ON STATIC SPHERICALLY SYMMETRIC SOLUTIONS OF EINSTEIN'S EQUATIONS WITH TWO PERFECT FLUIDS AS SOURCE

by

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ABSTRACT

ON STATIC SPHERICALLY SYMMETRIC SOLUTIONS OF EINSTEIN'S EQUATIONS WITH TWO PERFECT FLUIDS AS SOURCE

We first point out that in the context of static spherically symmetric solutions of Einstein's Equations, the "radially imperfect fluids" introduced recently are equivalent to superpositions of noninteracting normal and tachyonic perfect fluid sources, also discussed recently. We discuss the given solutions, challenging especially the black hole interpretations, and give more examples of this type. We then ask what we can find about solutions with two noninteracting perfect fluids as source, where more than one component of the fluid four-velocities is nonzero, unlike in the single-fluid case. We find that such solutions exist; that is, even in a static spacetime, the source fluids can be moving, if there is more than one.

ÖZET

EINSTEIN DENKLEMLERİNİN İKİ MÜKEMMEL AKIŞKAN KAYNAKLI DURAĞAN KÜRESEL SİMETRİK ÇÖZÜMLERİ ÜZERİNE

Einstein Denklemleri'nin durağan küresel simetrik çözümleri çerçevesinde, literatürde yakın zamanda "radially imperfect fluids" tabiri ile bahsedilen akışkanların, normal ve (yine yakın zamanda tartışılan) takiyonik mükemmel akışkanların etkileşimsiz olarak birarada bulunması şeklinde anlaşılabileceğine dikkat çekilmektedir. Verilen sözkonusu cözümler tartışılmakta, özellikle kara delik yorumları sorgulanmakta ve bu tip başka örnekler de türetilmektedir. Daha sonra, tek akışkan durumunun aksine, dört boyutlu hızının birden fazla bileşeni sıfırdan farklı olan mükemmel akışkanlar irdelenmektedir. Bu tür çözümlerin var oldugu, yani durağan bir uzay-zamanda bile, eger birden fazla akışkan varsa, bunların hareketli olabilecekleri bulunmuştur.

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LIST OF SYMBOLS

F	Mass function
$g_{\mu\nu}$	Metric tensor
$G_{\mu\nu}$	Einstein tensor
M	Schwarzschild radius
p	Pressure of the fluid
p_N	Pressure of the normal fluid
p_T	Pressure of the tachyonic fluid
R	Trace of the Ricci tensor
$R_{\mu\nu}$	Ricci curvature tensor
$T_{\mu\nu}$	Stress-energy-momentum tensor
$T_{\mu\nu}^{\mathrm{pf}}$	Stress-energy-momentum tensor of the perfect fluid
u_μ	Four-velocity of the fluid
U_{eff}	Effective potential
V_μ	Unit spacelike vector
κ	Coupling constant
ρ	Density of the fluid
ρ_N	Density of the normal fluid
ρ_T	Density of the tachyonic fluid

LIST OF ACRONYMS/ABBREVIATIONS

EFE	Einstein Field Equations
EoS	Equation of state
ESU	Einstein static universe
GR	General Relativity
KS	Kantowski-Sachs
ND	Normal-dynamic
NS	Normal-static
OV	Oppenheimer-Volkoff
SEM	Stress-energy-momentum
SSSPF	Static spherically symmetric perfec fluid
TD	Tachyonic-dynamic
TS	Tachyonic-static
TOV	Tolman-Oppenheimer-Volkoff
WEC	Weak energy condition

1. INTRODUCTION

According to general relativity, gravitation is a manifestation of the curvature of spacetime; and spacetime is curved by matter and energy. The theory is mathematically represented by Einstein Field Equations (EFE)

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.1}$$

which were published by Albert Einstein in 1915. κ, the coupling constant of the theory, is given by

$$\kappa = \frac{8\pi G}{c^4}. \tag{1.2}$$

The Einstein tensor $G_{\mu\nu}$ reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.3}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci curvature tensor defined by the components of $g_{\mu\nu}$ and their derivatives up to second order, R is the trace of the Ricci tensor, also called the scalar curvature. $T_{\mu\nu}$ is the stress-energy-momentum (SEM) tensor and serves as the source of the gravitational field.

One can use three different approaches in order to find an exact solution for the EFE [1]:

- One can first specify the SEM tensor (i.e. the right-hand-side of the equation), sometimes incorporating the metric tensor $g_{\mu\nu}$, too; and then try to solve for the metric components $g_{\mu\nu}$. This method has strong physical motivation, but one has to deal with a set of nonlinear differential equations.
- One can specify the metric components $g_{\mu\nu}$, so the Einstein tensor $G_{\mu\nu}$ first. EFE

then reduces to the definition of the SEM tensor in some sense. This is rather an easy method, since one eliminates the derivatives on the left-hand-side and EFE can be solved algebraically now. This approach, on the other hand, raises the question of the physical validity of the solution found: The SEM tensor may not correspond to any known matter or field.

- One can also specify both the metric and the SEM tensor partly, and then use EFE in order to determine the components of $g_{\mu\nu}$ and $T_{\mu\nu}$ completely. This hybrid approach can yield physically valid and useful results. However, it might also here be difficult to solve EFE because it still contains the derivatives of unknown functions on the left-hand-side.

The EFE are very complicated, since they are ten *nonlinear*, coupled PDE's. Hence, to find solutions one has to make some simplifying assumption for the left-hand-side and/or the right-hand-side. Some of the most popular of these are staticity and spherical symmetry about the the left-hand-side, and the perfect fluid assumption about the right-hand side. We will use the staticity and spherical symmetry assumptions throughout this work; and also take the source as a combination of two perfect fluids.

The next section discusses the staticity, spherical symmetry and perfect fluid assumptions in more detail.

1.1. Static spherically symmetric perfect fluid solutions

1.1.1. Staticity and spherical symmetry

The study of static spherically symmetric solutions has a history almost as long as GR itself. In fact, the first solutions of GR, the famous Schwarzschild solution, described by the line element

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2 d\Omega^2, \tag{1.4}$$

the Schwarzschild exterior solution and the Einstein static universe (discussed in Section 1.1.4), belong to this category.

The staticity and spherical symmetry assumptions about the the left-hand-side of the EFE lead to the ansatz

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \tag{1.5}$$

for the line element, as is well known [2, Section 23.2]. Specifying the 22 element of the metric as r^2 amounts to a coordinate choice –the so-called Schwarzschild coordinates–. For this line element, the Einstein tensor becomes

$$G_{00} = \frac{B}{r^2}\left(1 - \frac{1}{A} + \frac{rA'}{A^2}\right) \tag{1.6}$$

$$G_{11} = \frac{1}{r^2}\left(1 - A + \frac{rB'}{B}\right) \tag{1.7}$$

$$G_{22} = \frac{r}{2A}\left[-\frac{A'}{A} + \frac{B'}{B} - \frac{rA'B'}{2AB} - \frac{rB'^2}{2B^2} + \frac{rB''}{B}\right] \tag{1.8}$$

$$G_{33} = G_{22}\sin^2\theta \tag{1.9}$$

where $A(r)$ and $B(r)$ are written as A and B for brevity, prime denotes r-derivative, and the last relation comes from spherical symmetry. These forms for the Einstein tensor mean that there are three independent components of the EFE for the static spherically symmetric case.

To get the Schwarzschild solution referred to above, one sets these elements of the Einstein tensor to zero, i.e. assumes vacuum. Birkhoff's theorem [2, Section 32.2] states that the Schwarzschild metric is the unique spherically symmetric vacuum solution of EFE. Note that the existence of a solution is by no means obvious, since we have two unknowns A(r) and B(r) for the three independent EFE components.

1.1.2. The perfect fluid

Another most-often used simplifying assumption about the right-hand-side of EFE is the perfect fluid SEM tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{1.10}$$

Here, ρ and p are the energy density and pressure, respectively, as measured by an observer moving with the perfect fluid, and u_μ is the fluid's four-velocity. Such a SEM tensor can be created by an actual fluid or effectively by a scalar field [3, 4]. The use of this $T_{\mu\nu}$ together with ansatz (1.5) describes, for example, the interiors of static spherically symmetric stars.

The use of a perfect fluid as the source brings the number of unknowns to four, namely $A(r)$, $B(r)$, $\rho(r)$ and $p(r)$. Since the number of independent equations is still three, extra information is needed to uniquely fix the solution. The alternatives of providing that extra information mirror the options 1 and 2 listed on p.1; and discussed in the following two sections.

1.1.3. The Mathematical or Tolman approach

This approach, pioneered by Tolman in his 1939 paper [5], consists of providing the extra information by specifying either of the functions $A(r)$ or $B(r)$. For example, Tolman's Solution IV starts with the assumption

$$B = C_1 r^2 + C_2 \tag{1.11}$$

From $G_{22} = \kappa T_{22}$, we find $p(r)$ in terms of $A(r)$ and substitute into $G_{11} = \kappa T_{11}$ to yield

$$A = \frac{2C_1 r^2 + C_2}{(C_1 r^2 + C_2)(1 + C_3 r^2)}. \tag{1.12}$$

Putting this back into $p(r)$ gives

$$p = \frac{C_1 + C_2C_3 + 3C_1C_3r^2}{\kappa(2C_1r^2 + C_2)}. \tag{1.13}$$

Finally from $G_{00} = \kappa T_{00}$,

$$\rho = \frac{3C_2(C_1 - C_2C_3) + C_1(2C_1 - 7C_2C_3)r^2 - 6C_1^2C_3r^4}{\kappa(2C_1r^2 + C_2)^2}. \tag{1.14}$$

Returning to the approach; if, as Tolman also does, we solve for the pressure from the 11 and 22 components of the EFE, we get an equation, called the pressure isotropy equation, involving $A(r)$, $B(r)$ and their derivatives; and r. Hence in principle, if we specify one of $A(r)$ or $B(r)$, we can solve for the other. The 00 and 11 components of the EFE then give the pressure and the density. But, as mentioned in item 2 on p.1, the found SEM tensor may not correspond to any known matter or field; we discuss the reasonability issues in Section 1.1.5 and 1.1.6.

1.1.4. The Physical or Oppenheimer-Volkoff approach

In this approach, pioneered by Oppenheimer and Volkoff in their 1939 paper [6] adjacent to the Tolman paper of the above section, the extra information is provided in terms of some properties of the fluid, in the form of an equation of state (EoS) $f(p, \rho) = 0$ between p and ρ. The integrability of the paranthesis in G_{00} is used to define a function $F(r)$,

$$F(r) = \kappa \int \rho r^2 dr \tag{1.15}$$

which here is $\kappa/4\pi$ times the "mass function" defined in the literature. Then B'/B is also expressed in terms of F via EFE-11 and finally substitution for A, B and their derivatives in EFE-22 gives

$$A = \frac{r}{r-F} \tag{1.16}$$

$$\frac{B'}{B} = \frac{\kappa p r^2 + 1}{r-F} - \frac{1}{r} \tag{1.17}$$

$$p' = -\frac{(\kappa p r^3 + F)}{2r(r-F)}(\rho + p) \tag{1.18}$$

Equation 1.18 is the well-known Oppenheimer-Volkoff (OV) equation. In this equation now one would put p in terms of ρ via an equation of state, then ρ in terms of F', via (1.15), eventually getting a differential equation for F. After solving for F, A and B would be found via (1.16) and (1.17), giving a metric for that equation of state.

Some simple solutions, including the three early ones mentioned in Section 1.1.1, can be understood easily now: For the Schwarzschild solution, we have $\rho = p = 0$, hence $F =$ constant from (1.15), and A and B can easily be found from (1.16) and (1.17). For the Schwarzschild interior solution, we have $\rho =$ constant, hence $F = Cr^3$, then (1.18) gives $p(r)$ and (1.17) again gives B. The Einstein Static Universe (ESU) is homogeneous, so $\rho =$ constant and $p =$ constant; again $F = Cr^3$, but this time (1.18) determines the p/ρ ratio (disregarding the trivial cosmological constant case), and (1.17) again gives B.

Since in the OV-approach the properties of the fluid are built-in into the solution via the EoS, the solutions are expected to be typically more reasonable than those found by the mathematical or Tolman approach; however some issues remain (see Section 1.1.5 and 1.1.6.)

A peripheral issue is the name by which one should refer to Equation1.18. Recently, some authors started to refer to this equation as the Tolman-OppenheimerVolkoff (TOV) equation, instead of OV. Well-known textbooks on GR, e.g MTW [2] call it OV; and since the relevant 1939 papers use in some sense diametrically opposite approaches to providing the extra information for unique determination of the solution, we do not believe that associating Tolman's name too with the OV equation is justified.

1.1.5. Issues of reasonability of the source

The density and pressure functions found in solutions obtained with the mathematical or Tolman approach do not necessarily correspond to any known matter or field. In this case, it is valid to ask if such matter could exist in nature. This question, however, is outside the realm of GR, it should be answered e.g. by Quantum Field Theory; and the answer is lacking.

In the absence of a satisfactory answers, some conditions have been suggested that "reasonable matter" should satisfy. These are collectively called "energy conditions" (e.g. [7]). For instance, the weak energy condition (WEC) states that the energy density should be nonnegative according to every observer; taking the form $\rho \geq 0$, $\rho + p \geq 0$ for a perfect fluid. Another consideration is causality, e.g. the speed of sound in a fluid should not exceed the speed of light.

The mathematical or Tolman approach described above does not guarantee that the fluid satisfies any of these conditions. In fact, in a study [8], tests of acceptability along these lines, such as positivity of energy density and pressure, regularity at origin, and subluminal sound speed, were applied to 127 listed candidates for SSSPF solutions, and only 16 were found to be acceptable. Hence, most of the solutions obtained using this approach are regarded as mathematical curiosities, and not physically very relevant.

While the use of an EoS in the physical, or OV approach would incorporate some aspects of reasonability in the source of the solution, most popularly used EoS's do not have the energy conditions built-in (although they could incorporate the speed-of-sound condition). For example, the EoS $p = w\rho$ used often in cosmology does not guarantee that$\rho \geq 0$ or $\rho + p \geq 0$, hence may lead to a solution that violates the WEC.

On the other hand, the dark energy and phantom fluids, recently popularized in cosmology, violate most or all energy conditions; so it is not really clear how seriously one should take them.

1.1.6. Issues of signature and reality

In either approach above, it is not even guaranteed that the signature of the metric stays Minkowskian; and this is an even more important constraint than the energy conditions, since it has to do with the local structure of the space-time. In terms of metric functions of the ansatz (1.5) implies $A(r)B(r) > 0$.

