LOCAL TOPOLOGICAL STRUCTURE IN THE LUC COMPACTIFICATION OF A LOCALLY COMPACT GROUP

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ABSTRACT

LOCAL TOPOLOGICAL STRUCTURE IN THE LUC COMPACTIFICATION OF A LOCALLY COMPACT GROUP

In this thesis we first construct the LUC-Compactification of a topological group. We present $G^{\mathcal{LUC}}$ with two different approaches, first as the set of multiplicative means on the space of LUC functions on G, and when G is locally compact, as a quotient space of the set of ultrafilters on G.

Then local topological structure of $G^{\mathcal{LUC}}$ is investigated and a neighborhood basis for elements of $G^{\mathcal{LUC}}$ is characterized. Results on the injectivity property of multiplication on $G^{\mathcal{LUC}}$ are obtained, and a special condition on G, under which injectivity property can be extended is also examined.

Finally a subclass, the slowly oscillating functions, of LUC-functions is defined to decompose a special subspace of $G^{\mathcal{LUC}}$. Then the decomposition is extended to discrete cancellative semigroups.

ÖZET

YEREL KOMPAKT BİR GRUBUN LUC KOMPAKTİFİKASYONUNUN YEREL TOPOLOJİK YAPISI

Bu tezde ilk once bir topolojik grubun LUC-kompaktifikasyonu sunulmuştur. $G^{\mathcal{LUC}}$ iki farklı tanımla kurulmuştur. Önce, LUC fonksiyonları üzerindeki çarpımsal ortaların kümesi olarak, ve eğer G yerel kompakt ise G grubunun üzerindeki ultrafiltreler kümesinin bir bölüm grubu olarak yapılandırılmıştır.

Daha sonra $G^{\mathcal{LUC}}$'nin yerel topolojik özellikleri incelenmiştir, ve $G^{\mathcal{LUC}}$'nin elemanlarının bir komşuluk bazı karakterize edilmiştir. Ayrica $G^{\mathcal{LUC}}$ üzerinde tanımlanan çarpma işleminin birebirlik özelliğiyle ilgili sonuçlar elde edilmiştir, ve bazı özel durumlarda bu sonuçların genişletilebileceği gösterilmiştir.

Son olarak, LUC fonksiyonlarının bir altkümesi olan yavaş salınımlı fonksiyonlar tanımlanmıştır. Bu fonksiyonlar $G^{\mathcal{LUC}}$'nin özel bir alt uzayının ayrıştırılmasında kullanılmışlardır. Daha sonra bu ayrışım, ayırtık kısaltmalı yarıgruplara genişletilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$\hat{A}$	Basic clopen set in the Stone-Čech compactification
$\overline{A}$	Closure of A in $G^{\mathcal{LUC}}$
$\|A\|$	Cardinality of the set A
$\mathcal{BC}(X)$	Space of scalar-valued bounded continuous functions on X
$\mathcal{B}(X)$	Space of scalar-valued bounded functions on X
$\mathbf{C}$	The set of complex numbers
$cl_X A$	The closure of A in X
$\mathcal{C}(X)$	Space of scalar-valued continuous functions on X
$e(a)$	Principal ultrafilter generated by a
$E(S)$	The set of idempotents in the semigroup S
$\mathcal{F}$	A linear subspace of $\mathcal{B}(X)$
$\mathcal{F}^*$	The set of linear functionals on $\mathcal{F}$
$\mathbf{F}$	A filter on a space X
$\mathcal{F}_r$	The subset of real-valued members in a space of functions $\mathcal{F}$
$f^{-1}[S]$	Inverse image of the set S by under the function f
$f \wedge g$	Maximum of f and g
G^*	The set of elements of LUC-compactification of G, which are not in G
G^{AP}	AP-compactification of G
$G^{\mathcal{LUC}}$	LUC-compactification of G
$Inj_S(E)$	The set of elements x in S, whose left multiplication by x is injective on E
$K(S)$	The minimal ideal of the semigroup S
$\mathcal{LC}(S)$	The set of norm continuous functions on S
$L(S)$	The set of left multiplication maps on S
$L_S f$	Left orbit of f
$L_s f$	Composition of the map f with λ_s
$\mathcal{LUC}(S)$	The set of uniformly continuous functions on S
$\mathcal{M}(\mathcal{F})$	The set of means on $\mathcal{F}$

$\mathcal{MM}(\mathcal{F})$	The set of multiplicative means on $\mathcal{F}$, spectrum of $\mathcal{F}$
$\mathbf{N}$	The set of natural numbers
$\mathcal{N}(1)$	The set of open (relatively compact) neighborhoods of 1 in G
R	The set of real numbers
$R(S)$	The set of right multiplication maps on S
$R_S f$	Right orbit of f
$R_s f$	Composition of the map f with ρ_s
$s^{-1}t$	Set of elements u such that $su = t$
T_ν	Left introversion operator determined by ν
ts^{-1}	Set of elements u such that $us = t$
$\mathcal{U}(S)$	The set of uniform ultrafilters on S
U_ν	Right introversion operator determined by ν
$\|x\|$	Norm of x
$[x]$	Equivalence class of x with respect to the equivalence relation $\simeq$
$[x]_\sim$	Equivalence class of x with respect to the equivalence relation $\sim$
Z(f)	Zero set
βG	The Stone-Čech compactification of G
βG_d	The Stone-Čech compactification of G with discrete topology
$\beta G/\sim$	Quotient space of βG
$\epsilon(s)$	Evaluation map at s
κ	A cardinal number
$\kappa(A)$	Minimal number of compact sets required to cover A
$\Lambda(S)$	Topological center of a right topological semigroup S
λ_s	Left multiplication by s
ρ_s	Right multiplication by s
$\sigma(X, Y)$	Weak topology on X induced by the set of functions Y
χ_A	Characteristic function of the set A
ψ^*	Dual map of ψ

AP	Almost periodic
LC	Left norm continuous
LUC	Left uniformly continuous
Q.E.D.	End of a proof

1. INTRODUCTION

In 1930 Tychonoff proved that every completely regular, Hausdorff space X has a compactification, by embedding X in a product of intervals. The Stone-Čech Compactification was produced independently by M. Stone and E. Čech in 1937. Both Stone and Čech showed independently that X has a unique compactification βX such that every continuous mapping from X into a compact space extends uniquely to βX.

In 1966 the Left Uniformly Continuous functions were introduced, with which the Left Uniformly Continuous Compactification $G^{\mathcal{LUC}}$ is defined. When G is locally compact the compact semigroup $G^{\mathcal{LUC}}$ is its largest semigroup compactification. In 1977 Veech showed an injectivity property of multiplication in $G^{\mathcal{LUC}}$. Pym, in [1] characterized the local structure of $G^{\mathcal{LUC}}$ and gave a different proof of Veech's Theorem. In [2], Budak and Pym, extended Veech's Theorem and Local Structure Theorem on $G^{\mathcal{LUC}}$.

In 1993, the concept of slowly oscillating functions were defined by Higson and Roe in coarse geometry. In 2003 these functions were used by Protasov to study the algebraic structure of the Stone-Čech Compactification of a countable discrete group G.

In 1991, Hindman and Davenport proved Van Douwen's right ideal theorem which decomposes the subgroup of uniform ultrafilters in βS, when S is an infinite cancellative discrete semigroup. In [3] Filali and Salmi generalized Van Douwen's right ideal theorem to non-compact locally compact groups using slowly oscillating functions.

After the historical introduction of the subjects that will be studied in this thesis, we give some preliminary results and notations in Chapter 2.

In Chapter 3, we construct the LUC-Compactification of a locally compact group, G, first as the set of multiplicative means on the C*-algebra of bounded left uniformly continuous functions on G, and then as a quotient of the Stone-Čech Compactification βG_d, where G_d is the same algebraic group G with the discrete topology.

In Chapter 4, first the Local Structure Theorem is proven, which gives a description of a basic neighborhood of the points of $G^{\mathcal{LUC}}$, then its relationships with Veech's Theorem are investigated and finally some variants and extensions to both Veech's Theorem and Local Structure Theorem are given.

The final chapter, Chapter 5 is devoted to a special class of bounded left uniformly continuous functions, namely the slowly oscillating functions. First we construct a class of slowly oscillating functions, which separates elements of a particular form in a special subsemigroup of $G^{\mathcal{LUC}}$, then using that class of functions, we present a generalization of Van Douwen's right ideal theorem.

2. PRELIMINARIES

In this chapter, we present basic definitions and theorems which will be used in the following chapters of this thesis.

First we give some definitions and results on semigroups. The proofs can be found in [4] or in [5].

A *semigroup* is a pair $(S, \cdot)$ where S is a non-empty set and $\cdot : S \times S \to S$ is an associative operation on S. The operation on S is usually called *multiplication* and in this case $\cdot(s,t) = s \cdot t$ is called the *product* of s and t. For convenience, we denote $s \cdot t$ with st.

If S and T are semigroups, a mapping $\phi : S \to T$ satisfying

$$\phi(st) = \phi(s)\phi(t) \qquad \forall s, t \in S$$

is called an *homomorphism*. A bijective homomorphism is called an *isomorphism*.

An element $u \in S$ satisfying $su = s$ for all $s \in S$ is said to be a *right identity*. An element $z \in S$ satisfying $sz = z$ for all $s \in S$ is said to be a *right zero*. Similarly $left$ *identity* and $left$ *zero* elements are defined. If $u \in S$ is both a right and a left identity, then u is called an *identity*. An element $e \in S$ satisfying $ee = e$ is called an *idempotent*. The set of all idempotents of S is denoted by $E(S)$.

A semigroup S is called a group if there is an element $e \in S$ such that e is an identity for S and for each $x \in S$ there is a $y \in S$ such that $xy = yx = e$. Then y is called the *inverse* of x, and denoted by x^{-1}.

An element $x \in S$ is *right cancellable* ($left$ *cancellable*) if and only if whenever $y, z \in S$ and $yx = zx$ ($xy = xz$), we have $y = z$. x is called *cancellable* if it is both right cancellable and left cancellable. The semigroup S is called *right cancellative* (*left cancellative*) if and only if every $x \in S$ is right cancellable (left cancellable). S is *cancellative* if and only if S is both right cancellative and left cancellative.

A semigroup S is *commutative* if $st = ts$ for all $s, t \in S$. The *algebraic center* is the set $Z(S) = \{s \in S : st = ts, \forall t \in S\}$.

For $s \in S$ the maps $\lambda_s : S \to S$, $t \mapsto \lambda_s(t) = st$; $\rho_s : S \to S$, $t \mapsto \rho_s(t) = ts$ are called *left* and *right multiplication maps* by s, respectively.

A nonempty subset T of S is said to be;

(i) A *subsemigroup* of S if $T \cdot T \subseteq T$,

(ii) A *left ideal* of S if $S \cdot T \subseteq T$,

(iii) A *right ideal* of S if $T \cdot S \subseteq T$,

(iv) A (*two − sided*) *ideal* of S if it is both a left and a right ideal,

(v) A *minimal left ideal* (respectively, *minimal right ideal*) of S if it is a left (respectively, right) ideal and it has no proper left (respectively, right) ideal of S.

We now note some basic results on ideals of a semigroup S;

(i) If L is a minimal left ideal of S, then a subset T of S is a minimal left ideal of S if and only if there is some $a \in S$ such that $T = La$.

(ii) If S has a minimal left ideal, then every left ideal of S contains a minimal left ideal.

(iii) If a semigroup S has a minimal ideal, then it is unique and denoted by $K(S)$.

(iv) If S has a minimal left ideal, then $K(S)$ exists and it is the union of all minimal left ideals of S.

(v) If S has a minimal left ideal L and a minimal right ideal R then $K(S) = L{\cdot}R = L \cap R$ and $K(S)$ is a group.

A semigroup S is *left* (respectively, *right*) *simple* if and only if S is a minimal left (respectively, *right*) ideal of S.

Theorem 2.1: Let S be a semigroup with a minimal left ideal containing an idempotent and let $e \in E(S)$. The following statements are equivalent:

(i) Se is a minimal left ideal,

(ii) Se is left simple,

(iii) eSe is a group,

(iv) eSe is a maximal subgroup of S containing e,

(v) eS is a minimal right ideal,

(vi) eS is right simple,

(vii) $e \in K(S)$,

(viii) $K(S) = SeS$.

An idempotent that satisfies conditions of Theorem 2.1 is called *minimal.*

Next, we present some result on topological semigroups, which can be found in [4] or in [5].

A *right topological semigroup* is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\rho_x : S \to S$ is continuous. A $left$ *topological semigroup* is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\lambda_x : S \to S$ is continuous. A *semitopological semigroup* is both a right and left topological semigroup. A *topological semigroup* is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and $\cdot : S \times S \to S$ is jointly continuous. A *topological group* is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a group, (S, τ) is a topological space, $\cdot : S \times S \to S$ and $In : S \to S, s \mapsto s^{-1}$ are both continuous.

If S is a right topological semigroup then the set $\Lambda(S) = \{s \in S : \lambda_s \text{ is continuous}\}$ is the *topological center* of S.

Let (X, τ) be a topological space. A subset $A \subseteq X$ is called $\sigma -$ *compact* if A can be written as a countable union of compact sets. A is called *discrete* if all subsets of A are open in X.

We now give a brief description of the Stone-Čech Compactification βG, of a discrete group G. We present two constructions of βG first as the maximal ideal space on the ring of bounded real-valued continuous functions and then as the space of ultrafilters on G.

Definition 2.2: Let X be a completely regular topological space. A $Stone-\breve{C}ech$ $compactification$ of X is a pair (φ, Z) such that
(i) Z is a compact space,
(ii) φ is an embedding of X into Z,
(iii) $\varphi[X]$ is dense in Z, and
(iv) given any compact space Y and any continuous function $f : X \to Y$ there exists a continuous function $g : Z \to Y$ such that $g \circ \varphi = f$.
Then, Z is denoted by βX.

The details of first construction can be found in [6] or in [7].

Let X be a discrete topological space. Let $\mathcal{C}(X)$ denote the ring of all real-valued continuous functions on X, and let $\mathcal{BC}(X)$ be the subring consisting of all bounded members of $\mathcal{C}(X)$. Let $\mathcal{M}(\mathcal{BC}(X))$ denote the collection of all maximal ideals of $\mathcal{BC}(X)$. We consider the Stone topology on the set $\mathcal{M}(\mathcal{BC}(X))$.

For any $f \in \mathcal{C}(X)$ the set $Z = f^{-1}(\{0\})$ is called a *zero set* in X. The family of all zero sets in X is denoted by $Z(X)$.

A subspace Y of X is said to be $C-$ *embedded* in X if every function in $\mathcal{C}(Y)$ can be extended to a function in $\mathcal{C}(X)$. Likewise, Y is $BC-$ *embedded* in X if every function in $\mathcal{BC}(Y)$ can be extended to a function in $\mathcal{BC}(X)$.

Proposition 2.3: If Y is a compact space, the maximal ideals of $\mathcal{BC}(Y)$ are in a one-to-one correspondance with the points of Y and are given by

$$M_p = \{f \in \mathcal{BC}(Y) \colon f(p) = 0\} \text{ for a point } p \in Y.$$

Moreover, this correspondance is a homeomorphism.

From the observation that $\mathcal{BC}(X)$ is isomorphic to $\mathcal{BC}(\beta X)$ we conclude:

Corollary 2.4: βX is homeomorphic with the maximal ideal space $\mathcal{M}(\mathcal{BC}(X))$.

Now, we construct βG as the space of ultrafilters on G. The proofs can be found in [5].

Definition 2.5: A non-empty family $\mathbf{F}$ of subsets of a set X is called a *filter* on X provided

(i) $\emptyset \notin \mathbf{F}$,

(ii) If $A, B \in \mathbf{F}$ then $A \cap B \in \mathbf{F}$,

(iii) If $A \in \mathbf{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathbf{F}$.

An *ultrafilter* on X is a filter which is not properly contained in any other filter on X. Put $\beta G = \{p : p \text{ is a ultrafilter on } G\}$. For, $A \subseteq G$, define $\hat{A} = \{p \in \beta G : A \in p\}$. And for $a \in G$, the *principal ultrafilter* corresponding to a is $e(a) = \{A \subseteq G : a \in A\}$.

Lemma 2.6: For any discrete topological space G, βG is a topological space with the basis $B = \{\hat{A} : A \subseteq G\}$, which is also a basis for the closed sets. Furhermore, the topological space βG satisfies

(i) βG is a compact Hausdorff space,

(ii) The mapping $e : G \to \beta G$ is injective and $e[G]$ is a dense subset of βG whose points are precisely the isolated points of βG.

Theorem 2.7: $(e, \beta G)$ is a Stone-Čech compactification of G, when G is a topological space.

If, in addition G is also a semigroup, the operation on G can be extended to βG.

Theorem 2.8: Let G be a discrete space and let $\cdot$ be a binary operation defined on G. There is a unique binary operation $* : \beta G \times \beta G \to \beta G$ such that

(i) For every $s, t \in G$, $s * t = s \cdot t$,

(ii) For each $q \in \beta G$, the function $\rho_q : \beta G \to \beta G$, $p \mapsto \rho_q(p) = p * q$ is continuous.

(iii) For each $s \in G$, the function $\lambda_s : \beta G \to \beta G$, $p \mapsto \lambda_s(p) = s * p$ is continuous.

(iv) $*$ is associative on βG.

The operation on βG has a characterization in terms of limits. The statements in the following remark follow immediately from the fact that λ_s is continuous for each $s \in G$ and ρ_q is continuous for each $q \in \beta G$.

Remark: Let $\cdot$ be a binary operation on a discrete space G. Let $s, t \in G$,

(i) If $s \in G$ and $q \in \beta G$, then $s * q = \lim_{t \to q} st$.

(ii) If $p, q \in \beta G$, then $p * q = \lim_{s \to p}(\lim_{t \to q} st)$.

3. LUC COMPACTIFICATION

Our aim is to describe the LUC compactification of a topological group. When S is a semitopological semigroup, the LUC compactification denoted by $S^{\mathcal{LUC}}$ is the largest compactification of S satisfying the joint continuity property. We will construct $S^{\mathcal{LUC}}$ first as the space of multiplicative means on the set of left norm continuous functions defined on S, and if S is a topological group, we will construct it as a quotient of the Stone-Čech compactification of S, considered with discrete topology.

3.1. Background Development

Definition 3.1: Let S be a semigroup. We define the sets $R(S) = \{\rho_t : t \in S\}$ and $L(S) = \{\lambda_t : t \in S\}$. As a consequence of the identities $\rho_{ts} = \rho_s \circ \rho_t$ and $\lambda_{ts} = \lambda_t \circ \lambda s$, the sets $R(S)$ and $L(S)$ are semigroups under composition of mappings.

Definition 3.2: Let S be a semigroup. For any function f on S we define the maps $R_s f = f \circ \rho_s$ and $L_s f = f \circ \lambda_s$, $s \in S$.

Definition 3.3: Let A be a linear associative algebra over the complex numbers. A is called a *normed algebra* if there is a norm on A such that $\|xy\| \leq \|x\|\|y\|$ for any $x, y \in A$.

A mapping $x \mapsto x^* : A \to A$ is called an *involution* if it satisfies:

(i) $(x^*)^* = x$;

(ii) $(x + y)^* = x^* + y^*$;

(iii) $(xy)^* = y^* x^*$;

(iv) $(\lambda x)^* = \overline{\lambda} x^*$.

An algebra with an involution * is called a $^* - algebra$. A *-algebra which is also a Banach space is called a $C^* - algebra$ if it satisfies $\|x^* x\| = \|x\|^2$ for all $x \in A$.

Let A and B be C^*-algebras, and let ϕ be an *algebra homomorphism* of A into B (that is, $\phi(x+y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$, and $\phi(\lambda x) = \lambda\phi(x)$ for any $x, y \in A$ and $\lambda \in \mathbf{C}$). ϕ is called a $^* - homomorphism$ if $\phi(x^*) = \phi(x)^*$ for any $x \in A$.

Remark: If S is a semigroup, let $\mathcal{B}(S)$ denote the space of all bounded complex-valued functions on S. Then $\mathcal{B}(S)$ is a C^*-algebra under the usual operations of *addition* and *multiplication*, under the $uniform\ norm$ $\|f\|= \sup\{|f(s)| : s \in S\}$ and under $involution$ $f \mapsto f^*$, where $f^*(s)=\overline{f(s)}$, $s \in S$. The mappings R_s and L_s are $* - homomorphisms$ of $\mathcal{B}(S)$ such that

$$R_{st} = R_s R_t \qquad and \qquad L_{st} = L_t L_s \qquad \forall s, t \in S.$$

If S is a right (left) topological semigroup, then, for each $s \in S$, R_s (L_s) maps $\mathcal{C}(S)$, the C^*-subalgebra of all continuous functions in $\mathcal{B}(S)$, into itself.

Lemma 3.4: Let X and Y be topological spaces with X compact, and let $g : X \times Y \to \mathbf{C}$ be a bounded function such that the function $x \mapsto g(x, y)$ is continuous for each $y \in Y$.
(i) g is (jointly) continuous if and only if the mapping $y \mapsto g(\cdot, y) : Y \to \mathcal{C}(X)$ is norm continuous.
(ii) If Y is also compact, g is separately continuous if and only if the mapping $y \mapsto g(\cdot, y) : Y \to \mathcal{C}(X)$ is $\sigma(\mathcal{C}(X)$,$\mathcal{C}(X)^*)$ continuous.

Proof: **(i)** If $y \mapsto g(\cdot, y)$ is not norm continuous, then there exists $\varepsilon > 0$, and a net $\{y_\alpha\}_{\alpha \in I} \subseteq Y$ converging to $y \in Y$ such that

$$\sup_{x \in X} |g(x, y_\alpha) - g(x, y)| \geq \varepsilon \qquad \forall \alpha$$

For each α, choose x_α such that $|g(x_\alpha, y_\alpha) - g(x_\alpha, y)| \geq \varepsilon$. Since X is compact, we may assume $\{x_\alpha\}_{\alpha \in I}$ converges to some $x \in X$. Then g cannot be jointly continuous otherwise we would have $|g(x, y) - g(x, y)| \geq \varepsilon$.

