_3 = 0.4697 m, l_4 = 0.4945 m, l_5 = 0.4383 m, and k_{a1} = 4.8399 N/m, k_{a2} = 7.9668 N/m, k_{a3} = 30.7913 N/m, k_{a4} = 14.1465 N/m, k_{a5} = 86.2487 N/m, and c_{a1} = 0.0497 N s/m, c_{a2} = 0.0756 N s/m, c_{a3} = 0.2242 N s/m, c_{a4} = 0.1225 N s/m, and c_{a5} = 0.2251 N s/m.

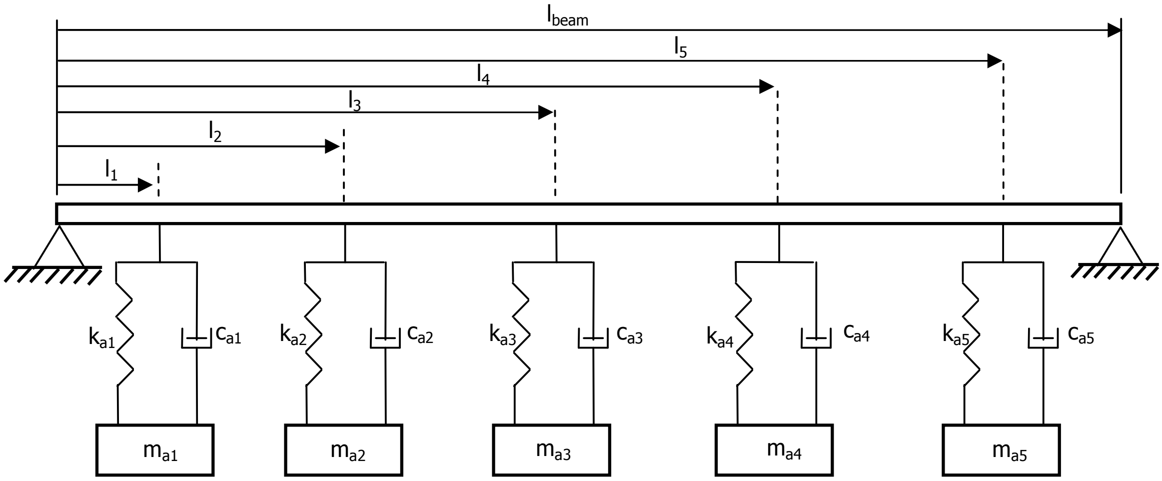

Figure 3.12. Five absorbers attached to a pinned-pinned beam (six mode shapes)

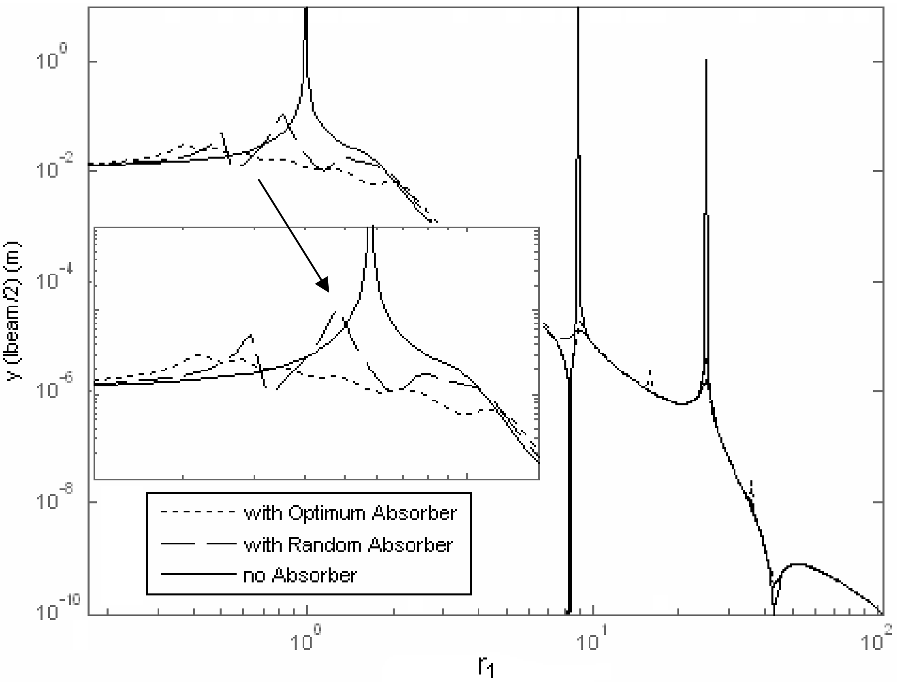

Figure 3.13. Displacement of the beam at x=l_{beam}/2 – Case Study 3

3.4. Case Study 4 (with Five Modes)

The optimum absorber parameters depend upon several factors: the point of forcing, the frequency of the applied force, and the point at which the response is to be minimized. In this case study, the beam system is modeled with five mode shapes and two absorbers (Figure 3.14). The external force is applied at $x = 0.75\,l_{beam}$. The optimization goal is to minimize the vibration amplitude at $x = 0.25l_{beam}$.

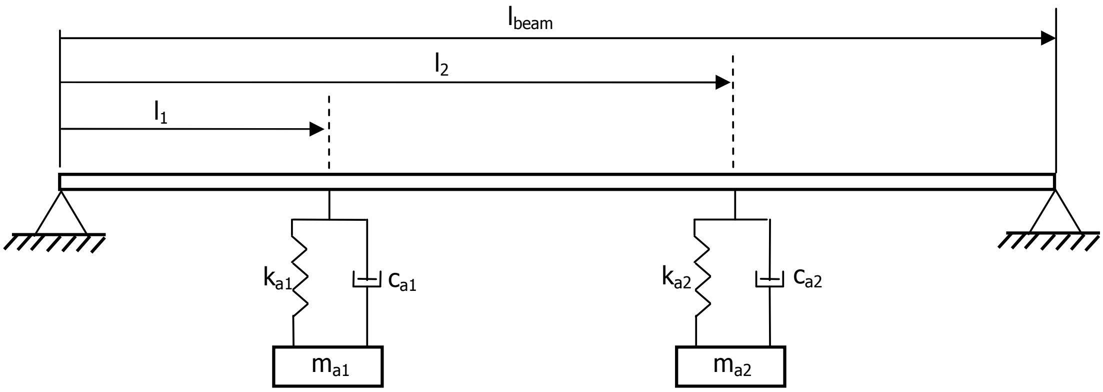

Figure 3.14. Two absorbers attached to a pinned-pinned beam (five mode shapes)

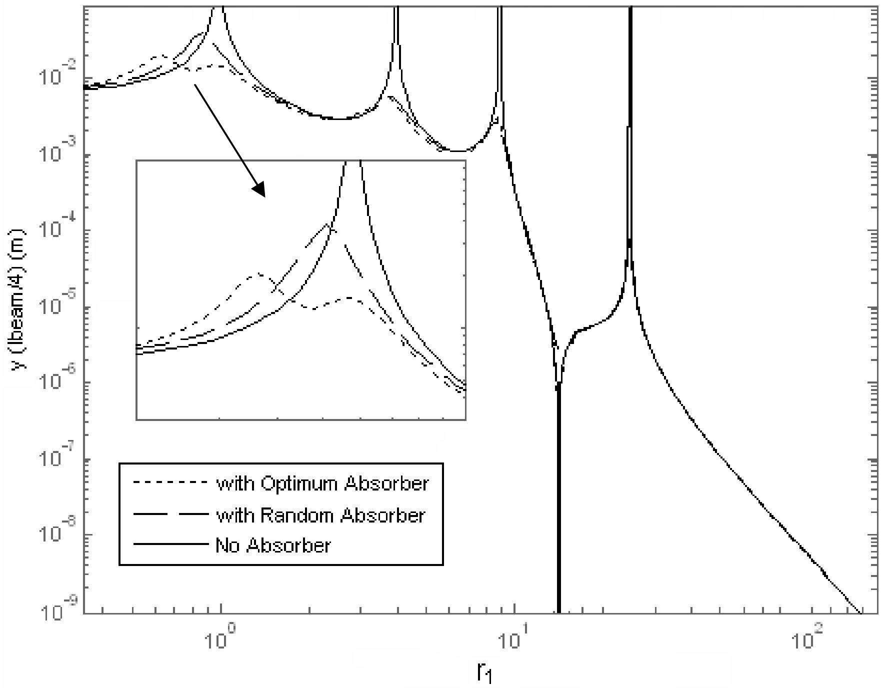

Figure 3.15. Displacement of the beam at x=l_{beam}/4

A comparison between cases with no absorber, randomly selected absorbers, and the optimal absorbers is shown in Figure 3.15. The optimal values for the absorbers are found as follows: l_1 = 0.5057 m, l_2 = 0.7596 m, k_{a1} = 15.2700 N/m, k_{a2} = 61.8525 N/m, c_{a1} = 0.1843 N s/m, and c_{a2} = 1.1590 N s/m.

For non-dimensional representation of the solution, the following input conditions are considered. The same beam material used in case study 1 is selected. An external force (F) is applied at $x = 0.5\,l_{\text{beam}}$. The aim is to minimize the vibration amplitudes at $x = l_{\text{beam}}/2$. The optimum absorber parameters, k_1, c_1, β, Y are plotted for various values of μ_1 and μ_2 in Figures 3.16, 3.17, 3.18 and 3.19, respectively. The square of H_2 norm of the system with respect to μ_1 and μ_2 is plotted in Figure 3.20.

As mass ratio μ_1 increases and μ_2 decreases, the optimum value of spring constant (k_{a1}) and the optimum value of damping constant (c_{a1}) increase. It can be concluded when the second absorber mass increase, the first absorber spring and damping constant decrease. Due to the existence and high mass ratio of the second absorber (μ_2), the first absorber parameters (k_{a1}, c_{a1}) decrease. For fixed μ_1 value, there is an indirect correlation between the μ_2 and k_{a1}, c_{a1} values.

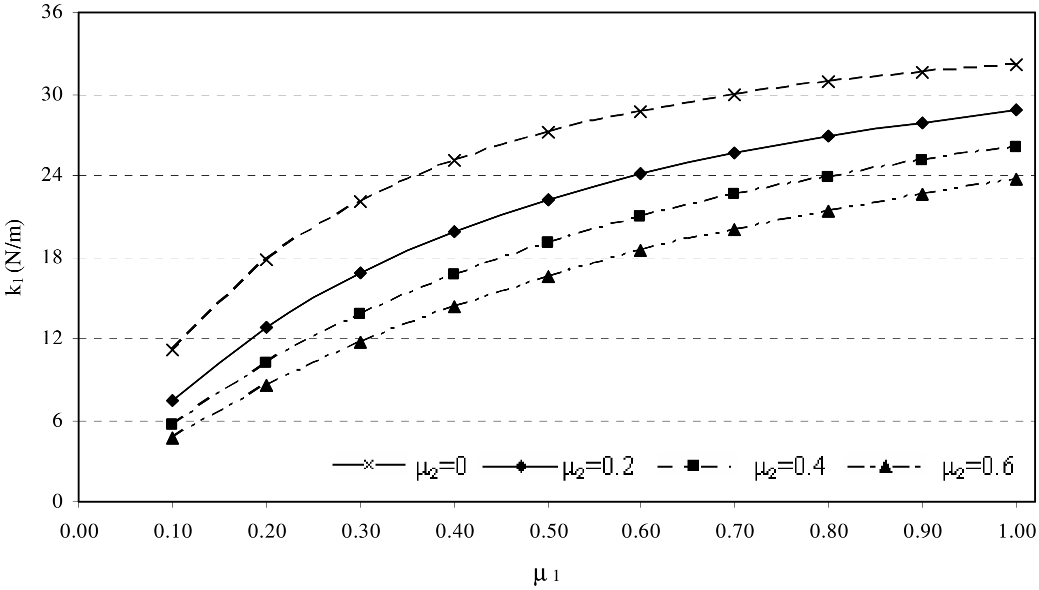

Figure 3.16. Optimum value of absorber stiffness (k_{a1}) vs. mass ratio (μ_1 and μ_2)

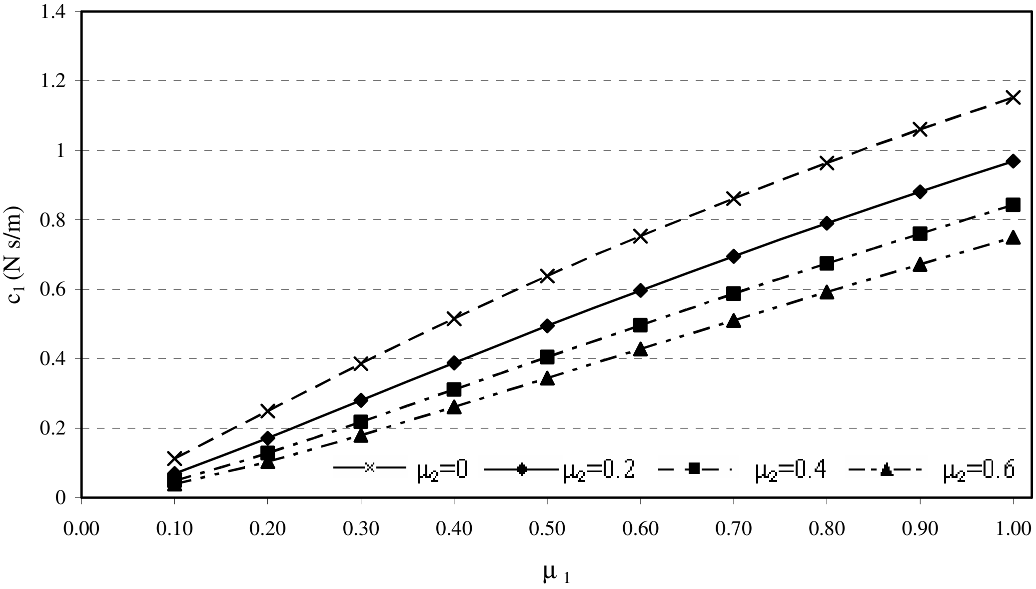

Figure 3.17. Optimum value of absorber damping (c_{a1}) vs. mass ratio (μ_1 and μ_2)

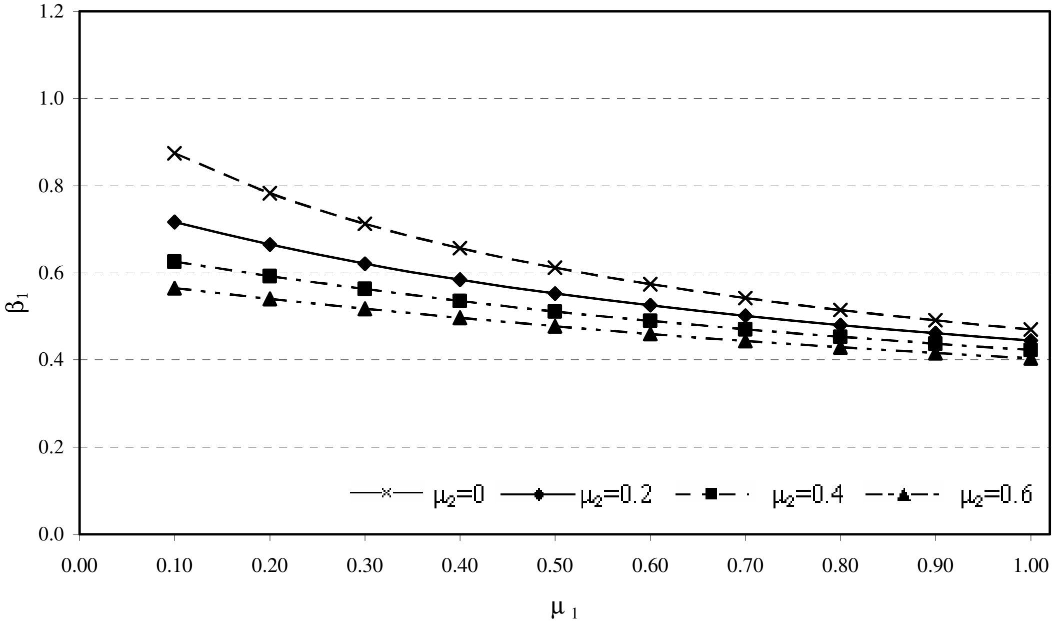

Figure 3.18. Optimum value of tuning ratio (β_1) vs. mass ratio (μ_1 and μ_2)

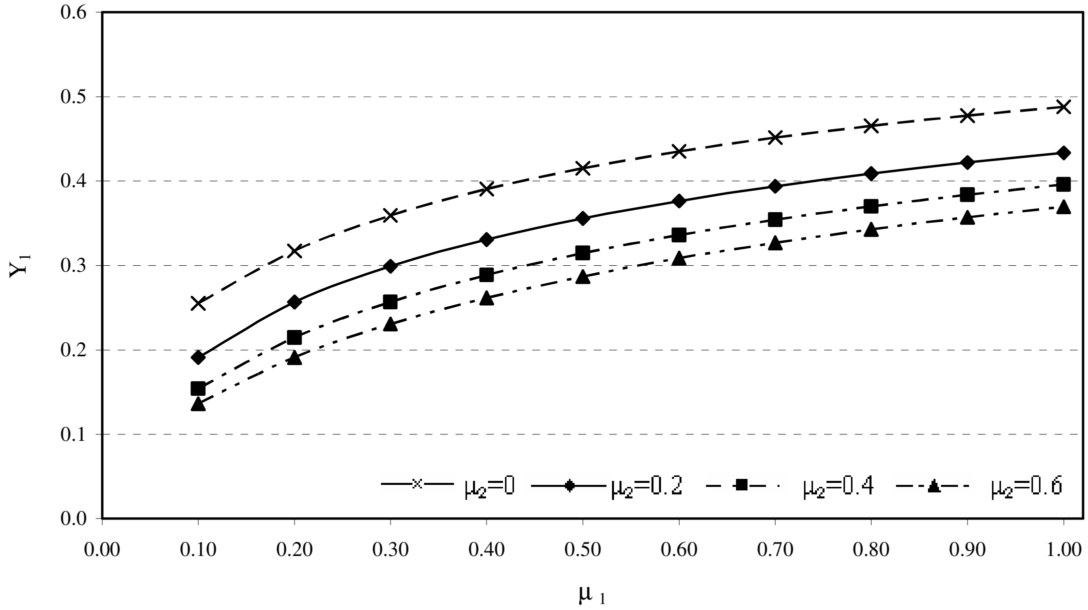

Figure 3.19. Optimum value of damping (Y_1) vs. mass ratio (μ_1 and μ_2)

As mass ratio μ_1 increases and μ_2 decreases, the optimum value of tuning ratio and the optimum value of damping ratio of the first absorber decrease. For fixed μ_1 value, there is an indirect correlation between the μ_2 and β_1, Y_1 values.