In a static space-time, the fluid is, in general, taken to be at rest, i.e. $u^0 \neq 0$ and we obtain

$$-B(r)(u^0)^2 = -1 \tag{1.19}$$

from the normalization of the four velocity

$$u_\mu u^\mu = -1 \tag{1.20}$$

However, both metric functions $A(r)$ and $B(r)$ could be negative without changing the signature. Then, the ansatz (1.5) describes nonstatic spacetimes; however, (1.20) would yield an imaginary u^0, which is obviously unacceptable. The resolution is to replace (1.20) with

$$u_\mu u^\mu = 1, \tag{1.21}$$

since one can always normalize the four velocity to ± 1. The fluid is tachyonic in this case.

In a recent work [9], it was shown that in the context of the ansätze (1.5) and (1.10), that the four-velocity of the source fluid should be real, and can only have one nonzero component, and it does *not* have to be u^0. Together with the signature argument and the possibilities of normal and tachyonic fluids, [9] points out that we have a 2x2 matrix of possibilities: either u_0 or u_1 is nonzero; the fluid may be normal or tachyonic. The four possible cases are

- fluid normal, $u_0 \neq 0$: normal-static (NS)/ standard case
- fluid tachyonic, $u_0 \neq 0$: tachyonic-static (TS) case
- fluid normal, $u_1 \neq 0$: normal-dynamic (ND)/ Kantowski-Sachs (KS) case
- fluid tachyonic, $u_1 \neq 0$: tachyonic-dynamic (TD) case.

We will often refer to these cases in the rest of the work.

In light of the foregoing discussion, one should note that the solutions of the OV equation are valid only for positive $A(r)$ and $B(r)$; and for normal fluids (the NS case). Alternative equations can be derived for the other cases.

1.2. Radially Imperfect Fluid Solutions

A generalization of the form (1.10) for $T_{\mu\nu}$ was introduced in recent works [10, 11] in the context of static spherically symmetric solutions of Einstein's Equations. This type of source is called "radially imperfect fluid" in [11] and is defined as having the SEM tensor

$$\kappa T_{\mu\nu} = T^{\mathrm{pf}}_{\mu\nu} + qV_\mu V_\nu \tag{1.22}$$

where $T^{\mathrm{pf}}_{\mu\nu}$ is the SEM tensor of a perfect fluid, and V_μ is a unit spacelike vector in the direction of anisotropy. The variable q then represents the fluid's possible response to the anisotropy [10]. (1.22) is the most general [12] SEM tensor[1] compatible with the "static" spherically symmetric ansatz (1.5). The author of [11] then goes on to derive several simple solutions assuming a radially imperfect fluid as the source, and gives interpretations of the line elements found. We repeat the solutions below.

[1]But note that the definitions of q and p are different in [11] vs. in [12].

1.2.1. Exactly vanishing perfect fluid

This case is defined by $\rho = p = 0$. The author finds (his Equations. 8 and 9)

$$A(r) = \frac{r}{r-b} \tag{1.23}$$

$$f(r) = a\left(\sqrt{1-\frac{b}{r}}\right) - c\left[1-\sqrt{1-\frac{b}{r}}\log\left(\frac{\sqrt{r}+\sqrt{r-b}}{\sqrt{d}}\right)\right] \tag{1.24}$$

where $B(r) = f(r)^2$, and (his Equation 10)

$$q(r) = \frac{c}{r^2 f} \tag{1.25}$$

It is shown that $B(r)$ has roots (see Figure 1.1), and their existence is interpreted as the solution spacetimes representing black holes.

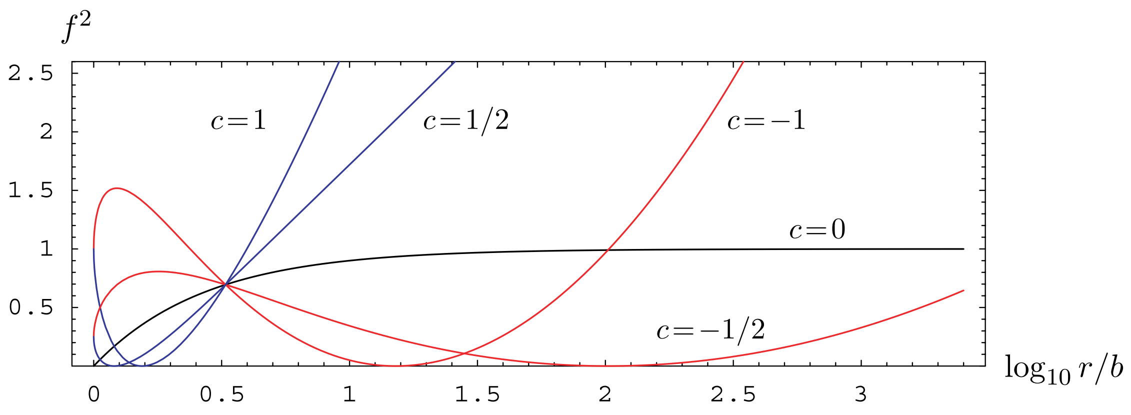

Figure 1.1. Figure 1 of [11]; author's subtitle is "Space-times with black holes"; however, we disagree. Shows a plot of $g_{00} = B(r)$ vs. $\log(r/b)$ for the case of "imperfect fluid only", i.e. Equation 1.24, with the choice $a = 1$, $d = b$. Compare with Figure 2.1.

1.2.2. Cosmological constant

This case is defined by $\rho = -p = \Lambda$. The author finds (his Equation 13)

$$A(r) = \frac{1}{1 - b/r - \Lambda r^2/3} \tag{1.26}$$

and again defining $B(r) = f(r)^2$, the equation (his 14.)

$$2r^2(3b - 3r + \Lambda r^3)f'' + r(3b - 6r + 4\Lambda r^3)f' - (3b + 4\Lambda r^3)f = 0 \tag{1.27}$$

whose solutions may or may not have roots (see Figure 1.2), and the spacetimes with roots of $B(r)$ are interpreted as representing black holes.

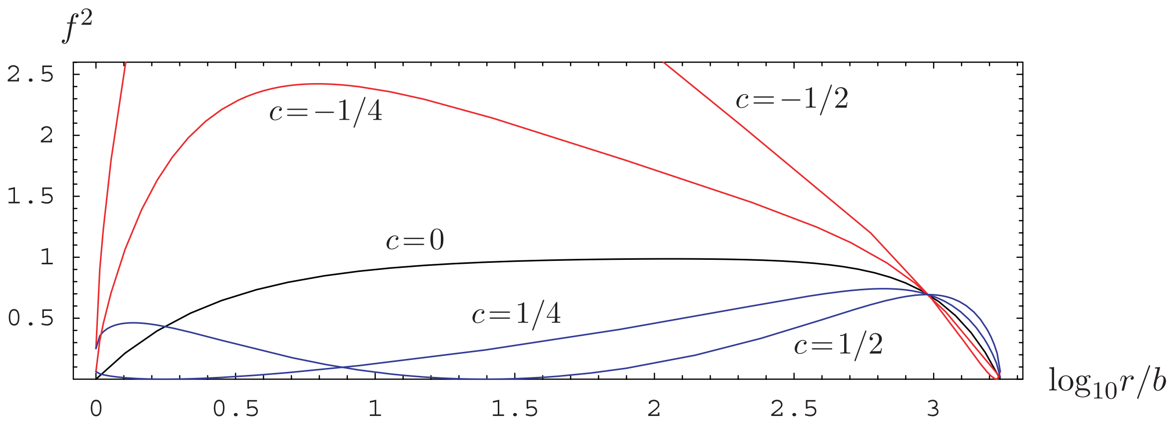

Figure 1.2. Figure 3 of [11]; author's subtitle is "Space-times with black holes and cosmological constant"; however, we disagree. Shows a plot of $g_{00} = B(r)$ vs. $\log(r/b)$ for the case of "cosmological constant", i.e. solutions of Equation 1.27, with the choice $a = 1$, $\Lambda b^2 = 10^{-6}$.

1.3. Purpose and plan of the thesis

In this thesis, we will discuss that in the context of static spherically symmetric solutions of Einstein's Equations, the "radially imperfect fluids" introduced recently are equivalent to superpositions of normal and tachyonic perfect fluid sources, also discussed recently. We will also discuss what we can find about solutions with two perfect fluids as source, where more than one component of the fluid four-velocities is nonzero, unlike in the single-fluid case.

In Chapter 2, we show the equivalence of radially imperfect fluids introduced in [11] to mixtures of normal and tachyonic perfect fluids discussed in [9] seperately. We reinterpret solutions found in [11], and find some new solutions. We also challenge the interpretations of the solutions in [11] in regions with negative $A(r)$ and/or near the roots of $B(r)$, in particular, the "black hole" interpretations.

In Chapter 3, we will ask what other possibilities exist for superposition of two perfect fluids. We point out that in a two-fluid solution, the fluid four-velocities may have more than one nonzero component. This means that even though the spacetime is static, the source fluids need not be; but of course, the motion of the two fluids must be coordinated with each other. We exhibit some such solution in that chapter.

2. ‘RADIALLY IMPERFECT FLUID’ SOLUTIONS AS SUPERPOSITIONS OF ‘STATIC’ NORMAL AND TACHYONIC PERFECT FLUIDS

For timelike u_μ and spacelike r-coordinate, (the static case), the form for the SEM tensor of radially imperfect-fluid introduced in [11] and discussed above in Section (1.2) is equivalent to

$$T_{00} = \rho(r)B(r), \quad T_{11} = [p(r) + q(r)]A(r), \quad T_{22} = p(r)r^2, \quad T_{33} = p(r)r^2 \sin^2\theta. \quad (2.1)$$

Let us recall that this was the most general [12] SEM tensor compatible with the “static” spherically symmetric ansatz (1.5); and also recall footnote 1. We do not consider spacelike u_μ (see e.g. [13] for this possibility), since v_μ showing the direction of anisotropy is necessarily spacelike, and hence in this case q would just amount to a redefinition of ρ.

For timelike u_μ and timelike r-coordinate, $A(r)$ and $B(r)$ are negative (dynamic case), $u_\mu = (0, \sqrt{-A(r)}, 0, 0)$ and $v_\mu = (\sqrt{-B(r)}, 0, 0, 0)$, hence the imperfect-fluid SEM tensor becomes

$$T_{00} = -[q(r) + p(r)]B(r), \quad T_{11} = -\rho(r)A(r), \quad T_{22} = p(r)r^2, \quad T_{33} = p(r)r^2 \sin^2\theta \quad (2.2)$$

On the other hand, we give the expressions found in [9] in Table 2.1 for the components of the SEM tensor in terms of p and ρ of *single* normal or tachyonic perfect fluids in the same kind of spacetime. Comparing these expressions to (2.1) and

Table 2.1. Expressions for the components of the stress-energy-momentum tensor in the four cases discussed in [9]. These are all possible cases compatible with (1.10) and (1.5); and $T_{33} = T_{22}\sin^2\theta$.

	NS	TD	ND	TS
T_{00}	ρB	$-(\rho+2p)B$	$-pB$	$-pB$
T_{11}	pA	pA	$-\rho A$	$(\rho+2p)A$
T_{22}	pr^2	pr^2	pr^2	pr^2

(2.2), we see that we can make the identifications

$$\rho = \rho_N - p_T \tag{2.3}$$

$$q = \rho_T + p_T \tag{2.4}$$

$$p = p_N + p_T \tag{2.5}$$

for both the static and the dynamic cases; where ρ, p and q are associated with the radially imperfect fluid, and the labels N and T stand for the normal and tachyonic perfect fluids, respectively. In other words, a mixture of normal and tachyonic perfect fluids with *one nonzero component of* u_μ *each* can mimic a radially imperfect fluid. Note that this cannot be done with two normal or two tachyonic perfect fluids.

Of course, there is the question of possible interaction between the two fluids. Any such interaction will presumably be represented by a cross-term in the Lagrangian,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}} \tag{2.6}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the individual fluid Lagrangians. Since the SEM tensor is defined via the variation of the total Lagrangian with respect to the metric, we have

$$T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + T^{(\text{int})}_{\mu\nu} \tag{2.7}$$

Since by Equations 2.1 and 2.2 we are taking the SEM tensor of the two-fluid system to be the sum of the SEM tensors of the two fluids, we have automatically assumed that the interaction term in the SEM tensor vanishes, i.e. that the fluids are noninteracting. In that case, the conservation of energy-momentum, ${T^{\mu\nu}}_{;\nu} = 0$ can be applied to each fluid separately, instead of only the total SEM tensor. Actually, the two conditions are not independent, since their sum is satisfied already via the EFE and the (contracted) Bianchi identity. Therefore, we will refer to ${T^{\mu\nu}}_{;\nu} = 0$ for one of the component fluids only, as the "noninteraction condition".

The noninteraction condition has only one nonzero component; $\mu = 1$, given by

$$\frac{1}{A}\left[\frac{dp_N}{dr} + \frac{1}{2}(\rho_N + p_N)\frac{B'}{B}\right] \tag{2.8}$$

for normal fluids, and

$$\frac{1}{A}\left[\frac{d}{dr}(\rho_T + 2p_T) + \frac{1}{2}(\rho_T + p_T)\left(\frac{4}{r} + \frac{B'}{B}\right)\right] \tag{2.9}$$

for tachyonic fluids. Of course, only one of the above equations can be set to zero in finding a solution, as mentioned above. But one should also remark that it is possible to trade the second noninteraction equation with one of the components of EFE.

In all the cases above, the Einstein Equations give three equations. If the spacetime contains a single perfect fluid, we have four variables to solve for (A, B, ρ, p), and the noninteraction condition is not independent. The radially imperfect fluid increases the number of variables to five (q added) and the above-mentioned two-fluid mixture to six $(A, B, \rho_N, p_N, \rho_T, p_T)$, but also adding the noninteraction condition to the equations. In any case, without extra constraints, an infinity of solutions can be found[2] . The question then becomes, if the found p and ρ are physically acceptable (e.g. satisfy some proposed "energy conditions" [7]) as discussed in Section 1.1.5. Let us emphasize that this question is outside General Relativity.

Or, one can take the properties of the perfect fluid(s) into account (if known),

[2]It is even possible for two spacetimes to have same geometry, but to host different fluid mixtures.

via an *equation of state* (EoS) $f(p, \rho) = 0$ for each fluid, increasing the number of equations. For both the single-fluid case and the two-fluid cases of this chapter, this allows the system to be solved uniquely, up to integration constants. In the radially imperfect fluid case, on the other hand, it is not clear if the equation of state would give one equation or two.

In the rest of this chapter, we reproduce the solutions of [11] in our formalism, and find a few more.