Conversely, assume $y \mapsto g(\cdot, y)$ is norm continuous, and let $\{x_\alpha\}_{\alpha \in I}$ be a net in X converging to $x \in X$, and $\{y_\beta\}_{\beta \in J}$ be a net in Y converging to $y \in Y$. Then from the inequality,

$$\begin{aligned} |g(x_\alpha, y_\beta) - g(x, y)| &\leq |g(x_\alpha, y_\beta) - g(x_\alpha, y)| + |g(x_\alpha, y) - g(x, y)| \\ &\leq \|g(\cdot, y_\beta) - g(\cdot, y)\| + |g(x_\alpha, y) - g(x, y)| \end{aligned}$$

it follows that $\{g(x_\alpha, y_\beta)\}_{\alpha \in I, \beta \in J}$ converges to $g(x, y)$.

(ii) Suppose Y is compact and g is separately continuous. Then $y \mapsto g(\cdot, y)$ is continuous in the topology on $\mathcal{C}(S)$ of pointwise convergence, and therefore $g(\cdot, Y)$ is compact in this topology; hence on $g(\cdot, Y)$, this topology agrees with the topology $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$. Thus $y \mapsto g(\cdot, y)$ is $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$ continuous.

Conversely, if $y \mapsto g(\cdot, y)$ is $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$ continuous, then in particular, $y \mapsto g(x, y)$ is continuous for each $x \in X$ and g is separately continuous. *Q.E.D.*

Corollary 3.5: Let S be a compact right topological semigroup.
(i) S is a topological semigroup if and only if $s \mapsto R_s f: S \to \mathcal{C}(S)$ is norm continuous for each $f \in \mathcal{C}(S)$. In this case, $s \mapsto L_s f$ is norm continuous for each $f \in \mathcal{C}(S)$.
(ii) S is a semitopological semigroup if and only if $s \mapsto R_s f: S \to \mathcal{C}(S)$ is $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$ continuous for each $f \in \mathcal{C}(S)$. In this case, $s \mapsto L_s f$ is $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$ continuous for each $f \in \mathcal{C}(S)$.

Proof: **(i)** Since S is compact its topology is generated by $\mathcal{C}(X)$. That is, S is a topological semigroup if and only if $(s, t) \mapsto f(st)$ from $S \times S$ into $\mathbf{C}$ is continuous for each $f \in \mathcal{C}(S)$. By Lemma 3.4(i) the latter condition is equivalent to the norm continuity of $t \mapsto R_t f$.

(ii) S is a semitopological semigroup if and only if $(s, t) \mapsto f(st)$ from $S \times S$ into $\mathbf{C}$ is separately continuous for each $f \in \mathcal{C}(S)$. By Lemma 3.4(ii) the latter condition is equivalent to the $\sigma(\mathcal{C}(X),\mathcal{C}(X)^*)$ continuity of $s \mapsto R_s f$. *Q.E.D.*

3.2. Means on Function Spaces

We will consider the general properties of means on a function subspace $\mathcal{F}$ of $\mathcal{B}(S)$. Our main result, Theorem 3.12 will describe the algebraic and topological structure of the set of means on $\mathcal{F}$.

Definition 3.6: Let $\mathcal{F}$ be a linear subspace of $\mathcal{B}(S)$ and let $\mathcal{F}_r$ denote the set of all real-valued members of $\mathcal{F}$. A *mean* on $\mathcal{F}$ is a linear function μ on $\mathcal{F}$ with the property that for all $f \in \mathcal{F}_r$,

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \qquad (1)$$

The set of all means on $\mathcal{F}$ is denoted by $\mathcal{M}(\mathcal{F})$. If $\mathcal{F}$ is also an algebra and if $\mu \in \mathcal{M}(\mathcal{F})$ satisfies

$$\mu(fg) = \mu(f)\mu(g)$$

for all $f \in \mathcal{F}$, then μ is said to be *multiplicative*. The set of all multiplicative means on $\mathcal{F}$ is called the *spectrum* of $\mathcal{F}$ and denoted by $\mathcal{MM}(\mathcal{F})$.

Note that (1) requires that $\mu(f)$ be real for each real valued member $f \in \mathcal{F}$.

Proposition 3.7: Let $\mathcal{F}$ be a conjugate closed linear subspace of $\mathcal{B}(S)$ containing the constant functions. Then a mean μ on $\mathcal{F}$ satisfies:

(i) μ is positive, that is, if $f \in \mathcal{F}_r$ and $f \geq 0$, then $\mu(f) \geq 0$;

(ii) $\mu(1)=1$;

(iii) μ is a bounded linear functional on $\mathcal{F}$ with $\|\mu\| = 1$;

(iv) For all $f \in \mathcal{F}$, $\mu(Re(f)) = Re(\mu(f))$, $\mu(Im(f)) = Im(\mu(f))$, and $\mu(\overline{f}) = \overline{\mu(f)}$;

(v) $\mu(f)$ is in the closed convex hull of $f[S]$ for all $f \in \mathcal{F}$.

Conversely, a linear functional μ on $\mathcal{F}$ that satisfies any two of the properties (i), (ii) and (iii) is a mean.

Proof: Properties (i) and (ii) are direct consequences of the definition of mean. To establish (iii) and (iv) we prove the following more general fact:
"If ν is a positive linear functional on $\mathcal{F}$, then ν is bounded, $\|\nu\| = \nu(1)$, and $\nu(f)$ is real for all real-valued $f \in \mathcal{F}$."

To verify this, we first note that if $f \in \mathcal{F}_r$, then $1 - \|f\|^{-1}f \geq 0$, hence, since ν is positive, $\nu(f)$ is real and $\nu(f) \leq \|f\|\nu(1)$.

For an arbitrary $f \in \mathcal{F}$, choose $c \in \mathbf{C}$ such that $|c| = 1$ and $|\nu(f)| = c\nu(f)$. If $g = Re(cf)$ and $h = Im(cf)$, then $|\nu(f)| = \nu(cf) = \nu(g) + i\nu(h)$ and since $\nu(g)$ and $\nu(h)$ are real,

$$|\nu(f)| = |\nu(g)| \leq \|g\|\nu(1) \leq \|cf\|\nu(1) = |c|\|f\|\nu(1) = \|f\|\nu(1).$$

Therefore, ν is bounded and $\|\nu\| = \nu(1)$. This proves the assertion and establishes (iii) and (iv).

To show (v), let D be any closed disc in $\mathbf{C}$ containing $f[S]$. If D has centre c and radius r, then by (ii) and (iii)

$$|\mu(f) - c| = |\mu(f - c)| \leq \|\mu\|\|f - c\| \leq r$$

hence $\mu(f) \in D$. Since the closed convex hull of $f[S]$ is the intersection of all such discs, (v) follows.

Conversely, suppose that μ is a linear functional on $\mathcal{F}$ satisfying either (i) and (ii) or (i) and (iii). Then by the general fact established in the first part of the proof, $\mu(1) = 1$ and $\mu(f)$ is real for $f \in \mathcal{F}_r$. Inequality (1) is now immediate.

To complete the proof it suffices to show that any linear functional μ on $\mathcal{F}$ that satisfies (ii) and (iii) also satisties (i). Let $f \in \mathcal{F}$, $f \geq 0$ and assume without loss of generality that $\|f\| \leq 1$. Then $\|1 - f\|$=$\sup_{s \in S} 1 - f(s) \leq 1$, hence

$$\mu(1 - f) \leq \|\mu\|\|1 - f\| \leq 1 = \mu(1)$$

which shows that $\mu(f) \geq 0$. *Q.E.D.*

Lemma 3.8: Let μ be a bounded linear functional on the real Banach space $\mathcal{B}(S)_r$. Then there exist positive linear functionals μ^+ and μ^- on $\mathcal{B}(S)_r$ such that $\mu = \mu^+ - \mu^-$.

Proof: For $f \geq 0$ define $\mu^+(f) = \sup_{0 \leq g \leq f} \mu(g)$. Then $\mu^+(f) \geq 0$, $\mu^+(f) \leq \|\mu\|\|f\|$ and $\mu^+(cf) = c\mu^+(f)$ if $c \geq 0$.

To show that μ^+ is additive, let f_1,f_2 be nonnegative functions in $\mathcal{B}(S)_r$ and let g_1,$g_2 \in \mathcal{B}(S)_r$ satisfy $0 \leq g_i \leq f_i$, $i = 1, 2$. Then, $0 \leq g_1 + g_2 \leq f_1 + f_2$. Hence, $\mu^+(f_1 + f_2) \geq \mu(g_1) + \mu(g_2)$. Taking the suprema over all such g_1 and g_2 gives $\mu^+(f_1 + f_2) \geq \mu^+(f_1) + \mu^+(f_2)$.

Also, if $0 \leq g \leq f_1 + f_2$ then $0 \leq f_1 \wedge g \leq f_1$ and $0 \leq g - (f_1 \wedge g) \leq f_2$. So, $\mu(g) = \mu(f_1 \wedge g) + \mu(g - (f_1 \wedge g)) \leq \mu^+(f_1 + f_2)$. Taking the supremum over all such g gives $\mu^+(f_1) + \mu^+(f_2) \geq \mu^+(f_1 + f_2)$.

Thus, $\mu^+(f_1 + f_2) = \mu^+(f_1) + \mu^+(f_2)$.

For an arbitrary $f \in \mathcal{B}(S)_r$ set $\mu^+(f) = \mu^+(f_1) - \mu^+(f_2)$, where f_1 and f_2 are nonnegative members of $\mathcal{B}(S)_r$ such that $f = f_1 - f_2$. Then μ^+ is well defined, positive, linear functional on $\mathcal{B}(S)_r$, hence bounded.

Next set $\mu^- = \mu^+ - \mu$. Then μ^- will also have the required properties. *Q.E.D.*

Theorem 3.9: Let $\mathcal{F}$ be a conjugate closed linear subspace of $\mathcal{B}(S)$ containing the constant functions. If $\mu \in \mathcal{F}^*$, then there exist $\mu_j \in \mathcal{M}(\mathcal{F})$ and nonnegative real numbers a_j, $1 \leq j \leq 4$ such that

$$\mu = a_1\mu_1 - a_2\mu_2 + i[a_3\mu_3 - a_4\mu_4].$$

Proof: By the Complex Hahn-Banach Theorem, it suffices to show the Theorem for the case $\mathcal{F} = \mathcal{B}(S)$. For $f \in \mathcal{B}(S)_r$ let $\mu_r(f)$ and $\mu_i(f)$ denote, respectively, the real and imaginary parts of $\mu(f)$. Then μ_r and μ_i are bounded linear functionals on the real Banach space $\mathcal{B}(S)_r$, hence by Lemma 3.8, there exist positive linear functionals μ_r^+, μ_r^-, μ_i^+, and μ_i^- on the real Banach space $\mathcal{B}(S)_r$ such that for every $f \in \mathcal{B}(S)_r$

$$\mu_r(f) = \mu_r^+(f) - \mu_r^-(f) \qquad \textit{and} \qquad \mu_i(f) = \mu_i^+(f) - \mu_i^-(f).$$

Therefore, $\mu(f) = \mu_r^+(f) - \mu_r^-(f) + i[\mu_i^+(f) - \mu_i^-(f)]$ (2)

If $g, h \in \mathcal{B}(S)_r$ and $f = g + ih$, then we have:

$$\mu(f) = \mu(g) + i\mu(h) = \eta_1(f) - \eta_2(f) + i[\eta_3(f) - \eta_4(f)] \qquad (3)$$

where,

$$\begin{aligned} \eta_1(f) &= \mu_r^+(g) + i\mu_r^+(h); \\ \eta_2(f) &= \mu_r^-(g) + i\mu_r^-(h); \\ \eta_3(f) &= \mu_i^+(g) + i\mu_i^+(h); \\ \eta_4(f) &= \mu_i^-(g) + i\mu_i^-(h). \end{aligned}$$

Each η_j is easily seen to be positive linear functional on $\mathcal{B}(S)$, hence by Proposition 3.7 each η_j is bounded. Letting $\mu_j = \|\eta_j\|^{-1}\eta_j$ if $\|\eta_j\| \neq 0$ and choosing $\mu_j \in \mathcal{M}(\mathcal{B}(S))$ arbitrary if $\|\eta_j\| = 0$, we see that the desired decomposition of μ follows from (3) on setting $a_j = \|\eta_j\|$.

$$\begin{aligned}
\mu(f) &= \mu(g) + i\mu(h) \\
&= \eta_1(f) - \eta_2(f) + i[\eta_3(f) - \eta_4(f)] \\
&= \|\eta_1\|(\frac{1}{\|\eta_1\|}\eta_1)(f) - \|\eta_2\|(\frac{1}{\|\eta_2\|}\eta_2)(f) + i[\|\eta_3\|(\frac{1}{\|\eta_3\|}\eta_3)(f) - \|\eta_4\|(\frac{1}{\|\eta_4\|}\eta_4)(f)] \\
&= a_1\mu_1(f) - a_2\mu_2(f) + i[a_3\mu_3(f) - a_4\mu_4(f)].
\end{aligned}$$

Q.E.D.

Definition 3.10: Let $\mathcal{F}$ be a conjugate closed linear subspace of $\mathcal{B}(S)$ containing the constant functions. For each $s \in S$ define

$$\epsilon(s)(f) = f(s) \qquad f \in \mathcal{F}.$$

The mapping $\epsilon : S \to \mathcal{M}(\mathcal{F})$ is called the *evaluation mapping*, and $\epsilon(s)$ is called *evaluation* at s. If $\mathcal{F}$ is also an algebra, then $\epsilon[S] \subseteq \mathcal{MM}(\mathcal{F})$, hence we may write $\epsilon : S \to \mathcal{MM}(\mathcal{F})$.

In the setting being developed here, the natural topology to give $X = \mathcal{M}(\mathcal{F})$ or $X = \mathcal{MM}(\mathcal{F})$ is the relative *weak* topology* $\sigma(X, \mathcal{F})$.

Definition 3.11: Let $\mathcal{F}$ be a conjugate closed linear subspace (respectively, subalgebra) of $\mathcal{B}(S)$ containing the constant functions, and let $X = \mathcal{M}(\mathcal{F})$ (respectively, $X = \mathcal{MM}(\mathcal{F})$) be furnished with the relative *weak** topology. For each $f \in \mathcal{F}$ define the function $\hat{f} \in \mathcal{C}(X)$

$$\hat{f}(\mu) = \mu(f) \qquad \mu \in X.$$

Further we define $\hat{F} = \{\hat{f} : f \in \mathcal{F}\}$.

Remark: The mapping $f \mapsto \hat{f} : \mathcal{F} \to \mathcal{C}(X)$ is linear (and multiplicative if $\mathcal{F}$

is an algebra and $X = \mathcal{MM}(\mathcal{F})$), preserves complex conjugation and is an isometry since for any $f \in \mathcal{F}$

$$\begin{aligned} \|\hat{f}\| &= \sup_{\mu \in X} |\mu(f)| \leq \sup_{\|\mu\| \leq 1} |\mu(f)| = \|f\| \\ &= \sup_{s \in S} |f(s)| = \sup_{s \in S} |\hat{f}(\in (s))| \leq \|\hat{f}\| \end{aligned}$$

where $\epsilon : S \to X$ denotes the evaluation mapping, so that $\hat{f}(\epsilon(s)) = f(s)$ for all $f \in \mathcal{F}$ and $s \in S$. This last identity may be written in terms of the dual map $\epsilon^* : \mathcal{C}(X) \to \mathcal{B}(S)$ as $\epsilon^*(\hat{f}) = f$.

Now we describe the topological and geometric structure of $\mathcal{M}(\mathcal{F})$:

Theorem 3.12: Let $\mathcal{F}$ be a conjugate closed linear subspace of $\mathcal{B}(S)$ containing the constant functions. Then,
(i) $\mathcal{M}(\mathcal{F})$ is convex and weak*-compact;
(ii) The convex hull of $\epsilon[S]$, $co(\epsilon[S])$ is weak*-dense in $\mathcal{M}(\mathcal{F})$;
(iii) $\mathcal{F}^*$ is the weak*-closed linear span of $\epsilon[S]$;
(iv) If $\mathcal{F}$ is also an algebra, then $\mathcal{MM}(\mathcal{F})$ is weak*-compact and $\epsilon[S]$ is weak*-dense in $\mathcal{MM}(\mathcal{F})$;
(v) If S is a topological space and $\mathcal{F} \subseteq \mathcal{C}(S)$, then $\epsilon : S \to \mathcal{M}(\mathcal{F})$ is weak*-continuous.

Proof: **(i)** It follows directly from the definition that $\mathcal{M}(\mathcal{F})$ is convex and weak*-closed. Since $\mathcal{M}(\mathcal{F})$ is contained in the closed unit ball of $\mathcal{F}^*$ (by Proposition 3.7(iii)), Alaoglu's theorem implies that $\mathcal{M}(\mathcal{F})$ is weak*-compact.

(ii) Note first that by direct verification $co(\epsilon[S]) \subseteq \mathcal{M}(\mathcal{F})$. If there exists some $\mu \in \mathcal{M}(\mathcal{F})$, not in the weak*-closure of $co(\epsilon[S])$, then using the basic separation theorem for compact, convex sets and the fact that the topological dual of $(\mathcal{F}^*, \sigma(\mathcal{F}^*, \mathcal{F}))$ is just $\mathcal{F}^*$, we obtain a real-valued function $f \in \mathcal{F}$ such that

$$\sup_{s \in S} |f(s)| \leq \sup_{\nu \in co(\epsilon[S])} |\nu(f)| < \mu(f)$$

contradicting the definition of mean.

(iii) follows immediately from (ii) and Theorem 3.9.

(iv) Let $X = \mathcal{MM}(\mathcal{F})$ have the relative weak* topology. As X is a closed subset of $\mathcal{M}(\mathcal{F})$, it is compact. To show that $\epsilon[S]$ is dense in X, it suffices, by the complete regularity of X, to show that if $g \in \mathcal{C}(S)$ vanishes identically on $\epsilon[S]$, then g is the zero function. Note first that for any $f \in \mathcal{F}$, we have from the last remark,

$$\|\hat{f}\| = \sup_{s \in S} |\hat{f}(\epsilon(s)) - g(\epsilon(s))| \leq \|\hat{f} - g\|$$

whence $\|g\| \leq \|g - \hat{f}\| + \|\hat{f}\| \leq 2\|g - \hat{f}\|$ (4).

Since $\hat{\mathcal{F}}$ is a conjugate closed subalgebra of $\mathcal{C}(X)$ that contains the constant functions and separates points of X, $\hat{\mathcal{F}}$ must be dense in $\mathcal{C}(X)$ by Stone-Weierstrass Theorem. It follows from (4) that g is identically zero on X.

(v) is obvious. *Q.E.D.*

Corollary 3.13: Let $\mathcal{F}$ be a C^*-subalgebra of $\mathcal{B}(S)$ containing the constant functions. If X denotes the space $\mathcal{MM}(\mathcal{F})$ with the relative weak* topology, and if $\epsilon : S \to X$ denotes the evaluation mapping, then the mapping $f \mapsto \hat{f} : \mathcal{F} \to \mathcal{C}(X)$ is an isometric isomorphism with inverse $\epsilon^* : \mathcal{C}(X) \to \mathcal{F}$.

3.3. Semigroups of Means

Let S be a semitopological semigroup and $\mathcal{F}$ be a conjugate closed, norm closed linear subspace of $\mathcal{C}(S)$ containing the constant functions. For any $s \in S$ and for any function f on S we have defined

$$L_s f = f \circ \lambda_s \qquad \textit{and} \qquad R_s f = f \circ \rho_s$$

Now we call L_sf and R_sf the *left* and *right translates* of f by s, respectively. The *left* and *right orbits* of f are the sets

$$L_Sf = \{L_sf : s \in S\} \qquad \textit{and} \qquad R_Sf = \{R_sf : s \in S\}$$

Definition 3.14: Let $\mathcal{F}$ be a linear subspace of $\mathcal{B}(S)$. $\mathcal{F}$ is called *left* (*right*) *translation invariant* if $L_s\mathcal{F} \subseteq \mathcal{F}$ ($R_s\mathcal{F} \subseteq \mathcal{F}$) for all $s \in S$. $\mathcal{F}$ is *translation invariant* if it is both left and right translation invariant.

Proposition 3.15: If $\mathcal{F}$ is a translation invariant subspace of $\mathcal{B}(S)$, then R_s and L_s are bounded linear operators on $\mathcal{F}$ such that $\|R_s\| \leq 1$, $\|L_s\| \leq 1$, and $R_{st} = R_sR_t$, $L_{st} = L_tL_s$ for $s, t \in S$.

Proof: For any $f \in \mathcal{F}$, $\|R_sf\| = \sup_{t\in S}|R_sf(t)| = \sup_{t\in S}|f(ts)| \leq \sup_{t\in S}|f(t)| = \|f\|$ and similarly $\|L_sf\| \leq \|f\|$. Then it follows that $\|R_s\| \leq 1$ and $\|L_s\| \leq 1$. Since we have $\rho_{st} = \rho_t\rho_s$ and $\lambda_{st} = \lambda_s\lambda_t$, the equations $R_{st} = R_sR_t$ and $L_{st} = L_tL_s$ follow immediately. *Q.E.D.*

Definition 3.16: Let $\mathcal{F}$ be a translation invariant subspace of $\mathcal{B}(S)$. Let $\nu \in \mathcal{F}^*$. We define $T_\nu : \mathcal{F} \to \mathcal{B}(S)$ by $f \mapsto T_\nu f : S \to \mathbf{C}$ where $(T_\nu f)(s) = \nu(L_sf)$ for $s \in S$. T_ν is called the *left introversion operator* determined by ν.

