MILNOR OPEN BOOKS AND MILNOR FILLABLE CONTACT STRUCTURES

by

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ABSTRACT

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It was proven by Caubel, Némethi, Popescu-Pampu [1] in 2004 that any 3-dimensional Milnor fillable manifold inherits a unique contact structure from any of its fillings up to contactomorphism. The proof runs as follows: in the contact boundary of a germ of an isolated singularity, to every holomorphic function f with an isolated singularity, a so-called Milnor open book decomposition is assigned. Then it is shown that for any choice of f, this open book is compatible with the contact structure filled by the singularity. The key fact is that in a Milnor fillable 3-manifold, there is a Milnor open book which is determined by the topology of the manifold. In other words, diffeomorphic Milnor fillable 3-manifolds admit isomorphic open book decompositions compatible with the filled contact structures on them. Following Giroux's result, it is deduced that the Milnor fillable contact structures on a Milnor fillable 3-manifold are contactomorphic. In this thesis, we give this proof in detail.

ÖZET

MİLNOR AÇIK KİTAP AYRIŞMALARI VE MİLNOR DOLUMLU KONTAK YAPILAR

2004 yılında Caubel, Némethi ve Poprescu-Pampu [1] 3 boyutlu Milnor dolumlu bir manifoldun iki farklı yalıtılmış tekillikten alabileceği kontak dağılımların kontaktomorfik olduklarını göstermiştir. İspat şu şekilde yapılmıştır: öncelikle bir yalıtılmış tekilliğin kontak kenarı içinde Milnor açık kitap ayrışması kurulmuş ve bu ayrışmanın yalıtılmış tekillikten gelen kontak dağılımla uyumlu olduğu gösterilmiştir. En önemli iddia ise bu Milnor açık kitap ayrışmasının 3 boyutlu Milnor dolumlu bir manifold için manifoldun topolojisi tarafından belirlenebileceği. Bu sayede difeomorfik 3 boyutlu Milnor dolumlu manifoldlar içinde, farklı tekilliklerden gelen kontak dağılımları destekleyen izomorfik açık kitaplar olduğu gösterilmiştir. Bu yapılara Giroux'nun 3 boyutlu manifoldlar üzerinde açık kitaplar ve kontak dağılımlar arasındaki eşleşme üzerine yaptığı çalışma uygulandığında, esas teorem ispatlanmıştır. Bu tezde, bu ispat detaylı hesapları ile birlikte veriliyor.

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1. INTRODUCTION

Let V be an *analytic variety*, that is the zero locus $f^{-1}(0)$ of a holomorphic function $f : \mathbb{C}^k \to \mathbb{C}^m$ $(k > m)$. If V cannot be written as a union of two disjoint non-empty analytic varieties, then it is called *irreducible*. We say that $x \in V$ is an *isolated singularity* if there is a neighbourhood $X \subseteq V$ of x such that the rank of the complex matrix $df(p)$ is smaller than m for $p = x$ and equals m for $p \in X \setminus \{x\}$. In this case, set $n := k - m$. Then V is said to be an n-dimensional analytic variety with the isolated singularity x. Note also that $X \setminus \{x\}$ is an n-dimensional complex manifold (see [2], p. 11). In particular, when $n = 2$, x is called a *surface singularity*. When $\text{rank}[df(x)] = m$ as well, x is called a *non-singularity*.

If (X, x) is a germ of an n-dimensional irreducible complex analytic variety with an isolated singularity at x, then there exists an analytic embedding $(X, x) \subseteq (\mathbb{C}^N, 0)$ for N large enough, and X intersects with the sphere $S_\epsilon \subset \mathbb{C}^N$ centred at the origin with radius $\sqrt{\epsilon}$ transversally for $\epsilon > 0$ small enough. This intersection defines a $2n-1$ dimensional closed, oriented, smooth manifold M in X with a contact structure ξ on it. The diffeomorphism type of M does not depend on the embedding and the sufficiently small ϵ and it is called the *abstract boundary* (or the *link*) of the isolated singularity (X, x), denoted by $M(X)$. Moreover, in [3], A. N. Varčenko shows that the contact isotopy type of (M, ξ) does not depend on the embedding and the sufficiently small ϵ as well. This isotopy type, denoted by $(M(X), \xi(X))$, is called the *contact boundary* of (X, x).

Definition 1.1. *A closed, connected, oriented, smooth manifold M is called* Milnor fillable *if there exists a germ (X, x) of an irreducible complex analytic variety with an isolated singularity x such that M is diffeomorphic to the abstract boundary $M(X)$. The germ (X, x) is called a* Milnor filling *of M.*

Remark 1.1. If M is equipped with a contact structure ξ a priori, we use the same terminology and say that (M, ξ) is *Milnor fillable* if it is contactomorphic to the contact boundary $(M(X), \xi(X))$. In this case, ξ is said to be a *Milnor fillable contact structure.*

It is possible that two Milnor fillable manifolds are diffeomorphic but not contactomorphic. For instance, S^5, as a contact boundary of a 3-dimensional non-singularity, admits infinitely many different Milnor fillable contact structures up to contactomorphism (see [4]). However, the main theorem of this thesis guarantees that this is impossible for the contact boundary of an isolated surface singularity.

Theorem 1.1. [1] *Any Milnor fillable 3-manifold admits a unique Milnor fillable contact structure up to contactomorphism.*

In Chapter 2, first we give definitions for notions of contact structures and open book decompositions and their compatibility on smooth manifolds. We review the work of Giroux which states that on a closed, oriented 3-manifold, two contact structures are isotopic if they are compatible with the same open book. Then we list notions and facts from algebraic geometry which help us to make observations on surface singularities and their abstract boundaries. The key fact is that the abstract boundary of an isolated surface singularity is obtained as the plumbing of some circle bundles according to the minimal good resolution graph of the singularity.

We start Chapter 3 by giving the justification of the claims mentioned above under the condition that (X, x) is not embedded but immersed in $(\mathbb{C}^N, 0)$. For any given holomorphic function f on (X, x) which has an isolated singularity and vanishes at x, we construct an open book decomposition $(N(f), \theta(f))$ in $M(X)$ and claim that its isotopy type depends on neither the immersion nor the sufficiently small ϵ. Then we prove that the contact structure $\xi(X)$ and the open book decomposition $(N(f), \theta(f))$ are compatible on the contact boundary $M(X)$. Note that in Chapter 3, we talk about the Milnor fillable manifolds of any (odd) dimension.

In Chapter 4, we restrict our attention to surface singularities. Considering the contact boundary $M(\mathcal{S})$ of a surface singularity $(\mathcal{S}, 0)$ as a plumbed manifold and using some tools from algebraic geometry, in $M(\mathcal{S})$, we construct a horizontal open book determined by an appropriate holomorphic function f on (X, x) as above. We claim that this horizontal open book is not just isomorphic to the open book $(N(f), \theta(f))$

but also determined by the topology of $M(\mathcal{S})$. Hence, in diffeomorphic Milnor fillable 3-manifolds, we obtain isomorphic Milnor open book decompositions which are compatible with the Milnor fillable contact structures. Theorem 1.1 follows from a combination of the work of Milnor and Giroux.

2. PRELIMINARIES

This chapter is devoted to introductory notions and results in the literature of the contact geometry and algebraic geometry. Interested readers are suggested to see [5, 9, 12, 14] as general references for the subjects of this chapter.

2.1. Contact Structures

Definition 2.1. *Let M be a $2n+1$ dimensional smooth manifold with a $2n$ dimensional smooth hyperplane distribution ξ in the tangent bundle TM. If ξ satisfies the maximal non-integrability condition, then ξ is called a* contact distribution (*or* contact structure) *on M, and the pair (M, ξ) is said to be a* contact manifold.

Note that an equivalent condition for ξ to be a contact distribution is the following: For any 1-form α which locally defines the distribution ξ, i.e. $\ker \alpha = \xi$, we have $\alpha \wedge (d\alpha)^n \neq 0$ at any point p in M. In this case, α is said to be a *contact form* (locally). If the smooth manifold M is oriented, the contact form α is said to be *positive* if the volume form $\alpha \wedge (d\alpha)^n$ defines the orientation of M.

Observe that two 1-forms α_1 and α_2 on a smooth manifold M have the same kernel if and only if $\alpha_1 = \lambda \alpha_2$ for some smooth $\lambda : M \to \mathbb{R} \setminus \{0\}$.

Example 2.1. Consider the following:

(i) Let $(x_1, y_1, \ldots, x_n, y_n, z)$ be the coordinates on $\mathbb{R}^{2n+1}$. Then $\alpha = dz + \sum_{i=1}^n x_i dy_i$ is a contact form since $\alpha \wedge (d\alpha)^n$ is a volume form on $\mathbb{R}^{2n+1}$. The distribution $\xi = \ker \alpha$ is called the *standard contact structure* on $\mathbb{R}^{2n+1}$. Moreover, α is said to be the *standard contact form* on $\mathbb{R}^{2n+1}$.

(ii) Next we give an example of a contact structure on S^3 considering it embedded in $\mathbb{R}^4$. Let (x_1, x_2, x_3, x_4) be the coordinates on $\mathbb{R}^4$, and let α_0 be the restriction of the 1-form $\alpha = -x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4$ to S^3. Consider the smooth vector fields

$$\begin{aligned}
X_1 &:= -x_2\partial_{x_1} + x_1\partial_{x_2} - x_4\partial_{x_3} + x_3\partial_{x_4},\\
X_2 &:= \quad x_4\partial_{x_1} + x_3\partial_{x_2} - x_2\partial_{x_3} - x_1\partial_{x_4},\\
X_3 &:= -x_3\partial_{x_1} + x_4\partial_{x_2} + x_1\partial_{x_3} - x_2\partial_{x_4}.
\end{aligned}$$

on $\mathbb{R}^4$ which restrict to smooth tangential vector fields on S^3. An easy calculation shows that

$$\alpha_0 \wedge d\alpha_0(X_1, X_2, X_3) = 2.$$

Therefore, α_0 is a contact form on S^3. Moreover, $\alpha_0(X_2) = \alpha_0(X_3) = 0$ and so the contact distribution $\xi = \ker \alpha_0$ is generated by X_2 and X_3.

Definition 2.2. *Let α be a contact form on a smooth manifold M. The smooth vector field R_α on M which satisfies the equations*

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, v) = 0$ *for any tangent vector field v of M*

is called the Reeb vector field *associated to α.*

Lemma 2.1. *The Reeb vector field does exist and is unique.*

Proof. Let $\dim M = 2n + 1$. For a point $p \in M$ set $\xi_p = \ker \alpha_p$. Since α is contact, $(d\alpha)^n \neq 0$ on ξ_p and so $d\alpha|_{\xi_p}$ is a symplectic form on ξ_p. Therefore, the kernel, say L, of $d\alpha_p$ in T_pM does not lie in ξ_p and is 1-dimensional. Obviously, a tangent vector R_p at p solves the second equation above if and only if $R_p \in L$. Moreover, there exists a unique tangent vector R_p in L such that $\alpha_p(R_p) = 1$. Now define

$$R_\alpha : M \to TM \text{ with } p \mapsto R_p.$$

This is the unique vector field on M which solves the given equations and its smoothness follows from the fact that α is a smooth form. □

Example 2.2. Following are the Reeb vector fields associated to contact forms given in Example 2.1:

(i) The Reeb vector field associated to the standard contact form on $\mathbb{R}^{2n+1}$ is ∂_z.

(ii) The vector field X_1 on S^3 is the Reeb vector field associated to the contact form α_0 on S^3.

Definition 2.3. *Two contact manifolds* (M_1, ξ_1), (M_2, ξ_2) *are said to be* contactomorphic *if there exists a diffeomorphism* $f : M_1 \to M_2$ *such that* $f_*(\xi_1) = \xi_2$*. If* α_1 *and* α_2 *are two contact forms defining* ξ_1 *and* ξ_2 *respectively, then this amounts to saying that* $f^*(\alpha_2)$ *and* α_1 *have the same kernel* ξ_1*, i.e.* $f^*(\alpha_2) = \lambda\alpha_1$ *for some smooth function* $\lambda : M \to \mathbb{R} \setminus \{0\}$*. In this case,* f *is said to be a* contactomorphism.

Notice that every diffeomorphism $f : M_1 \to M_2$ induces a contact distribution $f_*(\xi_1)$ on M_2, however, for f to be a contactomorphism this induced contact distribution must coincide with ξ_2.

Definition 2.4. *Two contact distributions* ξ_0*,* ξ_1 *on a smooth manifold* M *are said to be* isotopic *if there exists a smooth family* $\{f_t : M \to M |\ t \in [0, 1]\}$ *of diffeomorphisms such that* $f_0 = id$ *and* $f_{1*}(\xi_0) = \xi_1$*. In this case, the family* $\{f_t |\ t \in [0, 1]\}$ *is called a* contact isotopy.

Theorem 2.1 (Gray's Theorem)**.** *Let* $\{\xi_t |\ t \in [0, 1]\}$ *be a smooth family of contact distributions on a smooth manifold* M*. Then there exists a contact isotopy* $\{f_t : M \to M |\ t \in [0, 1]\}$ *on* M *such that* $f_{t*}(\xi_0) = \xi_t$*.*

For a proof of Gray's Theorem, see Section 2.2 in [5].

2.2. Open Book Decompositions

Definition 2.5. *Let M be an n-dimensional, closed, connected, oriented, smooth manifold and N be a closed, oriented submanifold of M with codimension 2. If $\theta : M \setminus N \to S^1$ is a smooth, locally trivial fibration which coincides with the angular coordinate in a tubular neighbourhood $N \times D^2$ of N in M, then (N, θ) is called an* (embedded) open book decomposition *in M. In this case, N is called the* binding *and every fibre of θ a* page *of the open book. Note that in an open book decomposition, every page admits the binding as its boundary. If the orientation of N coincides with the co-orientation induced by $d\theta$ on N from M, then the open book (N, θ) is called* compatible *with the orientations.*

Theorem 2.2 (Alexander, [8])**.** *Let M be a 3-dimensional closed, connected, oriented, smooth manifold. Then M is diffeomorphic to an abstract open book decomposition $M(h) := F(h) \cup_{id} (\partial F \times D^2)$, where:*

- *F is an oriented surface with boundary $\partial F = S^1$,*
- *$h : F \to F$ is an orientation preserving diffeomorphism which restricts to identity in a collar neighbourhood of ∂F, and*
- *$F(h) := (F \times [0, 1])/(x, 0) \sim (h(x), 1)$.*

Remark 2.1. (a) Observe that $M(h)$ is obtained by gluing $F(h)$, which is called the *mapping torus*, and the solid torus $\partial F \times D^2$ along their boundary using the identity map to identify $\partial F(h) = S^1 \times \partial F$ with $\partial(\partial F \times D^2) = \partial F \times S^1$.
(b) Alexander's theorem assigns an embedded open book decomposition (N, θ) to every such 3-manifold M by identifying it with the abstract open book $M(h)$. In this decomposition, the binding N corresponds to $\partial F \times \{0\} \subset \partial F \times D^2$ and every page is diffeomorphic to the interior of F.