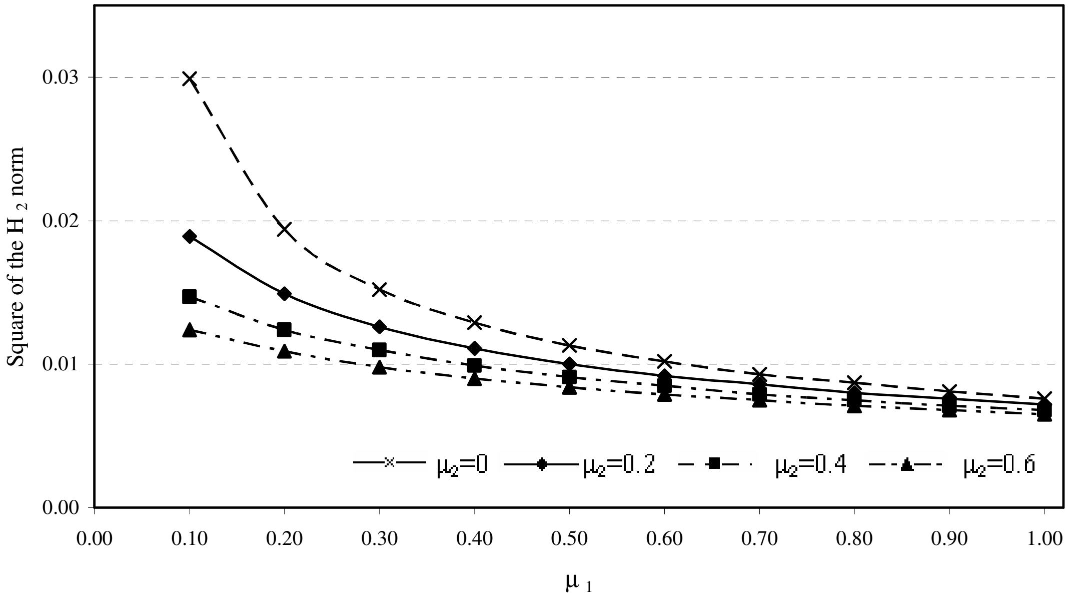

Figure 3.20. Minimized square of the H_2 norm ($\|G_{tot}\|_2^2$) vs. mass ratio (μ_1 and μ_2)

In order to suppress the vibration amplitudes to a high level, the H_2 norm is targeted to be as small as possible. For achieving this target, μ_1 and μ_2 values should be kept as large as possible. But, the effect rate of μ_1 and μ_2 onto the H_2 norm of the system decreases for the high values of μ_1 and μ_2.

3.5. Case Study 5 (with Reversed Input Parameters of Case Study 4)

In this case study, the system modeled with five mode shapes and two absorbers as in case study 4 is used with the reversed input parameters. Two absorbers are attached to a pinned-pinned beam with five mode shapes as displayed in Figure 3.14. The external force is applied at $x = 0.25\,l_{beam}$. The aim is to minimize the vibration amplitudes at $x = 0.75 l_{beam}$. A comparison between cases with no absorber, randomly selected absorbers, and the optimal absorbers is shown in Figure 3.21. The optimal values for the absorbers are as follows: l_1 = 0.4944 m, l_2 = 0.2407 m, k_{a1} = 15.2539 N/m, k_{a2} = 59.9944 N/m, c_{a1} = 0.1843 N s/m, and c_{a2} = 1.1544 N s/m.

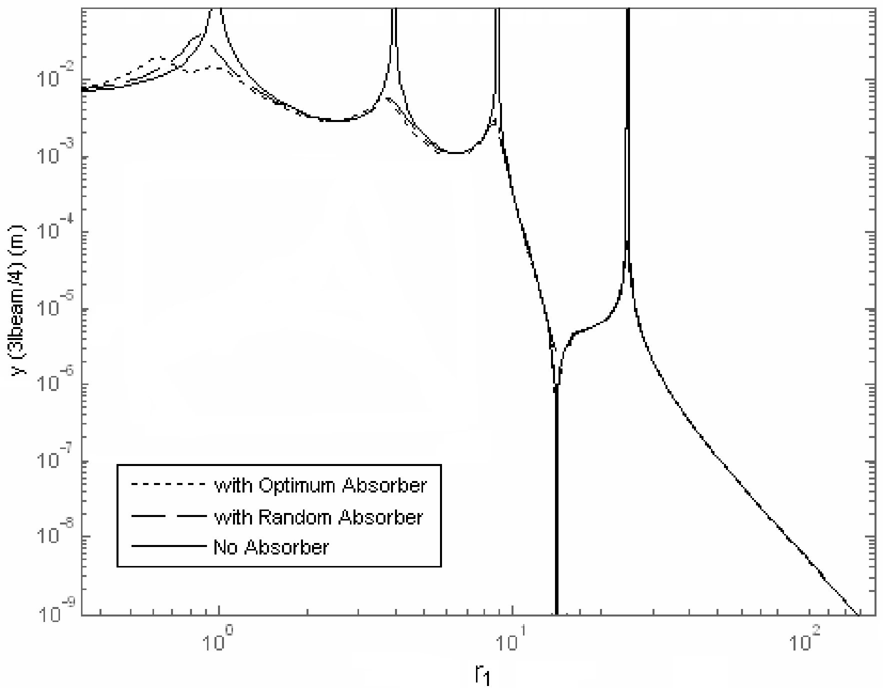

Figure 3.21. Displacement of the beam at x=3l_{beam}/4

3.6. Case Study 6 (Cantilever Beam with Constrained Minimization)

The same beam material used in case study 1 is selected, but different boundary conditions are used in this scenario. A fixed-free beam (cantilever beam) with three absorbers (Figure 3.22) is used to verify the constrained minimization approach. The external force is applied at $x = 0.8\,l_{\mathrm{beam}}$. The aim is to minimize the below weighted sum of the vibration amplitudes of the beam where

The objective function is $1.0\,y(l_{beam}) + 4.0\,y(l_{beam}/2)$ subject to c_{ai} < 3 N s/m and k_{ai} < 2 N/m.

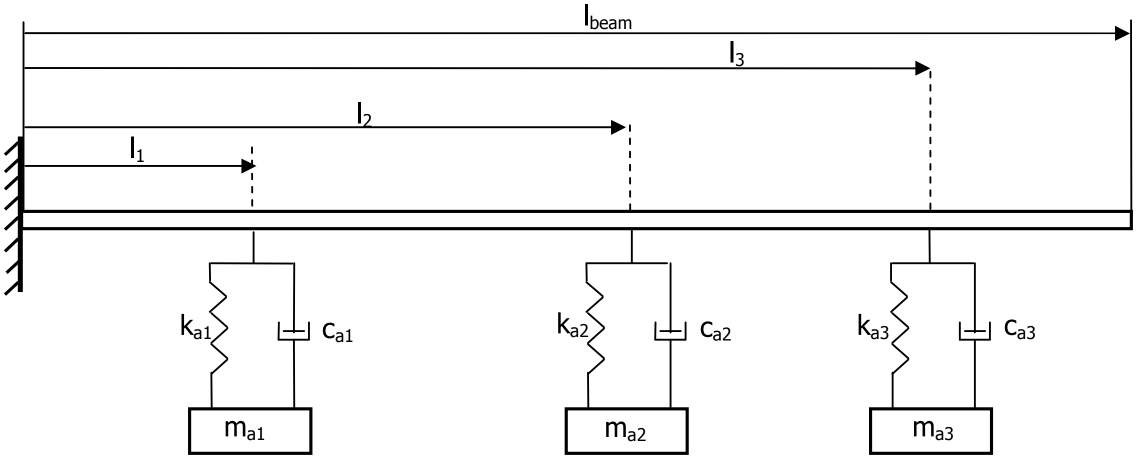

Figure 3.22. Three absorbers attached to a fixed–free beam (four mode shapes)

The beam is modeled with four modes, defined as follows:

$$\mathrm{MS} = \begin{pmatrix} \cosh(1.87510407x/l_{\mathrm{beam}}) - \cos(1.87510407x/l_{\mathrm{beam}}) \\ -0.7341(\sinh(1.87510407x/l_{\mathrm{beam}}) - \sin(1.87510407x/l_{\mathrm{beam}})); \\ \cosh(4.69409113x/l_{\mathrm{beam}}) - \cos(4.69409113x/l_{\mathrm{beam}}) \\ -1.0185(\sinh(4.69409113x/l_{\mathrm{beam}}) - \sin(4.69409113x/l_{\mathrm{beam}})); \\ \cosh(7.85475744x/l_{\mathrm{beam}}) - \cos(7.85475744x/l_{\mathrm{beam}}) \\ -0.9992(\sinh(7.85475744x/l_{\mathrm{beam}}) - \sin(7.85475744x/l_{\mathrm{beam}})); \\ \cosh(10.99554073x/l_{\mathrm{beam}}) - \cos(10.99554073x/l_{\mathrm{beam}}) \\ -1.0000(\sinh(10.99554073x/l_{\mathrm{beam}}) - \sin(10.99554073x/l_{\mathrm{beam}})) \end{pmatrix} \tag{3.7}$$

A mixed goal can be defined, with different weighing functions on various displacements of the cantilever beam.

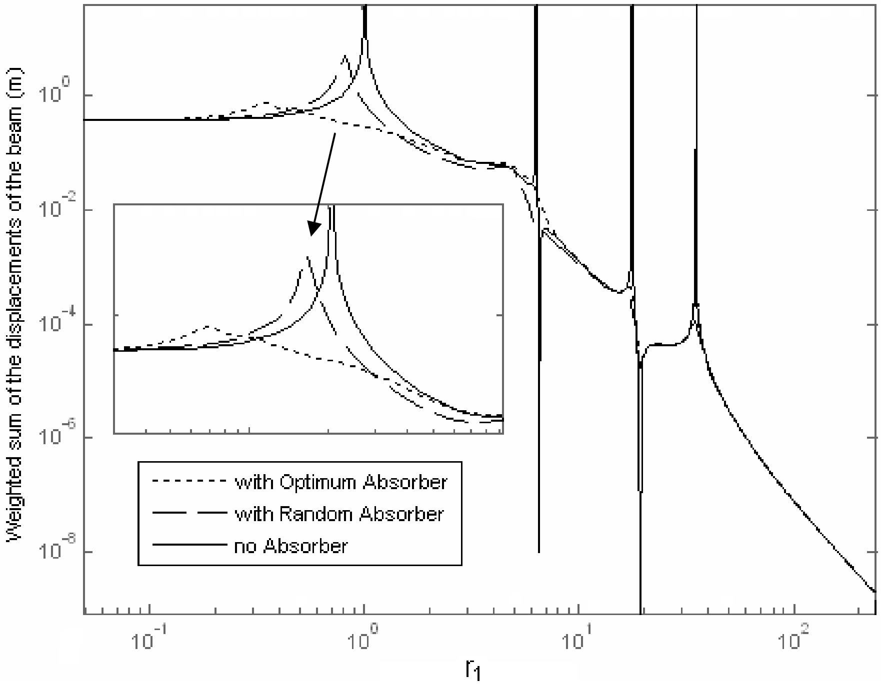

Figure 3.23. Displacement of the beam at $1.0\,y(l_{beam}) + 4.0\,y(l_{beam}/2)$

The optimal absorber parameters are obtained as: l_1 = 1.000 m, l_2 = 1.000 m, l_3 = 1.000 m, k_{a1} = 0.5230 N/m, k_{a2} = 2.0000 N/m, k_{a3} = 1.0333 N/m, c_{a1} = 0.0266 N s/m, c_{a2} = 0.1371 N s/m, and c_{a3} = 0.0639 N s/m. Figure 3.23 presents cases with no absorber, with randomly selected absorber and the optimal absorbers.

If the numerical results are given for a cantilever beam with an external harmonic force at the free end and the absorber parameters are optimized such that the resonance amplitudes at the free end are minimized, it should be noted that the excitation has the greatest effect at the free end and also the largest displacement of the beam occurs at this point.

3.7. Case Study 7 (Simply Supported Rectangular Plate)

In this section, one absorber attached to a simply supported plate, which is modeled with initial 4x4 mode shapes is examined (Figure 3.24).

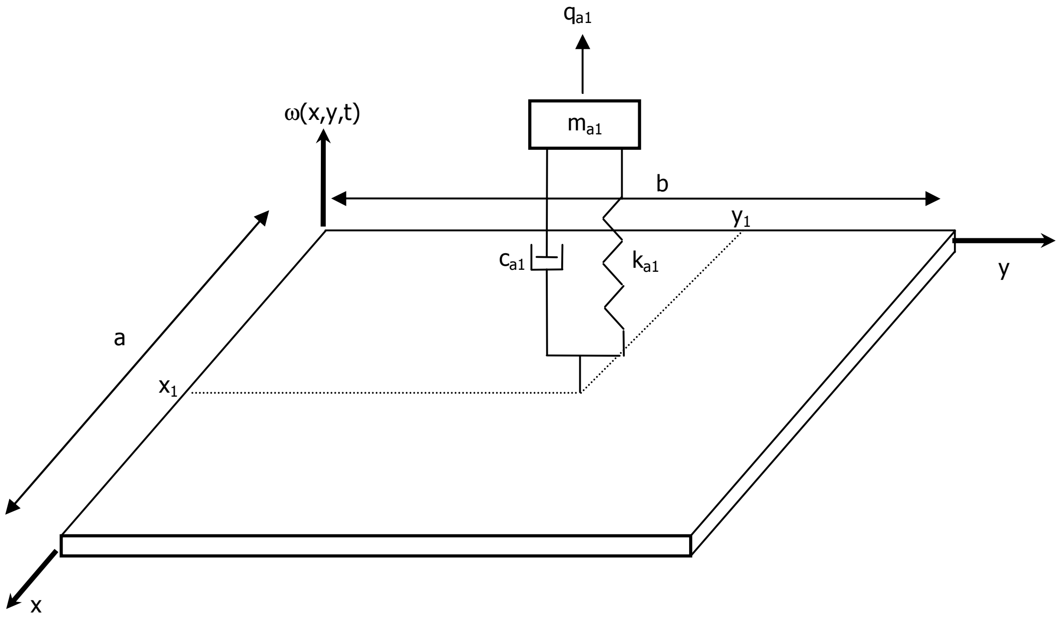

Figure 3.24. One absorber attached to a rectangular plate (4x4 modes)

The aim of optimization is to minimize the vibration amplitude at $[x, y] = [a/2, b/2]$. The plate material is an aluminum alloy. An external force is applied at $[x, y] = [a/2, b/2]$ and the input weighting function is of the form

$$W_f(s) = \left| \frac{\omega_f^{\ 2}}{s^2 + 2\,\xi\,\omega_f\ s + \omega_f^{\ 2}} \right|. \tag{3.8}$$

Other input parameters include E = $70*10^9$ Pa, a = 5 m, b = 5 m, h = 0.001 m, ρ = 2710 kg/m^3, ν = 0.3, $m_{absorber}$ = 0.5 * m_{plate}, ξ = 0.18 rad/s, and $\omega_f = 1$ rad/s. The modes considered are

$$MS = \sum_{i=1}^{4}\sum_{j=1}^{4} \sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right). \tag{3.9}$$

A comparison between the optimal absorber parameters and randomly selected absorber parameters is displayed in Figure 3.25. The randomly selected parameters are $x_1 = 1.0$ m, $y_1 = 1.0$ m, $k_{a1} = 500$ N/m, $c_{a1} = 40$ N s/m. The optimum parameters are found as follows: x_1 = 2.5 m, y_1 = 2.5 m, k_{a1} = 13.7428 N/m, and c_{a1} = 12.1877 N s/m.

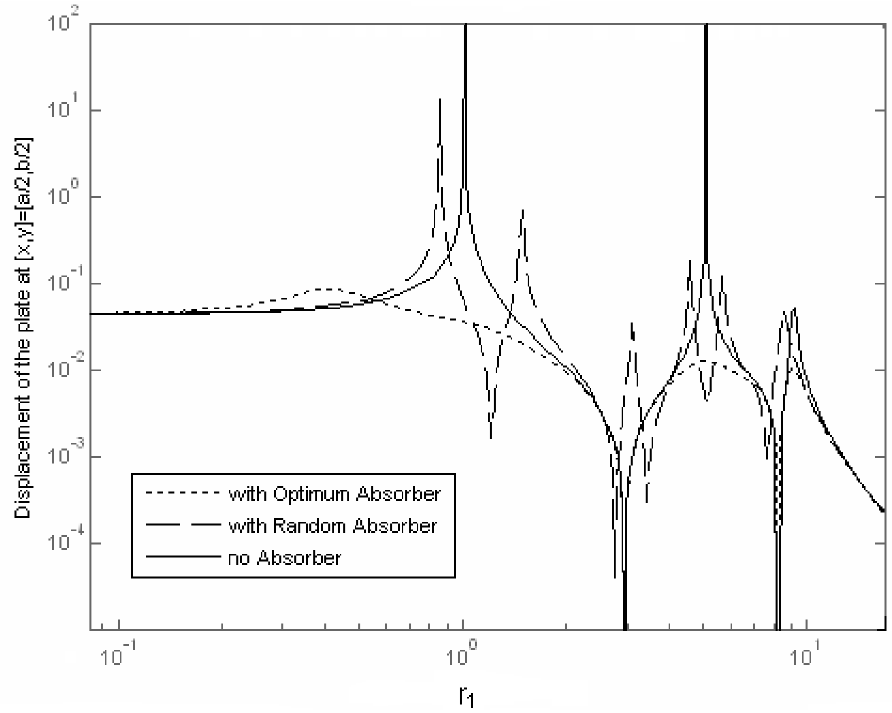

Figure 3.25. Displacement of the plate at [x,y]=[a/2,b/2] – one absorber

3.8. Case Study 8 (with 4x4 Mode Shapes and Three Absorbers)

This scenario uses the same plate and forcing as in Case Study 7. Three absorbers are attached to the plate, the vibrations of which are modeled with 4x4 modes. The system is displayed in Figure 3.26. The optimization goal is to minimize the vibration amplitude at $[x, y] = [a/2, b/2]$, subject to 0.2 N s/m $< c_{ai} <$ 50 N s/m and 5 N s/m $< k_{ai} <$ 10,000 N/m. Twelve different parameters must be optimized, four for each absorber.