2.1. "Radially imperfect fluid only" solutions

In [11], this case is called "vanishing perfect fluid" since it is described by $\rho = p = 0$. The author uses SEM tensor (2.1), i.e. assumes spacelike r, hence positive A and B. We first rederive and streamline this solution, and then consider the case of timelike r, which turns out to be somewhat trivial.

2.1.1. The Static Case

Since $\rho = 0$, and (2.1) is assumed, $G_{00} = \kappa T_{00}$ gives $A(r) = \frac{r}{r-b}$ where b is an arbitrary constant. Then $G_{22} = \kappa T_{22}$ gives

$$B(r) = \left[a\sqrt{1 - \frac{b}{r}} - d\left(1 - \sqrt{1 - \frac{b}{r}} \ln\left(\frac{\sqrt{r} + \sqrt{r-b}}{\sqrt{b}}\right)\right)\right]^2 \tag{2.10}$$

in agreement [see Equation 1.24] with the solution in Section 2.1 of [11], where we have redefined the parameter a to make the denominator of the argument of the logarithm $\sqrt{b}$. As in [11], $G_{11} = \kappa T_{11}$ gives

$$q(r) = \rho_T + p_T = \frac{d}{\kappa r^2 \sqrt{B(r)}}. \tag{2.11}$$

Here, a and d are also arbitrary constants, and the solution is valid for $r > b$ [otherwise the spacetime is not static and Equation 2.2 is valid instead of Equation 2.1].

By rescaling t, one of a or d can be set equal to 1 without loss of generality. Unless $d = 0$ (that case reduces to the Schwarzschild solution, as noted in [11]) the choice $d = 1$ allows $B(r)$ to be written more compactly as

$$B(r) = \left[\sqrt{1 - \frac{b}{r}} \ln\left(\frac{\sqrt{r} + \sqrt{r-b}}{\sqrt{r_1}}\right) - 1\right]^2. \tag{2.12}$$

To understand the sources in terms of normal-tachyonic fluid mixtures, consider that the conditions $\rho = 0$ and $p = 0$ use up two of the three arbitrarily specifiable functions associated with solutions in this section. If we use our remaining freedom to choose $\rho_N(r) = f(r)$, the sources are given by

$$\rho_N(r) = -p_N(r) = p_T(r) = f(r), \qquad \rho_T(r) = q(r) - f(r). \tag{2.13}$$

i.e. the normal component must have the same EoS as the cosmological constant, which, *only* by the noninteraction condition, (2.8) leads to constant ρ_N, say Λ, and p_N. Therefore, in our formalism, Solution 2.1 of [11] can be reproduced by a cosmological constant plus a noninteracting tachyonic perfect fluid with constant $p_T = \Lambda$ (which is *not* equivalent to another cosmological constant). In fact, the line element with metric function (2.12) is the same as solution TS3 of [9], where the source is only a pressureless tachyonic fluid, i.e. corresponds to the $f(r) = 0$ case of the present problem. Note that the line element does not contain $f(r)$ or Λ explicity.

Now, let us analyze the spacetime.

- *Structure of the Space-time:* In (2.12), the number of parameters of the solution is reduced to two, and if written as function of r/b, to one, $k = \sqrt{r_1/b}$. Hence, all possible $B(r)$ functions can be shown as a one-parameter family of curves, shown in Figure 2.1 (compare with Figure 1.1).
  As in [11], we observe that for all values of k, $B(r)$ has a root $r = r_2$, with $r_2 > b$, where q, the imperfect fluid component diverges, hence the spacetime is singular.

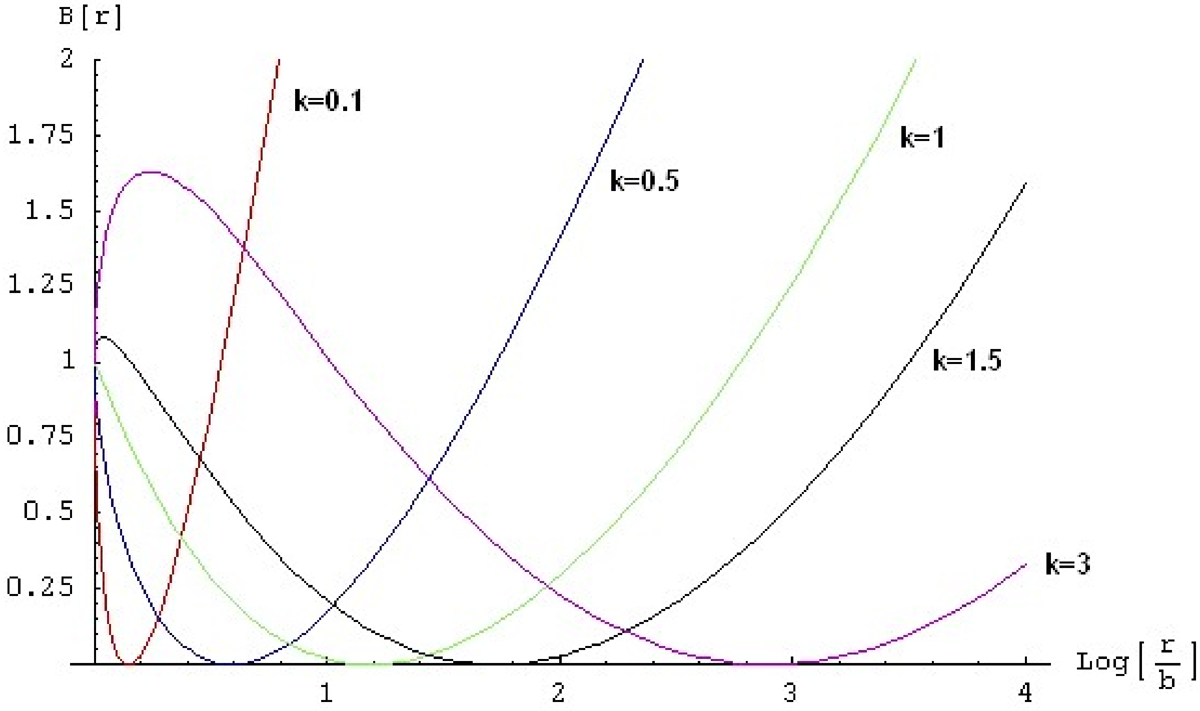

Figure 2.1. Plot of $g_{00} = B(r)$ vs. r/b for the case of "imperfect fluid only", i.e. Equation 2.10, with the choice $d = 1$, corresponding to a rescaling of t. This figure is an alternative to Figure 1.1, i.e. Figure 1 of [11]. In that figure, two parameter choices are made *before* plotting; it is not clear how curves with $d \neq b$ or $a = 0$ will look like.

On the other hand, we do not agree with the interpretation of [11] that this spacetime represents, or contains, a black hole. A spherically symmetric black hole spacetime is a composite of a static region for $r > r_H$ and a dynamic region for $r < r_H$, joined smoothly; the existence of the horizon is the *defining* feature of a black hole. The present solution features neither two such regions nor such a surface.

- *Radial motion of test particles:* The Lagrangian for radially moving test particles is

$$L = -B(r)\dot{t}^2 + A(r)\dot{r}^2 \tag{2.14}$$

where we left out the traditional factor 1/2. The time-independence of this Lagrangian gives the conserved quantity

$$E = B(r)\dot{t} \tag{2.15}$$

Putting $\dot{t}$ back into the Lagrangian, and setting that equal to -1 (for massive test particles), we get the effective-potential problem

$$\dot{r}^2 - \frac{E^2}{A(r)B(r)} + \frac{1}{A(r)} = 0 \tag{2.16}$$

where we identify the effective potential as

$$U_{\text{eff}}(r) = \frac{1 - E^2/B(r)}{A(r)} \tag{2.17}$$

and the effective energy as $E_{\text{eff}} = 0$ (not to be confused with the energy-per-unit-mass-like quantity derived from time invariance).

Plots of $U_{\text{eff}}(r)$ for various values of the parameters are is shown in Figures 2.2 and 2.3. The first feature to be noted is an infinitely deep well somewhere outside $r = b$. Of course, this corresponds to r_2, the root of $B(r)$. The second feature is that U_{eff} goes to zero as $r \longrightarrow b$. We observe that $r = b$ is repulsive (i.e. the slope

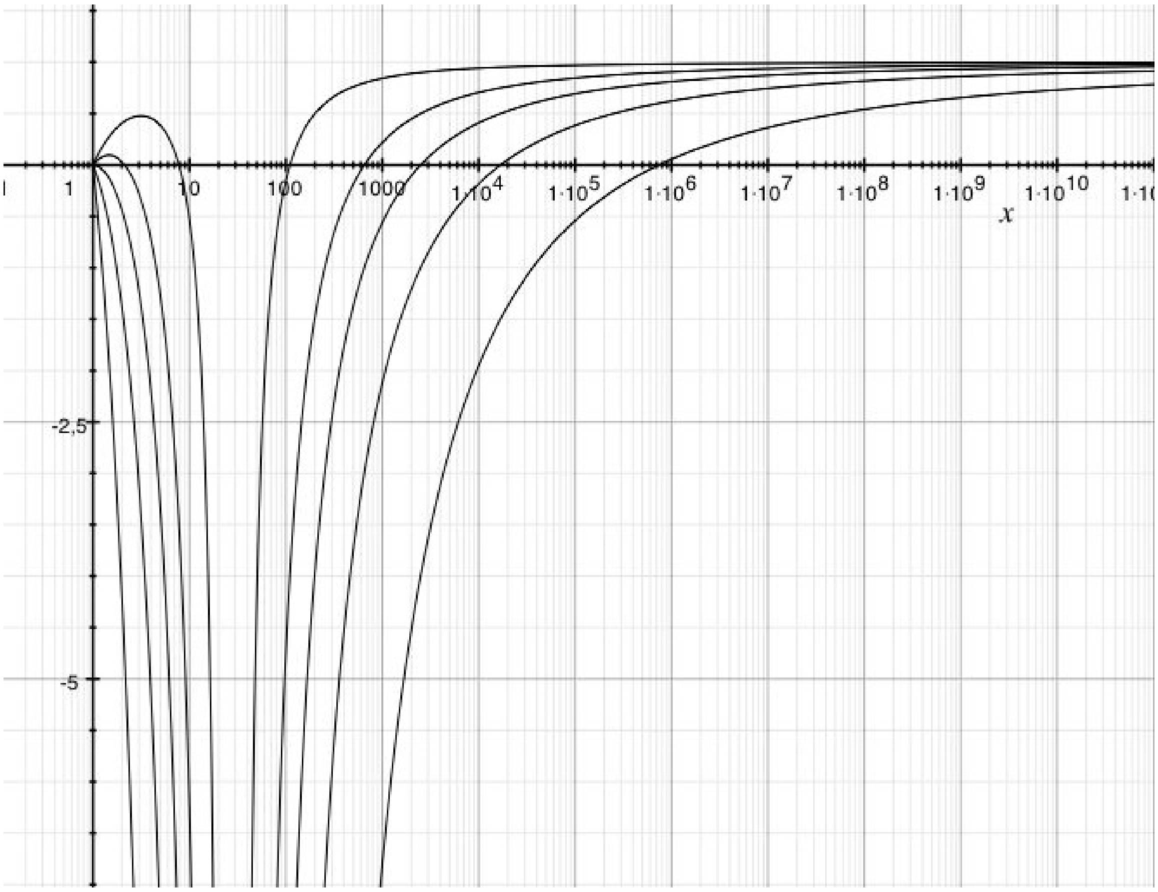

Figure 2.2. Plots of the effective potential (2.17) vs. r/b for $r_1 = b$ and $E^2 = 0.1,\ 0.5,\ 1,\ 2$ and 5 (innermost to outermost graphs). Note that the effective energy is zero, hence regions with positive $U_{\rm eff}$ are excluded.

of $U_{\rm eff}(r)$ is generally negative on the left branch of any graph); and for some parameter choices, $U_{\rm eff}$ rises above zero near $r = b$. Since the effective energy is zero, regions with positive $U_{\rm eff}$ are excluded, so that for most infalling particles, $r = b$ is a turning point, whereas some infalling particles cannot reach it at all. The condition for $U_{\rm eff}$ to rise above zero near $r = b$ is that E^2 is smaller that the maximum value $U_{\rm eff}(r)$ in the interval $b < r < r_2$. When that maximum is larger than 1, and E^2 is less than the maximum, but larger than 1, we get an interesting case: Radial oscillations become possible near $r = b$ (see Figure 2.4).

Obviously, none of this, especially the repulsive nature of the "center" is like a black hole. On the other hand, the surface $r = r_2$ is attractive for both particles inside it and particles outside it.

- *Investigation of Possible Singularities:* The solution has two special surfaces;

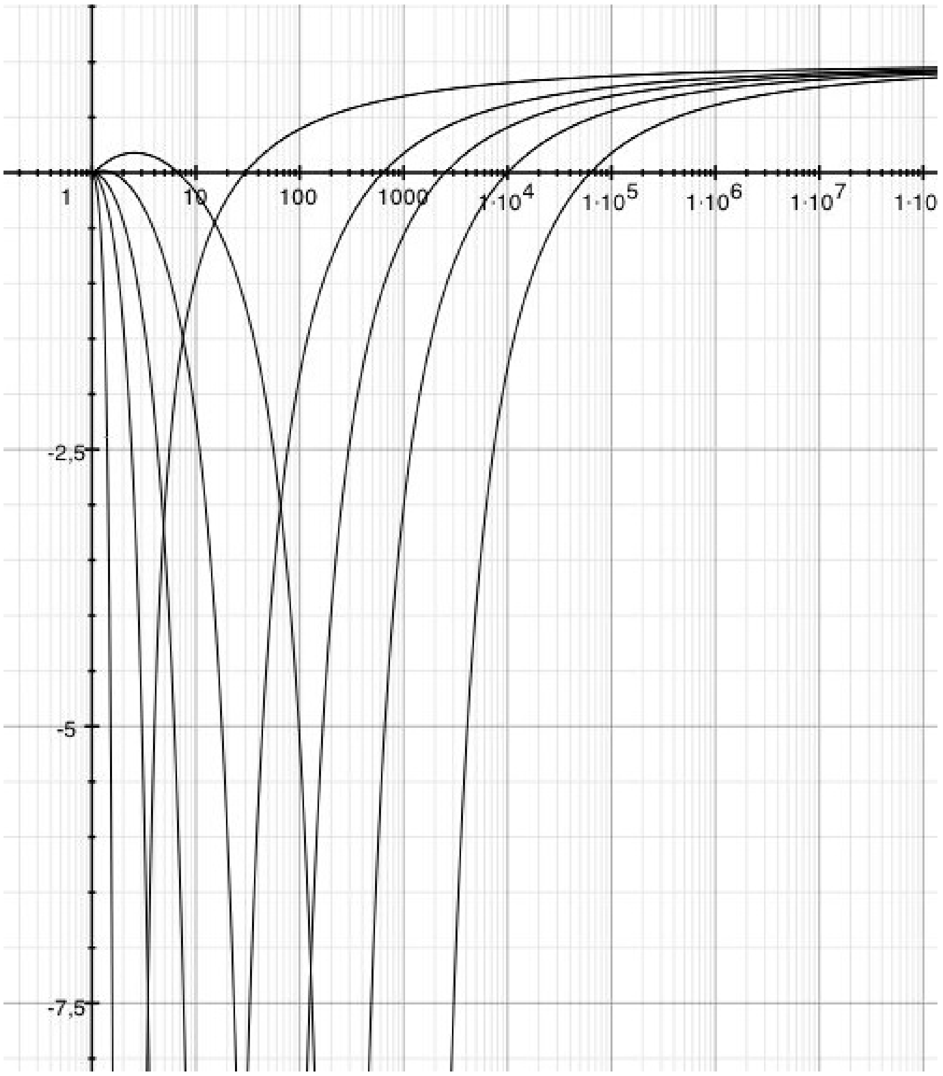

Figure 2.3. Plots of the effective potential (2.17) vs. r/b for $E^2 = 1$ and $r_1/b = 0.1,\ 0.5,\ 1,\ 2$ and 5 (left to right in both branches). Note that the effective energy is zero, hence regions with positive $U_{\rm eff}$ are excluded.