Definition 2.6. *Let M and M' be smooth manifolds with the embedded open book decompositions (N, θ) and (N', θ') respectively. Then (N, θ) is said to be* isomorphic *to (N', θ') if there exists a diffeomorphism $f : (M, N) \to (M', N')$ which preserves* *pages and their co-orientations. Formally, this amounts to saying that there exists an orientation preserving diffeomorphism* $\phi : S^1 \to S^1$ *such that the following diagram commutes.*

$$\begin{array}{ccc} M \setminus N & \xrightarrow{f} & M' \setminus N' \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta'} \\ S^1 & \xrightarrow{\phi} & S^1 \end{array}$$

Remark 2.2. Any given open book decomposition of a closed, connected, oriented, smooth 3-manifold M is isomorphic to an abstract open book $M(h)$. For further information see [5], Section 4.4.

2.3. Open Book Decompositions versus Contact Forms

Definition 2.7. *Let* M *be a smooth manifold with an open book decomposition* (N, θ). *A positive contact form* α *on* M *is said to be* compatible *with the open book* (N, θ) *if* α *satisfies the following properties:*

- α *is a positive contact form on the binding* N,
- $d\alpha$ *is a positive symplectic form on each page of the open book.*

A contact structure ξ *on* M *is called* compatible *with* (N, θ) *if* ξ *admits a defining contact form which is compatible with* (N, θ).

Next we list some important results from the literature concerning the compatibility of open books and contact structures on 3-manifolds.

Lemma 2.2 (Giroux, [6]). *Let* M *be an oriented, smooth manifold with a positive contact form* α. *Suppose that in* M, *there exists an open book decomposition* (N, θ) *and a tubular neighbourhood* $V \cong N \times D^2$ *of* N *such that:*

- θ *is the normal angular coordinate in* V,

- *for any $z \in D^2$, α restricts to a contact form on the submanifold $N \times \{z\}$ of V,*
- *$d\alpha$ restricts to a positive symplectic form on each fibre F' of θ in $M \setminus intV$.*

Then the open book (N, θ) and the contact structure $\xi = \ker \alpha$ are compatible.

Theorem 2.3 (Thurston-Winkelnkemper)**.** *Let Y be a closed, oriented 3-manifold with an open book decomposition (N, θ). Then there exists a contact form α on Y which is compatible with (N, θ).*

For a proof of Theorem 2.3 see [7], page 140. The following lemma is due to Giroux and we give a proof for it since our main theorem will follow from it.

Lemma 2.3 (Giroux)**.** *Let Y be a 3-dimensional closed, connected, oriented, smooth manifold. If ξ_0 and ξ_1 are two contact structures on Y both compatible with an open book decomposition (N, θ) in Y, then ξ_0 and ξ_1 are isotopic.*

Proof. Following the Alexander's theorem, (N, θ) is isomorphic to an abstract open book $Y(h)$. Let (θ, r, ϕ) be the coordinates in a tubular neighbourhood of the binding in $Y(h)$, and let α_0 and α_1 be the positive defining forms of ξ_0 and ξ_1 respectively, which are compatible with (N, θ). Then they both are positive contact forms on the binding, that is for $i = 0, 1$; $\alpha_i(\partial_\theta) > 0$ on the binding $(\theta, 0, 0)$. Note that we can denote the binding by $\{r = 0\}$ as well.

Now fix a smooth real-valued function k of r such that $k(0) = 0$, $k(r) = r^2$ near 0, $k(r) = 1$ away from 0, and $k'(r) \geq 0$ everywhere. Now for any $c \in \mathbb{R}_{\geq 0}$, $t \in [0, 1]$, and $i = 0, 1$; define

- $\alpha_{i,c} := \alpha_i + ck(r)d\phi$, and
- $\alpha_{t,c} := (1 - t)\alpha_{0,c} + t\alpha_{1,c}$.

Since ϕ represents the angular coordinate near the binding, forms above are defined in a tubular neighbourhood of the binding. However, the angular coordinate ϕ coincides with θ in such a neighbourhood. So we can extend $d\phi$ and the forms above outside of this tubular neighbourhood by setting $d\phi = \theta^*(\tau)$, where τ is the orientation form on the circle.

Observe that for any $c \geq 0$, $i = 0, 1$; $d\alpha_{i,c} = d\alpha_i + ck'(r)dr \wedge d\phi$. Therefore,

$$\alpha_{i,c} \wedge d\alpha_{i,c} = \alpha_i \wedge d\alpha_i + ck'(r)\alpha_i \wedge dr \wedge d\phi + ck(r)d\phi \wedge d\alpha_i.$$

First of all, since α_i is a positive contact form on Y, $\alpha_i \wedge d\alpha_i > 0$ on the oriented basis. Note that α_i defines the orientation on the binding. So $\alpha_i \wedge dr \wedge d\phi$ is an orientation form on Y. Since c and $k'(r)$ are non-negative, it follows that $ck'(r)\alpha_i \wedge dr \wedge d\phi \geq 0$ on the oriented basis. Finally, since $d\alpha_i$ is an orientation form on the pages, it follows that $d\phi \wedge d\alpha_i$ is an orientation form on Y. The fact that c and $k(r)$ are non-negative implies that $ck(r)d\phi \wedge d\alpha_i \geq 0$ on the oriented basis as well. Therefore, $\alpha_{i,c} \wedge d\alpha_{i,c}$ is a volume form on Y, and so $\alpha_{i,c}$ is a (positive) contact form on Y for any choice of i and c. By construction, for any $c \geq 0$, $\alpha_{i,c}$ is also isotopic to α_i.

A similar calculation shows that there exists a $c \geq 0$ large enough for which $\alpha_{t,c} \wedge d\alpha_{t,c}$ is non-vanishing on Y for any choice of t. Fix such a $c \geq 0$. The smooth family $\{\alpha_{t,c} : t \in [0,1]\}$ of contact forms defines a contact isotopy between $\alpha_{0,c}$ and $\alpha_{1,c}$. Since α_0 is isotopic to $\alpha_{0,c}$, and $\alpha_{1,c}$ is isotopic to α_1, it follows that α_0 and α_1 are isotopic. Therefore, ξ_0 and ξ_1 are isotopic. $\square$

In particular, to check that two contact structures on a 3-manifold M are contactomorphic, it is sufficient to show that they are compatible with the same open book decomposition in M.

2.4. Sheaves and Cohomology

Definition 2.8. *Let X be a topological space. A* sheaf $\mathcal{F}$ *on X assigns to each open set $U \subseteq X$ a group $\mathcal{F}(U)$, whose elements are called* sections, *and to each pair of open subsets $U, V \subseteq X$ with $U \subseteq V$ a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the* restriction map *and denoted by $\cdot|_U$, such that*

- *For open sets $U \subseteq V \subseteq W$ of X, $r_{W,V} = r_{V,U} \circ r_{W,V}$,*
- *For open sets $U, V \subseteq X$ and sections $\sigma \in \mathcal{F}(U)$, $\mu \in \mathcal{F}(V)$ such that $\sigma|_{U \cap V} = \mu|_{U \cap V}$, there exists a section $\rho \in \mathcal{F}(U \cup V)$ for which $\rho|_U = \sigma$ and $\rho|_V = \mu$,*
- *For open sets $U, V \subseteq X$, if $\sigma \in \mathcal{F}(U \cup V)$ with $\sigma|_U = \sigma|_V = 0$, then $\sigma = 0$.*

Example 2.3. The most basic examples of sheaves on a complex manifold A are defined on an open subset U of A as follows:

(i) $\mathcal{O}(U)$, the additive group of holomorphic maps on U;

(ii) $\mathcal{O}^*(U)$, the maltiplicative group of non-zero holomorphic maps on U;

(iii) $\mathcal{M}(U)$, the additive group of meromorphic maps on U;

(iv) $\mathcal{M}^*(U)$, the multiplicative group of not identically 0 meromorphic maps on U;

(v) $\mathcal{J}_V(U)$, the additive group of holomorphic functions on U vanishing on $U \cap V$, where V is an analytic variety in A.

If p is an element of complex the complex manifold A, we define the *stalk* of holomorphic functions on A at p by $\mathcal{O}_{A,p} := \bigcup_U \mathcal{O}(U)$, where the union is taken over all open subsets of A containing p.

Definition 2.9. *Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X with the restriction maps r and r' respectively.* A map of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ *is a collection of group homomorphisms $\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U) |\ U \subseteq X \text{ open}\}$ such that for any open sets $U \subseteq V \subseteq X$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\alpha_V} & \mathcal{G}(V) \\ {\scriptstyle r_{V,U}}\downarrow & & \downarrow{\scriptstyle r'_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \end{array}$$

The *kernel* of a map α of sheaves is defined as $\ker(\alpha)(U) := \{\ker \alpha_U\}$ on each open set $U \subseteq X$, which is a sheaf on X as well.

Above we define the kernel of a map of sheaves in a very natural way. However, for the *cokernel* of a map of sheaves similar set up may not work. In other words, if we set $\mathrm{coker}(\alpha)(U) := \mathrm{coker}(\alpha_U)$ for any open set $U \subseteq X$, then coker may not be a sheaf on X (see [15], p. 36). Instead, we define the *cokernel* of a map $\alpha : \mathcal{F} \to \mathcal{G}$ of sheaves on a topological space X as follows: $\mathrm{coker}(\alpha)(U)$ is given by an open cover $\{U_\beta\}$ of U with sections $\sigma_\beta \in \mathcal{G}(U_\beta)$ such that $\left(\sigma_\beta\big|_{U_\beta \cap U_\gamma} - \sigma_\gamma\big|_{U_\beta \cap U_\gamma}\right) \in \alpha_{U_\beta \cap U_\gamma}\left(\mathcal{F}(U_\beta \cap U_\gamma)\right)$ for any β, γ.

Definition 2.10. *A sequence $\mathcal{E} \xrightarrow{\alpha_1} \mathcal{F} \xrightarrow{\alpha_2} \mathcal{G}$ of maps of sheaves is said to be* short exact *if $\mathcal{E} = \ker(\alpha_1)$ and $\mathcal{G} =$coker(α_2). In this case, $\mathcal{E}$ is called a* subsheaf *of $\mathcal{F}$ and $\mathcal{G}$ is called the* quotient sheaf *of $\mathcal{F}$ by $\mathcal{E}$, denoted by $\mathcal{F}/\mathcal{E}$.*

Example 2.4. If V is an analytic variety in a complex manifold A, then the sheaf $\mathcal{O}_V$, with extension by 0, can be considered as a sheaf on A. Then $\mathcal{J}_V \xrightarrow{i} \mathcal{O}_A \xrightarrow{r} \mathcal{O}_V$ is a short exact sequence of sheaves, where i is the inclusion and r is the restriction map.

Next we list two important observations on the cohomology of sheaves. For further information see Chapter 3 in [15]; see also Section B.9 in [14]. Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be sheaves on a topological space X. Then

(i) $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.
(ii) Any short exact sequence $0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$ of sheaves induces a long exact sequence of cohomologies:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{E}) &\longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \\ \longrightarrow H^1(X, \mathcal{E}) &\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \\ \longrightarrow \dots & \end{aligned}$$

2.5. Divisors and Their Intersection Numbers

Let A be a complex manifold of complex dimension n. A subset V of A is said to be an *analytic hypersurface* if around every point $p \in V$ there exists a neighbourhood $U \subseteq A$ with a holomorphic function f on U such that $f^{-1}(0) = U \cap V$. An analytic hypersurface is said to be *irreducible* if it cannot be written as a union of two disjoint non-empty analytic hypersurfaces.

Remark 2.3. Note that in the above definition the holomorphic function $f \in \mathcal{O}_{A,p}$ can be chosen so that for any other function g whose zero locus is $U \cap V$, f divides g. Such an f is said to be the *local defining function* of V near p and it is unique upto multiplication by a holomorphic function non-vanishing at p. Moreover, V is irreducible if and only if for any $p \in V$, the local defining function of V near p is irreducible in the UFD $\mathcal{O}_{A,p}$ (see [15], p. 130).

Definition 2.11. *A finite integral linear combination $D = \sum a_i V_i$ of irreducible analytic hypersurfaces is said to be a* divisor *on A. A divisor D is called* effective, *denoted by $D \geq 0$, if each a_i is non-negative. We denote the set of all divisors on A by $Div(A)$ and this is a group with the formal sum operation.*

Definition 2.12. *For any meromorphic function h on A, let $div(h) := \sum ord_V(h)V$, where the sum is taken over all irreducible hypersurfaces of A and $ord_V(h)$, called the* discrete valuation, *is*

- *the negative of the order of poles of h, if h has poles along V,*
- *the order of the zeros of h, if h has zeros along V,*
- *0, if h is non-zero and holomorphic on V.*

Remark 2.4. (a) Note that the sum defining $\mathrm{div}(h)$ is finite.
(b) It follows from the definition that for any holomorphic function h, $\mathrm{div}(h)$ is effective.
(c) By definition, we assign to every meromorphic function h on A a divisor $\mathrm{div}(h)$ on A in a way that $\mathrm{div}(h) = 0$ if and only if $h \in \mathcal{O}^*(A)$. Indeed, every divisor D on A is also a divisor of some meromorphic function h which is unique up to multiplication by non-vanishing holomorphic functions on A (see [15], p. 131). In this way, we construct a group homomorphism between $\mathrm{Div}(A)$ and $\mathcal{M}^*(A)/\mathcal{O}^*(A)$.
(d) Two divisors D and D' on A are said to be *linearly equivalent* if there exists a meromorphic function h on A such that $D' = D + \mathrm{div}(h)$.