Similarly, the mapping $U_\nu : \mathcal{F} \to \mathcal{B}(S)$ is defined by $(U_\nu f)(s) = \nu(R_sf)$ for $f \in \mathcal{F}$ and $s \in S$ is called the *right introversion operator* determined by ν.

Proposition 3.17: If $\mu \in \mathcal{M}(\mathcal{F})$, then T_μ and U_μ are bounded linear operators on $\mathcal{F}$ such that $\|T_\mu\| \leq 1$ and $\|U_\mu\| \leq 1$.

Proof: By Propositions 3.7 and 3.15,

$$\|T_\mu f\| \;=\; \sup_{s\in S}|(T_\mu f)(s)|$$

$$
\begin{aligned}
&= \sup_{s\in S} |\mu(L_s f)| \\
&\leq \sup_{s\in S} \|\mu\|\|L_s f\| \\
&\leq \sup_{s\in S} \|\mu\|\|L_s\|\|f\| \\
&\leq \|f\|.
\end{aligned}
$$

Hence, $\|T_\mu\| \leq 1$. Similarly, $\|U_\mu\| \leq 1$. *Q.E.D.*

Definition 3.18: Let $\mathcal{F}$ be a conjugate closed, translation invariant subspace (respectively, subalgebra) of $\mathcal{B}(S)$ containing the constant functions. $\mathcal{F}$ is said to be:
(i) *left introverted* (respectively, *left m-introverted*) if $T_\mu\mathcal{F} \subseteq \mathcal{F}$ for all $\mu \in \mathcal{M}(\mathcal{F})$ (respectively, $\mathcal{MM}(\mathcal{F})$);
(ii) *right introverted* (respectively, *right m-introverted*) if $U_\mu\mathcal{F} \subseteq \mathcal{F}$ for all $\mu \in \mathcal{M}(\mathcal{F})$ (respectively, $\mathcal{MM}(\mathcal{F})$);
(iii) *introverted* (respectively, *m-introverted*) if $\mathcal{F}$ is both left and right introverted (respectively, left and right m-introverted).

Definition 3.19: Let $\mathcal{F}$ be left translation invariant subspace (respectively, subalgebra) of $\mathcal{B}(S)$. If $\mathcal{F}$ is left introverted (respectively, $\mathcal{F}$ is an algebra and is left m-introverted) for $\mu, \nu \in \mathcal{M}(\mathcal{F})$ (respectively, $\mathcal{MM}(\mathcal{F})$) the *product* $\mu\nu \in \mathcal{F}^*$ is defined by

$$\mu\nu = \mu \circ T_\nu$$

(that is, for any $f \in \mathcal{F}$, $\mu\nu(f) = \mu(T_\nu(f))$.)

Proposition 3.20: Let $\mu, \nu \in \mathcal{MM}(\mathcal{F})$. Then
(i) $\mu\nu \in \mathcal{MM}(\mathcal{F})$;
(ii) $T_{\mu\nu} = T_\mu \circ T_\nu$.

Proof: **(i)** For any $\nu \in \mathcal{MM}(\mathcal{F})$ and $s \in S$, we have

$$\begin{aligned}(T_\nu(fg))(s) &= \nu(L_s(fg)) \\ &= \nu((L_s f)(L_s(g)) \\ &= \nu(L_s f)\nu(L_s g) \\ &= (T_\nu f)(s)(T_\nu g)(s).\end{aligned}$$

Hence, for any $\mu, \nu \in \mathcal{MM}(\mathcal{F})$ and $f, g \in \mathcal{F}$

$$\mu\nu(fg) = \mu(T_\nu(fg)) = \mu((T_\nu f)(T_\nu g)) = \mu(T_\nu f)\mu(T_\nu g) = \mu\nu(f)\mu\nu(g).$$

Thus, $\mu\nu \in \mathcal{MM}(\mathcal{F})$.

(ii) For $\nu \in \mathcal{MM}(\mathcal{F})$, $f \in \mathcal{F}$ and $s, t \in S$ we have

$$(T_\nu L_t f)(s) = \nu(L_s L_t f) = \nu(L_{ts} f) = (T_\nu f)(ts) = (L_t T_\nu f)(s)$$

Hence, $T_\nu L_t = L_t T_\nu$. So, for any $\mu, \nu \in \mathcal{MM}(\mathcal{F})$ $f \in \mathcal{F}$ and $t \in S$,

$$(T_{\mu\nu} f)(t) = \mu\nu(L_t f) = \mu(T_\nu L_t f) = \mu(L_t T_\nu f) = T_\mu((T_\nu f))(t)$$

Therefore, $T_{\mu\nu} = T_\mu \circ T_\nu$. *Q.E.D.*

Definition 3.21: A norm closed, conjugate closed, translation invariant, left introverted subspace of $\mathcal{B}(S)$ containing the constant functions is called an *admissible subspace* of $\mathcal{B}(S)$. A translation invariant, left m-introverted C^*-subalgebra of $\mathcal{B}(S)$ containing the constant functions is called an *m-admissible subalgebra* of $\mathcal{B}(S)$.

Theorem 3.22: If $\mathcal{F}$ is an m-admissible subalgebra of $\mathcal{B}(S)$, then, with respect to the *weak**-topology and the operation $(\mu, \nu) \mapsto \mu\nu = \mu \circ T_\nu$, $\mathcal{MM}(\mathcal{F})$ is a compact right topological semigroup, $\epsilon[S] \subseteq \Lambda(\mathcal{MM}(\mathcal{F}))$ and $\epsilon : S \to \mathcal{MM}(\mathcal{F})$ is a homomorphism.

Proof: To prove that $\mathcal{MM}(\mathcal{F})$ with $(\mu, \nu) \mapsto \mu\nu$ is a semigroup, we need to show that the multiplication is associative. Let $\mu, \nu, \eta \in \mathcal{MM}(\mathcal{F})$, then we have

$$(\mu\nu)\eta = (\mu\nu) \circ T_\nu = (\mu \circ T_\nu) \circ T_\nu = \mu \circ (T_\nu \circ T_\eta) = \mu \circ T_{\nu\eta} = \mu(\nu\eta).$$

Next, let $\mu, \nu \in \mathcal{MM}(\mathcal{F})$ and suppose the net $(\mu_i)_{i\in I}$ converges to μ in the *weak** $-$ *topology*. Then, for any $f \in \mathcal{F}$ and $T_\nu f \in \mathcal{F}$, we have

$$\mu_i\nu(f) = \mu_i(T_\nu f) \to \mu(T_\nu f) = \mu\nu(f)$$

Thus $\mu_i\nu \to \mu\nu$. Hence, $\mathcal{MM}(\mathcal{F})$ is a right topological semigroup.

Now, let $s \in S$. Take $\mu \in \mathcal{MM}(\mathcal{F})$ and suppose that $\mu_\alpha \to \mu$. Then for any $f \in \mathcal{F}$, $L_s f \in \mathcal{F}$,

$$\epsilon(s)(\mu_\alpha(f) = \epsilon(s)(T_{\mu_\alpha} f) = T_{\mu_\alpha} f(s) = \mu_\alpha(L_s f) \to \mu(L_s f).$$

Also,

$$\mu(L_s f) = T_\mu f(s) = \epsilon(s)(T_\mu f) = \epsilon(s)\mu(f)$$

Thus, $\epsilon(s)\mu_\alpha \to \epsilon(s)\mu$, which means $\epsilon[S] \subseteq \Lambda(\mathcal{MM}(\mathcal{F}))$.

To prove that ϵ is a homomorphism, note that for any $s, t \in S$, $f \in \mathcal{F}$,

$$(T_{\epsilon(s)} f)(t) = \epsilon(s)(L_t f) = L_t f(s) = f(ts) = (R_s f)(t).$$

So, for any $f \in \mathcal{F}$ $T_{\epsilon(s)} f = R_s f$. Then, we get,

$$[\epsilon(s)\epsilon(t)](f) = \epsilon(s)(T_{\epsilon(t)} f) = \epsilon(s)(R_t f) = R_t f(s) = f(st) = \epsilon(st)(f).$$

Q.E.D.

3.4. Compactifications of Semitopological Semigroups

Definition 3.23: A *semigroup compactification* of a semitopological semigroup S is a pair (ψ, X) where

(i) X is a compact, Hausdorff, right topological semigroup;

(ii) $\psi : S \to X$ is a continuous homomorphism;

(iii) $\psi[S]$ is dense in X;

(iv) $\psi[S] \subseteq \Lambda(X)$.

Theorem 3.24: Assume that (ψ, X) is a compactification of a semitopological semigroup S.

(i) If $\theta : T \to S$ is a continuous homomorphism from a semitopological semigroup T onto a dense subsemigroup of S, then $(\psi \circ \theta, X)$ is a compactification of T.

(ii) If $\pi : X \to Y$ is a continuous homomorphism onto a compact right topological semigroup Y, then $(\pi \circ \psi, Y)$ is a compactification of S.

Proof: **(i)** is straightforward.

(ii) Conditions (i),(ii) and (iii) of the definition are clear. To prove that $(\pi \circ \psi)[S] \subseteq \Lambda(Y)$, we first show that $\pi(\Lambda(X)) \subseteq \Lambda(Y)$. Let $x \in \Lambda(X)$, $y \in Y$ and $(y_\alpha)_{\alpha \in I}$ be a net in Y converging to y. Since π is onto, for each $\alpha \in I$, there is an $x_\alpha \in X$ such that $y_\alpha = \pi(x_\alpha)$. Since X is compact, we may assume that $(x_\alpha)_{\alpha \in I}$ is convergent. Then we have

$$\begin{aligned} \lim_\alpha \lambda_{\pi(x)}(y_\alpha) &= \lim_\alpha \pi(x) y_\alpha = \lim_\alpha \pi(x x_\alpha) \\ &= \pi(\lambda_x(\lim_\alpha x_\alpha)) = \pi(x \lim_\alpha x_\alpha) \\ &= \pi(x)\pi(\lim_\alpha x_\alpha) = \pi(x) y \\ &= \lambda_{\pi(x)}(y) \end{aligned}$$

That is, $\lambda_{\pi(x)}$ is continuous. Hence, $\pi(x) \in \Lambda(Y)$. So, we conclude $\pi[\Lambda(X)] \subseteq \Lambda(Y)$. Since $\psi[S] \subseteq \Lambda(X)$, and $\pi(\Lambda(X)) \subseteq \Lambda(Y)$, we get that $(\pi \circ \psi)[S] \subseteq \Lambda(Y)$. *Q.E.D.*

Definition 3.25: Let (ψ, X) and (ϕ, Y) be compactifications of a semitopological semigroup S.

(i) A continuous homomorphism π of X onto Y such that $\pi \circ \psi = \phi$ is called a *homomorphism* of (ψ, X) onto (ϕ, Y). If such a homomorphism exists, then (ϕ, Y) is said to be a *factor* of (ψ, X), and (ψ, X) is said to be an *extension* of (ϕ, Y).

(ii) If (ψ, X) is an extension of (ϕ, Y) and (ϕ, Y) is an extension of (ψ, X), then we say that (ψ, X) is *isomorphic* to (ϕ, Y).

Definition 3.26: Let (ψ, X) be a compactification of a semitopological semigroup S. The map $\psi^* : \mathcal{BC}(X) \to \mathcal{BC}(S)$ defined by $\psi^*(f) = f \circ \psi$ is called the *dual map*.

Theorem 3.27: If (ψ, X) is a compactification of a semitopological semigroup S, then $\psi^*[\mathcal{BC}(X)]$ is an m-admissible subalgebra of $\mathcal{BC}(S)$. Conversely, if $\mathcal{F}$ is an m-admissible subalgebra of $\mathcal{BC}(S)$, then there exists a compactification (ψ, X) of S with the property that $\psi^*[\mathcal{BC}(X)] = \mathcal{F}$.

Proof: Let $\mathcal{F} = \psi^*[\mathcal{BC}(X)]$. First, we need to prove that $\mathcal{F}$ is a C^*-subalgebra of $\mathcal{BC}(S)$. Since ψ is a homomorphism, it is clearly a homomorphism and for any $f \in \mathcal{BC}(X)$

$$\psi^*(\overline{f}) = \overline{f} \circ \psi = \overline{f \circ \psi} = \overline{\psi^* f}$$

which gives that ψ^* is a *-homomorphism. Since a *-homomorphic image of a C^*-algebra is also a C^*-algebra, $\mathcal{F}$ is a C^*-subalgebra of $\mathcal{BC}(S)$.

Next, we prove that $\mathcal{F}$ is translation invariant. We have to show that for all $s \in S$, $R_s \circ \psi^* = \psi^* \circ R_{\psi(s)}$ and $L_s \circ \psi^* = \psi^* \circ L_{\psi(s)}$. Let $g \in \mathcal{BC}(X)$ and $s \in S$. Then for all $t \in S$, we have

$$\begin{aligned}(g \circ \rho_{\psi(s) \circ \psi})(t) &= g(\rho_{\psi(s)}(\psi(t))) = g(\psi(t)\psi(s)) \\ &= g(\psi(ts)) = g(\psi(\rho_s(t)))\end{aligned}$$

$$= \quad (g \circ \psi \circ \rho_s)(t).$$

Thus, $g \circ \rho_{\psi(s)} \circ \psi = g \circ \psi \circ \rho_s$ for all $s \in S$. Then

$$R_s(\psi^* g) = \psi^* g \circ \rho_s = g \circ \psi \circ \rho_s = g \circ \rho_{\psi(s)} \circ \psi = R_{\psi(s)} g \circ \psi = \psi^*(R_{\psi(s)} g)$$

Since g is arbitrary, the identity $R_s \circ \psi^* = \psi^* \circ R_{\psi(s)}$ follows and the identity $L_s \circ \psi^* = \psi^* \circ L_{\psi(s)}$ can be shown similarly.

To show that $\mathcal{F}$ is left m-introverted, let $\mu \in \mathcal{MM}(\mathcal{F})$ and choose a net $(s_\alpha)_{\alpha \in I}$ in S such that $\epsilon(s_\alpha) \to \mu$ in the $\sigma(\mathcal{F}^*, \mathcal{F})$-topology, where $\epsilon : S \to \mathcal{MM}(\mathcal{F})$ is the evaluation mapping. Since X is compact, we may assume that $x = \lim_\alpha \psi(s_\alpha)$ exists in X. Then, for all $f \in \mathcal{BC}(X)$ and $s \in S$,

$$\begin{aligned} T_\mu(\psi^* f)(s) &= \mu(L_s \psi^* f) = \lim_\alpha \epsilon(s_\alpha)(L_s \psi^* f) \\ &= \lim_\alpha (L_s \psi^* f)(s_\alpha) = \lim_\alpha (\psi^* f)(s s_\alpha) \\ &= \lim_\alpha (f \circ \psi)((s s_\alpha) = f(\lim_\alpha \psi(s s_\alpha)) \\ &= f(\lim_\alpha \psi(s)\psi(s_\alpha)) = f(\psi(s) \lim_\alpha \psi(s_\alpha)) \\ &= f(\psi(s)x) = (R_x f)(\psi(s)) \\ &= \psi^*(R_x f)(s). \end{aligned}$$

Thus, $T_\mu(\psi^* f) = \psi^*(R_x f)$. Since $R_x f \in \mathcal{BC}(X)$, we get $T_\mu(\psi^* f) \in \mathcal{F}$. That is, $\mathcal{F}$ is left m-introverted and hence $\mathcal{F}$ is m-admissible.

Conversely, if $\mathcal{F}$ is an m-admissible subalgebra of $\mathcal{BC}(X)$, then by Theorems 3.12 and 3.22, the pair $(\psi, X) = (\epsilon, \mathcal{MM}(\mathcal{F}))$ is a compactification of S. By Corollary 3.13, the map $f \mapsto \hat{f} : \mathcal{F} \to \mathcal{BC}(X)$ is an isometric isomorphism and for any $f \in \mathcal{F}$, $\epsilon^*(\hat{f}) = f$. So, ϵ^* is the inverse map of $f \mapsto \hat{f} : \mathcal{F} \to \mathcal{BC}(X)$. Hence, $\epsilon^*[\mathcal{BC}(X)] = \mathcal{F}$. *Q.E.D.*

Definition 3.28: Let S be a semitopological semigroup and $\mathcal{F}$ be an m-admissible subalgebra of $\mathcal{BC}(S)$. Any compactification (ψ, X) of S such that $\psi^*\mathcal{BC}(S) = \mathcal{F}$ is called an *$\mathcal{F}$-compactification* of S. The compactification $(\epsilon, \mathcal{MM}(\mathcal{F}))$ is called the *canonical $\mathcal{F}$-compactification* of S, and will be denoted by $(\epsilon, \mathcal{S}^{\mathcal{F}})$.

Definition 3.29: Let S be a semitopological semigroup and P a property of compactifications (ψ, X) of S. A *P-compactification* of S is a compactification that has the given property P. A P-compactification of S that is an extension of every other P-compactification of S is called a *universal P-compactification* of S.

3.5. Left Norm Continuous and Left Uniformly Continuous Functions

Definition 3.30: Let S be a semitopological semigroup. A function $f \in \mathcal{BC}(S)$ is said to be *left norm continuous* if the mapping

$$s \mapsto L_s f : S \to \mathcal{BC}(S)$$

is norm continuous. The set of all left norm continuous functions on S is denoted by $\mathcal{LC}(S)$.

Theorem 3.31: Let S be a semitopological semigroup. Then $\mathcal{LC}(S)$ is a left m-introverted, translation invariant C^*-subalgebra of $\mathcal{BC}(S)$ containing the constant functions, that is $\mathcal{LC}(S)$ is m-admissible.

Proof: First we show that $\mathcal{LC}(S)$ is left translation invariant. Let $t \in S$. Let $f \in \mathcal{LC}(S)$, we need to show that $u \mapsto L_u L_t f$ is norm continuous for any $u \in S$. Let $u \in S$ and let $(u_i)_{i \in I}$ be a net in S such that $u_i \to u$. Then

$$\begin{aligned} \|L_{u_i} L_t f - L_u L_t f\| &= \sup_{r \in S} |f(u_i(tr)) - f(u(tr))| \\ &= \sup_{r \in S} |f((u_i t)r) - f((ut)r)| \\ &= \sup_{r \in S} |L_{u_i t} f(r) - L_{ut} f(r)| \end{aligned}$$

$$= \|L_{u_i t}f - L_{ut}f\|$$

Since λ_t is continuous, $u_i t \to ut$ and as $f \in \mathcal{LC}(S)$, the result follows.

Next, we will show that $\mathcal{LC}(S)$ is left m-introverted, let $f \in \mathcal{LC}(S)$ and $\mu \in \mathcal{MM}(\mathcal{LC}(S))$. We want to show that $T_\mu \in \mathcal{LC}(S)$. Let $u \in S$ and V be a neighborhood of u. Then for any $v \in V$,

$$\begin{aligned}
\|L_v T_\mu f - L_u T_\mu f\| &= \sup_{t\in S} |L_v T_\mu f(t) - L_u T_\mu f(t)| \\
&= \sup_{t\in S} |T_\mu f(vt) - T_\mu f(ut)| \\
&= \sup_{t\in S} |\mu(L_{vt}f) - \mu(L_{ut}f)| \\
&= \sup_{t\in S} |\mu(L_t L_v f - L_t L_u f)| \\
&= \sup_{t\in S} |\mu(L_t(L_v f - L_u f))| \\
&= \sup_{t\in S} |T_\mu(L_v f(t) - L_u f(t))| \\
&\leq \|T_\mu\| \|L_v f - L_u f\| \\
&\leq \|L_v f - L_u f\|
\end{aligned}$$

Since $f \in \mathcal{LC}(S)$, we get the result. *Q.E.D.*

Definition 3.32: A compactification (ψ, X) of a semitopological semigroup S is said to have the *joint continuity property* if the mapping

$$(s, x) \mapsto \psi(s)x : S \times X \to X$$

is continuous.

Now, we state a result on joint continuity, whose proof can be found in [4, Appendix B]

Lemma 3.33: Let X be a topological space, Y a compact topological space and let $f : X \times Y \to \mathbf{C}$ be a bounded, separately continuous function. Define a function

$F : X \to \mathcal{BC}(Y)$ by $F(x)(y) = f(x, y)$ $(x \in X,\, y \in Y)$. Then f is jointly continuous at every point of $\{x\} \times Y$ if and only if F is norm continuous at x.

Proposition 3.34: Let (ψ, X) be a compactification of S with the joint continuity property and let $g \in \mathcal{BC}(X)$. Then the mapping $s \mapsto L_{\psi(s)}g : S \to \mathcal{BC}(X)$ is norm continuous, that is, $g \in \mathcal{LC}(X)$. In other words, $\mathcal{BC}(X) \subseteq \mathcal{LC}(X)$.

Proof: Define $f : S \times X \to \mathbf{C}$ by $f(s, x) = g(\psi(s)x)$ and $F : S \to \mathcal{BC}(X)$ by $F(s) = L_{\psi(s)g}$. Then, $F(s)(x) = L_{\psi(s)}g(x) = g(\psi(s)x) = f(s, x)$. Since X is compact and g is continuous, f is a bounded function. Since g and $(s, x) \mapsto \psi(s)x$ are continuous, f is jointly continuous. Therefore by Lemma 3.33, F must be norm continuous, that is the mapping $s \mapsto L_{\psi(s)}g$ is norm continuous. *Q.E.D.*

Remark: By Theorems 3.27 and 3.31, a semitopological semigroup S has a canonical $\mathcal{LC}$-compactification $(\epsilon, S^{\mathcal{LC}})$.

We next prove the universal property of $\mathcal{LC}$-compactifications.

Theorem 3.35: Let S be a semitopological semigroup. Then $(\epsilon, S^{\mathcal{LC}})$ is the compactification of S that is universal with respect to the joint continuity property.