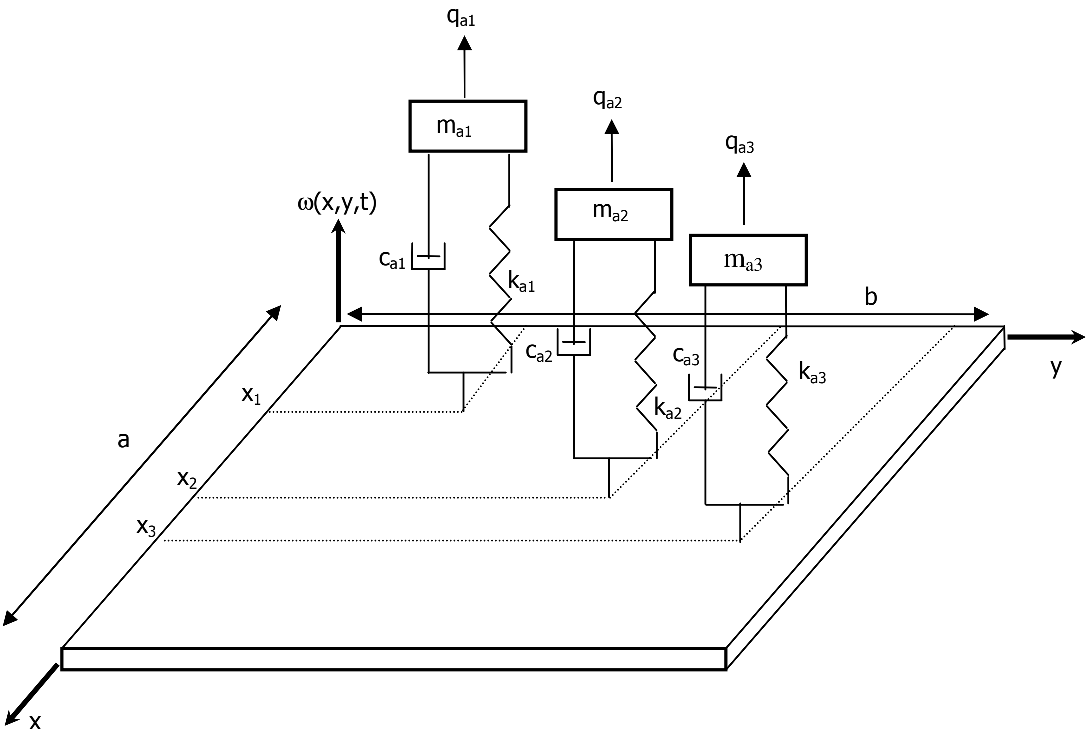

Figure 3.26. Three absorbers attached to a simply supported rectangular plate (4x4 modes)

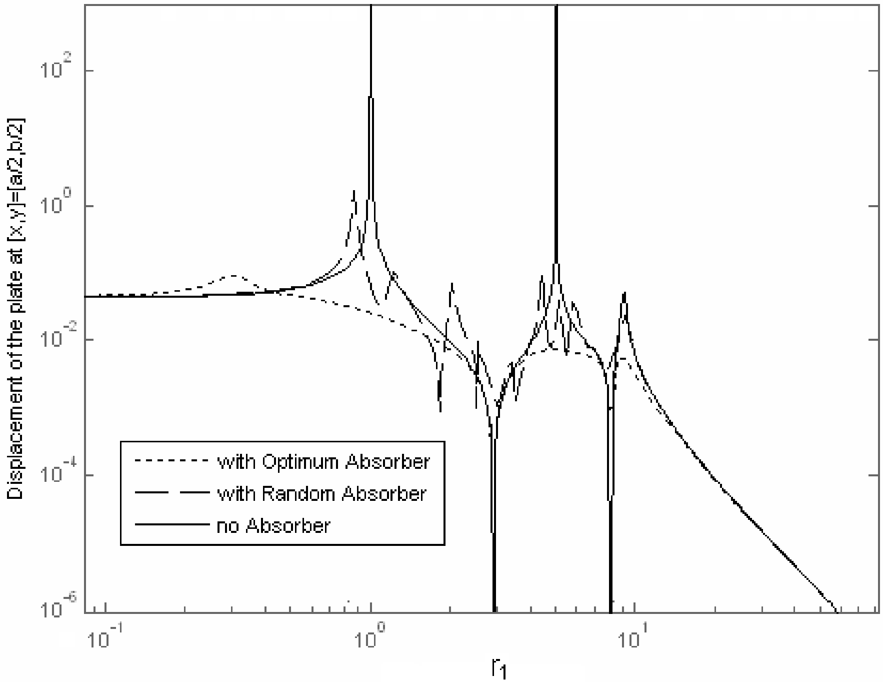

Figure 3.27. Displacement of the plate at [x,y]=[a/2,b/2] – three absorbers

A comparison between cases with no absorber, randomly selected absorbers, and optimal absorbers is shown in Figure 3.27. The optimal values for the absorbers are obtained as follows: x_1 = 2.5 m, y_1 = 2.5 m, k_{a1} = 15.5960 N/m, c_{a1} = 4.0077 N s/m, x_2 = 2.5 m, y_2 = 2.5 m, k_{a2} = 5.2720 N/m, c_{a2} = 2.2812 N s/m, x_3 = 2.5 m, y_3 = 2.5 m, k_{a3} = 5.00 N/m, and c_{a3} = 4.4485 N s/m.

3.9. Case Study 9 (with Constrained Minimization)

In this section, the distributed force is applied to the system defined in case study 8. The force is applied over the rectangular area defined by points $[x, y] = [1, 2.2]$ and $[x, y] = [3, 4.5]$. This case uses a damping ratio of $\xi = 0.18$ rad/s and a forcing frequency of $\omega_f = 1$ rad/s. The aim is to minimize the vibration amplitudes at $[x, y] = [a/2, b/2]$. The absorber parameters are constrained as in case study 8.

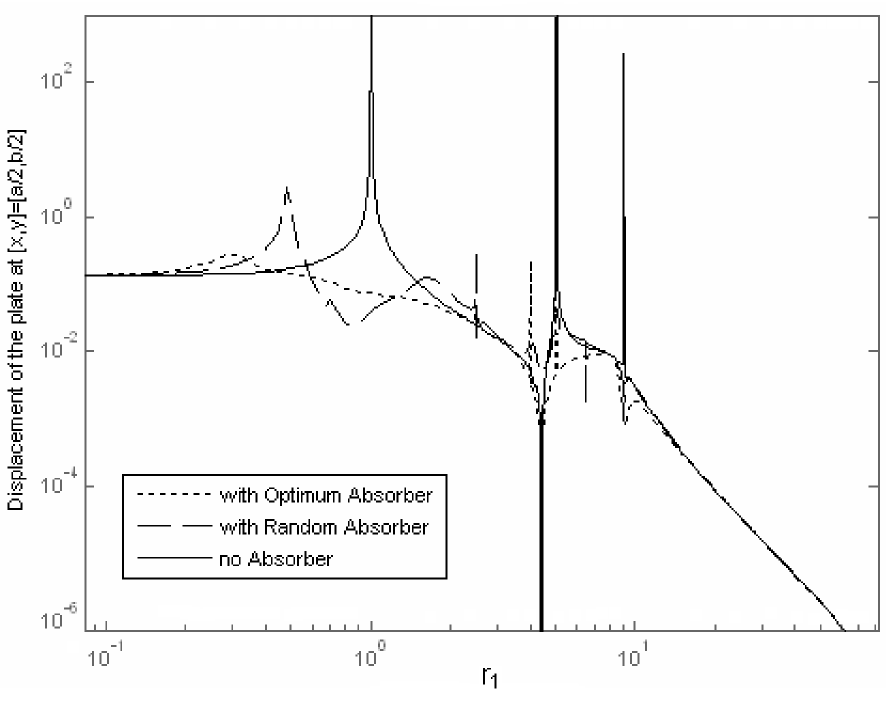

Figure 3.28. Displacement of the plate at [x,y]=[a/2,b/2] – with constrained minimization

The optimal values for the absorbers are obtained as follows: x_1 = 2.5744 m, y_1 = 2.3803 m, k_{a1} = 9.1833 N/m, c_{a1} = 4.9430 N s/m, x_2 = 2.6045 m, y_2 = 2.3301 m, k_{a2} =

12.0615 N/m, c_{a2} = 2.3877 N s/m, x_3 = 2.5029 m, y_3 = 2.4946 m, k_{a3} = 5.00 N/m, and c_{a3} = 4.6431 N s/m. A comparison between cases with no absorber, randomly selected absorbers, and optimal absorbers is shown in Figure 3.28.

3.10. Comparison with the Literature – Continuous Systems

3.10.1. Comparison with the Study of Candir and Ozguven [39]

Candir and Ozguven [39] have investigated the first and second resonances of beams with two dynamic vibration absorbers.

In one of their case studies, numerical results are given for a cantilever beam with an external force at the free end. A dynamic vibration absorber is most effective when attached at the point of maximum response for the resonance of interest. For the first and second resonances of a cantilever beam, this point is the free end ($l_1 = l_2 = l_{beam}$). Hence, numerical results are presented for the case where both absorbers are attached to the free end. In a case where it is difficult to attach both absorbers to the same point or very close to each other, the second absorber can be attached to an appropriate point with a large displacement in the second mode, such as $l_2 = 0.48\, l_{beam}$, while the first absorber is kept at $l_1 = l_{beam}$.

When the same case study is carried out via the approach studied in Chapter 2, the optimum values for the two absorbers and the regarding goal function value are concluded as ($l_1 = l_2 = l_{beam}$); respectively. The optimization results are converged to the two global minimum points. The obtained values are: l_1 = 1.000 m, l_2 = 1.000 m, k_{a1} = 6.1977 N/m, k_{a2} = 1.9430 N/m, c_{a1} = 0.0384 N s/m, c_{a2} = 0.7774 N s/m and $\|G_{tot}\|_2^2 = 0.1604$.

As an alternative solution where it is not possible to apply two absorbers at the same point; the first absorber is concentrated on the free end and the second absorber on the largest displacement of the second mode. The results are ($l_1 = l_{beam}$, $l_2 = 0.4829\, l_{beam}$);

l_1 = 1.000 m, l_2 = 0.4829 m, k_{a1} = 1.9594 N/m, k_{a2} = 5.5516 N/m, c_{a1} = 0.7918 N s/m, c_{a2} = 0.0406 N s/m, and $\|G_{tot}\|_2^2 = 0.1609$.

Table 3.5. Optimal absorber parameters of present study and Candir and Ozguven's study [39]

	Present Study	**Candir and Ozguven's study [39]**
l_1 (m)	0.4829	0.48

The results found with the approach defined in Chapter 2 are the same as the results displayed in the investigation of Candir and Ozguven [39].

3.10.2. Comparison with the Study of Warburton and Ayorinde [56]

Warburton and Ayorinde [56] have presented results for a uniform simply supported beam. A transverse harmonic force is applied at mid-span. To preserve symmetry, either an absorber with properties k_{a1}, m_{a1} and c_{a1} is attached at mid-span or two identical absorbers with properties ½ k_{a1}, ½ m_{a1} and ½ c_{a1} are placed with the same distance to the mid-span, symmetrically. In the study [56], the following parameters are introduced:

$$\mu_{eff} = \frac{m_{a1}}{m_{beam}} \text{ (Mass Ratio)} \tag{3.10}$$

$$\omega_1^2 = \frac{k_{a1}}{m_{a1}} \tag{3.11}$$

$$\omega_{beam}^2 = \frac{k_{beam}}{m_{beam}} \tag{3.12}$$

$$f_{opt} = \frac{\omega_1}{\omega_{beam}} \tag{3.13}$$

$$\gamma_{1,opt} = \frac{c_{a1}}{2m_{a1}\omega_1} \text{ (Absorber Damping)} \tag{3.14}$$

The aim is to minimize the vibration amplitude at $x = \mathrm{l_{beam}}/2$. The beam material is an aluminum alloy, with a cross section of 0.004 m × 0.004 m. The inputs are E = $70*10^9$ Pa, I = $2.1333*10^{-11}$ m^4, m_{beam} = 0.0434 kg. From their Figures 6 and 7 [56], for the value of $\mu_{eff} = 0.2$, $\gamma_{1,opt}$ and f_{opt} values are found as: $\gamma_{1,opt} = 0.23$, $f_{opt} = 0.81$.

$$m_{a1} = \mu_{eff} m_{beam} = 0.0087\ \text{kg} \tag{3.15}$$

$$\omega_1 = f_{opt}\omega_{beam} = 46.90\ \text{rad/s} \tag{3.16}$$

$$k_{a1} = \omega_1^2 m_{a1} = 19.092\ \text{N/m} \tag{3.17}$$

$$c_{a1} = 2\gamma_{1,opt} m_{a1}\omega_1 = 0.187\ \text{N s/m} \tag{3.18}$$

Figure 3.29 compares the cases of no absorber, randomly selected absorbers, and optimal absorbers obtained by applying the method of Chapter 2 to this system. The optimal parameters obtained by both works are displayed in Table 3.6.

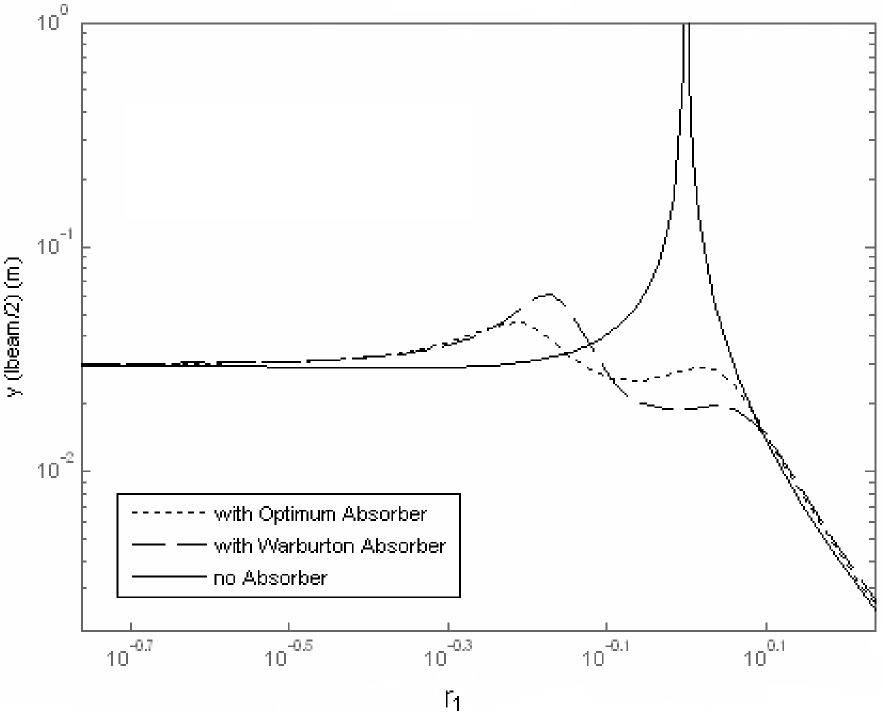

Figure 3.29. Displacement of the beam studied by Warburton and Ayorinde [56] at x=l_{beam}/2

Table 3.6. Optimal absorber parameters of the present study compared to those obtained by Warburton and Ayorinde [56]

	Present Study	**Warburton and Ayorinde's study [56]**
k_{a1} (N/m)	14.4966	19.092
c_{a1} (N s/m)	0.1801	0.187
$\lVert G_{tot} \rVert_2^2$	0.0232	0.0250

3.10.3. Comparison with the Study of Rade and Steffen [57]

Rade and Steffen [57] have performed standard laboratory vibration tests on a steel, uniform, free-free beam. Excitation is introduced in the vertical direction with an impact hammer at x=0, and time-domain acceleration responses are collected using piezoelectric accelerometers.

The beam's cross-section is 0.0366 m × 0.0114 m. The other material and experimental parameters are $E = 200*10^9$ Pa, $I = 4.5187*10^{-9}$ m^4, $m_{beam} = 2.54$ kg, and $l_{beam} = 0.763$ m.

The intent of this experiment is to design a single absorber to be attached at $x = 3l_{beam}/10$ that would control vibrations at the same coordinate. Using a dynamic vibration absorber, the optimal parameters are obtained as: $k_{a1} = 2.9*10^6$ N/m and $c_{a1} = 50.9$ N s/m. The results are displayed in Table 3.7.

In this example, instead of broadband attenuation, specific frequency range is considered. Due to the frequency band examined (520 – 680 Hz), the inclusion of the vibration absorber enables the resonant amplitudes of the third vibration mode to be suppressed. Since the optimization is pursued over specific frequency bands, the method defined by Rade and Steffen [57] does not guarantee the control of the vibration amplitude outside those bands.

Table 3.7. Optimal absorber parameters of the present study compared to those obtained by Rade and Steffen [57]

	Present Study	**Rade and Steffen's study [57]**
k_{a1} (N/m)	2.725×10^{6}	2.9×10^{6}
c_{a1} (N s/m)	89.9	50.9
$\left\Vert G_{tot} \right\Vert_2^2$	2.1019×10^{-10}	4.3333×10^{-10}

4. VIBRATION SUPPRESSION OF TRANSLATIONAL MASS-SPRING-DAMPER SYSTEMS USING PASSIVE RECEPTANCE METHOD

This chapter describes a new, independent method for calculating optimal parameters based on using the Sherman-Morrison matrix inversion formula [58]. The method also analyzes and provides the receptance of a system with more than one absorber.

4.1. Calculating the Dynamic Stiffness via Linear Graphs

The linear graph technique, comprehensively explained by Deo [59], is a powerful tool for representing the elements and unified structure of a given physical system. Linear graphs are related to topology, the branch of mathematics that deals with the properties of figures which are related to the way in which their various parts are interconnected [60].

To better visualize the incidence relations, oriented graphs are generally represented by a diagram, where nodes are represented as points and branches as directed line segments between their end nodes, consistent with the ordered pairing of nodes with which they are associated. In a graph representing a dynamical system, each branch corresponds to an energy port of a lumped element and each node corresponds to a distinct across variable location in the system, through which system elements are connected [61]. A constitutive equation is associated with each node and expressed in terms of the "through" and "across" variables, their product being the power provided or dissipated by the component [62].

The procedure expressed by Mcphee [63, 64, 65] defines a set of "through" and "across" variables that fully describe the state of the physical system and gives the constitutive equation for each element in terms of these variables. A "through" variable is any physical quantity that can be measured by an instrument placed in series with an element; the current in an electrical system and the force between bodies in a mechanical system are two examples. An "across" variable is any quantity that can be measured by an instrument placed across the end of an element. This would be voltage in an electrical network and displacement in a multi-body system.

In modeling of physical systems via linear graph, the three quantities, namely; the graph orientation, the velocity direction and the actual forcing configuration can be selected arbitrarily. However, once any two out of three quantities are selected, the remaining one becomes fixed.

In order to verify the receptance approach that will be detailed in Section 4.2, the linear graph methodology is used in this study. Through both methodologies, the same dynamic stiffness expression is found. To obtain the dynamic stiffness, i.e., the inverse of the receptance, the physical systems can be modeled by a linear graph. Moreover, this chapter will describe a methodology to move between receptance and linear graph representations.

4.1.1. Series and Parallel Branches

To calculate the dynamic stiffness of a system using linear graph representation, the series and parallel branches are considered.

The dynamic stiffness coefficients associated with branches in series are added in reciprocal. That is, the inverse of the total dynamic stiffness (Z_{tot}) is equal to the sum of the inverse stiffnesses of each branch:

$$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5} \cdots \tag{4.1}$$

The dynamic stiffness of a compound system composed of parallel branches is simply found by adding together their dynamic stiffness values.

$$Z_{tot} = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \cdots \tag{4.2}$$

4.1.2. Examples of Linear Graphs

In order to obtain the dynamic stiffness, the problem is investigated in an example as shown in Figure 4.1.