Definition 2.13. *Let A be an n-dimensional algebraic variety or complex manifold with local coordinates $(z_1, \ldots, z_n)$, and s be an n-form not identically 0 on A. Then $s = J \cdot dz_1 \wedge \ldots \wedge dz_n$, where J is the determinant of some n by n transformation* *matrix. Observe that J is a meromorphic function on A. The* canonical divisor *on A, denoted by K_A, is defined as $K_A := div(s) := \sum ord_V(J)V$, where the sum is taken over all irreducible hypersurfaces.*

Remark 2.5. (a) The discrete valuation of J is well-defined since if one chooses different local coordinates $(w_1, \ldots, w_n)$, this amounts to multiplying J by the determinant of an n by n invertible matrix. So the resulting determinant has the same zeros and poles with J.
(b) If we choose a different n-form s', then $s' = hs$ for some not identically 0 meromorpfic function h on A. So $\mathrm{div}(s') = \mathrm{div}(s) + \mathrm{div}(h)$, which implies that $\mathrm{div}(s)$ and $\mathrm{div}(s')$ are linearly equivalent. In fact, the canonical divisor K_A represents this equivalence class, which is called the *canonical class.* So the canonical divisor is well-defined up to linear equivalence.

Before defining the intersection number of divisors, we give one more example of a sheaf: For a divisor $D = \sum a_i V_i$ on a complex manifold A, define $\mathcal{O}_A(D)$ on an arbitrary open subset U as the set of all meromorphic functions on U whose discrete valuations on each V_i intersecting U are greater than or equal to a_i.

Now we define the intersection number of divisors in a complex surface A. First, assume that C and D are two irreducible analytic curves in A. If C and D intersect transversally at a point p we define the *intersection index* $\iota_p(C \cdot D)$ of C and D at p to be $+1$ if the oriented bases of C and D in order form an oriented basis for $T_p(A)$, and -1 otherwise. Observe that on the complex surface A, $\iota_p(C \cdot D) = \iota_p(D \cdot C)$ since the oriented bases of C and D in reverse order defines the same orientation on A at p. Now define the intersection number as

$$C \cdot D = \sum_{p \in C \cap D} \iota_p(C \cdot D).$$

Definition 2.14. *The* intersection *of a pair $D = \sum_i a_i V_i$, $D' = \sum_j b_j W_j$ of divisors* *in a complex surface A is defined to be*

$$D \cdot D' = \sum_{i,j} a_i b_j (V_i \cdot W_j).$$

Remark 2.6. The sum above is always finite. So intersection number is a well-defined function from $\mathrm{Div}(A) \times \mathrm{Div}(A)$ to $\mathbb{Z}$.

Till now, we gave highly abstract definitions. However, the key point is that intersection numbers are, in general, used to put some conditions on divisors. Following, we list some results which are related to canonical divisors, intersection number etc.

(i) Let Y be a non-singular analytic hypersurface in a complex manifold X. Then the *Adjunction Formula* states that $K_Y = (K_X + Y)_Y$, that is the canonical divisor of Y can be obtained as follows: Take a divisor D on X which is linearly equivalent to $(K_X + Y)$ and does not contain Y, and then intersect it with Y.

(ii) In particular, applying the adjunction formula to a non-singular analytic curve C in a complex surface A we get the *Genus Formula* $(K_A + C) \cdot C = 2g(C) - 2$, where $g(C)$ is the number of genera of C (see e.g. [14], p. 33).

2.6. The Resolutions and Resolution Graphs of Surface Singularities

Definition 2.15. *Given an isolated surface singularity $(\mathcal{S}, 0)$, let Σ be a 2-dimensional complex manifold and $p : \Sigma \to \mathcal{S}$ be a proper analytic map. Then*

- (Σ, p) *is said to be a* resolution *of* $(\mathcal{S}, 0)$ *if* $p : \Sigma \setminus \{p^{-1}(0)\} \to \mathcal{S} \setminus \{0\}$ *is an isomorphism.*
- (Σ, p) *is said to be a* good resolution *if it is a resolution and* $E := p^{-1}(0)$*, which is called the* exceptional divisor *is* normal crossing*, that is* $E = E_1 \cup \ldots \cup E_r$ *where* E_i*'s are smooth irreducible curves intersecting each other transversally and* $E_i \cap E_j \cap E_k = \emptyset$ *for* i, j, k *different.*

Remark 2.7. Any resolution can be completed to a good resolution by extra *blowing-ups.*

Note that we also denote the divisor $\sum_{i=1}^{r} E_i$ on Σ by E.

Given a resolution (Σ, p) of an isolated surface singularity $(\mathcal{S}, 0)$, the associated *resolution graph* $\Gamma(p)$ is constructed in the following way: Consider smooth irreducible components E_i, $i \in \{1, \ldots, r\}$, of the exceptional divisor E as vertices of $\Gamma(p)$. If E_i and E_j intersects each other in Σ, join the vertex E_i to E_j with an edge. Then we get a connected graph. Moreover, each vertex is labeled with two integers v_i and g_i, where v_i is the valency of E_i (as a vertex) and g_i is the genus of E_i (as a smooth compact surface).

Definition 2.16. *An integral linear combination* $C = \sum_{i=1}^{r} a_i E_i$ *is called a* cycle *on* Σ. *To the resolution graph* $\Gamma(p)$, *we assign an* intersection form $I(\Gamma(p))$ *on the set of cycles in the following way: First, for any pair* (E_i, E_j) *of vertices define* $I(\Gamma(p))$ *to be their intersection number* $E_i \cdot E_j$ *in the sense of the previous section, and then extend it to the set of cycles bilinearly. We simply denote* $I(\Gamma(p))(C, C')$ *as* $C \cdot C'$.

Remark 2.8. Note that the intersection form $I(\Gamma(p))$ is negative definite, i.e. for any cycle C on Σ, $C \cdot C$, denoted by C^2, is negative (see e.g. [14], Chapter A).

Lemma 2.4. *The sequence* $0 \longrightarrow \mathcal{O}_\Sigma(-E) \xrightarrow{i} \mathcal{O}_\Sigma \xrightarrow{r} \mathcal{O}_E \longrightarrow 0$ *is exact, where* i *is the inclusion and* r *is the restriction map.*

Proof. First, observe that $\mathcal{O}_\Sigma(-E)$ is the set of all holomorphic functions vanishing on E. Then $r(\mathcal{O}_\Sigma(-E)) = 0$ in $\mathcal{O}_E$ since holomorphic functions on Σ vanishing on E restricts to 0 on E. Moreover, if $f \in \mathcal{O}_\Sigma \setminus \mathcal{O}_\Sigma(-E)$ then f is non-zero at some point in E which implies $r(f) = f|_E \neq 0$ in $\mathcal{O}_E$. Therefore, $\mathcal{O}_\Sigma(-E) = \ker(r)$. Moreover, r is onto because every holomorphic map f on E can be extended to a holomorphic map $\hat{f}$ on Σ with $r(\hat{f}) = f$. Thus, the sequence is exact. □

Remark 2.9. By tensoring with $\mathcal{O}_\Sigma(-D)$, we can realize

- $\mathcal{O}_\Sigma(-D-E)$ as $\mathcal{O}_\Sigma(-D) \otimes \mathcal{O}_\Sigma(-E)$,
- $\mathcal{O}_\Sigma(-D)$ as $\mathcal{O}_\Sigma(-D) \otimes \mathcal{O}_\Sigma$, and
- $\mathcal{O}_E(-D)$ as $\mathcal{O}_\Sigma(-D)|_E \otimes \mathcal{O}_E$.

Lemma 2.4, with the above identifications, gives rise to following:

Corollary 2.1. *The sequence* $0 \longrightarrow \mathcal{O}_\Sigma(-D-E) \xrightarrow{i} \mathcal{O}_\Sigma(-D) \xrightarrow{r} \mathcal{O}_E(-D) \longrightarrow 0$ *is exact, where i is the inclusion and r is the restriction map.*

Following theorem is a good example which presents the relation between the intersection number of divisors and the sheaf cohomology. Note that it plays an important role in the proof of Theorem 1.1.

Theorem 2.4 (Ramanujam, [11])**.** *Let D be a a divisor on Σ. If $D \cdot E_i \geq K_\Sigma \cdot E_i$ for any i, then $H^1(\mathcal{O}_\Sigma(D)) = 0$.*

Lemma 2.5. *For any holomorphic function f on $\mathcal{S}$ $div(fop) \cdot E_i = 0$.*

For a topological proof of Lemma 2.5 see [12], page 17.

Definition 2.17. *Let $p : (\Sigma, E) \to (\mathcal{S}, 0)$ be a good resolution, and let f be a holomorphic function on $\mathcal{S}$ vanishing at 0. The part of the divisor $div(f \circ p)$ which is supported by E is called the* exceptional divisor *of f, and denoted by $div(f \circ p)_e$. Moreover, the divisor $div(f \circ p)_s :=div(f \circ p) - div(f \circ p)_e$ is called the* strict transform *of f. We say that p is a* good resolution of f *if $div(f \circ p)$ is a normal crossing divisor on Σ.*

2.7. Plumbing Manifolds Associated to Resolution Graphs

Let Γ be the good resolution graph of a surface singularity $(\mathcal{S}, 0)$. Then Γ is connected and every vertex E_i has a non-zero valency v_i. Moreover, each vertex is a weighted by two integers e_i, the self intersection of the divisor E_i, and g_i, the genus of the surface E_i. Following Neumann's construction (see [13], Ch. 1), we define a 3-manifold $M(\Gamma)$ associated to Γ as follows:

- For any i, let S_i be a closed genus g_i surface with v_i open discs removed.
- Take a circle bundle $p_i : M_i \to S_i$ with the Euler number e_i.
- If E_i and E_j are adjacent vertices, then glue p_i to p_j over the torus which exists on the boundary of one of the removed open discs.

The manifold $M(\Gamma)$ is called the *plumbed manifold* associated to Γ.

Remark 2.10. In [13], Neumann gives a more general definition for a plumbed manifold associated to an arbitrary dual weighted graph Γ. In his construction, the plumbing graph Γ is allowed to contain loops which is equivalent to the self intersection of the irreducible components E_i of the exceptional divisor E. Moreover, he assigns a $+$ or $-$ sign to each edge which determines how to glue the circle bundles corresponding to the vertices joined by the edge. Nevertheless, we do not deal such situations since by the definition of a resolution, each E_i is smooth and so intersects itself nowhere, and E_i and E_j has positive intersection. From this perspective, Neumann's construction is equivalent to our definition of plumbed manifolds associated to resolution graphs.

2.8. Horizontal Open Book Decompositions

Definition 2.18. *Let Γ be a connected plumbing graph with r vertices. Consider the plumbed 3-manifold $M(\Gamma)$. For $i \in \{1, \dots, r\}$, choose n_i generic fibres from the circle bundle $p_i : M_i \to S_i$, where n_i is a non-negative integer. The collection of this fibres, denoted $N(n_1, \dots, n_r)$, is called a* vertical link *in $M(\Gamma)$. An open book decomposition in $M(\Gamma)$ whose binding is a vertical link and pages are transversal to the fibres of $p_i : M_i \to S_i$ is said to be a* horizontal *open book decomposition.*

Proposition 2.1. *Let Γ be a connected weighted plumbing graph with r vertices whose associated intersection form $I(\Gamma)$ is non-degenerate. Let $\underline{n} = (n_1, \dots, n_r)$ be an r-tuple of strictly positive integers. Then any two horizontal open books in $M(\Gamma)$ with the binding $N(\underline{n})$ are isomorphic.*

For a proof of Proposition 2.1 see [1], page 13.

3. THE LINK OF AN ISOLATED SINGULARITY

3.1. The Contact Boundary of an Isolated Singularity

Let (X, x) be a germ of an n-dimensional irreducible complex analytic variety with isolated singularity x. Consider the set $m_{X,x}$ of holomorphic functions on X vanishing at x, which is an ideal of the ring $\mathcal{O}_{X,x}$ of holomorphic functions on X. Set $X^* := X \setminus \{x\}$. Since x is an isolated singularity, X^* is an n-dimensional complex manifold.

Now we define an operator $J : TX^* \to TX^*$ as the fiberwise multiplication by the complex number i on TX^*. For $p \in X^*$, identifying T_pX^* with $\mathbb{C}^n$, we can write

$$J(p, (z_1, z_2, \ldots, z_n)) = (p, (iz_1, iz_2, \ldots, iz_n)), \text{ or}$$
$$J(p, (a_1,b_1, a_2, b_2, \ldots, a_n, b_n)) = (p, (-b_1, a_1, -b_2, a_2, \ldots, -b_n, a_n)).$$

Then define the differential operator d^c on differentiable functions $F : X^* \to \mathbb{R}$ by $d^cF = dF \circ J$.

Remark 3.1. For any differentiable function $F : X^* \to \mathbb{R}$, $d^cF = i \cdot (\partial - \bar{\partial})F$, where in local coordinates $(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$,

$$\partial F = \sum_{k=1}^{n} \frac{\partial F}{\partial z_k} dz_k := \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial F}{\partial x_k} - i\frac{\partial F}{\partial y_k}\right)(dx_k + idy_k)$$
$$\bar{\partial} F = \sum_{k=1}^{n} \frac{\partial F}{\partial \bar{z}_k} d\bar{z}_k := \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial F}{\partial x_k} + i\frac{\partial F}{\partial y_k}\right)(dx_k - idy_k).$$

Definition 3.1. *A differentiable function $F : X^* \to \mathbb{R}$ is said to be* strictly plurisubharmonic (spsh) *if $-dd^cF(v, Jv) > 0$ for any non-zero tangent vector v to X^*.*

For $\phi_1, \phi_2, \ldots, \phi_N \in m_{X,x}$, define a holomorphic function $\Phi : (X, x) \to (\mathbb{C}^N, 0)$ and a real analytic function $\rho : (X, x) \to (\mathbb{R}, 0)$ by

$$\Phi(z) = (\phi_1(z), \ldots, \phi_N(z)), \text{ and}$$
$$\rho(z) = |\Phi(z)|^2 = \sum_{i=1}^{N} |\phi_i(z)|^2.$$

Set $M_{\rho,\epsilon} := \rho^{-1}(\epsilon)$.

Remark 3.2. We assume that Φ is a *finite analytic morphism*, i.e. the preimage of every element in $\mathbb{C}^N$ is a finite set. This implies in particular that $\rho^{-1}(0)$ contains finitely many elements. So $0 \in \mathbb{C}^N$ is not an accumulation point of the critical values of $\rho|_{X^*}$, and $M_{\rho,\epsilon}$ is a smooth, compact, codimension 1 submanifold in X^* for any $\epsilon > 0$ sufficiently small (see [10], p. 22).