Proof: In order to prove that $(\epsilon, S^{\mathcal{LC}})$ has the joint continuity property, we will show that if $h \in \mathcal{BC}(S^{\mathcal{LC}})$ the mapping $H : S \times S^{\mathcal{LC}} \to \mathbf{C}$ given by $H(s, \mu) = h(\epsilon(s)\mu)$ is continuous. Let $f = \epsilon^*(h)$. Then $f \in \mathcal{LC}(S)$. Let $s, s_0 \in S$, $\mu, \mu_0 \in S^{\mathcal{LC}}$. Choose a net $(s_i) \to \mu$ in *weak**-topology. Then,

$$\epsilon(ss_i) = \epsilon(s)\epsilon(s_i) \to \epsilon(s)\mu$$

Since h is continuous, $f(ss_i) = h(\epsilon(ss_i)) \to h(\epsilon(s)\mu)$.
Also $f(ss_i) = L_s f(s_i) = \epsilon(s_i)(L_s f) \to \mu(L_s f)$.

Thus, $H(s,\mu) = h(\epsilon(s)\mu) = \mu(L_s f)$. Hence, we have

$$\begin{aligned} |H(s,\mu) - H(s_0,\mu_0)| &= |\mu(L_s f) - \mu_0(L_{s_o} f)| \\ &\leq |\mu(L_s f) - \mu(L_{s_of})| + |\mu(L_{s_o} f) - \mu_0(L_{s_o} f)| \\ &\leq \|\mu\|\|L_s f - L_{s_o} f\| + |(\mu - \mu_0)(L_{s_o} f)| \end{aligned}$$

Hence, H is continuous.

Now, let (ψ, X) be any compactification of S with the joint continuity property. Let $g \in \mathcal{BC}(X)$. By proposition 3.34 the mapping $s \mapsto L_{\psi(s)} g$ from S into $\mathcal{BC}(X)$ is norm continuous. For any $s, t \in S$,

$$\begin{aligned} \|L_s\psi^* g - L_t\psi^* g\| &= \sup_{r\in S} |L_s\psi^* g(r) - L_t\psi^* g(r)| \\ &= \sup_{r\in S} |\psi^* g(sr) - \psi^* g(tr)| \\ &= \sup_{r\in S} |g(\psi(sr)) - g(\psi(tr))| \\ &= \sup_{r\in S} |g(\psi(s)\psi(r)) - g(\psi(t)\psi(r))| \\ &\leq \sup_{x\in X} |g(\psi(s)x) - g(\psi(t)x)| \\ &= \sup_{r\in S} |L_{\psi(s)} g(x) - L_{\psi(t)} g(x)| \\ &= \|L_{\psi(s)} g - L_{\psi(t)} g\|. \end{aligned}$$

Hence, $\psi^* g \in \mathcal{LC}(S)$. Therefore, $\psi^*[\mathcal{BC}(X)] \subseteq \mathcal{LC}(S)$. Thus, we get that $(\epsilon, S^{\mathcal{LC}})$ is an extension of (ψ, X). *Q.E.D.*

Definition 3.36: Let S be a topological group. A function f in $\mathcal{B}(S)$ is called *left uniformly continuous* if for any $\varepsilon > 0$ there is a neighborhood V of 1 such that $xy^{-1} \in V$ implies that $|f(x) - f(y)| < \varepsilon$. The set of all left uniformly continuous functions on S is denoted by $\mathcal{LUC}(S)$.

Theorem 3.37: Let S be a topological group. Then $\mathcal{LUC}(S) = \mathcal{LC}(S)$.

Proof: Let $f \in \mathcal{LUC}(S)$. Let $\varepsilon > 0$ be given. Then there is a neighborhood

V of 1 such that $xy^{-1} \in V$ implies that $|f(x) - f(y)| < \varepsilon$. Let $s \in S$. Then Vs is a neighborhood of s and $t \in Vs$ means $ts^{-1} \in V$ which implies for any $r \in S$, $(tr)(sr)^{-1} \in V$. Then, for any $r \in S$, we have $|f(tr) - f(sr)| < \varepsilon$. Therefore, $\|L_t f - L_s f\| = \sup_{r \in S} |f(tr) - f(sr)| \leq \varepsilon$.

Conversely, if $f \in \mathcal{LC}(S)$, then the mapping $s \mapsto L_s f : S \to \mathcal{BC}(S)$ is norm continuous at 1, so there is a neighborhood V of 1 such that $t \in V$ implies that $\|L_t f - L_1 f\| < \varepsilon$. Hence, if $xy^{-1} \in V$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(xy^{-1}y) - f(y)| \leq \sup_{r \in S} |f(xy^{-1}r) - f(r)| \\ &= \|L_{xy^{-1}} f - L_1 f\| \leq \varepsilon. \end{aligned}$$

Q.E.D.

Remark: If S is a topological group, the compactification $S^{\mathcal{LC}}$ will be the $\mathcal{LUC}$-compactification of S and will be denoted by $S^{\mathcal{LUC}}$.

3.6. $G^{\mathcal{LUC}}$ as a Quotient of βG_d

In this section, we define the LUC-compactification of a topological group with a second approach. Let G be a topological group. Let G_d be the same group with the discrete topology. Recall that $G^{\mathcal{LUC}}$ is the compactification of G in which multiplication is continuous in the left hand variable and for which the map $G \times G^{\mathcal{LUC}} \to G^{\mathcal{LUC}}$ is continuous. We will now construct it as a quotient of βG_d.

Note that the Stone-Čech compactification βG_d can be seen as the space of ultrafilters on G_d, where the ultrafilter $p \in \beta G_d$ is determined by its members $P \in p$ ($P \subseteq G_d$). Equivalently, we can determine p by its clopen neighborhoods of the form $cl_{\beta G_d}(P)$ in the extremally disconnected space βG_d. Hence $P \in p$ if and only if $p \in cl_{\beta G_d}(P)$.

The Stone-Čech compactification βG_d can be made into a semigroup by

$$pq = \lim_i (\lim_j x_i y_j)$$

for $p, q \in \beta G_d$ and (x_i), (y_j) nets in G_d with $(x_i) \to p$, $(y_j) \to q$. Then,
(i) pq does not depend on the nets chosen;
(ii) This multiplication is associative;
(iii) It is continuous in the p-variable;
(iv) It is continuous in the q-variable only when $p \in \beta G_d$.

First we present a lemma which will be used frequently.

Lemma 3.38: Let X and Y be compact spaces,
(i) If $f : X \to Y$ is a continuous function, then for any $A \subseteq X$, $f(\overline{A}) = \overline{f(A)}$.
(ii) If $\pi : X \to Y$ is surjective and continuous, $\varphi : X \to X$ is continuous, and $\psi : Y \to Y$ is such that $\pi\varphi = \psi\pi$, then ψ is continuous.

Proof: **(i)** $\overline{A}$ is a closed subset of the compact space X, so, it is compact, and f is continuous. Hence, $f[\overline{A}]$ is compact. Also $\overline{f[A]}$ is a closed subset of the compact space Y, so, it is also compact.

Let $y \in f(\overline{A})$ so that $f(x) = y$ for some $x \in \overline{A}$. Then there exists a net $(x_i)_{i \in I} \subseteq A$ such that $x_i \to x$. Since f is continuous, $f(x_i) \to f(x) = y$. Therefore, $y \in \overline{f(A)}$.

Conversely, let $y \in \overline{f(A)}$. There exists a net $(y_i) \subseteq f(A)$ such that $(y_i) \to y$, which means there exists a net $(x_i) \subseteq A \subseteq \overline{A}$, such that $f(x_i) = y_i$ for all i. Since $\overline{A}$ is compact, there is a $x \in \overline{A}$ with $x_i \to x$. Thus, $y_i = f(x_i) \to f(x) = y \in f(\overline{A})$.

(ii) Let $K \subseteq Y$ be closed, then K is compact and $\psi^{-1}(K) = \pi(\varphi^{-1}(\pi^{-1}(K)))$. Since π is continuous, $\pi^{-1}(K)$ is closed in X, hence compact. Since φ is continuous, $\varphi^{-1}(\pi^{-1}(K))$ is closed in X, hence compact. Hence, $\pi(\varphi^{-1}(\pi^{-1}(K)))$ is compact, and consequently, closed in X, by the continuity of π. Therefore, ψ is continuous. *Q.E.D.*

We will construct $G^{\mathcal{LUC}}$ as a quotient of βG_d by an equivalence relation. Let $\mathcal{N}(1)$ be the set of open neighborhoods of the identity 1 of G. Define, for $p, q \in \beta G_d$

$$p \sim q \Leftrightarrow UP \cap UQ \neq \emptyset$$

for all $U \in \mathcal{N}(1)$, $P \in p$, and $Q \in q$.

Lemma 3.39: Let $p, q \in \beta G_d$. Then the followings are equivalent,
(i) $p \sim q$;
(ii) $UP \in q$ for each $U \in \mathcal{N}(1)$, $P \in p$;
(iii) $q \in \bigcap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1),\ P \in p\}$

Proof: **(i $\Leftrightarrow$ ii)** Suppose that $p \sim q$. Then for all $U \in \mathcal{N}(1)$, $P \in p$, and $Q \in q$ we have

$$\begin{aligned} UP \cap UQ \neq \emptyset &\Leftrightarrow \exists x \in UP \cap UQ \\ &\Leftrightarrow x = u_1 p = u_2 q \qquad for\ some\ u_1, u_2 \in U \\ &\Leftrightarrow u_2^{-1} u_1 p = q \qquad where\ u_2^{-1} u_1 p \in U^{-1}UP\ and\ q \in Q \\ &\Leftrightarrow U^{-1}UP \cap Q \neq \emptyset. \end{aligned}$$

Note that the map $x \mapsto x^{-1}x$ is continuous in a topological group. So, given a neighborhood V of 1, there is an open neighbourhood U of 1 such that $U^{-1}U \subseteq V$. Therefore, the neighborhoods of the form $U^{-1}U$ form a base of neighborhoods of 1 for $U \in \mathcal{N}(1)$. Therefore, $U^{-1}UP \cap Q \neq \emptyset$ if and only if $UP \cap Q \neq \emptyset$ for every U and this holds for every $Q \in q$ if and only if $UP \in q$ for all U, since q is an ultrafilter.

(ii $\Leftrightarrow$ iii) $UP \in q$ means $q \in cl_{\beta G_d}(UP)$, by the remark in the beginning of this section. *Q.E.D.*

An equivalence relation R on a topological space X is said to be closed if the canonical mapping of X onto X/R is closed. But for a compact space an equivalence relation R is closed if and only if the graph of R in $X \times X$ is closed in X (see [8]Proposition 8, Chapter 1 §10.4).

Lemma 3.40: $\sim$ is a closed equivalence relation.

Proof: Reflexivity and symmetry follow immediately from the definition. If $p \sim q$ and $q \sim r$ then for $U \in \mathcal{N}(1)$, $P \in p$, $R \in r$, by Lemma 3.39 UP, $UR \in q$ so that $UP \cap UR \neq \emptyset$, which means $p \sim r$. Therefore, $\sim$ is an equivalence relation.

Let $(p_i), (q_i)$ be nets in βG_d such that for all i, $p_i \sim q_i$ and suppose $p_i \to p \in \beta G_d$ and $q_i \to q \in \beta G_d$, then for $P \in p$, $Q \in q$, we have eventually $P \in p_i$ and $Q \in q_i$, since P and Q are neighborhoods of p and q, respectively. Thus, for $U \in \mathcal{N}(1)$, $UP \cap UQ \neq \emptyset$, so that $p \sim q$. Therefore, $\sim$ is a closed equivalence relation. *Q.E.D.*

From the theory of equivalence relations, we immediately conclude that $\beta G_d/\sim$ is a compact Hausdorff space. For $E \subseteq G$, let $\overline{E}$ be the closure of E in $\beta G_d/\sim$, and $h : \beta G_d \to \beta G_d/\sim$ be the canonical map. By Lemma 3.39, we have $q \in \cap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1),\ P \in p\}$ if and only if $p \sim q$. Therefore, the equivalence class of $p \in \beta G_d$ is $[p]_\sim = \bigcap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1),\ P \in p\}$.

Let W be a neighborhood of the equivalence class of $p \in \beta G_d$, in βG_d, then $h(W)$ is a neighborhood of $h(p)$ in $\beta G_d/\sim$. It should be noticed that this statement does not imply that for any neighborhood V of $p \in \beta G_d$, $h(V)$ is a neighborhood of $h(p)$. And by Lemma 3.38(i), $h(cl_{\beta G_d}(UP)) = \overline{h(UP)}$ is a neighborhood of $h(p)$ for each $U \in \mathcal{N}(1)$, $p \in P$. We now prove that these neighborhoods actually form a base.

Lemma 3.41: **(i)** For $p \in \beta G_d$, the sets $h(cl_{\beta G_d}(UP)) = \overline{h(UP)}$ form a neighborhood base of $h(p)$;

(ii) h is injective on $G_d \subseteq \beta G_d$. The topology induced on $h(G_d)$ is the topology of G.

Proof: **(i)** By the above paragraph, the sets $\overline{h(UP)}$ are neighborhoods of $h(p)$. If W is any open neighborhood of $h(p)$, then $h^{-1}(W)$ is an open neighborhood of the equivalence class $\bigcap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1),\ P \in p\}$. Since UP is directed by $\supseteq$, the finite intersection property in the compact space βG_d shows that there are U and P such that $cl_{\beta G_d}(UP) \subseteq h^{-1}(W)$, hence $\overline{h(UP)} \subseteq W$.

(ii) To show that h is injective on G_d, for $p \in G_d$ we take $P = \{p\}$. When $U \subseteq G_d$ we have $cl_{\beta G_d}U \cap G_d = U$, so that $cl_{\beta G_d}(U\{p\}) \cap G_d = Up$. Injectivity of h on G_d immediately follows. Since the collection of sets, $\{Up : U \in \mathcal{N}(1)\}$ gives a neighborhood base for p in the group topology of G, $h(G_d)$ is homeomorphic to G. *Q.E.D.*

We now know that G_d is injectively embedded in $\beta G_d/\sim$, so we shall consider G as a subset of $\beta G_d/\sim$, hence for $g \in G$ and $p \in \beta G_d$ we may write $gh(p) = h(gp) = h(g)h(p)$. We also want $\beta G_d/\sim$ to be a semigroup.

Lemma 3.42: $\beta G_d/\sim$ is a semigroup in which multiplication is continuous in the left-hand variable and the product map $G \times (\beta G_d/\sim) \rightarrow \beta G_d/\sim$ is continuous.

Proof: To show that the quotient space $\beta G_d/\sim$ is semigroup it is enough to show that multiplication in βG_d is compatible with $\sim$. We will prove this in two steps, first for multiplication on the right, then for multiplication on the left.

Let $r \in \beta G_d$. We will show that right multiplication by r sends equivalence classes into equivalence classes. Let $p \in \beta G_d/\sim$, and put $t = pr$. For $T \in t$, continuity of multiplication on the right in βG_d by r shows that there is $P \in p$ with $cl_{\beta G_d}(P)r \subseteq cl_{\beta G_d}(T)$. Then for any $u \in G_d$, continuity of multiplication on the left by u shows that $cl_{\beta G_d}(uP)r \subseteq cl_{\beta G_d}(uT)$ and consequently for any $u \in \mathcal{N}(1)$, $cl_{\beta G_d}(UP)r \subseteq cl_{\beta G_d}(UT)$, by Lemma 3.38(i). Intersecting over all such P and then U, T shows that r maps the equivalence class of p into the equivalence class of t.

Next, we show that multiplication by $g \in G_d$ on the left also preserves equivalence classes. With $p \in \beta G_d$ and $t = gp$, repeating the above argument, we get $cl_{\beta G_d}(UgP) \subseteq cl_{\beta g_d}(UT)$. Then, by continuity of multiplication on the left by g, we have $g(cl_{\beta G_d}(g^{-1})UgP) = cl_{\beta G_d}(UgP)$ and also as U runs through $\mathcal{N}(1)$, so also does $g^{-1}Ug$. Hence, $g\bigcap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1), P \in p\} \subseteq \bigcap\{cl_{\beta G_d}(UT) : U \in \mathcal{N}(1),\, T \in t\}$.

Now, we will show that multiplication on the left by $s \in \beta G_d$ is compatible with $\sim$. Let $(g_i)_{i\in I}$ be a net in G_d with $g_i \to s$. If $p \sim q$, then we have $g_i p \sim g_i q$ for all $i \in I$ and since $\sim$ is closed we conclude that $sp \sim sq$. Therefore, multiplication is compatible with $\sim$.

Since multiplication on the right in $\beta G_d/\sim$ is the quotient of multiplication on the right in βG_d writing

$$h(p)h(q) = h(pq) = h(h^{-1}(p)h^{-1}(q))$$

and applying Lemma 3.38(ii), we get multiplication on the right in $\beta G_d/\sim$ is continuous.

G is a topological group, so every element $g \in G$ has an inverse $g^{-1} \in G$. And since multiplication on the left by g^{-1} is continuous, to show that $G \times (\beta G_d/\sim) \to \beta G_d/\sim$ is continuous, it is enough to show continuity at each point of the form $(1, h(p))$. Let $\overline{h(UP)}$ be a basic neighborhood of $h(p)$. Let $V \in \mathcal{N}(1)$ be such that $V^2 \subseteq U$. Then $\overline{h(VP)}$ is also a neighborhood of $h(p)$ and since h is a continuous homomorphism, we get

$$V \times \overline{h(VP)} \to V\overline{h(VP)} \subseteq \overline{h(V^2P)} \subseteq \overline{h(UP)}. \qquad Q.E.D.$$

If G is locally compact, that is, if every neighborhood of 1 contains a compact neighborhood, then for a compact neighborhood V, we observe from Lemma 3.38(i) that $V\overline{h(P)} = \overline{h(VP)}$. So, we have:

Proposition 3.43: Let G be a locally compact group. Then the sets $V\overline{h(P)}$ with V an open (or a compact) neighborhood of 1 in G and $P \in p$ form a neighborhood base of $h(p)$ in $\beta G_d/\sim$.

Recall that $G^{\mathcal{LUC}}$ is the largest semigroup compactification with the joint continuity property. Now, we identify $G^{\mathcal{LUC}}$ with $\beta G_d/\sim$.

Theorem 3.44: $\beta G_d/\sim$ is isomorphic with $G^{\mathcal{LUC}}$.

Proof: Let (ψ, S) be a compactification of G with the joint continuity property. Let $f : \beta G_d \to S$ be the continuous extension of ψ regarded as a map from G_d to S. Then,

$$\begin{aligned}
f(pq) &= f(\lim_i \lim_j x_i y_j) \\
&= \lim_i \lim_j f(x_i y_j) \qquad \mathit{as\, f\, is\, continuous} \\
&= \lim_i \lim_j f(x_i) f(y_j) \qquad \mathit{as\, \psi\, is\, a\, homomorphism} \\
&= \lim_i f(x_i) \lim_j f(y_j) \\
&= \lim_i f(x_i) f(q) \qquad \mathit{as\, \psi\, is\, continuous} \\
&= f(p) f(q)
\end{aligned}$$

Hence, f is a homomorphism on βG_d.

Let $p \in \beta G_d$. Let W be any closed neighborhood of $f(p)$ in S. Then, by the joint continuity property, there are neighborhoods $U \in \mathcal{N}(1)$ and V of $f(p)$ in S such that $\psi(U)V \subseteq W$. Since f is continuous, there is $P \in p$ with $f(cl_{\beta G_d} P) \subseteq V$. Then $f(UP) \subseteq \psi(U)V$, and since W is closed $f(cl_{\beta G_d} UP) \subseteq W$. Now, the image of the equivalence class of p satisfies $f(\bigcap\{cl_{\beta G_d}(UP) : U \in \mathcal{N}(1),\, P \in p\}) \subseteq f(cl_{\beta G_d}(UP)) \subseteq W$, which means $f(p) \in \bigcap\{W : W$ is a neighbourhood of $f(p)\}$. We conclude that f maps the equivalence class of p to the point $f(p)$. Thus, f is compatible with the relation $\sim$ and gives quotient map $\beta G_d/\sim \longrightarrow S$, which means (ψ, S) is a factor of $(h, \beta G_d/\sim)$. *Q.E.D.*

4. LOCAL STRUCTURE THEOREMS AND VEECH'S THEOREM

4.1. Local Structure Theorems

In this section we shall prove the Local Structure Theorem and give some variants and extensions of it. From now on G will be a locally compact group and $\mathcal{N}(1)$ will be the set of open, relatively compact neighborhoods of the identity of G.

Definition 4.1: A subset $P \subseteq G$ (considered with its group topology) is called *U-discrete* for $U \in \mathcal{N}(1)$ if $Up_1 \bigcap Up_2 = \emptyset$ when $p_1, p_2 \in P$ and $p_1 \neq p_2$. $P \subseteq G$ is called $uniformly\ discrete$ if it is U-discrete for some $U \in \mathcal{N}(1)$.

A uniformly discrete space is discrete with its relative topology. So, we may speak of its Stone-Čech compactification βP. Also βP can be thought of as a subspace of βG_d. In section 3.5 we have characterized $G^{\mathcal{LUC}}$ using the Left Uniformly Continuous functions on G and in chapter 2, we have noted that βG_d can also be determined by the continuous bounded functions on G. Also for any discrete subset $P \subseteq G$, both βP and $\overline{P} \subseteq G^{\mathcal{LUC}}$ inherit these properties.

Proposition 4.2: The image of βP under the natural homomorphism $h : \beta G_d \rightarrow \beta G_d$ is $\overline{P}$, the closure of P regarded as a subset of $P \subseteq G \subseteq \beta G_d / \sim$.

Proof: Let P be U-discrete, for some $U \in \mathcal{N}(1)$ and put $P = \{p_i \colon i \in I\}$. Consider the function $\psi = \sum_{i \in I} a_i \rho_{p_i} \varphi$, where φ is a continuous function on U with $\varphi(1) = 1$, $\varphi = 0$ outside of U, and the set $\{a_i : i \in I\}$ is bounded. First we prove that ψ is a left uniformly continuous function.

