BLACK HOLES, COSMOLOGY AND BRANS-DICKE THEORIES IN 4+1 DIMENSIONAL UNIVERSE

by

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ABSTRACT

BLACK HOLES, COSMOLOGY AND BRANS-DICKE THEORIES IN 4+1 DIMENSIONAL UNIVERSE

We consider the embedding of $3+1$ dimensional cosmology in $4+1$ dimensional Jordan-Brans-Dicke theory. We show that exponentially growing and power law scale factors are implied. We solve the Brans-Dicke equations and we realize that only a linear warp factor satisfies these equations. In both exponentially expanding and power law expanding cases we find out that the scalar field is a function of the warp factor and the scale factor. Whereas the $4+1$ dimensional scalar field is approximately constant for each case, the effective $3+1$ dimensional scalar field is constant for exponentially growing scale factor and time dependent for power law scale factor. We calculate the effective gravitational constant and realize that same results are valid.

We construct a static solution for $4+1$ dimensional bulk such that the $3+1$ dimensional world has a linear warp factor and describes the Schwarzschild-dS_4 black hole. The $4+1$ dimensional space-time is taken to be flat. For zero mass this four dimensional universe and Friedmann–Robertson–Walker universe are related with an explicit coordinate transformation. Also we realize that if there is a contribution from the mass term, both energy momentum and cosmological constant vanish in the bulk. We explain the four dimensional cosmological constant originates from the hidden brane and its effects cause the localized mass densities in our visible brane. Using hierarchy between the Planck mass and electroweak energy scale, we obtain the size of the extra dimension as close to Hubble length but smaller than it. We emphasize that for linear warp factors the effect of bulk on the brane world shows up as the dS_4 background which is favored by the Big Bang cosmology.

ÖZET

KARA DELİKLER, KOZMOLOJİ VE 4+1 BOYUTLU EVRENDE BRANS-DICKE TEORİSİ

3+1 boyutlu kozmolojinin 4+1 boyutlu Jordan-Brans-Dicke teorisine gömülmesi ele alınmıştır. Ölçek çarpanının exponansiyel olarak ve kuvvet şeklinde artabileceğini gösterdik. Brans-Dicke denklemlerini çözdük ve sadece lineer eğrilik faktörünün bu denklemleri sağladığını saptadık. Ölçek çarpanının exponansiyel artması durumunda da, kuvvet şeklinde artması durumunda da skaler alanın eğrilik faktörünün ve ölçek çarpanının fonksiyonu olduğunu bulduk. Her iki durum için 4+1 boyutlu skaler alan yaklaşık olarak sabit olmasına karşın, etkin 3+1 boyutlu skaler alan exponansiyel artan ölçek çarpanı için sabit, kuvvet şeklinde artan ölçek çarpanı için zamana bağlıdır. Etkin gravitasyonel sabiti hesapladık ve aynı sonuçların geçerli olduğunu gördük.

4+1 boyutlu evren içinde lineer eğrilik faktörüne sahip Schwarzschild-dS_4 kara delik olarak tanımlanan 3+1 boyutlu evren için statik (durgun) çözüm bulduk. 4+1 boyutlu uzay-zamanı düz olarak aldık. Sıfır kütle için bu dört boyutlu uzay ve Friedmann-Robertson-Walker evreni koordinat dönüşümü ile ilişkilendirdik. Ayrıca eğer kütle terimi olursa hem enerji momentumun hemde kozmolojik sabitin gövde içinde sıfır olduğunu gördük. Dört boyutlu kozmolojik sabitin görünmeyen evrenden kaynaklandığını ve etkilerinin bizim görünür evrenimizdeki lokalize olmuş kütle yoğunluklarına neden olduğunu açıkladık. Planck kütlesi ve elektrozayıf enerji skalası arasındaki hiyerarşiyi kullanarak ek boyutun büyüklüğünün Hubble uzunluğuna yakın fakat ondan küçük olduğunu elde ettik. Lineer eğrilik faktörü için gövde evren üzerine etkisinin büyük patlama kozmolojisi tarafından desteklenen dS_4 şeklinde uzay olduğunu vurguladık.

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LIST OF SYMBOLS/ABBREVIATIONS

G_4	Newton's constant in $3+1$ dimensional space-time
G_{MN}	Einstein Tensor
g_{MN}	Metric of $4+1$ dimensional space-time
M	$4+1$ dimensional Planck mass
M_P	$3+1$ dimensional Planck mass
R	Ricci Scalar
R_{MN}	Ricci Tensor
R_{MNPR}	Riemann Tensor
T_{MN}	Einstein Tensor
$\Gamma^{\lambda}_{\mu\nu}$	Christoffel symbol
Λ	Cosmological constant

ADD	Arkani Hamed-Dimopoulos-Dvali
AdS	Anti-de Sitter
BD	Brans-Dicke
EH	Einstein-Hilbert
EW	Electroweak
FRW	Friedmann-Robertson-Walker
JBD	Jordan-Brans-Dicke
KK	Kaluza-Klein
LHC	Large Hadron Collider
RS	Randall-Sundrum
RWFL	Robertson-Walker-Friedmann-Lemaitre
SdS	Schwarzschild-de Sitter
WEP	Weak Equivalence Principle
WKB	Wentzel-Kramers-Brillouin

1. INTRODUCTION

Unifying all the forces in nature is one of the greatest problem in physics. Especially, the difficulty is to reconcile quantum theory and general relativity. Since general relativity is a highly nonlinear theory, gravity can not be quantized. Although string theory and loop quantum gravity try to unify them, there is no fully satisfactory quantum theory of gravity yet. Here we will not try to review these subjects in detail. A string is a one dimensional object which lives in space-time. The concept can be generalized to a p-brane which is a p dimensional object which lives in space-time. One can formulate a theory in which the 3+1 dimensional space-time we observe is a 3-brane in higher dimensional space-time. In some theories, an observer on the p-brane predicts the existence of extra dimensions only by gravitational interactions. These extra dimensions can roll up to small circles and they are visible only at very high energies. Certainly this theory is a logical possibility which has not been empirically verified until now.

Originally, the idea of extra dimension has been first introduced by Nordstrom, Kaluza and Klein in the 1920s in order to unify gravity and electromagnetism. Many years later, in the 1970s, string theorists revitalized this idea in order to quantize gravity. In the 1980s Arkani-Hamed, Dimopoulos and Dvali proposed (3+1+n)-dimensional flat space-time, in which the extra n dimensions are compact with radius R. In this model all standard model particles are confined in the brane, however only gravity propagates on the extra dimensions. In this model the large volume of the extra dimensions can help to solve the hierarchy problem by the relation

$$M_P^2 = M^{2+n} R^n, \tag{1.1}$$

where the traditional four-dimensional Planck scale, M_P, is an effective energy scale derived from higher dimensional fundamental Planck scale M and R is the radius of the compactified extra dimensions. Since gravity can be tested down to sizes of around a millimeter, R could be as large as a fraction of a millimeter. This requires that there must be more than one extra dimension.

Meanwhile, Randall and Sundrum proposed non-flat (warped) bulk geometries. They consider (4+1)-dimensional AdS space-time with warped extra dimension where all standard model particles are confined on the brane and only gravity can propagate in the fifth dimension. Their mechanism offers a solution of the hierarchy problem which is generated by the appropriate inter-brane distance (the radion),

$$M_P^2 = e^{2kr_c} M^3/k, \tag{1.2}$$

where M_P is the Planck mass we measure on our brane, M is the five dimensional Planck mass, k is a constant and r_c is the compactification radius. Thus, if M is around the electroweak scale TeV, we need $kr_c = 50$, in order to get Planck scale on our brane. In this theory radion needs to be stabilized, therefore this mechanism is not truly understood yet.

The introduction of extra dimensions change all the existing theories. All standard cosmology needs modification in the presence of extra dimension. Similarly, the properties of black holes are also effected by the introduction of higher dimensional space-time and Brans-Dicke theory as well. In this thesis we try to explain these two topics from the $4+1$ dimensional space-time point of view.

The main body of this thesis was published in two articles corresponding to chapters 6 and 7. The purpose of the remaining parts is to provide an introduction which has already been carried out in the literature.

In section 2 we briefly review standard cosmology and particularly emphasize the issues which are related with our work. Starting with the cosmological principle we consider the Robertson-Walker-Friedmann universe. Then we write the Robertson-Walker metric in different forms by using coordinate transformations. We particularly pay attention to de-Sitter universe in section 2.3 to construct the basis for our article which is directly related to de-Sitter space-time. In section 3, we discuss the extra dimensions in general and then we especially focus on the large extra dimensional models of ADD and RS. In chapter 4, we review the Schwarzschild black holes in $3+1$ dimensional space-time briefly and then we mention some earlier works on higher dimensional black holes. Firstly, Chambling et al. studied the collapse to a black hole in the Randall-Sundrum brane world scenario. Their model originates from the idea that if matter on a 3-brane collapses under gravity without rotating to form a black hole, then the metric on the brane world should be close to the Schwarzschild metric in order to pass observational tests. This suggestion also coincides with our black hole model.

In chapter 5, we discuss standard general relativity and Jordan-Brans-Dicke generalization of general relativity obtained by replacing the inverse of the Newton gravitational constant by a scalar field. We explain the general relativity limit of BD theory in general as well as other models which fail to converge to general relativity at the corresponding limit.

In chapter 6, we present our work “Brane-world cosmology in Jordan-Brans-Dicke theory”. We construct five dimensional BD action (6.6) with a scalar potential. We consider our universe to be $3+1$ brane embedded in $4+1$ dimensional space-time with an unknown warp factor (6.8). We obtain the BD equations in the bulk and via the jump condition we find out the equation for matter on the brane. The next step is to solve BD equations in the bulk. In the beginning, BD equations restrict the Brans-Dicke scalar field $\phi(t, W)$ to factorize into a function of time t and a function of the extra dimension W. Then we consider the scalar factor $a(t)$ in two different cases: in the first case the universe is exponentially expanding and in the second one the universe is power law expanding. The first claim causes $3+1$ dimensional brane world to be de-Sitter space -time which is embedded in $4+1$ dimensional bulk with a linear warp factor $f(W)$. This result yields that the scalar field $\phi(t, W)$ is a function of scale factor and warp factor only. Then we discuss the energy density in the bulk. If the energy momentum tensor vanishes in the fifth coordinate, the other components of energy momentum tensor also vanish. This means there is an empty universe in $4+1$ dimensional space-time, actually our five dimensional space-time also reduces to Minkowski universe. In this empty bulk, the scalar potential is time independent and depends only fifth coordinate W and approaches to zero for the limit of infinite Brans-Dicke parameter ω. On the other hand, at this limit, the scalar field on the brane becomes a constant. For the case of nonvanishing energy momentum tensor in the fifth dimension, we get a cosmological constant dominated era, $\rho_{\text{bulk}} = -p_{\text{bulk}}$ as $\omega \to \infty$. The scalar potential again becomes time independent and scalar field is constant on the brane for this limit. Therefore applying this general relativistic limit, we derive an expression for the gravitational coupling constant.

In the second part of this chapter we investigate the solutions for the power law expanding scale factor $a\left(t\right)$. Solving the BD equation we get the same linear warp factor and the scalar field to be a function of scale factor and the warp factor. This power law expanding scale factor provides a radiation dominated universe, $\rho_{\text{bulk}} = 3p_{\text{bulk}}$ and energy density in the fifth coordinate to be zero. As BD parameter ω goes to infinity, scale factor $a\left(t\right)$ becomes $a_0\left(t/t_0\right)^{1/4}$, scalar potential vanishes and scalar field approaches but does not become a constant. Moreover for $a\left(t\right) = a_0\left(t/t_0\right)$, BD equations satisfy an empty bulk with vanishing scalar potential and the scalar field is same as the radiation dominated bulk. These two results represent decelerating cosmology for radiation dominated universe and expanding cosmology with constant velocity for empty universe. In the final part of this chapter we derive the effective four dimensional gravitational constant by integrating the five dimensional action over the extra dimension W. For the exponentially increasing scale factor we get time independent gravitational constant. On the other hand, for the power law scale factor, the effective gravitational constant becomes time dependent.

In chapter 7 we present our work "Schwarzschild-de Sitter Black Holes in $(4+1)$ dimensional bulk". We take our universe to be Schwarzschild-de Sitter black hole embedded in empty $4+1$ dimensional space-time (7.3) with the linear warp factor. The case for zero mass represents FRW universe in $3+1$ dimensional space-time. We establish the coordinate transformation for all the cases $k = -1, 0, 1$ of the FRW metric. Our main aim is to investigate this topic with $M \neq 0$. The setup is similar to the RS brane world model but this time empty bulk and curved brane is taken into account.

We suppose that the visible brane is located at $w = w_1$ and the hidden brane is at $w = 0$. We get the Einstein tensor on the bulk and the induced Einstein tensor on the visible and hidden branes separately. There is a contribution not only to the Einstein tensor but also to the four dimensional Planck mass from the warp factor on the brane at $w = w_1$. To get the mass parameter on the brane we use Higgs action and then we get the relation $m = \left(1 - \frac{|w_1|}{w_0}\right) m_0$ between the fundamental mass m_0 in higher dimensional theory and the physical mass m on the visible world. To get observable TeV mass scale from the Planck mass scale which is of the order of 10^{16} TeV, a huge fine tuning 10^{-15} is needed. This result causes the place of visible brane w_1 to be near the curvature singularity w_0. Then using the observable value of the cosmological constant, we conclude that location of a brane at any place throughout the extra dimension varies the mass density only on this brane and cosmological constant remains at the same value on every brane located in the bulk. Applying this consideration and using the observational data we get a very large mass density 10^3kg/m^3 for our brane world at $w = w_1$. This result corresponds to localized stellar energy densities on the visible brane world. In the final part of this chapter we derive the correction to gravitational potential. We determine the wave function by using the fluctuations of the metric and then we get the modified Newton's potential. However the correction term conflicts with the experimental measurements.

2. COSMOLOGY

2.1. The Cosmological Principle

Cosmology is based on a cosmological principle or Copernican principle which simplifies the mathematical calculations. The cosmological principle is based on the idea that at large scales, the universe is maximally symmetric and presents the same aspect from every point , except for local irregularities. Its validity on such scales is manifested by the observations coming from the cosmic microwave background (CMB). In 1965 Penzias and Wilson noticed a uniform background of 'cosmic photons' corresponding to blackbody radiation of 2.7 K [1]. Solar system or other galaxies causes the deviations from regularity, these local irregularities in the universe are on the order of $< 10^{-5}$ which is adequate for an approximation of isotropic and homogeneous universe. Here isotropy states that the space looks same in all directions and homogeneity claims that space is same throughout the manifold. Therefore if a space is isotropic everywhere then it is homogeneous. On the other hand, a homogeneous space may not be isotropic at the same space-time. In short, isotropy everywhere implies homogeneity but homogeneity does not imply isotropy.

Cosmological observations imply that universe is expanding with acceleration. In 1929 Hubble discovered that the spectra of nearby galaxies shifted towards the red which is interpreted as Doppler shift from receding galaxies [2]. This experiment became an end to the theory of a static universe.

To find an explanation for the observed structures, we must look at the very early universe. The blackbody nature of CMB and expanding universe is related with the very hot and dense state known as the Big Bang in the very early universe. Big Bang is the initial singularity where universe was filled with very high energy density. This motivates investigations of inflationary universe.

All the properties of spatial isotropy, homogeneity, expanding space-time and Big

Bang nature of the universe combined with general relativity can be well described by the Friedmann-Roberson-Walker line element. We introduce this topic following the treatment of [3] and [4].

2.2. Friedmann-Robertson-Walker Metrics

Maximally symmetric (isotropic and homogeneous) metric has constant curvature and obeys the equation

$$R_{\mu\nu\alpha\beta} = K\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right), \tag{2.1}$$

where for n dimensional manifold, K is a normalized measure of the Ricci curvature $K = \frac{R}{n(n-1)}$ and Ricci scalar R is constant in this manifold. The geometries of the above spaces depend on weather the curvature is $+$, $-$ or 0. For three dimensional space, contracting (2.1) with $g^{\mu\alpha}$

$$\begin{aligned} g^{\mu\alpha}R_{\mu\nu\alpha\beta} &= R_{\nu\beta} \\ &= Kg^{\mu\alpha}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right) \\ &= K\left(3g_{\nu\beta} - \delta^{\mu}_{\nu}g_{\mu\beta}\right) \\ &= 2Kg_{\nu\beta}. \end{aligned} \tag{2.2}$$

The isotropic spatial part of the line element must be spherically symmetric and can be expressed as

$$d\sigma^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.3}$$

where r is invariant under rotations, the angular part of the metric $\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is also invariant under rotations. The radial coordinate r has been chosen so that a circle with center at the origin has circumference $2\pi r$. This metric is isotropic about $r = 0$. The non-vanishing component of the Ricci tensor for such a spherically symmetric

metric is

$$\begin{aligned} R_{11} &= \frac{2}{r}\partial_1\lambda && (2.4)\\ R_{22} &= R_{33}\csc^2\theta = e^{-2\lambda}\left(r\partial_1\lambda - 1\right) + 1. && (2.5)\end{aligned}$$

We use the condition for a constant curvature (2.2) and solve for $\lambda(r)$:

$$\lambda(r) = -\frac{1}{2}\ln\left(1 - Kr^2\right), \tag{2.6}$$

which yields the metric of the 3−space of constant curvature

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2, \tag{2.7}$$

where K can be positive, negative or zero and the spherical part in the parenthesis in Equation (2.3) is the sphere S^2 and written as $d\Omega_2^2$. For $K \neq 0$, we define k by $K = |K|\, k$ and introduce a rescaled radial coordinate

$$r^* = |K|^{1/2}\, r, \tag{2.8}$$

then Equation (2.7) becomes

$$d\sigma^2 = \frac{1}{|K|}\left[\frac{dr^{*2}}{1 - kr^{*2}} + r^{*2} d\Omega_2^2\right]. \tag{2.9}$$

The curvature k is $-1, 0$ or $+1$ depending on the Ricci curvature K.

Mathematically, the $k = -1$ case corresponds to constant negative curvature on 3-space and this space is called open, where if

$$\begin{aligned} r &= \sinh\chi, && (2.10)\\ dr &= \cosh\chi d\chi = \left(1 + r^2\right)^{1/2} d\chi, && (2.11)\end{aligned}$$

and the line element becomes

$$d\sigma^2 = \frac{1}{|K|}\left[d\chi^2 + \sinh^2\chi d\Omega_2^2\right], \tag{2.12}$$

which has open topology. Here we dropped the stars on the radial coordinate r.

The $k = 0$ case corresponds to no curvature on 3−space and is called flat. The line element is

$$d\sigma^2 = \frac{1}{|K|}\left[dr^2 + r^2 d\Omega_2^2\right], \tag{2.13}$$

which is clearly three dimensional Euclidean space.

Finally the $k = +1$ case corresponds to positive curvature of space and is called closed. If

$$r = \sin\chi, \tag{2.14}$$

$$dr = \cos\chi = \left(1 - r^2\right)^{1/2} d\chi, \tag{2.15}$$

then the line element becomes

$$d\sigma^2 = \frac{1}{|K|}\left(d\chi^2 + \sin^2\chi d\Omega_2^2\right), \tag{2.16}$$

which is the 3 sphere.

Cosmological observations show that the universe is evolving in time. Thus the space-time metric takes the form of

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\, d\sigma^2, \\ &= -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2\right], \end{aligned} \tag{2.17}$$

where $1/|K|$ is absorbed into the function $a(t)$. Here t is time and the scale factor $a(t)$ describes the expansion of universe in time.

We can transform radial coordinate r to a new radial parameter as

$$r = \frac{\bar{r}}{\left(1 + \frac{1}{4}k\bar{r}^2\right)}, \tag{2.18}$$

in which Equation (2.9) takes the form of

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}{\left(1 + \frac{k}{4}\bar{r}^2\right)^2}\right]. \tag{2.19}$$

where the second therm in parenthesis constitutes the homogeneous and isotropic properties of space. This spatially homogeneous and isotropic universe is evolving in time and is called the Friedmann-Robertson-Walker (FRW) Universe.

2.3. Metrics of Spaces of Constant Curvature in FRW Form

As we have discussed in the previous section, the $3+1$ dimensional manifold of constant curvature are locally characterized by the condition

$$R_{\mu\nu\alpha\beta} = \frac{R}{12}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right). \tag{2.20}$$

The space of constant curvature with $R = 0$ is *Minkowski space-time* which is the simplest empty space-time in General and Special Relativity. It is the manifold R^4 with a flat Lorentz metric. In terms of the coordinates (x^4, x^1, x^2, x^3) on R^4, the metric is,

$$ds^2 = -\left(dx^4\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2. \tag{2.21}$$

If one uses spherical polar coordinates (t, r, θ, ϕ), the metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right), \tag{2.22}$$

with the new coordinates $x^4 = t, \quad x^1 = r \sin\theta \sin\phi, \quad x^2 = r \sin\theta \cos\phi, \quad x^3 = r\cos\theta$. Here $0 < r < \infty, \quad 0 < \theta < \pi$ and $0 < \phi < 2\pi$. This metric is singular for $r = 0$, and $\sin\theta = 0$, but this is a coordinate singularity and it can be eliminated by using suitable coordinates like (2.21).

An alternative coordinate system is given by choosing the so called advanced and retarded null coordinates v, w defined by $t = \frac{1}{2}(v + w)$ and $r = \frac{1}{2}(v - w)$, in which the metric becomes,

$$ds^2 = -dvdw + \frac{1}{4}(v - w)^2 d\Omega_2^2, \tag{2.23}$$

where $-\infty < v < \infty$ and $-\infty < w < \infty$. The absence of terms dv^2 and dw^2 corresponds to the fact that the surfaces $v =$constant, $w =$constant are null.

Another form of the four dimensional Minkowski metric is written as

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{1}{2}(t' + r')\right)\sec^2\left(\frac{1}{2}(t' - r')\right)\left[-dt'^2 + dr'^2 + \sin^2 r' d\Omega_2^2\right], \tag{2.24}$$

with the coordinate transformations $t = \frac{1}{2}\left(\tan\left(\frac{1}{2}(t' + r')\right) + \tan\left(\frac{1}{2}(t' - r')\right)\right)$ and $r = \frac{1}{2}\left(\tan\left(\frac{1}{2}(t' + r')\right) - \tan\left(\frac{1}{2}(t' - r')\right)\right)$ for $-\pi < t' + r' < \pi, \quad -\pi < t' - r' < \pi, \quad r' \geq 0$.

The space for $R > 0$ is called *de Sitter space-time,* which has the topology of $R^1 \times S^3$. It is visualized as the hyperboloid

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2, \tag{2.25}$$

embedded in a five dimensional Minkowski space with the metric

$$-dv^2 + dw^2 + dx^2 + dy^2 + dz^2 = ds^2 \tag{2.26}$$

One can introduce coordinates (t, χ, θ, ϕ) on the hyperboloid by the relations

$$\begin{aligned} v &= \sinh t, \quad w = \cosh t \ \cos\chi, \quad x = \cosh t \ \sin\chi \ \cos\theta, \\ y &= \cosh t \ \sin\chi \ \sin\theta \ \cos\phi, \quad z = \cosh t \ \sin\chi \ \sin\theta \ \sin\phi \end{aligned} \tag{2.27}$$

in these coordinates, the metric has the form

$$ds^2 = -dt^2 + \cosh^2 t \left(d\chi^2 + \sin^2 \chi d\Omega^2\right) \tag{2.28}$$

and for $\chi = \arcsin r$.

$$ds^2 = -dt^2 + \cosh^2 t \left(\frac{dr^2}{1 - r^2} + r^2 d\Omega_2^2\right) \tag{2.29}$$

Then for $r = \bar{r} / \left(1 + \frac{\bar{r}^2}{4}\right)$, this metric can be written in the closed form of the Robertson-Walker metric,

$$ds^2 = -dt^2 + \cosh^2 t \left[\frac{d\bar{r}^2 + \bar{r}^2 d\Omega_2^2}{\left(1 + \frac{1}{4}\bar{r}^2\right)^2}\right]. \tag{2.30}$$

Another important coordinate transformation of (2.26) can be made by the relations

$$\begin{aligned} v &= \sinh\tau + \frac{1}{2}\rho \exp\tau, \quad w = \cosh\tau + \frac{1}{2}\rho\exp\tau, \quad x = \rho\exp\tau \ \cos\theta, \\ y &= \rho\exp\tau \ \sin\theta \ \cos\phi, \quad z = \rho\exp\tau \ \sin\theta \ \sin\phi, \end{aligned} \tag{2.31}$$

the metric becomes the well known form of the de Sitter metric

$$ds^2 = -d\tau^2 + \exp(2\tau) \left(d\rho^2 + \rho^2 d\Omega_2^2\right). \tag{2.32}$$

By making the coordinate transformations,

$$\tau = t' + \frac{1}{2}\ln\left(1 - r'^2\right), \quad \rho = r' \exp\left(-t'\right)\frac{1}{\left(1 - r'^2\right)^{1/2}} \tag{2.33}$$

this metric can be transformed into

$$ds^2 = -\left(1 - r'^2\right)dt'^2 + \left(1 - r'^2\right)^{-1}dr'^2 + r'^2 d\Omega_2^2. \tag{2.34}$$

This metric can be obtained from the Schwarzschild-de Sitter metric

$$ds^2 = -\left(1 - \frac{2M}{r'} - \frac{r'^2}{k^2}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r'} - \frac{r'^2}{k^2}\right)}dr'^2 + r'^2 d\Omega_2^2$$

by putting $M = 0$. The inverse coordinate transformation from the de Sitter type (2.34) to Schwarzschild-de Sitter metric (2.32) can be done by the following relations

$$\begin{aligned} r' &= \rho \exp(\tau) \\ t' &= \tau - \frac{1}{2}\ln\left(1 - \rho^2 \exp(2\tau)\right). \end{aligned} \tag{2.35}$$

Then Equation (2.34) is transformed directly into closed spacelike sections of FRW metric (2.28) with the following transformation,

$$r' = \cosh t \; \sin\chi \tag{2.36}$$

$$t' = \ln\left\{\frac{\sinh t + \cosh t \; \cos\chi}{\left(1 - \cosh^2 t \; \sin^2\chi\right)^{1/2}}\right\} \tag{2.37}$$

The space of constant curvature with $R < 0$ is called *anti-de Sitter space-time.* It has the topology $S^1 \times R^3$, and represented as the hyperboloid

$$-v^2 - w^2 + x^2 + y^2 + z^2 = 1, \tag{2.38}$$

in the five dimensional space R^5

$$ds^2 = -dv^2 - dw^2 + dx^2 + dy^2 + dz^2. \tag{2.39}$$

If we introduce coordinates $\{t, \rho, \theta, \phi\}$ on the hyperbola via

$$\begin{aligned} v &= \sin t \ \cosh\rho, \quad w = \cos t \ \cosh\rho, \quad x = \sinh\rho \ \cos\theta \\ y &= \sinh\rho \ \sin\theta \ \cos\phi, \quad z = \sinh\rho \ \sin\theta \ \sin\phi, \end{aligned} \tag{2.40}$$

yielding the metric in the form

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2. \tag{2.41}$$

2.4. Cosmological Equations, the Friedmann-Lemaitre Equation

The application of Einstein's equation into FRW line element yields the Friedmann-Lemaitre equations. Since the space part of the universe is isotropic and homogeneous, energy and matter is considered as a perfect fluid. Based on this idea, starting with the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_4 g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda_4 g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.42}$$

the components of the Einstein tensor constructed from the FRW metric (2.17) is

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \tag{2.43}$$

$$G_{ii} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}, \tag{2.44}$$

where the dot denotes a derivative with respect to time and $a(t)$ is the scale factor. The energy momentum tensor

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \tag{2.45}$$

where the four velocity is $U_\mu = (1, 0, 0, 0)$ i.e the cosmological fluid is at rest in these coordinates. Then energy-momentum tensor becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} p & \\ 0 & & & \end{pmatrix}. \tag{2.46}$$

The first Friedmann's equation is obtained by using 00 component of Einstein equation (2.43)

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho + \frac{\Lambda_4}{3}. \tag{2.47}$$

Using ii component of Einstein equation (2.43) and (2.44) the second Friedmann's equation is obtained as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda_4}{3} \tag{2.48}$$

By homogeneity and isotropy energy density and pressure can only be functions of time. With one index raised the energy momentum tensor (2.46) takes the form

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p), \tag{2.49}$$

and trace of this matrix is

$$T^\mu{}_\mu = T = -\rho + 3p. \tag{2.50}$$

Since the Einstein tensor satisfies the equation

$$\nabla_\mu G^{\mu\nu} = 0, \tag{2.51}$$

the energy momentum tensor also satisfies

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.52}$$

However this is not an energy conservation law, since it is not in the form of $\partial_\mu T^{\mu\nu} = 0$. This nonconservation of energy is due to the fact that matter exchanges energy with gravitational field but there is no energy momentum tensor for the gravitational field. On the other hand for the FRW metric, the component of $\nu = 0$ of Equation (2.52) is called the energy conservation law, which is

$$\begin{aligned} 0 &= \nabla_\mu T^{\mu 0} \\ &= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}\left(\rho + p\right), \end{aligned} \tag{2.53}$$

where $\frac{\dot{a}}{a}$ is the rate of expansion and characterized by the Hubble parameter,

$$H = \frac{\dot{a}}{a}. \tag{2.54}$$

The present value of the Hubble parameter is the Hubble constant H_0 and the normalized Hubble expansion rate $h \equiv H_0/100$ is used to avoid uncertainties in the determination of H_0. The currently accepted value of H_0 is 71$\pm$7km/sec. Here the unit 1Mpc$= 3,09 \times 10^{24}$cm. The Hubble length is

$$d_H = H_0^{-1} c = 3.00 \times h^{-1}\text{Mpc}, \tag{2.55}$$

often considered as the radius of the universe. The Hubble time

$$t_H = H_0^{-1} = 9.78 \times 10^9 h^{-1}\text{yr}. \tag{2.56}$$

is an approximate measure of the age of the universe.