On X^*, we define:

$$\begin{aligned} \alpha &:= -d^c\rho, \text{ a 1-form} \\ \omega &:= d\alpha, \text{ a 2-form} \\ g(u, v) &:= \omega(u, Jv), \text{ a 2-tensor} \\ h &:= g + i\omega, \text{ a 2-tensor} \\ \xi_\rho &:= \ker(d\rho) \cap \ker(d^c\rho), \text{ a distribution.} \end{aligned}$$

Here are some important observations on ξ_ρ:

- ξ_ρ is a complex hyperplane distribution on X^*. In fact, if a tangent vector v to

X^* is in ξ_ρ, that is if

$$d^c\rho(v) = d\rho(Jv) = 0, \text{ and} \tag{3.1}$$

$$d\rho(v) = 0, \tag{3.2}$$

then Jv is clearly in $\ker(d\rho)$ by Equation 3.1. Besides, Jv is in $\ker(d^c\rho)$ since

$$d^c\rho(Jv) = d\rho(J^2v) = d\rho(-v) = 0$$

by Equation 3.2. This means Jv is also in ξ_ρ. Hence, J restricted to ξ_ρ is an almost complex structure.

- Since ξ_ρ is in $\ker(d\rho)$, it is also tangent to $M_{\rho,\epsilon}$. In fact, defining

$$\xi_{\rho,\epsilon} := \xi_\rho|_{M_{\rho,\epsilon}} = \ker(\alpha|_{M_{\rho,\epsilon}})$$

gives a hyperplane distribution in the real tangent bundle $T(M_{\rho,\epsilon})$ with the defining form $\alpha|_{M_{\rho,\epsilon}}$.

Now, since we have a hyperplane distribution $\xi_{\rho,\epsilon}$ on an odd dimensional manifold $M_{\rho,\epsilon}$, naturally we may ask under what conditions this distribution (or its defining form) is contact. The following lemma answers that question.

Lemma 3.1. *The following conditions are equivalent:*

(i) ρ *is spsh.*

(ii) Φ *restricted to* X^* *is an immersion.*

(iii) $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ *is a contact manifold for every* $\epsilon > 0$ *sufficiently small.*

Proof. First, we give an explicit formula for $-dd^c\rho$ in local coordinates. Then we use this formula to prove $(i \Leftrightarrow ii)$ and $(ii \Leftrightarrow iii)$ in order.

Let $\tau((z_1, \ldots, z_N)) := \sum_{k=1}^{N} |z_k|^2$ on $\mathbb{C}^N$. Then $d\tau = \sum_{k=1}^{N}(2x_k dx_k + 2y_k dy_k)$ for $(x_k, y_k) = z_k$. Fix a point p in X^*. Let $v \in T_pX^*$ and $\Phi_*|_p(v) = (a_1, b_1, \ldots, a_N, b_N) \in T_{\Phi(p)}\mathbb{C}^N$ in local coordinates. Note that Φ_* is a complex linear map on T_pX^*. Then

$$
\begin{aligned}
\alpha_p(v) = -d\rho_p(Jv) = -d(\tau \circ \Phi)_p(iv) = -d\tau \circ \Phi_*|_p(iv) = -d\tau(i\Phi_*|_p(v))\\
&= -\sum_{k=1}^{N}(2x_k dx_k + 2y_k dy_k)(i(a_1, b_1, \ldots, a_N, b_N)\\
&= -\sum_{k=1}^{N}(2x_k dx_k + 2y_k dy_k)(-b_1, a_1, \ldots, -b_N, a_N)\\
&= -\sum_{k=1}^{N}(2y_k dx_k - 2x_k dy_k)(a_1, b_1, \ldots, a_N, b_N)\\
&= \left(\sum_{k=1}^{N}(2x_k dy_k - 2y_k dx_k)\right) \circ \Phi_*|_p(v).
\end{aligned}
$$

which implies that on X^*, $\alpha = \Phi^*(\alpha_0)$, where on $\mathbb{C}^N$ we set

$$
\alpha_0 := -d^c\tau = -d\tau \circ J = \sum_{k=1}^{N}(2x_k dy_k - 2y_k dx_k).
$$

Therefore, $\omega = -dd^c\rho = d(\Phi^*(\alpha_0)) = \Phi^*(d\alpha_0)$, where

$$
\begin{aligned}
d\alpha_0 &= d\left(\sum_{k=1}^{N}(2x_k dy_k - 2y_k dx_k)\right)\\
&= 2\sum_{k=1}^{N}(dx_k \wedge dy_k - dy_k \wedge dx_k)\\
&= 4\sum_{k=1}^{N}(dx_k \wedge dy_k).
\end{aligned}
$$

Hence,

$$-dd^c\rho = \left(\sum_{k=1}^{N} 4(dx_k \wedge dy_k)\right) \circ \Phi_*.$$

That is for v,$w \in T_pX^*$ with $\Phi_*|_p(v) = (a_1, b_1, \ldots, a_N, b_N)$
and $\Phi_*|_p(w) = (c_1, d_1, \ldots, c_N, d_N)$ in local coordinates, we have

$$\begin{aligned}
-dd^c\rho|_p(v, Jw) &= 4\sum_{k=1}^{N}(dx_k \wedge dy_k)(\Phi_*|_p(v), \Phi_*|_p(Jw)) \\
&= 4\sum_{k=1}^{N}(dx_k \wedge dy_k)((a_1, b_1, \ldots, a_N, b_N), i(c_1, d_1, \ldots, c_N, d_N)) \\
&= 4\sum_{k=1}^{N}(dx_k \wedge dy_k)((a_1, b_1, \ldots, a_N, b_N), (-d_1, c_1, \ldots, -d_N, c_N)) \\
&= 4\sum_{k=1}^{N}(a_kc_k + b_kd_k).
\end{aligned}$$

$(i \Rightarrow ii)$: If $\Phi|_{X^*}$ is not an immersion, then for some point $p \in X^*$ there is a non-zero tangent vector $v \in T_pX^*$ with $\Phi_*|_p(v) = 0$. This implies at the point p,

$$-dd^c\rho(v, Jv) = 4\sum_{k=1}^{N}(dx_k \wedge dy_k)(\Phi_*(v), \Phi_*(Jv)) = 0,$$

i.e. ρ is not spsh.

$(ii \Rightarrow i)$: Conversely, if $\Phi|_{X^*}$ is an immersion, then for $p \in X^*$ and for any non-zero tangent vector $v \in T_pX^*$, $\Phi_*|_p(v)$ (say $(a_1, b_1, \ldots, a_N, b_N)$) is non-zero, and so

$$-dd^c\rho|_p(v, Jv) = 4\sum_{k=1}^{N}(a_k^2 + b_k^2) > 0,$$

i.e. ρ is spsh.

$(ii \Rightarrow iii)$: Assume $\Phi|_{X^*}$ is an immersion. At an arbitrary point $p \in M_{\rho,\epsilon}$,

$$\left(\alpha \wedge (d\alpha)^{n-1}\right)_p (u, v_1, Jv_1, \dots, v_{n-1}, Jv_{n-1}) = 2\alpha(u) \sum_{k=1}^{n-1} -dd^c\rho(v_k, Jv_k) \neq 0$$

for u is a vector in $T_p(M_{\rho,\epsilon}) \setminus (\xi_{\rho,\epsilon})_p$ and $\{v_1, Jv_1, \dots, v_{n-1}, Jv_{n-1}\}$ is a set of $\mathbb{R}$-linearly independent vectors in $(\xi_{\rho,\epsilon})_p$. Obviously, such vectors do exist. So $\alpha \wedge (d\alpha)^{n-1}$ is non-zero at any $p \in M_{\rho,\epsilon}$, i.e. α is a contact form on $M_{\rho,\epsilon}$.

$(iii \Rightarrow ii)$: Conversely, assume that $\Phi|_{X^*}$ is not an immersion. Without loss of generality, set $X = \rho^{-1}([0, \epsilon_0))$ for sufficiently small $\epsilon_0 > 0$. Then at some point $p \in M_{\rho,\epsilon}$ (for some $\epsilon < \epsilon_0$), there is a tangent vector u to X^* such that $\Phi_*|_p(u) = 0$. It follows that $\Phi_*|_p(Ju) = 0$ as well. Consider the 2-dimensional linear subspace U_p of T_pX^* spanned by $\{u, Ju\}$. Obviously, $U_p \cap T_p(M_{\rho,\epsilon})$ has dimension at least 1. Fix a non-zero vector $v \in U_p \cap T_p(M_{\rho,\epsilon})$. Then $\Phi_*|_p(v)$ is also equal to 0. Complete $\{v\}$ to a basis $\{v, v_1, \dots, v_{2n-2}\}$ for $T_p(M_{\rho,\epsilon})$. It follows that $(\alpha \wedge (d\alpha)^{n-1})_p(v, v_1, \dots, v_{2n-2}) = 0$, i.e. α is not a contact form on $M_{\rho,\epsilon}$. This completes the proof. $\square$

From now on, we assume that $\Phi : (X, x) \to (\mathbb{C}^N, 0)$ restricts to an immersion on X^*. By Lemma 3.1, the analytic function ρ associated to Φ is spsh and $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ is a contact manifold. The objects associated to ρ have following properties:

- α is a positive contact form on $M_{\rho,\epsilon}$.
- ω is a symplectic form compatible with the complex structure J on X^*. That is, for any tangent vector $u, v \in T_pX^*$ at an arbitrary point p, $\omega_p(u, v) = \omega_p(Ju, Jv)$.
- g is a Riemannian metric on X^*.
- h is a Hermitian metric on X^*.

In the above construction, the contact manifold $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ seems highly dependent on the chosen immersion Φ and the positive number ϵ. In [3], A. N. Varčenko proves that when Φ is an embedding, the contact isotopy type of $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ is independent of Φ and ϵ. A generalization of this is:

Lemma 3.2. *The contact isotopy type of the contact manifold $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ is independent from the chosen immersion Φ and the sufficiently small positive number ϵ.*

Proof. This proof indeed has two parts. However, here we are going to prove only one of them: The contact isotopy type does not depend on a sufficiently small $\epsilon > 0$. We note that both parts use Gray's Theorem.

Fix a finite analytic morphism $\Phi : (X, x) \to (\mathbb{C}^N, 0)$ which restricts to an immersion on X^*. Let $\epsilon_0 > \epsilon_1 > 0$ be sufficiently small. For $t \in [0, 1]$, set $\epsilon_t := (1-t)\epsilon_0 + t\epsilon_1$. By Lemma 3.1, $(M_{\rho,\epsilon_t}, \xi_{\rho,\epsilon_t})$ is a contact manifold for any $t \in [0, 1]$. Note that there exists a smooth family of diffeomorphisms $\{g_t : M_{\rho,\epsilon_0} \to M_{\rho,\epsilon_t} | \ t \in [0, 1]\}$ with $g_0 = id_{M_0}$ (see e.g. [10], p. 23). It follows that $\{\hat{\xi}_t := g_t^*(\xi_t) | \ t \in [0, 1]\}$ is a smooth family of of contact distributions on M_0. Thus, by Gray's Theorem, there exists a contact isotopy $\{\phi_t : M_0 \longrightarrow M_0\}$ with $\phi_{t*}(\xi_0) = \hat{\xi}_t$. This completes the proof. $\square$

Definition 3.2. *The isotopy type of the contact manifold $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ is called the* contact boundary *of (X, x) and denoted by $(M(X), \xi(X))$. It is a well-defined notion since the isotopy depends on neither Φ (and ρ) nor ϵ.*

3.2. The Milnor Open Book in the Contact Boundary

Now, for any holomorphic function $f \in m_{X,x}$ which defines an isolated singularity at x, we are going to construct an open book decomposition associated to f in the contact boundary $(M(X), \xi(X))$ of (X, x).

Before constructing the open book, we give a definition for the notion of a singularity of a holomorphic function $f \in m_{X,x}$. Let V be an $n = k - m$ dimensional analytic variety in $\mathbb{C}^k$ with an isolated singularity x, and let $g = (g_1, \ldots, g_m) : \mathbb{C}^k \to \mathbb{C}^m$ be a holomorphic function such that $V = g^{-1}(0)$. Let (X, x) be a germ of the singularity x. An element f of $m_{X,x}$ can be realized as a holomorphic function on $\mathbb{C}^k$ which vanishes at x. Let $G := (g, f) : \mathbb{C}^k \to \mathbb{C}^{m+1}$. Then G is also holomorphic. Now consider the rank of the complex matrix $dG(p)$ for any point $p \in X$. If the rank of $dG(p)$ is smaller than $m+1$ for $p = x$ and equal to $m+1$ for $p \neq x$, then f is said to have an isolated singularity at x.

Example 3.1. Consider the 2-dimensional complex analytic variety $V := \{x^2+y^3+z^5 = 0\} \subset \mathbb{C}^3$ with an isolated singularity at 0. Fix a small neighbourhood X of 0 in V. Let $f : \mathbb{C}^3 \to \mathbb{C}$ with $f(x, y, z) = 2x + 3y + z$. Then f vanishes at x, and so the restriction of it on X is an element of $m_{X,0}$. Without loss of generality, denote this as $f \in m_{X,0}$. Then the derivative of the function $G = (g, f) : \mathbb{C}^3 \to \mathbb{C}^2$ equals the complex matrix

$$\begin{pmatrix} 2x & 3y^2 & 5z^4 \\ 2 & 3 & 1 \end{pmatrix}.$$

Observe that the rank of this matrix is smaller than 2 at 0 and equals 2 at other points in X. Hence, f has an isolated singularity at 0.

Now let (X, x) be a germ of an isolated singularity x. Let $f \in m_{X,x}$ have an isolated singularity at x. Then for any $\epsilon > 0$ small enough, $M_{\rho,\epsilon}$ and $f^{-1}(0)$ intersects each other transversally in X^*. So $N_{\rho,\epsilon}(f) := M_{\rho,\epsilon} \pitchfork f^{-1}(0)$ is a smooth, naturally oriented, codimension 2 submanifold in $M_{\rho,\epsilon}$.