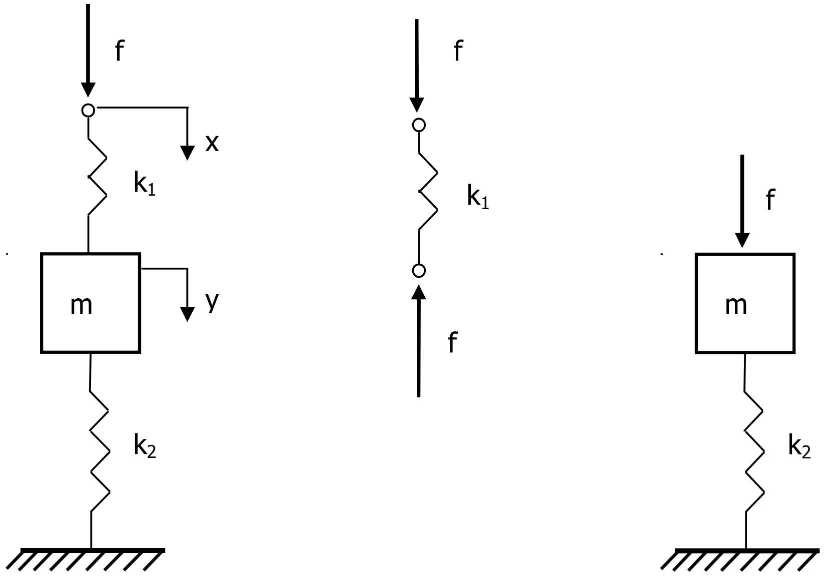

Figure 4.1. A mass-spring system

The equations of motion for this system are

$$f = k_1(x - y) \tag{4.3}$$

$$-k_2 y + f = m\ddot{y} \tag{4.4}$$

Taking the Laplace Transform, subsequently;

$$F(s) = k_1(X(s) - Y(s)) \tag{4.5}$$

$$-k_2 Y(s) + F(s) = ms^2 Y(s) \tag{4.6}$$

These equations can now be combined and simplified:

$$F(s) = (ms^2 + k_2)Y(s) = k_1(X(s) - Y(s)), \tag{4.7}$$

$$Y(s) = \left[\frac{k_1}{ms^2 + k_1 + k_2}\right] X(s). \tag{4.8}$$

Substituting (4.8) into (4.5) yields

$$F(s) = k_1\left[1 - \frac{k_1}{ms^2 + k_1 + k_2}\right]X(s)\text{, or} \tag{4.9}$$

$$F(s) = \left[\frac{k_1(ms^2 + k_2)}{ms^2 + k_1 + k_2}\right]X(s)\,. \tag{4.10}$$

The dynamic stiffness is therefore

$$Z_{tot} = \left[\frac{k_1(ms^2 + k_2)}{ms^2 + k_1 + k_2}\right]. \tag{4.11}$$

The linear graph of this system is displayed in Figure 4.2.

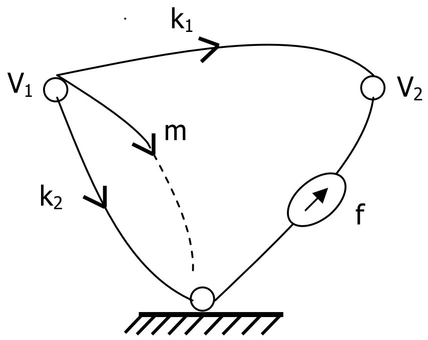

Figure 4.2. Linear graph of the mass-spring system

$$Z_{tot} = \left[\frac{1}{Z_1} + \frac{1}{Z_2}\right]^{-1} \tag{4.12}$$

$$Z_{tot} = \left[\frac{1}{k_1} + \frac{1}{ms^2 + k_2}\right]^{-1} \tag{4.13}$$

$$Z_{tot} = \left[\frac{k_1(ms^2 + k_2)}{ms^2 + k_1 + k_2}\right] \tag{4.14}$$

It is observed that Eqs. (4.11) and (4.14) are the same.

The problem is investigated in a more complex example of translational mass-spring-damper system working in vertical plane shown in Figure 4.3.

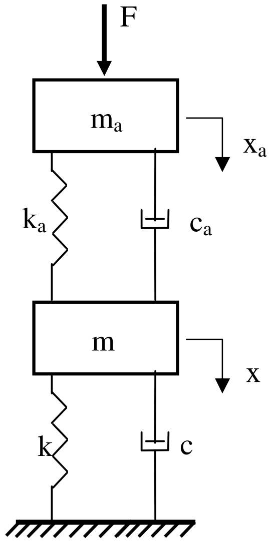

Figure 4.3. A mass-spring-damper system

The equations of motion are:

$$k_a(x - x_a) + c_a(\dot{x} - \dot{x}_a) + F = m_a\ddot{x}_a \quad \text{and} \tag{4.15}$$

$$-kx - c\dot{x} - k_a(x - x_a) - c_a(\dot{x} - \dot{x}_a) = m\ddot{x}\,. \tag{4.16}$$

These equations are used following the same sequence of steps described in the previous example. The dynamic stiffness between $X_a(s)$ and $F(s)$ (where $X_a(s)$ and $F(s)$ are the Laplace transform of the $x_a(t)$ and $f(t)$, respectively) is found to be

$$Z_{X_aF} = \frac{(m_a s^2 + c_a s + k_a)(ms^2 + cs + k + c_a s + k_a) - (c_a s + k_a)^2}{ms^2 + cs + k + c_a s + k_a} \tag{4.17}$$

The linear graph of the system is shown in Figure 4.4. Z_1 is in series with Z_2, and the compound system composed of Z_1 and Z_2, is in parallel with Z_3. Z_{X_aF} is the dynamic stiffness of the whole system.

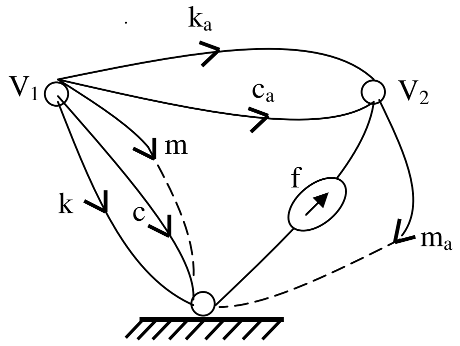

Figure 4.4. Linear graph of the mass-spring-damper system

$$Z_{X_aF} = \left[\frac{1}{Z_1} + \frac{1}{Z_2}\right]^{-1} + Z_3 \tag{4.18}$$

where

$$Z_1 = ms^2 + cs + k \tag{4.19}$$

$$Z_2 = c_a s + k_a \tag{4.20}$$

$$Z_3 = m_a s^2 \tag{4.21}$$

Substituting Eqs. (4.19) – (4.21) into Eq. (4.18) and simplifying, Z_{X_aF} is obtained as:

$$Z_{X_aF} = \frac{mc_a s^3 + mk_a s^2 + cc_a s^2 + ck_a s + c_a ks + kk_a + m_a ms^4 + m_a cs^3 + m_a ks^2 + m_a c_a s^3 + m_a k_a s^2}{ms^2 + cs + k + c_a s + k_a} \tag{4.22}$$

It is observed that, as expected, Eqs. (4.17) and (4.22) are the same. The same dynamic stiffness expression between $X_a(s)$ and $F(s)$ is obtained with both linear graph methodology and the equations of motion.

The next example consists of translational mass-spring-damper system working in the horizontal plane. The dynamic stiffness is calculated based on the linear graph methodology. The system is shown in Figure 4.5, and its corresponding linear graph is given in Figure 4.6.

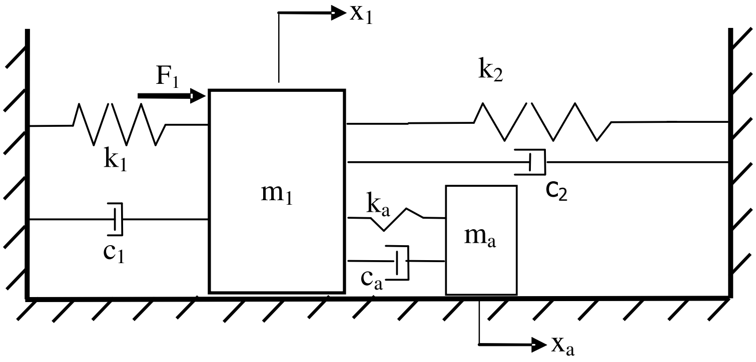

Figure 4.5. One mass with one absorber system

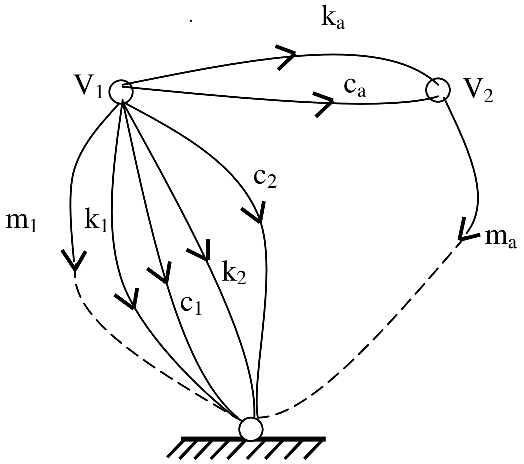

Figure 4.6. Linear graph of "one mass with one absorber" system

The dynamic stiffness coefficients are

$$Z_1 = m_1 s^2 + c_1 s + k_1 + c_2 s + k_2 , \tag{4.23}$$

$$Z_2 = c_a s + k_a , \text{ and} \tag{4.24}$$

$$Z_3 = m_a s^2 . \tag{4.25}$$

$$Z_{X_1F} = Z_1 + \left[\frac{1}{Z_2} + \frac{1}{Z_3} \right]^{-1} \tag{4.26}$$

Substituting Eqs. (4.23) – (4.25) into Eq. (4.26) and simplifying, the total dynamic stiffness between X_1 and F is

$$Z_{X_1F} = m_1 s^2 + c_1 s + k_1 + c_2 s + k_2 + c_a s + k_a - \frac{(c_a s + k_a)^2}{m_a s^2 + c_a s + k_a}. \tag{4.27}$$

Naturally, the same result can be achieved by analyzing equations of motion:

$$-k_1 x_1 - c_1 \dot{x}_1 - k_2 x_1 - c_2 \dot{x}_1 + k_a (x_a - x_1) + c_a (\dot{x}_a - \dot{x}_1) + F_1 = m_1 \ddot{x}_1 \tag{4.28}$$

$$-k_a (x_a - x_1) - c_a (\dot{x}_a - \dot{x}_1) = m \ddot{x}_a. \tag{4.29}$$

Taking the Laplace Transform yields:

$$\left[m_1 s^2 + c_1 s + k_1 + c_2 s + k_2 + c_a s + k_a\right] x_1 - \left[c_a s + k_a\right] x_a = F_1 \text{ and} \tag{4.30}$$

$$\left[m_a s^2 + c_a s + k_a s\right] x_a = \left[c_a s + k_a\right] x_1. \tag{4.31}$$

Substituting x_a in terms of x_1 into Eq. (4.30) gives

$$\left[m_1 s^2 + c_1 s + k_1 + c_2 s + k_2 + c_a s + k_a - \frac{(c_a s + k_a)^2}{m_a s^2 + c_a s + k_a}\right] x_1 = F_1 . \tag{4.32}$$

Once again, the linear graph methodology and the equations of motion result in the same dynamic stiffness:

$$Z_{X_1F} = m_1 s^2 + c_1 s + k_1 + c_2 s + k_2 + c_a s + k_a - \frac{(c_a s + k_a)^2}{m_a s^2 + c_a s + k_a} . \tag{4.33}$$

4.2. Translational Mass-Spring-Damper System with m Masses and n absorbers

This section describes a new method for calculating optimal absorber parameters of discrete systems, based on the Sherman-Morrison matrix inversion formula. The general case for m masses with n absorbers is shown in Figure 4.7. It should be noted that each main mass does not necessarily have an absorber attached. As an example, a system with two masses and two absorbers is analyzed.

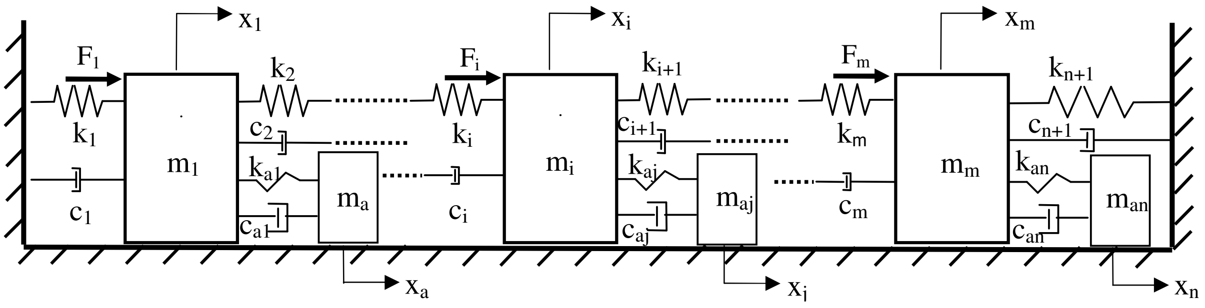

Figure 4.7. A discrete system with m masses and n absorbers

$\mathbf{K_{add}}$ and $\mathbf{C_{add}}$ (additive matrices that come upon the addition of the absorbers to the main system) can be easily found from both the equation of motion of the main system (Eq. (4.34)) and also the equation of motion of the absorbers (Eq. (4.39)). Displaying the equation of motion in matrix form, we obtain

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} c_1+c_2+c_{a1} & -c_2 \\ -c_2 & c_2+c_3+c_{a2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2+k_{a1} & -k_2 \\ -k_2 & k_2+k_3+k_{a2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$+\begin{bmatrix} -c_{a1} & 0 \\ 0 & -c_{a2} \end{bmatrix}\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix}+\begin{bmatrix} -k_{a1} & 0 \\ 0 & -k_{a2} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{4.34}$$

The matrices $\mathbf{K_{add}}$ and $\mathbf{C_{add}}$ are thus

$$\mathbf{C_{add}}=\begin{bmatrix} -c_{a1} & 0 \\ 0 & -c_{a2} \end{bmatrix}, \tag{4.35}$$

$$\mathbf{K}_{\mathbf{add}} = \begin{bmatrix} -k_{a1} & 0 \\ 0 & -k_{a2} \end{bmatrix}. \tag{4.36}$$

In general, the equations of motion of the main masses and absorbers are constructed separately:

$$\mathbf{M}\ddot{x} + \mathbf{C}\dot{x} + \mathbf{K}x + \mathbf{C}_{\mathbf{add}}\dot{\tilde{x}} + \mathbf{K}_{\mathbf{add}}\tilde{x} = \mathbf{F} \tag{4.37}$$

$$\tilde{\mathbf{M}}\ddot{\tilde{x}} + \tilde{\mathbf{C}}\dot{\tilde{x}} + \tilde{\mathbf{K}}\tilde{x} + \mathbf{C}_{\mathbf{sec}}\dot{x} + \mathbf{K}_{\mathbf{sec}}x = \overline{\mathbf{0}} \tag{4.38}$$

K_{sec}, C_{sec} are the additive matrices that exist in the absorbers' equation of motion and they are the multipliers of the main system's generalized coordinates. In this example, the equation of motion of the absorbers is

$$\begin{aligned} &\begin{bmatrix} m_{a1} & 0 \\ 0 & m_{a2} \end{bmatrix}\begin{bmatrix} \ddot{x}_a \\ \ddot{x}_b \end{bmatrix} + \begin{bmatrix} c_{a1} & 0 \\ 0 & c_{a2} \end{bmatrix}\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} + \begin{bmatrix} k_{a1} & 0 \\ 0 & k_{a2} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} -c_{a1} & 0 \\ 0 & -c_{a2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &+ \begin{bmatrix} -k_{a1} & 0 \\ 0 & -k_{a2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{4.39}$$

4.2.1. Formulation for a System with m Masses and n Absorbers

In the general formulation, it is assumed that at most one absorber is attached to each main mass.

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & \ddots & & 0 & & \\ & & & m_i & & & \\ & & & & m_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & m_m \end{bmatrix} \tag{4.40}$$

$$\mathbf{C}=\begin{bmatrix} c_1+c_2+c_{a1} & -c_2 & & & & & & \\ -c_2 & c_2+c_3+c_{a2} & -c_3 & & & 0 & & \\ & -c_3 & \cdots\cdots\cdots & -c_i & & & & \\ & & -c_i & c_i+c_{i+1}+c_{aj} & -c_{i+1} & & & \\ & & & -c_{i+1} & c_{i+1}+c_{i+2}+c_{aj+1} & & & \\ & 0 & & & & \cdots\cdots\cdots & & \\ & & & & & & & -c_{m-1} \\ & & & & & & -c_{m-1} & c_{m-1}+c_m+c_{an} \end{bmatrix}$$

(4.41)

$$\mathbf{K}=\begin{bmatrix} k_1+k_2+k_{a1} & -k_2 & & & & & & \\ -k_2 & k_2+k_3+k_{a2} & -k_3 & & & 0 & & \\ & -k_3 & \cdots\cdots\cdots & -k_i & & & & \\ & & -k_i & k_i+k_{i+1}+k_{aj} & -k_{i+1} & & & \\ & & & -k_{i+1} & k_{i+1}+k_{i+2}+k_{aj+1} & & & \\ & 0 & & & & \cdots\cdots\cdots & & \\ & & & & & & & -k_{m-1} \\ & & & & & & -k_{m-1} & k_{m-1}+k_m+k_{an} \end{bmatrix}$$

(4.42)

By substituting $x = e^{i\omega t}$ and its derivatives into Eqs. (4.37) and (4.38), we obtain

$$\left[-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}\right]x+\left[i\omega\mathbf{C}_{\mathbf{add}}+\mathbf{K}_{\mathbf{add}}\right]\tilde{x}=\mathbf{F} \tag{4.43}$$

$$\left[-\omega^2\tilde{\mathbf{M}}+i\omega\tilde{\mathbf{C}}+\tilde{\mathbf{K}}\right]\tilde{x}+\left[i\omega\mathbf{C}_{\mathbf{sec}}+\mathbf{K}_{\mathbf{sec}}\right]x=\bar{\mathbf{0}} \tag{4.44}$$

Substituting Eq. (4.44) into Eq. (4.43) and simplifying, the output equation is as follows:

$$\left(\left[-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}\right]-\left[i\omega\mathbf{C}_{\mathbf{add}}+\mathbf{K}_{\mathbf{add}}\right]\left[-\omega^2\tilde{\mathbf{M}}+i\omega\tilde{\mathbf{C}}+\tilde{\mathbf{K}}\right]^{-1}\left[i\omega\mathbf{C}_{\mathbf{sec}}+\mathbf{K}_{\mathbf{sec}}\right]\right)x=\mathbf{F} \tag{4.45}$$

The relationship between x and F is known as the dynamic stiffness Z of the system:

$$\mathbf{Zx}=\mathbf{F} \tag{4.46}$$

$$\mathbf{Z} = \mathbf{Z_x} + \mathbf{Z_{add}} = \left(\left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right] - \left[i\omega\mathbf{C_{add}} + \mathbf{K_{add}}\right]\left[-\omega^2\tilde{\mathbf{M}} + i\omega\tilde{\mathbf{C}} + \tilde{\mathbf{K}}\right]^{-1}\left[i\omega\mathbf{C_{sec}} + \mathbf{K_{sec}}\right]\right) \tag{4.47}$$

where

$$\mathbf{Z_x} = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right] \text{ and} \tag{4.48}$$

$$\mathbf{Z_{add}} = -\left[i\omega\mathbf{C_{add}} + \mathbf{K_{add}}\right]\left[-\omega^2\tilde{\mathbf{M}} + i\omega\tilde{\mathbf{C}} + \tilde{\mathbf{K}}\right]^{-1}\left[i\omega\mathbf{C_{sec}} + \mathbf{K_{sec}}\right]. \tag{4.49}$$