The three equations (2.47), (2.48), and (2.53) are the basic equations governing the dynamics of a Roberson-Walker- Friedmann-Lemaitre (RWFL) universe. The continuity equation (2.53) can be derived from the others. On the other hand, provided that for $\dot{a} \neq 0$, (2.48) can be derived from (2.47) and (2.53).

2.5. Evolution of RWFL Universe

To integrate the Friedmann-Lemaitre equations, pressure is related to the energy density by an equation of state

$$p = w\rho, \tag{2.57}$$

where w is a constant. The conservation of energy equation (2.53) becomes

$$\frac{\dot{\rho}}{\rho} = -3\left(1+w\right)\frac{\dot{a}}{a}, \tag{2.58}$$

then energy density is

$$\rho = \rho_i \left(\frac{a_i}{a}\right)^{3(w+1)}, \tag{2.59}$$

where ρ_i and a_i are the initial values. The cosmological fluid can be matter dominated where the medium includes collisionless and nonrelativistic particles (stars, galaxies) which have zero pressure $p = 0$. This is only an approximation since in this era the pressure is present but negligible in comparison with the energy density ρ. Using equation of state for zero pressure, $w = 0$, the energy density is related to the scale factor as

$$\rho_M \sim a^{-3}. \tag{2.60}$$

A universe in which most of the energy density is in the form of radiation is known as radiation dominated era. Here $w = 1/3$ and energy density is

$$\rho_R \sim a^{-4}. \tag{2.61}$$

Additionally, the cosmological perfect fluid could have an equation of state $p = -\rho$, namely $w = -1$ where

$$\rho_\Lambda \sim a^0. \tag{2.62}$$

Since the universe expands the energy density in radiation falls off faster than that in matter. These two decrease as the universe evolves and expands in time, then if there is a nonzero vacuum energy, this energy will dominate over the long term. Thus it will be natural that, at very early times when the universe was much smaller, the energy density in radiation was dominant, after some time matter became dominant, and eventually as expansion continues the universe becomes cosmological constant dominated.

Another useful quantity is the critical density ρ_c obtained by using (2.47), for $\Lambda_4 = 0$, $k = 0$

$$\rho_c \equiv \frac{3H^2}{8\pi G_N}, \tag{2.63}$$

and the density parameter is

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G_N}{3H^2}\rho. \tag{2.64}$$

The Equation (2.47) with $\Lambda_4 = 0$ is written as

$$\Omega - 1 = \frac{k}{H^2 a^2}, \tag{2.65}$$

which describes the geometry of the universe such that

$$\begin{array}{lllll} \rho < \rho_c & \text{therefore} & \Omega < 1 & \text{then} & k < 0 \Leftrightarrow & \text{open universe} \\ \rho = \rho_c & \text{therefore} & \Omega = 1 & \text{then} & k = 0 \Leftrightarrow & \text{flat universe,} \\ \rho > \rho_c & \text{therefore} & \Omega > 1 & \text{then} & k > 0 \Leftrightarrow & \text{closed universe} \end{array}$$

The value of ρ_c today is

$$\rho_c = 1.88 \times 10^{-29} h_0^2 \frac{\text{g}}{\text{cm}^3}, \tag{2.66}$$

and the recent measurements shows that Ω is very close to unity. Including Λ_4, Equation (2.47) becomes

$$\begin{aligned} 1 &= -\frac{k}{H^2 a^2} + \frac{8\pi G_N}{3H^2}\rho + \frac{\Lambda_4}{3H^2} \\ &= -\frac{8\pi G_N}{3H^2}\frac{3k}{8\pi G_N a^2} + \frac{8\pi G_N}{3H^2}\rho + \frac{8\pi G_N}{3H^2}\frac{\Lambda_4}{8\pi G_N} \\ &= \frac{\rho_k}{\rho_c} + \frac{\rho}{\rho_c} + \frac{\rho_\Lambda}{\rho_c}, \end{aligned} \tag{2.67}$$

in terms of density parameter

$$1 = \Omega_k + \Omega_M + \Omega_{\Lambda_4}, \tag{2.68}$$

where the energy density parameter Ω_k is the contribution of spatial curvature of the universe, Ω_M represents the matter ($w = 0$) or radiation ($w = 1/3$) part of the ingredients and Ω_{Λ_4} is the contribution of the cosmological constant. Since universe expands, the relative influence of matter, curvature and cosmological constant are altered, since the corresponding densities evolve at different rates depending on the scale factor $a\left(t\right)$, such that

$$\Omega_k \propto a^{-2} \tag{2.69}$$

$$\Omega_M \propto a^{-3(w+1)} \tag{2.70}$$

$$\Omega_{\Lambda_4} \propto a^0. \tag{2.71}$$

Observations show that our universe is nearly flat, therefore $\Omega_k = 0$ and other constituents of universe is $\Omega_M = 0.3 \pm 0.1$ and $\Omega_{\Lambda_4} = 0.7 \pm 0.1$. This cosmological constant dominant universe is also supported by theoretical results. Namely, the second Friedmann equation (2.48) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho_M - 2\rho_{\Lambda_4}\right), \tag{2.72}$$

where since the matter is pressureless, $p_M = 0$ and since the contribution from the radiation is very small, $\rho_R = 3p_R \simeq 0$. For the accelerating universe $\ddot{a} > 0$ must be satisfied. This condition can be satisfied only if ρ_{Λ_4} is sufficiently large compared to ρ_M. Therefore, both Friedmann-Lemaitre equations and observational results show that cosmological constant dominates the universe. The concept "cosmological constant" is a possibility for dark energy which is required if the universe is expanding with acceleration.

3. EXTRA DIMENSIONS AND BRANES

3.1. History of Extra Dimensions

The formalism to describe geometry of manifolds with arbitrary dimensions is a century and a half old, dating back to the 1850's with the work of Bernhard Riemann. Since development of modern physics came after Riemann, it is appropriate to start the history with Riemann. His work in differential geometry provided the mathematical basis for Einstein's theories of special and general relativity. In special relativity, as opposed to Isaac Newton's (1643-1727) view of the world, time and space are not absolute concepts. If one observer uses a set of rulers and a watch to describe events in space and time, then a second observer moving with respect to the first will find that, using identical rulers and watch, the same events appear to take place at different positions and times. The transformation between the two observers' coordinate systems is called Lorentz transformations after Hendrik A. Lorentz (1853-1928), and mixes up measurements of displacements and time. The concept of space-time was invented by Hermann Minkowski in 1909 when he realized that if he formally thought of time as an imaginary coordinate, so that events are points in a 3+1 dimensional space-time with coordinates (x_1, x_2, x_3, ict)(where c is the speed of light), then Lorentz transformations can be thought of as rotations in space-time. Furthermore, Maxwell's theory of electromagnetism took a particularly simple form in the language of space-time. Today, all models of particle physics and gravity can be described in terms of fields living in space-time.

In 1915 Einstein completed his theory of general relativity, according to which all bodies should fall the same way in a gravitational field. This was explained by postulating that all bodies move along trajectories which minimize the distance between points in space-time. So the motion of an object falling under gravity depends only on the geometry of space-time, and not on the composition of that object. Energy and momentum carried by all matter in the universe are related with gravity and determines the geometry of space-time via Einstein equations.

Coming back to extra dimensions, in 1914 Gunnar Nordström discovered that he could unite the physics of electromagnetism and gravity by postulating the existence of a fourth spatial dimension [5]. However, recall that general relativity did not exist yet, so Nordström had not produced the correct theory of gravity, but only an approximation. Theodor Kaluza discovered in 1921 that by making certain assumptions, the equations of general relativity with an extra spatial dimension contains Maxwell's equations for an electromagnetic field [6]. However, there was no explanation given for Kaluza's assumption that none of the fields in the universe should depend on the extra dimension. In 1926 Oskar Klein provided an explanation for Kaluza's assumption, namely that, the extra dimension forms a circle much smaller than the distance scales we commonly observe [7]. Quantum mechanically, the statement is that all energy eigenstates will have masses which are inversely proportional to the size of the circle, and there's no way to produce such states without a lot of energy. The theory of gravity where extra dimensions form a compact space is called Kaluza-Klein theory.

Kaluza-Klein theory had still certain problems in its interpretation as a theory which unifies gravity and electromagnetism. Most notably this theory predicted an infinite tower of charged, massive spin 2 particles with charges $e_n = n\sqrt{2}\kappa m$, where $m_n = |n|m$ and κ is Newton's gravitational constant. If we identify the fundamental unit of charge with the charge of electron, then we are forced to make the particles with masses of the order of Planck mass which is beyond the range of any experimental observations. Since these extra dimensions are unobservable there is no problem with this huge value of mass. However these massive Kaluza-Klein modes have the unusual gyromagnetic ratio $g = 1$, which gives unacceptable high energy behavior for Compton scattering [8]. As a result, many physicists left the idea of extra dimensions for the realm of curiosity. In the 1960's string theory was invented as a model of strong interactions which bind quarks together. The idea was that the particles which were discovered in particle accelerators were to be thought of as vibrations of a relativistic string. The theory had several problems, though: it had a tachyon, which signalled an instability; it had a massless spin-two field in the spectrum that was not observed in accelerators; and worst of all the theory seemed to suffer from an anomaly which would have made the entire theory inconsistent. In superstring theory, space-time supersymmetry helps in cancelling these inconsistencies. In 1971 Ramond-Neveu-Schwarz formalism of supersymmetry removed the tachyon. This model possesses manifest world-sheet supersymmetry but not space-time supersymmetry [9]. However using this formalism, it is difficult to cancel these inconsistencies. Then, in 1984 Michael Green and John Schwartz provided manifest space-time supersymmetry [10]. Making the theory supersymmetric made sense, but only in 10 dimensions. That is acceptable, though, because Kaluza and Klein already taught us that if all of the extra dimensions are compact and small enough, then we would not have noticed them. The massless spin-two field was suggested as a type of field which describe gravity, and so string theory became a quantum theory of gravity. A decade later, Joe Polchinski and others proposed that string theory contained in it objects which extended in various numbers of dimensions [11]. These objects are called p-branes if they extend in p spatial dimensions. It was found that many such objects have the property that open strings can end on the p-branes. Studying quantum mechanics of a string stuck to a brane it was found that the oscillations of the string contain a massless particle that could be interpreted as the electromagnetic field. This suggests another way in which we might not see the presence of extra dimensions. For some reason or another, we might just be stuck to a braneworld that extends in three spatial dimensions. In 1996, Petr Horáva and Edward Witten studied a (10+1)-dimensional theory of gravity in which one of the spatial dimensions exists only in an interval bounded by two 9-branes [12]. This theory was related to one of the five consistent string theories that have been developed. Two years later, Nima Arkani-Hamed, Savas Dimopoulos and Georgi Dvali (ADD) considered a brane which extends in three spatial dimensions sitting in a higher-dimensional space-time [13]. The extra dimensions might have been of much larger spatial extent than we have studied in particle physics experiments and we would still not have seen them in any experiment or observation. Since then, particle physics and gravity experiments have been able to place constraints on the sizes of these dimensions.

In 1999, Lisa Randall and Raman Sundrum pointed out two models in which the space-time is 5-dimensional AdS space-time and two or one 3-branes are located at orbifold fixed points [15, 16]. In the first model there are two branes, one of the branes is our 'visible' universe and supports the standard model matter fields, the other brane will be referred to as the 'hidden' brane. Mass scales on the two branes are related to each other by an exponential factor arising from the particular form of the background metric. Thus, by an exponential suppression, small scales on one brane can be generated from large scales on the other. In particular, this mechanism offers a solution to the hierarchy problem in which all scales are derived from a single scale. In the second model, they consider infinitely large extra dimensions in a warped space-time geometry. Braneworld might still be the same as ordinary 3+1 dimensional gravity. (Here we introduce this topic following the reference [17] directly.)

3.2. What is a Brane?

After the introduction to extra dimensions and branes, we try to answer the questions "What is a brane?", "How can the large extra dimensions exist without getting into conflict with observations?" and "How the effective four dimensional world that we observe would be arising from the higher dimensional theory?".

The word "brane" originates from the word "membrane" which is a surface. In general we consider a (p+1)-dimensional surface in n-dimensional space. Such objects are called p-branes. In the models we consider in this thesis, standard model particles are localized on a 3-brane which corresponds to the space we observe. In theory, brane could be embedded into more dimensions where only gravity could propagate. A brane could have more or less spatial dimensions than the usual 2 dimensional membrane, namely, a p-brane has p spatial dimensions, a 0-brane may be viewed as a pointlike particle or a black hole, a 1-brane is a string, a 2-brane is just a usual membrane and a 3-brane which is the most important one has 3 spatial dimensions just like our observed world. In String theory the term D-brane is used to describe a surface where an open string can end and then this open string localizes all the fields on the brane. This brane world theory has originated from the Horáva-Witten theory.

3.3. Horavá-Witten Idea

For brane world cosmology Horavá-Witten theory is particularly interesting. In this scenario, the strong coupling limit of $E_8 \times E_8$ Heterotic string on R^{10} is the same as the low energy limit of 11-dimensional M theory on $R^{10} \times S^1/Z_2$. Clearly, the eleventh dimension is curled up to a circle S^1, and this circle has a mirror symmetry Z_2. The coset space S^1/Z_2 is called an orbifold. If θ denotes the coordinates on S^1, there are two fixed points at $\theta = 0$ and $\theta = \pi$, at which the coset space becomes singular. This singularity can be avoided by placing a $(9+1)$-dimensional hypersurface at each singular point and they correspond to 9-branes. Each 9-brane has a tension. Since the condition for stability is required, the sum of these two tensions must be zero, then they are considered as a negative and positive tension brane. Cosmologically, sign of the tension determines whether gravity on a brane is attractive or repulsive. If six spatial dimensions can be compactified to end up with two parallel 3-branes, the resulting space-time becomes 5-dimensional and S^1 plays the role of the fifth dimension which might be larger than the six compactified dimensions [12].

This setup is very common in brane cosmology, where one of the two 3-brane is identified with our universe.

3.4. The Hierarchy Problem

First of all, it is useful to mention the energy scales in short before we define the hierarchy problem. Energy unit can be converted into mass so that

$$1\text{eV} = 1.783 \times 10^{-36}\text{kg}, \tag{3.1}$$

where $E = mc^2$ is used, and $e = 1.6 \times 10^{-19}$C, $c = 2.98 \times 10^8$m/s. We use the convention $c = \hbar = 1$. Then, energy scale can be converted into the length scale as

$$1\text{eV} = \left(2 \times 10^{-7}\text{m}\right)^{-1} = 5 \times 10^6\text{m}^{-1}, \tag{3.2}$$

The most common energy scales are

$$1eV = 10^{-3}\text{keV} = 10^{-6}\text{MeV} = 10^{-9}\text{GeV} = 10^{-12}\text{TeV}. \tag{3.3}$$

In standard physics, there are four fundamental forces: strong, electromagnetic, weak, and gravitational and their relative strengths are $\alpha_s \simeq 1$, $\alpha_{EM} = 10^{-3}$, $\alpha_W = 10^{-6}$ and $\alpha_g = 10^{-39}$ respectively. For gauge interactions, instead, an important scale is the 'electroweak scale' defined as the energy at which the running coupling constants α_{EM} and α_W are of the same size, and the electro-magnetic and the weak force are unified to a $SU(2) \times U(1)$ gauge theory. The numerical value of the electroweak unification scale is 10^3GeV.

All of these scales show that the gravity is much weaker than the other fundamental forces. In particle physics, hierarchy problem is the large separation between the electroweak unification scale $\sim 10^3$GeV and the gravitational unification scale with $M_P \sim 10^{19}$GeV.

To solve this problem, some theories use the idea that 3-spatial dimensions in which we live could be a 3-spatial dimensional "brane" embedded in a much larger extra dimensional space. This theory leads to the generation of hierarchy by the geometry of the additional dimensions.

Although the weak and strong forces extend down to scales of $\sim 10^{-18} - 10^{-15}$mm, we have almost no knowledge of gravity at distances less than a millimeter. It is thus conceivable that gravity may behave quite differently from the 3-dimensional Newtonian theory at small distances. This leads to the possibility that matter and non-gravitational forces are confined to our 3-dimensional subspace whereas gravity may propagate throughout a higher dimensional volume. Gravity would no longer follow the inverse-square force law at distances smaller than the size of extra dimensions. For $r \gtrsim 1$ mm gravity must behave as if there were only three spatial dimensions. Thus, in these models, gravity appears to us very weak at macroscopic scales because its intensity is spread in the "hidden" extra dimensions.

3.5. Large Extra Dimensions

In this thesis we study the case of large extra dimension, so we give some brief introduction on the most important ideas that have been done on this subject.

3.5.1. The Arkani Hamed-Dimapoulos-Dvali Theory

This theory is based on the idea that our four-dimensional universe would be a three-brane living in a flat $(4+n)$-dimensional space-time where n is the number of compactified extra dimensions with torus topology of common radius R. Here ordinary matter would be confined to our four-dimensional universe while gravity would live in the whole extended space-time. One of the advantages of this picture would be to relax the constraint on the size of extra dimensions so that they can be sufficiently large to be experimentally observable. In general, if the four–dimensional Planck scale is M_P and $(4+n)$–dimensional Planck scale M, two test particles m_1 and m_2, at distances r apart feel a gravitational potential of

$$V(r) = \frac{m_1 m_2}{M^{2+n}} \frac{1}{r^{1+n}}, \quad \text{for} \quad r \ll R, \tag{3.4}$$

and

$$\begin{aligned} V(r) &= \frac{m_1 m_2}{M^{2+n}} \frac{1}{R^n r}, \quad \text{for} \quad r \gg R \\ &= \frac{m_1 m_2}{M_P^2} \frac{1}{r}, \end{aligned} \tag{3.5}$$

this yields the simple relation

$$M_P^2 = M^{2+n} R^n, \tag{3.6}$$

where R is the size of the extra dimension. On sizes much larger than R, $(4+n)$-dimensional gravity behaves effectively as our usual four-dimensional gravity. The simple but crucial remark of ADD is that a fundamental mass M of the order of the electroweak scale can explain the huge four-dimensional Planck mass M_P we observe, provided that the extra dimensions are very large. This can be satisfied only if ordinary matter is confined to a three-brane and gravity propagates to the extra dimension. The behavior of gravity is much weaker and the usual Newton's law has been verified experimentally only on scales above about 0.1 mm, this means gravity is tested only down to sizes of around a millimeter. This leaves a reason for extra dimensions as large as this millimeter experimental bound. Moreover, the four-dimensional Planck mass M_P is in this model only a "projection" of the higher dimensional (fundamental) Planck mass M, which can be lower than M_P, thus offering a new perspective on the hierarchy problem. The most important result of the ADD proposal is a possible resolution to the hierarchy problem, that is the large discrepancy between the Planck scale at 10^{19}GeV and the electroweak scale at 10^3 GeV. The fundamental Planck mass could be comparable to the electroweak scale as long as the volume of the extra dimensional space is large enough. To realize this, their proposal requires more than one extra dimension. More clearly, substituting the values of fundamental Planck mass $M \sim 10^3$GeV and $M_P \sim 10^{19}$GeV in Equation (3.6),

$$R \sim \left(\frac{M_P}{M}\right)^{2/n} \frac{1}{M} = 10^{\frac{32}{n}} 10^{-17}\text{cm}, \tag{3.7}$$

One extra dimension $n = 1$, gives unacceptably large value of $R = 10^{17}$cm which is grater than the astronomical unit of 1.5×10^{13}cm. Therefore there can not be one flat large extra dimension if one would like to lower M to the TeV scale. An interesting case is $n = 2$ in which radius of extra dimension is $R = 1$mm. This value of extra dimensions are very close to range of the deviation from Newton's gravity law. For $n > 2$, extra dimensions become smaller and smaller then their gravitational measurements become impossible. The importance of this theory is that, if the fundamental gravity scale is indeed in the TeV range, one expects that extra dimensions should observed in collider experiments at energies approaching this scale [13, 14].

3.5.2. The Randall-Sundrum Model

Randall and Sundrum (RS) suggested a brane world model with a four dimensional Minkowski space-time which is embedded into five dimensional Anti de-Sitter (AdS) space-time. The model imposes an S^1/Z_2 line segment to generate boundary condition. The five dimensional AdS space-time is a maximally symmetric solution to Einstein equation with negative cosmological constant. In this model fifth dimension is "highly curved" with a warp factor. The brane itself remains static and flat therefore it preserves 4D Lorentz invariance. The ansatz for the most general metric satisfying these properties is given by

$$ds^2 = g^{(5)}_{MN} dx^M dx^N = a^{-A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{3.8}$$

The amount of curvature (warping) along the extra dimension depends on the function $e^{-A(y)}$, which is therefore called the warp-factor.

When we consider the distance scales much larger than the brane thickness, we may view the brane as a delta function source of the gravitational field. If the energy density per unit three volume is σ, the five dimensional gravitational action becomes

$$\begin{aligned} S &= S_{\text{gravity}} + S_{y=0} + S_{y=y_1}, & (3.9)\\ S_{\text{gravity}} &= \int d^4x dy \sqrt{-g^{(5)}} \left(-\Lambda + 2M^3 R\right), & (3.10)\\ S_{y=0,y_1} &= \int d^4x \sqrt{g} \left(\sigma_{0,y_1} + L_{\text{brane}}\right) & (3.11) \end{aligned}$$

for two branes located at $y = 0$ and $y = y_1$. Here R and Λ is the five dimensional Ricci scalar and five dimensional cosmological constant respectively and M is the five dimensional fundamental Plank mass. This action leads to Einstein equations

$$\sqrt{-g^{(5)}} G_{MN} = -\frac{1}{4M^3} \left[\sqrt{-g^{(5)}} g^{(5)}_{MN} \Lambda + \sqrt{-g} g_{\mu\nu} \delta^\mu_M \delta^\nu_N \sigma_y \left(\delta\left(y\right) + \delta\left(y - y_1\right)\right) \right]. \tag{3.12}$$

The non-vanishing components of Einstein tensor for the metric (3.8) are

$$G_{\mu\nu} = -g_{\mu\nu}\left(3A'' - 6A'^2\right), \tag{3.13}$$

$$G_{yy} = g_{yy}6A'^2. \tag{3.14}$$

Using the relations (3.12) and (3.14),

$$6A'^2 = -\frac{\Lambda}{4M^3}, \tag{3.15}$$

$$3A'' = \frac{\sigma_0}{4M^3}\delta(y) + \frac{\sigma_{y_1}}{4M^3}\delta(y - y_1). \tag{3.16}$$

Integration of (3.15) gives

$$A = |y|\sqrt{-\frac{\Lambda}{24M^3}} = k|y|, \tag{3.17}$$

for the orbifold symmetry $y \to -y$. Here k is the curvature constant. Computing second derivative of (3.17)

$$A'' = 2k\left(\delta(y) - \delta(y - y_1)\right), \tag{3.18}$$

therefore matching (3.16) with (3.18), the brane tensions are found as equal but opposite signs,

$$\sigma_0 = -\sigma_{y_1} = 24M^3k, \quad \Lambda = -\frac{\sigma_0^2}{24M^3}, \tag{3.19}$$

Thus there is a static flat solution only if the above two fine tuning conditions are satisfied.

Randall and Sundrum produced two important papers [15] and [16] in which two different brane world scenarios had been worked out. In the first model which is called as Randall- Sundrum I model (RSI), they consider two branes and in the second one (Randall-Sundrum II model, RSII), they investigate an infinitely large extra dimension.

It is more convenient to study these theories separately.

The RSI Model. In this model matter fields are localized on the negative tension brane where the induced metric is exponentially small compared to the other brane. Considering a scalar field, for example the Higgs scalar, on the brane, the action would be

$$S^{\text{Higgs}} = \int d^4x\sqrt{-g}\left[g_{\mu\nu}D^{\mu}HD^{\nu}H - V\left(H\right)\right], \quad V\left(H\right) = \lambda\left(H^{\dagger}H - v_0^2\right)^2, \tag{3.20}$$

where v_0 is the mass parameter. If the size of the extra dimension is r_c, then the induced metric at the negative tension brane is

$$g_{\mu\nu}\mid_{y=r_c} = e^{-2kr_c}\eta_{\mu\nu}. \tag{3.21}$$

Substituting this in the above action (3.20) we get

$$S^{\text{Higgs}} = \int d^4x e^{-4kr_c}\left[e^{-2kr_c}\eta_{\mu\nu}\partial^{\mu}H\partial^{\nu}H - \lambda\left(H^{\dagger}H - v_0^2\right)^2\right]. \tag{3.22}$$

After the renormalization of Higgs field $\tilde{H} = e^{-kr_c}$, the action (3.22) becomes

$$S^{\text{Higgs}} = \int d^4x\left[\eta_{\mu\nu}\partial^{\mu}\tilde{H}\partial^{\nu}\tilde{H} - \lambda\left(\tilde{H}^{\dagger}\tilde{H} - e^{-2kr_c}v_0^2\right)^2\right], \tag{3.23}$$

this is exactly the action for a normal Higgs scalar, but the mass parameter is warped with e^{-kr_c}, then the mass parameter is redefined as

$$v_{\text{Higgs}} = e^{-kr_c}v_{0.} \tag{3.24}$$

This result can be generalized for a mass parameter m_o,

$$m \equiv e^{-kr_c}m_0. \tag{3.25}$$

This shows that all mass scales are exponentially suppressed on the negative tension brane, but not on the positive tension brane. Therefore the positive tension brane is often referred as the Planck brane, since the fundamental mass scale is unsuppressed and of the order of the Planck scale. On the other hand, the negative tension brane is referred as the TeV brane, since the relevant mass scale there is TeV. Namely, if e^{kr_c} is of order 10^{15}, the physical mass scale TeV is produced from the fundamental mass scale, Planck scale, 10^{19}GeV.