Furthermore, since f is non-zero and smooth on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$, it follows that $\arg(f)$ restricted to $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ is a smooth function

$$\theta_\epsilon(f) := \arg(f) : M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f) \to S^1.$$

Proposition 3.1. *For $\epsilon > 0$ sufficiently small, the pair $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ is an open book in the contact boundary $M_{\rho,\epsilon}$ which is compatible with the orientations.*

We start the proof of Proposition 3.1 with some definitions and lemmas.

Definition 3.3. *The* gradient ∇F *of a real-valued smooth function* F *on* X^* *is the unique tangent vector field of* X^* *which satisfies* $dF(\cdot) = g(\nabla F, \cdot)$. *Similarly, the* gradient $\nabla\phi$ *of a complex-valued smooth function* $\phi \in m_{X,x}$ *is the unique tangent vector field of* X^* *which satisfies* $d\phi(\cdot) = h(\nabla\phi, \cdot)$.

Lemma 3.3. *Let* $\phi \in m_{X,x}$. *Then*

(i) $\nabla|\phi|^2 = 2\phi\nabla\phi$,

and in $X^* \setminus \phi^{-1}(0)$*;*

(ii) $d(\arg\phi) = \mathrm{Im}\,(d\phi/\phi)$ *and* $\nabla(\arg\phi) = i\left(\nabla\phi/\bar{\phi}\right)$,

(iii) $\nabla|\phi|^2 = -2i|\phi|^2\nabla(\arg\phi)$,

(iv) $\nabla(\log\phi) = \nabla(\log|\phi|) = -i\nabla(\arg\phi)$.

Proof. Let $\phi(z) = x(z) + iy(z)$ in the sense that $x(z)$ is the real part, and $y(z)$ is the imaginary part of $\phi(z)$.

(i) Then $|\phi|^2 = x^2(z) + y^2(z)$. We know that h is a Hermitian metric. Therefore,

$$\begin{aligned} g(2\phi\nabla\phi, \cdot) + i\omega(2\phi\nabla\phi, \cdot) = h(2\phi\nabla\phi, \cdot) = 2\bar{\phi}h(\nabla\phi, \cdot) &= 2\bar{\phi}d\phi \\ &= (2x - i2y)(dx + idy) \\ &= (2xdx + 2ydy) + i(2xdy - 2ydx). \end{aligned}$$

Hence, $d|\phi|^2 = (2xdx + 2ydy) = g(2\phi\nabla\phi, \cdot)$, which implies that $2\phi\nabla\phi = \nabla|\phi|^2$.

(ii) Notice that for $z \in X^* \setminus \phi^{-1}(0)$, locally $\arg\phi(z) = \arctan\left(y(z)/x(z)\right) + c$ for some constant c. Since

$$\mathrm{Im}\left(\frac{d\phi}{\phi}\right) = \mathrm{Im}\left(\frac{dx + idy}{x + iy}\right) = \mathrm{Im}\left[\frac{1}{x^2 + y^2}(x - iy)(dx + idy)\right] = \frac{1}{x^2 + y^2}(xdy - ydx),$$

then the equality follows since

$$d(\arg\phi) = \frac{x^2}{x^2+y^2}\frac{-y}{x^2}dx + \frac{x^2}{x^2+y^2}\frac{1}{x}dy = \frac{1}{x^2+y^2}(xdy - ydx) = \mathrm{Im}\left(\frac{d\phi}{\phi}\right).$$

Furthermore, note that

$$\frac{i}{\bar{\phi}} = \frac{i}{x-iy} = \frac{i(x+iy)}{x^2+y^2} = \frac{-y+ix}{x^2+y^2}.$$

Now since

$$\begin{aligned} g\left(i\left(\frac{\nabla\phi}{\bar{\phi}}\right), \cdot\right) + i\omega\left(i\left(\frac{\nabla\phi}{\bar{\phi}}\right), \cdot\right) &= h\left(i\left(\frac{\nabla\phi}{\bar{\phi}}\right), \cdot\right) = \overline{\left(\frac{i}{\bar{\phi}}\right)}h(\nabla\phi, \cdot) \\ &= \frac{-y-ix}{x^2+y^2}d\phi = \frac{-y-ix}{x^2+y^2}(dx+idy) \\ &= \frac{1}{x^2+y^2}\cdot(xdy - ydx) + i\frac{1}{x^2+y^2}\cdot(-ydy - xdx), \end{aligned}$$

it follows that

$$g\left(i\left(\frac{\nabla\phi}{\bar{\phi}}\right), \cdot\right) = \frac{1}{x^2+y^2}(xdy - ydx) = d(\arg\phi),$$

and therefore, $\nabla(\arg\phi) = i(\nabla\phi/\bar{\phi})$.

(iii) This is a corollary of (1) and (2):

$$\nabla|\phi|^2 = 2\phi\nabla\phi = 2\phi\frac{\bar{\phi}}{i}\nabla(\arg\phi) = -2i|\phi|^2\nabla(\arg\phi).$$

(iv) Since $\log\phi = \log|\phi| + i\arg\phi$, it follows that

$$\begin{aligned} g(\nabla(\log\phi),\cdot) + i\omega(\nabla(\log\phi),\cdot) &= h(\nabla(\log\phi),\cdot) = d(\log\phi) \\ &= d(\log|\phi|) + id(\arg\phi) \\ &= g(\nabla(\log|\phi|),\cdot) + ig(\nabla(\arg\phi),\cdot) \\ &= g(\nabla(\log|\phi|),\cdot) + i\omega(-i\nabla(\arg\phi),\cdot), \end{aligned}$$

which implies $g(\nabla(\log\phi),\cdot) = g(\nabla(\log|\phi|),\cdot)$ and $\omega(\nabla(\log\phi),\cdot) = \omega(-i\nabla(\arg\phi),\cdot)$. Since ω (and so g) is non-degenerate, we get $\nabla(\log|\phi|) = \nabla(\log\phi) = -i\nabla(\arg\phi)$. $\square$

Now let us continue our main discussion and make some more observations on $\arg(f)$ for the fixed $f \in m_{X,x}$. Since x is the isolated singularity of it, f is a submersion in a small neighbourhood U of x in X^*, i.e. df_p is onto for any $p \in U$. Then by Equation (i) of Lemma 3.3, $d\arg(f)_p = \mathrm{Im}(df_p/f(p))$ is also onto for any $p \in U \setminus f^{-1}(0)$, i.e. $\arg(f)$ is a submersion on $U \setminus f^{-1}(0)$. However, we are interested in whether $\arg(f)$ restricted to $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ is a submersion or not. Following lemma gives a necessary and sufficient condition for $d(\theta_\epsilon(f))$ to be onto at a point.

Lemma 3.4. *The set $\{p \in M_{\rho,\epsilon} | \ \exists r \in \mathbb{R} \setminus \{0\},\ \nabla\arg(f) = r\nabla\rho \text{ at } p\}$ equals the set of all critical points of $\theta_\epsilon(f)$.*

Proof. Let $p \in M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ such that $(\nabla\arg(f))_p = r(\nabla\rho)_p$ for some $r \in \mathbb{R} \setminus \{0\}$. Then on T_pX^*,

$$d\arg(f) = g(\nabla\arg(f),\cdot) = g(r\nabla\rho,\cdot) = rd\rho.$$

Since $\ker(d\rho)_p = T_p(M_{\rho,\epsilon})$, it follows that $d\arg(f)$ is identically 0 on $T_p(M_{\rho,\epsilon})$. Therefore, p is a critical point of $\theta_\epsilon(f)$.

On the other hand, if $p \in M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ is a critical point of $\theta_\epsilon(f)$, then $d\arg(f)$ is identically 0 on $T_p(M_{\rho,\epsilon})$. So $T_p(M_{\rho,\epsilon}) \subseteq \ker(d\arg(f))_p$. Furthermore, since $\arg(f)$ is submersion at $p \in X^*$, it follows that $\ker(d\arg(f))_p$ has dimension $2n-1$ in T_pX^*. Thus, $\ker(d\arg(f))_p = T_p(M_{\rho,\epsilon}) = \ker(d\rho)_p$. But this implies that there exists $r \in \mathbb{R} \setminus \{0\}$ such that on T_pX^*,

$$d\arg(f) = rd\rho \Rightarrow g(\nabla\arg(f), \cdot) = g(r\nabla\rho, \cdot) \Rightarrow \nabla\arg(f) = r\nabla\rho.$$

$\square$

This lemma plays an important role in the construction. Once we show that $\nabla\arg(f)$ and $\nabla\rho$ are $\mathbb{R}$-linearly independent at any point $p \in M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ for $\epsilon > 0$ sufficiently small, it will follow that $\theta_\epsilon(f)$ has no critical points. Therefore, the preimage of any point in S^1 under $\theta_\epsilon(f)$ will be a smooth, naturally oriented submanifold of real codimension 1 in $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$. The linear independence of $\nabla\arg(f)$ and $\nabla\rho$ is a corollary of the following two lemmas. Notice that we have some more work to do around $N_{\rho,\epsilon}(f)$ to show that $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ is an open book decomposition.

Lemma 3.5. *Let $p : (\mathbb{R}_+, 0) \to (X, x)$ be a non-constant real analytic arc. Set $\dot{p} := dp/dt$. Then*

$$\lim_{t\to 0} h\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) = 1.$$

Proof. To prove this lemma, first we transform the limit expression given in (X, x) to a limit expression in $(\Phi(X), 0)$ by pushing forward under Φ. Then we compute the limit in $\mathbb{C}^N$.

Again let $\tau = \sum_{i=1}^N |z_i|^2$ on $\mathbb{C}^N$. Then τ induces a symplectic form $\omega_0 = -dd^c\tau$ with the associated Riemannian metric g_0 and Hermitian metric $h_0 = g_0 + i\omega_0$ on $\mathbb{C}^N$.

Now set $q := \Phi \circ p$ and let ∇_0 be the gradient with respect to g_0. The fact that $\rho = \big(\tau|_{\Phi(X)}\big) \circ \Phi$ implies $h = \Phi^*(h_0|_{\Phi(X)})$, and $\Phi_*(\nabla\rho) = \nabla_0(\tau|_{\Phi(X)})$, and $\Phi_*(\dot{p}) = \dot{q}$. Indeed, the geometric objects on X are defined via pull-back under Φ from $\Phi(X)$. That is, the Riemannian metric g on X^* equals $\Phi^*(g_0|_{\Phi(X)})$ etc. Hence, we get:

$$\begin{aligned} h\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) &= \Phi^* h_0 \left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) \\ &= h_0\left(\Phi_*\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t))\right), \Phi_*\left(\frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right)\right) \\ &= h_0\left(\frac{\nabla_0(\tau|_{\Phi(X)})}{\big\|\nabla_0(\tau|_{\Phi(X)})\big\|}(q(t)), \frac{\dot{q}}{\|\dot{q}\|}(t)\right). \end{aligned}$$

Observe that above Φ_* maps unit vectors to unit vectors. This is because Φ_* preserves the length of the vectors. In fact, the *length* of a vector v tangent to X^* at $\hat{x}$ is defined as: $\|v_{\hat{x}}\| := \sqrt{g_{\hat{x}}(v,v)} = \sqrt{g_0(\Phi_*|_{\hat{x}}(v), \Phi_*|_{\hat{x}}(v))} := \|\Phi_*|_{\hat{x}}(v)\|_0$.

Now we try to replace the vector field $\nabla_0(\tau|_{\Phi(X)})$ with a more convenient one. Note that $\nabla_0(\tau|_{\Phi(X)})$ is the restriction of $\nabla_0(\tau)$ onto $T(\Phi(X))$. Moreover, on the ambient space $\mathbb{C}^N$, $(\nabla_0(\tau))_z = 2z$. So

$$\frac{\nabla_0(\tau)}{\|\nabla_0(\tau)\|}(q(t)) = \frac{q(t)}{\|q(t)\|} \tag{3.3}$$

as both are unit vectors in the direction of $q(t)$. Observe that on the arc $q(t)$, as t tends to 0, $(\nabla_0(\tau))_{q(t)} = 2q(t)|_{q(t)}$ approaches the tangent vector $(\nabla_0(\tau|_{\Phi(X)}))_{q(t)}$. In other words,

$$\lim_{t\to 0} \frac{\nabla_0(\tau)}{\|\nabla_0(\tau)\|}(q(t)) = \frac{\nabla_0(\tau|_{\Phi(X)})}{\big\|\nabla_0(\tau|_{\Phi(X)})\big\|}(q(t)).$$

Following Equation 3.3, we can rearrange the last equation as

$$\frac{\nabla_0(\tau|_{\Phi(X)})}{\left\|\nabla_0(\tau|_{\Phi(X)})\right\|}(q(t)) = \frac{\nabla_0(\tau)}{\|\nabla_0(\tau)\|}(q(t)) + o(1) = \frac{q(t)}{\|q(t)\|} + o(1)$$

with little-o notation. This implies

$$\begin{aligned}
h\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) &= h_0\left(\frac{\nabla_0(\tau|_{\Phi(X)})}{\left\|\nabla_0(\tau|_{\Phi(X)})\right\|}(q(t)), \frac{\dot{q}}{\|\dot{q}\|}(t)\right) \\
&= h_0\left(\frac{q}{\|q\|}(t) + o(1), \frac{\dot{q}}{\|\dot{q}\|}(t)\right) \\
&= h_0\left(\frac{q}{\|q\|}(t), \frac{\dot{q}}{\|\dot{q}\|}(t)\right) + o(1).
\end{aligned}$$

Let $q(t) = \sum_{k=Q}^{\infty} z_k t^k$ (for some $Q \in \mathbb{N}$ and $z_Q \in \mathbb{C}^N \setminus \{0\}$) be the series expansion for the analytic arc q. In little-o notation write $q(t) = z_Q t^Q + o(t^Q)$ so that $\dot{q}(t) = Q z_Q t^{Q-1} + o(t^{Q-1})$. Therefore, as t tends to 0, the angle between $q(t)$ and z_Q tends to 0. Similarly, as t tends to 0, the angle between $\dot{q}$ and Qz_Q tends to 0. This implies again in little-o notation:

$$\begin{aligned}
\frac{q}{\|q\|}(t) &= \frac{z_Q}{\|z_Q\|} + o(1), \text{ and} \\
\frac{\dot{q}}{\|\dot{q}\|}(t) &= \frac{z_Q}{\|z_Q\|} + o(1).
\end{aligned}$$

Thus,

$$\begin{aligned} h\left(\frac{\nabla\rho}{\|\nabla\rho\|}(p(t)), \frac{\dot{p}(t)}{\|\dot{p}(t)\|}\right) &= h_0\left(\frac{q}{\|q\|}(t), \frac{\dot{q}}{\|\dot{q}\|}(t)\right) + o(1) \\ &= h_0\left(\frac{z_Q}{\|z_Q\|} + o(1), \frac{z_Q}{\|z_Q\|} + o(1)\right) + o(1) \\ &= h_0\left(\frac{z_Q}{\|z_Q\|}, \frac{z_Q}{\|z_Q\|}\right) + o(1) \\ &= g_0\left(\frac{z_Q}{\|z_Q\|}, \frac{z_Q}{\|z_Q\|}\right) + o(1) = 1 + o(1) \end{aligned}$$

converges to 1 as t approaches 0.