$\mathbf{Z_x}$ is composed of a matrix containing the parameters of the main system and the diagonal matrix containing those of the absorber system:

$$\mathbf{Z_x} = \mathbf{Z_{main}} + \begin{bmatrix} i\omega c_{a1} + k_{a1} & & & & & & & 0 \\ 0 & i\omega c_{a2} + k_{a2} & & & & & & \\ \vdots & 0 & i\omega c_{a3} + k_{a3} & & & & & \\ \vdots & \vdots & 0 & \cdots\cdots & & & & \\ \vdots & \vdots & \vdots & 0 & \cdots\cdots & & & \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots\cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots\cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & i\omega c_{an} + k_{an} \end{bmatrix} \tag{4.50}$$

where

$$\mathbf{Z_{main}} = -\omega^2\mathbf{M_{main}} + i\omega\mathbf{C_{main}} + \mathbf{K_{main}}. \tag{4.51}$$

In the case of two masses with two absorbers, the dynamic stiffness is

$$\mathbf{Z_x} = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right] = -\omega^2\mathbf{M_{main}} + i\omega\mathbf{C_{main}} + \mathbf{K_{main}} + \mathbf{Z_1}\mathbf{Z_1^T} + \mathbf{Z_2}\mathbf{Z_2^T} \tag{4.52}$$

where the matrices are

$$\mathbf{M_{main}} = \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \ \mathbf{C_{main}} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}; \ \mathbf{K_{main}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (4.53)$$

$$\mathbf{Z_1} = \begin{bmatrix} \sqrt{i\omega c_{a1} + k_{a1}} \\ 0 \end{bmatrix}; \ \mathbf{Z_2} = \begin{bmatrix} 0 \\ \sqrt{i\omega c_{a2} + k_{a2}} \end{bmatrix}. \quad (4.54)$$

For the general case of m masses with n absorbers, the following formulation is found:

$$\mathbf{Z} = \mathbf{Z_{main}} + \mathbf{Z_{add}} + \mathbf{Z_1 Z_1^T} + \mathbf{Z_2 Z_2^T} + \mathbf{Z_3 Z_3^T} + \ldots\ldots\ldots \mathbf{Z_m Z_m^T} \quad (4.55)$$

The system receptance is then written as follows:

$$\boldsymbol{\alpha} = \mathbf{Z}^{-1} = \left[\mathbf{Z_{main}} + \mathbf{Z_{add}} + \mathbf{Z_1 Z_1^T} + \mathbf{Z_2 Z_2^T} + \mathbf{Z_3 Z_3^T} + \ldots\ldots\ldots \mathbf{Z_m Z_m^T}\right]^{-1}. \quad (4.56)$$

4.2.2. Sherman-Morrison Matrix Inversion Formula

The Sherman-Morrison formula is proposed in literature as one of the useful methods that compute the approximate inverse applications. The effectiveness of this method has been reported in many works [66].

Specifically, the Sherman-Morrison formula is an explicit and exact expression for the inverse of a rank-one perturbation of a known matrix. The inverse of the unperturbed matrix is modified by a correction that is easy to evaluate [67].

The formula can also be used for expressing the response in a form that isolates the non-linear term [68].

$$\left[\mathbf{A} + \mathbf{u}\mathbf{v}^{\mathbf{T}}\right]^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^{\mathbf{T}}\mathbf{A}^{-1}}{1 + \mathbf{v}^{\mathbf{T}}\mathbf{A}^{-1}\mathbf{u}} \quad (4.57)$$

Applying the Sherman-Morrison formula to Eq. (4.56) yields the following results: The number of absorbers in the system, here, is designated by i.

$i = 0$ (no absorber):

$$\boldsymbol{\alpha}_0 = \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}}\right]^{-1} \tag{4.58}$$

$i = 1$ (one absorber):

$$\boldsymbol{\alpha}_1 = \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\right]^{-1} = \boldsymbol{\alpha}_0 - \frac{\boldsymbol{\alpha}_0\mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\boldsymbol{\alpha}_0}{1 + \mathbf{Z}_1^{\mathbf{T}}\boldsymbol{\alpha}_0\mathbf{Z}_1} \tag{4.59}$$

$i = 2$ (two absorbers):

$$\begin{aligned}
\boldsymbol{\alpha}_2 &= \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \mathbf{Z}_2\mathbf{Z}_2^{\mathbf{T}}\right]^{-1} \\
&= \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\right]^{-1} - \frac{\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\right]^{-1}\mathbf{Z}_2\mathbf{Z}_2^{\mathbf{T}}\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\right]^{-1}}{1 + \mathbf{Z}_2^{\mathbf{T}}\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}}\right]^{-1}\mathbf{Z}_2} \\
&= \boldsymbol{\alpha}_1 - \frac{\boldsymbol{\alpha}_1\mathbf{Z}_2\mathbf{Z}_2^{\mathbf{T}}\boldsymbol{\alpha}_1}{1 + \mathbf{Z}_2^{\mathbf{T}}\boldsymbol{\alpha}_1\mathbf{Z}_2}
\end{aligned} \tag{4.60}$$

$i = n$ (n absorbers):

$$\begin{aligned}
\boldsymbol{\alpha}_{\mathbf{n}} &= \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \mathbf{Z}_2\mathbf{Z}_2^{\mathbf{T}} + \dots\dots\dots + \mathbf{Z}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n-1}}^{\mathbf{T}} + \mathbf{Z}_{\mathbf{n}}\mathbf{Z}_{\mathbf{n}}^{\mathbf{T}}\right]^{-1} \\
&= \left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \dots + \mathbf{Z}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n-1}}^{\mathbf{T}}\right]^{-1} \\
&- \frac{\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \dots + \mathbf{Z}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n-1}}^{\mathbf{T}}\right]^{-1}\mathbf{Z}_{\mathbf{n}}\mathbf{Z}_{\mathbf{n}}^{\mathbf{T}}\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \dots + \mathbf{Z}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n-1}}^{\mathbf{T}}\right]^{-1}}{1 + \mathbf{Z}_{\mathbf{n}}^{\mathbf{T}}\left[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{add}} + \mathbf{Z}_1\mathbf{Z}_1^{\mathbf{T}} + \dots + \mathbf{Z}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n-1}}^{\mathbf{T}}\right]^{-1}\mathbf{Z}_{\mathbf{n}}} \\
&= \boldsymbol{\alpha}_{\mathbf{n-1}} - \frac{\boldsymbol{\alpha}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n}}\mathbf{Z}_{\mathbf{n}}^{\mathbf{T}}\boldsymbol{\alpha}_{\mathbf{n-1}}}{1 + \mathbf{Z}_{\mathbf{n}}^{\mathbf{T}}\boldsymbol{\alpha}_{\mathbf{n-1}}\mathbf{Z}_{\mathbf{n}}}
\end{aligned} \tag{4.61}$$

In general, by applying the Sherman-Morrison formula sequentially, the system receptance is found in terms of the receptance of the system with one less absorber:

$$\boldsymbol{\alpha}_{\mathbf{j}} = \boldsymbol{\alpha}_{\mathbf{j-1}} - \frac{\boldsymbol{\alpha}_{\mathbf{j-1}}\mathbf{Z}_{\mathbf{j}}\mathbf{Z}_{\mathbf{j}}^{\mathbf{T}}\boldsymbol{\alpha}_{\mathbf{j-1}}}{1 + \mathbf{Z}_{\mathbf{j}}^{\mathbf{T}}\boldsymbol{\alpha}_{\mathbf{j-1}}\mathbf{Z}_{\mathbf{j}}} \tag{4.62}$$

The main outcomes can be summarized as follows:

- In general, the mass, damping and stiffness matrices are symmetric, and the parameters of the absorbers have no effect on the off-diagonal terms.

- The parameters of the absorbers have no effect on the main system's mass matrix.

- The damping and stiffness matrices have similar patterns due to the presence of both damper and spring between each of the masses.

- The parameters of the absorbers ($\mathrm{k_{a1}, c_{a1}, ..., k_{an}, c_{an}}$) appear on the n x n $^{\text{th}}$ elements of the main system damping and stiffness matrices if the n$^{\text{th}}$ mass has direct attachment with the absorber.

- If the number of absorbers is equal to the number of main masses in the system, then

$$\mathbf{C_{sec}} = \mathbf{C_{add}},\ \mathbf{K_{sec}} = \mathbf{K_{add}}, \tag{4.63}$$

otherwise

$$\mathbf{C_{sec}} = \mathbf{C_{add}^T},\ \mathbf{K_{sec}} = \mathbf{K_{add}^T}. \tag{4.64}$$

- In the matrices $\mathbf{C_{add}}$ and $\mathbf{K_{add}}$, $\mathrm{c_{a1}, c_{a2}, ..., c_{an}}$ and $\mathrm{k_{a1}, k_{a2}, ..., k_{an}}$ terms respectively appear a number of times equal to the number of attachments between absorbers and main system masses.

- In the $\mathbf{Z_x}$ matrix, terms such as $i\omega c_{a1} + k_{a1}$, $i\omega c_{a2} + k_{a2}, ..., i\omega c_{an} + k_{an}$ appear a number of times equal to the number of attachments between absorber and main system masses.

- Absorber parameters can exist in off-diagonal terms if and only if the absorbers' spring and damper are directly attached to more than one of the main masses. This situation, however, is not physically realizable.

- The number of $\mathbf{Z_1 Z_1^T}$, $\mathbf{Z_2 Z_2^T}$... terms increases with the number of attachments between main masses and absorbers.

- The Sherman-Morrison formula cannot be applied in a single step formulation for the general case of m masses with n absorbers.

4.3. Optimization of Absorber Parameters in Discrete Mechanical Systems

This section develops a new method for optimizing the absorber parameters in a discrete mechanical system. With the aid of the method, the parameters of the system absorbers can be easily and efficiently optimized, based on the minimization of the amplitude of the vibration over a specified frequency range or at resonance frequency, etc. Any objective function can be used, including a mixed goal with different weights applied to various displacements of the main masses. The transfer function of the main system is denoted $\mathbf{G_{sys}}(s)$. The weighting functions of the input forces and output displacements are $\mathbf{W_f}(s)$ and $\mathbf{W_y}(s)$ respectively. The desired output with weighing functions of the input and output can be defined (see Figure 2.5).

As described previously, the equations of motion of both the main system and absorber system are derived. The equations of motion reveal the form of sub-matrices (M_{main}, C_{main}, K_{main}, C_{add}, K_{add}, $C_{\sec}$, $K_{\sec}$), which are used to construct the dynamic stiffness matrices of the system (in particular Z_{main} and Z_{add}).

For the general case of m masses with n absorbers, the system receptance is obtained sequentially using the Sherman-Morrison formula as described in Eqs. (4.58) through (4.62). The displacement expression follows directly from the system receptance.

Once the displacement expression is obtained in terms of the absorber parameters and driving frequency, the desired value of the displacement which can be any specified constant (including zero) or the average value of the displacement over the frequency range that is of interest, is obtained. In order to minimize the fluctuations of the vibration amplitudes in the frequency range of interest, the goal function is chosen to be the area as the absolute difference between the displacement curve and its average value. This goal function is calculated by the Gaussian quadrature numerical integration method [69, 70, 71].

The Gaussian quadrature numerical integration method is utilized on the plot of the displacement versus the frequency. Finally, the goal function is optimized using the Matlab optimization toolbox. The method described in this chapter can accomplish the following tasks:

- Find the optimum absorber parameters according to the predefined goal (operation frequency range, resonance frequency, etc.)

- Analyze discrete systems of m masses and n absorbers, where m and n are any positive integers, provided each mass is attached to no more than one absorber.

- Express the receptance explicitly (receptance of a main system with 1 absorber, 2 absorbers ..... n absorbers are derived sequentially).

- Display plots of receptance and displacement versus frequency.

5. CASE STUDIES FOR VIBRATION SUPPRESSION OF TRANSLATIONAL MASS-SPRING-DAMPER SYSTEMS

Numerical studies are examined to validate the method of Chapter 4. First, a system with two masses and two absorbers is studied (Figure 5.1). The aim is to minimize the displacements of both main masses (m_1 and m_2) with equal weighting.

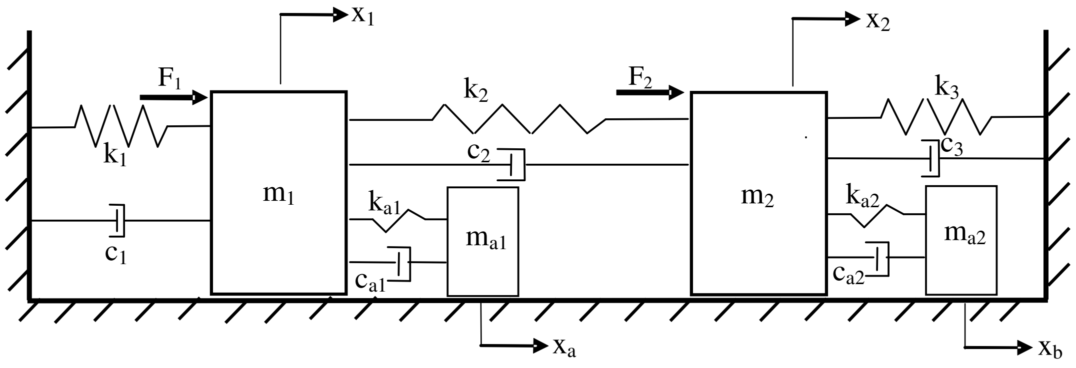

Figure 5.1. Two main masses with two absorbers system

5.1. Case Study 1

The linear graph of the system is shown in Figure 5.2; an external force is applied to the first main mass (m_1). The dynamic stiffness between x_1 and F is found via the linear graph method.

$$
\begin{aligned}
Z_1 &= m_1 s^2 + c_1 s + k_1 \\
Z_2 &= m_2 s^2 + c_3 s + k_3 \\
Z_3 &= m_{a1} s^2 \\
Z_4 &= m_{a2} s^2 \\
Z_5 &= c_2 s + k_2 \\
Z_{a1} &= c_{a1} s + k_{a1} \\
Z_{a2} &= c_{a2} s + k_{a2}
\end{aligned}
\tag{5.1}
$$

$$Z_{x_1F} = \left[\frac{1}{c_{a1}s + k_{a1}} + \frac{1}{m_{a1}s^2} \right]^{-1} + m_1 s^2 + c_1 s + k_1$$
$$+ \left[\frac{1}{c_2 s + k_2} + \left[\left[\frac{1}{m_{a2}s^2} + \frac{1}{c_{a2}s + k_{a2}} \right]^{-1} + m_2 s^2 + c_3 s + k_3 \right]^{-1} \right]^{-1} \tag{5.2}$$

The same result can be verified by solving the relation between x_1 and F_1 using the equations of motion of the main and absorber systems as follows.

$$-k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + k_{a1}(x_a - x_1) + c_{a1}(\dot{x}_a - \dot{x}_1) + F_1 = m_1\ddot{x}_1 \tag{5.3}$$
$$-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - k_3x_2 - c_3\dot{x}_2 + k_{a2}(x_b - x_2) + c_{a2}(\dot{x}_b - \dot{x}_2) = m_2\ddot{x}_2 \tag{5.4}$$
$$-k_{a1}(x_a - x_1) - c_{a1}(\dot{x}_a - \dot{x}_1) = m_{a1}\ddot{x}_a \tag{5.5}$$
$$-k_{a2}(x_b - x_2) - c_{a2}(\dot{x}_b - \dot{x}_2) = m_{a2}\ddot{x}_b \tag{5.6}$$

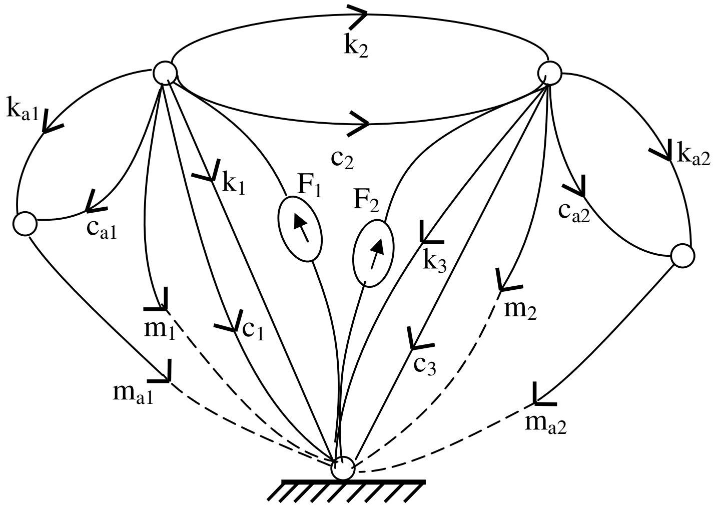

Figure 5.2. Linear graph of “two masses with two absorbers” system

The receptance is found by applying the formulation for m masses with n absorbers (derived in Section 4.2). Also, the dynamic stiffness expression can be found by applying the linear graph methodology. The linear graph methodology is used in this study in order to validate the approach defined in Section 4.2. The same dynamic stiffness expression is found through both approaches.

The input weighting function is

$$W_f(s) = \left[\left| \frac{4000}{(s+40)^2 + 225} \right| ;0 \right]. \tag{5.7}$$

The other input parameters are m_1 = 3 kg, m_2 = 2 kg, c_1 = 0.8 N s/m, c_2 = 2 N s/m, c_3 = 3 N s/m, k_1 = 4000 N/m, k_2 = 2000 N/m, k_3 = 5000 N/m, m_{a1} = 0.6 kg, and m_{a2} = 0.7 kg.