Since φ has a compact support, given $\varepsilon > 0$, there exists a neighborhood W of 1 such that if $yx^{-1} \in W$ then $|\varphi(x) - \varphi(y)| < \varepsilon$.

Therefore, every bounded function f on P gives rise to a function ψ_f on G by putting $a_i = f(pi)$ and this is a left uniformly continuous function. Since the bounded functions, determine βP, this gives a map $\beta P \to \overline{P} \subseteq G^{\mathcal{LUC}}$.

Every $x \in \beta P$ gives a complex valued homomorphism $f \mapsto f(x)$ on $\mathcal{C}(\beta P)$ and every such homomorphism has an extension on $\mathcal{C}(G^{\mathcal{LUC}})$. Hence, we have a map $F : \beta P \to G^{\mathcal{LUC}}$ such that for any $i \in I$, $F(p_i) = p_i$. $F(\beta P)$ is compact and contains P, so also contains $\overline{P}$. Since any element of βP is a limit of a net of elements of P, for any q, $F(q)$ is a limit of points in $F[P]$, which means $F(q) \in \overline{P}$. Therefore, $F(\beta P) = \overline{P}$.

On the other hand, if $q, t \in \beta P$ and $q \neq t$, there is a bounded function f on P such that $f(q) \neq f(t)$. Define ψ_f on $G^{\mathcal{LUC}}$ using f as above. Then $\psi \circ F = f$ and $\psi \circ F(q) = f(q) \neq f(t) = \psi o F(t)$. Hence, $F(q) \neq F(t)$. Thus, F is also an injection. Hence, $\beta P = \overline{P}$. *Q.E.D.*

Theorem 4.3: Let $U \in \mathcal{N}(1)$, and $P \subseteq G$ be U-discrete.
(i) $(u, q) \mapsto uh(q) : U \times \beta P \to U\overline{P} \subseteq \beta G_d / \sim$ is continuous and injective.
(ii) If G is locally compact and $cl_G(U)$ is compact, then $U\overline{P}$ is open and homeomorphic with $U \times \beta P$.

Proof: **(i)** Since multiplication is continuous on U, it is enough to show that for all $p \in \overline{P}$, $(1, p) \mapsto p \in U\overline{P}$ is continuous. For any $p \in P \subseteq G^{\mathcal{LUC}}$, by Proposition 3.43 a basic open neighborhood is of the form $V\overline{h(Q)}$ for $V \in \mathcal{N}(1)$ and $Q \in p$. Therefore, $(1, p) \mapsto p$ is continuous from $U \times P \to UP$. Then since, P is dense in βP and in $\overline{P}$, we get $(1, p) \mapsto p$ is continuous on $U \times \beta P \to U\overline{P}$.

Injectivity: Let (u, q) and (v, r) be distinct points in $U \times \beta P$. If $q \neq r$, there are disjoint subsets of Q and R of P with $Q \in q$ and $R \in r$. Since, right multiplication by u and v are both continuous and U is open in group topology of G, there are neighborhoods W_1, W_2 of 1 such that $W_1 v$, $W_2 v \subseteq U$, so if we let $W = W_1 \cap W_2$, we have a neighborhood of 1 such that Wu, $Wv \subseteq U$.

Now, we claim that $uQ \cup vR$ is W-discrete: First observe that $Wu \subseteq U$ and $Wv \subseteq U$ implies that $WuQ \subseteq UQ$ and $WvR \subseteq UR$, respectively. Let $a, b \in uQ \cup vR$ be distinct. If $a = uq_1$, $b = uq_2$ for some $q_1, q_2 \in Q$, then

$$Wa \cap Wb = Wuq_1 \cap Wuq_2 \subseteq Uq_1 \cap Uq_2 = \emptyset.$$

Or if $a = vr_1$, $b = vr_2$ for some $r_1, r_2 \in R$, then

$$Wa \cap Wb = Wvr_1 \cap Wvr_2 \subseteq Ur_1 \cap Ur_2 = \emptyset.$$

Or if $a = uq$, $b = vr$ for some $q \in Q$ and $r \in R$, then

$$Wa \cap Wb = Wuq \cap Wvr \subseteq Uq \cap Ur = \emptyset.$$

Since $Q, R \subseteq P$ and P is U-discrete.

Therefore, uq and vr are contained in disjoint $\sim$-equivalence classes, which means

$$uh(q) = h(uq) \neq h(vr) = vh(r).$$

On the other hand, if $q = r$, then we must have $u \neq v$. Since $u \neq v$, $uv^{-1} \neq 1$, so there exists $W_3 \in \mathcal{N}(1)$ such that $uv^{-1} \notin W_3$ and by a similar argument as above we choose $W \in \mathcal{N}(1)$ symmetric such that $Wu, Wv \subseteq U$ and $uv^{-1} \notin W^2$. Then $uP \cup vP$ is W-discrete, since $Wup_1 \cap Wvp_2 \subseteq Up_1 \cap Up_2 = \emptyset$. Hence, again

$$uh(q) \neq vh(q).$$

(ii) First, we prove that $U\overline{P}$ is a neighborhood of any of its points ux such that $u \in U$ and $x \in \overline{P}$. Let $W \in \mathcal{N}(1)$ satisfy $Wu \subseteq U$. We take $p \in \beta G_d$ with $h(p) = x$, and $P \in p$ such that $uP \in up$. Then by Proposition 3.43, $Wu\overline{P} = Wu\overline{h(P)} = W\overline{h(uP)}$ is a neighborhood of $h(up) = ux$ contained in $U\overline{P}$. Hence, $U\overline{P}$ is open.

Now, let $(u,p) \in U \times \beta P$, V be an open neighborhood of u such that $\overline{V} \subseteq U$, and $Q \subseteq P$ satisfy $p \in \beta Q$. The map $(v,q) \mapsto vh(q)$; $\overline{V} \times \beta Q \to \overline{VQ}$ is a continuous bijection between compact spaces and therefore a homeomorphism. From the above argument, this map sends the open neighborhood $V \times \beta Q$ of (u,p) to the open neighborhood $V\overline{Q}$ of up. Thus, the map $U \times \beta Q \to U\overline{P}$ is a homeomorphism. *Q.E.D.*

Our next theorem, The Local Structure Theorem describes the neighborhoods of points in $G^{\mathcal{LUC}}$. It is essentially a version of Theorem 4.3 and will enable us to use the properties of βG_d in our future proofs.

The Local Structure Theorem 4.4: Let G be a locally compact group. Let $x \in G^{\mathcal{LUC}}$. Then there exists $U \in \mathcal{N}(1)$ and a U-discrete $P \subseteq G$, with $x \in \overline{P}$ such that $U\overline{P} \cong U \times \beta P$ is an open neighborhood of x. Furthermore, if $p \in \beta G_d$ is any element with $h(p) = x$, we may take P (considered now as a subset of G_d) with $P \in p$.

Proof: Let $V \in \mathcal{N}(1)$ be symmetric, and $p \in \beta G_d$ satisfy $h(p) = x$. Let $Q \in p$ ($Q \subseteq G$), and take any maximal $\overline{V}$-discrete set $Q' \subseteq Q$ (which exists by Zorn's Lemma). Thus, $q_1 \neq q_2$ in Q' implies $\overline{V}q_1 \cap \overline{V}q_2 = \emptyset$. Since Q' is maximal with this property we have for all $q \in Q$, $\overline{V}q \cap \overline{V}Q' \neq \emptyset$, which means there exist $v_1, v_2 \in \overline{V}$ and $q' \in Q'$ with $v_1 q = v_2 q'$, so, $q = v_1^{-1} v_2 q'$ and $v_1^{-1} v_2 \in \overline{V}^{-1}\overline{V} = \overline{V}^2$, since V is symmetric. Hence, $q \in \overline{V}^2 Q'$. Therefore, $p \in Q \subseteq \overline{Q} \subseteq \overline{V}^2\overline{Q'}$, by joint continuity property, and there exists $v \in \overline{V}^2$ and $p' \in \overline{Q'}$ such that $p = vp'$.

Now, for any $q_1, q_2 \in Q'$ with $q_1 \neq q_2$, we have,

$$vVv^{-1}vq_1 \cap vVv^{-1}vq_2 = v(Vq_1 \cap Vq_2) = \emptyset$$

since Q is V-discrete. Hence vQ' is vVv^{-1}-discrete. Let $U = vVv^{-1}$ and $P = vQ'$ in Theorem 4.3. So, $U \times \beta P \to UP$ is a homeomorphism, $U\overline{P}$ is open and contains $x = h(p)$, which means $U\overline{P}$ is a neighborhood of x. *Q.E.D.*

Next, we will consider three special cases. The first one is the case of σ-compact groups:

Lemma 4.5: Let P be uniformly discrete in a σ-compact, locally compact group G. Then, $\overline{P} \setminus P \subseteq G^{\mathcal{LUC}} \setminus G$ and P is countable.

Proof: Since the relative topology of P as a subset of G is discrete, P does not contain any cluster points in G. So, $\overline{P} \setminus P \subseteq G^{\mathcal{LUC}} \setminus G$ follows. Let $G = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. Then for each $n \in \mathbf{N}$, $P \cap K_n$ must be finite, because otherwise P would have a cluster point and could not be uniformly discrete. Hence, $P = \bigcup_{n=1}^{\infty} P \cap K_n$ gives that P is countable. *Q.E.D.*

Corollary 4.6: If G is σ-compact, every point in $G^{\mathcal{LUC}}$ (respectively, in $G^* = G^{\mathcal{LUC}} \setminus G$) has an open neighborhood in $G^{\mathcal{LUC}}$ (respectively, in G^*) homeomorphic to $U \times \beta\mathbf{N}$ (respectively, $U \times \mathbf{N}^*$), for some neighborhood U of 1 in G.

Proof: Follows immediately from The Local Structure Theorem and Lemma 4.5. *Q.E.D.*

Our next case is IN groups:

Definition 4.7: A topological group G is said to be an *IN group* if G has a relatively compact neighborhood base of the identity, which is also invariant under inner automorphisms, that is $gVg^{-1} = V$ for all $g \in G$ and for all $V \in \mathcal{N}(1)$.

Corollary 4.8: **(i)** Let V be an open relatively compact neighborhood of 1 in a σ-compact IN group G. Let $x \in G^{\mathcal{LUC}}$. Then for any V-discrete subset P of G with $x \in \overline{P}$ (and such sets exist) $V\overline{P}$ is an open neighborhood of x. In particular, every point of $G^{\mathcal{LUC}}$ has a neighborhood homeomorphic with $V \times \beta\mathbf{N}$.
(ii) Suppose in addition that V generates G. Then for any compact $K \subseteq G$ there is a uniformly discrete $P \subseteq G$ such that multiplication is a homeomorphism $VK \times \beta P \rightarrow VK\overline{P}$ onto an open neighborhood of x.

Proof: **(i)** Let V be a relatively compact invariant neighborhood of 1 and let $p \in \beta G_d$ be such that $h(p) = x$ and $Q \in p$ ($Q \subseteq G$). Let again, Q' be a maximal $\overline{V}$-discrete set, then we have for any $q \in Q$, $\overline{V}q \cap \overline{V}Q' \neq \emptyset$, so, $q \in \overline{V}^2 Q'$. Hence, $p \in \overline{Q} \subseteq \overline{V}^2\overline{Q'}$, and $p = vp'$ where $v \in \overline{V}^2$, $p' \in \overline{Q'}$. Now, vQ' is vVv^{-1}-discrete, so if we let $P = vQ'$ and $U = V$ as in The Local Structure Theorem, from 4.3(i) we get, $U \times \beta P \to U\overline{P}$ is a homeomorphism onto an open subset. Therefore, $U\overline{P}$ is a neighborhood of x, homeomorphic with $U \times \beta P$. And by Corollary 4.6, $U \times \beta P$ is homeomorphic to $U \times \beta\mathbf{N}$.

(ii) Assume that V generates G. Then since K is compact, $K \subseteq \bigcup_{n=1}^{\infty} V^n = G$ implies that there is $n \in \mathbf{N}$ with $K \subseteq V^n$. The set V^{n+1} is in $\mathcal{N}(1)$ and contains VK. Take P to be V^{n+1}-discrete and apply (i) to get the result. *Q.E.D.*

Our last case is the case of discrete groups, which will be considered again in Section 4.2:

Corollary 4.9: Let G be a discrete group. Take any $g \in G$, $g \neq 1$, and $x \in \beta G_d$. Then there is $P \subseteq G$ with $x \in \overline{P}$ such that $\{g, 1\} \times \beta P \to g\overline{P} \cup \overline{P}$ is a homeomorphism onto an open subset of $G^{\mathcal{LUC}} = \beta G$.

Proof: $V = \{g, 1\}$ is a compact and open neighborhood of 1 in G. Let $p \in \beta G_d$ be such that $h(p) = x$. Let $Q \in p$, $Q \subseteq G$. As in Theorem 4.4 and Corollary 4.8, take any maximal $\overline{V} = V$-discrete set $Q' \subseteq Q$. Since Q' is maximal, for any $q \in Q$, $Vq \cap VQ' \neq \emptyset$, so $q \in V^2Q'$, therefore, $Q \subseteq V^2Q'$. But, $V^2 = \{1, g, g^2\}$, so for each $v \in V^2$, v commutes with V. Hence, if we put $U = vVV^{-1} = V$ and $P = vQ'$, by Theorem 4.3, we get $U \times \beta P \to U\overline{P}$ is a homeomorphism onto the open subset $U\overline{P}$ of $G^{\mathcal{LUC}}$, and contains $h(p) = x$. *Q.E.D.*

Now, we turn to the general case of locally compact groups:

The Three Sets Lemma 4.10: Let P be a set, $Q \subseteq P$, $f : Q \to P$ be an injective function such that $f(q) \neq q$ for all $q \in Q$. Then there is a partition

$Q = F_0 \cup F_1 \cup F_2$ ($F_i = \emptyset$ is also possible) with $f(F_i) \cap F_i = \emptyset$, $(0 \leq i \leq 2)$.

Proof: For any $q \in Q$, define the *orbit* of q by

$$O(q) = \{f^r(q) : f^r(q) \textit{ is defined and in } Q\}$$

(if $r < 0$, $f^r(q)$ means the unique p for which $f^{-r}(p) = q$ if it exists). Since f is injective,orbits are either identical or disjoint. As the orbits partition Q, it is enough to write each $O(q)$ as $F_0 \cup F_1 \cup F_2$. First fix one q in each orbit.

If $O(q)$ is a chain (finite or infinite), $O(q) = \{\ldots f^{r-1}(q), f^r(q), f^{r+1}(q)\ldots\}$, then write $F_0 = \{f^r(q) : r \text{ is even}\}$, $F_1 = \{f^r(q) : r \text{ is odd}\}$, $F_2 = \emptyset$. So, $f[F_0] \subseteq F_1 \cup (P \setminus Q)$, $f[F_1] \subseteq F_0 \cup (P \setminus Q)$, hence the conditions are satisfied.

If $O(q)$ is not a chain, then because f is injective, $O(q) = \{q = f^r(q), f^1(q), ..., f^{r-1}(q)\}$ must be a cycle (here $r \geq 2$ since $f(q) \neq q$). If r is even define F_0, F_1, F_2 as in the previous case. If r is odd, write $F_0 = \{q, f^2(q), ..., f^{r-3}(q)\}$, $F_1 = \{f(q), f^3(q), ..., f^{r-2}(q)\}$, $F_2 = \{f^{r-1}(q)\}$. Then, $f[F_0] \subseteq F_1$, $f[F_1] \subseteq F_0 \cap F_2$ and $f[F_2] = \{q\} \subseteq F_0$. *Q.E.D.*

Our next step is to prove a more general version of Corollary 4.9.

The Two Point Local Structure Lemma 4.11: Let $x \in G^{\mathcal{LUC}}$, and $g \in G$. Take $p \in \beta G_d$ with $h(p) = x$. Then there is an open neighborhood V of 1 in G and a V-discrete set $P \subseteq G$ with $P \in p$ such that $(gV \cup V) \times \beta P \to gV\overline{P} \cup V\overline{P} \subseteq G^{\mathcal{LUC}}$ is a homeomorphism onto an open set containing gx and x.

Proof: The case $g = 1$ is The Local Structure Theorem 4.4, so we may assume $g \neq 1$. Let $W_0 \in \mathcal{N}(1)$ be such that there is a W_0-discrete set P_0 with $P_0 \in p$. By taking W_0 smaller if necessary, we may assume $g \notin W_0$. By The Local Structure Theorem, $W_0\overline{P_0}$ is an open neighborhood of x, and gP_0 is gW_0g^{-1}-discrete, so that also $gW_0\overline{P_0} = gW_0gg^{-1}\overline{P_0}$ is an open neighborhood of $gx \in g\overline{P}$.

Now, we have $gW_0\overline{P_0} \cong gW_0 \times \beta P_0$ and $W_0\overline{P_0} \cong W_0 \times \beta P_0$. So, our Lemma would be immediate if $gW_0\overline{P_0} \cap W_0\overline{P_0} = \emptyset$. We shall show we can achieve this by making W_0 and P_0 smaller.

First note for a symmetric W,

$$\begin{aligned} gWg^{-1}g\overline{P} \cap W\overline{P} = \emptyset \ &\Leftrightarrow \ gw_1g^{-1}gp_1 = w_2p_2; \qquad w_1, w_2 \in W,\, p_1, p_2 \in \overline{P} \\ &\Leftrightarrow \ w_1g^{-1}gp_1 = g^{-1}w_2p_2 \\ &\Leftrightarrow \ g^{-1}gp_1 = w_1^{-1}g^{-1}w_2p_2 \\ &\Leftrightarrow \ gp_1 = gw_1^{-1}g^{-1}w_2p_2 \\ &\Leftrightarrow \ g\overline{P} \cap gWg^{-1}W\overline{P} \neq \emptyset. \end{aligned}$$

That is, for $P \subseteq P_0$, there is $W \in \mathcal{N}(1)$ with $gW\overline{P} \cap W\overline{P} = gWg^{-1}g\overline{P} \cap W\overline{P} = \emptyset$ if and only if there is $V = gWg^{-1}W$ with $g\overline{P} \cap V\overline{P} = g\overline{P} \cap gWg^{-1}W\overline{P} = \emptyset$.

Let V be symmetric and open such that $V \subseteq W_0$ and $g^{-1}Vg \subseteq W_0$, then $g \notin V$ (otherwise $g^{-1}g \in W_0$ will imply that $g \in W_0$). Define $Q = \{q \in P_0 : gq \in VP_0\}$. Then $x \in \overline{P_0} = (\overline{P_0 \setminus Q}) \cap \overline{Q}$.

When $x \in (\overline{P_0 \setminus Q})$, because $V\overline{P_0}$ is open $gx \notin V\overline{P_0}$. In this case, we can put $P = P_0 \setminus Q$ to find $g\overline{P} \cap V\overline{P} \subseteq g\overline{P_0 \setminus Q} \cap V\overline{P_0} = \emptyset$

Now, we consider the case $x \in \overline{Q}$. We define a map $\pi : V\overline{P_0} \to \overline{P_0}$ by translating the projection $V \times \beta P_0 \to \beta P_0$ using the homeomorphism $V \times \beta P_0 \cong V\overline{P_0}$.

Next, define the function $f : \overline{Q} \to \overline{P_0}$ by $q \mapsto \pi(gq)$. Then

$$f[Q] = \{\pi(gq) : q \in P_0 \ \textit{and} \ gq \in VP_0\} \subseteq P_0$$

and since f is continuous it is determined by its values on Q. Note that, $\pi(x) = x$.

Observe that if $q_1, q_2 \in Q$, then

$$\begin{aligned} f(q_1) = f(q_2) &\Leftrightarrow \pi(gq_1) = \pi(gq_2) \\ &\Leftrightarrow gq_1 \in Vgq_2 \\ &\Leftrightarrow q_1 \in g^{-1}Vgq_2 \\ &\Leftrightarrow q_1 = q_2 \end{aligned}$$

since Q is W_0-discrete and $g^{-1}Vg \subseteq W_0$. Therefore as a map of $Q \to P_0$, f is injective. And since $f(q) = q \Leftrightarrow gq \in Vq$, and $g \notin Vq$. Hence, $f(q) \neq q$ for $g \in Q$. Therefore, we can apply The Three Sets Lemma 4.10 to f and partition Q as $F_0 \cup F_1 \cup F_2$. Then the closures $\overline{F_0}$, $\overline{F_1}$, $\overline{F_2}$ partition $\overline{Q}$ and since $V\overline{Q} \cong V \times \beta Q$, the sets $V\overline{F_0}$, $V\overline{F_1}\, V\overline{F_2}$ partition $V\overline{Q}$. The point x is in one of the sets, $\overline{F_i}$ say. Put $P = F_i$. Then $f(P) = \{\pi(gp) : p \in P\}$ is a subset of another set , F_j, say. This means that $gP \subseteq VF_j$, and therefore that VP and gP are disjoint. *Q.E.D.*

Now, we come to the most general Local Structure Theorem:

The Compact Set Local Structure Theorem 4.12: Let $K \subseteq G$ be compact, $x \in G^{\mathcal{LUC}}$. Take $p \in \beta G_d$ with $h(p) = x$. Then there is an open neighborhood V of K and a uniformly discrete set $P \subseteq G$ with $P \in p$ (so, $x \in \overline{P}$) such that the multiplication map $V \times \beta P \to V\overline{P}$ is a homoemorphism onto an open set $V\overline{P}$ containing Kx.

Proof: We may assume that the identity 1 of G is in K. By The Two Point Local Structure Lemma 4.11 for each $g \in G$, there exists an open neighborhood V_g of 1 and a V_g-discrete $P_g \in p$ such that $(gV_g \cup V_g) \times \beta P_g \to (gV_g \cup V_g)\overline{P_g}$ is a homeomorphism onto an open subset of $G^{\mathcal{LUC}}$. Note that $g = 1$ is also possible and if $g \neq 1$, then $gV_g \cap V_g = \emptyset$.