□

Lemma 3.6. *Let $\phi \in m_{X,x}$. For any $\theta \in \left(0, \frac{\pi}{2}\right)$, there exists a neighbourhood U_θ of x in X such that in $U_\theta \setminus \phi^{-1}(0)$ the following holds:*

$$\nabla(\arg\phi) = i\lambda\nabla\rho \text{ with } \lambda \in \mathbb{C}^* \Rightarrow |\arg\lambda| < \theta.$$

To prove Lemma 3.6 we will use the famous lemma of Milnor of which statement is as follows:

Curve Selection Lemma (Milnor, [2], Lemma 3.1)**.** Let $V \subseteq \mathbb{R}^m$ be a real algebraic set, and let U be an open subset of $\mathbb{R}^m$ defined by finitely many polynomial inequalities

$$U := \{y \in \mathbb{R}^m : \ g_1(y) > 0, \dots, g_l(y) > 0\}.$$

If $U \cap V$ contains points arbitrarily close to 0, i.e. $0 \in \overline{U \cap V}$, then there exists a real analytic curve $p : [0, \epsilon) \to \mathbb{R}^m$ such that $p(0) = 0$ and $p(t) \in U \cap V$ for $t > 0$.

Remark 3.3. Milnor proves the curve selection lemma under the condition that V is an algebraic set in $\mathbb{R}^m$, and the functions $g_1, \dots, g_l$ which define U are polynomials on $\mathbb{R}^m$. However, his proof is also valid for the analytic case with minor modifications. Note that the curve selection lemma can also be read as: in an analytic variety V with an isolated singularity x, if a property is satisfied at points arbitrarily close to x, then this property can be observed on a non-constant real analytic curve $p : [0, \epsilon) \to V$ whose limit, as t tends to 0, is x.

Proof of Lemma 3.6. Let $\theta \in \left(0, \frac{\pi}{2}\right)$ such that for any neighbourhood U of x in X, there is a $\lambda \in \mathbb{C}^*$ with $\nabla(\arg\phi) = i\lambda\nabla\rho$, but $|\arg\lambda| \geq \theta$ in $U \setminus \phi^{-1}(0)$. Then, by the curve selection lemma, there is a non-constant real analytic arc $p : (\mathbb{R}_+, 0) \to (X, x)$ on which

$$\nabla \arg\phi(p(t)) = i\lambda(t)\nabla\rho(p(t)), \text{ and } |\arg(\lambda(t))| \geq \theta$$

for t small enough. Then by Equation (ii) of Lemma 3.3, for any t sufficiently small, at the point $p(t)$ we get:

$$\frac{d}{dt}\phi(p(t)) = h\left(\nabla\phi, \frac{dp}{dt}\right) = h(-i\bar{\phi}\nabla(\arg\phi), \dot{p}) = h(\bar{\phi}\lambda\nabla\rho, \dot{p}) = \phi\bar{\lambda}h(\nabla\rho, \dot{p})$$

which is equivalent to

$$\bar{\lambda}(t) = \frac{d}{dt}\phi(p(t)) \cdot \phi(p(t))^{-1} \cdot h\left(\nabla\rho(p(t)), \dot{p}(t)\right)^{-1}.$$

But this implies

$$\lambda(t) = \frac{d}{dt}\overline{\phi(p(t))} \cdot \left(\overline{\phi(p(t))}\right)^{-1} \cdot \|\nabla\rho\|^{-1} \cdot \|\dot{p}\|^{-1} \cdot \overline{h\left(\frac{\nabla\rho}{\|\nabla\rho\|}, \frac{\dot{p}}{\|\dot{p}\|}\right)^{-1}}. \tag{3.4}$$

Equation 3.4 implies that $\lambda : \mathbb{R}_+ \to \mathbb{C}^*$ can be expressed as the quotient of two analytic functions for $t \in \mathbb{R}_+$ sufficiently small. Therefore, we deduce that λ has a Laurent expansion

$$\lambda(t) = at^A + o(t^A), \quad a \neq 0, \quad A \in \mathbb{Z}.$$

Next we try to understand the series expansions of the functions in the left-hand side of Equation 3.4. Note that since $\phi(p(t))$ is complex analytic and converges to 0 as t

tends to 0, it has a series expansion of the form:

$$\phi(p(t)) = bt^B + o(t^B),\ b \in \mathbb{C}^*,\ B \in \mathbb{N} \setminus \{0\}, \tag{3.5}$$

and since $\|\nabla\rho\|$ and $\|\dot{p}\|$ are real analytic and always non-negative, we have

$$\|\nabla\rho\|^{-1} \cdot \|\dot{p}\|^{-1} = ct^C + o(t^C),\ c \in \mathbb{R}_+,\ C \in \mathbb{Z}.$$

Furthermore, Equation 3.5 implies

$$\frac{d}{dt}\overline{\phi(p(t))} = B\bar{b}t^{B-1} + o(t^{B-1}),\ b \in \mathbb{C}^*,\ B \in \mathbb{N} \setminus \{0\}.$$

Finally, by Lemma 3.5, the last factor in the right-hand side of Equation 3.4 converges to 1 as t tends to 0.

Now Equation 3.4 implies that the leading coefficients of the series expansions above satisfies the relation: $a = bc\bar{b}B = |b|^2Bc > 0$, where b is complex, and c and B are positive real numbers. Therefore, the leading coefficient a of the series expansion λ is a positive real number. But this implies that as t tends to 0, $\lambda(t)$ approaches a real number, i.e. $\arg(\lambda(t))$ converges to 0. Observe that this contradicts our assumption that for t near 0, $|\arg(\lambda(t))| \geq \theta$, where θ is a fixed positive angle. □

Corollary 3.1. *For $\epsilon > 0$ sufficiently small, $\nabla \arg(f)$ and $\nabla\rho$ are linearly independent over $\mathbb{R}$ on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$.*

Proof. Fix $\theta = \frac{\pi}{4}$ and U_θ as in Lemma 3.6. In U_θ, if $\nabla \arg(f) = i\lambda\nabla\rho$ for some $\lambda \in \mathbb{C}^*$, then $-\frac{\pi}{4} < arg\lambda < \frac{\pi}{4}$. So $\frac{\pi}{4} < arg(i\lambda) < \frac{3\pi}{4}$ which implies $i\lambda$ is not a real number. Since for $\epsilon > 0$ sufficiently small $M_{\rho,\epsilon} \subset U_\theta$, it follows that $\nabla \arg(f)$ is not a real multiple of $\nabla\rho$ on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$. □

Now we are going to generalize calculations of Chapter 4 in [2] to the space $M_{\rho,\epsilon}$.

Lemma 3.7. *For any $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\epsilon > 0$ sufficiently small, there exists a smooth tangential vector field v on $M_{\rho,\epsilon} \backslash N_{\rho,\epsilon}(f)$ such that the Hermitian product $h(\nabla \arg(f), v)$ is non-zero and has argument less than θ in absolute value.*

Proof. Fix $\theta \in \left(0, \frac{\pi}{2}\right)$. Choose U_θ as in Lemma 3.6. Then for $\epsilon > 0$ sufficiently small, $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ is in U_θ. Now let p be an arbitrary point in $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$.

Case 1. *Suppose that $\nabla\rho$ and $\nabla \arg(f)$ are linearly independent over $\mathbb{C}$ at p. Then the system of linear equations*

$$h(\nabla\rho, v) = 0 \tag{3.6}$$

$$h(\nabla \arg(f), v) = 1 \tag{3.7}$$

has a solution for v in T_pX^*. By Equation 3.6, $g(\nabla\rho, v) = d\rho(v) = 0$, which implies that $v \in \ker d\rho = T(M_{\rho,\epsilon})$, and by Equation 3.7, $|\arg[h(\nabla \arg(f), v)]| = 0 < \theta$.

Case 2. *Suppose that $\nabla\rho$ and $\nabla \arg(f)$ are linearly dependent over $\mathbb{C}$ at p. Let $\nabla \arg(f) = i\lambda\nabla\rho$ for some $\lambda \in \mathbb{C}^*$. Then set $v = i\nabla\rho$. Since*

$$d\rho(v) = g(\nabla\rho, i\nabla\rho) = \omega(\nabla\rho, -\nabla\rho) = 0,$$

again it follows that $v \in \ker d\rho = T(M_{\rho,\epsilon})$, i.e. v is tangent to $M_{\rho,\epsilon}$. Moreover,

$$h(\nabla \arg(f), i\nabla\rho) = h(i\lambda\nabla\rho, i\nabla\rho) = \bar{\lambda}g(i\nabla\rho, i\nabla\rho) + i\bar{\lambda}\omega(i\nabla\rho, i\nabla\rho) = \bar{\lambda}\|\nabla\rho\|^2$$

has the same argument with $\bar{\lambda}$. But by Lemma 3.6, λ (and so $\bar{\lambda}$) has argument less than θ.

In this way, on a neighbourhood U of every point $p \in M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$, we get a smooth vector field v_U satisfying the required properties. With partitions of unity techniques, we can construct a global vector field v on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ with the same properties. □

Now for $\theta = \frac{\pi}{4}$, choose a smooth vector field $v(p)$ as in Lemma 3.7. Define

$$\begin{aligned} w(p) &:= \frac{v(p)}{g(\nabla \arg(f), v(p))}, \text{ so that} \\ g(\nabla \arg(f), w(p)) &= \frac{g(\nabla \arg(f), v(p))}{g(\nabla \arg(f), v(p))} = 1. \end{aligned}$$

Observe that $|\omega(\nabla \arg(f), w(p))| < 1$ by Lemma 3.7. For $p \in M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$, let γ_p be the unique maximal integral curve of w, i.e. $\gamma_p(0) = p$ and $\gamma_p'(0) = w(p)$. Equation (iv) of Lemma 3.3 implies that

$$\begin{aligned} |d(\log|f|)(w(p))| = |g(\nabla(\log|f|), w(p))| &= |g(-i\nabla \arg(f), w(p))| \\ &= |\omega(\nabla \arg(f), w(p))| < 1. \end{aligned}$$

This implies that the derivative of $\log|f|$ on γ_p is bounded, that is $\log|f| \nrightarrow -\infty$ on γ_p. Equivalently, $|f|$ does not tend to 0 on γ_p. Moreover, since

$$d\arg(f)(w(p)) = g(\nabla \arg(f), w(p)) = 1,$$

it follows that the derivative of $\arg(f)$ on γ_p is always 1. Equivalently,

$$\arg(f)(\gamma_p(t)) = t + c,$$

where the initial condition $\gamma_p(0) = p$ implies

$$c = \arg(f)(\gamma_p(0)) = \arg(f)(p).$$

In other words, $\arg(f)$ maps γ_p to a curve $\hat{\gamma} \subset S^1$ which winds around S^1 in the positive direction with the unit velocity. Notice that in the above construction $\gamma_p(t)$ smoothly depends on p and t. Now for any t, define a map $h_t : M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f) \to M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ by $h_t(p) = \gamma_p(t)$. Observe that $h_0 = id$ on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$.

For $\theta \in S^1$, set $F_\theta := arg(f)^{-1}(\theta)$ in $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$. It follows that for any t, h_t is a diffeomorphism which maps F_θ to $F_{(\theta+t)}$.

Lemma 3.8. *The smooth map $\theta_\epsilon(f) : M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f) \to S^1$ is a locally trivial fibration.*

Proof. Let $\theta \in S^1$ and $U \subset S^1$ be a small neigborhood of θ. Define

$$\varphi : U \times F_\theta \to \theta_\epsilon(f)^{-1}(U) \text{ by}$$
$$(t + \theta, p) \mapsto h_t(p).$$

It is obvious that φ is a diffeomorphism. We just need to show that $(\arg(f) \circ \varphi) = \pi_1$, where π_1 is the projection of $U \times F_\theta$ onto the first component. Let $(t + \theta, p) \in U \times F_\theta$. Then

$$\begin{aligned}\arg(f) \circ \varphi(t + \theta, p) &= \arg(f)(h_t(p)) = \arg(f)(\gamma_p(t)) = t + \arg(f)(p) = t + \theta \\ &= \pi_1(t + \theta, p).\end{aligned}$$

This completes the proof. ☐

Now we know that $\theta_\epsilon(f)$ is a locally trivial fibration. To achieve our goal, we need to prove one more claim which is stated in the following lemma.

Lemma 3.9. *The locally trivial fibration $\theta_\epsilon(f)$ coincides with the angular coordinate in a tubular neighbourhood of $N_{\rho,\epsilon}(f)$.*

Proof. Let $\eta > 0$ be small enough so that all the fibres $f^{-1}(t)$ intersect $M_{\rho,\epsilon}$ transversally for $|t|^2 < \eta$. Then $V_\eta := M_{\rho,\epsilon} \cap f^{-1}(\{|t|^2 < \eta\})$ is a tubular neighbourhood of the binding $N_{\rho,\epsilon}(f)$ in $M_{\rho,\epsilon}$, that is $V_\eta \cong N_{\rho,\epsilon}(f) \times D^2$. In this neighbourhood (outside of $N_{\rho,\epsilon}(f)$), the local coordinates can be given as $(x_1, \ldots, x_{2n-3}, |f|, \arg(f))$, where $(x_1, \ldots, x_{2n-3})$ represents the local coordinates on $N_{\rho,\epsilon}(f)$ and $(|f|, \arg(f))$ represents the polar coordinates on D^2. In particular, $\arg(f)$ coincides with the angular coordinates in this neighbourhood. ☐

This discussion gives a proof for Proposition 3.1.