Figure 2.4. Close-up of the region near $r = b$ of a plot of the effective potential (2.17) vs. r/b for $E^2 = 1.5$ and $r_1/b = 5$. Compare with Figures 2.2 and 2.3.

$r = b$ and $r = r_2$. The $r = b$ surface, even though nonsingular, is a boundary of the spacetime, since the solution is valid for $r > b$ only. The trajectories with constant θ and ϕ on this surface are timelike, unlike those on black hole horizons, which are null. The above investigation of timelike radial geodesics showed that radially ingoing particles will turn back at $r = b$ or before; the "center" is *repulsive*, again unlike a black hole.

The $r = r_2$ surface is also unlike a black hole horizon: Firstly, it is singular as mentioned above (and in [11]); secondly, it separates two *static* regions of spacetime. For observers/particles in the $b < r < r_2$ region, it is an attractive naked singularity *surrounding* the space, so in a sense it is another boundary of the spacetime. For observers/particles in the $r > b$ region, the $r = r_2$ surface is also an attractive naked singularity, bounding their spacetime from inside.

The reason for the implicit assumption $b > 0$ in the beginning of this subsection was that in the Schwarzschild case it corresponds to negative mass and a naked singularity at the origin. But since the $r = r_2$ is also a naked singularity, we have no reason any more not to consider $b < 0$. In that case, for $r_1 < |b|$, $B(r)$ has no root and the spacetime is like the corresponding Schwarzschild case; for $r_1 > |b|$, $B(r)$ has one root and the spacetime is divided into two regions, one bounded by two naked singularities and one bounded by one naked singularity inside. Since there are no dynamic regions, no part of the spacetime can be called a black hole spacetime in the $b < 0$ case, either.

Therefore we conclude that no part of this spacetime can be called a black hole.

2.1.2. The Dynamic Case

Now we consider the SEM tensor (2.2) with ρ and p set equal to zero. One can solve for B'/B from $G_{11} = 0$; and putting this into $G_{22} = 0$ gives $A(r) = \dfrac{r}{r-b}$ again. But this time, $B(r) = \frac{C}{A(r)}$, so the solution is the (dynamic part of the) Schwarzschild spacetime! The $A(r)$ expression tells us that this solution is valid for $r < b$. Of course, [via $G_{00} = -qB(r)$], we get $q = 0$.

Note that this spacetime cannot be patched to the static spacetime discussed above, since its boundary at $r = b$ is null [also, a discontinuity in $q(r)$ would result].

Why does q vanish in the dynamic case, while it did not in the static case? The mathematical answer lies in the sequence of operations leading to the solution. In both cases we have three equations, five functions, two of which are specified; hence the solution is unique up to integration constants. Since q appears only in one equation, we have to leave that one to the end. That means one less integration in the dynamic case, since the 00 equation is integrated in the static case, while the 11 equation is algebraically solved in both cases. Since $A(r)$ has one integration constant, and one multiplicative constant in $B(r)$ is irrelevant, the total of two integration constants do not provide enough degree of freedom for a third independent function. Since we know that $q = 0$ is a valid solution (the Köttler/SdS solution), it must be the unique solution.

Interpretation of this dynamic solution in terms of normal-tachyonic fluid mixtures is similar to the last subsection, but since we want $q = 0$, we have a cosmological constant-like normal and a tachyonic fluid each, with ρ's equal and opposite at each point in spacetime.

2.2. "Cosmological constant" solutions

This case is described by $\rho = \Lambda = -p$. In [11], Section 2.2, the author again finds the static solution; here we consider both cases. Again, the dynamic case turns out to be somewhat trivial.

2.2.1. The Static Case

Using SEM tensor (2.1), the equation $G_{00} = \kappa T_{00}$ gives $A(r) = (1 - b/r - \kappa\Lambda r^2/3)^{-1}$, where b is a constant. The condition that the spacetime is static, i.e. $A(r) > 0$, will limit the region of validity, depending on the signs of, and relation between, b and Λ: $\frac{1}{A(r)}$ will have (it is easier to analyze $\frac{r}{A(r)} = r - b - \kappa\Lambda r^3/3$)

(a) for $b > 0$ & $\Lambda > 0$, (a0) no, (a1) one, or (a2) two positive roots;

(b) for $b > 0$ & $\Lambda < 0$, one positive root;

(c) for $b < 0$ & $\Lambda > 0$, one positive root;

(d) for $b < 0$ & $\Lambda < 0$, no positive roots.

Here the case (a1) corresponds to the merging (degeneracy) of the roots of the two-root case (a2). Then the validity regions are: (a0) none, (a1) none, (a2) $r_1 < r < r_2$, (b) $r > r1$, (c) $r < r1$, (d) all r, where r_1 and r_2 are the positive roots, if they exist.

Putting the above $A(r)$ into $G_{22} = \kappa T_{22}$ gives, after defining $B(r) = f^2(r)$ as in [11], Equation 14 of the same work:

$$2r^2(3b - 3r + \kappa\Lambda r^3)f'' + r(3b - 6r + 4\kappa\Lambda r^3)f' - (3b + 4\kappa\Lambda r^3)f = 0 \tag{2.18}$$

which is solved in [11] for $b = 0$ and $\Lambda < 0$; after which $q(r)$ is again found to be proportional to $1/\left(r^2\sqrt{B(r)}\right)$ as in Equation 2.11, leading to the guess that this relation might be more generally true, which can be verified.

An alternative guess is that Equation 2.18 might be written as a total derivative: We get[3]

$$\left[2(3b - 3r + \kappa\Lambda r^3)f'\right]' + \left[\left(\frac{3b}{r} - 2\kappa\Lambda r^2\right) f\right]' = 0 \tag{2.19}$$

which after integration can be brought into the form

$$\left[\sqrt{\frac{r}{3r - 3b - \kappa\Lambda r^3}}\, f\right]' = \frac{C_1\sqrt{r}}{2(3r - 3b - \kappa\Lambda r^3)^{3/2}} \tag{2.20}$$

[3]This is a guess rather than a derivation. It involves multiplying Equation 2.18 by r^{n-2} and hoping that the result can be written as

$$\left[r^n(3b - 3r + \kappa\Lambda r^3)f'\right]' + \left[r^{n-1}(Ab + Br + C\kappa\Lambda r^3)f\right]' = 0$$

but this means matching six numbers [3, -6, 4, -3, 0, -4 of Equation 2.18] using the four parameters n, A, B, C; so it did not have to work.

from which one can write $f(r)$ as an integral:

$$f(r) = \sqrt{\frac{r - b - \kappa\Lambda r^3/3}{r}} \left(\frac{C_1}{6} \int \frac{\sqrt{r}}{(r - b - \kappa\Lambda r^3/3)^{3/2}} dr + C_2 \right). \tag{2.21}$$

Now, $f'(r)$ can be solved from the integrated form of Equation 2.19 and put into $G_{11} = \kappa T_{11}$, allowing the determination of q in terms of f, r, the integration constant of Equation 2.19, and potentially, b and Λ. However, the explicit b and Λ dependences cancel, and we find

$$q(r) = \frac{C_1}{3\kappa r^2 f(r)} \tag{2.22}$$

as in [11], but without the restriction $\Lambda < 0$. The only restriction is the condition of positivity of $A(r)$ mentioned in the beginning of this subsection[4] .

- *Structure Of The Space-time:* The function $f(r)$, therefore the metric element $B(r)$ may have a root: The integral in Equation 2.21 is a monotonically increasing function, which diverges at the roots of $A(r)$, if any (However, note that $f(r)$ does not diverge.). If the interval of validity is delimited by two such roots, as in case (a2) above, the integral will go from $-\infty$ to $+\infty$ in this interval, therefore *must* have a root (one such case is depicted in Figure 1.2, that is, Figure 3 of [11]). However, if $A(r)$ has no roots, $f(r)$ will have no roots if the ratio C_1/C_2 is chosen appropriately. The other cases also may give rise to a root of $f(r)$ in the interval of validity. Of course, as in Section 2.1, we can rescale time to make a choice for one of C_1 or C_2 without loss of generality; and possible roots of neither $B(r)$ nor $A(r)$ (unless $C_1 = 0$) are horizons. As discussed below, they are boundaries, possibly also naked singularities.

[4]The relatively easy evaluation of the integral in Equation 2.21 for $b = 0$ is also free from the negativity condition on Λ used in [11], and subject only to positivity of $A(r)$. This can be seen by the substitution $1 - \kappa\Lambda r^2/3 = 1/x^2$:

$$I = \int \frac{\sqrt{r}}{(r - \kappa\Lambda r^3/3)^{3/2}} dr = \int \frac{dr}{r(1 - \kappa\Lambda r^2/3)^{3/2}} \int \frac{3dx}{r^2\kappa\Lambda x^3 \left(\frac{1}{x^2}\right)^{3/2}} = \int \frac{x^2}{x^2 - 1} dx = x - \tanh^{-1}(x)$$

.

- *Investigation of Possible Singularities:* Unless $C_1 = 0$, the roots of $A(r)$, if any, are not horizons, they are boundaries: As in Section 2.1, again trajectories with constant θ and ϕ on this (these) surface(s) are timelike. In cases where $B(r) = f^2(r)$ has a root, that surface is not a horizon either, since it separates two static regions. It is a naked singularity, effectively making the two regions separate universes.

Hence, no part of solutions of this section (except the trivial case $C_1 = 0$) can be called a black hole, contrary to the statement in the caption[5] of Figure 3 of [11] (duplicated here as Figure 1.2).

2.2.2. The Dynamic case

This case gives the solution of $\rho = -p = \Lambda$ for $A(r) < 0$ where the spacetime is not static. As in Subsection 2.1.2, the relevant equations are Equations 2.2. Following the same sequence of operations, we again find the same mathematical form for $A(r)$ as in the corresponding static case, and $B(r) = \frac{C}{A(r)}$; so the solution is the (dynamic part of the) Köttler (aka Schwarzshild-de Sitter) spacetime. Again, and for the same reasons, this gives $q = 0$, and the spacetime cannot be patched to the above-discussed static spacetime. In fact, the spacetime may have no boundaries (horizons) at all.

Interpretation in terms of normal-tachyonic fluid mixtures is easily accomplished by using $\rho = -p = \Lambda$, $q = 0$, Equations 2.3-2.5: We again need a cosmological constant-like normal and a tachyonic fluid each, but with $\rho_T = \Lambda - \rho_N$ at each point in spacetime.

2.3. Other possible static "normal-tachyonic fluid" solutions

As discussed in the beginning of this chapter, for the case of superposition of two noninteracting fluids with only u^0 or u^1 nonzero, we need either to have two more

[5] The convergence of different $f(r)$ curves near $r \approx 3000b$ in that figure is an artifact of fixing the parameter a (C_2 in our notation) and has no physical significance.

equations or to make two choices to arrive at a "solution" without arbitrary functions. The two other equations could be an equation of state for each fluid, or we could make some choices to make our equations simpler and analytically solvable, or a mixture. Let us reemphasize that solutions in this section can double as radially imperfect fluid solutions.

The equations of state that both have simple physical meaning, and simplify the solutions are

- $p_N = 0$ (pressureless normal fluid)
- $\rho_N = \rho_0 = \text{const.}$ (incompressible normal fluid, unless the pressure vanishes).

Occasionally, it might be possible to use

- the proportional-EoS, $p = w\rho$ for either fluid.

Some other choices that might simplify the solutions are

- $B = B_0 = \text{const.}$ (simplifies the 11 and especially 22 components of EFE, and also the noninteraction condition for the normal fluid)
- $A = A_0 = \text{const.}$ (simplifies the 00 and 22 components of EFE)
- $A = \frac{r}{r - C}$, same as in the Schwarzschild solution (simplifies the 00 component of EFE)
- $\frac{B'}{B} - \frac{A'}{A} = 0$ (simplifies the 22 component of EFE)
- $1 + \frac{r}{2}\frac{B'}{B} = 0$ (simplifies the 22 component of EFE)
- $p_N = p_0 = \text{const.}$ (simplifies the noninteraction condition for the normal fluid)
- $\rho_N + p_N = 0$ (simplifies the noninteraction condition for the normal fluid)
- $\rho_T + p_T = 0$ (simplifies the noninteraction condition for the tachyonic fluid; we did not include this among the EoS's in the previous paragraph since the physical meaning of ρ_T and p_T is not clear).
- $\rho_T + 2p_T = \epsilon_0 = \text{const.}$ (simplifies the noninteraction condition for the tachyonic fluid; it was not included in the previous paragraph for the same reason as above).

- $B = \left(\frac{r_0}{r}\right)^4$ (simplifies the noninteraction condition for the tachyonic fluid).

Actually, any pair of choices for $A(r)$ and $B(r)$ will give p_N as a single integral,

$$\frac{dp_N}{dr} = -\frac{1}{2}(G^2_{\ 2} - G^0_{\ 0})\frac{B'}{B} \tag{2.23}$$

as can be seen by solving for $\rho_N + p_N$ from the EFE and substituting into the noninteraction equation; and then the other pressure and the densities can be found. However, it is challenging to choose $A(r)$ and $B(r)$ such that the right-hand-side of Equation 2.23 is integrable in terms of elementary functions.