Another important task is to find out the effective scale of gravity on the brane. To do this, we need to determine the r_c dependence of the gravitational action. The five dimensional Einstein-Hilbert (EH) action is

$$\begin{aligned} S &= -M^3 \int d^5x\sqrt{-g^{(5)}}R^{(5)} \\ &= -M^3 \int d^4x dy e^{-4k|y|}\sqrt{-g}e^{2k|y|}R^{(4)}, \end{aligned} \tag{3.26}$$

considering 4D EH action

$$S = M_P^2 \int d^4x\sqrt{-g}R^{(4)}, \tag{3.27}$$

Planck mass becomes

$$M_P^2 = M^3 \int_{y=-r_c}^{y=r_c} dy e^{-2k|y|} = \frac{M^3}{k}\left(1 - e^{-2kr_c}\right). \tag{3.28}$$

This equation shows that the dependence of the extra dimension becomes smaller for large values of r_c. Namely, the exponential warp factor has very little effect in determining the Planck scale. Therefore the hierarchy problem is not solved by considering the effective gravitational action. On the other hand, in the $r_c \to \infty$ limit this equation does not diverge, thus there is no problem with considering infinitely large extra dimension. Randall-Sundrum model II investigates this limit, namely one brane with an infinitely large extra dimension [15].

The RSII Model. RS model solves the hierarchy problem. To find out how this model solves the hierarchy problem we need to study the properties of gravity in this five dimensional background. This involves derivation of bulk gravitational waves, then this give rise the Kaluza-Klein (KK) modes on the brane. Moreover, obtaining these modes provide us to get the four dimensional gravity localized on the brane. The name of this process is the KK reduction on the brane. In order to find the KK expansion, we rewrite the RS metric (3.8),

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{3.29}$$

by introducing $e^{k|y|}/k = w$, then this metric becomes conformally flat

$$ds^2 = \frac{1}{k^2 w^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dw^2\right). \tag{3.30}$$

For a brane at $w = w_0$, introduce the coordinate $z = w - w_0$, therefore the Z_2 symmetric metric (on both sides of the brane) is written as

$$ds^2 = \frac{1}{k^2\left(|z| + w_0\right)^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right). \tag{3.31}$$

Therefore fluctuations of this metric are

$$ds^2 = \frac{1}{k^2\left(|z| + w_0\right)^2}\left[\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu + dz^2\right], \tag{3.32}$$

where $h_{\mu\nu}$ are the deviation of five dimensional brane world. In RS model, there are no sources of the gravitational field except for the bulk cosmological constant and the two branes, then $h_{\mu\nu}$ become transverse and trace-free in the vacuum, namely

$$\partial_\mu h^\mu_\nu = 0, \qquad h^\mu_\mu = 0. \tag{3.33}$$

Here we consider the small perturbations on the background space-time in conformally flat form, then we want to find the Einstein equation. In general, as derived in [18], for a conformal metric,

$$\begin{aligned} ds^2 &= \tilde{g}_{MN} dx^M dx^N \\ &= \Omega^2 g_{MN} dx^M dx^N, \end{aligned} \tag{3.34}$$

Einstein equation is

$$\begin{aligned} \tilde{G}_{MN} &= G_{MN} + 3[\nabla_M \ln\Omega \nabla_N \ln\Omega - \nabla_M \nabla_N \ln\Omega \\ &\quad + g_{MN}\left(\nabla_B \nabla^B \ln\Omega + \nabla_B \ln\Omega \nabla^B \ln\Omega\right)]. \end{aligned} \tag{3.35}$$

The Einstein tensor for linear perturbations about flat space-time [18],

$$\begin{aligned} \delta G_{MN} &= \frac{1}{2}(\partial^B \partial_N h_{MB} + \partial^B \partial_M h_{NB} - \partial^B \partial_B h_{MN} - \partial_M \partial_N h^B_B \\ &\quad - \eta_{MN}\left(\partial^B \partial^C h_{BC} - \partial^B \partial_B h^C_C\right)). \end{aligned} \tag{3.36}$$

and using perturbed metric $g_{MN} = \eta_{MN} + h_{MN}$, and Christoffel symbol $\delta\Gamma^B_{MN} = \frac{1}{2}\eta^{BC}\left(\partial_M h_{CN} + \partial_N h_{MC} - \partial_C h_{MN}\right)$ in the second and third part of (3.35), the perturbed Einstein tensor becomes

$$\begin{aligned} \delta\tilde{G}_{MN} &= \frac{1}{2}[\partial^B \partial_N h_{MB} + \partial^B \partial_M h_{NB} - \underline{\partial^B \partial_B h_{MN}} - \partial_M \partial_N h^B_B \\ &\quad - \eta_{MN}\left(\partial^B \partial^C h_{BC} - \partial^B \partial_B h^C_C\right)] \\ &\quad + 3[\frac{1}{2}\eta^{BC}\left(\partial_M h_{NB} + \partial_N h_{MB} - \underline{\partial_B h_{MN}}\right)\partial_C \ln\Omega \\ &\quad + \underline{h_{MN}\partial_B \partial^B \ln\Omega} + \frac{\eta_{MN}}{2}\left(\partial_C h^B_B + \partial_B h^B_C - \partial^B h_{BC}\right)\partial^C \ln\Omega \\ &\quad + \underline{h_{MN}\partial_B \ln\Omega \partial^B \ln\Omega}]. \end{aligned} \tag{3.37}$$

Only the underlined terms are nonzero because of the gauge conditions (3.33) or h_{MN} has only nonzero components $h_{\mu\nu}$ and moreover the conformal factor Ω is depends only on the fifth dimension z.

The stress energy tensor is

$$\begin{aligned}\tilde{T}_{MN} &= \tilde{T}^B_M \tilde{g}_{BN} \\ &= \tilde{T}^B_M \left(\tilde{\eta}_{BN} + \tilde{h}_{BN}\right)\end{aligned} \tag{3.38}$$

where tilde is related with the whole metric with conformal factor as seen easily from the definition of metric (3.34). Then the variation of the stress energy tensor is

$$\begin{aligned}\delta\tilde{T}_{MN} &= \tilde{T}^B_M \tilde{h}_{BN} \\ &= \kappa^{-2} G^B_M h_{BN} + 3\kappa^{-2}[\left(\nabla_M \ln\Omega \nabla^B \ln\Omega - \nabla_M \nabla^B \ln\Omega\right) h_{BN} \\ &\quad + \delta^B_M \left(\nabla_B \nabla^B \ln\Omega + \nabla_B \ln\Omega \nabla^B \ln\Omega\right) h_{BN}] \\ &= 0 + 3\kappa^{-2}[\left(\partial_M \ln\Omega \partial^B \ln\Omega - \partial_M \partial^B \ln\Omega + 0\right) h_{BN} \\ &\quad + \left(\partial_B \partial^B \ln\Omega + 0 + \partial_B \ln\Omega \partial^B \ln\Omega\right) h_{MN}],\end{aligned} \tag{3.39}$$

where since the metric η_{MN} is flat, Einstein tensor and Christoffel symbols give zero contribution to this equation. Then the final equation for the stress energy tensor is,

$$\delta T_{MN} = 3\kappa^{-2}\left[h_{NB}\left(\partial_M \ln\Omega \partial^B \ln\Omega - \partial_M \partial^B \ln\Omega\right) + h_{MN}\left(\underline{\partial_C \partial^C \ln\Omega} + \underline{\partial_C \ln\Omega \partial^C \ln\Omega}\right)\right], \tag{3.40}$$

again the only the underlined terms are nonzero because of the same reason we mention above. Then using the equation of motion $\delta G_{MN} = \kappa^2 \delta T_{MN}$, two terms cancel, therefore the remaining parts become

$$\partial^B \partial_B h_{MN} + 3\partial_B h_{MN} \partial_C \ln\Omega = 0, \tag{3.41}$$

if we redefine the perturbed metric as $h_{MN} = \Omega^{-3/2} \breve{h}_{MN}$,

$$\left[-\partial^M \partial_M + \frac{9}{4}\partial_M \ln\Omega \partial^M \ln\Omega + \frac{3}{2}\partial_M \partial^M \ln\Omega\right] \breve{h}_{MN} = 0. \tag{3.42}$$

Using the fact that, $\partial^M \partial_M = \Box + \partial_z^2$, where $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_{z'}^2$ and looking for solution of the form $\breve{h}_{\mu\nu}(x,z) = \breve{h}_{\mu\nu}(x)\,\psi(z)$ with $\Box\breve{h}_{\mu\nu}(x) = m^2\breve{h}_{\mu\nu}(x)\,,$ where m is the four dimensional Kaluza-Klein mass of the function then since $\Omega = \Omega(z)$ only, we have the Schrodinger equation for the wave function $\psi(z)$

$$\left(-\partial_z^2 + V(z)\right)\psi(z) = m^2\psi(z)\,, \tag{3.43}$$

where

$$\begin{aligned} V(z) &= \left(\frac{9}{4}\partial_z \ln\Omega\partial_z \ln\Omega + \frac{3}{2}\partial_z^2 \ln\Omega\right) \\ &= .\frac{15}{4}\frac{1}{\left(|z| + w_0\right)^2} - \frac{3\delta(z)}{\left(|z| + w_0\right)}. \end{aligned} \tag{3.44}$$

This potential has the shape of a volcano, since there is a peak in it due to the first term, but then there is a delta function which is like the crater of volcano. Now all the equations are simple and in the well known quantum mechanical form. After this stage, we can describe the effective four dimensional theory of gravity. For the condition of localization of gravity on the brane, (3.43) must have the normalizable zero-energy ground state. This zero energy ground state corresponds to $m = 0$, then the zero mode wave function becomes

$$\psi(z) = \Omega^{3/2} = \frac{1}{k^{3/2}\left(|z| + w_0\right)^{3/2}}, \tag{3.45}$$

therefore the massless 4D graviton exists. If this function is normalizable, the condition for the localization of gravity is satisfied. Then the normalization of the wave function is

$$\int_{-z_0}^{z_0} \frac{dz}{k^3\left(|z| + w_0\right)^3} < \infty, \tag{3.46}$$

therefore the gravity is localized on the brane. Really we can realize the normalizability of the wave function from the relation (3.43). Here, as $|z| \to \infty$, if $V(z) > 0$, the wave function $\psi(z)$ is always normalizable, on the contrary as $|z| \to \infty$, if $V(z) < 0$, $\psi(z)$

is not normalizable. The most important thing on this integral is that it is converging even in the limit of $z_0 \to \infty$. This means, even if the size of extra dimension becomes infinitely large, gravity can be localized on the brane. Since the gravitational zero mode wave function (3.45) is peaked around the positive tension brane which is $z = 0$, gravity is localized around the positive tension brane. This is also the case even if we transform extra dimension z back to the y coordinate system,

$$ds^2 = e^{-2k|y|} \left(\eta_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu + dy^2. \tag{3.47}$$

In this theory, gravity localizes on the positive tension brane, therefore the observers living far away from the positive tension brane, for example on the negative tension brane, will feel the exponentially suppressed gravitational wave function. On the other hand particle physics interactions are independent of the position along the extra dimension. This solves the hierarch problem in such a way that; since the strength of the gravity is decreased by the exponential warp factor along the extra dimension, it will be much weaker than the particle physics interactions, then we feel the fundamental strength of interactions.

To get the higher dimensional gravity namely, we find the continuum KK modes of graviton. In RS theory, however there is a volcano potential which is also felt by these continuum modes, thus their wave function will be strongly suppressed by this volcano potential at the positive tension brane (the Planck brane). This means in order to get TeV brane they have to tunnel through the large barrier. For the tunneling rate, we can use WKB approximation of the wave function (3.43). For the continuum mode with mass m, tunneling amplitude is,

$$T\left(m\right) \sim e^{-2\int\limits_{z_0}^{z_1} dz\sqrt{V(z)-m^2}}, \tag{3.48}$$

where z_0 and z_1 are the turning points in this volcano potential for a particle with mass m. For large z, potential (3.44) becomes $\sim 1/z^2$, then for the continuum zero mode,

$m \rightarrow 0$,

$$T(0) \sim e^{-\int\limits_0^\infty dz/z} \rightarrow 0, \tag{3.49}$$

which means there is no gravitational zero mode namely there is no higher dimensional gravity therefore KK modes recover 4D gravity even in the large extra dimension limit. Really this not the case in the other higher dimensional theories: when there is a compact extra dimension, there is usually a graviton zero mode and KK tower. In these theories, as the size of the extra dimension gets infinitely large, the zero mode becomes more and more decoupled due to its huge spread in the extra dimension, on the other hand the KK modes become lighter and lighter, and eventually form a continuum that reproduces higher dimensional gravity. Here, however since we have a large potential barrier, their wave function is strongly suppressed at the Planck brane therefore 4D gravitational interactions is recovered when the size of extra dimension is very large. Therefore KK modes in the RS model is not turn gravity into a higher dimensional theory of gravity.

In order to get the correction to Newton's Law, we consider the gravitational potential between two pointlike sources of mass M_1 and M_2 located at the $z = 0$ brane (Planck brane). KK mode contributes a Yukowa-like correction to the four dimensional gravitational potential between two masses M_1, M_2,

$$U(r) \sim \frac{G_N M_1 M_2}{r} + \frac{1}{M^3} \int\limits_0^\infty dm \frac{M_1 M_2 e^{-mr}}{r} \psi_m^2(0), \tag{3.50}$$

where the first term comes from the exchange of the zero mode which would cause 4D gravitational interactions, but the second term is due to the exchange of the KK modes. To get the KK modes, we solve the Equation (3.43) for $m \neq 0$,

$$\left(-\partial_z^2 + \frac{15}{4\left(|z| + w_0\right)^2}\right) \psi(z) = m^2 \psi(z), \tag{3.51}$$

has the solution of [1]

$$\psi\left(m\right)=a_m\left(\left|z\right|+w_0\right)^{1/2}Y_2\left(m\left(\left|z\right|+w_0\right)\right)+b_m\left(\left|z\right|+w_0\right)^{1/2}J_2\left(m\left(\left|z\right|+w_0\right)\right) \quad (3.52)$$

then KK zero modes are

$$\psi\left(0\right)\sim\left(mw_0\right)^{1/2}, \quad (3.53)$$

therefore the gravitational potential becomes

$$U\left(r\right) \sim \frac{G_N M_1 M_2}{r}+\frac{1}{M^3}\int\limits_0^\infty dm\frac{M_1 M_2 e^{-mr}}{r}\left(mw_0\right) \quad (3.54)$$

$$= \frac{G_N M_1 M_2}{r}\left(1+\frac{C}{r^2}\right), \quad (3.55)$$

where C is a constant. where C is a calculable constant of order one. This shows that the full correction due to the KK modes is small for large r. Therefore for very large radial distance we get 4D gravitational potential irrespectively of the compactification. Thus in the presence of localized gravity there is no need to compactify the extra dimension. Also, due to the wave function suppression of the KK modes the production of the continuum KK modes would be suppressed on the Planck brane [16].

[1]in general if the potential is in the form of $V\left(z\right)=\frac{\alpha(a+1)}{(|z|+w_0)}$, Bessel functions give the solution as $\psi\left(z\right)=a_m\left(z+w_0\right)^{1/2}Y_{\alpha+\frac{1}{2}}\left(m\left(z+w_0\right)\right)+b_m\left(z+w_o\right)^{1/2}J_{\alpha+\frac{1}{2}}\left(m\left(z+w_0\right)\right).$

4. BLACK HOLES

4.1. The Schwarzschild Solution

Schwarzschild metric is the stationary spherically symmetric vacuum solutions to Einstein's field equations. Since the gravitational field produced by a point particle is spherically symmetric the Schwarzschild metric will be the simplest model considering the behavior of gravity. The general form of a spherically symmetric and static line element is

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where the metric of the unit two sphere is $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Here the condition for a static space-time causes the metric to be time independent and no cross terms of time coordinate appear in the line element. Since we want to find a time independent solution, line element must be time reversal invariant namely the condition of invariance under $t \to -t$ makes $dtdx^i$ terms vanish. The condition for the spherically symmetric space-time requires that the spherical part of the line element is kept in the form of $d\Omega^2$. The nonvanishing components of Ricci tensor are

$$R_{tt} = \frac{2f'}{rfg^2} - \frac{f'g'}{fg^3} + \frac{f''}{fg^2}, \tag{4.2}$$

$$R_{rr} = \frac{2g'}{rg^3} + \frac{f'g'}{fg^3} - \frac{f''}{fg^2}, \tag{4.3}$$

$$R_{\theta\theta} = R_{\phi\phi} = -\frac{f'}{rfg^2} - \frac{1}{r^2g^2} + \frac{1}{r^2} + \frac{g'}{rg^3}. \tag{4.4}$$

and the curvature scalar is

$$R = \frac{2}{r^2} - \frac{2}{r^2g^2} - \frac{4f'}{rfg^2} + \frac{4g'}{rg^3} + \frac{2f'g'}{fg^3} - \frac{2f''}{fg^2}. \tag{4.5}$$

Since this space-time must be a vacuum outside a spherical body, we must consider Einstein's equations in vacuum, then

$$R_{\mu\nu} = 0. \tag{4.6}$$

These leave

$$\begin{aligned} \frac{f'}{f} + \frac{g'}{g} &= 0, && (4.7) \\ f &= g^{-1}. && (4.8) \end{aligned}$$

Substituting this equation into (4.4) and equating to zero, we get

$$2ff'r + f^2 = 1, \tag{4.9}$$

or equivalently

$$\left(f^2 r\right)' = 1. \tag{4.10}$$

Integrating, we get

$$f^2 r = r + c,$$

choosing the constant c as $-2GM$ for correct Newtonian limit,

$$f^2 = g^{-2} = \left(1 - \frac{2G_N M}{r}\right). \tag{4.11}$$

Therefore the resultant Schwarzschild metric (4.1) becomes

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{4.12}$$

In section 5.3, we have proven that, in the weak field limit $g_{00} = -(1+2\Phi)$, additionally in Newtonian theory, a point mass M at the origin gives rise to a potential $\Phi = -G_N M/r$. Therefore we need only identify that $c = -2G_N M$.

In fact, a time dependent spherically symmetric vacuum solution does not exists. In the literature, this is known as "Birkhoff's theorem" which states that a spherically symmetric vacuum solution in the exterior region is necessarily static.

4.2. Singularities

Singularities are very important in general relativity. Singularity occurs when the metric coefficients become infinite as we approach some point. However this definition is not always useful. There are two kinds of singularities which are the coordinate singularities and physical singularities. Coordinate singularity results from a breakdown of a specific coordinate system rather than the underlying manifold. If we choose an appropriate coordinate system, all the metric coefficients would have finite values in this new coordinate system. Therefore coordinate singularity is a removable singularity namely it is not a real singularity. To get the sufficient condition for the real singularities, we must construct coordinate-independent scalar quantities such as the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ or the square of the Ricci tensor $R^{\mu\nu}R_{\mu\nu}$ or the square of Riemann tensor $R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda}$. If one of these scalars becomes infinite at some place, this is enough to convince us that this place is a real physical singularity which is not removable. This kind of singularity is called as intrinsic, curvature, essential or real singularity.

For the line element (4.12), the metric coefficients become infinite at $r = 0$ and $r = 2G_N M$. However in the case of the condition for physical singularity, if we can calculate the square of Riemann tensor which is also called as Kretschmann scalar, we get

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = \frac{48G_N^2 M^2}{r^6} \tag{4.13}$$

which is finite at $r = 2G_N M$. Since square of Riemann tensor is a scalar, its value remains the same in all coordinate systems. This scalar quantity convinces us that $r = 0$ is a real singularity, since square of the Riemann tensor blows up at this point. On the other hand $r = 2G_N M$ is not an actual singularity where the curvature invariants do not blow up. If it is possible we transform this metric into a more appropriate coordinate system then this singularity disappears.

4.3. Schwarzschild Black Holes

As we discussed in the previous part, Schwarzschild metric breaks down at $r = 2G_N M$ which is known as the Schwarzschild radius. Consider the null geodesics, for constant θ and ϕ, $ds^2 = 0$. Then (4.12) becomes

$$0 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2, \tag{4.14}$$

from which we get

$$\frac{dt}{dr} = \pm \left(1 - \frac{2G_N M}{r}\right)^{-1}. \tag{4.15}$$

For $r \to \infty$, dt/dr becomes ± 1. That is, the null geodesics make angles of 45^0 with r and t axis as in flat space-time. We expected this result, since Schwarzschild line element (4.12) becomes asymptotically flat for large r. On the other hand, as we approach $r = 2G_N M$, we get $dt/dr \to \infty$ and also $t \to \infty$, which means we need an infinite amount of time to reach the Schwarzschild radius or an infalling observer as observed from a stationary observer never crosses $r = 2G_N M$.

Schwarzschild solution is a vacuum solution exterior to some spherical body of radius $r > 2G_N M$, on the other hand for the solution of radius $r < 2G_N M$ a different metric would describe the interior body and would correspond to some distribution of matter resulting in a nonzero energy momentum tensor. Indeed, the spherically symmetric static perfect fluid solution is known as the interior Schwarzschild solution. Therefore, Schwarzschild radius is a boundary that divides the manifold into two components. This surface of sphere is known as the "event horizon" from where no matter or radiation can escape. Even light from inside the horizon can never reach the observer, and anything that passes through the horizon from the observer's side is never seen again. Such a massive object surrounded by event horizon is defined as the Schwarzschild black hole. They are "black" because they neither emit nor reflect light and they are "hole" because nothing entering can ever escape. Since nothing escapes from the event horizon, a black hole has highly powerful gravitational field. Moreover if sufficiently large amount of mass is present within a small enough volume, all paths through space are warped inwards towards the center of the volume.In fact, black holes can be of almost any mass. Since gravity increases in strength as volume is decreased, almost any object sufficiently compressed will become a black hole.

For relatively small gravitating bodies like a planet and the sun, this radius is inside the surface of the mass distribution. Since the empty space condition $T_{\mu\nu} = 0$ is no longer holds inside the surface of these bodies, the Schwarzschild metric is not used to describe these regions. In this description of such gravitating bodies the Schwarzschild solution is only acceptable outside the mass distribution and the Schwarzschild radius is not involved in the description of this region.

Black holes have three independent internal properties: mass, angular momentum and electric charge. Schwarzschild solution describes the nonrotating and uncharged black holes. On the other hand black holes would be charged objects, then Reissner Nordström solution represents the electrically charged nonrotating black holes [5]. Additionally, the solution for an uncharged rotating black hole was investigated by Kerr [19] and then a more advanced description of black hole were discovered by Kerr-Newmann which explains charged rotating black holes.

4.4. Higher Dimensional Black Holes

Starting on this subject, we want to ask why black holes are so important and what we want to get with the determination of properties of black holes. In this context, Steven B. Giddings writes, "Black holes are perhaps the most profound and mysterious objects we have imagined. Being able to create and study them should teach us a lot. In particular, it can teach us about how quantum mechanics can be reconciled with gravity; it could allow us to explore extra dimensions of space and time; and it may tell us something about an ultimate unified theory of physics."

In this thesis, we will discuss the properties of black holes in higher dimensional space-time with the same hopes as Giddings. From the point of view of standard physics in 3+1 dimensions, black holes are well described by the Kerr-Newman family of solutions. However, the true nature of gravity is not well described at larger and smaller scales yet. To solve this problem, there are many different considerations. But theories are all based on the same suggestions: to solve the hierarchy problem, there might be extra dimensions. As mentioned earlier, hierarchy problem is the difficulty in answering the problem of hierarchy between the characteristic scale of gravity and electroweak scale. Clearly, the four dimensional characteristic scale of gravity (Planck scale) $M_P = G_N^{-1/2} \sim 10^{19}$GeV is 16 orders of magnitude larger than the electroweak scale, $M_{EW} = G_{EW}^{-1/2} \sim 100$GeV. As discussed, this problem could be solved by assuming the existence of extra dimensions in the universe [13]. In this theory all ordinary matter, is restricted to live on a $(3+1)$-dimensional hypersurface, a 3-brane. The 3-brane, playing the role of our $(3+1)$ dimensional world, is then embedded in the higher dimensional space-time, usually called the bulk, in which only gravity can propagate. Unfortunately, this problem is not totally solved by the scaling of these parameters. This subject became more serious after the discovery of dark energy. Dark energy implies that the universe appears to be accelerating rather than decelerating. Thus we must take into account the Hubble constant which is of the order of $\sim 10^{-42}$GeV. This brings forth a big ratio $\sim 10^{61}$ between the Planck scale and the dark energy scale. Then the hierarchy problem becomes more fatal. We have discussed this subject in detail under the title of "Hierarchy Problem".

If Planck scale is lower to electroweak scale, it becomes clear that black holes might be experimentally accessible in accelerators [20, 45]. After this consideration, Giddings [20, 23], Kanti [24], Frolov[25] and Maartens [26, 27] studied on higher dimensional black holes widely. Especially, they focused on the arguments for the production of black hole in high energy collisions.

Similar to $3+1$ dimensional black holes, the subject of $4+1$ dimensional black holes have been studied and argued widely from a different point of view. This theory starts with the work of Chamblin et al. [46]. They assume that, when matter trapped on the brane undergoes gravitational collapse, a black hole is formed. They discuss black holes (collapsed matter) in the Rundall-Sundrum (RS) brane world scenario. The metric on the brane world should be close to the Schwarzschild metric at astrophysical scales in order to agree with the observationally tested predictions in general relativity. They gave a black cigar solution whose restriction to the brane is the Schwarzschild solution. This black hole has a center on the brane and extends along the extra dimension. Introducing the coordinate transformation $z = le^{y/l}$, they write the RS metric in conformally flat form

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + \eta_{ij}dx^i dx^j\right). \tag{4.16}$$

If induced metric on the brane at a fixed z is Schwarzschild, RS metric becomes

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 - U\left(r\right)dt^2 + \frac{1}{U\left(r\right)}dr^2 + r^2 d\Omega_2^2\right) \tag{4.17}$$

with the $U\left(r\right) = 1 - 2M/r$.

This metric describes a "black cigar" in five dimensional space-time, on the other hand, from the point of view of an observer on the brane it appears as if it is an uncharged, non-rotating Schwarzschild black hole. For this metric, Ricci scalar and square of Ricci tensor are finite. However, square of Riemann tensors diverge at the AdS horizon $z \to \infty$, therefore this metric becomes unstable near AdS horizon. From this result, it seems likely that, there might be other solutions which give rise to the Schwarzschild solution on the domain wall.

After the introduction of extra dimensions, all the existing theories in standard cosmology need some modifications or extensions in order to accommodate the effects of extra dimensions. Similarly, the properties of black holes are also bound to change in the context of a higher- dimensional theory.

4.5. Mini Black Holes

Although mini black holes are not our main topic, since they have a crucial importance in the higher dimensional theory, we need to discuss mini black holes in short.

In principle, in collider experiments, one may produce mini black holes and those black holes evaporate quickly by emitting Hawking radiation which we could detect using detectors. In fact, this is not an easy procedure because in order to produce black holes in collider experiments one needs a center of mass energy above the Planck scale (10^{19}GeV) which is inaccessible at present or near future. After the introduction of large extra dimension [13, 14], this situation changed. As we expressed above, the traditional Planck scale M_P is derived from the fundamental higher dimensional one, M, by the relation $M_P = M^{2+d}R^d$. Here R is the compactification radius of extra dimensions. According to this relation, for $M \sim$ TeV, compactification radius R will be in the range from 10^{-1}mm to 10^3fm for $d = 2$ to $d = 7$. In this model $d = 1$ is excluded since size of the extra dimension becomes the size of the solar system. The upper bound of extra dimension is defined as the smallest length scale down to which the Newtonian potential, and its $1/r$ dependence could be measured. During the last years, experiments require that the extra dimensions have radii not larger than 0.1mm, which disfavors the case of $d = 2$. In scales smaller than 0.1mm, gravitational potential would have a different dependence on r, gravity becomes stronger at this scale. Namely, due to the large size of extra dimensions, the scale of quantum gravity (Planck scale) becomes the same order as the electroweak interaction scale (of the order of a few TeVs). This, in turn, opens up the new possibility of TeV-size black hole production at high energy collision processes in cosmic ray and at future colliders [20, 45]. These experiments will be performed at CERN by using the Large Hadron Collider (LHC) experiment setting.