Remark 3.4. In [1], it is also stated that for the fixed $f \in m_{X,x}$, the isotopy type of the open book $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ does not depend on either Φ or sufficiently small $\epsilon > 0$. This isotopy type, denoted by $(N(f), \theta(f))$, is called the *Milnor open book* of f. The pair $(M(X), N(f))$ is called the *link* of f. Although this is an important observation, without this isotopy type we can prove Theorem 1.1.

3.3. Compatibility

Let (X, x) be a germ of a complex analytic variety with the isolated singularity x. In the previous two sections, we fixed a spsh ρ (induced by an immersion Φ) on X^* and a sufficiently small $\epsilon > 0$. Then we constructed the contact boundary $(M_{\rho,\epsilon}, \xi_{\rho,\epsilon})$ in (X, x) with an open book decomposition $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ in $M_{\rho,\epsilon}$ associated with an arbitrary $f \in m_{X,x}$ which has an isolated singularity at x. In this construction, we gave precise formulae for the contact form α defining $\xi_{\rho,\epsilon}$ and the fibration $\theta_\epsilon(f)$. This was actually because we wish to verify the compatibility of $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ with $\xi_{\rho,\epsilon}$.

Theorem 3.1. *The Milnor open book $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ of $M_{\rho,\epsilon}$ is compatible with the contact structure $\xi_{\rho,\epsilon}$ of $M_{\rho,\epsilon}$.*

We are going to prove Theorem 3.1 by showing that $(N_{\rho,\epsilon}(f), \theta_\epsilon(f))$ and $\xi(X)$ satisfy the hypotheses of Lemma 2.2. The proof is highly technical. So we start with some lemmas which pack up the technical details.

Lemma 3.10. *The Reeb vector field of $\alpha = -d^c\rho$ is $i(\nabla\rho/\|\nabla\rho\|^2)$.*

Proof. Let R be the Reeb vector field of α. First, observe that $\iota_R\omega = \omega(R, \cdot)$ is a non-zero 1-form on X^*: otherwise, if $\omega(R, \cdot) = 0$ at some point $p \in X^*$, then $\ker \omega_p$ contains the non-zero vector R_p which contradicts the fact that ω is non-degenerate. Now, since $\omega(R, \cdot) = 0$ on $T(M_{\rho,\epsilon})$ by definition, it follows that $\ker(\iota_R\omega) = T(M_{\rho,\epsilon}) = \ker(d\rho)$ in TX^*. So $\iota_R\omega$ and $d\rho$ are 1-forms on X^* which define the same kernel. This implies

$\iota_R\omega = kd\rho$ for some smooth non-zero real valued function k on X^*. Therefore,

$$\omega(R,\cdot) = \iota_R\omega = kd\rho = g(k\nabla\rho,\cdot) = \omega(k\nabla\rho, i\cdot) = \omega(-ik\nabla\rho,\cdot).$$

Since ω is non-degenerate on X^*, it follows that $R = -ik\nabla\rho$. Furthermore, since

$$1 = \alpha(R) = -d^c\rho(R) = -d\rho(iR) = -d\rho(k\nabla\rho) = -kg(\nabla\rho,\nabla\rho) = -k\|\nabla\rho\|^2,$$

it follows that $k = -1/\|\nabla\rho\|^2$. $\square$

Lemma 3.11. *The contact distribution $\xi_{\rho,\epsilon}$ on $M_{\rho,\epsilon}$ is the orthogonal complement of $\mathbb{C}\cdot R = \mathbb{C}\cdot\nabla\rho$ in $TX^*|_{M_{\rho,\epsilon}}$ with respect to the Hermitian metric h.*

Proof. First, note that the equality $\mathbb{C}\cdot R = \mathbb{C}\cdot\nabla\rho$ follows from Lemma 3.10. Then notice that a tangent vector v to X^* is in the orthogonal complement of $\mathbb{C}\cdot R$ with respect to h if and only if for all $\lambda\in\mathbb{C}$,

$$\begin{aligned} & h(\lambda R, v) = \bar\lambda h(R,v) = \bar\lambda\omega(R,iv) + i\bar\lambda\omega(R,v) = 0 \\ \Leftrightarrow\ & \omega(R,v) = \omega(R,iv) = 0 \end{aligned}$$

Now, if v is a vector in $\xi_{\rho,\epsilon}\subset T(M_{\rho,\epsilon})$, then iv is also in $\xi_{\rho,\epsilon}\subset T(M_{\rho,\epsilon})$. Since R is the Reeb vector field on $M_{\rho,\epsilon}$, it follows that $\omega(R,iv) = \omega(R,v) = 0$.

On the other hand, if v is in $TX^*|_{M_{\rho,\epsilon}}\setminus\xi_{\rho,\epsilon}$, then iv is also in $TX^*|_{M_{\rho,\epsilon}}\setminus\xi_{\rho,\epsilon}$. So either v or iv is not in $T(M_{\rho,\epsilon})$. Without loss of generality, suppose $iv\notin T(M_{\rho,\epsilon}) = \ker(\iota_R\omega)$. Then $\omega(R,iv) = \iota_R\omega(iv)\neq 0$. This completes the proof. $\square$

Now fix $f\in m_{X,x}$. For simplicity, set $\theta := \arg(f)$, and for $c\in\mathbb{R}^+$ set $\alpha_c := e^{-c|f|^2}\alpha$ on $M_{\rho,\epsilon}$. Observe that α_c is also a positive contact form defining $\xi_{\rho,\epsilon}$ on $M_{\rho,\epsilon}$.

Let R_c denote the Reeb vector field associated to α_c.

Lemma 3.12. *For any $c > 0$, on $M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f)$ we have:*

$$d\theta(R_c) = e^{c|f|^2}(d\theta(R) + 2c|f|^2 \| \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta\|^2),$$

where $\operatorname{pr}_{\xi_{\rho,\epsilon}} : TX^*|_{M_{\rho,\epsilon}} \to \xi_{\rho,\epsilon}$ denotes the projection onto $\xi_{\rho,\epsilon}$ parallel to $\mathbb{C} \cdot R$.

Proof. Set $H := e^{-c|f|^2}$. Note that $R_c = k(R + S_c)$ for some smooth $k : M_{\rho,\epsilon} \to \mathbb{R} \setminus \{0\}$ and for some smooth section S_c of $\xi_{\rho,\epsilon}$. Indeed, $R_c = k(R+S_c)$ is a differential equation on $M_{\rho,\epsilon}$ which has unique solutions for k and S_c. Since

$$1 = \alpha_c(R_c) = H\alpha(k(R + S_c)) = kH(\alpha(R) + \alpha(S_c)) = kH,$$

one gets $k = 1/H = e^{c|f|^2}$. Now, $d\alpha_c = dH \wedge \alpha + Hd\alpha$, and for any tangent vector v in $\xi_{\rho,\epsilon} = \ker \alpha \subset T(M_{\rho,\epsilon})$,

$$0 = d\alpha_c(R_c, v) = (dH \wedge \alpha + Hd\alpha)\left(\frac{1}{H}(R + S_c), v\right) = -\frac{dH}{H}(v) + d\alpha(S_c, v).$$

This proves that for any v in $\xi_{\rho,\epsilon}$,

$$\frac{dH}{H}(v) = d\alpha(S_c, v) = \omega(S_c, v) = \iota_{S_c}\omega(v),$$

which is equivalent to saying that

$$(\iota_{S_c}\omega)|_{\xi_{\rho,\epsilon}} = \frac{dH}{H}\bigg|_{\xi_{\rho,\epsilon}} = \frac{Hd(-c|f|^2)}{H}\bigg|_{\xi_{\rho,\epsilon}} = -cd(|f|^2)|_{\xi_{\rho,\epsilon}}.$$

But on $X^* \setminus f^{-1}(0)$, by Equation (iii) of Lemma 3.3 we have:

$$d|f|^2 = g(\nabla |f|^2, \cdot) = \omega(-2i|f|^2 \nabla\theta, i\cdot) = \omega(-2|f|^2\nabla\theta, \cdot).$$

In particular, $\omega(S_c, \cdot)|_{\xi_{\rho,\epsilon}} = \omega(2c|f|^2\nabla\theta, \cdot)|_{\xi_{\rho,\epsilon}}$. Moreover, Lemma 3.11 implies that

$$\omega(S_c, \cdot)|_{\xi_{\rho,\epsilon}} = \omega(2c|f|^2\nabla\theta, \cdot)|_{\xi_{\rho,\epsilon}} = \omega(2c|f|^2 \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta, \cdot)|_{\xi_{\rho,\epsilon}}.$$

Since ω is also non-degenerate on $\xi_{\rho,\epsilon}$, it follows that $S_c = 2c|f|^2 \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta$. Then

$$\begin{aligned} d\theta(S_c) &= 2c|f|^2 d\theta(\operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta) = 2c|f|^2 g(\nabla\theta, \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta) = 2c|f|^2 g(\operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta, \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta) \\ &= 2c|f|^2 \| \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta\|^2. \end{aligned}$$

Thus, we get: $d\theta(R_c) = d\theta(k(R + S_c)) = e^{c|f|^2}(d\theta(R) + 2c|f|^2 \| \operatorname{pr}_{\xi_{\rho,\epsilon}} \nabla\theta\|^2)$. □

Proof of Theorem 3.1. Choose $V_\eta \cong N_{\rho,\epsilon}(f) \times D^2$ as in the proof of Lemma 3.9. For any t such that $|t|^2 < \eta$, set $V_t := f^{-1}(t)$. Observe that $V_t \setminus \{x\}$ is a complex submanifold in X^* for each t, and ρ restricts to a spsh function on $V_t \setminus \{x\}$. Let $N_{\rho,\epsilon}(f) \times \{z\}$ be a fibre in V_η. Then it equals some $V_t \cap M_{\rho,\epsilon}$ for a unique t. But $V_t \cap M_{\rho,\epsilon}$ is a level set of ρ restricted to $V_t \setminus \{x\}$. It follows that $\alpha = -d^c\rho$ restricts to a positive contact form on $V_t \cap M_{\rho,\epsilon}$. Hence, for any $z \in D^2$, α restricted to $N_{\rho,\epsilon} \times \{z\}$ is a positive contact form. For any $c > 0$, α_c satisfies the same property as well.

Next we show that there exists a $c > 0$ such that $d\alpha_c$ is a positive symplectic form on each fibre F' of θ in $(M_{\rho,\epsilon} \setminus \mathrm{int}V_\eta)$. Observe that on $(M_{\rho,\epsilon} \setminus \mathrm{int}V_\eta)$,

- $d\alpha_c$ is non-degenerate on the pages if and only if $d\theta(R_c) \neq 0$, and
- $d\alpha_c$ is positive on the pages, i.e. $(d\alpha)^{n-1}$ defines the orientation of each page, if

and only if $d\theta(R_c) > 0$.

So we find $c > 0$ such that $d\theta(R_c) > 0$. Since the Reeb vector field R and the 1-form $d\theta$ are smooth, $d\theta(R)$ is a smooth real-valued function on the compact set $(M_{\rho,\epsilon} \setminus \mathrm{int} V_\eta)$. So there exists an $m > 0$ such that $d\theta(R) > -m$ on $(M_{\rho,\epsilon} \setminus \mathrm{int} V_\eta)$. Set $Z_\epsilon := \{d\theta(R) \leq 0\}$ which is a closed (and so compact) subset of $(M_{\rho,\epsilon} \setminus \mathrm{int} V_\eta)$. Set

$$k := \min_{Z_\epsilon}(|f|^2 \|pr_{\xi_{\rho,\epsilon}} \nabla\theta\|^2).$$

which clearly exits and non-negative on the compact set Z_ϵ. Now we claim that $k \neq 0$.

Otherwise, if $k = 0$, since $|f|^2 > 0$ on $Z_\epsilon \subset (M_{\rho,\epsilon} \setminus N_{\rho,\epsilon}(f))$, then there exists $p \in Z_\epsilon$ such that $(pr_{\xi_{\rho,\epsilon}} \nabla\theta)_p = 0$. This implies that the non-zero vector $(\nabla\theta)_p$ is in the orthogonal complement of $(\xi_{\rho,\epsilon})_p$ with respect to h, which is $\mathbb{C} \cdot (\nabla\rho)_p$ by Lemma 3.11. Thus, there exists a $\lambda \in \mathbb{C}^*$ such that $(\nabla\theta)_p = i\lambda(\nabla\rho)_p$. But by Lemma 3.6, for $\epsilon > 0$ sufficiently small, $|\arg\lambda| < \frac{\pi}{2}$ and equivalently $\mathrm{Re}(\lambda) > 0$. So at the point p,

$$\begin{aligned} d\theta(R) = g(\nabla\theta, R) = g\left(i\lambda\nabla\rho, i\frac{\nabla\rho}{\|\nabla\rho\|^2}\right) &= \frac{1}{\|\nabla\rho\|^2} g(i\lambda\nabla\rho, i\nabla\rho) \\ &= \frac{1}{\|\nabla\rho\|^2}(g(\mathrm{Re}(\lambda)\nabla\rho, \nabla\rho) + \omega(\mathrm{Im}(\lambda)i\nabla\rho, i\nabla\rho)) \\ &= \mathrm{Re}(\lambda)\frac{g(\nabla\rho, \nabla\rho)}{\|\nabla\rho\|^2} = \mathrm{Re}(\lambda) > 0, \end{aligned}$$

which contradicts the fact that $d\theta(R) \leq 0$ on Z_ϵ. Hence, $k > 0$.

Set $c := m/k$ which is a positive real number. Consider the contact form α_c. On Z_ϵ we have:

$$d\theta(R_c) = e^{c|f|^2}(d\theta(R) + 2c|f|^2\|pr_{\xi_{\rho,\epsilon}} \nabla\theta\|^2) > e^{c|f|^2}\left(-m + 2\frac{m}{k}k\right) = me^{c|f|^2} > 0,$$

and on $(M_{\rho,\epsilon} \setminus \mathrm{int}V_\eta) \setminus Z_\epsilon$ we have:

$$d\theta(R_c) = e^{c|f|^2}(d\theta(R) + 2c|f|^2\|pr_{\xi_{\rho,\epsilon}}\nabla\theta\|^2) \geq e^{c|f|^2}d\theta(R) \geq d\theta(R) > 0.$$

Hence, $(d\theta)(R_c) > 0$ on $(M_{\rho,\epsilon} \setminus \mathrm{int}V_\eta)$. Since $\xi_{\rho,\epsilon}$ admits a positive defining contact form α_c, which satisfies the hypotheses of Lemma 2.2, it follows that the open book $(N_{\rho,\epsilon}(f), \theta(f))$ and the contact structure $\xi_{\rho,\epsilon}$ are compatible on $M_{\rho,\epsilon}$.