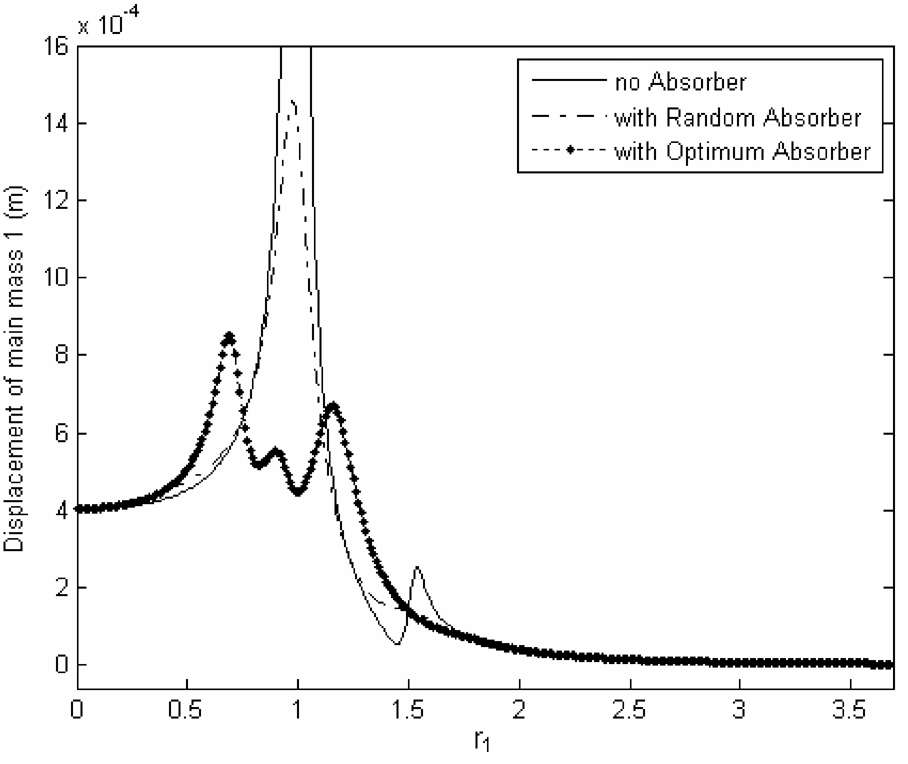

Figure 5.3. Displacement of main mass 1

The displacements of mass 1 (m_1) and mass 2 (m_2) are plotted in Figures 5.3 and 5.4 respectively. The results are compared to the cases of no absorber and absorbers with randomly selected parameters. In both plots, the randomly selected parameters are c_{a1} = 10 N s/m, c_{a2} = 20 N s/m, k_{a1} = 200 N/m, and k_{a2} = 400 N/m.

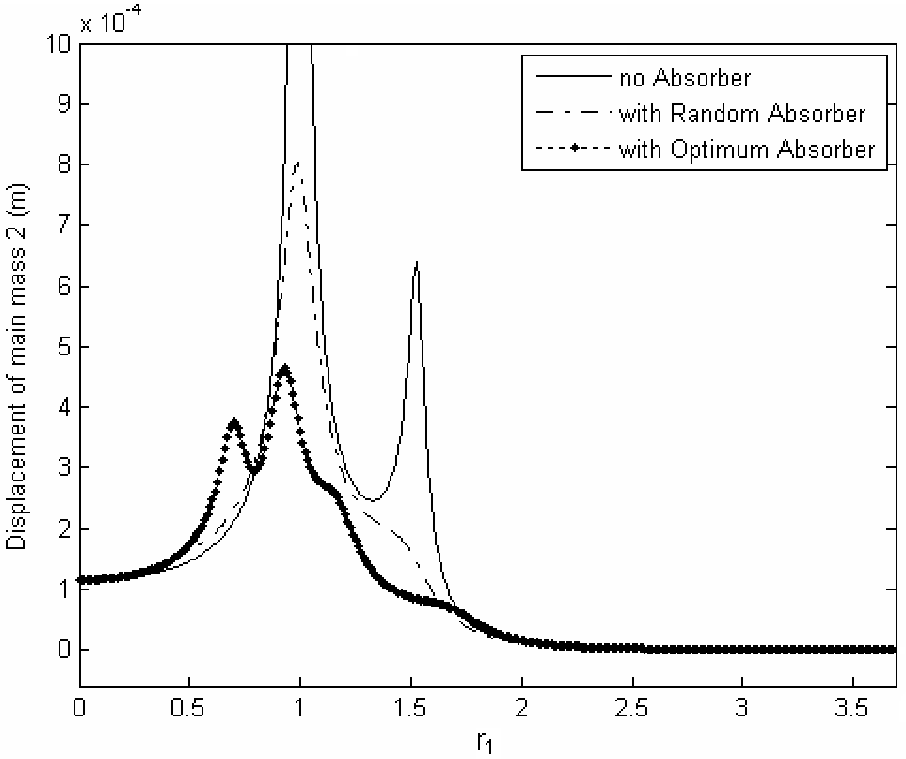

Figure 5.4. Displacement of main mass 2

In the optimization, the tolerances are very narrow so that, regardless of the initial estimate, the algorithm converged to the same parameters. The optimal absorber parameters are found as follows: c_{a1} = 5.5 N s/m, c_{a2} = 10.7 N s/m, k_{a1} = 585.3 N/m, and k_{a2} = 1469.1 N/m.

5.2. Case Study 2 (with Different Weighting Functions)

The same system displayed in Figure 5.1 is used as a multi-input multi-output (MIMO) system. Two input forces, applied to the two main masses with their respective output main mass displacements, are analyzed. The input weighting function is;

$$W_f(s) = \left[\left|\frac{4000}{(s+40)^2+225}\right|; \left\lfloor\frac{100}{s+100}\right\rfloor\right]. \tag{5.8}$$

This case uses the same values the input parameters as in Section 5.1 (Case Study 1). A mixed-goal with different weighting functions can be defined on the displacements of the main masses. First, a higher weight is assigned to displacement x_2 than to displacement x_1 (the objective function is the area under a weighted sum of the displacement curves). The reverse situation, where x_1 is given more weight than x_2, is then analyzed.

Figures 5.5 and 5.6 show the displacements of mass 1 and mass 2 respectively where the goal function is expressed as ($0.2x_1 + 0.8x_2$). In the reverse case, the goal function is expressed as ($0.8x_1 + 0.2 x_2$). The displacements of mass 1 and mass 2 are shown in Figures 5.7 and 5.8, respectively.

It can be concluded from Figures 5.5-5.8 that the displacement of main mass which has the higher weighting coefficient (in other words high effect on the goal function) is the one that should be more suppressed in order to reach the smallest goal function. As a result, the optimization is set to work on the predefined aim of minimizing the amplitudes in the desired weights.

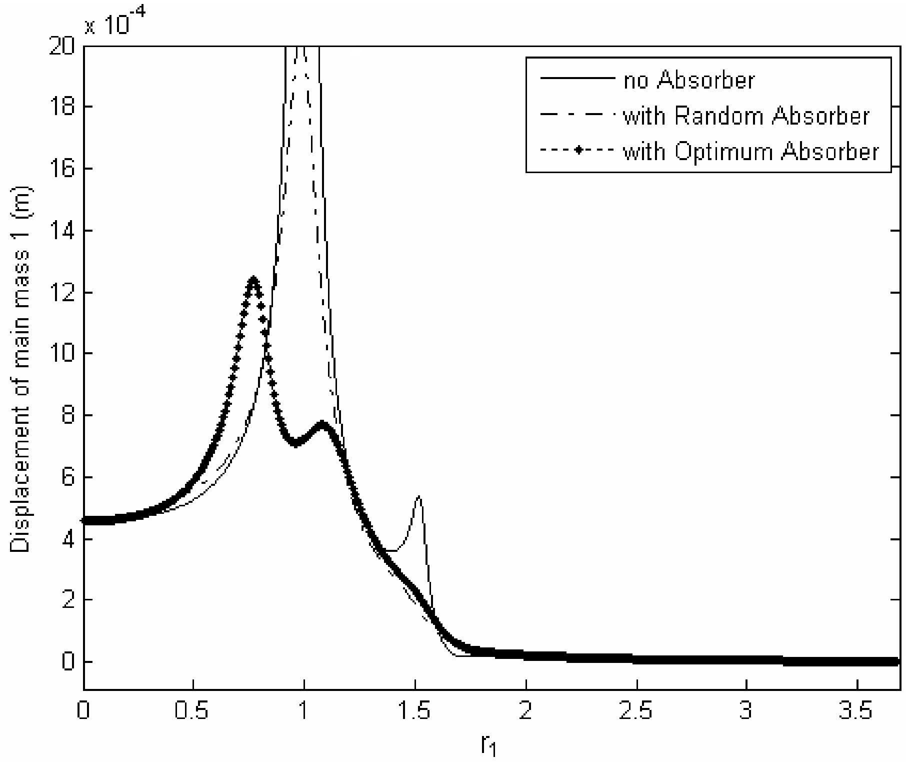

Figure 5.5. Displacement of main mass 1 – objective function : $0.2x_1+0.8x_2$

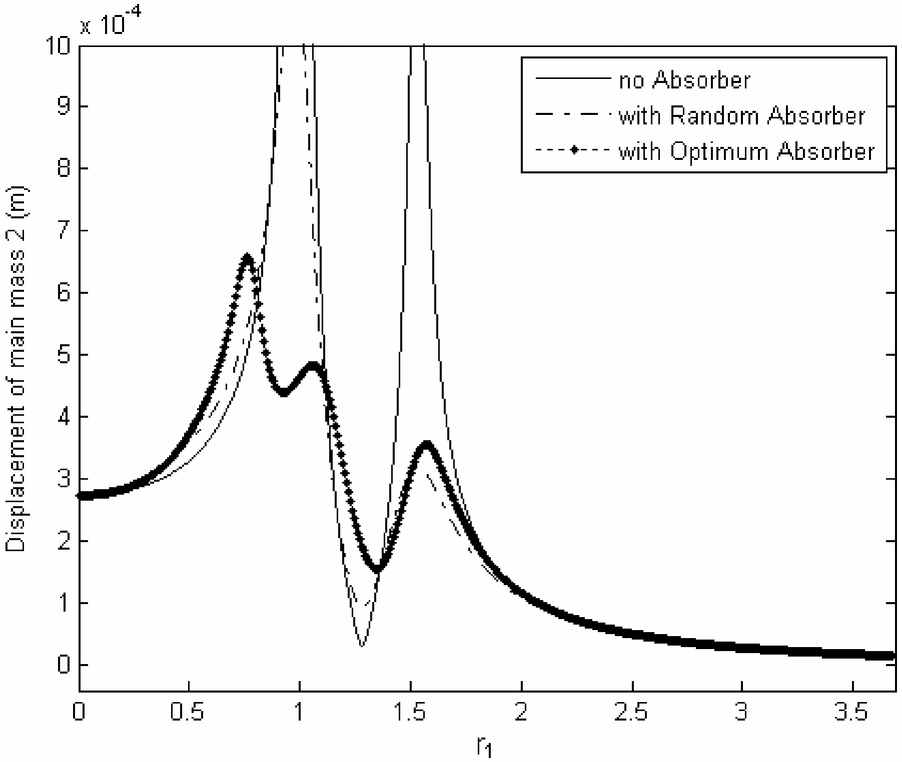

Figure 5.6. Displacement of main mass 2 – objective function : $0.2x_1+0.8x_2$

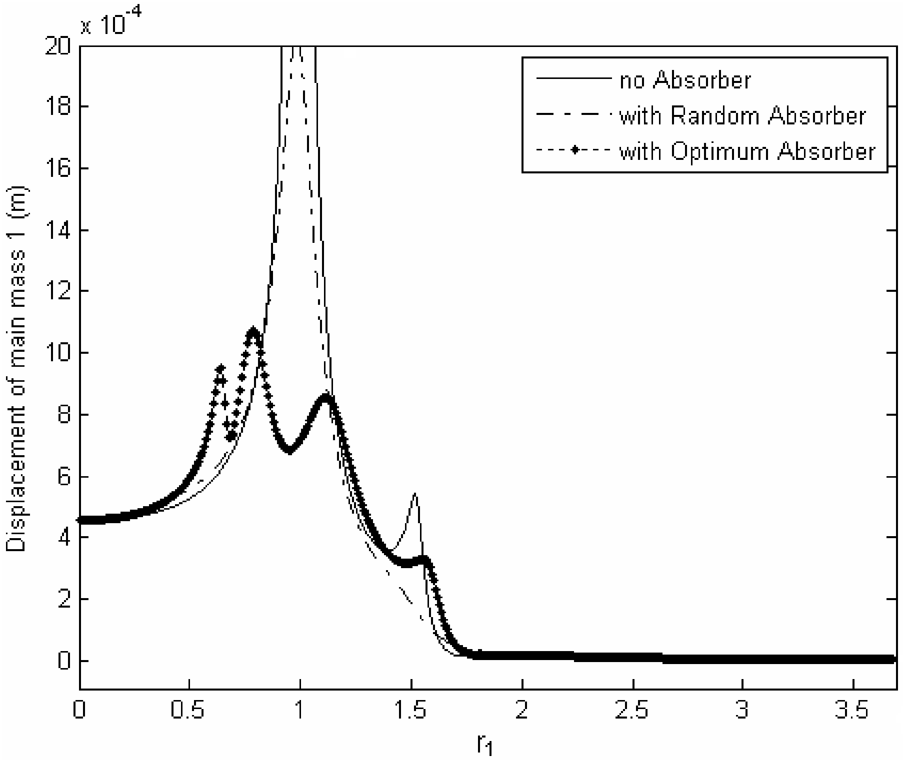

Figure 5.7. Displacement of main mass 1 – objective function : $0.8x_1+0.2x_2$

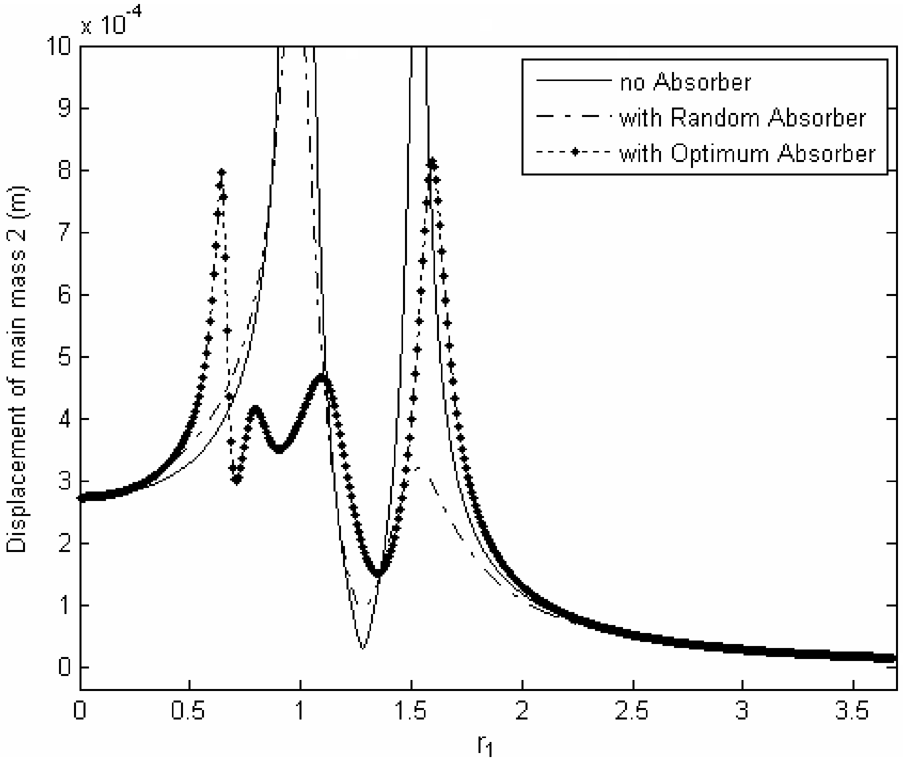

Figure 5.8. Displacement of main mass 2 – objective function : $0.8x_1+0.2x_2$

5.3. Case Study 3 (with constrained minimization)

Figure 5.9 shows an example with constrained minimization. A system of four main masses with three absorbers is used. The aim is to minimize the vibration of main mass 3, subject to $c_{ai} < 10$ N s/m and $k_{ai} < 1{,}000$ N/m.

Figures 5.10 and 5.11 show the displacements obtained with no absorber and optimal absorbers, respectively. The optimum parameters are found as: c_{a1} = 4.9 N s/m, c_{a2} = 4.9 N s/m, c_{a3} = 3.8 N s/m, k_{a1} = 574.9 N/m, k_{a2} = 336.5 N/m, and k_{a3} = 213.5 N/m.