In this theory, two elementary partons (quarks) approaching each other with a very high kinetic energy in the center of mass system close to the new fundamental scale $M \sim 1$TeV. At those high energies, these particles can come very close to each other. If the impact parameter is small enough, which will happen to a certain fraction of the particles, we have two particles plus their large kinetic energy in a very small region of space-time. If this region is smaller than the new Schwarzschild radius connected with the energy of the partons, the system will collapse and form a black hole. This collision is probably the most inelastic process. Due to the high velocity of the moving particles, space-time before and after collision is almost flat.

In fact, from General Relativistic arguments, two point like particles in a head on collision with zero impact parameter will always form a black hole, no matter how large or small their energy. However, at small energies, this is impossible due to the uncertainty principles. Then this process needs very high energy to allow the required close approach. This threshold must be of order M.

After the production, the black holes will undergo an evaporation process. According to Hawking radiation,

$$T = 10^{-6}\frac{M_{\odot}}{M}\,[K]\,, \tag{4.18}$$

temperature of the small black holes will be higher than the big black holes. This high temperature causes a very short lifetime such that this tiny black hole will radiate and decay close to the collision region. Observation of these black holes can be carried out by looking at the energy of the system. Once the collision energy crosses the threshold for black hole production, black holes are formed and there will be a sharp cut off in the total energy of the other particles formed in the reaction. Thus, black holes will give a clear signal [28].

5. GRAVITY

Using a scalar potential field, the first gravity theory was developed by Newton. After developing special relativity, Einstein considered gravity as a scalar theory. He assumed that gravity could be described by a 4-scalar. But he reached the result that this theory does not work. Since this potential becomes constant along the trajectory of every particle, the gravitational force must necessarily be zero on every particle. Then, Einstein described gravity in terms of a tensor in his general relativity theory. In fact, the physics of gravitation is described by general relativity.

5.1. Newtonian Gravity

The first theory of gravity was introduced by Sir Isaac Newton in 1686. His famous Law of Gravitation states that every pointlike object in the universe attracts every other object with a force directed along the line of centers for the two objects that is inversely proportional to the square the separation between the two objects,

$$\mathbf{F} = -G_N \frac{mM}{r^2} \hat{r}. \tag{5.1}$$

The constant of proportionality G_N is known as the universal gravitational constant. It is termed as "universal", because it is thought to be the same at all places and all times. According to Newton's second law, if this force acts on a mass m, this particle will accelerate in this direction namely,

$$\mathbf{F} = m\mathbf{a}. \tag{5.2}$$

In the language of gravitational scalar potential Φ, this potential is related with the mass density ρ with,

$$\nabla^2 \Phi = 4\pi G \rho \tag{5.3}$$

and acceleration of a body in a gravitational potential Φ is

$$\mathbf{a} = -\nabla\Phi \tag{5.4}$$

Newton's description of gravitational force proved to be satisfactory for almost two and a half centuries. In pure Newtonian theory light is massless and therefore should not undergo any gravitational deflection. If mass energy equivalence of special relativity is used together with Newtonian gravity the predicted deflection of light by gravity is half the deflection actually observed. The observations was performed after Einstein calculated this phenomenon (here observations were in totally in accordance with his results). Then, observations showed that Newton's gravitational law is not exactly correct. For example, the advance of the perihelion of Mercury is partly due to the influence of nearby planets as calculated using Newtonian gravity. However a discrepancy remained.

5.2. Special Relativistic Gravity

Einstein's first successful field theory was a special relativistic reformulation of Maxwell's electromagnetic theory. Then he tried to reformulate the Newtonian Mechanics by defining the force as a four-vector, F^{μ}

$$F^{\mu} = \frac{d}{d\tau}\left(m_i \frac{dx^{\mu}}{d\tau}\right), \tag{5.5}$$

where τ is the proper time and m_i is the constant inertial mass. Then the condition

$$\eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -1 \tag{5.6}$$

must be satisfied. Using the assumption of constant mass, (5.5) and (5.6) imply that

$$\eta_{\mu\nu}F^{\mu}\frac{dx^{\nu}}{d\tau} = 0. \tag{5.7}$$

In the special relativistic reformulation (5.3) and (5.4) can simply extended into the four dimensional case and then Φ will be a 4-scalar,

$$\Box\Phi = 4\pi G\rho, \tag{5.8}$$

and

$$F^{\mu} = -m_i \Phi'^{\mu}, \tag{5.9}$$

then (5.7) will be

$$\frac{dx^{\mu}}{d\tau}\frac{d\Phi}{dx^{\mu}} = \frac{d\Phi}{d\tau} = 0. \tag{5.10}$$

This means, the potential is constant along the path of every particle, so the gravitational force must be zero on every particle. It seems unacceptable. Here, the problem is that, Newtonian gravity cannot be incorporated into special relativity in the same way as Maxwell's electromagnetism.

After that Einstein developed his theory of general relativity where the metric tensor playing the role of gravitational potential. This put an end to the search for a purely scalar special relativistic gravitational theory.

5.3. Gravity in General Relativity

To explain gravity, it is convenient to start with the basics of general relativity; the Principle of Equivalence. The earliest form of this principle which called the "Weak Equivalence Principle (WEP)", states that the inertial mass and the gravitational mass of any object are equal. Clearly, it means that all the bodies at the same space-time point in a given gravitational field will undergo the same acceleration independent of the composition of the object. The important property of WEP is that, in a small region of space-time, there is no way to distinguish between uniformly accelerated frame and a gravitational field for a freely falling particles.

The basics of general relativity are that free particles move along the geodesics and curvature of space-time describe gravity. The geodesic equation for a particle is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{5.11}$$

For the Newtonian limits of this equations, three requirements must be satisfied: particles move slowly with respect to speed of light, the gravitational field is weak and finally the field is static namely, unchanging with time.

First of all, moving slowly means

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{5.12}$$

where τ is proper time and $i = 1, 2, 3$ which corresponds to the spatial components of a metric. Therefore the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0, \tag{5.13}$$

where $\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right)$. For the condition of static field, the relevant Christoffel symbol becomes

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}. \tag{5.14}$$

Then the third requirement says that for the weakness of gravity, the metric can be decomposed into the Minkowski form and a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.15}$$

where $|h_{\mu\nu}| \ll 1$. From the definition of an inverse of the metric $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$ and $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{5.16}$$

Putting all together we obtain that, $\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda}h_{00}$.Then the geodesic equation (5.13) becomes,

$$\frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda}h_{00}\left(\frac{dt}{d\tau}\right)^2, \tag{5.17}$$

for $\mu = 0, \quad \partial_0 h_{00} = 0$ and then $\frac{d^2t}{d\tau^2} = 0$. This means that $\frac{dt}{d\tau}$ =constant. For the spacelike sections, Equation (5.17) is

$$\frac{d^2x^{i}}{d\tau^2} = \frac{1}{2}\partial_{i}h_{00}\left(\frac{dt}{d\tau}\right)^2, \tag{5.18}$$

then

$$\frac{d^2x^{i}}{dt^2} = \frac{1}{2}\partial_{i}h_{00}. \tag{5.19}$$

This equation is the same as Newton's theory of gravitation which is $\mathbf{a} = -\nabla\Phi$, where Φ is the scalar potential. Namely, if we compare these equations, (5.19) satisfies that

$$h_{00} = -2\Phi, \tag{5.20}$$

this means

$$g_{00} = -\left(1 + 2\Phi\right). \tag{5.21}$$

Therefore, it is proven that, the curved space-time can describe gravity in the Newtonian limit.

5.3.1. The Gravitational Field Equations

In general relativity the metric is a dynamical variable describing the gravitational field. Its dynamics is governed by Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{5.22}$$

which relate the energy-momentum tensor $T_{\mu\nu}$ to the curvature of space-time, the Einstein tensor $G_{\mu\nu}$. Here $R_{\mu\nu}$ is Ricci tensor and R is the Ricci scalar. The Newton constant G_N is related with the Planck mass M_P by

$$8\pi G_N = M_P^{-2}. \tag{5.23}$$

The right hand side of Einstein equation (5.22) describes the energy-momentum while the left hand side is related with the geometry of space-time. These equations are also nonlinear, so that two known solutions cannot be superposed to find a third. Therefore it is very difficult to solve Einstein's equations in any sort of generality, and it is necessary to make some simplifying assumptions. The most popular simplification is to assume that the metric has a significant degree of symmetry. Therefore, in cosmology, the universe is considered as isotropic and homogenous, and matter can be regarded as a continuous medium. Then this continuous medium is assumed as a perfect fluid. In the orthonormal basis, the energy momentum tensor takes the form

$$T_{00} = \rho \quad , \quad T_{ij} = p\delta_{ij}, \tag{5.24}$$

where ρ and P describe the energy density and pressure respectively.

In more general case the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5.25}$$

where Λ is the cosmological constant. The cosmological constant is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In detail, in the vacuum, Einstein tensor $G_{\mu\nu} = 0$ and then the remaining part of (5.25) gives $8\pi G T_{\mu\nu} = \Lambda g_{\mu\nu}$. Then Λ can be interpreted as the energy density of the vacuum [3].

5.3.2. Action Principles

In classical mechanics, if a single particle moves in one dimension with coordinate $q(t)$, the action S is written as

$$S = \int_{t_1}^{t_2} dt L(q, \dot{q}), \tag{5.26}$$

where $L(q, \dot{q})$ is the Lagrangian. For a point particle

$$L = K - V. \tag{5.27}$$

where K is the kinetic energy and V is the potential energy of the system. The dynamics of the particle is determined by the action principle. The action principle states that the particle will travel from its position at time t_1 to its position at time t_2, so that it will travel along that path for which the value of the integral (5.26) is stationary. Here the term “stationary” means that the value of action S along the given path has the same value for nearby paths so that the first derivative of the action functional vanishes,

$$\delta S = 0. \tag{5.28}$$

This principle leads to the Euler-Lagrange equations,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{5.29}$$

For a field theory, a single coordinate $q(t)$ and $\dot{q}(t)$ is replaced by a set of space-time dependent fields $\phi^i(x^\mu)$ and their space-time derivatives $\partial_\mu \phi^i(x^\mu)$, then Lagrangian becomes a functional of these fields,

$$L = \int d^3x \mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right), \tag{5.30}$$

and the action

$$S = \int dt L = \int d^4x \mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right), \tag{5.31}$$

where $\mathcal{L}$ is the Lagrange density. Again using the action principle, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)}\right) = 0. \tag{5.32}$$

These are known as the Euler-Lagrange equations for field theory in flat space-time.

In general relativity, Lagrangian density $\mathcal{L}$ is assumed to be a function of the metric $g_{\mu\nu}$ and its first and possibly higher derivatives, that is

$$\mathcal{L} = \mathcal{L}\left(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \partial_\sigma \partial_\lambda g_{\mu\nu}, ...\right) \tag{5.33}$$

To form the action integral, it is necessary to convert the Lagrangian density into a scalar density

$$\mathcal{L} = \sqrt{-g}\widehat{\mathcal{L}}, \tag{5.34}$$

where $\widehat{\mathcal{L}}$ is a scalar and it leads to partial differential equations that are second order in the space-time coordinates then the action becomes

$$S = \int d^4x \sqrt{-g}\widehat{\mathcal{L}}$$

According to principle of action, S must be stationary under the variations of the $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, namely

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0. \tag{5.35}$$

The Euler-Lagrange equations for the curved space-time is

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \nabla_\lambda \left(\frac{\partial \mathcal{L}}{\partial \left(\nabla_\lambda g^{\mu\nu} \right)} \right) = 0 \tag{5.36}$$

In curved space-time partial derivative becomes covariant derivative. These equations are significant since they constitute the field equations of the theory [3].

5.3.3. Geodesic Action Principles

Geodesic defines the shortest path between two points in a certain geometry. In general relativity, geodesic is the path followed by a test particle in free fall. The geodesic equation can be thought of as the generalization of Newton's law $\mathbf{F} = m\mathbf{a}$, for the case $\mathbf{F} = 0$, to curved space-time.

The path length ds in general relativity is given by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{5.37}$$

where $g_{\mu\nu}$ is the metric tensor which simply defines the geometry of space-time. If τ is the proper time, the action can be written in the form of

$$S = \int d\tau \left(-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2}. \tag{5.38}$$

Using calculus of variations, we want to find the path which makes the path length stationary under a change in path $\delta x(\tau)$ satisfying $\delta x(\tau_1) = \delta x(\tau_2) = 0$ where τ_1 and τ_2 are the initial and final values of the parameter τ along the path respectively. It means δS vanishes for any variations of $\delta x(\tau)$,

$$\delta S = 0. \tag{5.39}$$

The infinitesimal variations of the path is

$$\begin{aligned} x^{\mu} &\rightarrow x^{\mu} + \delta x^{\mu} \\ g_{\mu\nu} &\rightarrow g_{\mu\nu} + (\partial_{\sigma} g_{\mu\nu})\,\delta x^{\sigma} \end{aligned} \tag{5.40}$$

where in the second line, we use Taylor expansion in curved space-time.

$$\delta S = -\int \frac{d\tau}{2}\left(\partial_{\sigma} g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\delta x^{\sigma} + g_{\mu\nu}\frac{d(\delta x^{\mu})}{d\tau}\frac{dx^{\nu}}{d\tau} + g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{d(\delta x^{\nu})}{d\tau}\right)\left(-g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right)^{-1/2}, \tag{5.41}$$

The second and third term can be integrated by parts, for example

$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{d(\delta x^{\nu})}{d\tau} = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\delta x^{\nu}\right) + \frac{dg_{\mu\nu}}{d\tau}\frac{dx^{\mu}}{d\tau}\delta x^{\nu} + g_{\mu\nu}\frac{d^2x^{\mu}}{d\tau^2}\delta x^{\nu}. \tag{5.42}$$

Using the chain rule on the derivative $g_{\mu\nu}$,

$$\frac{dg_{\mu\nu}}{d\tau} = \partial_{\sigma} g_{\mu\nu}\frac{dx^{\sigma}}{d\tau}, \tag{5.43}$$

and after rearranging some dummy indices, (5.41) then becomes

$$\delta S = \int \left(g_{\mu\sigma}\frac{d^2x^{\mu}}{d\tau^2} + \frac{1}{2}\left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right)\left(-g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right)^{-1/2}\delta x^{\sigma} d\tau, \tag{5.44}$$

where the total derivatives vanish at the boundary, hence the first term in (5.42) on the right hand side gives no contribution to the integral. Since we are searching for stationary points, we want δS to vanish for any variation δx^{σ}, which implies,

$$g_{\mu\sigma}\frac{d^2x^{\mu}}{d\tau^2} + \frac{1}{2}\left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0, \tag{5.45}$$

At the end, contracting the (5.45) by the inverse metric $g^{\sigma\lambda}$, the geodesic equation becomes

$$\frac{d^2x^\lambda}{d\tau^2} + \frac{1}{2}g^{\sigma\lambda}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{5.46}$$

where the Christoffel connection is defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{5.47}$$

The connection on the curved space-time gives us a way of relating vectors in the tangent spaces of nearby points.

Mathematically, the geodesic is a curve along which the tangent vector is parallel transported. The tangent vector to a path $x(\lambda)$ is $V = dx/d\lambda$. V is kept parallel to itself and satisfies $\nabla V = 0$. In other words, not only V is kept parallel to itself with constant magnitude along the curve, but locally the curve continues to point in the same direction all along the path. A geodesic is the natural extension of the definition of a "straight line" to a curved manifold [3].

5.3.4. Einstein-Hilbert Action

To construct the action for general relativity, we must use the metric $g_{\mu\nu}$. Using the Ricci scalar for scalar $\widehat{\mathcal{L}}$, Hilbert constructed the simplest possible choice for a Lagrangian,

$$S_H = \int d^4x\sqrt{-g}R, \tag{5.48}$$

known as the Hilbert action (or sometimes Einstein-Hilbert action). Here Ricci scalar is constructed from the metric and it has no higher than second order in its derivatives. Since the only gravitational part is included, this action can give the geometry of space-time. Nevertheless, it is not a complete description of the whole theory of Einstein's general relativity. To get a more general action, we need an additional term for the matter fields. Therefore the action might be

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_N} R + \widehat{\mathcal{L}}_M \right). \tag{5.49}$$

Let us look at how this action recovers the Einstein equations (5.22). To verify this result, we vary the action with respect to inverse metric $g^{\mu\nu}$. Using the definition $R = g^{\mu\nu} R_{\mu\nu}$,

$$\delta S = \int d^4x \left[\delta\sqrt{-g} \frac{R}{16\pi G_N} + \sqrt{-g}\delta g^{\mu\nu} \frac{R_{\mu\nu}}{16\pi G_N} + \sqrt{-g} g^{\mu\nu} \frac{\delta R_{\mu\nu}}{16\pi G_N} \right] + \delta S_M, \tag{5.50}$$

where the matter part is $S_M = \int d^4x \sqrt{-g}\widehat{\mathcal{L}}$ in short. To perform this small variation step by step, we start with the first term on the right hand side.

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g \tag{5.51}$$

where g is the determinant of metric $g_{\mu\nu}$, $g = \det g_{\mu\nu}$. For any positive definite square matrix A, the basic rule is

$$\det A = e^{\mathrm{Tr}\{\ln A\}}, \tag{5.52}$$

thus we have

$$\begin{aligned} \delta\left(\det A\right) &= e^{\mathrm{Tr}\{\ln A\}} \delta\left(\mathrm{Tr}\{\ln A\}\right) \\ &= \det A \mathrm{Tr}\left(A^{-1}\delta A\right). \end{aligned} \tag{5.53}$$

Applying to our calculation; the matrix $A = g_{\mu\nu}$ and $\det A = g$. Therefore $\delta g = g g^{\mu\nu} \delta g_{\mu\nu}$. The desired result follows immediately that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}. \tag{5.54}$$

Using the definition of the metric $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$,

$$\delta g^{\mu\lambda} g_{\lambda\nu} + g^{\mu\lambda}\delta g_{\lambda\nu} = 0, \tag{5.55}$$

(5.54) becomes

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{5.56}$$

The second term is already in the form of some expression multiplied by $\delta g^{\mu\nu}$, thus there is no need for any effort on this term.

For the third term, let us derive the result of $\delta R_{\mu\nu}$. Ricci tensor is identified as

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\nu\mu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}, \tag{5.57}$$

then

$$\begin{aligned} \delta R_{\mu\nu} &= \delta\left(\partial_\lambda \Gamma^\lambda_{\nu\mu}\right) - \delta\partial_\nu\left(\Gamma^\lambda_{\lambda\mu}\right) + \delta\Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} + \\ & \quad \Gamma^\lambda_{\lambda\sigma}\delta\Gamma^\sigma_{\nu\mu} - \delta\Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\lambda\mu} - \Gamma^\lambda_{\nu\sigma}\delta\Gamma^\sigma_{\lambda\mu}. \end{aligned} \tag{5.58}$$

Using the definition of covariant derivative,

$$\nabla_\lambda\left(\delta\Gamma^\rho_{\mu\nu}\right) = \partial_\lambda\left(\delta\Gamma^\rho_{\mu\nu}\right) + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\mu\sigma}, \tag{5.59}$$

Equation (5.58) becomes

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^\lambda_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\lambda\mu}\right). \tag{5.60}$$

Using the property of metric compatibility for any vector field

$$g_{\mu\nu}\nabla_\rho V^\nu = \nabla_\rho\left(g_{\mu\nu}V^\nu\right) = \nabla_\rho V_\mu, \tag{5.61}$$

we obtain that,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right). \tag{5.62}$$

Performing again the covariant derivative and multiplying with the root of the determinant of the metric

$$\begin{aligned}
\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}\left[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \Gamma^\lambda_{\lambda\beta}g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - \partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) - \Gamma^\lambda_{\nu\beta}g^{\mu\nu}\delta\Gamma^\beta_{\lambda\mu}\right] \\
&= \sqrt{-g}\Big[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \left(\frac{1}{2}g^{\nu\lambda}g_{\nu\lambda,\beta}\right)g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - \partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) \\
&\quad - \left(\frac{1}{2}g^{\nu\sigma}g_{\nu\sigma,\beta}\right)g^{\mu\beta}\delta\Gamma^\lambda_{\lambda\mu}\Big] \\
&= \sqrt{-g}\Big[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \frac{\left(\sqrt{-g}\right)_{,\beta}}{\sqrt{-g}}g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - \partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) \\
&\quad - \frac{\left(\sqrt{-g}\right)_{,\beta}}{\sqrt{-g}}g^{\mu\beta}\delta\Gamma^\lambda_{\lambda\mu}\Big] \\
&= \partial_\lambda\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) - \partial_\nu\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right),
\end{aligned} \tag{5.63}$$

changing dummy indices

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\lambda\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu} - \sqrt{-g}g^{\mu\lambda}\delta\Gamma^\lambda_{\lambda\mu}\right), \tag{5.64}$$

then integrating this term over the volume element

$$\int d^4x\sqrt{-g}g^{\mu\nu}\frac{\delta R_{\mu\nu}}{16\pi G_N} = \frac{1}{16\pi G_N}\int d^4x\partial_\lambda\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu} - \sqrt{-g}g^{\mu\lambda}\delta\Gamma^\lambda_{\lambda\mu}\right) = 0. \tag{5.65}$$

Using the Stokes' theorem, this integral of the total divergence with respect to the volume element vanishes at the boundary then it gives zero contribution to δS (5.50).

Finally, the last term in (5.50) which contains the matter fields gives the energy-momentum tensor, $T_{\mu\nu}$,

$$\frac{\delta \widehat{\mathcal{L}}_M}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}T_{\mu\nu}, \tag{5.66}$$

by definition.

Remembering again the action principle which states that the coordinate of a system follow an extremum path, i.e. the action is stationary under a small variation,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{5.67}$$

and turning back to (5.50), we obtain that

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 8\pi G_N T_{\mu\nu} = 0. \tag{5.68}$$

This result recovers the complete Einstein's equations (5.22). Additionally, this result implies that the coupling constant in front of the Hilbert action must be exactly $1/16\pi G_N$ [3].

5.4. Scalar-Tensor Theory

The scalar tensor theory, for first time was introduced by P. Jordan [47] and Y. Thiry [48] and then developed by C.Brans and R.H.Dicke [49] some years later. In this theory the scalar field acts as the source of the local gravitational coupling with $G_N \sim \varphi^{-1}$ and consequently the gravitational constant is not in fact a constant but is determined by the total matter in the universe. The strength of gravity (as measured by the local value of Newton's constant) might be different from place to place and time to time. Really, the idea of the possibility of whether G_N is truly a constant or would be a variable was discussed by Dirac [50] in 1938. His work is known as the Dirac's numbers in which Dirac had noticed that there were some interesting numerical coincidences of physical fundamental constants and then he expressed them in a dimension free manner. Namely

$$\begin{aligned}
\alpha = e^2/\kappa m_a^2 &\approx 10^{40}, \\
T_u/T_a &\approx 10^{40}, \\
T_a/\kappa &\approx 10^{40}, \\
M_u/m_a &\approx 10^{80},
\end{aligned} \tag{5.69}$$

then these identities lead to

$$\frac{1}{\kappa} \approx \frac{M_u}{R_u}, \tag{5.70}$$

where e is the electron charge, $T_u = R_u$ is the present age of the universe or inverse Hubble constant in distance scale, $T_a = R_a = e^2/m_a$ is the atomic time in distance scale, κ is the gravitational constant, M_u mass of the universe and m_a is some atomic mass. This equation raises the possibility of both κ would be a constant or vary with different values of M/R. Namely, κ can be determined by the mass distribution in the universe or κ can vary with time since the value of Hubble constant changes with time [51]. In other words, $1/\kappa$ might be a field quantity so, Jordan introduced a scalar field ϕ which plays the role of the reciprocal Newtonian gravitational constant κ (where $\kappa = 16\pi G_N$ in short). Beginning with the standard Einstein action (5.49) as,

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}R + \widehat{\mathcal{L}}_M\right), \tag{5.71}$$

and replacing Newtonian constant G_N by the inverse of scalar field φ, $16\pi G_N \to \varphi^{-1}$, the action (5.71) becomes

$$S = \int d^4x\sqrt{-g}\left(\varphi R + \widehat{\mathcal{L}}_M\right). \tag{5.72}$$

This equation is not complete, because there must be some additional action for the new field ϕ,

$$S = \int d^4x\sqrt{-g}\left(\varphi R + \widehat{\mathcal{L}}_M + \mathcal{L}_\varphi\right). \tag{5.73}$$

In standard field theory the Lagrangian density for a scalar field $\mathcal{L}_\varphi$ is given by

$$\mathcal{L}_\varphi = \varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu}. \tag{5.74}$$

Since $S = \int d^4x\sqrt{-g}\widehat{\mathcal{L}}$ is dimensionless in units where $\hbar = c = 1$, $\mathcal{L}$ must have length dimension -4, since φ has length dimension -2, we need the factor φ in the denominator leading to scalar field Lagrangian $\mathcal{L}$ to be

$$\mathcal{L}_\varphi = -\omega\varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu}/\varphi, \tag{5.75}$$

where ω is dimensionless coupling constant, for the dimensional consistency. Therefore, the action becomes

$$S = \int d^4x\sqrt{-g}\left(\varphi R - \frac{\omega}{\varphi}\varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu} + \widehat{\mathcal{L}}_M\right), \tag{5.76}$$

which is known as Jordan-Brans-Dicke (JBD) action. In fact, the more general Einstein equation includes a cosmological constant. Since BD theory is the prototype of gravitational theories alternative the general relativity, when we consider the general relativity limit, we may need an additional term which reduces to the cosmological constant. Hence the scalar field potential $V(\varphi)$ is added to the action (5.76). Therefore the more general BD action becomes

$$S = \int d^4x\sqrt{-g}\left(\varphi R - \frac{\omega}{\varphi}\varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu} - V(\varphi) + \widehat{\mathcal{L}}_M\right). \tag{5.77}$$

The scalar field potential $V(\varphi)$ may reduce to a constant or to a mass term. Generally, it is included when BD theory is used in theory of the early universe or in quintessential scenarios of the present universe. Moreover, in high energy physics, scalar potential corresponds to small interactions when the quantum corrections are considered.

The matter part of the action (5.77), $\widehat{\mathcal{L}}_M$ is decoupled from the scalar field φ. If not, the geodesic equation will change, and weak equivalence principle will not be satisfied. Weak equivalence principle states that all bodies at the same space-time point in a given gravitational field undergo the same acceleration. Coupling this variable scalar field to matter part of this action would causes the bodies of different mass to have different gravitational acceleration in identical gravitational fields.

Using the action principle, variations of the action with respect to metric $g^{\mu\nu}$ and the scalar field φ will vanish,

$$\begin{aligned} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}} &= 0, \\ \frac{\delta S}{\delta \varphi} &= 0, \end{aligned} \tag{5.78}$$

which yields BD field equations (Apendix C),

$$\varphi G_{\mu\nu} - \varphi_{,\mu;\nu} + g_{\mu\nu}\Box\varphi + \frac{\omega}{\varphi}\left(\frac{g_{\mu\nu}}{2}\partial_\lambda\varphi\partial^\lambda\varphi - \partial_\mu\varphi\partial_\nu\varphi\right) + \frac{1}{2}g_{\mu\nu}V(\varphi) = \frac{1}{2}T_{\mu\nu}, \tag{5.79}$$

and wave equation,

$$\varphi R + 2\omega\Box\varphi - \frac{\omega}{\varphi}\partial_\lambda\varphi\partial^\lambda\varphi - \varphi\frac{dV(\varphi)}{d\varphi} = 0. \tag{5.80}$$

By taking the trace of (5.79) and solving for R,

$$\varphi R = 3\Box\varphi + \frac{\omega}{\varphi}\partial_\lambda\varphi\partial^\lambda\varphi + 2V - \frac{1}{2}T, \tag{5.81}$$

where the trace $T = T^\mu_\mu$ of the energy momentum tensor $T_{\mu\nu}$. Using (5.80) and eliminating R from (5.81), we obtain that

$$\Box\varphi = \frac{1}{2\omega + 3}\left(\frac{1}{2}T + \varphi\frac{dV}{d\varphi} - 2V\right). \tag{5.82}$$

where ω is a dimensionless coupling constant which determines the coupling between gravity and the BD scalar field. This theory approaches to general relativity as $\omega \to \infty$ [51].