Let U be an open neighborhood of K such that $U \subseteq \bigcup_{g \in K} gV_g$ and $cl_G(U)$ is compact. Then since multiplication and taking inverses are continuous on G, $\overline{U}^{-1}\overline{U}$ is a compact set. Let $g_1V_{g_1}, \ldots, g_kV_{g_k}$ be a finite cover of $\overline{U}^{-1}\overline{U}$. Put $P = P_{g_1} \cap \ldots \cap P_{g_k}$.

Since p is an ultrafilter $P \in p$.

Now, if $u, v \in \overline{U}$, then $v^{-1}u \in \overline{U}^{-1}\overline{U}$, and therefore there is r with $v^{-1}u \in g_rV_{g_r}$. For any $y, z \in \overline{P}$, we have $y, z \in \overline{P_{g_r}}$ and therefore if $(u, y) \neq (v, z)$, then since P_{g_r} is V_{g_r}-discrete we have $g_r^{-1}v^{-1}uy \neq g_r^{-1}1z$, which implies $v^{-1}uy \neq z$, then $uy \neq vz$.

Thus $\overline{U} \times \beta P \to \overline{UP}$ is injective. Now $\overline{U} \subseteq \overline{U}^{-1}\overline{U} \subseteq g_1V_{g_1} \cup \ldots \cup g_kV_{g_k}$. For each $1 \leq r \leq k$, the map $g_rV_{g_r} \times \beta P_{g_r} \to g_rV_{g_r}\overline{P_{g_r}}$ is a homeomorphism onto an open subset of βP_{g_r}. Therefore, the map $U \times \beta P \to U\overline{P}$ is also a homeomorphism onto an open set containing Kx. *Q.E.D.*

The Compact Set Local Structure Theorem 4.12 gives us a result for every compact subset of G. Before we end this section it is natural to ask whether there is a parallel result for the whole of G. For any $P \subseteq G$ with more than one element multiplication is not injective as a map $G \times P \to G$. Let $g_1, g_2 \in P$ with $g_1 \neq g_2$, put $x_1 = 1$, identity of G and $x_2 = g_1g_2^{-1}$. Then $(x_1, g_1) \neq (x_2, g_2)$ but $x_1g_1 = 1g_1 = g_1g_2^{-1}g_2 = x_2g_2$. However, we can achieve a result which is valid for a restricted class of groups.

Theorem 4.13: Let G be a locally compact and σ-compact group. Let $x \in G^* = G^{\mathcal{LUC}} \setminus G$. Then there is an open set W^o in G^* with $x \in \overline{W^o}$ such that the multiplication map $G \times W^o \to GW^o \subseteq G^*$ is a homeomorphism onto an open subset of G^*.

Proof: Let $p \in \beta G_d$ be such that $h(p) = x$. Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets with $K_n \nearrow G$ such that there is an open $U \in \mathcal{N}(1)$ with $K_nU \subseteq K_{n+1}$ for all $n \in \mathbf{N}$. From the Compact Set Local Structure Theorem 4.12, for each n, there is a uniformly discrete set $P_n \in p$ such that $K_n \times \beta P_n \to K_n\overline{P_n}$ is a homeomorphism. Since p is an ultrafilter, we can assume $(P_n)_{n=1}^{\infty}$ is decreasing (otherwise, let $O_1 = P_1, Q_2 = P_2 \cap Q_1, ..., Q_n = P_n \cap Q_{n-1}, ...$ instead of P_n). And, we can also choose P_n with $P_n \cap K_n = \emptyset$, for all $n \in \mathbf{N}$.

Put $W = \bigcap_{n=1}^{\infty} \overline{P_n}$, then $W \cap G = \emptyset$. Also, since W is a zero set in βP_1, it follows from [6, Theorem 3.28] that p is in the closure of the interior (in P_1^*) of W.

From The Compact Set Local Structure Theorem 4.12, we get that $K_n \times \overline{P_n} \to K_n\overline{P_n}$ is a homeomorphism, and since $K_{n-1}U$ is open and contained in K_n the set $K_{n-1}U\overline{P_n}$ is an open neighborhood of x in $G^{\mathcal{LUC}}$. Therefore, $G \times W = (\bigcap_{n=1}^{\infty} K_n) \times W \to GW$ is injective.

Next, we show that the map $G \times W \to GW$ is a homeomorphism. Let $(g_i, w_i)_{i\in I}$ be a net in $G \times W$ with $(g_i w_i) \to gw \in GW$. Since $g \in G$, there is an $n \in \mathbf{N}$, $g \in K_n$. Then $gw \in K_nW \subseteq K_nU\overline{P_{n+1}} \subseteq K_{n+1}\overline{P_{n+1}}$, and so for all large $i \in I$, $g_iw_i \in K_{n+1}\overline{P_{n+1}}$. By injectivity, $g_i \in K_{n+1}$, $w_i \in W$ for all such i. Using compactness we may assume $g_i \to g_0$, $w_i \to w_0$, for some $g_0, w_0 \in G^{\mathcal{LUC}}$. The joint continuity property shows that $(g_iw_i) \to g_0w_0 \in G^*$, and then uniqueness of limits shows that $g_0w_0 = gw$. From injectivity, $g_0 = g$, $w_0 = w$. Thus, we can conclude $(g_iw_i) \to (g, w) \in G \times W$. Therefore, our map is a homeomorphism.

Let W^o be the interior of W in P_1^*. We know that $K_nU\overline{P_n} \cong K_nU \times \beta P_n$ is an open subset of $G^{\mathcal{LUC}}$. We intersect with G^* to find $K_nU\overline{P_n} \cap G^* \cong K_nU \times P_n^*$, and since W^o is open in P_n^*, we conclude K_nUW^o is open in G^*. Then $\bigcap_{n=1}^{\infty} K_nUW^o = GW^o$ is open and contains K. *Q.E.D.*

4.2. Veech's Theorem

In this section, we will see how our results relate to Veech's Theorem. First we prove directly Veech's Theorem for discrete groups, to illustrate the role of the Three Sets Lemma.

Veech's Theorem for Discrete Groups 4.14: Let G be a discrete group. For $s \in G^* = G^{\mathcal{LUC}} \setminus G$, if $gs = s$ for some $g \in G$, then $g = 1$.

Proof: Let $s \in G^*$ and $g \in G$ be such that $g \neq 1$. If the subgroup H, generated by g is finite, say $H = \{g, g^2, ..., g^n = 1\}$, then choose one point from each right coset of H in G, and call the resulting set A. Then $A \subseteq G$ and the sets $A, gA, \ldots, g^{n-1}A$ are disjoint in G. So, $cl(A) \cup cl(gA) \cup \ldots \cup cl(g^{n-1}A)) = \beta G = G^{\mathcal{LUC}}$, gives a partition into open and closed subsets of βG. So, $s \in cl(g^i A)$ for some $0 \leq i \leq n-1$. Then $gs \in cl(g^{i+1}A)$ if $0 \leq i < n-1$ or $gs \in cl(A)$ if $i = n-1$. Therefore, $gs \neq s$.

If H is infinite, say $H = \{\ldots, g^{-1}, 1, g, g^2, \ldots\}$, put

$$\begin{aligned} H_0 &= \{..., g^{-2}, 1, g^2, ...\} \\ H_1 &= \{..., g^{-1}, g, g^3, ...\} \end{aligned}$$

Choose one point from each coset of H, and call the resulting set A. Then $G = AH_0 \cup AH_1$ is a disjoint union, and $cl(AH_0) \cup cl(AH_1) = \beta G = G^{\mathcal{LUC}}$ is a disjoint union of closed and open sets. Then $s \in cl(AH_i)$, $i \in \{1, 2\}$, and consequently $gs \in cl(AH_j)$, $j \in \{1, 2\}$ with $j \neq i$. Therefore, $gs \neq s$. *Q.E.D.*

Veech's Theorem 4.15: Let G be a locally compact group. Then for every $g \in G$ with $g \neq 1$ and each $s \in G^{\mathcal{LUC}}$, $gs \neq s$.

Proof: Immediate from Compact Set Local Structure Theorem 4.12. *Q.E.D.*

Our aim is to combine the results of section 4.1 with Veech's Theorem. First we prove an equivalent statement for Veech's Theorem.

Theorem 4.16: Let G be a locally compact group. Then the followings are equivalent:

(i) For every $g \in G$ with $g \neq 1$ and each $s \in G^{\mathcal{LUC}}$, $gs \neq s$.

(ii) The map $g \mapsto gs$ is injective on G for all $s \in G^{\mathcal{LUC}}$.

Proof: **(i $\Rightarrow$ ii)** If $g_1 s = g_2 s$, then $g_2^{-1} g_1 s = s$, by (1) we get $g_2^{-1} g_1 = 1$, that is $g_1 = g_2$.

(ii $\Rightarrow$ i) If $g \neq 1$ is an element of G, then since the map $g \mapsto gs$ is injective,

$gs \neq s$. *Q.E.D.*

The Local Structure Theorem 4.4, obviously implies a "local Veech", namely for each $s \in G^{\mathcal{LUC}}$, the map $g \mapsto gs$ is injective on a neighborhood V of the identity 1 of G. Also, as we have noted, Veech's Theorem is a consequence of the Compact Set Local Structure Theorem 4.12, since it implies that on each compact subset of G, $g \mapsto gx$ is injective. Hence, since G is locally compact $x \mapsto gx$ is injective on G itself. On the other hand, with a similar argument, Veech's Theorem is a consequence of the Two Point Structure Lemma 4.11. Conversely, Veech's Theorem implies that s and gs have disjoint neighborhoods, for $s \in G^{\mathcal{LUC}}$, $g \neq 1$ in G, these neighborhoods can be taken of the form given in the Local Structure Theorem 4.4, and then 4.11 follow immediately. Thus, Veech's Theorem is equivalent to Two Point Structure Lemma, 4.11.

For discrete groups, by the remarks of the last paragraph together with Corollary 4.9, we get Veech's Theorem. Conversely, the proof of Corollary 4.9 would have been trivial, if we had used Veech's Theorem, that $gx \neq x$, since then these two points would have disjoint neighborhoods and consequently $g\overline{P} \cap \overline{P} = \emptyset$.

Similarly, Corollary 4.8(ii) shows that Veech's Theorem holds for IN groups generated by an invariant neighborhood.

4.3. Extending Veech to a Set Larger Than G

Let S be a semigroup, for $E \subseteq S$, define $Inj_S(E) = \{x \in S : y \mapsto yx$ is injective on E$\}$. If $x \in Inj_S(S)$, then x is right cancellative, and Veech's Theorem says that $Inj_{G^{\mathcal{LUC}}}(G) = G^{\mathcal{LUC}}$. In this section we ask, whether G is the largest subset E of $G^{\mathcal{LUC}}$ for which $Inj_{G^{\mathcal{LUC}}}(E) = G^{\mathcal{LUC}}$. For a special class of σ-compact groups, there is an open set U such that $G \subset U$ and for all $s \in G^{\mathcal{LUC}}$ $u \mapsto us$; $U \to Us \subseteq G^{\mathcal{LUC}}$ is injective.

Definition 4.17: Let G be a topological group. A pair (S, ϕ) is called an *almost periodic compactification* of G if

(i) S is a compact topological group,

(ii) $\psi : G \to S$ is a continuous homomorphism,

(iii) $cl_S\psi(G) = S$,

(iv) If K is any compact group and $f : G \to K$ is a continuous homomorphism, then there is a continuous homomorphism $\hat{f} : S \to K$ such that $\hat{f} \circ \phi = f$.

In this case, S is denoted by G^{AP}.

Theorem 4.18: Let G be a locally compact, σ-compact group for which the natural homomorphism $\psi : G \to G^{AP}$ is not surjective. Then there is a set U open in $G^{\mathcal{LUC}}$ with $G \subseteq U$ and $G \neq U$ for which $u \mapsto us$: $U \to Us \subseteq G^{\mathcal{LUC}}$ is injective for all $s \in G^{\mathcal{LUC}}$.

In the next lemma, we find an equivalence of the hypothesis of Theorem 4.18.

Lemma 4.19: Let G be a locally compact, σ-compact group. Then the followings are equivalent:

(i) The continuous homomorphism $\psi : G \to G^{AP}$ is not surjective;

(ii) There is a compact metrizable group G^M and a continuous homomorphism $\varphi : G \to G^M$ with $\varphi(G)$ dense in G^M but $\varphi(G) \neq G^M$.

(For information on metrizability see Appendix A.)

Proof: **(i $\Rightarrow$ ii)** Let $G = \bigcup_{n=1}^{\infty} K_n$, with each K_n compact. By (i), there is $x \in G^{AP}$ such that $x \notin \psi(G)$. Then, if 1 is the identity of G^{AP}, $1 \notin x^{-1}\psi(G)$. Hence, $x \notin x^{-1}\psi(K_n)$ for each $n \in \mathbf{N}$. Since the map $g \mapsto x^{-1}\psi(g)$; $G \to G^{AP}$ is continuous, $x^{-1}\psi(K_n)$ is compact for each $n \in \mathbf{N}$. Then for each n, $U_n = G^{AP} \backslash x^{-1}\psi(K_n)$ is an open neighborhood of 1 in G^{AP}. From the Theorem 8.7 of [9], we find that a compact normal subgroup H such that $H \subseteq \bigcap_{n=1}^{\infty} U_n$ and G^{AP}/H is metrizable. Then, $x^{-1}\psi(G) \cap H = \emptyset$ which means $\psi(G) \cap xH = \emptyset$, and since H is a normal subgroup, $xH \cap \psi(G)H = \emptyset$. Thus in the quotient G^{AP}/H, the images of x and $\psi(G)$ do not intersect. Hence if we let φ to be the composition $g \mapsto \psi(g) \mapsto \psi(g)H$; $G \to G^{AP} \to G^{AP}/H$, then the image of x is not in $\varphi(G)$, but since the quotient map $G^{AP} \to G^{AP}/H$ is onto and $\psi(G)$ is dense in G^{AP}, $\varphi(G)$ is dense in G^{AP}/H.

(ii $\Rightarrow$ i) Let $\psi : G \to G^{AP}$ be the canonical map. Given $\varphi : G \to G^M$, let $\hat{\varphi} : G^{AP} \to G^M$ be the extension of φ to G^{AP}. Since $\varphi(G)$ is dense in G^M, $\hat{\varphi}$ is surjective. Then $\varphi = \hat{\varphi} \circ \psi$ implies that, if ψ is surjective, so will φ be. *Q.E.D.*

We prove two more lemmas, before we begin the proof of 4.18.

Lemma 4.20: Let S be a compact topological group and $u, v \in S$. If there is $s \in S$ such that $us = vs$, then in each minimal left ideal L of S, there is a minimal idempotent e, for which $ue = ve$.

Proof: Assume that $us = vs$ for some $s \in S$. Let L be a minimal left ideal and $t \in L$. Then $ust = vst$ and $st \in L$. Since L is a union of groups, let e be the identity of the group containing st, and $(st)^{-1}$ be the inverse of st in that group, that is $st(st)^{-1} = e$. Therefore, $ue = ust(st)^{-1} = vst(st)^{-1} = ve$. *Q.E.D.*

Lemma 4.21: Let D be a discrete topological space. Let A, B be countable subsets of βD. If $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, then $\overline{A} \cap \overline{B} = \emptyset$.

Proof: Suppose that $A = \{a_n\}_{n=1}^{\infty}$ and $B = \{b_n\}_{n=1}^{\infty}$. $A \cap \overline{B} = \emptyset$ implies that each a_n has a clopen neighborhood U_n in βD disjoint from B. And $\overline{A} \cap B = \emptyset$ implies that each b_n has a clopen neighborhood V_n disjoint from A. Let $G_n = U_n \setminus \bigcap_{i=1}^{n} V_i$ and $H_n = V_n \setminus \bigcap_{i=1}^{n} U_i$, then $(\bigcup_{n=1}^{\infty} G_n) \cap (\bigcup_{n=1}^{\infty} H_n) = \emptyset$, and so $\overline{(\bigcup_{n=1}^{\infty} G_n)} \cap \overline{(\bigcup_{n=1}^{\infty} H_n)} = \emptyset$. Since $A \subseteq \bigcup_{n=1}^{\infty} G_n$ and $B \subseteq \bigcup_{n=1}^{\infty} H_n$, it follows that $\overline{A} \cap \overline{B} = \emptyset$. *Q.E.D.*

Proof of Theorem 4.18: Let G be a locally compact, σ-compact group such that $\psi : G \to G^{AP}$ is not surjective. Then by Lemma 4.19 we obtain a metrizable space G^M and a continuous homomorphism $\varphi : G \to G^M$ such that $\overline{\varphi(G)} = G^M$ but $\varphi(G) \neq G^M$. Let $\hat{\varphi} : G^{\mathcal{LUC}} \to G^M$, and $\hat{\varphi}(G^{\mathcal{LUC}}) = G^M$.

Let $a \in G^M \setminus \varphi(G)$. Let V be an open, relatively compact neighborhood of 1 in G ($V \in \mathcal{N}(1)$). Since $\overline{\varphi(G)} = G^M$, there is a sequence $(x_n)_{n \in N} \subseteq G$ with $\varphi(x_n) \to a$. We may assume that $(x_n)_{n \in N}$ is V-discrete, ($(x_n)_{n \in N}$ has a subsequence which is V-

discrete, otherwise $(x_n)_{n\in N}$ would lie in a finite union of translates of V and would therefore have a cluster point in G, we may replace $(x_n)_{n\in N}$ by this subsequence if necessary). Put $X = \{x_1, x_2, \ldots\}$. Let x be any cluster point of X, in the compact space $G^{\mathcal{LUC}}$. Then $\varphi(x) = a \notin \varphi(G)$. By The Local Structure Theorem 4.4, $V\overline{X}$ is an open neighborhood of x in $G^{\mathcal{LUC}}$, homeomorphic with $V \times \beta X$. Let $U = G \cup V\overline{X}$. Let L be a minimal left ideal in $G^{\mathcal{LUC}}$. We want to show that $u \mapsto us$; $U \to Us \subseteq G^{\mathcal{LUC}}$ is injective for every $s \in G^{\mathcal{LUC}}$. By Lemma 4.20, if there is $us = vs$, $u, v \in U$, $s \in G^{\mathcal{LUC}}$, there is a minimal idempotent e in L, with $ue = ve$. So, it is enough to prove that $u \mapsto ue$ is injective on U for every idempotent e in L. Let $e \in L$ be an idempotent. Since G^M is a group, 1 is the only idempotent in G^M, hence $\varphi(e) = 1$.

We prove $u \mapsto ue$ is injective in three cases:

1) Veech's Theorem tells that $u \mapsto ue$ is injective on G.

2) Let $g \in G$ and $vx_1 \in V\overline{X} \setminus G = U \setminus G$, such that $v \in V \subseteq G$ and $x_1 \in \overline{X}$. Since $vx_1 \notin G$, we must have $x_1 \in \overline{X} \setminus X$. Then,

$$\varphi(vx_1e) = \varphi(v)\varphi(x_1)\varphi(e) = \varphi(v)a1 \notin \varphi(G)$$

since $a \notin \varphi(G)$ and $\varphi(v) \in \varphi(G)$. Also, we have

$$\varphi(ge) = \varphi(g)\varphi(e) = \varphi(g) \in \varphi(G)$$

Thus, $vx_1e \neq ge$.

3) Our last case is $v_1x_1, v_2x_2 \in V\overline{X} \setminus G = U \setminus G$, with $v_1, v_2 \in V$, $x_1, x_2 \in \overline{X} \setminus X$ and $v_2x_2 \neq v_1x_1$. We will show that $v_2x_2e \neq v_1x_1e$. To this end, we need further cases:

3i) If $x_1 = x_2$, then we have $v_1 \neq v_2$. By Veech's Theorem $g \mapsto gx_1e$ is injective on G, so

$$v_1x_1e \neq v_2x_1e = v_2x_2e$$

3ii) Assume that $x_1 \neq x_2$. Assume for a contradiction that $v_1x_1e \neq v_2x_2e$. since $\overline{X}$ is homeomorphic with βX, there is $X_1 \subseteq X$ and $X_2 \subseteq X$ such that $x_1 \in \overline{X_1}$, $x_2 \in \overline{X_2}$ and $\overline{X_1} \cap \overline{X_2} = \emptyset$. By The Local Structure Theorem 4.4, there is a neighborhood $W\overline{Q}$ of $v_1x_1e \neq v_2x_2e$ with $v_1x_1e \in \overline{Q}$, Q is V-discrete and W is a symmetric neighborhood of 1 in G such that $W \subseteq V$. Translating the map $W \times \beta Q \to \beta Q$, we define $\pi : W\overline{Q} \to \overline{Q}$ as the canonical projection.

Since $y \mapsto ye$ is continuous on $G^{\mathcal{LUC}}$, there is a neighborhood N_1 of v_1x_1 such that $N_1e \subseteq W\overline{Q}$. Also since $v_1x_1 \in V\overline{X} \cong V\mathrm{x}\beta X$, we may assume that the neighborhood N_1 is of the form $W_1 \times \overline{Y_1}$, where $W_1 \subseteq V$ is an open neighborhood of v_1, and $Y_1 \subseteq X_1$ with $x_1 \in \overline{Y_1}$. Similarly, there is a neighborhood N_2 of the form $W_2 \times \overline{Y_2}$ with $N_e \subseteq W\overline{Q}$. Furthermore, $\overline{Y_1}e$, $\overline{Y_2} \subseteq W\overline{Q}$.

Next, we work with $\overline{Y_1}$, $\overline{Y_2}$ as spaces rather than continuing with v_1x_1, v_2x_2. Let $y \in \overline{Y_1} \setminus Y_1$ and $g \in G$, it is impossible that $ye = ge$. Since $y \in \overline{X} \setminus X$, if $ye = ge$, we would have $a = \varphi(y) = \varphi(g)$, which is impossible because $a \notin \varphi[G]$. We further get for such a y $\pi(ye) = \pi(ge)$ for any $g \in G$, for if this did happen there would be $w_1, w_2 \in W \subseteq G$ with $w_1ye = w_2ge$ or $ye = w_1^{-1}w_2ge$, since $w_1^{-1}w_2g \in G$, we have just observed that this is impossible. Then $\pi((\overline{Y_1} \setminus Y_1)e) \cap \pi(Y_2e) = \emptyset$.