In what follows, we present some solutions; considering only static ones. Therefore, the parameters of the solutions should be chosen and if necessary, the range of r should be restricted such that the positivity of A and B is ensured. We will explicitly mention such conditions in the first solution below as an example; it should be kept in mind that they are implied in all solutions.

2.3.1. Pressureless normal fluid of constant density

Of course, a pressureless fluid ("dust") cannot be static by itself, but here we show that it can be balanced by a tachyonic fluid appropriately distributed: The noninteraction condition, Equation 2.8 gives $B(r) = B_0$ =const. Then, superposing all three Einstein's Equations gives $\rho_0 + \rho_T + 3p_T = 0$, i.e. the needed EoS of the tachyonic fluid (Here, ρ_0 is the constant density of the normal fluid). Using this in $G_{11} = \kappa T_{11}$ gives A in terms of p_T and r, which one can substitute into $G_{00} = \kappa T_{00}$ and solve to get

$$p_T = \frac{-\rho_0}{2} + \frac{C}{r^4} \tag{2.24}$$

hence

$$\rho_T = \frac{\rho_0}{2} - \frac{3C}{r^4}. \tag{2.25}$$

This gives $A(r)$ as

$$A(r) = \frac{1}{1 - \frac{\kappa\rho_0}{2}r^2 - \frac{\kappa C}{r^2}}. \tag{2.26}$$

The above-mentioned restrictions associated with the static nature of the solution mean that here we must have

$$C < \frac{1}{2\kappa^2\rho_0} \tag{2.27}$$

and constrain r to lie between the two double-roots of the denominator of $A(r)$. Interestingly, the q of the corresponding imperfect fluid does not depend on ρ_0.

2.3.2. Pressureless fluids

Again, the noninteraction condition gives $B(r) = B_0$ =const; $G_{22} = \kappa T_{22}$ gives $A(r)$ is also a constant. From $G_{11} = \kappa T_{11}$, we find

$$\rho_T = \frac{C}{r^2} \tag{2.28}$$

and from $G_{00} = \kappa T_{00}$,

$$\rho_N = \frac{-C}{r^2}. \tag{2.29}$$

The two ρ's are equal and opposite, but note that this does not mean that the density of the corresponding imperfect fluid vanishes.

2.3.3. The $B =$ constant family of solutions

The noninteraction condition gives immediately that $p_N = p_0 =$ constant. Then, the 22 component of EFE leads to

$$p_T = -p_0 - \frac{A'}{2\kappa r A^2} \tag{2.30}$$

and the 00 component gives

$$\rho_N = -p_0 + \frac{1}{\kappa r^2}\left(1 - \frac{1}{A}\right) + \frac{A'}{2\kappa r A^2} = -p_0 + \frac{1}{2\kappa r^3}\left[r^2\left(1 - \frac{1}{A}\right)\right]' \tag{2.31}$$

finally, from the 11 component we get

$$\rho_T = p_0 - \frac{1}{\kappa r^2}\left(1 - \frac{1}{A}\right) + \frac{A'}{\kappa r A^2} = p_0 - \left[\frac{1}{\kappa r}\left(1 - \frac{1}{A}\right)\right]'. \tag{2.32}$$

Now we have to make a second choice to find a unique solution; obviously, any choice for $A(r)$ is easily implementable. Some simplifying choices are $A = A_0$= const, $A = r/(r - C)$, $\rho_N = \epsilon_1 - p_0$= const (equivalent to the proportional EoS $p_N = w\rho_N$ with constant w), $\rho_T = \epsilon_2 + p_0 =$ const, even $p_T = w\rho_T$. The solutions are, in the same order,

- $A = A_0, \quad B = B_0, \quad p_N = p_0$
  The left-hand-side of $G_{22} = \kappa T_{22}$ vanishes and we find $p_T = -p_0$. From $G_{00} = \kappa T_{00}$,

  $$\rho_N = -p_0 + \frac{1}{\kappa r^2}\left(1 - \frac{1}{A_0}\right) \tag{2.33}$$

  and from $G_{11} = \kappa T_{11}$, we have

  $$\rho_T = p_0 - \frac{1}{\kappa r^2}\left(1 - \frac{1}{A_0}\right). \tag{2.34}$$

- $A = \frac{r}{r - C}, \quad B = B_0, \quad p_N = p_0$

  The 22 component of EFE leads to

$$p_T = -p_0 + \frac{C}{2\kappa r^3} \tag{2.35}$$

Using this in 00 component of EFE gives

$$\rho_N = -p_0 + \frac{C}{2\kappa r^3} \tag{2.36}$$

and finally, from 00 component,

$$\rho_T = p_0 - \frac{2C}{\kappa r^3}. \tag{2.37}$$

- $\rho_N = \epsilon_1 - p_0, \quad B = B_0, \quad p_N = p_0$

  By summing equations $G_{00} = \kappa T_{00}$ and $G_{22} = \kappa T_{22}$, one can eliminate p_T and find

$$\frac{1}{\kappa r^2}\left(1 - \frac{1}{A} + \frac{rA'}{2A^2}\right) = \rho_N + p_T \tag{2.38}$$

since B is constant. Solving this for $A(r)$, we obtain

$$A = \frac{r^2}{r^2 + C_1 - \kappa\epsilon_1 r^4/2}. \tag{2.39}$$

We can substitute this into either 00 or 22 components of EFE and find

$$p_T = -p_0 - \frac{\epsilon_1}{2} - \frac{C_1}{\kappa r^4} \tag{2.40}$$

Putting all into $G_{11} = \kappa T_{11}$ gives

$$\rho_T = p_0 + \frac{\epsilon_1}{2} + \frac{3C_1}{\kappa r^4}. \tag{2.41}$$

- $\rho_T = \epsilon_2 + p_0, \quad B = B_0, \quad p_N = p_0$

  From $G_{11} - 2G_{22} = \kappa(T_{11} - 2T_{22})$, we can eliminate p_T and find

$$\frac{1}{\kappa r^2}\left(\frac{1}{A} - 1 + \frac{rA'}{A^2}\right) = \rho_T - p_N. \tag{2.42}$$

  We solve this and get

$$A = \frac{1}{1 + C_2 r - \kappa\epsilon_2 r^2}. \tag{2.43}$$

  Putting this into either $G_{11} = \kappa T_{11}$ or $G_{22} = \kappa T_{22}$ leads to

$$p_T = -p_0 + \frac{C_2}{2\kappa r} - \epsilon_2. \tag{2.44}$$

  Finally from $G_{00} = \kappa T_{00}$,

$$\rho_N = -p_0 - \frac{3C_2}{2\kappa r} + 2\epsilon_2. \tag{2.45}$$

- $p_T = w\rho_T, \quad B = B_0, \quad p_N = p_0$

  11 component of EFE becomes

$$\frac{1}{\kappa r^2}\left(\frac{1}{A} - 1\right) = \left(\frac{1 + 2w}{w}\right) p_T + p_0 \tag{2.46}$$

  and 22 component is

$$\left(-\frac{A'}{2\kappa r A^2}\right) = p_T + p_0 \tag{2.47}$$

  Solving these two equations together, we find

$$A = \frac{1}{1 + \kappa p_0 r^2 + C_3 r^{\frac{2w}{2w+1}}}. \tag{2.48}$$

and

$$p_T = \frac{wC_3}{(2w+1)\kappa r^2} r^{\frac{2w}{2w+1}} \tag{2.49}$$

From $p_T = w\rho_T$,

$$\rho_T = \frac{C_3}{(2w+1)\kappa r^2} r^{\frac{2w}{2w+1}}. \tag{2.50}$$

From 00 component, we solve for ρ_N which is

$$\rho_N = -3p_0 - C_3 \frac{r^{\frac{2w}{2w+1}}}{\kappa r^2}. \tag{2.51}$$

Note that for the case $2w + 1 = 0$, we must have C_3=0.

2.3.4. The $A =$ constant family of solutions

A good second choice to simplify this family of solutions is $\frac{B'}{B} = \frac{C}{r}$, so that $B \propto r^C$. Then, adding $G_{00} = \kappa T_{00}$ and $G_{22} = \kappa T_{22}$ we get

$$\kappa(\rho_N + p_N) = \frac{1}{r^2}\left(1 - \frac{1}{A_0} + \frac{C^2}{4A_0}\right). \tag{2.52}$$

Putting this into the noninteraction equation and integrating,

$$p_N = \frac{C}{4\kappa r^2}\left(1 - \frac{1}{A_0} + \frac{C^2}{4A_0}\right) + p_0; \tag{2.53}$$

hence from $G_{22} = \kappa T_{22}$,

$$p_T = \frac{C}{4\kappa r^2}\left(-1 + \frac{1}{A_0} + \frac{C(4-C)}{4A_0}\right) - p_0; \tag{2.54}$$

therefore from $G_{11} = \kappa T_{11}$,

$$\rho_T = \frac{1}{\kappa r^2}\frac{(4-C)(4-4A_0+4C-C^2)}{16A_0} + p_0; \tag{2.55}$$

and from $G_{00} = \kappa T_{00}$,

$$\rho_N = \frac{1}{\kappa r^2}\frac{(4-C)(-4+4A_0+C^2)}{16A_0} - p_0. \tag{2.56}$$

2.3.5. The $B \propto 1/r^2$ family of solutions

These are motivated by the fact that the condition $1 + \frac{r}{2}\frac{B'}{B} = 0$ simplifies the 22 component of EFE. This gives $B = r_0^2/r^2$, and

$$\frac{1}{\kappa r^2}\left(1 - \frac{1}{A} + \frac{rA'}{A^2}\right) = \rho_N - p_T \tag{2.57}$$

$$-\frac{1}{\kappa r^2}\left(\frac{1}{A} + 1\right) = p_N + 2p_T + \rho_T \tag{2.58}$$

$$\frac{1}{\kappa A r^2} = p_N + p_T \tag{2.59}$$

from the 00, 11 and 22 components of the EFE, respectively. Now $\rho_N + p_N$ can be calculated and substituted into the noninteraction equation to yield

$$\frac{dp_N}{dr} - \frac{1}{\kappa r^3}\left(1 + \frac{rA'}{A^2}\right) = 0. \tag{2.60}$$

For the second choice, one can choose in this equation either of $p_N(r)$ and $A(r)$, and solve for the other by a single integration. For example, putting $A = (r/r_0)^\alpha$ into Equation 2.60 gives

$$p_N = p_0 - \frac{2 + \alpha + 2\alpha(\frac{r}{r_0})^\alpha}{2(2+\alpha)\kappa r^2}. \tag{2.61}$$

We substitute this into (2.59) and find

$$p_T = -p_0 + \frac{2 + \alpha + 4(\frac{r}{r_0})^\alpha (1+\alpha)}{2(2+\alpha)\kappa r^2}. \tag{2.62}$$

From 00 and 11 components of EFE, we obtain

$$\rho_N = p_0 - \frac{(2+\alpha) + 2(\frac{r}{r_0})^\alpha (-4 - \alpha + \alpha^2)}{2(2+\alpha)\kappa r^2} \tag{2.63}$$

and

$$\rho_T = p_0 - \frac{3(2+\alpha) + 4(\frac{r}{r_0})^\alpha (3 + 2\alpha)}{2(2+\alpha)\kappa r^2} \tag{2.64}$$

respectively.

Another example is choosing $A = \dfrac{r}{r - C}$ (i.e. Schwarzschild's A). Using the same method above, we find the following solution

$$p_N = p_0 - \frac{1}{2\kappa r^2} - \frac{C}{3\kappa r^3} \tag{2.65}$$

$$p_T = -p_0 + \frac{3}{2\kappa r^2} - \frac{4C}{3\kappa r^3} \tag{2.66}$$

$$\rho_N = p_0 - \frac{9}{2\kappa r^2} + \frac{10C}{3\kappa r^3} \tag{2.67}$$

$$\rho_T = p_0 - \frac{3}{2\kappa r^2} + \frac{4C}{3\kappa r^3}. \tag{2.68}$$

A more interesting choice is $p_N = p_0 - \alpha r^\beta$. From the non-interaction equation, we find

$$A(r) = \frac{1}{\frac{\kappa\alpha\beta}{2+\beta} r^{2+\beta} - C_1 + \ln r}. \tag{2.69}$$

Putting this into 22 component of EFE gives

$$p_T = \frac{2\alpha(1+\beta)}{2+\beta} r^\beta - p_0 - \frac{C_1 - \ln r}{\kappa r^2} \tag{2.70}$$

As a result, Equations(2.57 and 2.58 yield

$$\rho_N = -p_0 + \alpha(1-\beta) r^\beta \tag{2.71}$$

and

$$\rho_T = \frac{-2\alpha(1+2\beta)}{2+\beta} r^\beta + p_0 + \frac{-1 + 3C_1 - 3\ln r}{\kappa r^2} \tag{2.72}$$

respectively. $p_N = p_0 - \alpha r^\beta$, which for positive p_0, α and β (preferably $\beta > 1$) is like the pressure of a star of normal matter, gives nonsingular (but negative) ρ_N, although B, ρ_T and p_T are singular at the origin.

2.3.6. The $\rho_N + p_N = 0$ family of solutions

This condition gives, via the noninteraction equation, $p_N = \epsilon_1 =$ constant, hence $\rho_N = -\epsilon_1$. Then, one can eliminate p_T between the 00 and 22 components of the EFE to get what we can call an AB-equation (the analog of the pressure isotropy equation in the single-normal-fluid case). Then one can make the second choice for either of $A(r)$ and $B(r)$, and (hope to) solve for the other from the AB-equation (The $B = B_0$ solution was already found in 2.3.3, it is the third one, with $\epsilon_1 = 0$). But solving the AB-equation is not very different from specifying $A(r)$ or $B(r)$ first; some covered in the above subsections; so we will not pursue these solutions any further.

2.3.7. The $\rho_T + p_T = 0$ family of solutions

This condition gives, via the noninteraction equation, $p_T = \epsilon_1 =$ constant, hence $\rho_T = -\epsilon_1$. Then, one can eliminate p_N between the 11 and 22 components of the EFE to get another AB-equation, and then, make the second choice for either of $A(r)$ and $B(r)$, or some combination, and (hope to) solve for the other from this equation, then solve for ρ_N and p_N. (The $B = B_0$ solution was already found in 2.3.3, it is the fourth one, with $C_2 = 0$). A particularly simple second choice is

$$\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0. \tag{2.73}$$

By subtracting 22 component of EFE from 11 component and solving, we find

$$A = \frac{B_0}{B_0 + C_1 r^2 + \frac{C_2}{r}} \tag{2.74}$$

which leads to

$$B = B_0 + C_1 r^2 + \frac{C_2}{r}. \tag{2.75}$$

From $G_{00} = \kappa T_{00}$,

$$\rho_N = \epsilon_1 - \frac{3C_1}{\kappa B_0}. \tag{2.76}$$

$G_{22} = \kappa T_{22}$ yields

$$p_N = -\epsilon_1 + \frac{3C_1}{\kappa B_0}. \tag{2.77}$$

Another simplification of the AB-equation, $1 + \frac{r}{2}\frac{B'}{B} = 0$, leads to negative (and constant) A, hence is unacceptable.