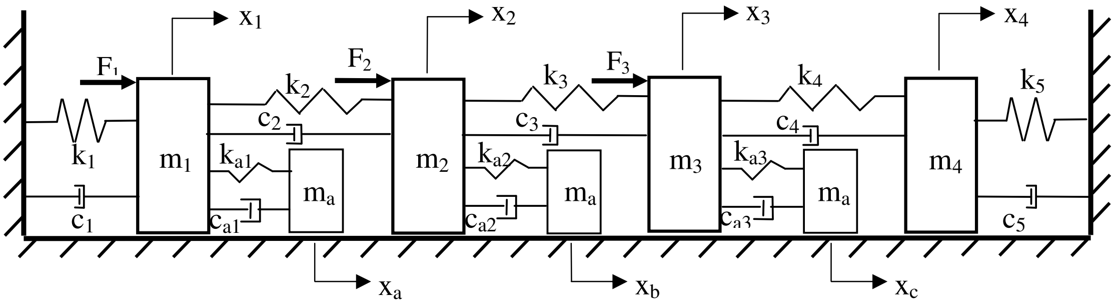

Figure 5.9. A system with four main masses and three absorbers

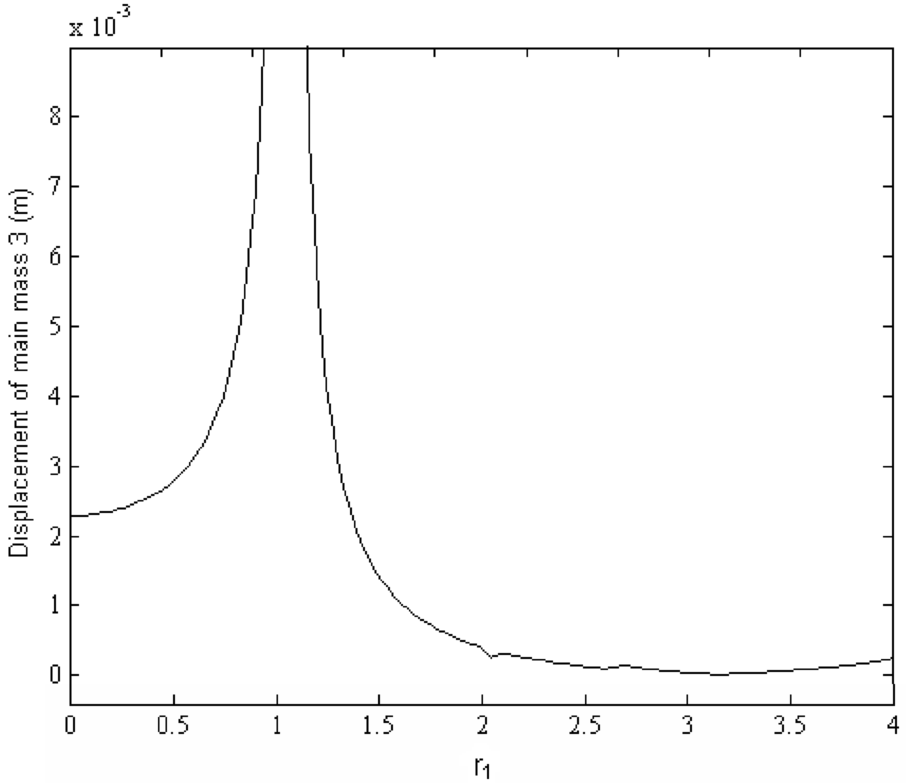

Figure 5.10. Displacement of main mass 3 – no absorber

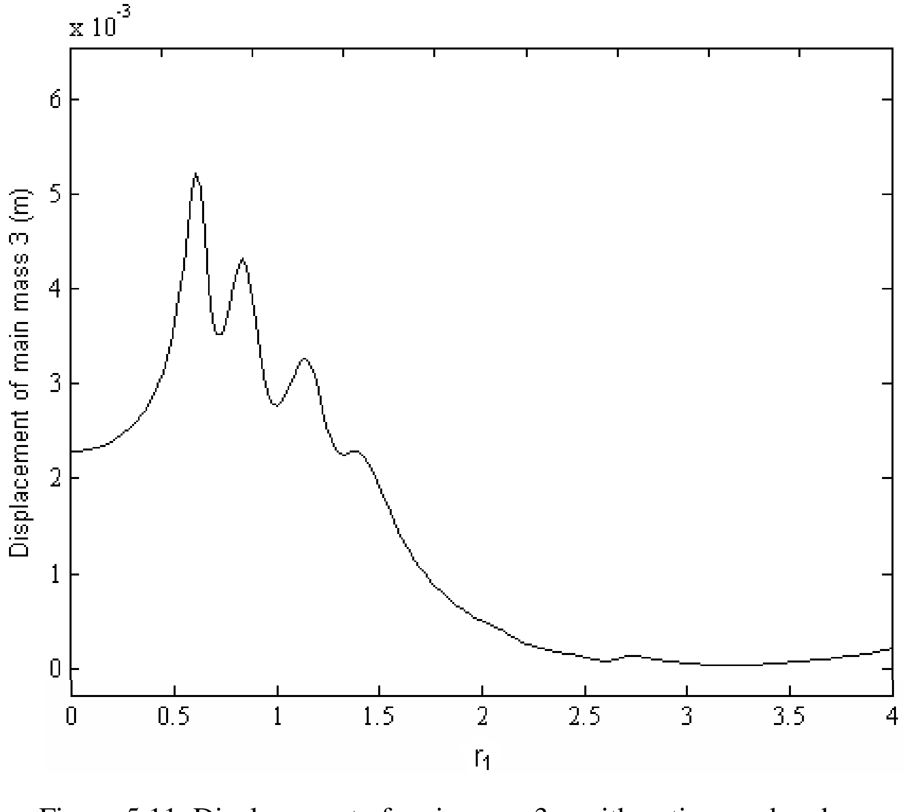

Figure 5.11. Displacement of main mass 3 – with optimum absorbers

5.4. Comparison with the Literature – Discrete Systems

5.4.1. Comparison with the Study of Ozer and Royston [31]

Ozer and Royston [31, 72] have investigated an optimization method based on the Sherman-Morrison matrix inversion formula for calculating the optimal absorber parameters of a damped multi degree of freedom system. Their method minimizes the displacement of particular masses or modes of vibration, regardless of their location relative to the placement of the absorber subsystem. Moreover, the performance index can be defined as a linear summation of the degrees of freedom. In the first case study of Ozer and Royston [31], a four degree of freedom system is used where the absorber is attached to the first mass. The target mass is chosen to be mass 1, the target mode is the first mode and the force is applied to the third mass. The aim is to find the vibration absorber parameters that result in zero vibration of the first mass at the first resonant frequency. It is observed that the Sherman-Morrison method is very effective at the resonant frequency of the first mode. The disadvantage of the method proposed by Ozer and Royston [31] is that the absorber acts like a narrow band absorber instead of having more broadband attenuation. The absorber parameters using the Sherman-Morrison method with Den Hartog's damping value are found as k_a = 87.7 N/m and c_a = 0.384 N s/m.

If the same case study and same data are used and the method defined in Chapter 4 is carried out, the optimum values for the absorber are concluded as k_a = 90.3211 N/m and c_a = 0.4249 N s/m.

Table 5.1. Optimal absorber parameters obtained by the present method and Ozer and Royston's study [31] – Case Study I

	Present Study	**Ozer and Royston's study [31]**
k_a (N/m)	90.3211	87.7
c_a (N s/m)	0.4249	0.384

The displacement of main mass one for the optimum absorber (as determined by the present study) and no absorber are displayed in Figure 5.12.

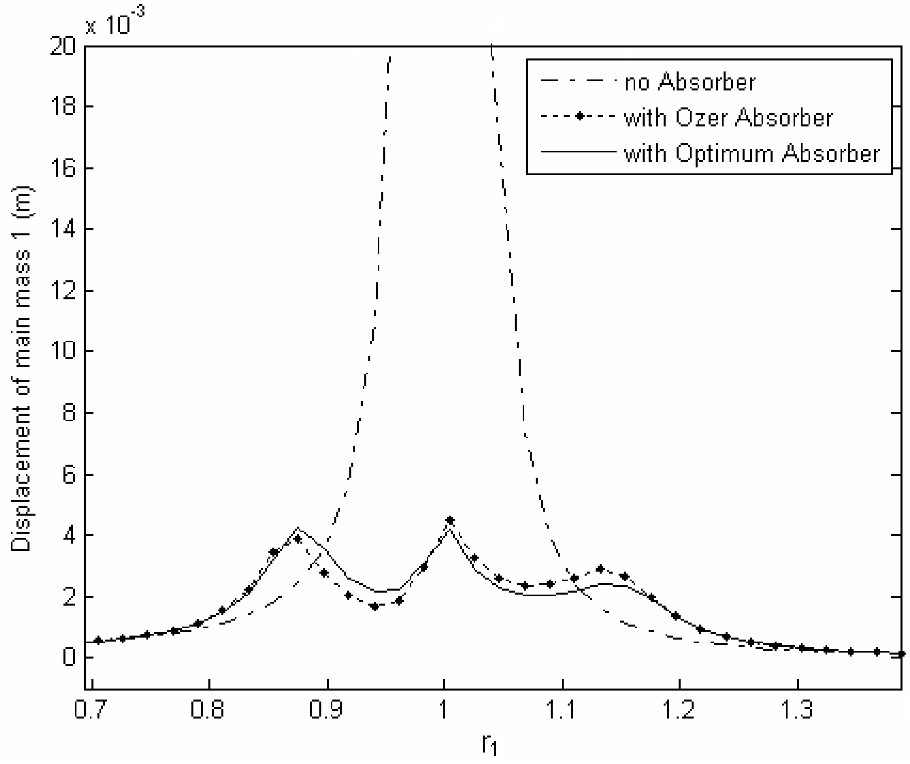

Figure 5.12. Displacement of main mass 1 – four masses with one absorber

In the second case study described by Ozer and Royston [31], a weighted linear sum of the displacements (a performance index denoted as J) is minimized instead:

$$J = 0.15x_1 + 0.35x_2 + 0.35x_3 + 0.15x_4 \tag{5.9}$$

As in the previous case, the vibration absorber is attached to the first mass and the force is applied to the third mass. The performance index is to be minimized around the first mode. The absorber parameters using the Sherman Morrison method with numerically calculated damping value are found as k_a = 173.15 N/m and c_a = 6.049 N s/m.

When the same case study (with the same input parameters) is carried out via the approach studied in Chapter 4, the optimum values for the absorber are concluded as k_a = 153.1774 N/m and c_a =3.8883 N s/m. The responses with optimum absorbers and without absorbers are shown in Figure 5.13.

Table 5.2. Optimal absorber parameters obtained by the present method and Ozer and Royston's study [31] – Case Study II

	Present Study	**Ozer and Royston's study [31]**
k_1 (N/m)	153.1774	173.15
c_1 (N s/m)	3.8883	6.049

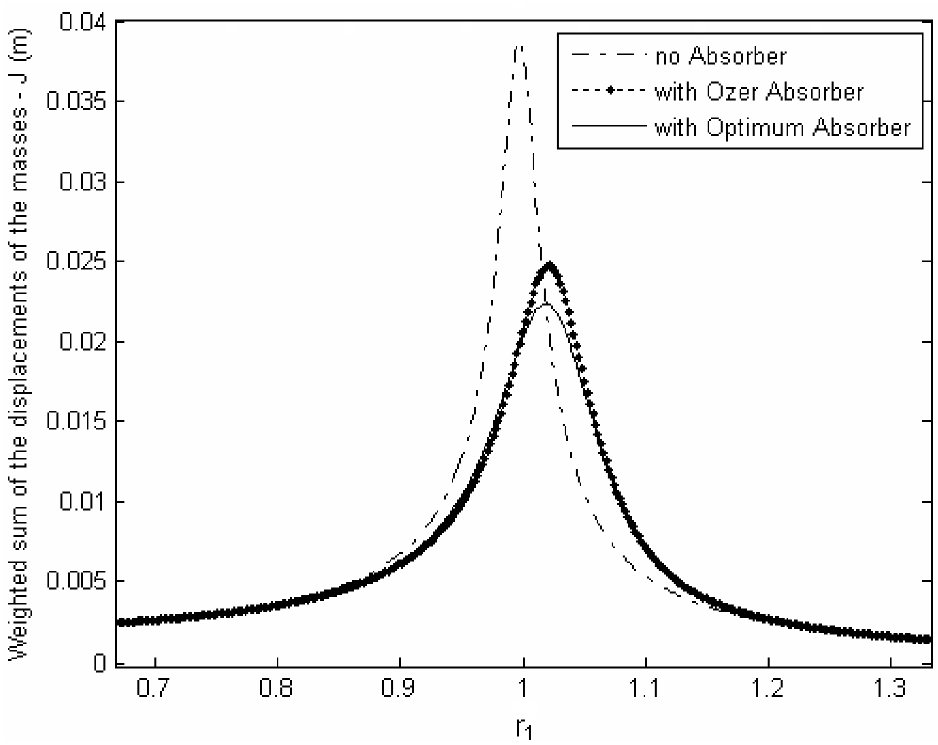

Figure 5.13. Weighted sum of the displacements of the masses

5.4.2. Comparison with the Study of Warburton and Ayorinde [56]

Warburton and Ayorinde [56, 73, 74] have investigated translational mass-spring-damper systems with damped absorbers. In the first numerical example of the study [56], a main system consisting of a mass m_1, spring of stiffness k_1 and viscous damper c_1 is considered. The main mass is subjected to harmonic force represented by $Fe^{i\omega t}$. The absorber system has mass m_a, a spring stiffness k_a, and viscous damping c_a. The authors undertook a computer study of the dependence of the maxima on the system parameters in order to find the optimum values of the tuning ratio f_{opt}, and absorber damping ratio $\gamma_{a,opt}$ for specified values of the mass ratio μ_{eff} and damping ratio γ_1.

The input values are defined as m_1 = 2 kg and k_1 = 200 N/m. For the values of $\mu_{eff} = 0.1$ and $\gamma_1 = 0.01$, Table 1 of their study [56] gives the optimal result as: $f_{opt} = 0.9051$ and $\gamma_{a,opt} = 0.187$. Based on these values,

$$m_a = \mu_{eff} m_1 = 0.2 \text{ kg} \tag{5.10}$$

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = 10 \text{ rad/s} \tag{5.11}$$

$$\omega_a = f_{opt}\omega_1 = 9.051 \text{ rad/s} \tag{5.12}$$

$$k_a = \omega_a^2 m_a = 16.3841 \text{ N/m} \tag{5.13}$$

$$c_1 = 2\gamma_1 m_1 \omega_1 = 0.4 \text{ N s/m} \tag{5.14}$$

$$c_a = 2\gamma_{a,opt} m_a \omega_a = 0.6770 \text{ N s/m}. \tag{5.15}$$

A comparison between the response curves of systems with no absorber, a randomly selected absorber, and the optimal absorber (as determined by the method of Chapter 4) is shown in Figure 5.14. The numerical results of Warburton and Ayorinde [56] are compared to this study's results in Table 5.3.

Table 5.3. Optimal absorber parameters obtained by the present method and Warburton and Ayorinde's study [56]

	Present Study	**Warburton and Ayorinde's study [56]**
k_1 (N/m)	17.2373	16.3841
c_1 (N s/m)	0.5641	0.6770

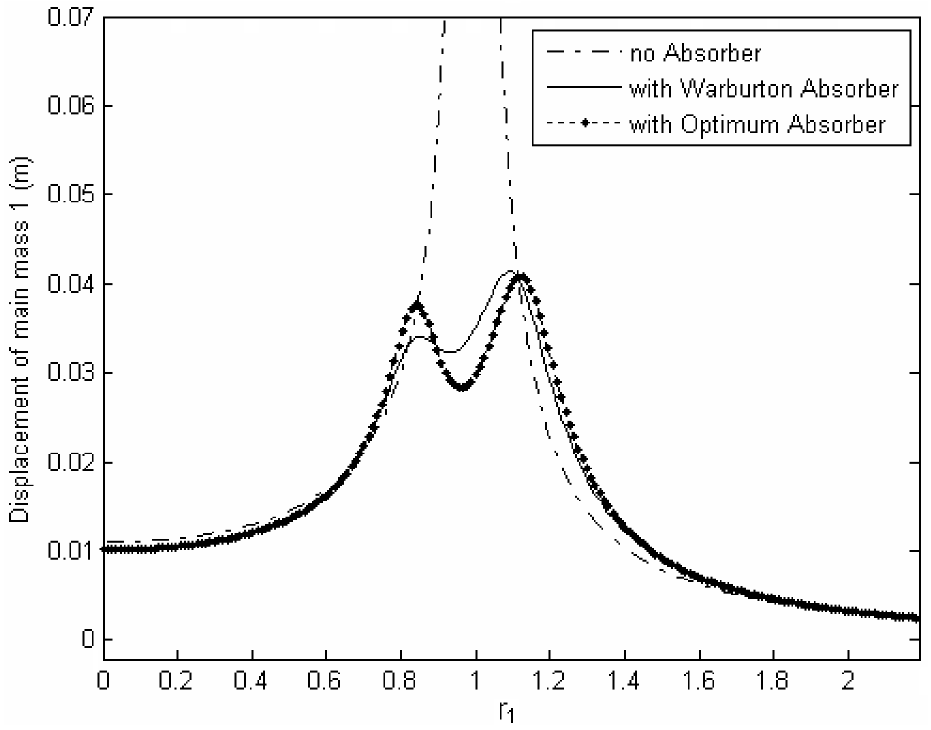

Figure 5.14. Displacement of main mass 1 – Warburton and Ayorinde's [56] case

6. MUTUAL APPLICATIONS OF VIBRATION SUPPRESSION METHODS OF CONTINUOUS AND DISCRETE SYSTEMS

In this chapter, the approach used in the vibration suppression of continuous systems will be investigated to determine whether it is also applicable to the vibration suppression of discrete systems. The method defined for discrete systems will also be applied to continuous systems. In general, a discrete system can be solved more easily than a continuous one. At the same time, the information obtained under a discrete model may not be as accurate as that obtained under a continuous model [75].

The translational mass-spring-damper system composed of three main masses and one absorber is analyzed (Figure 6.1).

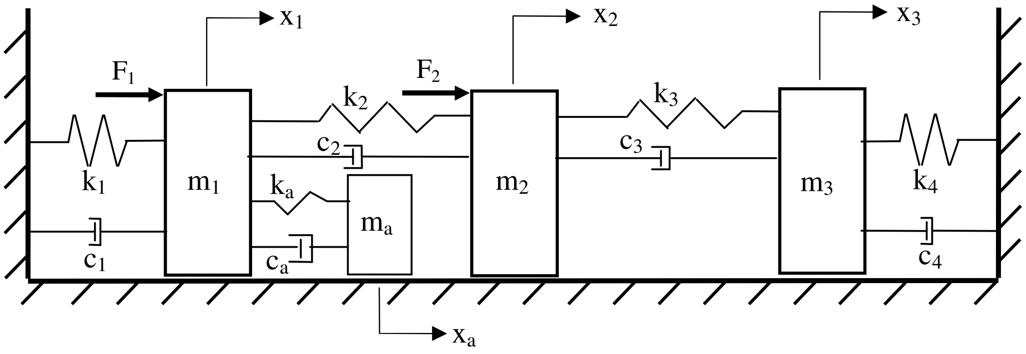

Figure 6.1. Three main masses with one absorber system

The equations of motion for the main system and absorber are displayed below.

$$-k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + k_a(x_a - x_1) + c_a(\dot{x}_a - \dot{x}_1) + F_1 = m_1\ddot{x}_1$$
$$-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + k_3(x_3 - x_2) + c_3(\dot{x}_3 - \dot{x}_2) + F_2 = m_2\ddot{x}_2$$
$$-k_3(x_3 - x_2) - c_3(\dot{x}_3 - \dot{x}_2) - k_4x_3 - c_4\dot{x}_3 = m_3\ddot{x}_3 \tag{6.1}$$

$$-k_a(x_a - x_1) - c_a(\dot{x}_a - \dot{x}_1) = m_a\ddot{x}_a \tag{6.2}$$

Define the following generic state variables:

$$\begin{aligned} X_1 &= x_1 & X_5 &= x_3 \\ X_2 &= \dot{x}_1 & X_6 &= \dot{x}_3 \\ X_3 &= x_2 & X_7 &= x_a \\ X_4 &= \dot{x}_2 & X_8 &= \dot{x}_a \end{aligned} \tag{6.3}$$

The resulting state equations are displayed below.

$$\begin{aligned}
\dot{X}_1 &= X_2 \\
\dot{X}_2 &= -\frac{c_1+c_2+c_a}{m_1}X_2 + \frac{c_2}{m_1}X_4 - \frac{k_1+k_2+k_a}{m_1}X_1 + \frac{k_2}{m_1}X_3 + \frac{c_a}{m_1}X_8 + \frac{k_a}{m_1}X_7 + \frac{F_1}{m_1} \\
\dot{X}_3 &= X_4 \\
\dot{X}_4 &= \frac{c_2}{m_2}X_2 - \frac{c_2+c_3}{m_2}X_4 + \frac{c_3}{m_2}X_6 + \frac{k_2}{m_2}X_1 - \frac{k_2+k_3}{m_2}X_3 + \frac{k_3}{m_2}X_5 + \frac{F_2}{m_2} \\
\dot{X}_5 &= X_6 \\
\dot{X}_6 &= \frac{c_3}{m_3}X_4 - \frac{c_3+c_4}{m_3}X_6 - \frac{k_3+k_4}{m_3}X_5 + \frac{k_3}{m_3}X_3 \\
\dot{X}_7 &= X_8 \\
\dot{X}_8 &= -\frac{c_a}{m_a}X_8 - \frac{k_a}{m_a}X_7 + \frac{c_a}{m_a}X_2 + \frac{k_a}{m_a}X_1
\end{aligned} \tag{6.4}$$

$\mathbf{G}_{\mathbf{sys}}(\mathrm{s})$ denotes the main system transfer function, and $\mathbf{W}_{\mathbf{f}}(\mathrm{s})$ is the a weighting function for the input forces.