$\square$

4. HORIZONTAL OPEN BOOKS IN THE CONTACT BOUNDARY OF A SURFACE SINGULARITY

In this chapter, we focus on 3-dimensional Milnor fillable manifolds which are naturally the abstract boundary of surface singularities. For an arbitrary germ $(\mathcal{S}, 0)$ of an isolated surface singularity, first we find an $f \in m_{\mathcal{S},0}$ which has an isolated singularity at 0 and is resolved by the the minimal good resolution of $(\mathcal{S}, 0)$. Then considering $M(\mathcal{S})$ as a plumbed manifold, we construct a horizontal open book decomposition in it induced by f.

4.1. A Special Function

Let $(\mathcal{S}, 0)$ be an isolated surface singularity with a good resolution $p : (\Sigma, E) \to (\mathcal{S}, 0)$, where $E = p^{-1}(0)$. Let E_i, $i = 1, \ldots, r$, be the smooth irreducible curves in E, i.e. $E = p^{-1}(0) = \bigcup_{i=1}^{r} E_i$. Note that we also denote the divisor $\sum_{i=1}^{r} E_i$ on Σ by E.

In this section, first we will find an effective divisor D which has positive intersection with each E_i. Then using sheaf theoretic tools, we will show that this divisor indeed can be chosen to be the exceptional divisor of an $f \in m_{\mathcal{S},0}$. We start with some lemmas.

Lemma 4.1. *There exists an effective divisor on Σ whose intersection with E_i is non-positive, for any $i \in \{1, \ldots, r\}$.*

Proof. Fix a non-zero $g \in m_{\mathcal{S},0}$. Then $g \circ p$ is a holomorphic map on Σ. Observe that for any $j \in \{1, \ldots, r\}$, $E_j \subset \{g \circ p = 0\}$ since $g \circ p(E_j) = g(0) = 0$. So

$$\mathrm{div}(g \circ p) = C' + \sum_{j=1}^{r} a_j E_j,$$

where $a_j \geq 1$ for any j, and C' is the strict transform of $\{g = 0\} \subset \mathcal{S}$. We claim that the last term is effective, and since C' has non-negative intersection with each E_i, it satisfies the required property by Lemma 2.5. In fact,

$$\left(\sum_{j=1}^{r} a_j E_j\right) \cdot E_i \leq 0.$$

$\square$

Remark 4.1. The divisor we have found in the proof above is not just effective but also strictly greater than 0.

Lemma 4.2. *There exists an effective divisor* $D = \sum m_j E_j \neq 0$ *on* Σ *such that* $-D \cdot E_i \geq v_i + 2g_i$ *for any* $i \in \{1, \ldots, r\}$.

Proof. Let $D' = \sum_{j=1}^{r} a_j E_j$ be a divisor as in Lemma 4.1. Consider the matrix

$$-\begin{pmatrix} E_1^2 & \ldots & E_r \cdot E_1 \\ \vdots & \ddots & \vdots \\ E_1 \cdot E_r & \ldots & E_r^2 \end{pmatrix},$$

which is negative of the matrix representation of negative definite intersection form on the set of cycles on Σ. So this matrix is positive definite. Then it has an inverse, say A. Now set

$$\begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} := \alpha A \begin{pmatrix} v_1 + 2g_1 \\ \vdots \\ v_r + 2g_r \end{pmatrix}$$

for some $\alpha \in \mathbb{Z}^+$ such that $c_1, \ldots, c_r$ are all integers. Then choose $m \in \mathbb{Z}^+$ large enough so that $m_j := ma_j + c_j \geq 0$ for any j. Since each a_j is positive, such an m exists. Then

$$D := \sum_{j=1}^{r} m_j E_j$$

is a non-zero effective divisor on Σ. Moreover,

$$-\begin{pmatrix} E_1^2 & \dots & E_r \cdot E_1 \\ \vdots & \ddots & \vdots \\ E_1 \cdot E_r & \dots & E_r^2 \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = -m \begin{pmatrix} E_1^2 & \dots & E_r \cdot E_1 \\ \vdots & \ddots & \vdots \\ E_1 \cdot E_r & \dots & E_r^2 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} + \alpha \begin{pmatrix} v_1 + 2g_1 \\ \vdots \\ v_r + 2g_r \end{pmatrix},$$

which implies that for any $i \in \{1, \dots, r\}$,

$$\begin{aligned} -D \cdot E_i &= \left(\sum_{j=1}^{r} -m_j E_j \right) \cdot E_i = -m \left(\sum_{j=1}^{r} a_j E_j \right) \cdot E_i + \alpha(v_i + 2g_i) \\ &= -m(D' \cdot E_i) + \alpha(v_i + 2g_i) \geq \alpha(v_i + 2g_i) \geq (v_i + 2g_i). \end{aligned}$$

□

Let us fix a divisor D as in Lemma 4.2.

Remark 4.2. Notice that if $v_i + 2g_i > 0$, then automatically $-D \cdot E_i > 0$. On the other hand, if $v_i + 2g_i = 0$ which is equivalent to $r = 1$ and $g_1 = 0$, then $m_1 > 0$ and so $-D \cdot E_i = -a_1 E_i^2 > 0$. Therefore, $-D \cdot E_i$ is always positive.

Lemma 4.3. *$H^1(\mathcal{O}_\Sigma(-D - E)) = 0$.*

Proof. By the genus formula, $v_i + 2g_i = E \cdot E_i + K_\Sigma \cdot E_i + 2$ for any $i \in \{1, \dots, r\}$. Then Lemma 4.2 implies that for any i, $(-D - E) \cdot E_i \geq (-D - E) \cdot E_i - 2 \geq K_\Sigma \cdot E_i$. The result follows from the Laufer-Ramanujam vanishing theorem (see Theorem 2.4). □

Lemma 4.4. *There exists a function $f \in m_{\mathcal{S},0}$ which defines an isolated singularity at 0 such that:*

- *div$(f \circ p)$ is a normal crossing divisor in Σ,*
- *div$(f \circ p)_e = D$,*
- *div$(f \circ p)_s \cdot E_i$ is positive for any $i \in \{1, \ldots, r\}$.*

Proof. Consider the part $H^0(\mathcal{O}_\Sigma(-D)) \longrightarrow H^0(\mathcal{O}_E(-D)) \longrightarrow H^1(\mathcal{O}_\Sigma(-D-E))$ of the long exact sequence of cohomologies induced by the exact sequence $\mathcal{O}_\Sigma(-D-E) \longrightarrow \mathcal{O}_\Sigma(-D) \longrightarrow \mathcal{O}_E(-D)$ of sheaves. Since $H^1(\mathcal{O}_\Sigma(-D-E)) = 0$, it follows that the projection map $H^0(\mathcal{O}_\Sigma(-D)) \xrightarrow{\pi} H^0(\mathcal{O}_E(-D))$ is onto. In other words, the restriction map $\cdot|_E : \mathcal{O}_\Sigma(-D) \longrightarrow \mathcal{O}_E(-D)$ is onto. Notice that $\mathcal{O}_E(-D)$ is generated by global sections. Now choose a section $s \in \mathcal{O}_\Sigma(-D)$ such that $\mathrm{div}(s|_E) = D$. If we define a function f on $\mathcal{S}$ by

$$f(x) = \begin{cases} s \circ p^{-1}(x) & , \text{if } x \neq 0 \\ 0 & , \text{if } x = 0 \end{cases}$$

for any $x \in \mathcal{S}$, then f is a holomorphic function on $\mathcal{S}$ vanishing at 0, and it has an isolated singularity at 0. Moreover, by definition, $s = f \circ p$. So $\mathrm{div}(f \circ p) = \mathrm{div}(s)$ is a normal crossing divisor with $\mathrm{div}(f \circ p)_e = \mathrm{div}(s)_e = \mathrm{div}(s|_E) = D$. Such an f automatically satisfies the condition that

$$\mathrm{div}(f \circ p)_s \cdot E_i = \mathrm{div}(f \circ p) \cdot E_i - D \cdot E_i = -D \cdot E_i > 0$$

for any $i \in \{1, \ldots, r\}$. This completes the proof. □

4.2. Horizontal Open Books in the Abstract Boundary

Let $(\mathcal{S}, 0)$ be a surface singularity and $p : (\Sigma, E) \to (\mathcal{S}, 0)$ be a good resolution with the dual resolution graph $\Gamma(p)$ with vertices E_i, $i \in \{1, \ldots, r\}$. Then each vertex E_i of $\Gamma(p)$ is weighted by two integers (e_i, g_i), where e_i is the self-intersection and g_i is the genus of E_i. So $\Gamma(p)$ also serves as a weighted plumbing graph. Consider the plumbed 3-manifold $M(\Gamma(p))$ obtained by plumbing $\Gamma(p)$.

By Lemma 4.4, there exists an $f \in m_{\mathcal{S},0}$ such that f has an isolated singularity at 0, $\mathrm{div}(f \circ p)$ is normal crossing divisor in Σ and $n_i :=\mathrm{div}(f \circ p)_s \cdot E_i$ is positive for any i. Let $N(\underline{n})$ be a vertical link in $M(\Gamma(p))$ associated to $\underline{n} := (n_1, \ldots, n_r)$.

Lemma 4.5. *The abstract boundary $M(\mathcal{S})$ associated to the surface singularity $(\mathcal{S}, 0)$ is diffeomorphic to $M(\Gamma(p))$. Moreover, the Milnor open book $(N(f), \theta(f))$ associated to f in $M(\mathcal{S})$ is isomorphic to a horizontal open book $(N(\underline{n}), \theta)$ in $M(\Gamma(p))$.*

For a proof of Lemma 4.5 see [13], page 333.

Proposition 4.1. *Let M be a Milnor fillable contact 3-manifold. There exists an open book decomposition (N, θ) in M, which is determined by the topology of M, such that for any Milnor filling $(\mathcal{S}, 0)$ of M there exists $f \in m_{\mathcal{S},0}$ which has an isolated singularity at 0 whose Milnor open book $(N(f), \theta(f))$ is isomorphic to (N, θ).*

Proof. Given a Milnor filling $(\overline{\mathcal{S}}, 0)$ of M with the minimal good resolution $\bar{p} : (\bar{\Sigma}, \bar{E}) \to (\overline{\mathcal{S}}, 0)$. Let Γ be the associated minimal good resolution graph. Choose a divisor D in $\bar{\Sigma}$ as in Lemma 4.2. Then there exists $\bar{f} \in m_{\overline{\mathcal{S}},0}$ as in Lemma 4.4. Let (N, θ) be the horizontal open book decomposition in M associated to $\bar{f}$. Note that since the minimal good resolution graph Γ is determined by the topology of M, so is (N, θ). Keep in mind that the intersection form $I(\Gamma)$ of Γ is non-degenerate.

Now, let $(\mathcal{S}, 0)$ be any Milnor filling of M. Then $(\mathcal{S}, 0)$ has the same minimal resolution graph Γ. Following same steps as above, we can find $f \in m_{\mathcal{S},0}$ (similar to $\bar{f}$)

of which horizontal open book decomposition, say (N', θ'), in M has the same number of vertical link components with (N, θ). By Proposition 2.1, (N, θ) is isomorphic to (N', θ') and by Lemma 4.5, (N', θ') is isomorphic to Milnor open book $(N(f), \theta(f))$. This completes the proof. □

4.3. Proof of the Main Theorem

Let M be a Milnor fillable 3-manifold with two Milnor fillings $(\mathcal{S}, 0)$ and $(\mathcal{S}', 0)$, and ξ and ξ' be the contact structures on M induced from $(\mathcal{S}, 0)$ and $(\mathcal{S}', 0)$ respectively. By Proposition 4.1, there exists an open book decomposition (N, θ) in M such that for some $f \in m_{\mathcal{S},0}$ and $f' \in m_{\mathcal{S}',0}$, (N, θ) is isomorphic to both $(N(f), \theta(f))$ and $(N(f'), \theta(f'))$ in M. By Theorem 3.1, $(N(f), \theta(f))$ and $(N(f'), \theta(f'))$ are compatible with ξ and ξ' respectively. Therefore, by Lemma 2.3, ξ and ξ' are isotopic on M. In particular, (M, ξ) and (M, ξ') are contactomorphic. Thus, any Milnor filling of M induces the same contact structure on M up to contactomorphism.

□

5. CONCLUSIONS

In this thesis, we obtained the following results:

(i) The level set $\rho^{-1}(\epsilon)$ of a strictly pluri-subharmonic function ρ induced by a holomorphic immersion Φ on a germ (X, x) of an isolated singularity of an irreducible analytic variety is a closed, connected, contact, oriented, codimension 1 manifold in X^*, which is called the contact boundary. The contact isotopy type of this manifold does not depend either on the immersion Φ or on ϵ.

(ii) For any holomorphic function f on (X, x) which has an isolated singularity and zero at x, there exists a Milnor open book decomposition in the contact boundary. The binding of this open book is the set of zeros of f in the contact boundary, and the locally trivial fibration is $\arg(f)$ restricted to the contact boundary outside of the binding.

(iii) For any holomorphic function f on (X, x) which has an isolated singularity and zero at x, the Milnor open book of f is compatible with the contact structure on the contact boundary.

(iv) If (X, x) is a surface singularity, then for a suitable choice of f, the Milnor open book associated to f is determined by the topology of the contact boundary of (X, x).

(v) The main result that any Milnor fillable 3-manifold admits a unique Milnor fillable contact structure up to contactomorphism followed from the third and the fourth result: Diffeomorphic Milnor fillable 3-manifolds contain isomorphic copies of the Milnor open book determined by their topology. These open books are compatible with the contact structure of the contact boundary. Finally, contact structures of the diffeomorphic contact boundaries are also contactomorphic.

Even though the statement of the Theorem 1.1 appears to have a contact geometry content, its proof uses the compatibility with Milnor open book decompositions, which is a topological notion. The compatibility of Milnor open books and Milnor fillable contact structures is also a quite important result and it has played a significant role in the study of Milnor fillable manifolds. Note also that proving the existence of a Milnor open book determined by the topology of the link of an isolated surface singularity requires a serious use of algebraic geometry regarding irreducible analytic varieties.

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