Now, assume for a contradiction that $\pi(Y_1e) \cap \pi(Y_2e) \neq \emptyset$. Then there are $y_1 \in Y_1 \subseteq G$, $y_2 \in Y_2 \subseteq G$ and $w_1, w_2 \in W$ with $w_1y_1e = w_2y_2e$. And there are $z \in Q$, $w_1', w_2' \in W$ with $w_1'z = y_1e$ and $w_2'z = y_2e$. By, Veech's Theorem, we get $w_1y_1 = w_2y_2$, but since $W \subseteq V$, this contradicts the fact that Y is in a V-discrete set. Therefore, $\pi(\overline{Y_1}e) \cap \pi(Y_2e) = \emptyset$. Similarly, $\pi(Y_1e) \cap \pi(\overline{Y_2}e) = \emptyset$. Note that, by Lemma 3.38(i) $\pi(\overline{Y_i}e) = \overline{\pi(Y_ie)}$ for $i = 1, 2$. Applying Lemma 4.21, we conclude $\overline{\pi(Y_1e)} \cap \overline{\pi(Y_2e)} = \emptyset$.

As, $\pi(v_1x_1e) = \pi(x_1e) \in \pi(\overline{Y_1}e)$ and $\pi(v_2x_2e) = \pi(x_2e) \in \pi(\overline{Y_2}e)$, we can deduce,

$$v_1x_1e \neq v_2x_2e$$

which contradicts with the assumption $v_1x_1e = v_2x_2e$. $Q.E.D.$

5. SLOWLY OSCILLATING FUNCTIONS IN THE LUC COMPACTIFICATION

In this chapter we generalize Van Douwen's right ideal theorem to first non-compact locally compact groups then to discrete cancellative semigroups, with the aid of slowly oscillating functions.

5.1. Slowly Oscillating Functions

Recall first that $\mathcal{LUC}(G)$ is the C^*-algebra of all bounded left uniformly continuous functions on G, that is $\mathcal{LUC}(G)$ consists of all bounded, complex-valued, continuous functions f on G such that the map $s \mapsto L_s f$ is norm-continuous.

Definition 5.1: Let S be an infinite discrete semigroup of cardinality κ. The *semigroup* of *uniform ultrafilters* is defined to be $\mathcal{U}(S)= \{x \in \beta S : \rho(x) = \kappa\}$ where $\rho(x) = min\{|A| : A \subseteq S \text{ and } x \in \overline{A}\}$.

We next generalize the concept of uniform ultrafilter to a locally compact group.

Definition 5.2: Let G be a locally compact group. For a $A \subseteq G$, let $\kappa(A)$ be the minimal number of compact sets in G, required to cover A. For simplicity denote $\kappa(G)$ by κ.

Let $x \in G^{\mathcal{LUC}}$, the *height* of x is defined by $\rho(x) = min\{\kappa(A) : A \subseteq S$ and $x \in \overline{A}\}$. Let $\mathcal{U}(G)$ denote the set of all x in $G^{\mathcal{LUC}}$ with $\rho(x) = \kappa$.

Remarks:**(i)** When G is discrete, $\mathcal{U}(G)$ corresponds to the set of uniform ultrafilters on G.

(ii) When G is σ-compact, $\mathcal{U}(G) = G^* = G^{\mathcal{LUC}} \setminus G$.

(iii) From the Local Structure Theorem 4.4, we conclude that $\mathcal{U}(G)$ is closed in $G^{\mathcal{LUC}}$.

(iv) Since right translations are continuous, $\mathcal{U}(G)$ is a left ideal in $G^{\mathcal{LUC}}$.

We now define the main tool of this chapter:

Definition 5.3: Let G be a locally compact group. A function f in $\mathcal{LUC}(G)$ is called *slowly oscillating* if, for any $\varepsilon > 0$ and for any compact subset F of G, there exists a subset A of G with $\kappa(A) < \kappa$ such that,

$$|f(st) - f(t)| < \varepsilon \qquad \textit{and} \qquad |f(ts) - f(t)| < \varepsilon$$

whenever $s \in F$ and $t \in G \setminus A$.

Proposition 5.4: Let G be a locally compact group. Let $f \in \mathcal{LUC}(G)$ be a slowly oscillating function. Then for all $x \in \mathcal{U}(G)$, y $\in$ G$^{\mathcal{LUC}}$, $s \in G$, we have

$$f(yx) = f(x) \qquad f(xs) = f(x).$$

Proof: Let $s \in G$ and $x \in \mathcal{U}(G)$. In the definition 5.3, put $F = \{s\}$, let $\varepsilon > 0$ be arbitrary, then there exists $A_\varepsilon \subseteq G$ with $\kappa(A_\varepsilon) < \kappa$ and

$$|f(st) - f(t)| < \varepsilon \qquad \textit{and} \qquad |f(ts) - f(t)| < \varepsilon$$

for all $t \in G \setminus A_\varepsilon$.

Let $x \in \mathcal{U}(G)$, then $\rho(x) = \kappa$, that is for all $B \subseteq G$ such that $x \in \overline{B}$, we have $\kappa(B) = \kappa$. So, $x \in \overline{G \setminus A_\varepsilon}$ for all $x \in \mathcal{U}(G)$. Hence,

$$|f(sx) - f(x)| \leq \varepsilon \qquad |f(xs) - f(x)| \leq \varepsilon.$$

But this holds for all $\varepsilon > 0$ and for all $A_\varepsilon \subseteq G$. Therefore,

$$f(sx) = f(x) \qquad f(xs) = f(x).$$

Next, let $y \in G^{\mathcal{LUC}}$, then there is a net $(y_\alpha)_{\alpha \in I} \subseteq G$ with $y_\alpha \to y$. For all $\alpha \in I$, we have $f(y_\alpha x) = f(x)$ and by continuity of multiplication as a map $G \times G^{\mathcal{LUC}} \to G^{\mathcal{LUC}}$, we conclude $f(x) = f(y_\alpha x) \to f(yx)$. Hence,

$$f(x) = f(yx).$$

Q.E.D.

The construction of the slowly oscillating functions is included in the proof of the following Lemma.

Lemma 5.5: Let G be a locally compact group. Let $X \subseteq G$ with $\kappa(X) = \kappa$. There exists a left uniformly discrete subset T of X such that $|T| = \kappa$ and the points in $\overline{T}$ can be separated by slowly oscillating functions.

Proof: We divide the proof into two parts, treating σ-compact and non-σ-compact groups separately.

THE σ-COMPACT CASE: Suppose first that $\kappa = \omega$. Let U be a compact symmetric neighborhood of the identity in G. There is an increasing compact cover $\{K_n\}_{n \in N}$ of G such that
(i) each compact subset of G is included in some K_n;
(ii) $K_1 = U$;
(iii) $K_n^3 \subseteq K_{n+1}$ for each $n \in \mathbf{N}$
(iv) $K_n^{-1} = K_n$ for each $n \in \mathbf{N}$
Conditions (ii) and (iii) imply that the identity, $1 \in K_n$ for each $n \in \mathbf{N}$. So, $U \subseteq K_n$ for each $n \in \mathbf{N}$ by induction. Then,
(v) $UK_nU \subseteq K_n^3 \subseteq K_{n+1}$;
If $n \geq m \geq 1$, $K_nK_m \subseteq K_nK_n \subseteq K_n^3 \subseteq K_{n+1} \subseteq K_{n+m}$. That is,
(vi) $K_nK_m \subseteq K_{n+m}$.

For convenience, put $K_n = \emptyset$ for every $n = 0, -1, -2, ...$ and note that conditions

(v) and (vi) hold for all integers n and m.

By induction, we construct a subset $T = \{t_n\}_{n=1}^{\infty}$ of X such that

$$K_n t_n K_n \cap K_m t_m K_m = \emptyset \qquad (1)$$

for every $n \neq m$. Indeed, suppose that $t_1, ..., t_{n-1}$ are all chosen. Then the set

$$\bigcup_{m=1}^{n-1} K_n^{-1} K_m t_m K_m K_n^{-1} = \bigcup_{m=1}^{n-1} K_n K_m t_m K_m K_n \subseteq \bigcup_{m=1}^{n-1} K_{n+m} t_m K_{n+m}$$

is compact. Since X is not compact, we can pick t_n from $X \setminus \bigcup_{m=1}^{n-1} K_n^{-1} K_m t_m K_m K_n^{-1}$.

For every $n \geq 2$, define

$$f_n = \begin{cases} 1 & \text{on } K_1 t_n K_1 \\ 1 - \frac{1}{n-1} & \text{on } (K_2 t_n K_2) \setminus (K_1 t_n K_1) \\ 1 - \frac{2}{n-1} & \text{on } (K_3 t_n K_3) \setminus (K_2 t_n K_2) \\ \vdots & \\ \frac{1}{n-1} & \text{on } (K_{n-1} t_n K_{n-1}) \setminus (K_{n-2} t_n K_{n-2}) \\ 0 & \text{off } K_{n-1} t_n K_{n-1} \end{cases}$$

Let I be any subset of $\{2, 3, \ldots\}$ and put $f = \sum_{n \in I} f_n$, f is well defined by (1). It is clearly bounded and measurable with respect to any regular Borel measure on G [see Appendix B for information on Borel measures].

Let φ be any measurable function on G such that

$$\varphi \geq 0 \qquad \mathit{supp}\ \varphi \subseteq U \qquad \int_G \varphi = 1$$

Since $\varphi \in L^1(G)$ and $f \in L^\infty(G)$, we have $h(x) = \int_G \varphi(t) f(tx) dt$ is in $\mathcal{LUC}(G)$. We will show that h is slowly oscillating and that functions such as h separate points in $\overline{T}$.

If $n \in I$, then

$$h(t_n) = \int_U \varphi(u) f(ut_n) du = \int_U \varphi = 1$$

because $f(ut_n) = f_n(ut_n) = 1$ for every $u \in U$. On the other hand, if $n \notin I$, then $h(t_n) = 0$. Therefore $h = 1$ on $\{t_n : n \in I\}$ and $h = 0$ on $\{t_n : n \notin I\}$. This shows that by choosing the index set I appropriately, we can separate points in $\overline{T}$ by functions of the same form as h.

To complete the case when G is σ-compact, we need to show that h is slowly oscillating. Let $0 < \varepsilon < 1$ and let F be a compact subset of G. We should find a compact set K such that for every $s \in F$ and for every $t \in G \setminus K$

$$|h(st) - h(t)| < \varepsilon \qquad \textit{and} \qquad |h(ts) - h(t)| < \varepsilon. \qquad (2)$$

By condition (i), F is included in some K_m, and so we may replace F by K_m. Choose an integer l such that $(m+1)/l < \varepsilon$, and put $K = \bigcap_{j=1}^{l} K_{j+m} t_j K_{j+m}$. Fix s in K_m and t in $G \setminus K$. We shall confirm only the first inequality in (2), the second is similar. First,

$$|h(st) - h(t)| \leq \int_U \varphi(u) |f(ust) - f(ut)| du.$$

We consider the difference $|f(ust) - f(ut)|$. If $f(vst) \neq 0$ for some $v \in U$, then $vst \in K_{n-1} t_n K_{n-1}$ for some $n \geq 2$, and so by condition (iv) $st \in U K_{n-1} t_n K_{n-1}$. Hence,

$$t \in K_m U K_{n-1} t_n K_{n-1}. \qquad (3)$$

Applying condition (vi) to (3), we see that $t \in K_{n+m} t_n K_{n+m}$. Then the choice of K implies that

$$n > l > m + 1. \qquad (4)$$

It follows from (3) that $UK_mt \subseteq UK_m^2UK_{n-1}t_nK_{n-1}$. But $UK_m^2U \subseteq K_{m+2} \subseteq K_l$ by conditions (iii) and (v), and so $UK_mt \subseteq K_lK_{n-1}t_nK_{n-1} \subseteq K_nt_nK_n$ by condition (iii) with (4). In particular, ut and ust belong to $K_nt_nK_n$, for every $u \in U$.

We have shown that if $f(vst) \neq 0$ for some $v \in U$, then there exists $n > l$ such that ut and ust are in $K_nt_nK_n$ for every $u \in U$. Similarly, if $f(vt) \neq 0$ for some $v \in U$, then there exists $n > l$ such that ut and ust are in $K_nt_nK_n$ for every $u \in U$. The remaining case is trivial: $f(vst) = f(vt) = 0$ for every $v \in U$.

Excluding the trivial case, we can assume that ut and ust are in $K_nt_nK_n$ with $n > l$ for every $u \in U$. Let $u \in U$. Then $ut \in (K_{k+1}t_nK_{k+1}) \setminus (K_kt_nK_k)$ for some $0 \leq k \leq n-1$ and

$$f(ut) = f_n(ut) = 1 - \frac{k}{n-1}. \qquad (5)$$

By conditions (v) and (vi),

$$\begin{aligned} ust &\in UK_mU((K_{k+1}t_nK_{k+1}) \setminus (K_kt_nK_k)) \\ &\subseteq (K_{k+m+2}t_nK_{k+m+2}) \setminus (K_{k-m-1}t_nK_{k-m-1}), \end{aligned}$$

and so

$$1 - \frac{k+m+1}{n-1} \leq f_n(ust) \leq 1 - \frac{k-m-1}{n-1}.$$

Combining with (5) gives

$$-\frac{m+1}{n-1} \leq f_n(ust) - f_n(ut) \leq \frac{m+1}{n-1}.$$

Therefore,

$$|f(ust) - f(ut)| \leq \frac{m+1}{n-1} \leq \frac{m+1}{l} \leq \varepsilon$$

for every $u \in U$. Then, we get

$$|h(st) - h(t)| \leq \int_G \varphi(u)\varepsilon du = \varepsilon,$$

as required.

THE NON-σ-COMPACT CASE: Suppose $\kappa > \omega$. Let $\{K_\alpha\}_{\alpha<\kappa}$ be a compact cover of G and suppose that K_0 has non-empty interior. For every $\alpha < \kappa$, let G_α be the subgroup algebraically generated by the set $\bigcup_{\beta\leq\alpha} K_\beta$. Then $\{G_\alpha\}_{\alpha<\kappa}$ is a cover of G such that
(i) each G_α is an open subgroup of G with $\kappa(G_\alpha) \leq max\{\omega, |\alpha|\}$,
(ii) $\bigcup_{\beta<\alpha} G_\beta \subseteq G_\alpha$ for every $\alpha < \kappa$. Details of constructing such a cover can be found in [10].

Applying condition (i) of the cover $\{G_\alpha\}_{\alpha<\kappa}$, we can construct by transfinite induction a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of X such that

$$G_\alpha t_\alpha G_\alpha \cap G_\beta t_\beta G_\beta = \emptyset \qquad \alpha \neq \beta. \qquad (6)$$

For every $\alpha < \kappa$, let f_α be the characteristic function of $G_\alpha t_\alpha G_\alpha$. Put

$$f = \sum_{\alpha\in I} f_\alpha,$$

where I is any subset of κ. Note that f is well defined by (6) and $f \in \mathcal{LUC}(G)$ since f is constant on each right coset of G_0,which is an open subgroup of G. As in the σ-compact case, functions of the same form as f separate points in $\overline{T}$, so it suffices to show that f is slowly oscillating.

Fix a compact set $F \subseteq G$. Since $\{G_\alpha\}_{\alpha<\kappa}$ is an increasing cover consisting of open sets, $F \subseteq G_\alpha$ for some $\alpha < \kappa$. Put $A = \bigcup_{\gamma\leq\alpha} G_\alpha t_\gamma G_\alpha$, and note that $\kappa(A) = \kappa(G_\alpha)$ by condition (i). Let $s \in F$ and $t \in G \setminus A$. If $f(t) = 1$, then $t \in G_\beta t_\beta G_\beta$ for some

$\beta \in I$. The choice of A implies that $\beta > \alpha$. So, $st \in G_\alpha G_\beta t_\beta G_\beta = G_\beta t_\beta G_\beta$ and $f(st) = f(t) = 1$. On the other hand, if $f(st) = 1$, then $t \in G_\alpha G_\beta t_\beta G_\beta$ for some $\beta \in I$. Again the choice of A implies that $\beta > \alpha$. Therefore, $t \in G_\beta t_\beta G_\beta$, and so $f(t) = f(st) = 1$. The third alternative, $f(st) = f(t) = 0$ is trivial. Consequently, for every $s \in F$ and for every $t \in G \setminus A$, we have $|f(st) - f(t)| = 0$. A similar argument shows that $|f(t) - f(ts)| = 0$, and so f is slowly oscillating. *Q.E.D.*

5.2. The Decomposition Theorem

In this section, out main result is Theorem 5.6, which generalizes, Van Douwen's Right Ideal Theorem to non-compact, locally compact groups. Recall first that $\mathcal{U}(G)$ is the closed left ideal in $G^{\mathcal{LUC}}$ consisting of all the points that are not in the closure of any subset of G with compact covering less than κ.

Theorem 5.6: Let G be a non-compact, locally compact group. Then there is a decomposition $\mathcal{I}$ of $\mathcal{U}(G)$ such that
(i) Each member of $\mathcal{I}$ is a closed left ideal in $G^{\mathcal{LUC}}$.
(ii) If $I \in \mathcal{I}$ and $x \in I$, then $\overline{xG} \subseteq I$.
(iii) Each member of $\mathcal{I}$ has empty interior in $\mathcal{U}(G)$.
(iv) $|\mathcal{I}| = 2^{2^\kappa}$.

Proof: We will define an equivalence relation on $\mathcal{U}(G)$ such that the family $\mathcal{I}$ of the equivalence classes satisfies properties (i)-(iv). For every $x, y \in \mathcal{U}(G)$ put

$$x \simeq y \ \ if \ \ f(x) = f(y) \ \ for \ \ every \ \ slowly \ \ oscillating \ \ function \ \ f \ \ on \ \ G.$$

It is easy to see that $\simeq$ is an equivalence relation. Since $\mathcal{U}(G)$ is closed in $G^{\mathcal{LUC}}$, it is compact. So, by equivalence relation theory [8], we conclude that $\simeq$ is a closed equivalence relation, since each slowly oscillating function is continuous. For every $x \in \mathcal{U}(G)$ denote the equivalence class containing x by $[x]$.

If $y \in \mathcal{U}(G)$, then $f(sy) = f(y)$ and $f(ys) = f(y)$ for every slowly oscillating function f and for every $s \in G$. Therefore, whenever $y \in [x]$ $Gy \subseteq [x]$ and $yG \subseteq [x]$. Hence, condition (ii) is immediate, because $[x]$ is closed. Since, right translation by y is continuous and $\overline{G} = G^{\mathcal{LUC}}$, we have $\overline{G}y = G^{\mathcal{LUC}}y \subseteq [x]$. So, each equivalence class is also a left ideal.

If we apply Lemma 5.5 to $X = G$, it follows that distinct points in $\overline{T} \cap \mathcal{U}(G)$ are in distinct equivalence classes. The points in $\overline{T} \cap \mathcal{U}(G)$ correspond to uniform ultrafilters on the set T [by 5, Theorem 3.58], so there is at least $2^{2^{\kappa}}$ equivalence classes.

Next let U be a compact neighborhood of the identity and let X be a maximal uniformly discrete subset of G with respect to U. Then by $G^{\mathcal{LUC}} = U^2\overline{X}$. it follows that each left ideal in $G^{\mathcal{LUC}}$ contains at least one point from $\overline{X}$. Since $|X| = \kappa$, there are at most $2^{2^{\kappa}}$ disjoint left ideals in $G^{\mathcal{LUC}}$ [Details of this argument can be found in 10].

Therefore, there are exactly $2^{2^{\kappa}}$ equivalence classes.

Finally, we shall show that each equivalence class has empty interior in $\mathcal{U}(G)$. By the regularity of $G^{\mathcal{LUC}}$, it suffices to show that if N is a closed neighborhood in $G^{\mathcal{LUC}}$ of a point in $\mathcal{U}(G)$, then $N \cap \mathcal{U}(G)$ is not included in any of the equivalence classes. Since $\kappa(N \cap G) = \kappa$, an application of Lemma 5.5 gives a uniformly discrete subset S of $N \cap G$ such that the points in $\overline{S} \cap \mathcal{U}(G)$ can be separated by slowly oscillating functions. In other words, each of the $2^{2^{\kappa}}$ points in $\overline{S} \cap \mathcal{U}(G)$, which is a subset of N, belongs to a distinct equivalence class. *Q.E.D.*

Theorem 5.7: Let G be a locally compact group. The topological centre of $G^{\mathcal{LUC}}$ is G.

Proof: Let $x \in G^{\mathcal{LUC}} \backslash G$. Then there exists a net $(x_\alpha)_{\alpha \in I}$ in G such that $x_\alpha \to x$. If we let H to be the subgroup of G generated by $(x_\alpha)_{\alpha \in I}$, then $x \in \overline{H}$ and $\kappa(H) = \rho(x)$. The closure of H in $G^{\mathcal{LUC}}$ may be identified with $H^{\mathcal{LUC}}$, and $x \in \mathcal{U}(H)$. Let $y \in \mathcal{U}(H)$ such that x and y are in distinct left ideals I_x and I_y in the decomposition of $\mathcal{U}(H)$. Let

$(s_\alpha)_{\alpha \in J}$ be a net in H converging to y. Then the net $(xs_\alpha)_{\alpha \in J}$ is in I_x by condition (ii) of Theorem 5.6. On the other hand, $xy \in I_y$, so xs_α does not converge to xy, because I_x is closed and disjoint from I_y. *Q.E.D.*

Lemma 5.8: Let G be a locally compact group. Suppose that x is in the topological centre of $G^{\mathcal{LUC}} \setminus G$. Then x is in the topological centre of $G^{\mathcal{LUC}}$.