2.3.8. The $B \propto 1/r^4$ family of solutions

The condition $B = (r_0/r)^4$ gives, via the noninteraction equation, $\rho_T + 2p_T = \epsilon_1 =$ constant, and also leads to simplified EFE

$$\frac{1}{\kappa r^2}\left(1 - \frac{1}{A} + \frac{rA'}{A^2}\right) = \rho_N - p_T \tag{2.78}$$

$$-\frac{1}{\kappa r^2}\left(\frac{3}{A} + 1\right) = \epsilon_1 + p_N \tag{2.79}$$

$$\frac{1}{2\kappa A r}\left(\frac{A'}{A} + \frac{8}{r}\right) = p_N + p_T \tag{2.80}$$

Here, any choice for A (including Schwarzschild's A) will give the unknown pressures and densities without the need of an integration; or one can choose a condition for one of the pressures and densities and find the others, and $A(r)$, possibly needing an integration. It is even possible to solve using the EoS $p_N = w\rho_N$ for normal matter. Some solutions are

- $A = A_0, \quad B = \left(\frac{r_0}{r}\right)^4$
  Since $\rho_T + 2p_T = \epsilon_1$, $G_{11} = \kappa T_{11}$ gives

$$p_N = -\epsilon_1 - \left(\frac{3}{A_0} + 1\right)\frac{1}{\kappa r^2}. \tag{2.81}$$

Using this in $G_{22} = \kappa T_{22}$ leads to

$$p_T = \epsilon_1 + \left(\frac{7}{A_0} + 1\right)\frac{1}{\kappa r^2} \tag{2.82}$$

hence,

$$\rho_T = -\epsilon_1 - \left(\frac{7}{A_0} + 1\right)\frac{2}{\kappa r^2}. \tag{2.83}$$

Finally, from $G_{00} = \kappa T_{00}$,

$$\rho_N = \epsilon_1 + \left(\frac{3}{A_0} + 1\right)\frac{2}{\kappa r^2}. \tag{2.84}$$

- $p_N = 0, \quad B = \left(\frac{r_0}{r}\right)^4$

  From 11 component of EFE, we solve for $A(r)$ and obtain

$$A = -\frac{3}{1 + \kappa\epsilon_1 r^2}. \tag{2.85}$$

We substitute this into $G_{22} = \kappa T_{22}$ to find

$$p_T = -\epsilon_1 - \frac{4}{3\kappa r^2} \tag{2.86}$$

and obviously,

$$\rho_T = 3\epsilon_1 + \frac{8}{3\kappa r^2}. \tag{2.87}$$

As a result, $G_{00} = \kappa T_{00}$ yields

$$\rho_N = 0. \tag{2.88}$$

This solution only features tachyonic matter.

- $p_N = p_0 - \alpha r^\beta, \quad B = \left(\frac{r_0}{r}\right)^4$

  One can use $\rho_T + 2p_T = \epsilon_1$ in 11 component of the EFE and get

$$A = \frac{3}{\kappa r^2(\alpha r^\beta - p_0 - \epsilon_1) - 1}. \tag{2.89}$$

We find

$$p_T = (2 - \frac{\beta}{6})\alpha r^\beta - 2p_0 - \epsilon_1 - \frac{4}{3\kappa r^2} \tag{2.90}$$

by putting $A(r)$ into $G_{22} = \kappa T_{22}$, and

$$\rho_T = 3\epsilon_1 - (4 - \frac{\beta}{3})\alpha r^\beta + 4p_0 + \frac{8}{3\kappa r^2} \tag{2.91}$$

from the non-interaction condition. Finally,

$$\rho_N = \alpha(1 - \frac{\beta}{2})r^\beta - p_0. \tag{2.92}$$

We started by choosing p_N to be like a star, as in 2.3.5, and again ρ_N turns out to be nonsingular.

- $p_N = w\rho_N, \quad B = \left(\frac{r_0}{r}\right)^4$

For this proportional-equation-of-state, we begin by adding 00 and 22 components of the EFE to eliminate p_T. We solve this equation for ρ_N and obtain

$$\rho_N = \frac{6A + 2A^2 + 3rA'}{2(1+w)\kappa r^2 A^2}. \tag{2.93}$$

We put this expression into $G_{11} = \kappa T_{11}$ to find

$$A = \frac{3}{C_1 r^{4+2/w} - \kappa\epsilon_0 r^2 - 1}. \tag{2.94}$$

Substituting (2.94) back into (2.93) gives

$$\rho_N = -\frac{C_1}{\kappa w} r^{2+2/w} \tag{2.95}$$

hence,

$$p_N = -\frac{C_1}{\kappa} r^{2+2/w}. \tag{2.96}$$

From either $G_{00} = \kappa T_{00}$ or $G_{22} = \kappa T_{22}$,

$$p_T = -\epsilon_0 - \frac{4}{3\kappa r^2} + C_1 \frac{5w-1}{3\kappa w} r^{2+2/w} \tag{2.97}$$

and finally, from the non-interaction condition,

$$\rho_T = 3\epsilon_0 + \frac{8}{3\kappa r^2} - C_1 \frac{2(5w-1)}{3\kappa w} r^{2+2/w}. \tag{2.98}$$

2.3.9. Unused choices and general choices

We did not use two of the choices on our list, since they lead to some of the other choices covered. $p_N =$ const. leads, by the noninteraction equation, to either $B =$ const, or $\rho_N + p_N = 0$. Similarly, $\rho_T + 2p_T =$ const. leads to either $\rho_T + p_T = 0$, or to $B \propto 1/r^4$.

As mentioned in the beginning of this section, if $A(r)$ and $B(r)$ can be chosen such that the right-hand-side of Equation 2.23 is analytically integrable, p_N can be found. Then, p_T, ρ_T and ρ_N can be found directly from the EFE, in that order. While it is not easy to choose such $A(r)$ and $B(r)$, we have found some suitable functions. The first one is

$$A = B = \frac{r}{r-C} \implies p_N = \frac{1}{4\kappa C}\left(-\frac{C^2}{3r^3} + \frac{1}{r} + \frac{1}{r-C}\right) + \frac{1}{2\kappa C^2}\ln\left(\frac{r-C}{r}\right) + p_0, \tag{2.99}$$

which gives

$$\rho_N = p_T = \frac{1}{2\kappa(C-r)}\left(-\frac{1}{C} + \frac{5C^2}{6r^3} - \frac{11C}{6r^2} + \frac{1}{2r}\right) + \frac{1}{2\kappa C^2}\ln\left(\frac{r-C}{r}\right) - p_0, \tag{2.100}$$

and

$$\rho_T = \frac{1}{2\kappa(C-r)}\left(\frac{1}{C} + \frac{11C^2}{6r^3} + \frac{23C}{6r^2} - \frac{1}{2r}\right) - \frac{1}{2\kappa C^2}\ln\left(\frac{r-C}{r}\right) + p_0, \tag{2.101}$$

Two more analytical solutions follow from assuming $B = (r-2C)^2$. For these, the choices for A and some matter variables that follow are

- $A = \dfrac{1}{1 - \frac{r}{3C_1} + \frac{C_2}{r^2}}, \quad p_N = p_0 = \text{const}, \quad \rho_N = -p_0,$

  $$p_T = -p_0 - \frac{C_2}{r^4} + \frac{2}{3C_1 r}, \tag{2.102}$$

- $$A = \frac{6C_1}{3 + 6C_1 - 2r + \frac{6C_1C_2}{r^2}}. \tag{2.103}$$

3. SOLUTIONS FEATURING TWO MOVING PERFECT FLUIDS

In this chapter, we will again deal with two perfect fluids in a static spherically symmetric spacetime, but drop the restriction used in the last chapter that only one component of any fluid four-velocity being nonzero. In fact, this restriction was imposed by the EFE when the source was a *single* perfect fluid, and does not apply when there are two.

We will take the stress-energy momentum (SEM) tensor for a mixture of two perfect fluids as the sum of the SEM tensors of individual fluids,

$$T_{\mu\nu} = (\rho_1 + p_1)u_\mu u_\nu + p_1 g_{\mu\nu} + (\rho_2 + p_2)v_\mu v_\nu + p_2 g_{\mu\nu} \tag{3.1}$$

where u_μ is the four-velocity of fluid 1 and v_μ is the four-velocity of fluid 2. The absence of cross-terms in (3.1) means that here also we are assuming that the fluids are noninteracting, by the same argument as in the beginning of Chapter 2.

Because the Einstein tensor for the static spherically symmetric metric is diagonal, i.e. Equations 1.6-1.9, the off-diagonal components of Equation 3.1 vanish by EFE, giving

$$T_{0i} = (\rho_1 + p_1)u_0 u_i + (\rho_2 + p_2)v_0 v_i = 0 \tag{3.2}$$

$$T_{ij} = (\rho_1 + p_1)u_i u_j + (\rho_2 + p_2)v_i v_j = 0, \quad i \neq j \tag{3.3}$$

where $i, j = 1, 2, 3$. These equations can be seen as a matrix equation for the vector

$\begin{pmatrix} u_i \\ v_i \end{pmatrix}$, hence for a given j, both u_i and v_i will vanish, unless the matrix

$$M = \begin{pmatrix} (\rho_1 + p_1)u_0 & (\rho_2 + p_2)v_0 \\ (\rho_1 + p_1)u_j & (\rho_2 + p_2)v_j \end{pmatrix} \tag{3.4}$$

is singular. Note that this can happen for one j only: Since $i \neq j$, the other space components of u_μ and v_μ will vanish by Equations 3.2 and 3.3, hence the matrix will become singular for the other possible values of j.

So, u_μ and v_μ can have only one nonzero space component each; if a j can be found such that the matrix is nonsingular. But, the nonzero ones cannot be 2^{nd} or 3^{rd} components, since this would conflict with Equation 1.9 and the EFE; hence they must be the first components. So we can have a static metric with two radially moving perfect fluids, as promised in the introduction.

We will first discuss the generic case where the matrix is not singular, then the special case where it is.

3.1. The generic case: Two moving fluids

In this case, the diagonal components of Einstein Equations are[6]

$$\frac{B}{\kappa r^2}\left(1 - \frac{1}{A} + \frac{rA'}{A^2}\right) = (\rho_1 + p_1)u_0^2 + (\rho_2 + p_2)v_0^2 - (p_1 + p_2)B \tag{3.5}$$

$$\frac{1}{\kappa r^2}\left(1 - A + \frac{rB'}{B}\right) = (\rho_1 + p_1)u_r^2 + (\rho_2 + p_2)v_r^2 + (p_1 + p_2)A \tag{3.6}$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A} + \frac{B'}{B} - \frac{rA'B'}{2AB} + \frac{rB'^2}{2B^2} + r\left(\frac{B'}{B}\right)'\right] = (p_1 + p_2)r^2 \tag{3.7}$$

[6]where we use index r rather than 1 on the velocities, to avoid possible confusion with label 1 referring to the first fluid.

and the 01 component is

$$(\rho_1 + p_1)u_0 u_r + (\rho_2 + p_2)v_0 v_r = 0. \tag{3.8}$$

As discussed above, we take the fluids to be noninteracting, therefore they will satisfy $T^{\mu\nu}{}_{;\nu} = 0$ separately. But as discussed in Chapter 2, on p.14, we can really use the "noninteraction equations" for one fluid only. Alternatively, we could use more of the doubled set of noninteraction equations, and leave out appropriate number of EFE component equations.

Assuming that ρ_i, p_i, u^μ and v^μ depend on r only[7] , the nontrivial components of the non-interaction condition are the 0^{th} and the 1^{st}, which can be brought into the forms

$$\left[(\rho_1 + p_1)u^0 u^r\right]' + \frac{1}{2}(\rho_1 + p_1)u^0 u^r \left[\frac{A'}{A} + \frac{3B'}{B} + \frac{4}{r}\right] = 0 \tag{3.9}$$

$$\frac{\left[(\rho_1 + p_1)(u^r)^2 ABr^2\right]'}{Br^2} + p_1' + (\rho_1 + p_1)\frac{B'}{2B} = 0. \tag{3.10}$$

for the first fluid, for example. Perhaps surprisingly, Equation 3.9 is easily integrable, yielding

$$(\rho_1 + p_1)u^0 u^r = \frac{C_1}{B^{3/2}A^{1/2}r^2} \tag{3.11}$$

where C_1 is a constant. The 01 component of EFE, Equation 3.8 tells us that the same is correct for the second fluid, with $C_1 \to -C_1$, hence we can drop the label 1.

Let us call Equations 3.5, 3.6, 3.7, 3.8, 3.11 and 3.10 the set EFE-NI2F-0 (See Figure 3.1).

Of course, the components of four-velocities are related by the normalization

[7] Actually, Equation 3.7 shows that $p_1 + p_2$ depends on r only; and then, dividing (3.5) by $-B$, (3.6) by A and adding shows that so does $\rho_1 + \rho_2$, after use of the normalization of four-velocities. This is valid for both tachyonic and normal (i.e. non-tachyonic) fluids.

$$\frac{B}{\kappa r^2}\left(1-\frac{1}{A}+\frac{rA'}{A^2}\right) = (\rho_1+p_1)u_0^2+(\rho_2+p_2)v_0^2-(p_1+p_2)B \qquad (3.5)$$

$$\frac{1}{\kappa r^2}\left(1-A+\frac{rB'}{B}\right) = (\rho_1+p_1)u_r^2+(\rho_2+p_2)v_r^2+(p_1+p_2)A \qquad (3.6)$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A}+\frac{B'}{B}-\frac{rA'B'}{2AB}+\frac{rB'^2}{2B^2}+r\left(\frac{B'}{B}\right)'\right] = (p_1+p_2)r^2 \qquad (3.7)$$

$$(\rho_1+p_1)u_0u_r+(\rho_2+p_2)v_0v_r = 0 \qquad (3.8)$$

$$(\rho_1+p_1)u^0u^r = \frac{C}{B^{3/2}A^{1/2}r^2} \qquad (3.11)$$

$$\frac{[(\rho_1+p_1)(u^r)^2ABr^2]'}{Br^2}+p_1'+(\rho_1+p_1)\frac{B'}{2B} = 0. \qquad (3.10)$$

Figure 3.1. The set EFE-NI2F-0.

conditions. So, let us introduce $f_1(r)$ and $f_2(r)$, trading them for the variables u_r and v_r:

$$u_r = f_1(r)\sqrt{A(r)}; \qquad v_r = f_2(r)\sqrt{A(r)} \qquad (3.12)$$

This makes

$$u_0 = \sqrt{f_1^2+\epsilon_1}\sqrt{B(r)}; \qquad v_0 = \sqrt{f_2^2+\epsilon_2}\sqrt{B(r)}, \qquad (3.13)$$

where ϵ_i is 1 for normal and -1 for tachyonic fluids. Substituting into the EFE, we get the set EFE-NI2F-1 (See Figure 3.2).