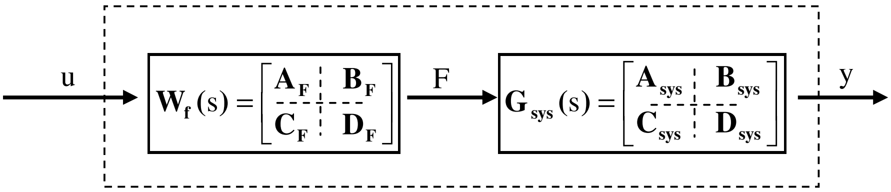

Figure 6.2. Transfer function diagram

The main system's ($G_{sys}(s)$) state space representation is found by the Equations (6.4) and it is displayed below:

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\\ \dot{X}_5\\ \dot{X}_6\\ \dot{X}_7\\ \dot{X}_8\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -\frac{k_1+k_2+k_a}{m_1} & -\frac{c_1+c_2+c_a}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 & \frac{k_a}{m_1} & \frac{c_a}{m_1}\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \frac{k_a}{m_a} & \frac{c_a}{m_a} & 0 & 0 & 0 & 0 & -\frac{k_a}{m_a} & -\frac{c_a}{m_a}\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7\\ X_8\end{bmatrix}$$

$$+\begin{bmatrix}0 & 0\\ \frac{1}{m_1} & 0\\ 0 & 0\\ 0 & \frac{1}{m_2}\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix} \tag{6.5}$$

In this framework, the displacement of interest (that of mass 2) can be written as follows:

$$y=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7\\ X_8\end{bmatrix} \tag{6.6}$$

The external forces are displayed below:

$$W_f(s)=\begin{bmatrix}\frac{4000}{(s+40)^2+225} & \frac{100}{s+100}\end{bmatrix} \tag{6.7}$$

$$\begin{bmatrix}\dot{x}_{F1}\\ \dot{x}_{F2}\\ \dot{x}_{F3}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -182500 & -9825 & -180\end{bmatrix}\begin{bmatrix}x_{F1}\\ x_{F2}\\ x_{F3}\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \tag{6.8}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 400000 & 4000 & 0 \\ 182500 & 8000 & 100 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{F1} \\ x_{F2} \\ x_{F3} \end{bmatrix} \tag{6.9}$$

$$y = G_{sys}(s) W_f(s) u = G_{total}(s)\, u \tag{6.10}$$

The total system's $G_{total}(s)$ state space representation is displayed below:

$$\begin{bmatrix} \dot{X} \\ \dot{x}_F \end{bmatrix} = \begin{bmatrix} A_{sys} & B_{sys} C_F \\ 0 & A_F \end{bmatrix} \begin{bmatrix} X \\ x_F \end{bmatrix} + \begin{bmatrix} 0 \\ B_F \end{bmatrix} u = A_{total} \begin{bmatrix} X \\ x_F \end{bmatrix} + B_{total}\, u$$

$$y = \begin{bmatrix} C_{sys} & 0 \end{bmatrix} \begin{bmatrix} X \\ x_F \end{bmatrix} = C_{total} \begin{bmatrix} X \\ x_F \end{bmatrix} \tag{6.11}$$

The optimization is performed on the square of the H_2 norm ($\|G_{tot}\|_2^2$) of the system:

$$\|G_{tot}\|_2^2 = C_{total} L C_{total}{}^T, \tag{6.12}$$

where L satisfies the Lyapunov equation,

$$A_{total} L + L A_{total}{}^T + B_{total} B_{total}{}^T = 0. \tag{6.13}$$

Thus, by minimizing the H_2 norm, the output power of the generalized system is minimized. The optimum absorber parameters are k_a = 459.56 N/m and c_a = 4.57 N s/m. The system response with an optimal absorber is shown in Figure 6.3, along with the cases of no absorber and a randomly selected absorber.

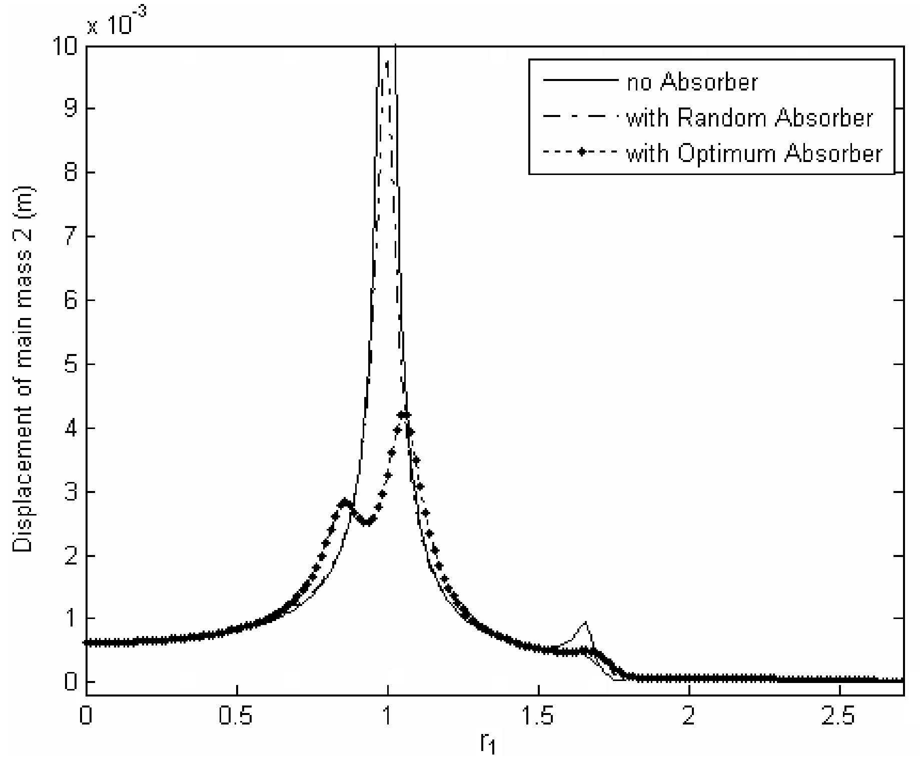

Figure 6.3. Three main masses with one absorber system

As a next step, the approach used in the vibration suppression of discrete systems is investigated to determine whether it is also applicable to the vibration suppression of continuous systems. One absorber attached to a fixed-free beam is taken for analyzing the relationship between discrete and continuous systems.

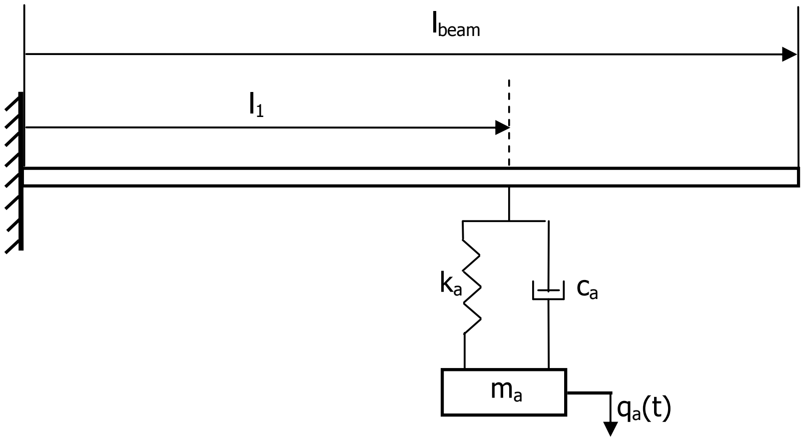

Figure 6.4. One absorber attached to a fixed-free beam (two mode shapes)

The equations of motion for the main system and the absorber are derived below. The following energy equations apply:

$$T(t) = \frac{1}{2} m_u \int_0^{l_{beam}} \left(\frac{\partial \omega}{\partial t} \right)^2 dx + \frac{1}{2} m_a \dot{q}_a{}^2(t) \tag{6.14}$$

$$V(t) = \frac{1}{2} EI \int_0^{l_{beam}} \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 dx + \frac{1}{2} k_a \left(\omega(l_1, t) - q_a(t) \right)^2 \tag{6.15}$$

$$Q_{nc1} = \frac{1}{2} c_a \left(\dot{\omega}^2(l_1, t) - \dot{q}_a(t) \right)^2 \tag{6.16}$$

The equation of motion for the beam is as follows:

$$\begin{bmatrix} m_u \int_0^{l_{beam}} u_1(x) u_1(x) dx & 0 & 0 \\ 0 & m_u \int_0^{l_{beam}} u_1(x) u_1(x) dx & 0 \\ 0 & 0 & m_u \int_0^{l_{beam}} u_1(x) u_1(x) dx \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}$$

$$+ \begin{bmatrix} c_a u_1(l_1) u_1(l_1) & c_a u_2(l_1) u_1(l_1) & c_a u_3(l_1) u_1(l_1) \\ c_a u_1(l_1) u_2(l_1) & c_a u_2(l_1) u_2(l_1) & c_a u_3(l_1) u_2(l_1) \\ c_a u_1(l_1) u_3(l_1) & c_a u_2(l_1) u_3(l_1) & c_a u_3(l_1) u_3(l_1) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$+ \begin{bmatrix} EI \int_0^{l_{beam}} \frac{d^2 u_1}{dx^2} \frac{d^2 u_1}{dx^2} dx + k_a u_1(l_1) u_1(l_1) & \cdots & EI \int_0^{l_{beam}} \frac{d^2 u_1}{dx^2} \frac{d^2 u_3}{dx^2} dx + k_a u_1(l_1) u_3(l_1) \\ \vdots & \ddots & \vdots \\ EI \int_0^{l_{beam}} \frac{d^2 u_3}{dx^2} \frac{d^2 u_1}{dx^2} dx + k_a u_3(l_1) u_1(l_1) & \cdots & EI \int_0^{l_{beam}} \frac{d^2 u_3}{dx^2} \frac{d^2 u_3}{dx^2} dx + k_a u_3(l_1) u_3(l_1) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{l_{beam}} f(x,t) u_1(x) dx \\ \int_0^{l_{beam}} f(x,t) u_2(x) dx \\ \int_0^{l_{beam}} f(x,t) u_3(x) dx \end{bmatrix} \tag{6.17}$$

The relationship between displacement and the driving force is known as the dynamic stiffness $\mathbf{Z}$ of the system:

$$\mathbf{Zx} = \mathbf{F} \tag{6.18}$$

where

$$\mathbf{Z} = \mathbf{Z_x} + \mathbf{Z_{add}} \tag{6.19}$$

and

$$\mathbf{Z_x} = \left[-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}\right] \tag{6.20}$$

Thus,

$$\begin{aligned} Z = -\omega^2 &\begin{bmatrix} m_u \int_0^{l_{beam}} u_1(x)u_1(x)dx & 0 & 0 \\ 0 & m_u \int_0^{l_{beam}} u_1(x)u_1(x)dx & 0 \\ 0 & 0 & m_u \int_0^{l_{beam}} u_1(x)u_1(x)dx \end{bmatrix} \\ &+ \begin{bmatrix} EI\int_0^{l_{beam}} \frac{d^2u_1}{dx^2}\frac{d^2u_1}{dx^2}dx & \cdots & EI\int_0^{l_{beam}} \frac{d^2u_1}{dx^2}\frac{d^2u_3}{dx^2}dx \\ \vdots & \ddots & \vdots \\ EI\int_0^{l_{beam}} \frac{d^2u_3}{dx^2}\frac{d^2u_1}{dx^2}dx & \cdots & EI\int_0^{l_{beam}} \frac{d^2u_3}{dx^2}\frac{d^2u_3}{dx^2}dx \end{bmatrix} + Z_{add} \end{aligned} \tag{6.21}$$

The last term can be written out as follows:

$$Z_{add} = \begin{bmatrix} i\omega c_a u_1(l_1)u_1(l_1) + k_a u_1(l_1)u_1(l_1) & \cdots & i\omega c_a u_3(l_1)u_1(l_1) + k_a u_3(l_1)u_1(l_1) \\ \vdots & \ddots & \vdots \\ i\omega c_a u_1(l_1)u_3(l_1) + k_a u_1(l_1)u_3(l_1) & \cdots & i\omega c_a u_3(l_1)u_3(l_1) + k_a u_3(l_1)u_3(l_1) \end{bmatrix} \tag{6.22}$$

Z_{add} can also be separated into row and column matrices.

$$Z_{add} = Z_1 Z_1^T + Z_2 Z_2^T$$
$$= \begin{bmatrix} i\omega c_a u_1(l_1) \\ i\omega c_a u_2(l_1) \\ i\omega c_a u_3(l_1) \end{bmatrix} \begin{bmatrix} u_1(l_1) & u_2(l_1) & u_3(l_1) \end{bmatrix} + \begin{bmatrix} k_a u_1(l_1) \\ k_a u_2(l_1) \\ k_a u_3(l_1) \end{bmatrix} \begin{bmatrix} u_1(l_1) & u_2(l_1) & u_3(l_1) \end{bmatrix} \quad (6.23)$$

For the case of n absorbers attached to a beam, the following formulation is found:

$$\mathbf{Z} = \mathbf{Z_x} + \mathbf{Z_1 Z_1^T} + \mathbf{Z_2 Z_2^T} + \mathbf{Z_3 Z_3^T} + \ldots\ldots\ldots \mathbf{Z_n Z_n^T} \quad (6.24)$$

The system receptance is displayed as follows:

$$\boldsymbol{\alpha} = \mathbf{Z}^{-1} = \left[\mathbf{Z_x} + \mathbf{Z_1 Z_1^T} + \mathbf{Z_2 Z_2^T} + \mathbf{Z_3 Z_3^T} + \ldots\ldots\ldots \mathbf{Z_n Z_n^T}\right]^{-1} \quad (6.25)$$

In general, by applying the Sherman-Morrison formula sequentially, the system receptance is found in terms of the receptance of the system with one less absorber:

$i = 1$ (one absorber):

$$\boldsymbol{\alpha_1} = \left[\mathbf{Z_{main}} + \mathbf{Z_{add}} + \mathbf{Z_1 Z_1^T}\right]^{-1} = \boldsymbol{\alpha_0} - \frac{\boldsymbol{\alpha_0} \mathbf{Z_1 Z_1^T} \boldsymbol{\alpha_0}}{1 + \mathbf{Z_1^T} \boldsymbol{\alpha_0} \mathbf{Z_1}} \quad (6.26)$$

The system response with an optimal absorber is shown in Figure 6.5, along with the cases of no absorber and a randomly selected absorber. The optimum parameter values are $l_1 = 1.000$ m, $k_1 = 1.7088$ N/m, and $c_1 = 0.1044$ N s/m .

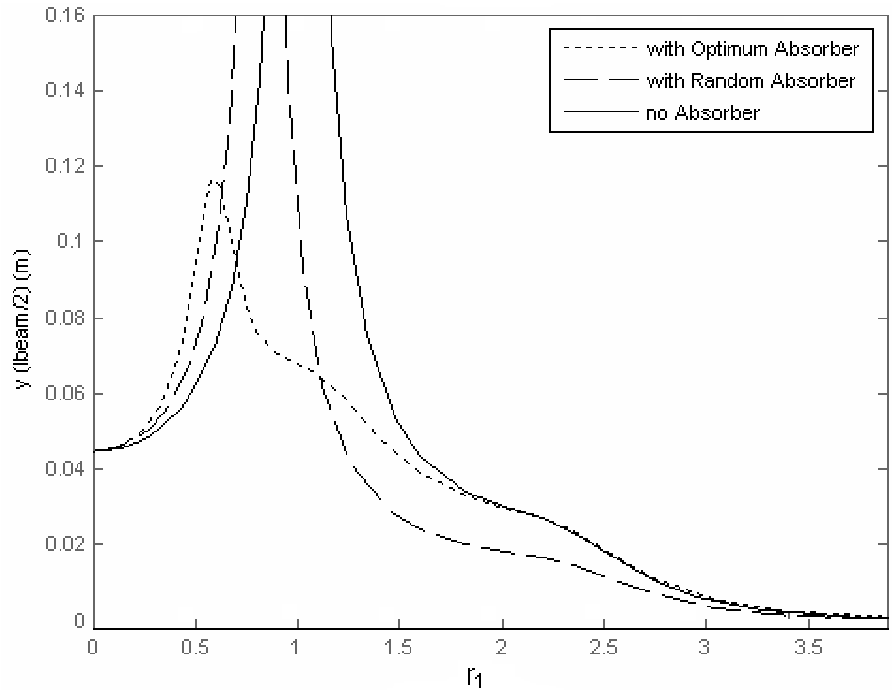

Figure 6.5. Displacement of the beam at x=0.5 l_{beam}

7. CONCLUSIONS

This thesis has developed two new analytical approaches to optimizing the characteristics of dynamic vibration absorbers: one for continuous systems, and the other for discrete systems.

An example of a continuous system, a uniform beam, is modeled with passive absorbers. A method for calculating the optimum parameters of n absorbers attached to a uniform beam, for which the first M modes are considered, has been developed. The method can deal with various boundary conditions and design constraints on the absorber properties. Although there are many papers on this subject in the vibration literature, to the best of the author's knowledge, there has not been any specific investigation to describe the general case of n absorbers and considering M initial modes. The method is also extended to the case of a thin rectangular plate with any number of absorbers and modes.

First, for the most general case, dissipation due to damping, the kinetic and potential energies of the system, and external forces are taken into account. Lagrange's equation is used in order to provide the state space representation of the system.

The state space representation of the corresponding system is derived in general. On the basis of the state space representation and the Lyapunov equation, the H_2 norm of the transfer function of the system is utilized in the optimization. The system output is minimized and displayed with a comparison of cases "without absorber", "with randomly selected absorber" and "with optimum absorber". Finally, the developed method is used to reproduce specific results found in the literature. The optimal absorber parameters of these cases are in good agreement.

As a realization of discrete system, a translational mass-spring-damper system is used to suppress vibration amplitudes. A new and efficient method for calculating the system receptance has been developed.

The receptance matrix (also called the frequency response matrix) relates the inputs of damped mechanical system (external forces) to its outputs (displacements). The literature does not contain any specific results comparable to the method developed in this thesis, which provides the receptance of a generic translational mass-spring-damper system with m masses and n absorbers.

The combined system's receptance can be obtained separately in terms of the main and absorber system parameters by using the well known Sherman-Morrison formula. However, this formula cannot be applied in a single step formulation for the general case of m masses with n absorbers. In general, by applying the Sherman-Morrison formula sequentially (in a series manner), the receptance of any system can be expressed in terms of the receptance of an equivalent system with one less absorber. The alternative method based on linear graphs validates this approach.

Physical constraints on the parameters are taken into consideration in order to be applicable to specified design criteria in the study of discrete systems. A desired output with weighting functions of input and output is defined. Weighting functions are also used to generate a more complex objective function.

Once the displacement is expressed in terms of the absorber parameters and driving frequency, an objective function for optimizing the former can be defined. In order to minimize the fluctuations of the vibration amplitudes in the frequency range of concern, the goal function is chosen to be the area as the absolute difference between the displacement curve and its average value. This goal function is obtained by the Gaussian quadrature numerical integration method. In practice, many iterations are required to evaluate this integral accurately enough for the optimization procedure to converge. The developed method is applied to the specific cases from the literature. It is found out that the results are compatible with each other.

To show that continuous and discrete systems are interrelated, and in fact that the two viewpoints are physically equivalent, the corresponding methods are applied to each type of system mutually. The translational mass-spring-damper system is revisited and its optimal absorber parameters are calculated by using the method developed for continuous system. Likewise, the method described for discrete systems is applied to a uniform beam.

As a summary; two basic models, both the continuous and the discrete system modeling for suppressing vibration, have been carried out and the related methodologies have been derived in this thesis.

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