Proof: Let $x \in \Lambda(G^{\mathcal{LUC}} \setminus G)$. Let $(y_\alpha)_{\alpha \in I}$ be a net in $G^{\mathcal{LUC}}$ converging to $y \in G^{\mathcal{LUC}}$. It is enough to show that every subnet of $(xy_\alpha)_{\alpha \in I}$ has a subnet converging to xy. Given a subnet of $(xy_\alpha)_{\alpha \in I}$, choose a subnet $(xy_\beta)_{\beta \in J}$ that converges to some z in the compact space $G^{\mathcal{LUC}}$.

Pick a right cancellable point w from $G^{\mathcal{LUC}} \setminus G$ [existence of which is given by 10]. The right translation by w is continuous, so $(xy_\beta)w \to zw$. But, the left translation with x is also continuous on $G^{\mathcal{LUC}} \setminus G$, so $x(y_\beta)w \to x(yw)$. Therefore, $zw = xyw$, and since w is right cancellable, $z = xy$. Hence, $(xy_\beta) \to xy$, as required. *Q.E.D.*

Theorem 5.7 and Lemma 5.8 together imply that, if there were an element in $\Lambda(G^{\mathcal{LUC}} \setminus G)$, it would be an element of $\Lambda(G^{\mathcal{LUC}}) = G$. So, we have:

Theorem 5.9: Let G be a locally compact group. The topological centre of $G^{\mathcal{LUC}} \setminus G$ is empty.

5.3. The Decomposition Theorem for Discrete Semigroups

The purpose of our last section is to show that our argument can be applied to prove the original right ideal theorem concerning discrete cancellative semigroups. In this case, slowly oscillating functions are defined similarly:

Definition 5.10: Let S be a discrete cancellative semigroup. A bounded function $f : S \to C$ is called *slowly oscillating* if for every $\varepsilon > 0$ and for every finite set $F \subseteq S$, there exists $A \subseteq S$ such that $|A| < |S|$ with

$$|f(st) - f(t)| < \varepsilon \qquad \textit{and} \qquad |f(ts) - f(t)| < \varepsilon$$

whenever $s \in F$ and $t \in S \setminus A$.

Definition 5.11: Let S be a semigroup. For $s, t \in S$ define the sets

$$s^{-1}t = \{u \in S : su = t\} \qquad \textit{and} \qquad ts^{-1} = \{u \in S : us = t\}$$

S is called
(i) *weakly left cancellative* if the set $s^{-1}t$ is finite for all $s, t \in S$.
(ii) *weakly right cancellative* if the set ts^{-1} is finite for all $s, t \in S$.
(iii) *weakly cancellative* if both $s^{-1}t$ and ts^{-1} are finite for all $s, t \in S$.

Notations: $s^{-1}B = \{u \in S\colon\ su \in B\}$ and $A^{-1}B = \bigcup_{a \in A} a^{-1}B$ where $s \in S$ and $A, B \subseteq S$.

Remarks: Suppose that S is weakly cancellative. Let $A, B \subseteq S$,
(i) If both A and B are finite, then the cardinalities of both $A^{-1}B$ and BA^{-1} are finite.
(ii) If A or B is infinite, then the cardinalities of both $A^{-1}B$ and BA^{-1} are at most $max\{|A|, |B|\}$.

Lemma 5.12: Suppose that S is a discrete weakly cancellative semigroup with $|S| = \kappa$. Let $X \subseteq S$ with $|X| = \kappa$. There exists a subset T of X such that $|T| = \kappa$ and the points in $\overline{T}$ (the closure taken in $\beta S = S^{\mathcal{LUC}}$) can be separated by slowly oscilating functions.

Proof: As in the case of locally compact groups, we divide the proof into two parts.

THE COUNTABLE CASE: Suppose first that $|S| = \omega$. Enumerate $S = \{s_n\}_{n=1}^{\infty}$

Put

$$\begin{aligned} K_1 &= \{s_1\} \\ K_2 &= \{s_1, s_1^2, s_1^{-1}s_1, s_1s_1^{-1}, s_2\} \\ &\vdots \\ K_{n+1} &= K_n \cup K_n^2 \cup K_n^{-1}K_n \cup K_nK_n^{-1} \cup \{s_{n+1}\} \\ &\vdots \end{aligned}$$

Since S is weakly cancellative, we have a cover $\{K_n\}_{n=1}^{\infty}$ of S, consisting of finite sets satisfying

$$K_n \cup K_n^2 \cup K_n^{-1}K_n \cup K_nK_n^{-1} \subseteq K_{n+1} \qquad \forall n \in \mathbf{N}$$

It follows that $K_n^{-1}K_m \subseteq K_{max\{n,m\}}^{-1}K_{max\{n,m\}} \subseteq K_{n+m}$.

For convenience, put $K_n = \emptyset$ for every $n = 0, -1, \ldots$ and note that

$$K_nK_m \subseteq K_{n+m} \qquad K_n^{-1}K_m \subseteq K_{n+m} \qquad K_nK_m^{-1} \subseteq K_{n+m}$$

for all integers n and m.

Then we construct by induction a subset $T = \{t_n\}_{n\in N}$ of S such that

$$K_nt_nK_n \cap K_mt_mK_m = \emptyset \qquad (1)$$

whenever $n \neq m$. Indeed suppose that $t_1, ..., t_{n-1}$ are all chosen. Then the set $\bigcup_{m=1}^{n-1} K_n^{-1}K_mt_mK_mK_n^{-1}$ is finite, because S is weakly cancellative. Since X is infinite, we can pick t_n from $X \setminus \bigcup_{m=1}^{n-1} K_n^{-1}K_mt_mK_mK_n^{-1}$.

For every $n \geq 2$, define

$$f_n = \begin{cases} 1 & \text{on } K_1 t_n K_1 \\ 1 - \frac{1}{n-1} & \text{on } (K_2 t_n K_2) \setminus (K_1 t_n K_1) \\ 1 - \frac{2}{n-1} & \text{on } (K_3 t_n K_3) \setminus (K_2 t_n K_2) \\ \vdots & \\ \frac{1}{n-1} & \text{on } (K_{n-1} t_n K_{n-1}) \setminus (K_{n-2} t_n K_{n-2}) \\ 0 & \text{off } K_{n-1} t_n K_{n-1} \end{cases}$$

Let I be any subset of $\{2, 3, \ldots\}$ and put $f = \sum_{n \in I} f_n$, f is well defined and bounded by (1), so it has a continuous extension to βS. The functions of the same form as f separate points in $\overline{T}$ because for every $n \geq 2$,

$$f(t_n) = \begin{cases} 1 & \text{if } n \in I \\ 0 & \text{if } n \notin I \end{cases}$$

To complete the countable case, it is enough to show that f is slowly oscillating. Let $0 < \varepsilon < 1$ and let F be a finite subset of S. Then $F \subseteq K_m$ for some m. Let l be a positive integer such that $m/l < \varepsilon$ and put

$$A = \bigcup_{j=1}^{l} (K_m^{-1} K_j t_j K_j) \cup (K_j t_j K_j) \cup (K_j t_j K_j K_m^{-1}).$$

Then A is finite. Let $s \in F \subseteq K_m$ and $t \in S \setminus A$. If $f(st) \neq 0$, then $st \in K_{n-1} t_{n-1} K_{n-1}$ for some n. Therefore, $t \in K_m^{-1} K_{n-1} t_{n-1} K_{n-1}$ and the choice of A implies that $n > l > m$. Therefore $K_m^{-1} K_{n-1} \subseteq K_n$ and $t \in K_n t_n K_n$. If $f(t) \neq 0$, then t and st are again in $K_n t_n K_n$ for some $n > l$. The remaining case is trivial, so we can assume that both t and st are in $K t_n K_n$ for some $n > l$.

There is a unique k such that $0 \leq k \leq n-1$ and $t \in (K_{k+1} t_n K_{k+1}) \setminus (K_k t_n K_k)$.

Then $st \in (K_{m+k+1}t_nK_{k+1}) \setminus (K_{k-m}t_nK_k)$, and so

$$1 - \frac{m+k}{n-1} \leq f(st) \leq 1 - \frac{k-m}{n-1}.$$

Since $f(t) = 1 - k/(n-1)$, we conclude

$$-\frac{m}{n-1} \leq f(st) - f(t) \leq \frac{m}{n-1}.$$

Thus $|f(st) - f(t)| < m/l < \varepsilon$. The other requirement for slow oscillation is confirmed similarly.

THE UNCOUNTABLE CASE: Suppose next that $|S| = \kappa > \omega$. We begin by constructing a cover $\{S_\alpha\}_{\alpha<\kappa}$ of S such that

$$S_\alpha^2 \subseteq S_\alpha, \qquad S_\alpha^{-1}S_\alpha \subseteq S_\alpha, \qquad S_\alpha S_\alpha^{-1} \subseteq S_\alpha,$$

$$|S_\alpha| \leq max\{\omega, |\alpha|\}, \qquad and \qquad \bigcup_{\beta<\alpha} S_\beta \subseteq S_\alpha$$

for every $\alpha < \kappa$. Let $\{s_\alpha\}_{\alpha<\kappa}$ be an enumeration of S. For each $\alpha < \kappa$, define S_α as follows: Put $Y_1 = \{s_\beta\}_{\beta\leq\alpha}$, and for every $n = 1, 2, \ldots$ put

$$Y_{n+1} = Y_n \cup Y_n^2 \cup Y_n^{-1}Y_n \cup Y_nY_n^{-1}.$$

Finally put $S_\alpha = \bigcup_{n=1}^{\infty} Y_n$. The required properties of $\{S_\alpha\}_{\alpha<\kappa}$ are easily confirmed.

By transfinite induction, we get a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of X such that

$$S_\alpha t_\alpha S_\alpha \cap S_\beta t_\beta S_\beta = \emptyset \qquad if \;\; \alpha \neq \beta.$$

We need to show that for each subset I of κ the function $f = \sum_{\alpha \in I} f_\alpha$ is slowly oscillating. Let $F \subseteq S$ be finite, and let $\alpha < \kappa$ be such that $F \subseteq S_\alpha$. Let

$$A = \bigcup_{\gamma < \alpha} (S_\alpha^{-1} S_\gamma t_\gamma S_\gamma) \cup (S_\gamma t_\gamma S_\gamma) \cup (S_\gamma t_\gamma S_\gamma S_\alpha^{-1}),$$

and note that $|A| \leq max\{\omega, \alpha\} < \kappa$. Let $s \in F$ and $t \in S \setminus A$. If $f(st) = 1$, then $t \in S_\alpha^{-1} S_\beta t_\beta S_\beta$ for some $\beta \in I$. By the choice of A, $\beta \geq \alpha$, so $t \in S_\beta t_\beta S_\beta$. On the other hand, if $f(t) = 1$, then $f(st) = 1$. The second requirement of slow oscillation is proven similarly. *Q.E.D.*

Definition 5.13: Let S be a discrete semigroup. An element $x \in \beta S$ is called a *uniform ultrafilter* if $x \notin \overline{A}$ for any $A \subseteq S$ with $|A| < |S|$. The set of all uniform ultrafilters on S is denoted by $\mathcal{U}(S)$. Note that this definition agrees with the notation introduced in Definition 5.1.

Proposition 5.14: Let S be a discrete semigroup. If S is weakly left cancellative, then $\mathcal{U}(S)$ is a closed left ideal in βS.

Proof: [See theorem 6.53 in 5].

Applying Lemma 5.12, we can use the proof of Theorem 5.6 to prove the following Decomposition Theorem.

Theorem 5.15: Let S be a discrete, weakly cancellative semigroup. Then there is a decomposition $\mathcal{I}$ of $\mathcal{U}(S)$ such that
(i) Each member of $\mathcal{I}$ is a closed left ideal in βS.
(ii) If $I \in \mathcal{I}$ and $x \in I$, then $xS \subseteq I$.
(iii) Each member of $\mathcal{I}$ has empty interior in $\mathcal{U}(S)$.
(iv) $|\mathcal{I}| = 2^{2^\kappa}$.

Proof: The result follows by the method used in the proof of Theorem 5.6. *Q.E.D.*

Theorem 5.16: Let S be a discrete weakly cancellative semigroup. The topological centre of βS is S. If S is also right cancellative, then the topological centre of $\beta S \setminus S$ is empty.

Proof: To prove the first statement we apply the argument used in Theorem 5.7. Then the second statement follows from the analogue of Lemma 5.8, which can be proved using a right cancellable point contained in $\beta S \setminus S$. Such a point exists by [10, Theorem 3.1 and Proposition 2.2], provided that S is right cancellative and weakly left cancellative. *Q.E.D.*

6. CONCLUSION

In this thesis we examined the construction of the LUC-compactification of a locally compact topological group in two different ways. Each construction equipped us with tools to examine the local topological properties of the LUC-compactification of G and the properties of the multiplication map in $G^{\mathcal{LUC}}$ which is an extension of the group operation of G. A newly defined, special class of LUC-functions, the slowly oscillating functions, helped us to further exploit the topological properties of $G^{\mathcal{LUC}}$.

APPENDIX A: LOCALLY COMPACT METRIZABLE GROUPS

First we state some preliminary results and definitions on topological groups for reference, proofs can be found in [9].

Definition A.1: A topological space X has the *Lindelof property* if every open cover of X admits a countable subcovering.

Theorem A.2: Let G be a topological group, and $\mathcal{U}$ be an open neighborhood basis at the identity e. Then,
(i) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2 \subseteq U$.
(ii) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.
(iii) for every $U \in \mathcal{U}$ and $x \in U$, there is a $V \in \mathcal{U}$ such that $xV \subseteq U$.
(iv) for every $U \in \mathcal{U}$ and $x \in U$, there is a $V \in \mathcal{U}$ such that $xVx^{-1} \subseteq U$.

Theorem A.3: Let G be a topological group. For every neighborhood U of e, there is a neighborhood V of e such that $\overline{V} \subseteq U$.

Theorem A.4: Let G be a topological group, let U be any neighborhood of e, and let F be any compact subset of G. Then there is a neighborhood V of e such that $xVx^{-1} \subseteq U$ for any $x \in F$.

Theorem A.5: Let $\mathcal{A}$ be a family of neighborhoods of e in a topological group G such that
(i) for each $U \in \mathcal{A}$, there is a $V \in \mathcal{A}$ such that $V^2 \subseteq U$,
(ii) for each $U \in \mathcal{A}$, there is a $V \in \mathcal{A}$ such that $V^{-1} \subseteq U$,
(iii) for each $U, V \in \mathcal{A}$, there is a $W \in \mathcal{A}$ such that $W \subseteq U \bigcap V$,
Let $H = \bigcap\{U : U \in \mathcal{A}\}$. Then H is a closed subgroup of G. If, in addition,
(iv) for every $U \in \mathcal{A}$, and $x \in G$, there is a $V \in \mathcal{A}$ such that $xVx^{-1} \subseteq U$, then H is a normal subgroup of G.

Definition A.6: A topological group G is said to be *compactly generated* if it contains a compact subset F for which the subgroup generated by F is G, that is

$$G = \{e\} \cup \bigcup_{n=1}^{\infty} (F \cup F^{-1})^n.$$

Theorem A.7: Let G be a locally compact topological group. Then the followings are equivalent:

(i) G is compactly generated.

(ii) There is an open subset U of G such that $\overline{U}$ is compact and U generates G.

(iii) There is a neighborhood U of e in G such that $\overline{U}$ is compact and U generates G.

Definition A.8: Let G be a topological group. G is *metrizable* if topology of G is compatible with some metric d on G.

Theorem A.9: Let G be a T_0 topological group. Then G is metrizable if and only if there exists a countable open basis at e.

Proof: See [9] Theorem 8.3.

Theorem A.10: Let G be a locally compact, compactly generated group with identity e. Then for every countable family $\{U_n\}_{n=1}^{\infty}$ of neighborhoods of e, there is a compact normal subgroup N of G such that $N \subseteq \bigcap_{n=1}^{\infty} U_n$ and G/N is metrizable and has a countable basis for its open sets.

Proof: By $A.7$, there is a neighborhood V_0 of e such that $\overline{V_0}$ is compact and V_0 generates G. Using $A.2$ and $A.4$, we construct a sequence $\{V_n\}_{n=1}^{\infty}$ of symmetric neighborhoods of e such that $V_n^2 \subseteq V_{n-1} \cap U_n$ and such that $xV_nx^{-1} \subseteq V_{n-1}$ for all $x \in \overline{V_0} \cup (\overline{V_0})^{-1}$ for all $n \in \mathbf{N}$. It follows that $\overline{V_n} \subseteq V_{n-1}$ for all $n \in \mathbf{N}$. Let $N = \bigcap_{n=0}^{\infty} V_n$.

Obviously $N \subseteq \bigcup_{n=1}^{\infty} U_n$ and by $A.5$, N is a closed subgroup of G. Since $N \subseteq \overline{V_0}$, N is also compact. Clearly, $xNx^{-1} \subseteq N$ for all $x \in \overline{V_0} \cup (\overline{V_0})^{-1}$. Since we have

$G = \bigcup_{n=1}^{\infty}(V_0 \cup V_0^{-1})^n$, it follows that $xNx^{-1} \subseteq N$ for all $x \in G$. Hence, N is a normal subgroup of G.

Let φ denote the natural mapping of G onto G/N. We now show that $\{\varphi(V_n)\}_{n=1}^{\infty}$ is a basis at N in G/N. For some n_0, we have $V_{n_0} \subseteq WN$. (Otherwise $\{\overline{V_n} \cap (WN)'\}_{n=1}^{\infty}$, which is a family of compact sets, has the finite intersection property and hence $\bigcap_{n=1}^{\infty}(\overline{V_n} \cap (WN)')$ is nonempty. This is impossible since

$$\bigcap_{n=1}^{\infty}(\overline{V_n} \cap (WN)') = (\bigcap_{n=1}^{\infty} \overline{V_n}) \cap (WN)' = (\bigcap_{n=1}^{\infty} V_n) \cap (WN)' = N \cap (WN)' = \emptyset.)$$

Since $V_{n_0} \subseteq WN$, we have $\varphi(V_{n_0}) \subseteq \{wN : w \in W\}$. Since G/N has a countable open basis at N, G/N is metrizable by $A.9$. The set $\varphi(V_0) \cup \varphi(V_0)^{-1}$ has compact closure in G/N by $A.3$. We also have

$$G/N = \bigcup_{n=1}^{\infty}(\varphi(V_0) \cup \varphi(V_0)^{-1})^n.$$

Thus, G/N is compactly generated, hence it is σ-compact and Lindelof. As a metric space satisfying the Lindelof property, G/N has a countable basis for its open sets. *Q.E.D.*

APPENDIX B: BOREL MEASURES ON LOCALLY COMPACT SPACES

We sketch a part of the Borel measures on locally compact groups. For further information see [12] and [13].

Let X be a fixed locally compact Hausdorff space. Let C be the class of all compact subsets of X, S the σ-algebra generated by C, and let U be the class of all open sets belonging to S. We shall call the sets of S the *Borel sets* of X. A real valued function on X is *Borel measurable* if it is measurable with respect to the σ-algebra S. A *Borel measure* is a measure μ defined on the class S of all Borel sets such that $\mu(K) < \infty$ for every $K \in C$.

A set $E \in S$ is *outer regular* with respect to μ if

$$\mu(E) = \inf\{\mu(O) : E \subseteq O \in U\},$$

a set $E \in S$ is *inner regular* with respect to μ if

$$\mu(E) = \sup\{\mu(K) : E \supseteq K \in C\}.$$

A set $E \in S$ is *regular* if it is both inner regular and outer regular. A measure μ is *regular* if every set $E \in S$ is regular with respect to μ.

Let G be a locally compact topological group. Let μ be a regular Borel measure on G. For $f, g : G \to \mathbf{C}$ Borel measurable functions, we define the *convolution* of f and g to be the function

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d\mu(y).$$

For, $1 \leq p < \infty$, we define

$$L^p(G) = \{f : G \to \mathbf{C} : f \ \textit{is measurable, and} \ \int_G |f(x)|^p d\mu(x) < \infty\}$$

and

$$L^\infty(G) = \{f : G \to \mathbf{C} : f \ \textit{is measurable and essentially bounded}\}.$$

Lemma B.1: Let G be a locally compact group. Suppose $1 \leq p < \infty$, $f \in L^p(G)$. Define $\lambda(x)(f)(y) = f(yx^{-1})$ for all $x, y \in G$. Then the map $G \to L^p(G); x \mapsto \lambda(x)(f)$ is uniformly continuous.

Theorem B.2: Let G be a locally compact group. If $f \in L^1(G)$ and $g \in L^\infty(G)$, then $f * g$ is bounded and continuous on G.

Proof: Let $x \in G$, we have

$$\begin{aligned} |(f*g)(x)| &\leq \int_G |f(xy^{-1})g(y)| d\mu(y) \\ &\leq \|g\|_\infty \int_G |f(xy^{-1})| d\mu(y) \\ &= \|g\|_\infty \int_G |f(y)| d\mu(y) \\ &\leq \|g\|_\infty \|f\|_1 \end{aligned}$$

Hence, $f * g$ is bounded. Next, let $x, z \in G$, then

$$\begin{aligned} |(f*g)(x) - (f*g)(z)| &= |\int_G f(xy^{-1})g(y)d\mu(y) - \int_G f(zy^{-1})g(y)d\mu(y)| \\ &\leq \int_G |f(xy^{-1}) - f(zy^{-1})||g(y)| d\mu(y) \\ &\leq \|g\|_\infty \|\lambda(-x)(f) - \lambda(-z)(f)\|_1 \end{aligned}$$

By $B.1$, we can make $\|\lambda(-x)(f) - \lambda(-z)(f)\|_1$ arbitrarily small, by taking x and z close enough. Hence, the continuity of $f * g$ follows. *Q.E.D.*

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