5.4.1. The Limit to General Relativity

BD theory reduces to general relativity in the limit of large ω. To explain this limit in a simple manner, we can consider the free scalar field, namely $V(\varphi) = 0$ or we can choose the scalar potential $V(\varphi)$ to consist of a pure mass term, then $\varphi\frac{dV}{d\varphi} - 2V$ vanishes. Then (5.82) becomes

$$\Box\varphi = \frac{T}{4\omega + 6}. \tag{5.83}$$

For finite T, (5.83) yields

$$\Box\varphi = O\left(\frac{1}{\omega}\right), \tag{5.84}$$

for a large ω. Therefore we can conclude that

$$\varphi = \langle\varphi\rangle + O\left(\frac{1}{\omega}\right). \tag{5.85}$$

As we have explained in the previous chapter, the average value of φ is

$$\langle\varphi\rangle = \varphi_0 = \frac{1}{16\pi G_N}. \tag{5.86}$$

Then substituting this value into (5.85), we obtain the scalar field

$$\varphi = \frac{1}{16\pi G_N} + O\left(\frac{1}{\omega}\right). \tag{5.87}$$

Using this equation in (5.79), we obtain the corresponding Einstein equation as $\omega \to \infty$:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{5.88}$$

The recent data from the time delay experiments in Solar system shows that the lower bound on ω is $\omega > 3300$ [52]. However, the principle of BD theory reducing to general relativity in the $\omega \to \infty$ limit is not always true. A number of exact BD solutions have been reported not to tend to the corresponding general relativity solutions in this limit [53, 54]. The anomaly comes from the vanishing trace of the energy-momentum tensor. If $T = T^{\mu}_{\mu} = 0$, $\Box\varphi$ is also zero and the order of the φ for $\omega \to \infty$ limit cannot be worked out from (5.83). For this case (5.81) becomes

$$\begin{aligned} R &= \frac{\omega}{\varphi^2}\partial_\lambda\varphi\partial^\lambda\varphi \\ &= \omega\partial_\lambda(\ln\varphi)\,\partial^\lambda(\ln\varphi). \end{aligned} \tag{5.89}$$

For large ω,

$$(\ln\varphi)_\lambda \sim \sqrt{\frac{R}{\omega}} = O\left(\frac{1}{\sqrt{\omega}}\right), \tag{5.90}$$

$$\ln\varphi = \text{const} + O\left(\frac{1}{\sqrt{\omega}}\right), \tag{5.91}$$

which yields,

$$\varphi = \varphi_0 + O\left(\frac{1}{\sqrt{\omega}}\right). \tag{5.92}$$

Although this result tends to a constant value for φ for large values of ω, the term $\frac{\omega}{\varphi^2}\left(\frac{g_{\mu\nu}}{2}\partial_\lambda\varphi\partial^\lambda\varphi - \partial_\mu\varphi\partial_\nu\varphi\right)$ in the field equations (5.79) gives rise to a non trivial contribution, and thus this equation becomes different from the corresponding general relativistic limit with the same energy momentum tensor for $T = 0$.

Another explanation of non-convergence of solution with $T = 0$ has been proposed using the conformal transformation theory [55, 57]. This theory is as follows: for the BD theory with a free scalar, $V = 0$ and vanishing trace of energy momentum tensor $T = 0$, the BD action (5.77) is invariant under a class of transformations. The change of the BD parameter $\omega \rightarrow \tilde{\omega}$ is equivalent to a mapping of BD theory into a same equivalence class. Increasing value of $\tilde{\omega}$ is equivalent to the limit $\omega \rightarrow \infty$, thus this limit can also be regarded as a parameter change that moves the theory within the same equivalence class. Hence this limit cannot yield general relativity which is outside of this equivalence class. On the contrary, when $T \neq 0$, the conformal invariance is broken and the mapping of $\omega \rightarrow \tilde{\omega}$ or $\omega \rightarrow \infty$, does not move BD theory into the same equivalence class. Therefore, general relativity can be obtained in this limit [57].

To explain this argument in detail, we consider the purely gravitational sector of the BD action (5.77). The scale transformation of the metric is

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{5.93}$$

where $\Omega\left(x^\alpha\right)$ is a non vanishing smooth function called a Weyl or conformal transformation. It leaves the light cones unchanged, but it affects the lengths of intervals and the norm of vectors. Under this transformation, the Ricci scalar R and determinant of metric $g_{\mu\nu}$ transform as

$$\tilde{R} = \Omega^{-2}\left(R + \frac{6\Box\Omega}{\Omega}\right), \qquad \tilde{g} = \Omega^8 g. \tag{5.94}$$

Substituting these results in the gravitational part of the action (5.77), we get

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left(\Omega^{-2}\varphi\tilde{R} - \frac{6\varphi\Box\Omega}{\Omega^5} + \frac{\omega}{\Omega^2\varphi}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right). \tag{5.95}$$

Using this transformed equation, if the conformal factor is chosen as $\Omega = \varphi^\alpha$, the new scalar field becomes,

$$\varphi \rightarrow \tilde{\varphi} = \varphi^{1-2\alpha}, \tag{5.96}$$

which yields

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{\varphi}\tilde{R} - \frac{\tilde{\omega}}{\tilde{\varphi}} \tilde{\varphi}_{,\mu}\tilde{\varphi}_{,\nu}\tilde{g}^{\mu\nu} \right), \tag{5.97}$$

where $\tilde{\omega} = \frac{\omega - 6\alpha(\alpha-1)}{(1-2\alpha)^2}$. Thus, the gravitational part of BD action is left unchanged under the conformal transformation for $\alpha \neq 1/2$. This transformation maps BD space-time into the same equivalence class. Therefore it is not possible to get general relativity from the BD theory in the limit of $\omega \rightarrow \tilde{\omega} = \infty$ with vanishing trace of energy momentum tensor $T = 0$. However, there is no theoretical results which show that BD theory always reduces to general relativity for $T \neq 0$. Besides, BD theory solutions have been obtained which do not converge to general relativity when $T \neq 0$ [58]. However, corresponding limit of these solutions are physically irrelevant since when $\omega \rightarrow \infty$, the scalar field $\varphi \rightarrow 0$, which means that the effective gravitational constant becomes infinite then this result is not physically acceptable. Moreover, BD field equations lead to the conclusion that $\varphi \rightarrow \varphi_0 + O\left(1/\sqrt{\omega}\right)$, even in the case of $T \neq 0$, for the Bianchi I type solution (homogeneous, but anisotropic), namely anisotropic cosmology fails to converge to the corresponding solution of general relativity theory [59]. Additionally, in this work, it is shown that, for the non-static spherically symmetric case, convergence to the general relativity does not work even if $T \neq 0$.

5.4.2. Brans-Dicke Theory in Higher Dimensions

As we have discussed before, there might exist new spatial dimensions beyond the four where the standard model particles exist. In these models, only gravity can propagate into the extra dimensions which we cannot observe. The main aim of this theory is to incorporate gravity and gauge interactions in a unique scheme in a reliable manner. Simply, Newton's gravitational constant is related with the energy scale of

Planck mass where quantum gravity effects would be important;

$$M_P c^2 = \left[\frac{\hbar c^5}{8\pi G_N}\right]^{1/2} \sim 2.4.10^{18}\mathrm{GeV}. \tag{5.98}$$

This value of Planck mass is too large to measure experimentally. Therefore, Newton's constant again needs some modifications and investigations on it. On the other hand, gravitational constant is described as a scalar field in BD theory which is alternative to Einstein general relativity. Similar to application to four dimensional space-time, BD theory can be generalized to the higher dimensional space-time and then effective gravity localized on the brane can be investigated. This consideration might be helpful to find out the mysteries of gravity.

Additionally, observations indicate that the expansion of our universe might be presently accelerating [60, 63]. The main content of the universe responsible for the acceleration is usually called as dark energy which constitutes about three fourths of the whole matter of our universe. To explain the dark energy, there are a number of quintessence models which involve a minimally coupled scalar field with potential [64, 66]. However, in these models, the scalar field is added by hand and hence its origin is not understood. On the other hand, in BD theory, this scalar field plays the role of Newton's constant as we have discussed before. However, this scalar field does not lead to cosmological acceleration. In this model, to get an accelerated universe, some authors have considered that the coupling constant ω must vary with time [67], or there must be an additional potential is added by hand [68, 69]. But in Kaluza-Klein theory, the model of accelerating universe is satisfied by the compactification of extra dimensions [70, 71]. Therefore, higher dimensional space-time can take into account the cosmological acceleration. Dereli and Obukhov [72] have investigated the exact static solution to the Einstein-Maxwell-Klein-Gordon theory with a cosmological term in (d+n) dimensions and have related the solutions with higher dimensional BD black holes.

The action of the n-dimensional BD theory is proposed as

$$S_n = \int d^n x \sqrt{g^{(n)}} \left[\varphi R^{(n)} - \frac{\omega}{\varphi} g^{AB} \left(\partial_A \varphi \right) \left(\partial_B \varphi \right) - V\left(\varphi \right) \right] + \int d^n x \sqrt{g^{(n)}} \widehat{\mathcal{L}}_M^{(n)}, \quad (5.99)$$

where $R^{(n)}$ is the curvature scalar associated with the n-dimensional space-time metric g_{AB}, φ is a scalar field, ω is a coupling constant, and finally, $\widehat{\mathcal{L}}_M^{(n)}$ represents the n-dimensional lagrangian of the matter fields. The indices $A, B = 0, 1, 2, 4, ..., (4 + d)$ and $\mu, \nu = 0, 1, 2, 3$ and signature of the metric is $(-, +, +, +....)$.

6. BRANE WORLD COSMOLOGY AND JORDAN-BRANS-DICKE THEORY

In spite of the successes of general relativity, which is now called the standard theory of gravitation, there are many other generalizations. As mentioned earler, among them the scalar-tensor theory has great importance. The scalar-tensor theory was conceived originally by P. Jordan [47] and Y. Thiry [48]. Jordan embedded a four dimensional curved manifold in five dimensional flat space-time. He presented a general Lagrangian for the scalar field living in four-dimensional curved space-time:

$$L_J = \sqrt{-g}\left[\varphi_J^{\gamma}\left(R - \omega_J \frac{1}{\varphi_J^2} g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J\right) + L_{\text{matter}}\left(\varphi_J, \Psi\right)\right], \tag{6.1}$$

where $\varphi_J(x)$ is Jordan's scalar field, γ and ω_J are constants, and Ψ represents matter fields. $\varphi_J R$ is the nonminimal coupling term which marked the birth of scalar-tensor theory and the term $L_{\text{matter}}(\varphi_J, \Psi)$ is for the matter part which depends on the scalar field.

Jordan's work was taken over particularly by C.Brans and R.H. Dicke [49]. As we discussed in section 5.4, they assumed that decoupling of scalar field from the matter part of the Lagrangian occurs. They defined their scalar field φ by

$$\varphi = \varphi_J^{\gamma} \tag{6.2}$$

and then they rewrite the Jordan's Lagrangian in JBD form as in Equation (5.76)

$$L_{JBD} = \sqrt{-g}\left(\varphi R - \omega \frac{1}{\varphi} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + L_{\text{matter}}\left(\Psi\right)\right), \tag{6.3}$$

where $\omega = \omega_J/\gamma^2$. They demanded that the matter part of the Lagrangian $\sqrt{-g}L_{\text{matter}}$ be decoupled from $\varphi(x)$ as their requirement that the weak equivalence principle be respected, in contrast to Jordan's model. To remove the singularity from the second term on the right hand side we introduce a new field ϕ:

$$\varphi = \frac{\phi^2}{8\omega}. \tag{6.4}$$

Then the Brans-Dicke (BD) action will be [73]

$$L_{BD} = \sqrt{-g}\left(\frac{\phi^2}{8\omega}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + L_{\text{matter}}\right). \tag{6.5}$$

where in order to get cosmic acceleration, either the parameter ω should be time dependent [54], or a potential term for the scalar field could be added [68, 69] to the Lagrangian.

On the other hand, string theory has led to a new type of object which is called a brane. This also gives a new perspective to cosmology so that our universe is confined to a four dimensional space-time subspace or 3-brane. The extra dimension may have large compact toroidal topology [13] or be unbounded with a warp factor, depending on the distance from the brane [15, 16]. Several works have studied higher dimensional BD theory to combine the advantages of higher dimensional cosmology with the BD theory [74, 78]. Moreover, considering the scalar field in the five dimensional bulk with Einstein gravity was proposed in the papers [79].

Our starting point is the paper of Bander who studied five dimensional bulk whose dynamics is governed by a scalar Liouville field coupled to gravity in the usual way [80]. He derived that the effective theory on the brane has a time dependent Planck mass and a cosmological constant and also found expanding scale factors with no acceleration. In our work, instead of the five dimensional action in Bander's work, we investigate the properties of the five dimensional bulk in Brans-Dicke theory [81]. Moreover, in both works, the five dimensional metric ansatze are similar. In our work we start with a general warp factor which becomes linear to satisfy the BD equations. The layout of this chapter is as follows. In section 6.1 we present the general framework for our five dimensional theory and compute the five dimensional Brans-Dicke equations. In section 6.2 we analyze the cosmological solutions. We find false vacuum energy

$(p_B = -\rho_B)$ for exponentially growing scale factors and radiation dominated universe $(p_B = \frac{1}{3}\rho_B)$ for power law scale factors in the bulk. In section 6.3 we derive the effective four dimensional scalar field and obtain its time dependence.

6.1. The Action and Equations of Motion

The action of our five dimensional BD theory is proposed as

$$S = \int d^5x\sqrt{-g}\Big(\frac{\phi^2}{8\omega}R - \frac{1}{2}\partial_A\phi\partial_B\phi g^{AB} - V\left(\phi\right)\Big) + S_{\text{matter}}, \tag{6.6}$$

where ω is the dimensionless Brans-Dicke parameter, ϕ is the scalar field and $V(\phi)$ is the scalar potential. S_{matter} represents the five dimensional action of the matter. The variation of the action with respect to g^{AB} gives field equation as

$$\frac{1}{8\omega}\left(\phi^2 G_{AB} - \phi^2_{,A;B} + g_{AB}\Box\phi^2\right) - \frac{1}{2}\partial_A\phi\partial_B\phi + \frac{g_{AB}}{4}\partial_C\phi\partial^C\phi + \frac{1}{2}g_{AB}V\left(\phi\right) = T_{AB}. \tag{6.7}$$

We choose a general five dimensional metric ansatz which can be written as [80]:

$$ds^2 = b\left(t\right)^2 dW^2 + f\left(W\right)^2\left[-dt^2 + a\left(t\right)^2\delta_{ij}dx^i dx^j\right], \tag{6.8}$$

where $i, j = 1, 2, 3$ and $f(W)$ is the warp factor which depends on the fifth coordinate, $a(t)$ is the cosmological scale factor and $b(t)$ is the time dependent scale factor of the fifth dimension. More generally, this metric has been studied in [78, 82]. In Mendes' work [78] the five dimensional brane cosmology with non-minimally coupled scalar field to gravity is interpreted in Jordan frame without a scalar potential. In our work we add a scalar potential to the action.

In the orthonormal basis $e^0 = fdt$, $e^i = fadx^i$ and $e^5 = bdW$, the stress-energy tensor can be considered as [82]

$$T^A_B = T^A_B\mid_{\text{bulk}} + T^A_B\mid_{\text{brane}}, \tag{6.9}$$

where $T^A_B\mid_{\rm bulk}$ is the energy momentum tensor of the bulk matter and

$$T^A_B\mid_{\rm bulk}=\mathrm{diag}\left(-\rho_B,p_B,p_B,p_B,q_B\right). \tag{6.10}$$

The second term $T^A_B\mid_{\rm brane}$ corresponds to the matter content in the brane $(W=0)$,

$$T^A_B\mid_{\rm brane}=\frac{\delta\left(W\right)}{b}\mathrm{diag}\left(-\rho,p,p,p,0\right). \tag{6.11}$$

If we substitute the Einstein tensor components in eq(6.7) we obtain the BD equations. In the coordinate basis, for component 00;

$$\begin{aligned}&\frac{1}{8\omega}\Big[3\Big(\frac{\dot a^2}{a^2}+\frac{\dot a\dot b}{ab}\Big)-\frac{f^2}{b^2}\Big(3\frac{f'^2}{f^2}+3\frac{f''}{f}+\frac{\partial_W^2\phi^2}{\phi^2}+3\frac{f'}{f}\frac{\partial_W\phi^2}{\phi^2}\Big)\\&+3\frac{\dot a}{a}\frac{\partial_t\phi^2}{\phi^2}+\frac{\dot b}{b}\frac{\partial_t\phi^2}{\phi^2}\Big]-\frac{f^2}{4b^2}\frac{\left(\partial_W\phi\right)^2}{\phi^2}-\frac{\left(\partial_t\phi\right)^2}{4\phi^2}-\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=\frac{T_{00}}{\phi^2}.\end{aligned} \tag{6.12}$$

For components ii;

$$\begin{aligned}&\frac{1}{8\omega}\Big[-\Big(2\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+2\frac{\dot a\dot b}{ab}+\frac{\ddot b}{b}\Big)+\frac{f^2}{b^2}\Big(3\frac{f'^2}{f^2}+3\frac{f''}{f}+\frac{\partial_W^2\phi^2}{\phi^2}\\&+3\frac{f'}{f}\frac{\partial_W\phi^2}{\phi^2}\Big)-\frac{\partial_t^2\phi^2}{\phi^2}-\frac{\dot b}{b}\frac{\partial_t\phi^2}{\phi^2}-2\frac{\dot a}{a}\frac{\partial_t\phi^2}{\phi^2}\Big]+\frac{f^2}{4b^2}\frac{\left(\partial_W\phi\right)^2}{\phi^2}-\frac{\left(\partial_t\phi\right)^2}{4\phi^2}\\&+f^2\frac{V\left(\phi\right)}{2\phi^2}=\frac{1}{a^2}\frac{T_{ii}}{\phi^2}.\end{aligned} \tag{6.13}$$

For component 55;

$$\begin{aligned}&\frac{1}{8\omega}\Big[-3\Big(\frac{\dot a^2}{a^2}+\frac{\ddot a}{a}\Big)+6\frac{f^2}{b^2}\frac{f'^2}{f^2}-\frac{\partial_t^2\phi^2}{\phi^2}-3\frac{\dot a}{a}\frac{\partial_t\phi^2}{\phi^2}+\frac{4f^2}{b^2}\frac{f'}{f}\frac{\partial_W\phi^2}{\phi^2}\Big]\\&-\frac{f^2}{4b^2}\frac{\left(\partial_W\phi\right)^2}{\phi^2}-\frac{\left(\partial_t\phi\right)^2}{4\phi^2}+\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=\frac{f^2}{b^2}\frac{T_{55}}{\phi^2}.\end{aligned} \tag{6.14}$$

For component 05;

$$\frac{1}{8\omega}\Big[\frac{3\dot{b}f'}{bf}-\frac{\partial_t\partial_W\phi^2}{\phi^2}+\frac{f'}{f}\frac{\partial_t\phi^2}{\phi^2}+\frac{\dot{b}}{b}\frac{\partial_W\phi^2}{\phi^2}\Big]-\frac{1}{2}\frac{\partial_t\phi\partial_W\phi}{\phi^2}=0. \qquad (6.15)$$

Assume that the 05 component of the energy-momentum tensor vanishes, which means that there is no flow of matter along the fifth dimension. Therefore the nonzero elements of the 5D stress-energy tensor are

$$T_{00} = f^2\rho_B + f^2\frac{\delta(w)}{b}\rho \qquad (6.16)$$

$$T_{ii} = a^2f^2p_B + a^2f^2\frac{\delta(w)}{b}p \qquad (6.17)$$

$$T_{55} = b^2q_B. \qquad (6.18)$$

Finally variation with respect to ϕ gives,

$$\frac{1}{4\omega}(\phi R)-\frac{\partial V(\phi)}{\partial\phi}+\Box\phi=0, \qquad (6.19)$$

which explicitly reads

$$\frac{1}{4\omega}R-\frac{\partial_t^2\phi}{f^2\phi}+\frac{4}{b^2}\frac{f'}{f}\frac{\partial_W\phi}{\phi}-\frac{3}{f^2}\frac{\dot{a}}{a}\frac{\partial_t\phi}{\phi}-\frac{\dot{b}}{bf^2}\frac{\partial_t\phi}{\phi}+\frac{\partial_W^2\phi}{b^2\phi}-\frac{1}{\phi}\frac{\partial V(\phi)}{\partial\phi}=0, \qquad (6.20)$$

where the Ricci scalar R is:

$$R=\frac{1}{f^2}\Big(\frac{6\ddot{a}}{a}+\frac{2\ddot{b}}{b}+\frac{6\dot{a}^2}{a^2}+\frac{6\dot{a}\dot{b}}{ab}\Big)-\frac{12f'^2}{f^2b^2}-\frac{8f''}{fb^2}. \qquad (6.21)$$

The metric and the BD field are continuous across the brane localized at $W=0$. However their derivatives can be discontinuous at the brane. Since we have Z_2 orbifold symmetry, second derivatives of scale factor and BD field will contain Dirac delta function in the second derivatives of the metric with respect to fifth dimension. Therefore for a function f, we have [78, 82]

$$f''=\widehat{f''}+[f']\,\delta(W), \qquad (6.22)$$

where $\widehat{f''}$ is the non-distributional part of the double derivative of f, and $[f']$ is the jump in the first derivative of f across $W = 0$, it is defined as

$$[f'] = f'\left(0^+\right) - f'\left(0^-\right). \tag{6.23}$$

Matching the Dirac delta functions in Equation (6.12),

$$\frac{1}{8\omega}\frac{f^2}{b^2}\left(3\frac{f''}{f} + 2\frac{\phi''}{\phi}\right) = -f^2\frac{\delta(w)}{b}\rho \tag{6.24}$$

$$\frac{1}{8\omega}\left(3\frac{[f']}{fb} + 2\frac{[\phi']}{\phi b}\right) = -\rho, \tag{6.25}$$

and in Equation (6.13)

$$\frac{1}{8\omega}\frac{f^2}{b^2}\left(3\frac{f''}{f} + 2\frac{\phi''}{\phi}\right) = f^2\frac{\delta(w)}{b}p \tag{6.26}$$

$$\frac{1}{8\omega}\left(3\frac{[f']}{fb} + 2\frac{[\phi']}{\phi b}\right) = p, \tag{6.27}$$

and finally in Equation (6.20)

$$-\frac{1}{4\omega}8\frac{f''}{f} + \frac{\phi''}{\phi} = 0 \tag{6.28}$$

$$-\frac{2}{\omega}\frac{[f']}{f} + \frac{[\phi']}{\phi} = 0, \tag{6.29}$$

using (6.25) and (6.29)we obtain that

$$\frac{[f']_0}{f_0 b_0} = -\frac{8\omega^2}{(3\omega+4)\,\phi^2}\rho \tag{6.30}$$

$$\frac{[\phi']_0}{\phi_0 b_0} = -\frac{16\omega}{(3\omega+4)\,\phi^2}\rho, \tag{6.31}$$

where the subscript '0' stands for the brane at $W = 0$. Using eq(6.25) and eq(6.27) we get the remarkable result that the cosmological constant dominates on the brane i.e.

$$\rho = -p = -\frac{\phi_0^2}{8\omega}\left(\frac{3\,[f']_0}{f_0 b_0} + \frac{2\,[\phi']_0}{\phi_0 b_0}\right). \tag{6.32}$$

Here choice of the scalar factor $a(t)$ does not make any difference on the equation of state $\rho = -p$.

To evaluate the jump condition we substitute (6.30) and (6.31) in eq(6.15),

$$\Big(-3\omega\frac{\dot{b}}{b}\rho + 4\frac{\dot{\phi}}{\phi}\rho - 8\frac{\dot{\phi}}{\phi}\rho + 4\frac{\dot{\phi}}{\phi}\rho + 4\frac{\dot{b}}{b}\rho + 4\dot{\rho} - 2\omega\frac{\dot{\phi}}{\phi}\rho - 4\frac{\dot{b}}{b}\rho\Big) + 8\omega\frac{\dot{\phi}}{\phi}\rho = 0, \tag{6.33}$$

where the derivation is

$$\begin{aligned} \frac{\partial_t \partial_W \phi^2}{\phi^2} &= 2\Big(\frac{\dot{\phi}}{\phi}\frac{\phi'}{\phi} + \frac{\dot{\phi}'}{\phi}\Big) \\ &= 2b\Big(\frac{\dot{\phi}}{\phi}\frac{\phi'}{\phi} + \frac{\dot{\rho}}{\rho}\frac{\phi'}{\phi} - 2\frac{\dot{\phi}}{\phi}\frac{\phi'}{\phi} + \frac{\dot{\phi}}{\phi}\frac{\phi'}{\phi} + \frac{\phi'}{\phi}\frac{\dot{b}}{b}\Big), \end{aligned} \tag{6.34}$$

where we use (6.31) in the second line, then we get the equation for the matter on the brane

$$4\dot{\rho} + 3\omega\frac{\dot{b}}{b}p + 6\omega\frac{\dot{\phi}}{\phi}\rho = 0, \tag{6.35}$$

where if $2\frac{\dot{b}}{b} = \frac{\dot{\phi}}{\phi}$ or in particular time derivatives of the b and ϕ are zero, we obtain that ρ and $p = -\rho$ are constant on the brane.

6.2. Solutions

Solutions of BD equations restrict the scalar field to be in the form $\phi(t, W) = B(t)\, C(W)$. Starting from this, to satisfy all of the BD equations we make two possible ansatze for $a(t)$. The first one is exponential growth in time and the other is power law expansion.