We now can use Equation 3.16 to eliminate ρ_2 from the 00 and 11 components, getting the set EFE-NI2F-2 (See Figure 3.3). Here $f_{12} = f_2\sqrt{f_1^2+\epsilon_1} - f_1\sqrt{f_2^2+\epsilon_2}$.

Then, we can use Equation 3.17 to eliminate also ρ_1 from the 00 and 11 components of EFE, and the 1$^{\text{st}}$ component of the non-interaction condition, getting the set EFE-2F-3 (See Figure 3.4).

$$\frac{1}{\kappa r^2}\left(1-\frac{1}{A}+\frac{rA'}{A^2}\right)=(\rho_1+p_1)(f_1^2+\epsilon_1)+(\rho_2+p_2)(f_2^2+\epsilon_2)-(p_1+p_2) \quad (3.14)$$

$$\frac{1}{\kappa r^2}\left(\frac{1}{A}-1+\frac{rB'}{AB}\right)=(\rho_1+p_1)f_1^2+(\rho_2+p_2)f_2^2+(p_1+p_2) \quad (3.15)$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A}+\frac{B'}{B}-\frac{rA'B'}{2AB}+\frac{rB'^2}{2B^2}+r\left(\frac{B'}{B}\right)'\right]=(p_1+p_2)r^2 \quad (3.7)$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1+(\rho_2+p_2)\sqrt{f_2^2+\epsilon_2}\,f_2=0 \quad (3.16)$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1=\frac{C}{Br^2} \quad (3.17)$$

$$\frac{[(\rho_1+p_1)f_1^2Br^2]'}{Br^2}+p_1'+(\rho_1+p_1)\frac{B'}{2B}=0. \quad (3.18)$$

Figure 3.2. The set EFE-NI2F-1.

$$\frac{1}{\kappa r^2}\left(1-\frac{1}{A}+\frac{rA'}{A^2}\right)=\frac{(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}}{f_2}f_{12}-(p_1+p_2) \quad (3.19)$$

$$\frac{1}{\kappa r^2}\left(\frac{1}{A}-1+\frac{rB'}{AB}\right)=-\frac{(\rho_1+p_1)f_1}{\sqrt{f_2^2+\epsilon_2}}f_{12}+(p_1+p_2) \quad (3.20)$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A}+\frac{B'}{B}-\frac{rA'B'}{2AB}+\frac{rB'^2}{2B^2}+r\left(\frac{B'}{B}\right)'\right]=(p_1+p_2)r^2 \quad (3.7)$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1+(\rho_2+p_2)\sqrt{f_2^2+\epsilon_2}\,f_2=0 \quad (3.16)$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1=\frac{C}{Br^2} \quad (3.17)$$

$$\frac{[(\rho_1+p_1)f_1^2Br^2]'}{Br^2}+p_1'+(\rho_1+p_1)\frac{B'}{2B}=0. \quad (3.23)$$

where $f_{12}=f_2\sqrt{f_1^2+\epsilon_1}-f_1\sqrt{f_2^2+\epsilon_2}$

Figure 3.3. The set EFE-NI2F-2.

$$\frac{1}{\kappa r^2}\left(1-\frac{1}{A}+\frac{rA'}{A^2}\right)=\frac{C}{f_1 f_2 B r^2}f_{12}-(p_1+p_2) \tag{3.21}$$

$$\frac{1}{\kappa r^2}\left(\frac{1}{A}-1+\frac{rB'}{AB}\right)=-\frac{C}{\sqrt{f_1^2+\epsilon_1}\sqrt{f_2^2+\epsilon_2}Br^2}f_{12}+(p_1+p_2) \tag{3.22}$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A}+\frac{B'}{B}-\frac{rA'B'}{2AB}+\frac{rB'^2}{2B^2}+r\left(\frac{B'}{B}\right)'\right]=(p_1+p_2)r^2 \tag{3.7}$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1+(\rho_2+p_2)\sqrt{f_2^2+\epsilon_2}\,f_2=0 \tag{3.16}$$

$$(\rho_1+p_1)\sqrt{f_1^2+\epsilon_1}\,f_1=\frac{C}{Br^2} \tag{3.17}$$

$$\left[\frac{f_1}{\sqrt{f_1^2+\epsilon_1}}\right]'+\frac{Br^2p_1'}{C}+\frac{1}{f_1\sqrt{f_1^2+\epsilon_1}}\frac{B'}{2B}=0. \tag{3.23}$$

Figure 3.4. The set EFE-NI2F-3.

Finally, Equation 3.23 suggests the introduction of

$$s_i=\frac{f_i}{\sqrt{f_i^2+\epsilon_i}}, \tag{3.24}$$

from which follows

$$f_i=\sqrt{\frac{\epsilon_i}{1-s_i^2}}s_i, \qquad \sqrt{f_i^2+\epsilon_i}=\sqrt{\frac{\epsilon_i}{1-s_i^2}}, \tag{3.25}$$

and most helpfully,

$$f_{12}=\sqrt{\frac{\epsilon_1\epsilon_2}{(1-s_1^2)(1-s_2^2)}}(s_2-s_1). \tag{3.26}$$

This transformation yields the set EFE-2F-4 (See Figure 3.5). Note that this set does not contain any square roots.

In all five sets above, the Einstein equations and the noninteraction condition provide six independent equations; but in all but the set EFE-NI2F-0, there are eight variables to describe the solution: A, B, p_1, p_2, ρ_1, ρ_2, f_1 (s_1) and f_2 (s_2). Hence two more equations are needed to determine the solution up to integration constants. They

$$\frac{1}{\kappa r^2}\left(1-\frac{1}{A}+\frac{rA'}{A^2}\right) = \frac{C}{Br^2}\frac{s_2-s_1}{s_1 s_2}-(p_1+p_2) \tag{3.27}$$

$$\frac{1}{\kappa r^2}\left(\frac{1}{A}-1+\frac{rB'}{AB}\right) = -\frac{C}{Br^2}(s_2-s_1)+(p_1+p_2) \tag{3.28}$$

$$\frac{r}{2\kappa A}\left[-\frac{A'}{A}+\frac{B'}{B}-\frac{rA'B'}{2AB}+\frac{rB'^2}{2B^2}+r\left(\frac{B'}{B}\right)'\right] = (p_1+p_2)r^2 \tag{3.7}$$

$$(\rho_1+p_1)\frac{\epsilon_1 s_1}{1-s_1^2}+(\rho_2+p_2)\frac{\epsilon_2 s_2}{1-s_2^2} = 0 \tag{3.29}$$

$$(\rho_1+p_1)\frac{\epsilon_1 s_1}{1-s_1^2} = \frac{C}{Br^2} \tag{3.30}$$

$$s_1'+\frac{Br^2 p_1'}{C}+\frac{1-s_1^2}{\epsilon_1 s_1}\frac{B'}{2B} = 0. \tag{3.31}$$

Figure 3.5. The set EFE-NI2F-4.

could be provided by the EoS's of the two fluids, for example.

However, as discussed in both previous chapters, this "physically-motivated approach" [9] a la Oppenheimer & Volkoff [6] is mathematically difficult: Solving even the single-fluid static case (the well-known OV equation discussed in 1.1.4) is usually not analytically possible for a given EoS. The alternative is the "mathematically motivated approach" [9] a la Tolman [5], that is, taking one of the four sets above and making choices for two of the variables so that the others can be solved for reasonably easily. For example, any choice of $A(r)$ and $B(r)$ allows calculation of $p_1 + p_2$ via Equation 3.7; which then can be put into Equations 3.27 and 3.27 to leave two equations that can be solved for s_1 and s_2 *algebraically*. Then, p_1 can be found if one can integrate Equations 3.31, the rest of the SEM variables again following algebraically. Of course, whichever variables one starts with, the acceptability question of the found pressure/density combinations, discussed in Section 1.1.5 remains.

Finally, one can take intermediate approaches, that is, one of the extra conditions might be physically motivated, the other mathematically. In the rest of this section, we present some example solutions for some (sets of) choices. We will usually choose the last set of equations, especially since the square roots have been eliminated in that set.

3.1.1. The case A=const, B=const.

For this case, Equation 3.7 gives

$$p_1 + p_2 = 0. \tag{3.32}$$

Using also this in (3.27) and (3.28) together, we get

$$\text{either } s_2 - s_1 = 0 \quad \text{or} \quad s_1 s_2 = 1 \tag{3.33}$$

- If $s_2 = s_1$, we must have

$$A = 1, \tag{3.34}$$

  hence flat spacetime. To see that this is compatible with the sources, note that the condition $s_2 = s_1$ leads to $f_1^2\epsilon_2 = f_2^2\epsilon_1$, hence to $\epsilon_1 = \epsilon_2$ (either both fluids normal or both tachyonic) and $f_1 = +f_2$ (since f_{12} also vanishes). Then Equation 3.29 means that ρ's also have opposite sign. Since the two fluids' four-velocities are the same (by virtue of $f_1 = f_2$), the total SEM tensor vanishes. Of course, in this case, at least one fluid must violate the energy conditions discussed in 1.1.5.

- If $s_1 s_2 = 1$, we have $f_1^2\epsilon_2 + f_2^2\epsilon_1 + \epsilon_1\epsilon_2 = 0$, therefore obviously ϵ_1 and ϵ_2 cannot both be +1; taking also into account that $f_i^2 > 1$ if $\epsilon_i = -1$ [see Equation 3.13], we see that they cannot be both -1 either. This means that with these starting choices we must have a normal and a tachyonic fluid each. Therefore, this case cannot be realized with actual fluids; at least one must be an effective fluid, for example an appropriate scalar field [13].
  Again, using $s_1 s_2 = 1$, we get from either of Equations 3.27 or 3.28

$$s_1 = \frac{-C_* \pm \sqrt{C_*^2 + 4}}{2} \quad \text{and} \quad s_2 = \frac{C_* \pm \sqrt{C_*^2 + 4}}{2} \tag{3.35}$$

where

$$C_* = \frac{B}{\kappa C}\left(1 - \frac{1}{A}\right). \tag{3.36}$$

By calling the normal fluid the first one, we easily get

$$p_N = p_0 = \text{const}, \quad \rho_N = \frac{CC_*}{Br^2} - p_0, \quad \text{and} \quad \rho_T = -\frac{CC_*}{Br^2} + p_0 \tag{3.37}$$

Note that if we had called the tachyonic fluid the first one, we would have gotten the same results, with the replacement $C \to -C$.

3.1.2. More B=const. solutions

What makes this choice attractive is that the 22 component of the EFE, Equation 3.7 simplifies enormously

$$-\frac{A'}{2\kappa r A^2} = (p_1 + p_2) \tag{3.38}$$

and can be substituted into the 00 and 11 components Equations 3.27 and 3.28, getting

$$\left(1 - \frac{1}{A} + \frac{rA'}{2A^2}\right) = \frac{\kappa C}{B}\frac{s_2 - s_1}{s_1 s_2} \tag{3.39}$$

$$\left(\frac{1}{A} - 1 + \frac{rA'}{2A^2}\right) = -\frac{\kappa C}{B}(s_2 - s_1) \tag{3.40}$$

Now we *could* make our second choice by equating *one* of the left-hand-sides above to some function of r, thereby determining $A(r)$, and then solve for s_1 and s_2. The simplest choice, zero, defaults to the $A = 1$ case above, and other likely choices do not seem to give simple answers. For example, if for the lhs of Equation 3.39 we choose

$C_1 r^\alpha$, we get

$$\frac{1}{A} = 1 - \frac{2C_1}{2+\alpha} r^\alpha + \frac{C_2}{r^2} \tag{3.41}$$

$$C_1 r^\alpha = \frac{\kappa C}{B} \frac{s_2 - s_1}{s_1 s_2} \tag{3.42}$$

$$\frac{C_1(\alpha - 2)}{2+\alpha} r^\alpha + \frac{2C_2}{r^2} = -\frac{\kappa C}{B}(s_2 - s_1) \tag{3.43}$$

which does not yield nice solutions for s_1 and s_2.