6.2.1. Exponential Expansion, $\mathbf{a}\left(\mathbf{t}\right)=\mathbf{a_0}\mathbf{e}^{\lambda\mathbf{t}}$

If we choose the scale factor $a(t)$ as exponentially growing, we see from (6.12-6.20) that, for component 00:

$$\frac{1}{8\omega}\Big[\Big(3\lambda^2+3\lambda\frac{\dot{b}}{b}+6\lambda\frac{\dot{B}}{B}+2\frac{\dot{b}}{b}\frac{\dot{B}}{B}\Big)-\frac{f^2}{b^2}\Big(3\frac{f'^2}{f^2}+2\frac{C'^2}{C^2}+6\frac{f'}{f}\frac{C'}{C}\Big)\Big]$$
$$-\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4}\frac{\dot{B}^2}{B^2}-\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=\frac{f^2}{\phi^2}\rho_B. \tag{6.36}$$

For component ii :

$$\frac{1}{8\omega}\Big[-\Big(3\lambda^2+2\lambda\frac{\dot{b}}{b}+\frac{\ddot{b}}{b}+4\lambda\frac{\dot{B}}{B}+2\frac{\dot{B}^2}{B^2}+2\frac{\ddot{B}}{B}\Big)+\frac{f^2}{b^2}\Big(3\frac{f'^2}{f^2}+2\frac{C'^2}{C^2}+6\frac{f'}{f}\frac{C'}{C}\Big)\Big]$$
$$+\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4}\frac{\dot{B}^2}{B^2}+\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=\frac{f^2}{\phi^2}p_B. \tag{6.37}$$

For component 55 :

$$\frac{1}{8\omega}\Big[-\Big(3\lambda^2+6\lambda\frac{\dot{B}}{B}+2\frac{\dot{B}^2}{B^2}+2\frac{\ddot{B}}{B}\Big)+\frac{f^2}{b^2}\Big(6\frac{f'^2}{f^2}+4\frac{f'}{f}\frac{C'}{C}\Big)\Big]$$
$$-\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4}\frac{\dot{B}^2}{B^2}+\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=\frac{f^2}{\phi^2}q_B. \tag{6.38}$$

For component 05 :

$$\frac{1}{8\omega}\Big[3\frac{\dot{b}}{b}\frac{f'}{f}-4\frac{\dot{B}}{B}\frac{C'}{C}+2\frac{\dot{B}}{B}\frac{f'}{f}+2\frac{\dot{b}}{b}\frac{C'}{C}\Big]-\frac{1}{2}\frac{\dot{B}}{B}\frac{C'}{C}=0. \tag{6.39}$$

Finally, for the ϕ derivative:

$$\frac{1}{4\omega}\Big[12\lambda^2+2\frac{\ddot{b}}{b}+6\lambda\frac{\dot{b}}{b}-\frac{f^2}{b^2}\Big(12\frac{f'^2}{f^2}\Big)\Big]-3\lambda\frac{\dot{B}}{B}-\frac{\ddot{B}}{B}-\frac{\dot{b}}{b}\frac{\dot{B}}{B}$$
$$+\frac{f^2}{b^2}\Big(4\frac{f'^2}{f^2}\Big)-\frac{f^2}{2}\frac{V\left(\phi\right)}{\phi^2}=0. \tag{6.40}$$

These equations consist of sum of several terms and we can solve them by making all of these terms constant in each equation. Therefore $b\left(t\right)$ must be constant, $b\left(t\right)=b_0$

and $B(t)$ must be in the exponential form also $B(t) = B_0 e^{\gamma t}$ and than for warp factor $f(W)$, we can easily read that

$$f(W) = \frac{W}{W_0}. \tag{6.41}$$

For a brane at $W = W_0$, we introduce the coordinate W' such that $\frac{W}{W_0} = 1 - \frac{W'}{W_0}$. The metric on both sides of brane can be written

$$ds^2 = b_0^2 dW^2 + \left(1 - \frac{|W|}{W_0}\right)^2 \left[-dt^2 + e^{2\lambda t} d\vec{x}^2\right], \tag{6.42}$$

and the brane is at $W = 0$. Here we dropped the prime for simplicity. This warp factor is the same as in Bander's work [80]. The brane we live in is embedded in the five-dimensional bulk space-time and the four dimensional part in the square parenthesis is the well known de-Sitter space-time. This metric is similar to a Randall- Sundrum type of model in some sense. Instead of the exponential warp factor we obtain the linear warp factor. However for small W it is known that

$$e^{-|W|} \simeq 1 - |W|, \tag{6.43}$$

and two the models are similar.

To satisfy the BD equations, we have obtained a linear warp factor for the exponentially expanding scale factor. From these results and BD equations, W dependent part of the scalar field must be equal to some power of the warp factor, namely

$$C(W) = c_0 \left(1 - \frac{|W|}{W_0}\right)^{\alpha}, \tag{6.44}$$

therefore since $B(t) = B_0 e^{\gamma t}$ and the scalar field $\phi = B(t)\, C(W)$, we can express it as

$$\phi = B_0 c_0 \left[e^{\beta t}\left(1 - \frac{|W|}{W_0}\right)\right]^{\alpha}, \tag{6.45}$$

where $\gamma = \beta\alpha$ and the power α and β will be derived below for different conditions.

Substituting these predicted values for f, B and C in Brans-Dicke field equations, we get for (6.36)

$$\frac{1}{8\omega}\Big[3\lambda^2+6\lambda\alpha\beta-\frac{1}{\left(b_0W_0\right)^2}\left(3+2\alpha^2+6\alpha\right)\Big]-\frac{\alpha^2}{4\left(b_0W_0\right)^2}$$
$$-\frac{\alpha^2\beta^2}{4}-\frac{V_0}{2}=\frac{f^2}{\phi^2}\rho_B, \tag{6.46}$$

for (6.37),

$$\frac{1}{8\omega}\Big[-\left(3\lambda^2+4\lambda\alpha\beta+4\alpha^2\beta^2\right)+\frac{1}{\left(b_0W_0\right)^2}\left(3+2\alpha^2+6\alpha\right)\Big]$$
$$+\frac{\alpha^2}{4\left(b_0W_0\right)^2}-\frac{\alpha^2\beta^2}{4}+\frac{V_0}{2}=\frac{f^2}{\phi^2}p_B, \tag{6.47}$$

for (6.38),

$$\frac{1}{8\omega}\Big[-\left(3\lambda^2+6\lambda\alpha\beta+4\alpha^2\beta^2\right)+\frac{1}{\left(b_0W_0\right)^2}\left(6+4\alpha\right)\Big]-\frac{\alpha^2}{4\left(b_0W_0\right)^2}$$
$$-\frac{\alpha^2\beta^2}{4}+\frac{V_0}{2}=\frac{f^2}{\phi^2}q_B, \tag{6.48}$$

for (6.36),

$$\frac{1}{8\omega}\Big[4\frac{\alpha^2\beta}{W_0}-2\frac{\alpha\beta}{W_0}\Big]+\frac{1}{2}\frac{\alpha^2\beta}{W_0}=0, \tag{6.49}$$

and finally for (6.40),

$$\frac{1}{4\omega}\Big[12\lambda^2-\frac{12}{\left(b_0W_0\right)^2}\Big]-3\lambda\alpha\beta-\alpha^2\beta^2+\frac{4}{\left(b_0W_0\right)^2}-\frac{V_0}{2}=0 \tag{6.50}$$

Where on the brane we live

$$\left(16\pi G\right)^{-1}=M_P^2=\frac{\phi^2}{8\omega}=\frac{\left(B_0c_0\right)^2e^{2\alpha\beta t}}{8\omega}, \tag{6.51}$$

where B_0c_0 is required to be within a few orders of magnitude of Planck mass [15].

In equations (6.12) and (6.13), since W dependent parts have opposite signs of each other, it looks like that the condition $p_B = -\rho_B$ might be satisfied. As we will show below, this condition can be exactly derived for exponentially expanding scale factor. Depending upon whether the fifth component of the energy momentum tensor T_{55} is zero or not, we will obtain $\rho_B = p_B = 0$ or $\rho_B = -p_B \neq 0$ respectively. We first discuss the solution where $T_{55} = 0$.

$T_{55} = 0$. For this case, using a computer program, the solutions of the BD equations (6.46-6.50) exists only if $\rho_B = 0$, $p_B = 0$ (empty universe), on the other hand there are two different results available for the other parameters as shown in following set of equations,

$$\rho_B = 0, \quad p_B = 0, \begin{cases} V_0 = 0, & \beta = \lambda, \quad \alpha = \frac{1}{2(1+\omega)} \\ V_0 = -\frac{(3\omega+4)\lambda^2}{2\omega(1+\omega)^2}\left(B_0c_0\right)^{2/\alpha}, & \beta = 0, \quad \alpha = \frac{1}{1+\omega} \end{cases} \tag{6.52}$$

where $b_oW_o\lambda = \pm 1$. Here in the first row of the Equation (6.52), the BD equations give a scalar field which depends not only on time but also on the fifth coordinate. On the other hand in the second row the scalar field only depends on the fifth coordinate and there is a scalar potential $V_0 \neq 0$. Therefore the scalar potential does not depend on time

$$V(\phi) = V_0\phi^{2-\frac{2}{\alpha}}, \tag{6.53}$$

where V_0 is a constant has dimension $L^{-2-\frac{3}{\alpha}}$. From these results as $\omega \to \infty$, $\alpha, V_0 \to 0$. Therefore $V(\phi) \to 0$. This means that at the large values of the BD parameter, the scalar field is constant

$$\frac{\phi^2}{8\omega} = M_P^2 = \frac{\left(B_0c_0\right)^2}{8\omega}, \tag{6.54}$$

with no scalar potential.

The $q = 0$ condition has been derived in [83] where it was found that empty and flat five dimensional universe where ${}^{(5)}R^{MN}{}_{PQ} = 0$ and $\Lambda_5 = 0$ gives rise to a four dimensional expanding universe with nonzero Riemann tensor and cosmological constant. This five dimensional space is a well known Minkowski universe

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \tag{6.55}$$

transformed into

$$ds^2 = b_0^2 dW^2 + \left(1 - \frac{|W|}{W_0}\right)^2 \left[-dt^2 + e^{2\lambda t}\left(dr^2 + r^2 d\Omega_2^2\right)\right], \tag{6.56}$$

by the following transformation

$$x_1 = b_0 \left(W_0 - |W|\right)\left(\sinh\left(\lambda t\right) + \frac{\lambda^2 r^2}{2} e^{\lambda t}\right), \tag{6.57}$$

$$x_2 = b_0 \left(W_0 - |W|\right)\left(\cosh\left(\lambda t\right) - \frac{\lambda^2 r^2}{2} e^{\lambda t}\right), \tag{6.58}$$

$$x_3 = b_0 \left(W_0 - |W|\right) \lambda r e^{\lambda t} \cos\theta, \tag{6.59}$$

$$x_4 = b_0 \left(W_0 - |W|\right) \lambda r e^{\lambda t} \sin\theta \cos\varphi, \tag{6.60}$$

$$x_5 = b_0 \left(W_0 - |W|\right) \lambda r e^{\lambda t} \sin\theta \sin\varphi, \tag{6.61}$$

after some calculations we get the factor $b_0 W_0 \lambda$ in front of the four dimensional part. This was already found as unity. Therefore the four dimensional curved space-time can be embedded in the five dimensional flat space-time by these coordinate transformations.

$T_{55} \neq 0$. From the BD equations (6.46-6.50), we obtain that $p_B = -\rho_B \neq 0$ and $\beta = 0$. As $\omega \to \infty$,

$$\rho_B = -p_B \to \frac{\left(B_0 c_0\right)^{2/\alpha}}{2\left(b_o W_o\right)^2} \frac{\alpha^2\left(\alpha+1\right)}{\left(\alpha-1\right)} \phi^{2-2/\alpha}, \tag{6.62}$$

$$q_B \rightarrow \frac{(B_0 c_0)^{2/\alpha}}{(b_0 W_o)^2} \frac{\alpha^2}{\alpha - 1} \phi^{2-2/\alpha}, \qquad (6.63)$$

$$V_0 \rightarrow \frac{(B_0 c_0)^{2/\alpha}}{(b_0 W_0)^2} \frac{\alpha^2 (3+\alpha)}{2(\alpha - 1)}, \qquad (6.64)$$

for all of the results $\beta = 0$ and $V(\phi) = V_0 \phi^{2-\frac{2}{\alpha}}$, therefore the scalar potential becomes again time independent for the exponentially expanding universe for $q_B \neq 0$.

In general, this behavior of the energy and momentum namely, $\rho = -p$ acts as a cosmological constant. In previous works [84, 85] this energy has been identified as the false vacuum energy density ρ_f. During the false vacuum phase the universe supercools. It is believed that as the universe expands it cools down and then it experiences a series of phase transitions. Since the cosmic expansion continues to drive the temperature downward, the universe enters a period of supercooling. As the universe supercools the energy density acts as an effective cosmological constant. If we suppose this phase as the false phase, the probability of a point remaining in the false phase during the bubble nucleation process is quite small as shown in [86]. Then the universe is dominated by the true vacuum and exits from the false vacuum.

In the true vacuum we can consider a power-law expansion.

6.2.2. Power-Law Expansion

The scale factors are:

$$a(t) = a_0 (t/t_0)^{\lambda}, \qquad (6.65)$$

$$b(t) = b_0 (t/t_0)^{\gamma}. \qquad (6.66)$$

These power law solutions restrict us to choose $B(t) = B_0 \left(\frac{t}{t_0}\right)^{\beta}$.Then the time component of BD equations(6.12) becomes

$$\frac{1}{8\omega}\left[\frac{1}{t^2}\left(3\lambda^2 + 3\lambda\gamma + 6\lambda\beta + 2\gamma\beta\right) - \frac{f^2}{b^2}\left(3\frac{f'^2}{f^2} + 2\frac{C'^2}{C^2} + 6\frac{f'}{f}\frac{C'}{C}\right)\right]$$

$$-\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4t^2}\beta^2-\frac{f^2}{2}\frac{V(\phi)}{\phi^2}=\frac{f^2}{\phi^2}\rho_B, \tag{6.67}$$

the Equation (6.13) is

$$\frac{1}{8\omega}\Big[-\frac{1}{t^2}\Big(3\lambda^2+2\lambda+2\lambda\gamma+\gamma^2-\gamma+2\beta^2-2\beta+2\gamma\beta+4\lambda\beta\Big)$$
$$+\frac{f^2}{b^2}\Big(3\frac{f'^2}{f^2}+2\frac{C'^2}{C^2}+6\frac{f'}{f}\frac{C'}{C}\Big)\Big]+\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4t^2}\beta^2+\frac{f^2}{2}\frac{V(\phi)}{\phi^2}=\frac{f^2}{\phi^2}p_B, \tag{6.68}$$

then the component of the extra dimension (6.14) becomes

$$\frac{1}{8\omega}\Big[-\frac{1}{t^2}\Big(6\lambda^2-3\lambda+\beta^2-\beta+6\lambda\beta\Big)+\frac{f^2}{b^2}\Big(6\frac{f'^2}{f^2}+8\frac{f'}{f}\frac{C'}{C}\Big)\Big]$$
$$-\frac{f^2}{4b^2}\frac{C'^2}{C^2}-\frac{1}{4t^2}\beta^2+\frac{f^2}{2}\frac{V(\phi)}{\phi^2}=\frac{f^2}{\phi^2}q_B, \tag{6.69}$$

the Equation (6.15) yields

$$\frac{1}{8\omega}\Big[3\gamma\frac{f'}{f}-4\beta\frac{C'}{C}+2\beta\frac{f'}{f}+2\gamma\frac{C'}{C}\Big]-\frac{\beta}{2}\frac{C'}{C}=0, \tag{6.70}$$

and finally the last equation of BD field equations (6.20) becomes

$$\frac{1}{4\omega}\Big[\frac{1}{t^2}\Big(12\lambda^2-6\lambda+2\gamma^2-2\gamma+6\lambda\gamma\Big)-12\frac{f^2}{b^2}\frac{f'^2}{f^2}\Big]-\frac{1}{t^2}\Big(\beta^2-\beta+3\lambda\beta+\gamma\beta\Big)$$
$$+\frac{f^2}{b^2}\Big(4\frac{f'}{f}\frac{C'}{C}+\frac{C'}{C}\Big)-\frac{1}{\phi}\frac{\partial V(\phi)}{\partial\phi}=0. \tag{6.71}$$

These equations restrict us to choose $\gamma = 1$, since all the terms must have some constant multiplied by the t^2 in the denominator. Then again we get same result for the warp factor, $f(W)=\left(1-\frac{|W|}{W_0}\right)$ and $C(W)=c_0\left(1-\frac{|W|}{W_0}\right)^{\alpha}$. On the other hand, the condition that the potential $V(\phi)$ must purely depends on ϕ yields $\beta=\alpha$. Then these results causes scalar field to be

$$\phi(t,W)=B_0c_0\left[\left(\frac{t}{t_0}\right)\left(1-\frac{|W|}{W_0}\right)\right]^{\alpha}, \tag{6.72}$$

where B_0 and c_0 are constants. More clearly, the last term $t^2 f^2 \frac{V(\phi)}{\phi^2}$ in the left hand side of equations(6.67-6.69) reveals the Equation (6.72), then scalar potential could be in the form of

$$V\left(\phi\right) = V_0 \phi^{\frac{2}{\alpha}(\alpha-1)}, \tag{6.73}$$

where since $B_0 c_0$ has dimension $L^{-3/2}$, therefore V_0 has dimension $L^{-3/\alpha-2}$. Here to make BD equations simpler we set $\frac{b_0 W_0}{t_0} = 1$.

Now we want to find a general result so we consider the equation of state as:

$$p_B = \nu \rho_B. \tag{6.74}$$

Putting all of these settings in the BD equations we find a nice result: here the interesting thing is that there is no solution other than $\nu = \frac{1}{3}$ for $p_B \neq 0$ and $\rho_B \neq 0$ and solutions are valid only for $q_B = 0$. Different values of the variables in equations (6.67-6.71) are satisfied only for a single value of ν which is $\frac{1}{3}$. Then this ratio between the pressure and energy density corresponds to the radiation dominated universe; and ω dependence of λ, α, and V_0 are:

$$\rho_B = 3 p_B, \tag{6.75}$$

$$\alpha_{\pm} = \frac{\pm\sqrt{3\omega+4}+1}{2\left(\omega+1\right)}, \tag{6.76}$$

$$\lambda_{\pm} = \frac{\omega \mp \sqrt{3\omega+4}}{4\left(\omega+1\right)}, \tag{6.77}$$

and finally

$$V_{0\pm} = -\frac{3\left(B_0 c_0\right)^{2/\alpha}}{t_0^2} \frac{\left(3\omega+4\right)\left(3\omega \pm \sqrt{3\omega+4}+5\right)}{32\omega\left(\omega+1\right)^2}. \tag{6.78}$$

All of these solutions do not give a specific value for ω. From the time-delay measurements, experimentally $\omega > 500$ [87] and more recently $\omega > 3300$ [52]. As

$\omega \to \infty$, $\alpha \to 0$, $\lambda_{\pm} \to \frac{1}{4}$ and $V_0 \to 0$. This means that at this limit, the scalar field becomes constant and the scalar potential vanishes.

For the power-law scale factor we obtain one more solution. BD equations give the empty universe, namely ρ_B, $p_B = 0$ and $\lambda = 1, V_0 = 0$ and $\alpha_{\pm} = \frac{\pm\sqrt{3\omega+4}+1}{2(\omega+1)}$ which are the same as previous value of α (6.76).

The solutions presented here represent decelerating cosmology for the radiation dominated universe and expanding cosmology with constant velocity for the empty universe. However astronomical observations show that the universe is not only expanding but also undergoing accelerated expansion [88]. It may be possible to obtain power law acceleration in BD theory if scale factors for external dimensions are time dependent. In string theory some cosmologies can achieve accelerating scale factors [89, 91].

The metric which we found in this part may be related with the Kasner space-time [92]. It has a cosmological singularity at $t = 0$ where the square of Riemann tensor diverges. On the brane we live ($W = 0$)

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{24\lambda^4}{t^4}. \qquad (6.79)$$

This is a physical singularity and it cannot be avoided by any coordinate transformation [93]. However, since the central part of the space-time is avoided in orbifold construction this has no importance for the brane world scenario [15].

6.3. The Effective Four Dimensional Gravitational Constant

Finally we calculate the four-dimensional effective gravitational constant on the brane and compare with our previous results eq(6.51). On the left hand side of the action in eq (6.6) the first term is:

$$\int d^5x\sqrt{g}\frac{\phi^2_{(5)}}{8\omega}R_{(5)} = \int d^5x\sqrt{g}M^3_{(5)}R_{(5)} = \int d^5x\sqrt{g}\frac{1}{16\pi G_{(5)}}R_{(5)}. \tag{6.80}$$

We can perform the W integral to obtain the effective gravitational constant. With the same manner in the [80] work, this equation reduces to

$$\int d^5x\sqrt{g}\frac{\phi^2_{(5)}}{8\omega}R_{(5)} = \int d^4x dW\sqrt{g^{(4)}}\frac{\phi^2_{(5)}}{8\omega}\left(1-\frac{|W|}{W_0}\right)^2 b\left(t\right)R\left(g^{(4)}_{ij}\left(x\right)\right). \tag{6.81}$$

For the exponentially increasing scale factor we have obtained time dependent scalar field that is $\phi_{(5)} = B_0c_0\left[e^{\beta t}\left(1-\frac{|W|}{W_o}\right)\right]^{\alpha}$. Then eq(6.81) becomes

$$\int d^5x\sqrt{g}\frac{\phi^2_{(5)}}{8\omega}R_{(5)} = \int d^4x 2\frac{b_0W_0}{8\omega}\frac{B_0^2c_0^2e^{2\alpha\beta t}}{(2\alpha+3)}\sqrt{g^{(4)}}R\left(g^{(4)}_{ij}\left(x\right)\right), \tag{6.82}$$

then since $\alpha \simeq 0$, the effective gravitational constant becomes

$$\frac{1}{16\pi G_{eff}} = M^2_{p(eff)} = \frac{\phi^2_{(4)}}{8\omega} = \frac{b_0W_0}{12\omega}\left(B_0c_0\right)^2, \tag{6.83}$$

which is independent of time. This is similar with what we have discussed in eq(6.54) and here B_oc_o is within a few orders of Planck mass.

For the power law scale factors the scalar field is $\phi_{(5)} = B_0c_0\left(\frac{t}{t_0}\left(1-\frac{|W|}{W_0}\right)\right)^{\alpha}$

$$\int d^5x\sqrt{g}\frac{\phi^2_{(5)}}{8\omega}R_{(5)} = \int d^4x\frac{b_0W_0}{4\omega}\left(\frac{t}{t_0}\right)^{2\alpha+1}\frac{B_0^2c_0^2}{(2\alpha+3)}\sqrt{g^{(4)}}R\left(g^{(4)}_{ij}\left(x\right)\right), \tag{6.84}$$

here again for $\alpha \simeq 0$, the four dimensional Brans-Dicke scalar field (or the inverse of the effective gravitational constant) is

$$\frac{1}{16\pi G_{eff}} = M_{p(eff)}^2 = \frac{\phi_{(4)}^2}{8\omega} = \frac{W_0}{12\omega}\left(B_0 c_0\right)^2 b\left(t\right). \tag{6.85}$$

Hence the four dimensional effective gravitational constant depends on time [80].

In this part we introduced a five dimensional BD action and studied the five dimensional metric with a warp factor. We showed that the field equations imply a linear warp factor. For an inflating scale factor we found that the energy density acts as an effective cosmological constant. For power law expansion of scale factor we obtained a radiation dominated universe. Additionally we have shown that the five dimensional scalar field is nearly but cannot be exactly constant. On the other hand the four dimensional effective scalar field is constant for exponentially growing scale factor and depends on time for the power law scale factors.

7. SCHWARZSCHILD-DE SITTER BLACK HOLES IN $4+1$ DIMENSIONAL BULK

In the universe, there exists four fundamental interactions; weak, strong, electromagnetic and gravitational. However, all the interactions in nature are not unified. Gravity shows some different aspects because of its long range and unquantizable behavior and its importance in cosmology. To get a natural explanation for the unification problem, the possibility that gravity might not be fundamentally four dimensional has been considered [13, 16]. Basically these scenarios say that our universe may be a brane embedded in some higher dimensional space. All matter and gauge interactions live on the brane while gravitational interactions are effective in the whole higher dimensional space. Besides cosmology, black holes explore the geometry of gravity and its quantum effects. Therefore gravity and black holes can be related with the brane-world cosmology. In many of the brane-world scenarios, the matter fields we observe are trapped on the brane and when they undergo gravitational collapse a black hole forms [94, 102]. This black hole becomes a higher dimensional object. This higher dimensional object can recover the usual physical properties of black holes and stars.

Most of the recent brane world scenarios originated from Randall–Sundrum (RS) models in which the four dimensional Minkowski universe is embedded in five dimensional AdS bulk. This bulk is compatible with a Friedmann–Robertson–Walker (FRW) brane. The most general vacuum bulk with a FRW brane is Schwarzschild-anti-de Sitter space-time [103, 105] where a domain wall moving in the 5 dimensional Schwarzschild-AdS space -time with the metric

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 \left[d\chi^2 + f_k(\chi)\, d\Omega_2^2 \right] \end{aligned} \tag{7.1}$$

is considered. The FRW brane moves radially along the fifth dimension with $r = a(t)$, where $a(t)$ is the FRW scale factor. Then, expansion of the universe is interpreted as the motion of the brane through the static bulk. Here the part in the square parenthesis is the metric of a unit three-dimensional sphere, plane or hyperboloid for $k = 1, 0, -1$ and $f_1(\chi) = \sin\chi$, $f_0(\chi) = \chi$, and $f_{-1}(\chi) = \sinh\chi$ respectively. The solution of field equations $R^5_{\mu\nu} = \Lambda^5 g^5_{\mu\nu}$ with negative cosmological constant $\Lambda^5 = -\frac{4}{l^2}$ is given by

$$f(r) = k + \frac{r^2}{l^2} - \frac{\mu}{r^2}, \tag{7.2}$$

where l and μ are constants. For $\mu = 0$, this reduces to AdS_5 and $\mu \neq 0$ generates the electric part of the Weyl Tensor. By this consideration the location of three branes in the five dimensional bulk can be investigated.

7.1. The Metric

Instead of these theories we have considered a four dimensional curved space-time with positive cosmological constant embedded in five dimensional flat and empty Minkowski Universe as shown in [83]. Starting from this point of view, in this part, we consider a usual four dimensional Schwarzschild metric embedded in five dimensional Ricci flat universe [106]. Our metric describes a black hole on a domain wall at fixed w. As such a space we take the five dimensional analogue of Schwarzschild-de Sitter space-time in the following form

$$ds^2 = \left(1 - \frac{|w|}{w_0}\right)^2 \left[-U(r)\,dt^2 + \frac{1}{U(r)}dr^2 + r^2 d\Omega_2^2\right] + dw^2, \tag{7.3}$$

where we choose

$$U(r) = 1 - \frac{2M}{r} - \frac{r^2}{k^2}. \tag{7.4}$$

If we choose pseudo-orthonormal basis one forms as

$$E^t = U(r)\left(1 - \frac{|w|}{w_0}\right) dt, \tag{7.5}$$

$$E^r = U^{-1}(r)\left(1 - \frac{|w|}{w_0}\right) dr, \tag{7.6}$$

$$
\begin{aligned}
E^{\theta} &= r\left(1-\frac{|w|}{w_0}\right)d\theta, &(7.7)\\
E^{\phi} &= r\sin\theta\left(1-\frac{|w|}{w_0}\right)d\phi, &(7.8)\\
E^{w} &= dw, &(7.9)
\end{aligned}
$$

and if the signature of metric η_{MN} is $(-,+,+,+,+)$, the five dimensional metric becomes

$$
ds^2 = \eta_{MN}E^M \otimes E^N. \tag{7.10}
$$

We get the nonvanishing components of the Riemann tensor,

$$
\begin{aligned}
-R_{wtwt} &= R_{wrwr} = R_{w\theta w\theta} = R_{w\phi w\phi} = \frac{2\delta(w)}{w_0\left(1-\frac{|w|}{w_0}\right)}, &(7.11)\\
-R_{trtr} &= R_{\theta\phi\theta\phi} = \left(1-\frac{|w|}{w_0}\right)^{-2}\left[\frac{1}{r^2}\left(\frac{2m}{r}+\frac{r^2}{k^2}\right)-\frac{1}{w_0^2}\right] &(7.12)\\
-R_{t\theta t\theta} &= -R_{t\phi t\phi} = R_{r\theta r\theta} = R_{r\phi r\phi} = \left(1-\frac{|w|}{w_0}\right)^{-2}\left[\frac{1}{r^2}\left(\frac{m}{r}-\frac{r^2}{k^2}\right)+\frac{1}{w_0^2}\right] &(7.13)
\end{aligned}
$$

Then we get the nonvanishing components of the Ricci tensor as

$$
\begin{aligned}
-R_{tt} &= R_{rr} = R_{\theta\theta} = R_{\phi\phi} = \frac{3}{\left(1-\frac{|w|}{w_0}\right)^2}\left(\frac{1}{k^2}-\frac{1}{w_0^2}\right)+\frac{2\delta(w)}{w_0\left(1-\frac{|w|}{w_0}\right)}, &(7.14)\\
R_{ww} &= \frac{8\delta(w)}{w_0\left(1-\frac{|w|}{w_0}\right)}. &(7.15)
\end{aligned}
$$

If we apply our consideration, to get the flat and empty five dimensional bulk (where $w \neq 0$ and $w \neq w_0$), for $M = 0$, we set the Riemann tensor (7.12) and (7.13) equal to zero, then the condition of $k = w_0$ must be satisfied. Therefore for $M = 0$, the bulk becomes flat and for $M \neq 0$ the bulk becomes Ricci flat (7.14). Then it is precisely the condition that the parameter w_0 in $U(r)$ (7.4) and in the warp factor in (7.3) are the same which causes the $4+1$ dimensional bulk to be Ricci flat.

This metric tells us that for constant w the cosmological constant is $\Lambda_{observed} =$

$3/w_0^2$. We set up the $3+1$ branes at $w = 0$ and $w = w_1 \leq w_0$. Then, on the branes this metric will be in the standard form of the Schwarzschild metric. Here note that in small scales the cosmological curvature is much smaller than the Schwarzschild curvature. If we denote the perturbed horizon as $r = 2M + \varepsilon$ for small ε,

$$1 - \frac{2M}{r} - \frac{r^2}{w_0^2} = 1 - \frac{2Mw_0^2 + 8M^3}{rw_0^2} - \frac{12M^2\varepsilon}{rw_0^2}, \tag{7.16}$$

the black hole horizon is determined to be at $r = 2M + 8M^3/w_0^2$,where $w_0 \gg M$. On the other hand from the square of the Riemann tensor for $\omega \neq 0$,

$$R^{MNPR}R_{MNPR} = \frac{48M^2}{r^6\left(1 - \frac{|w|}{w_0}\right)^4} \tag{7.17}$$

one finds curvature singularities at $r = 0$ and $w = \pm w_0$. However one expects that masses confined to the brane will generate the localized gravitational fields, so that these singularities are not physically important. Strictly speaking, the warp factor is given by $1 - |w|/w_0$, for $-w_1 \leq w \leq w_1$, and we impose periodicity of period $2w_1$ on w together with Z_2 symmetry $w \rightarrow -w$.

7.2. Case for $M = 0$

As we discussed above, our five dimensional metric gives the flat and empty universe for $M = 0$. The four dimensional part of this metric can also transform into the well known form of FRW metric. Namely the four dimensional part in square parenthesis in (7.3) transforms into

$$ds_4^2 = \left[-d\tau^2 + g(\tau)^2\left(d\chi^2 + f_k(\chi)^2 d\Omega_2^2\right)\right], \tag{7.18}$$

by the following transformations. Here τ and $g(\tau)$ correspond to cosmic time and scale factors of FRW universe respectively.

- $k = -1$

$$r = w_0 \sinh(\tau) \sinh(\chi) \tag{7.19}$$

$$t = \frac{w_0}{2} \ln \left(\frac{\cosh(\chi) \sinh(\tau) + \cosh(\tau)}{\cosh(\chi) \sinh(\tau) - \cosh(\tau)} \right) \tag{7.20}$$

$$g(\tau) = w_0 \sinh(\tau) \tag{7.21}$$

$$f_{-1}(\chi) = \sinh(\chi) . \tag{7.22}$$

- $k = 0$

$$r = \chi \exp(\tau/w_0) \tag{7.23}$$

$$t = \tau - \frac{w_0}{2} \left[\chi^2 \exp(2\tau/w_0) - 1 \right] \tag{7.24}$$

$$g(\tau) = \exp(\tau/w_0) \tag{7.25}$$

$$f_0(\chi) = \chi. \tag{7.26}$$

- $k = 1$

$$r = w_0 \cosh(\tau) \sin(\chi) \tag{7.27}$$

$$t = \frac{w_0}{2} \ln \left(\frac{\cosh(\tau) \cos(\chi) + \sinh(\tau)}{\cosh(\tau) \cos(\chi) - \sinh(\tau)} \right) \tag{7.28}$$

$$g(\tau) = w_0 \cosh(\tau) \tag{7.29}$$

$$f_1(\chi) = \sin(\chi) . \tag{7.30}$$

Brane world for FRW universe have been investigated in many papers. In our work we include the mass term M and try to find out brane world cosmology for this Schwarzschild-dS black hole (7.3).

7.3. Case For M$\neq 0$

For the case of $M \neq 0$, we can make the same transformation, however it will not make our work easier. If we put $M = 0$ in this transformation we obtain the same metric as in (7.18). This means that, our space-time becomes FRW universe far away from the black hole. Hence we can use the context of FRW cosmological model.

In this part we suppose that the visible world is located at $w = w_1$. Therefore the visible metric tensor will be $g_{\mu\nu}^{(\mathrm{vis})} = g_{\mu\nu}(x^\mu, w = w_1)$ and for the hidden brane $g_{\mu\nu}^{(\mathrm{hid})} = g_{\mu\nu}(x^\mu, w = 0)$. Then the action becomes

$$\begin{aligned} S &= S_{\mathrm{hid}} + S_{\mathrm{vis}} \\ &= \int d^4x \left(\sqrt{-g^{(\mathrm{hid})}} \left(L_{\mathrm{hid}} - V_{\mathrm{hid}} \right) + \sqrt{-g^{(\mathrm{vis})}} \left(L_{\mathrm{vis}} - V_{\mathrm{vis}} \right) \right) \end{aligned} \tag{7.31}$$

where V_{hid} and V_{vis} are the brane tensions which are the localized energy densities on the hidden brane and visible brane respectively. Here since we consider the Ricci flat $\left(R^{(5)} = 0\right)$ five dimensional bulk, this action does not include any contribution from the extra dimension. The setup is similar to the scenario of [16]. The Einstein tensor for this action is

$$\sqrt{-g^{(5)}} G_{MN} = -\frac{1}{4M^3} \left(V_{\mathrm{hid}} \sqrt{-g^{(\mathrm{hid})}} g_{\mu\nu}^{(\mathrm{hid})} \delta_M^\mu \delta_N^\nu \delta(w) + V_{\mathrm{vis}} \sqrt{g^{(\mathrm{vis})}} g_{\mu\nu}^{(\mathrm{vis})} \delta_M^\mu \delta_N^\nu \delta(w - w_1) \right) \tag{7.32}$$

At this stage, we look for the five dimensional Einstein tensor of metric (7.3),

$$G_{MN}^5 = 3\delta_M^\mu \delta_N^\nu \eta_{\mu\nu} \frac{1}{(1 - \frac{|w|}{w_0})} \left(1 - \frac{|w|}{w_0} \right)'', \tag{7.33}$$

applying Z_2 symmetry, this tensor becomes; for the brane at $w = 0$,

$$G_{MN}^5 = -6\delta_M^\mu \delta_N^\nu \eta_{\mu\nu} \frac{1}{(w_0 - |w|)} \delta(w) \quad \text{for} \quad -w_1 < w < w_1 \tag{7.34}$$

then for the brane at $w = w_1$

$$G_{MN}^5 = -6\delta_M^\mu \delta_N^\nu \eta_{\mu\nu} \frac{1}{(w_0 - |w|)} \left[-\delta(w - w_1) \right] \quad \text{for} \quad 0 < w < 2w_1 \tag{7.35}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, Capital Latin letters $M, N = 0, 1, 2, 3, 5$ denote full space-time and lower Greek $\mu, \nu = 0, 1, 2, 3$ run over the brane world. From (7.33), it is easily seen that there are no nongravitational fields ($T_{MN} = 0$) and no cosmological constant in the bulk. For the solution on the brane, we substitute this equation into (7.32), then we get the tensions on the visible and hidden brane as

$$V_{\rm vis} = \frac{-24M^3}{w_0\left(1 - \frac{|w_1|}{w_0}\right)} \qquad \text{and} \qquad V_{\rm hid} = \frac{24M^3}{w_0}. \tag{7.36}$$

These results are similar but not the same as RS model in which $V_{\rm vis}$ and $V_{\rm hid}$ are same with opposite sign. We can derive the induced brane gravity [96], which consists of four dimensional Einstein gravity on the brane. From the dimensionfull constants $\kappa_{5,}$ and κ_4 the Planck masses M, M_P are defined as

$$\kappa_5^3 = 8\pi G_{(5)} = M^{-3}, \qquad \qquad \kappa_4^2 = 8\pi G_{(4)} = M_P^{-2}. \tag{7.37}$$

The induced Einstein tensor on the hidden brane is

$$\frac{M_P^2}{2}\int d^4x\sqrt{g^4}G^4_{\mu\nu} = -\frac{6M^3}{2}\int d^4x\int_{-w_1}^{w_1}\frac{dw}{w_0}\sqrt{g^{(5)}}g_{\mu\nu}\left(1-\frac{|w|}{w_0}\right)^{-1}\delta\left(w\right), \tag{7.38}$$

then for the hidden brane world

$$G^4_{\mu\nu(\rm hid)} = -\frac{3}{w_0^2}\eta_{\mu\nu} \quad \text{and} \quad M_P^2 = \frac{V_{\rm hid}w_0^2}{12}, \tag{7.39}$$

and for the visible world

$$G^4_{\mu\nu(\rm vis)} = -\frac{3}{w_0^2\left(1-\frac{w_1}{w_0}\right)^2}\eta_{\mu\nu} \quad \text{and} \quad M_P^2 = \frac{V_{\rm vis}w_0^2}{12}\left(1-\frac{w_1}{w_0}\right)^2, \tag{7.40}$$

where M_P and M have dimensions (length)$^{-1}$ [96] and w_0 corresponds to the $1/k$ term in [16]. The Einstein tensor on the brane is found as exactly what we want to get. Here we use that, for the four dimensional metric at $w = 0$

$$ds^2 = -U\left(r\right)dt^2 + \frac{1}{U\left(r\right)}dr^2 + r^2 d\Omega_2^2, \tag{7.41}$$

and Einstein tensor is

$$G^4_{\mu\nu(\text{hid})} = -\frac{3}{w_0^2}\eta_{\mu\nu}. \tag{7.42}$$

Then for the metric at $w = w_1$

$$ds^2 = \left(1 - \frac{w_1}{w_0}\right)^2 \left[-U(r)\,dt^2 + \frac{1}{U(r)}dr^2 + r^2 d\Omega_2^2\right], \tag{7.43}$$

the Einstein tensor is

$$G^4_{\mu\nu(\text{vis})} = -\frac{3}{w_0^2\left(1 - \frac{w_1}{w_0}\right)^2}\eta_{\mu\nu}. \tag{7.44}$$

These two metrics are the usual well known Schwarzschild space-time on the visible and hidden brane world. Therefore we get that the cosmological constant of the hidden world is $\Lambda_{4(\text{hid})} = 3/w_0^2$. On the other hand for the visible brane we get the positive cosmological constant $\Lambda_{4(\text{vis})} = 3\left(w_0 - w_1\right)^{-2}$ and the five dimensional cosmological constant Λ_5 is zero everywhere. This may be interpreted as the four dimensional cosmological constant originating from the localized energy momentum tensor on the hidden brane affecting the visible brane through the bulk. In (7.36) the minus sign on the visible brane tension comes from the attractive gravitational force. This minus sign causes us to measure positive cosmological constant on the visible world.

To find the mass parameter on the 3-brane we find the effective action as in [16]. In our model the standard particles are located on the visible brane $w = w_1$ and this brane is linearly small compared to the hidden brane at $w = 0$. To discuss the physical scales, we can consider a scalar field on the visible brane. For example, for the Higgs field

$$S^{\text{Higgs}} = \int d^4x\sqrt{-g^{(\text{vis})}}\left[g^{\mu\nu}_{(\text{vis})}D_\mu H D_\nu H - \lambda\left(H^\dagger H - m_0^2\right)^2\right], \tag{7.45}$$

since the metric on visible brane is $g_{\mu\nu}^{(\mathrm{vis})} = \left(1 - \frac{|w_1|}{w_0}\right)^2 g_{\mu\nu}^{(4)}$, this action becomes

$$S^{\mathrm{Higgs}} = \int d^4x \sqrt{-g^{(4)}} \left(1 - \frac{|w_1|}{w_0}\right)^4 \left[\left(1 - \frac{|w_1|}{w_0}\right)^{-2} g_{(4)}^{\mu\nu} D_\mu H D_\nu H - \lambda \left(H^\dagger H - m_0^2\right)^2 \right]. \tag{7.46}$$

This Higgs field cannot canonically be normalized due to warp factor appeared into the action. To get canonically normalized field we redefine the Higgs field as $\tilde{H} = \left(1 - \frac{|w_1|}{w_0}\right)^{-1} H$, then we get

$$S^{\mathrm{Higgs}} = \int d^4x \sqrt{-g^{(4)}} \left[g_{(4)}^{\mu\nu} D_\mu \tilde{H} D_\nu \tilde{H} - \lambda \left(\tilde{H}^\dagger \tilde{H} - \left(1 - \frac{|w_1|}{w_0}\right)^2 m_0^2 \right)^2 \right]. \tag{7.47}$$

Therefore the action on the $(3+1)$ dimensional visible brane turns back to the same action in (7.45) except from the vacuum expectation values of mass parameter m. This mass parameter is affected from the linearly warped bulk metric, then if we define

$$m = \left(1 - \frac{|w_1|}{w_0}\right) m_0, \tag{7.48}$$

where m_0 is the mass parameter on the visible brane in the higher dimensional theory and m is the physical mass on the visible brane with the metric $g_{(4)}^{\mu\nu}$, the eigenvalue of mass scale on the visible brane becomes different from the hidden brane by the factor of warp factor. Since the mass scale on the hidden brane is unsuppressed and the fundamental mass scale is the Planck scale, this hidden brane can be referred as the Planck brane. Therefore hidden brane is referred as the TeV brane since this range of energy scale is more acceptable and observable in our universe. Here if the part in parenthesis is of order 10^{-15}, near the curvature singularity $w_1 \simeq w_0$ where the visible world is located, we get the TeV mass scale from the Planck mass scale, 10^{16}TeV. Then these results can be explained and summarized as the following two main points:

- Using (7.36), relation of the brane tensions become $V_{\mathrm{vis}} = -10^{15} V_{\mathrm{hid}}$.
- The observed cosmological constant $(10^{-52} m^{-2})$ comes from the effect of the hidden brane on our world. Really what we observe as a cosmological constant

$10^{-52}m^{-2}$ is the $\Lambda_{4(\text{hid})}$. From this consideration we find w_0 from the equation

$$10^{-52}m^{-2} = \frac{3}{w_0^2}, \tag{7.49}$$

then $w_1 \simeq w_0 = 10^{26}m$ is approximately just the Hubble length. Therefore we are located at the distance of nearly Hubble length from the hidden brane. On the other hand, the cosmological constant of the visible world which we cannot observe is $10^{-22}m^{-2}$. The factor 10^{30} in the two cosmological constants comes from the location of our world. We are much closer to singularity w_0 than the hidden brane. In fact, there is no cosmological constant more than one. To prevent this complication, we do not use the name $\Lambda_{(\text{vis})}$ anymore, we call it $G_{00}(w)$. We can summarize these results, by the relation

$$\Lambda_{(\text{hid})} = \Lambda = G_{00}(w)\left(1 - \frac{|w|}{w_0}\right)^2 = 10^{-52}m^{-2}, \tag{7.50}$$

which is satisfied for all w. Placing our brane at $w = w_1$ causes $G_{00}(w_1)$ to be $10^{-22}m^{-2}$. If we had placed our brane at any other w, G_{00} on the brane would change as indicated in this equation but Λ would not change. $G_{00}(w_0)$ becomes infinite on the singularity at $w = w_0$. However $0 \leq w \leq w_1 < w_0$ and the point $w = w_0$ is outside the branes and the bulk.

One of the main motivations for this work has been the argument that, since there is nothing except gravity in the fifth dimension, there should also be no cosmological constant in the fifth dimension. Gravity originates from the hidden brane and then it is localized and interacts with the matter on the brane we live. We cannot see the hidden brane. However we can measure its effect as a cosmological constant Λ. Additionally the distance from the hidden brane determines the localized mass density of the universe. Namely $G_{00}(w_1)$, the mass density in the visible brane depends on Λ and varies with the location of the universe in the bulk. This causes the energy density to be obtained as $\rho \simeq 10^3 kg/m^3$. This value is too large to be interpreted as an average energy density for our universe. However, interestingly enough this value is comparable to stellar energy densities. This indicates that a model where the cosmological constant is localized and creates localized masses may be viable.

7.4. Corrections to Newton's Potential

At first, in order to get a realistic theory we need to look for the localization of gravity on the brane. To do this, taking $(1-|w|/w_0)\, g_{\mu\nu}+h_{\mu\nu}(x,w)$, we study the fluctuations of the metric. Here $h_{\mu\nu}(x,w)$ is the perturbed metric and its transverse traceless components represents the graviton on the brane i.e. the conditions $\partial_\mu h^{\mu\nu}=0$ and $h^\mu_\mu=0$ must be satisfied [15, 16],[107, 112]. By separation of variables we write $h_{\mu\nu}$ in terms of the four-dimensional mass eigenstates $h_{\mu\nu}(x,w)=\phi_{\mu\nu}(t,x^i)\,\psi(m,w)$, where m corresponds to four-dimensional mass. This yields the wave function for gravity as we have discussed in section 3.5.2

$$\partial_w^2\psi+\left(-2\frac{f''(w)}{f(w)}-2\frac{f'(w)^2}{f(w)^2}+m^2 f(w)^{-2}\right)\psi=0, \tag{7.51}$$

where the warp factor $\left(1-\frac{|w|}{w_0}\right)$ is denoted as $f(w)$. Performing a coordinate transformation from w to the variable z defined by $\partial z/\partial w=\pm f(w)^{-1}$ and defining $u(z)$ by $\psi=f(w)^{1/2}u(z)$, we get the wave function as the solution of

$$\left(-\partial_z^2+V(z)\right)u(z)=m^2u(z), \tag{7.52}$$

where the potential $V(z)$ is

$$V(z)=\frac{3}{4}\frac{f'^2}{f^2}+\frac{3}{2}\frac{f''}{f}. \tag{7.53}$$

Here we denote the derivative with respect to z by prime and f is the z dependent warp factor as addressed below. Since our brane is located at $w=w_1$, we can write the warp factor as $f(w)=1-\frac{w_1}{w_0}+\frac{|w_1-w|}{w_0}$, then z dependence of the warp factor will be

$$f(z)=\left(1-\frac{w_1}{w_0}\right)^{-1}\exp\left(-|z|/w_0\right), \tag{7.54}$$

the transformed coordinate z is given by

$$z = \text{sign}\,(w_1 - w)\ln\left(\frac{1-\frac{w_1}{w_0}}{1-\frac{w_1}{w_0}+\frac{|w_1-w|}{w_0}}\right), \tag{7.55}$$

where our brane is placed at $z = 0$. Then we find the potential as

$$V\,(z) = \frac{9}{4w_0^2} - \frac{3}{w_0}\delta\,(z)\,. \tag{7.56}$$

Here the δ function in the potential allows a normalizable bound state mode and this shows that the 5D graviton is localized on the brane. In general for a potential $V(z)$ which allows bound states, (7.52) has $m = 0$ normalizable solutions iff $V\,(z) > 0$, as $z \to \infty$. In this case localization of gravity on the brane occurs. Here $m = 0$ mode corresponds to the gravitons on the brane.

The next point is to solve the Schrodinger equation (7.52). First of all, for the zero mode wave function, we find that $u_0\,(z) = (3/2w_0)^{1/2}\exp\left(-3w_0\,|z|\,/2\right)$. On the other hand, the massive modes are $u_m\,(z) = A\exp\left(-\sqrt{\frac{9}{4w_0^2} - m^2}\,|z|\right)$, where A is the normalization factor. As explained in [110], the massive modes for $0 < m^2 < 9/4w_0^2$ do not exist there and for $m^2 \geq 9/4w_0^2$ this wave function becomes plane wave. Then there is a mass gap $9/4w_0^2$ between the zero mode and the continuous modes. Normalizing the wave function for $m^2 \geq 9/4w_0^2$, we get the normalization factor A,

$$1 = \int_{\ln\left(1-\frac{w_1}{w_0}\right)}^{0} A^2 dz, \tag{7.57}$$

$$A^2 = \left[\ln\left(1-\frac{w_1}{w_0}\right)^{-1}\right]^{-1}. \tag{7.58}$$

At the end, the correction to Newton's Law is obtained as

$$U\,(r) = -G_N\frac{m_1 m_2}{r} - M^{-3}\int_{m_0}^{\infty} dm\frac{m_1 m_2 e^{-mr}}{r}\left(u_m\,(0)\right)^2$$

$$
\begin{aligned}
&= -G_N \frac{m_1 m_2}{r} - M^{-3} \int_{m_0}^{\infty} dm \frac{m_1 m_2 e^{-mr}}{r} A^2 \\
&= -G_N \frac{m_1 m_2}{r} - 2w_0 \left(1 - \frac{w_1}{w_0}\right) M_P^{-2} \frac{m_1 m_2}{r^2} e^{-m_0 r} A^2 \\
&= -G_N \frac{m_1 m_2}{r} (1 + C \frac{e^{-m_0 r}}{r}), \qquad (7.59)
\end{aligned}
$$

where as we have explained above $m_0 = 3/2w_0$, and then the constant C becomes

$$
C = 16\pi w_0 \left(1 - \frac{w_1}{w_0}\right) \left(-\ln\left(1 - \frac{w_1}{w_0}\right)\right)^{-1} \simeq 10^{11} m. \qquad (7.60)
$$

Here we used (7.36) and (7.40) and obtained that $M^{-3} = 2w_0 \left(1 - \frac{w_1}{w_0}\right) M_P^{-2}$. This value of C is severely in violation of experimental measurements.

In conclusion, in this part, we have presented a new brane world black hole solution such that the Schwarzschild-dS_4 space-time is embedded into $4+1$ dimensional bulk with the linear warp factor $(1 - |w|/w_0)$. Here the visible brane is located at $w = w_1 < w_0$. We have presented the coordinate transformation between the Schwarzschild-dS_4 and FRW space-time for $M = 0$. The same transformation can also be used for $M \neq 0$. Although such a transformation will result in a complicated metric, for $M = 0$ this metric will reduce to FRW. For $M \neq 0$, we found that in the bulk not only the energy momentum tensor but also the cosmological constant is zero. However the effect of the hidden brane to our world is just the cosmological constant which we now measure. The localized energy momentum tensor in the bulk also serves to cause a four dimensional cosmological constant on the brane. It was found that this cosmological constant depends on the distance scale w_0. Finally we have derived the range of fifth dimension from the hierarchy between the weak and Planck scales. This model is in violation of observed values concerning corrections to the Newtonian corrections and the matter-average energy density of the universe.

8. CONCLUSIONS

Cosmology has been the basis of our thesis. In this context we have studied on some topics in higher dimensional space-time. Therefore in chapter 2 we have presented a brief introduction to cosmology in which the homogeneous, isotropic and expanding behavior of space-time has been represented by the FRW metric. Using this model of our universe, we derived the cosmological equations and then gave some values of observed physical quantities.

The other main content of this thesis was the theory of extra dimensional space-time. Hence in chapter 3, we mentioned the earlier models which have some similarities with our works. We reviewed the Arkani-Hamed-Dimopoulos-Dvali and the Randall-Sundrum models with extra spacelike dimensions, recently proposed as a solution to the hierarchy problem.

We also introduced the Schwarzschild solution in chapter 4. We have formulated the spherically symmetric static vacuum solution in four dimensional space-time. We expressed the relation between the Schwarzschild black holes and gravity. We emphasized that the true nature of gravity is not well understood at larger and smaller scales, and the hierarchy problem originates from the characteristic scale of gravity. Then we emphasized that considering higher dimensional black holes may solve these problems.

In chapter 5, we studied gravity in Newtonian theory, in special relativity and in general relativity separately. In these theories the Newtonian gravitational constant is a universal constant. We also discussed derivation of gravitational field equations from the Einstein-Hilbert action. In addition we mentioned the scalar tensor theory in which the Newton's gravitational coupling is related by a scalar field therefore the gravitational constant is not considered as a constant anymore. We then introduced the BD theory which is an alternative theory to Einstein's general relativity. In the final part of this chapter, we expressed the BD theory in higher dimensional space-time.

In chapter 6, we have introduced a five dimensional BD action with a scalar potential. We have chosen our metric (6.8) to be a $3+1$dimensional space-time embedded in $4+1$ dimensional bulk with a warp factor $f(W)$. Then we obtained BD equations in five dimensional space-time. These equations contain Dirac delta functions which correspond to localized energy densities on the $3+1$ dimensional space-time. Using jump condition we have derived the equation for matter on the brane and then we got a cosmological constant dominated era for a certain condition. We then solve the BD equations for two different assumptions of scale factor $a(t)$. In the first case we assumed that the universe expands exponentially and for the second one it grows by a power law. For each type of evolution of the universe, we got the same linear warp factor and scalar field to be a functional of warp factor $f(W)$ and scale factor $a(t)$. Firstly, we have discussed the case of exponentially growing scale factor in BD equations. Nevertheless these equations were not enough to determine the matter content of the universe. Furthermore, BD equations have been considered separately either for $T_{55}=0$ or $T_{55}\neq 0$. For the first case, full BD equations satisfied only if there is an empty bulk. To that end, we have derived two formulas for the scalar potential; in the first solution, the scalar field depends on time but scalar potential vanishes. In the second solution the scalar potential survives but the scalar field becomes time independent. Then to discuss observed values of the BD parameter ω, since recent data shows $\omega > 3000$, we considered the limit of $\omega \to \infty$. At this limit, the scalar potential vanishes and scalar field becomes a constant which is consistent with general relativity. In this part, it is important to note that four dimensional curved (de Sitter) space-time can be embedded into the five dimensional flat space-time. Then we have derived the explicit coordinate transformations between the five dimensional Minkowski universe and our resultant metric (6.56). For the case $T_{55}\neq 0$, we have obtained a cosmological constant dominated universe and both scalar field and scalar potential becomes time independent at the limit of $\omega \to \infty$. We also discuss the power law growth of the scale factor in BD equations. These equations have been satisfied for two different equations of state. First of all, we have got a radiation dominated universe in which the scale factor becomes $\sim t^{1/4}$ at the large BD parameter limit. Moreover, scalar potential vanishes and scalar field approaches to a constant. On the other hand, in the other solution BD equations give empty universe and scale factor evolve as $\sim t$.

However these two types of expansion of universe are not supported by experimental data since observations show that the expansion of the universe accelerates. Finally, we have worked out the effective four dimensional gravitational constant. Then we conclude that, for exponentially increasing scale factor, the scalar field and therefore the effective gravitational constant becomes a constant. On the other hand, for the power law scale factor, effective gravitational constant becomes time dependent.

In chapter 7, we have studied a non-compact Z_2-symmetric empty five dimensional bulk and our universe has been considered to be a SdS space-time embedded into this $4+1$ dimensional space-time with a linear warp factor. Here the condition for the empty bulk can be satisfied only if we choose this warp factor to be a linear function which depends only on the extra coordinate [83]. Then the appropriate metric (7.3) has been introduced in section 7.1. We tried to discuss the case for $M \neq 0$ in the remaining part of this work. We found the Einstein tensor in $4+1$ dimensional bulk which contains Dirac delta function and no other terms contribute as we expect. This Dirac delta function describes matter localized on $3+1$ dimensional brane world. We considered two branes, one is located at $w = 0$ and the other which is our brane at $w = w_1$. Then we have found that all the physical quantities on the brane at w_1 are suppressed with some power of warp factor. To obtain the four dimensional physical mass we consider the Higgs action in higher dimensional theory on the visible brane. This physical mass is again suppressed with the warp factor. In order to solve the hierarchy problem, we assumed that this physical mass is of the order of experimentally observable energy scale TeV and the higher dimensional mass is of the order of Planck mass scale 10^{16}TeV. Then this consideration requires a huge fine-tuning condition which is in the order of 10^{-15}. For such a hierarchy, place of visible brane located at near the physical singularity w_0. Formulating the Einstein tensor on each brane, we then claimed that the hidden brane is a source of measured observable cosmological constant for the visible brane. More generally, the value of cosmological constant at $w = 0$ is also the same on every brane placed throughout the fifth dimension in the bulk. Hence the location of a brane only causes the mass density to vary on this brane. We therefore derived the values of $w_0 \simeq w_1$ as 10^{26}m and mass density $G_{00}\left(w_1\right)$ on our visible brane as 10^{-22}m^{-2}. This value of mass density corresponds to energy density

$\rho = 10^3\text{kg/m}^3$. However this value is too large compared to average energy density on our universe but this range of energy is comparable to localized stellar energy density on our $3+1$ dimensional space-time. We then expressed that a common cosmological constant survives on every brane placed in the bulk and it creates localized mass density at corresponding value of w.