Alternatively, we can directly solve Equations 3.39 and 3.40 for s_1 and s_2, then search for $A(r)$ functions that simplify the result(s). The solutions for s_1 are [in terms of $f(r) = 1/A(r)$],

$$\begin{aligned} s_1 &= \frac{1}{4C\kappa(-2+2f+rf')} \left\{ B\left[4 + [4(-2+f)f - r^2 f'^2\right] \right. \\ &\pm \left. \sqrt{4(-1+f)^2 - r^2 f'^2}\sqrt{4(B^2 + 4\kappa^2 C^2 + B^2(-2+f)f) - B^2 r^2 f'^2} \right\} \end{aligned} \tag{3.44}$$

To simplify this expression, we would like to get rid of the square-roots after the $\pm$ sign. One way to do this is to require that the two square-roots are proportional to each other. It turns out that a solution is

$$f(r) = \frac{1}{A(r)} = 1 + \frac{\kappa^2 C^2 C_1}{1 - B^2 C_1^2} r^2 + \frac{C_1}{r^2} \tag{3.45}$$

yielding

$$s_1 = \frac{BC_1 \pm 1}{C\kappa r^2} \tag{3.46}$$

and then

$$s_2 = \frac{-BC_1 \pm 1}{C\kappa r^2}. \tag{3.47}$$

The calculated s_1 gives p_1 via Equation 3.31 with $B = \text{const}$,

$$p_1 = p_0 - \frac{BC_1 \pm 1}{2\kappa B r^4} \tag{3.48}$$

Then, p_2 follows via Equation 3.38,

$$p_2 = \frac{C^2 C_1}{1 - C_1^2 B^2} - p_0 + \frac{-BC_1 \pm 1}{2\kappa B r^4} \tag{3.49}$$

and ρ_1 via Equation 3.30,

$$\rho_1 = \frac{1}{\epsilon_1 B}\left[\frac{C^2\kappa}{BC_1 \pm 1} - \frac{BC_1 \pm 1}{\kappa r^4}\right] \tag{3.50}$$

and finally ρ_2 via Equation 3.29,

$$\rho_2 = \frac{C^2}{BC_1 \mp 1}\left[\frac{\kappa}{\epsilon_2 B} + \frac{C_1}{BC_1 \pm 1}\right] + p_0 + \frac{BC_1 \pm 1}{\kappa B r^4}\left(\frac{1}{\epsilon_2} + \frac{1}{2}\right) \tag{3.51}$$

3.1.3. The case of pressureless fluids

For this case, the second term of Equation 3.31 disappears, therefore it can be easily integrated to give

$$B = C_1(1 - s_1^2)^{\epsilon_1} \tag{3.52}$$

hence

$$s_1 = \sqrt{1 - (B/C_1)^{\epsilon_1}} \tag{3.53}$$

since $1/\epsilon_1 = \epsilon_1$. We can also apply Equation 3.31 to the second fluid, and leave out one of the components of the EFE, as discussed in the beginning of Chapter 2. So we also have

$$s_2 = \sqrt{1 - (B/C_2)^{\epsilon_2}} \tag{3.54}$$

Although in principle one could put the expressions for s_1 and s_2 into the Equations 3.27 and 3.28 to get two equations for $A(r)$ and $B(r)$, the squareroots make the equations very difficult to solve. We can get a trivial solution by choosing $C_1 = C_2$ and $\epsilon_1 = \epsilon_2$: Then we have $s_2 - s_1 = 0$ giving the Schwarzschild solution. This is a vacuum solution, so again the SEM tensor vanishes as in the first "solution" in 3.1.1, and one fluid violates the energy conditions.

3.1.4. Various choices for $A(r)$ and/or $B(r)$

Following the Tolman-like prescription described on p. 49, we get the equations for s_1 and s_2. Although we can get simple-looking equations, unfortunately, the solutions are not very illuminating, with long quadratic expressions inside squareroots.

- $A(r) = B(r) = \frac{r_0}{r}$:

  The virtue of these choices is that they make the term enclosed by brackets in Equation 3.9 vanish. We get

$$p_1 + p_2 = \frac{1}{2\kappa r_0 r} \tag{3.55}$$

$$\frac{s_2 - s_1}{s_1 s_2} = \frac{1}{C\kappa}\left(-\frac{3}{2} + \frac{r_0}{r}\right) \tag{3.56}$$

$$s_2 - s_1 = \frac{1}{C\kappa}\left(\frac{1}{2} + \frac{r_0}{r}\right) \tag{3.57}$$

- $A = A_0 r^{\alpha}$, $B = B_0 r^{\beta}$:

  For these choices, G_{22} becomes proportional to a single power of r. The relevant equations become

$$p_1 + p_2 = \frac{\beta^2 - \alpha\beta - 2\alpha}{4\kappa A_0} r^{-2-\alpha} \tag{3.58}$$

$$\frac{s_2 - s_1}{s_1 s_2} = \frac{4A_0 r^\alpha + \beta^2 - \alpha\beta + 2\alpha - 4}{4\kappa A_0 C} B_0 r^{\beta-\alpha} \tag{3.59}$$

$$s_2 - s_1 = \frac{4A_0 r^\alpha + \beta^2 - \alpha\beta - 2\alpha - 4\beta - 4}{4\kappa A_0 C} B_0 r^{\beta-\alpha} \tag{3.60}$$

There do not seem to be any α or β values that significantly simplify these.

- $B = (r_0/r)^2$:

  Note that we did not specify $A(r)$ yet. As in Section 2.3, this choice for $B(r)$ simplifies G_{22} considerably. We obtain

$$p_1 + p_2 = \frac{1}{\kappa r^2 A} \tag{3.61}$$

$$\frac{s_2 - s_1}{s_1 s_2} = \frac{r_0^2}{C\kappa r^2 A^2}(A^2 + rA') \tag{3.62}$$

$$s_2 - s_1 = \frac{r_0^2}{C\kappa r^2 A}(A + 2) \tag{3.63}$$

  Even $A =$ constant does not give simple solutions; setting $A^2 + rA' = 0$ seems to be meaningless, even though it can be solved.

- A final note:

  Let us give names to the right-hand-sides of the two equations containing s_1 and s_2:

$$\frac{s_2 - s_1}{s_1 s_2} = Z(A, B, r) \tag{3.64}$$

$$s_2 - s_1 = Y(A, B, r) \tag{3.65}$$

  then we have

$$s_1 = \frac{-YZ \pm \sqrt{YZ(4 + YZ)}}{2Z} \tag{3.66}$$

  hence a simple solution may be obtained if it can be arranged that $Y(A, B, r)Z(A, B, r)$ = constant. This was not possible with the choices in this subsection.

3.2. The singular-matrix case

We have discussed in the beginning of the last section that the matrix M defined in Equation 3.4 can be nonsingular only for one j. Therefore in this section we are considering the case that is singular for all j. Hence the determinant vanishes:

$$\det M = (\rho_1 + p_1)(\rho_2 + p_2)(u_0 v_j - v_0 u_j) = 0 \tag{3.67}$$

This of course leads to three cases of each factor vanishing. But since the fluids can be relabeled, it is enough to consider the vanishing of only one $(\rho_1 + p_1)$ term; similarly we will not consider cases that are mirror images of cases already considered (with respect to $1 \leftrightarrow 2$).

Also, the vanishing of the matrix M constitutes one condition. So in each of the cases discussed below, another condition will need to be specified to uniquely fix the solution, up to integration constants.

3.2.1. The case $\rho_1 + p_1 = 0$

In this case, we have $p_1 = -\rho_1 = p_0 =$ constant by the nonineraction equation. Also, consideration of Equation 3.2 shows that $(\rho_2 + p_2)v_0 v_i = 0$, hence we have three possibilities:

(i) $\rho_1 + p_1 = 0,\ \rho_2 + p_2 = 0$
Then, we have

$$T_{\mu\nu} = (p_1 + p_2) g_{\mu\nu} \tag{3.68}$$

so this case is equivalent to a single cosmological constant fluid, hence the solution will be the well-known Köttler, (a.k.a. Schwarzschild-de Sitter) metric. In other words, p_2 and ρ_2 are also constant. The arbitrariness discussed on page 56 is in how the two fluids make up the equivalent fluid.

(ii) $\rho_1 + p_1 = 0,\ v_0 = 0,\ (\rho_2 + p_2) \neq 0$

Now considering (3.3), that is, $(\rho_2 + p_2)v_i v_j = 0$, we see that at most one v_i can be non-zero. Since (1.9) implies $v_3^2 = v_2^2 \sin^2\theta$, the non-zero component must be v_1. From normalization, we have $v_1^2 = A(r)$. Of course, this v_μ means that the second fluid is tachyonic, which in some sense takes us back to Chapter 2. Using Einstein Equations, we find an OV-like equation

$$\rho_2' + 2p_2' = -\frac{[-3F + 4r + \kappa r^3(\rho_2 + 2p_2 + p_0)]}{2r(r-F)}(\rho_2 + p_2) \qquad (3.69)$$

where

$$F(r) = -\kappa\int(p_0 + p_2)r^2 dr, \qquad A(r) = \frac{r}{r - F(r)} \qquad (3.70)$$

from $G_{00} = \kappa T_{00}$.

One simple solution to Equation 3.69 is $\rho_2 + 2p_2 + p_0$ =constant= $-C_1$. This solution implies

$$F(r) = \frac{4r - C_1\kappa r^3}{3} \quad \Rightarrow \quad A(r) = \frac{3}{C_1\kappa r^2 - 1} \qquad (3.71)$$

and

$$B(r) = C_2 r^{-4}, \qquad (3.72)$$

so that C_1 must be positive and the solution is a valid spacetime only for $r > r_C$. Of course, this is only one of the possible solutions of Equation 3.69. For another EoS of the second fluid, another solution can be found in principle; or one could specify another condition and use it in the EFE.

(iii) $\rho_1 + p_1 = 0,\ v_i = 0,\ (\rho_2 + p_2) \neq 0$

$v_i = 0$ holds for all i, so again this case really belongs to Chapter 2. From normalization, we must have $v_0^2 = B(r)$, and calculating the nonvanishing elements of SEM tensor gives,

$$T_{00} = (\rho_2 - p_0)B(r) \tag{3.73}$$

$$T_{ii} = (p_0 + p_2)g_{ii} \tag{3.74}$$

We see that fluid 1 and 2 together act like one fluid with an unspecified equation of state, therefore any known static one-fluid solution could work here.

3.2.2. The case $u_0 v_j - v_0 u_j = 0$

Here, unless $u_0 = 0$, we can solve for v_j in terms of u_0, v_0 and u_j. Therefore we must consider the cases of vanishing and nonvanishing of u_0 separately. But the case of vanishing u_0 will apparently also have two subcases: vanishing v_0 and vanishing u_j; but u_0 and u_j cannot vanish together; so we really have two cases. We will start with the generic case of nonvanishing u_0.

(i) Nonvanishing u_0:

We have

$$v_j = \frac{v_0}{u_0} u_j. \tag{3.75}$$

Putting this into Equation 3.2, we get

$$\frac{u_i}{u_0}\left[(\rho_1 + p_1)u_0^2 + (\rho_2 + p_2)v_0^2\right] = 0 \tag{3.76}$$

therefore either u_i or the term in the square brackets must vanish.

But if the u_i vanish, so do v_i, and we have the case of two normal static fluids; hence the two fluids must be identical, and identically distributed, a trivial solution.Hence we take square bracket to vanish. On the other hand, we can also use the relation (3.75) for T_{ij}:

$$T_{ij} = (\rho_1 + p_1)u_i u_j + (\rho_2 + p_2)\frac{v_0^2}{u_0^2}u_i u_j + (p_1 + p_2)g_{ij} \qquad (3.77)$$

where the first two terms vanish because square bracket above does. Also considering T_{00}, we conclude that here also, the two fluids together act like a cosmological constant.

(ii) Vanishing u_0, vanishing v_0:

Let us rewrite Equation 3.3:

$$T_{ij} = (\rho_1 + p_1)u_i u_j + (\rho_2 + p_2)v_i v_j = 0, \quad i \neq j \qquad (3.3)$$

Since u_0 and v_0 vanish, at least one of the u_i must be nonvanishing (similarly for v_i). Hence, we could divide by that u_i, if (ρ_1+p_1) is nonvanishing. But if (ρ_1+p_1) does vanish, either so must (ρ_2+p_2), or only one v_i must be nonvanishing. If both $(\rho + p)$'s do vanish, we are back to the cosmological constant case. Discarding that trivial case, then,

$$v_j = -\frac{(\rho_1 + p_1)u_i u_j}{(\rho_2 + p_2)v_i} \qquad (3.78)$$

for one i, at least (but $i \neq j$). Calling the value of that i as k, (3.3) becomes

$$T_{ij} = \frac{\rho_1 + p_1}{(\rho_2 + p_2)v_k^2}\left[(\rho_2 + p_2)v_k^2 + (\rho_1 + p_1)u_k^2\right]u_i u_j = 0, \quad i \neq j \qquad (3.79)$$

again we have the possibilities of the square bracket vanishing or only one nonzero u_i. The vanishing of the square bracket will bring us again to the cosmological constant case (the diagonal terms will be equal to the square bracket $+(p_1+p_2)g_{kk}$; $i = j = k$, to the terms in the above equation $+(p_1 + p_2)g_{ii}$ for other terms, hence the u_μ contributions will vanish, for T_{00} there will be no u_μ contribution either); therefore we focus on the only one nonzero u_i case.

4. SUMMARY, CONCLUSIONS AND PROSPECTS

In this work, we have started with the idea of "radially imperfect fluid" solutions [11] for static spherically symmetric spacetimes. The rationale for the introduction of this type of fluid is the possible nonminimal coupling between a fluid and spacetime curvature, resulting in modification of the fluid's stress-energy-momentum (SEM) tensor in response to the anisotropy of spacetime. It is also reported that these fluids' SEM tensor coincides with the most general SEM tensor possible for these spacetimes [10]. We do however, point out that this SEM tensor can also be thought of as a superposition of the SEM tensors of a normal and a tachyonic fluid where only one component of the four-velocity is nonzero for each fluid, and the fluids are not interacting. It was shown that single-fluid static spherically symmetric spacetimes can be classified into four types, static vs. dynamic and with normal vs. tachyonic fluid sources [9] (while actual fluids of course cannot have tachyonic four-velocity, the effective fluids [4] corresponding to scalar fields can [13]).

We in Chapter 2 rederived in this work some of the solutions found in [11], streamlining the results so that they can be understood better and for the complete range of their parameters, pointed out the above correspondence and found the distributions of normal and tachyonic fluids whose superpositions can serve as source for these spacetimes. We also argued that these spacetimes do not contain or represent black holes, contrary to the interpretation of the author of that work. A black hole must feature a horizon, which is a null surface separating a static and a dynamic region; these solutions do not contain dynamic regions within their domains of validity.

Then we found some other solutions of the same type. Ideally, one would solve the Einstein Field Equations (EFE) for an equation of state (EoS) for each fluid, but it is well-known that this is usually not analytically possible for even a single fluid. If the EoS's are left unspecified, the general solution will contain two arbitrary functions, but even in that vast space of solutions, it is hard to find those where all the metric and SEM tensor variables (pressures and densities) can be analytically written down.

We found about 20 solutions, and briefly interpreted them. Most of them feature singularities and/or boundaries.

Finally, in Chapter 2 we dropped the condition that only one component of the four-velocity should be nonzero for each fluid. We wrote down the six equations (four coming from the EFE now, and two from the noninteracting property of the two fluids), defined what we think are the analytically most suitable variables for the radial components of the four-velocities of the fluids, processed the equations for hopefully improved solvability; and then found some analytic solutions. In this case too, without th EoS's, the general solution features two arbitrary functions, but we were able to find much fewer solutions (3-4) except those that reduce to the case covered in Chapter 2, or even to much more trivial ones.

To summarize, we formulated equations suitable for search of static spherically symmetric solutions of Einstein's equations with two perfect fluids as source, and found about twenty analytical solutions. Future work that can be done includes

- finding more such solutions,
- interpreting each solution in more detail, searching for singularity-free ones (both in terms of metric- and pressures/densities),
- finding solutions starting from physically reasonable EoS's (which should facilitate finding singularity-free solutions),

and in further future, maybe

- discussing the case of *nonstatic* spherically symmetric spacetimes.

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