Following this work, in order to get the correction to Newton's potential, we have studied the evolutions of perturbations on our metric. Then we have obtained Schrodinger wave equation and potential term which includes Dirac delta function. This equation for potential shows that gravity is localized on the brane. We then derived the zero mode wave function and we got the modified gravitational potential. We saw that this model is in violation with the observed values of corrections to the Newtonian potential hence experimental confirmation is lacking.

In this thesis we considered two aspects of our $3+1$ dimensional universe embedded in $4+1$ dimensional flat space-time. The first model involved $4+1$ dimensional JBD cosmology. We showed that for exponential expansion of our universe the $4+1$ dimensional flat space-time naturally arises as a solution of the field equations. The second model involved embedding the $3+1$ dimensional Schwarzschild black hole in $4+1$ dimensional space-time such that in the limit $m = 0$ the $4+1$ dimensional space-time becomes flat. The results we have obtained for these two models naturally suggest further investigation of $4+1$ dimensional flat models into which $3+1$ dimensional curved universe is embedded as a brane. Another useful area of research could be to investigate embedding our curved universe into higher dimensional flat space-times. Our second model involved a variation of the Randall-Sundrum scenario such that the $4+1$ dimensional bulk is flat. The main disadvantage of this model is that the visible brane and hidden brane are apart a distance of the order 10^{26}m. Modification of this model such that this distance can be reduced to millimeter size is desirable. This also is a direction for further research.

APPENDIX A: NOTATIONS AND CONVENTIONS INVOLVING THE METRIC AND THE CURVATURE

- M, N label coordinates of the $4+1$ dimensional space-time, μ, ν specifies $3+1$ dimensional space-time and i, j denotes 3 dimensional space only.
- The metric signature is $-, +, ...+$, and the $4+1$ dimensional metric is

$$\eta_{MN} = \text{diag}(-1, +1, +1, +1, +1). \tag{A.1}$$

- The metric of the $4+1$ dimensional space-time is g_{MN} and the induced metric on the $3+1$ space-time is $g_{\mu\nu}$.
- We work in units $\hbar = c = 1$, such that there is only one dimension

$$[\text{energy}] = [\text{mass}] = [\text{ length}]^{-1} = [\text{time}]^{-1}.$$

Ordinary derivative of a scalar:

$$\partial_\alpha S = S_{,\alpha} \tag{A.2}$$

Covariant derivative:

$$\nabla_\alpha V^\nu = V^\nu{}_{;\alpha} \tag{A.3}$$

D'alambertion operator:

$$\Box = \nabla^\mu \nabla_\mu = \partial^\mu \partial_\mu = g^{\mu\nu} \partial_\mu \partial_\nu \tag{A.4}$$

Christoffel symbol:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma} \right) \tag{A.5}$$

Riemann tensor:

$$R^{\alpha}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\lambda} - \partial_\nu \Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} \tag{A.6}$$

$$R^{\alpha}_{\lambda\mu\nu} = g^{\alpha\beta} R_{\beta\lambda\mu\nu} \tag{A.7}$$

Symmetry properties of the Riemann tensor:

$$R_{\beta\lambda\mu\nu} = -R_{\lambda\beta\mu\nu} \tag{A.8}$$

$$R_{\beta\lambda\mu\nu} = -R_{\beta\lambda\nu\mu} \tag{A.9}$$

$$R_{\beta\lambda\mu\nu} = R_{\mu\nu\beta\lambda} \tag{A.10}$$

$$R_{\beta[\lambda\mu\nu]} = R_{\beta\lambda\mu\nu} + R_{\beta\nu\lambda\mu} + R_{\beta\mu\nu\lambda} = 0 \tag{A.11}$$

$$R_{[\beta\lambda\mu\nu]} = 0 \tag{A.12}$$

Ricci tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{A.13}$$

Symmetry property of the Ricci tensor:

$$R_{\mu\nu} = R_{\nu\mu} \tag{A.14}$$

Ricci (curvature) scalar:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu} \tag{A.15}$$

Bianchi identity:

$$\nabla_{[\alpha}R_{\beta\lambda]\mu\nu} = 0 \tag{A.16}$$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{A.17}$$

$$\nabla^{\mu}G_{\mu\nu} = 0 \tag{A.18}$$

APPENDIX B: PROOF OF SOME USEFUL IDENTITIES

Covariant derivative of a metric,

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{B.1}$$

Inverse of a metric:

$$g^{\mu\nu} g_{\mu\lambda} = \delta^\nu_\lambda \tag{B.2}$$

using this definition

$$\begin{aligned} g^{\mu\nu}{}_{,\alpha} g_{\mu\lambda} + g^{\mu\nu} g_{\mu\lambda,\alpha} &= 0 \\ g^{\mu\nu}{}_{,\alpha} g_{\mu\lambda} &= -g^{\mu\nu} g_{\mu\lambda,\alpha} \end{aligned} \tag{B.3}$$

For any square matrix A,

$$\det A = e^{\mathrm{Tr}\{\ln A\}} \tag{B.4}$$

thus we have

$$\begin{aligned} (\det A)_{,\alpha} &= e^{\mathrm{Tr}\{\ln A\}} \left(\mathrm{Tr}\{\ln A\}\right)_\alpha \\ &= \det A \mathrm{Tr}\left(A^{-1} A_{,\alpha}\right) \end{aligned} \tag{B.5}$$

Application to the metric $g_{\mu\nu}$,

$$g = \det g_{\mu\nu} \tag{B.6}$$

$$\frac{g_{,a}}{g} = \frac{(g_{\mu\nu})_{,\alpha}}{(g_{\mu\nu})} \tag{B.7}$$

$$\Gamma^{\lambda}_{\lambda\nu} = \frac{1}{2} g^{\sigma\lambda} \left(g_{\nu\sigma,\lambda} + g_{\sigma\lambda,\nu} - g_{\lambda\nu,\sigma} \right) \tag{B.8}$$

If the metric is diagonal,

$$\Gamma^{\lambda}_{\lambda\nu} = \frac{1}{2} g^{\lambda\lambda} g_{\lambda\lambda,\nu} \tag{B.9}$$

from the diagonality of a metric $g^{\mu\mu} = (g_{\mu\mu})^{-1}$

$$\begin{aligned} \Gamma^{\lambda}_{\lambda\nu} &= \frac{1}{2} (g_{\lambda\lambda})^{-1} g_{\lambda\lambda,\nu} \\ &= \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\right)_{,\nu} \end{aligned} \tag{B.10}$$

Variation of Christoffel symbol:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma} \right). \tag{B.11}$$

$$\begin{aligned} \delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} \delta g^{\sigma\lambda} \left(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma} \right) + \frac{1}{2} g^{\sigma\lambda} \left(\delta g_{\nu\sigma,\mu} + \delta g_{\sigma\mu,\nu} - \delta g_{\mu\nu,\sigma} \right) \\ &= \Gamma^{\beta}_{\mu\nu} g_{\beta\sigma} \delta g^{\sigma\lambda} + \frac{1}{2} g^{\sigma\lambda} \left(\delta g_{\nu\sigma,\mu} + \delta g_{\sigma\mu,\nu} - \delta g_{\mu\nu,\sigma} \right) \\ &= \frac{1}{2} g^{\lambda\sigma} \left(\delta g_{\nu\sigma;\mu} + \delta g_{\sigma\mu;\nu} - \delta g_{\mu\nu;\sigma} \right) \end{aligned} \tag{B.12}$$

Covariant derivative of a vector field V^{ν} :

$$\nabla_{\alpha} V^{\nu} = V^{\nu}{}_{;\alpha} = V^{\nu}{}_{,\alpha} + \Gamma^{\nu}_{\alpha\lambda} V^{\lambda} \tag{B.13}$$

$$\begin{aligned} \nabla_{\alpha} V^{\alpha} &= V^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\alpha\lambda} V^{\lambda} \\ &= V^{\alpha}{}_{,\alpha} + \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\right)_{,\lambda} V^{\lambda} \\ &= \frac{1}{\sqrt{-g}} \left(\sqrt{-g} V^{\alpha}\right)_{,\alpha} \end{aligned} \tag{B.14}$$

For a scalar field S:

$$\Box S = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} S^{,\alpha}\right)_{,a} \tag{B.15}$$

APPENDIX C: DERIVATION OF BRANS-DICKE EQUATIONS

1. Variation of BD action with respect to $g^{\mu\nu}$:

$$S = \int d^4x\sqrt{-g}\left(\frac{\phi^2}{8\omega}R - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V\left(\phi\right)\right) + S_{\text{matter}}, \tag{C.1}$$

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4, \tag{C.2}$$

where

$$\delta S_1 = \int d^4x\delta\left(\sqrt{-g}\frac{\phi^2}{8\omega}R\right) \tag{C.3}$$

$$\delta S_2 = \int d^4x\delta\left(\sqrt{-g}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\right) \tag{C.4}$$

$$\delta S_3 = \int d^4x\delta\left(\sqrt{-g}V\left(\phi\right)\right) \tag{C.5}$$

$$\delta S_4 = \delta S_{\text{matter}}. \tag{C.6}$$

- Performing the first variation, δS_1 :

$$\delta S_1 = \int d^4x\delta\left(\sqrt{-g}\right)\frac{\phi^2}{8\omega}R + \int d^4x\sqrt{-g}\frac{\phi^2}{8\omega}\delta R. \tag{C.7}$$

For the first part

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g \tag{C.8}$$

where g is the determinant of metric $g_{\mu\nu}$, $g = \det g_{\mu\nu}$. Applying to our calculation; the matrix $A = g_{\mu\nu}$ and $\det A = g$. Therefore $\delta g = gg^{\mu\nu}\delta g_{\mu\nu}$. The desired result follows immediately that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{C.9}$$

Using the definition of the metric $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$

$$\delta g^{\mu\lambda}g_{\lambda\nu} + g^{\mu\lambda}\delta g_{\lambda\nu} = 0, \tag{C.10}$$

(C.9) becomes

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{C.11}$$

For the second part, since $R = g^{\mu\nu}R_{\mu\nu}$

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{C.12}$$

The first term in the form of $\delta g^{\mu\nu}$ multiplied by an expression, there is not need for any effort on this term. For the second term, since

$$R_{\mu\nu} = \partial_\lambda\Gamma^{\lambda}_{\nu\mu} - \partial_\nu\Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} \tag{C.13}$$

derivation of this expression is

$$\begin{aligned} \delta R_{\mu\nu} &= \delta\left(\partial_\lambda\Gamma^{\lambda}_{\nu\mu}\right) - \delta\partial_\nu\left(\Gamma^{\lambda}_{\lambda\mu}\right) + \delta\Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \\ & \quad \Gamma^{\lambda}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \delta\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} \end{aligned} \tag{C.14}$$

Using the definition of covariant derivative,

$$\nabla_\lambda\left(\delta\Gamma^{\rho}_{\mu\nu}\right) = \partial_\lambda\left(\delta\Gamma^{\rho}_{\mu\nu}\right) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\mu\sigma} \tag{C.15}$$

Equation (C.14) becomes

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^{\lambda}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right) \tag{C.16}$$

Using the property of metric compatibility for any vector field

$$g_{\mu\nu}\nabla_\rho V^\nu = \nabla_\rho\left(g_{\mu\nu}V^\nu\right) = \nabla_\rho V_\mu \qquad \text{(C.17)}$$

we obtain that,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) \qquad \text{(C.18)}$$

Performing again the covariant derivative and multiplying with the root of the determinant of the metric

$$\begin{aligned}
\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \Gamma^\lambda_{\lambda\beta}g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} \\
&\quad -\partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) - \Gamma^\nu_{\nu\beta}g^{\mu\beta}\delta\Gamma^\lambda_{\lambda\mu}] \\
&= \sqrt{-g}\Big[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \left(\frac{1}{2}g^{\nu\lambda}g_{\nu\lambda,\beta}\right)g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} \\
&\quad -\partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) - \left(\frac{1}{2}g^{\nu\sigma}g_{\nu\sigma,\beta}\right)g^{\mu\beta}\delta\Gamma^\lambda_{\lambda\mu}\Big] \\
&= \sqrt{-g}\Big[\partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) + \frac{\left(\sqrt{-g}\right)_{,\beta}}{\sqrt{-g}}g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} \\
&\quad -\partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) - \frac{\left(\sqrt{-g}\right)_{,\beta}}{\sqrt{-g}}g^{\mu\beta}\delta\Gamma^\lambda_{\lambda\mu}\Big] \\
&= \partial_\lambda\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\right) - \partial_\nu\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\right) \qquad \text{(C.19)}
\end{aligned}$$

changing dummy indices

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\gamma\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\gamma_{\nu\mu} - \sqrt{-g}g^{\mu\gamma}\delta\Gamma^\lambda_{\lambda\mu}\right), \qquad \text{(C.20)}$$

$$\begin{aligned}
\phi^2\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \phi^2\partial_\gamma\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\gamma_{\nu\mu} - \sqrt{-g}g^{\mu\gamma}\delta\Gamma^\lambda_{\lambda\mu}\right) \\
&= \partial_\gamma\left[\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\gamma_{\nu\mu} - \sqrt{-g}g^{\mu\gamma}\delta\Gamma^\lambda_{\lambda\mu}\right)\phi^2\right] \\
&\quad - \left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\gamma_{\nu\mu} - \sqrt{-g}g^{\mu\gamma}\delta\Gamma^\lambda_{\lambda\mu}\right)\phi^2_{,\gamma} \qquad \text{(C.21)}
\end{aligned}$$

the term in the first line is a perfect divergence, so it will vanish at the boundary. Using the expression for variation of the Γ's

$$\delta\Gamma^{\gamma}_{\nu\mu} = \frac{1}{2}g^{\gamma\beta}(\delta g_{\nu\beta;\mu} + \delta g_{\beta\mu;\nu} - \delta g_{\mu\nu;\beta}) \tag{C.22}$$

we get

$$\begin{aligned}\phi^2\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= -\frac{\phi^2_{,\gamma}}{2}(\sqrt{-g}g^{\mu\nu}g^{\gamma\beta}(\delta g_{\nu\beta;\mu} + \delta g_{\beta\mu;\nu} - \delta g_{\mu\nu;\beta}) \\ &\quad -\sqrt{-g}g^{\mu\gamma}g^{\lambda\sigma}(\delta g_{\lambda\sigma;\mu} + \delta g_{\sigma\mu;\lambda} - \delta g_{\mu\lambda;\sigma}))\end{aligned} \tag{C.23}$$

changing dummy indices,

$$\begin{aligned}\phi^2\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= -\frac{\phi^2_{,\gamma}}{2}(\sqrt{-g}g^{\mu\nu}g^{\gamma\beta}(\delta g_{\nu\beta;\mu} + \delta g_{\beta\mu;\nu} - \delta g_{\mu\nu;\beta}) \\ &\quad -\sqrt{-g}g^{\beta\gamma}g^{\mu\nu}\left(\delta g_{\mu\nu;\beta} + \delta g_{\nu\beta;\mu} - \delta g_{\beta\mu;\nu}\right)) \\ &= -\phi^2_{,\gamma}\sqrt{-g}g^{\mu\nu}g^{\gamma\beta}\left(\delta g_{\nu\beta;\mu} - \delta g_{\mu\nu;\beta}\right)\end{aligned} \tag{C.24}$$

Then the expressions (B.1) and (B.2) yields

$$\begin{aligned}\phi^2\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= -\phi^2_{,\gamma}\sqrt{-g}g^{\gamma\beta}\left(g_{\mu\nu}\delta g^{\mu\nu}{}_{;\beta} - g_{\beta\nu}\delta g^{\beta\nu}{}_{;\beta}\right) \\ &= -\sqrt{-g}\left(\phi^2\right)^{,\beta}\left(\left(g_{\mu\nu}\delta g^{\mu\nu}\right)_{;\beta} - \left(g_{\beta\nu}\delta g^{\beta\nu}\right)_{;\beta}\right) \\ &= -\sqrt{-g}\Big[\left(\left(\phi^2\right)^{,\beta}g_{\mu\nu}\delta g^{\mu\nu}\right)_{;\beta} - \left(\phi^2\right)^{,\beta}{}_{;\beta}\,g_{\mu\nu}\delta g^{\mu\nu} \\ &\quad -\left(\left(\phi^2\right)^{,\beta}g_{\beta\nu}\delta g^{\beta\nu}\right)_{;\beta} + \left(\phi^2\right)^{,\beta}{}_{;\beta}\,g_{\beta\nu}\delta g^{\beta\nu}\Big]\end{aligned} \tag{C.25}$$

performing the same steps in (C.19) for covariant derivative, the first and the third terms vanish. The remaining parts become

$$\begin{aligned}\phi^2\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}\left[\left(\phi^2\right)^{,\beta}{}_{;\beta}\,g_{\mu\nu}\delta g^{\mu\nu} - \left(\phi^2\right)^{,\beta}{}_{;\beta}\,g_{\beta\nu}\delta g^{\beta\nu}\right] \\ &= \sqrt{-g}\delta g^{\mu\nu}\left[g_{\mu\nu}\Box\phi^2 - \phi^2_{,\nu;\mu}\right]\end{aligned} \tag{C.26}$$

Combining all of the results in the first and second parts we get for δS_1:

$$\begin{aligned}\frac{\delta S_1}{\delta g^{\mu\nu}} &= \int dx^4\sqrt{-g}\left(\phi^2\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+g_{\mu\nu}\Box\phi^2-\phi^2_{,\nu;\mu}\right)\\ &= \int dx^4\sqrt{-g}\left(\phi^2 G_{\mu\nu}+g_{\mu\nu}\Box\phi^2-\phi^2_{,\nu;\mu}\right)\end{aligned} \tag{C.27}$$

- Performing the second variation, δS_2 :

$$\begin{aligned}\delta S_2 &= \int d^4x\delta\left(\sqrt{-g}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\right)\\ &= \int d^4x\left[\delta\left(\sqrt{-g}\right)\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}+\sqrt{-g}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}\right]\\ &= \int d^4x\Big[\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\\ &\quad+\sqrt{-g}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}\Big],\end{aligned} \tag{C.28}$$

where we use the Equation (C.11) for the first part and there is no need any calculation for the second term, therefore

$$\frac{\delta S_2}{\delta g^{\mu\nu}} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right]. \tag{C.29}$$

- Performing the third variation, δS_3 :

$$\begin{aligned}\delta S_3 &= \int d^4x\delta\left(\sqrt{-g}V(\phi)\right)\\ &= \int d^4x\delta\left(\sqrt{-g}\right)V(\phi)\\ &= \int d^4x\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)V(\phi),\end{aligned} \tag{C.30}$$

therefore

$$\frac{\delta S_3}{\delta g^{\mu\nu}} = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g_{\mu\nu}V(\phi)\right). \tag{C.31}$$

- Performing the fourth variation, δS_4 :

$$\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}} = -\int d^4x\sqrt{-g}T_{\mu\nu}. \tag{C.32}$$

Using the action principle

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{C.33}$$

we obtain the BD field equations by adding equations (C.27), (C.29), (C.31) and (C.32),

$$\frac{1}{8\omega}\left(\phi^2 G_{\mu\nu} + g_{\mu\nu}\Box\phi^2 - \phi^2_{,\nu;\mu}\right) + \frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) = T_{\mu\nu} \tag{C.34}$$

2. Variation of BD action (C.1) with respect to ϕ :

$$\begin{aligned}
\delta S &= \int d^4x\sqrt{-g}\Big[\frac{2\phi}{8\omega}R\delta\phi - \frac{1}{2}\delta\left(\partial_\mu\phi\right)\partial_\nu\phi g^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\delta\left(\partial_\nu\phi\right)g^{\mu\nu} \\
&\quad -\frac{\partial V(\phi)}{\partial\phi}\delta\phi\Big] + \frac{\partial S_{\text{matter}}}{\partial\phi}\delta\phi \\
&= \int d^4x\sqrt{-g}\Big[\frac{\phi}{4\omega}R\delta\phi - \frac{1}{2}\partial_\mu\left(\delta\phi\right)\partial_\nu\phi g^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\left(\delta\phi\right)g^{\mu\nu} \\
&\quad -\frac{\partial V(\phi)}{\partial\phi}\delta\phi\Big] + 0 \\
&= \int d^4x\Big[\sqrt{-g}\frac{\phi}{4\omega}R\delta\phi - \partial_\mu\left(\frac{\sqrt{-g}}{2}\delta\phi\partial_\nu\phi g^{\mu\nu}\right) + \frac{1}{2}\partial_\mu\left(\sqrt{-g}\partial_\nu\phi g^{\mu\nu}\right)\delta\phi \\
&\quad -\partial_\nu\left(\frac{\sqrt{-g}}{2}\delta\phi\partial_\mu\phi g^{\mu\nu}\right) + \frac{1}{2}\partial_\nu\left(\sqrt{-g}\partial_\mu\phi g^{\mu\nu}\right)\delta\phi - \frac{\partial V(\phi)}{\partial\phi}\delta\phi\Big],
\end{aligned} \tag{C.35}$$

using Stokes theorem, second and fourth term do not give any contribution to the integral, therefore

$$\begin{aligned}
\frac{\delta S}{\delta\phi} &= \int d^4x\sqrt{-g}\Big[\frac{\phi}{4\omega}R + \frac{1}{2\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{1}{2\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\partial^\nu\phi\right) \\
&\quad -\frac{\partial V(\phi)}{\partial\phi}\delta\phi\Big] = 0.
\end{aligned} \tag{C.36}$$

Then

$$\frac{\phi}{4\omega}R + \Box\phi - \frac{\partial V\left(\phi\right)}{\partial\phi} = 0. \tag{C.37}$$

REFERENCES

1. Penzias, A. A. and R.W. Wilson, "A Measurement of Excess Antenna Temperature at 4080 Mc/s", *Astrophysical Journal,* Vol. 142, pp. 419-421, 1965.

2. Hubble, E., "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae", *Proc. Nat. Acad. Sci.*, Vol. 15, pp. 168-73, 1929.

3. Carroll S.M., *An Introduction to General Relativity, Spacetime and Geometry,* Addison-Welsey, San Francisco, 2004.

4. d'Inverdo R., *Introducing Einstein's General Relativity,* Oxford University Press, Oxford, 1992.

5. Nordström G, "Uber die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen (On the possibility of a unification of the electromagnetic and gravitational fields)", *Physik. Zeitschr.,* Vol. 15, pp. 504-506, 1914.

6. Kaluza, T., "On the Problem of Unity in Physics", *Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys.,* pp.966-972, 1921.

7. O. Klein, "Quantum Theory and Five Dimensional Theory of Relativity", *Z. Phys.* Vol. 37, pp. 895-906, 1926.

8. Duff M.J., "Kaluza-Klein Theory in Perspective", hepth/ 9410046,

9. Ramond P., "Dual Theory for Free Fermions", *Phys. Rev. D,* Vol. 3, pp. 2415-2418, 1971; Neveu A. and J.H. Schwarz, "Factorizable Dual Model of Pions", *Nucl. Phys. B,* Vol. 31, pp. 86-112, 1971.

10. Green M.B. and J.H. Schwarz, "Covariant Description of Superstrings", *Phys. Lett. B,* Vol. 136, pp. 367-370, 1984.

11. Polchinski J., "Dirichlet-Branes and Ramond-Ramond Charges", *Phys. Rev. Lett.,* Vol. 75, pp. 4724-4727, 1995.

12. P. Horáva and E. Witten, "Heterotic and Type I string Dynamics from Eleven Dimensions" *Nucl. Phys. B,* Vol. 460, pp. 506-524, 1996.

13. Arkani-Hamed N., S. Dimopoulos and G. Dvali, "The Hierarchy Problem and New Dimensions at a Millimeters", *Phys. Lett. B,* Vol. 429, pp. 263-272, 1998; "Phenomenology, Astrophysics and Cosmology of Theories with Submillimeter Dimensions and TeV Scale Quantum Gravity." *Phys. Rev. D* 59, 086004-1-21, 1999.

14. Antoniadis I., N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New Dimensions at a Millimeter to a Fermi and Superstrings at a TeV", *Phys. Lett. B,* Vol. 436, pp. 257-263, 1998.

15. Randall L. and R. Sundrum, "A Large Mass Hierarchy from a Small Extra Dimension", *Phys. Rev. Lett.* Vol. 83, pp. 3370-3373, 1999.

16. Randall L. and R. Sundrum, " An Alternative to Compactification". *Phys. Rev. Lett.* Vol. 83, pp. 4690-4693, 1999.

17. Erlich J., *History of Extra Dimensions,* http://physics.wm.edu/∼erlich/690S05/notes1.pdf, 2005.

18. Wald R.M., *General Relativity*, The University of Chicago Press, Chicago and London, 1984.

19. Kerr, R. P., "Gravitational field of a spinning mass as an example of algebraically special metrics". *Phys. Rev. Lett.*, Vol. 11, pp. 237-238, 1963.

20. Giddings S.B., "Black Holes in the Lab?", *Gen. Rel. Grav.*, Vol. 34, pp. 1775–1779, 2002.

21. Giddings S.B. and S. Thomas, “High Energy Colliders as Black Hole Factories: The End of Short Distance Physics ”, *Phys. Rev. D,* Vol. 65, 056010-1-12, 2002.

22. Giddings S.B. and E. Katz, “Effective Theories and Black Hole Production in Warped Compactifications”, *J.Math.Phys.,* Vol. 42, pp. 3082-3102, 2001.

23. Giddings S.B. and V. S. Rychkov, “Black Holes from Colliding Wavepackets”, *Phys. Rev. D,* Vol. 70, 104026-1-4, 2004.

24. Kanti P. and K. Tamvakis, “Quest for Localized 4D Black Holes in Brane Worlds”, *Phys. Rev. D*, Vol. 65, 084010-1-12, 2002.

25. Frolov A.V and V. P. Frolov, “Black Holes in a Compactified Space-time”, *Phys. Rev. D* 67, 124025-1-11, 2003.

26. Dadhich N, R. Maartens, P. Papadopoulos, and V. Rezania, “Black Holes on the Brane”, *Phys. Lett. B*, Vol. 487, pp. 1-6, 2000.

27. Maartens R., “ Brane-world Gravity”, *Living Rev. Relativity,* Vol. 7, pp1-99, 2004.

28. Hossenfelder S., “What Black Holes Can Teach Us”, *In ‘Focus on Black Hole Research’,* Nova Science Publishers, pp155-192, 2005.

29. Banks T. and W. Fishler, *A Model for High Energy Scattering in Quantum Gravity*, http://arxiv.org/abs/hep-th/9906038, 1999.

30. Dimopoulos S. and G. Landsberg, “Black Holes at the Large Hadron Collider”, *Phys. Rev. Lett.,* Vol. 87, 161602-1-4, 2001.

31. Eardley D.M. and S. B. Giddings, “Classical Black Hole Production in High-Energy Collisions”, *Phys. Rev. D,* Vol. 66, 044011-1-7, 2002.

32. Yoshino H. and Y. Nambu, “Black Hole Formation in the Grazing Collision of High-Energy Particles ”, *Phys. Rev. D,* Vol. 67, 024009-1-6, 2003.

33. Kohlprath E. and G. Veneziano, " Black Holes from High-Energy Beam–Beam Collisions", *JHEP,* Vol. 06, 057-1-15, 2002.

34. Cheung K., "Black Hole Production and Large Extra Dimensions", *Phys. Rev. Lett.,* Vol. 88, 221602-1-4, 2002.

35. Giudice G.F., R. Rattazzi and J. D. Wells, "Transplanckian Collisions at the LHC and Beyond", *Nucl. Phys. B,* Vol. 630, No1-2, pp. 293-325, 2002.

36. Chamblin A,. and G. C. Nayak, "Black Hole Production at the CERN LHC: String Balls and Black Holes from pp and Lead-lead Collisions ", *Phys. Rev. D,* Vol. 66, pp. 091901-1-5, 2002.

37. Cavaglia M., S. Das and R. Maartens, "Will We Observe Black Holes at the LHC?", *Class. Quant. Grav.,* Vol. 20, No.15, pp. L205-L213, 2003.

38. Cavaglia M., "Black Hole and Brane Production in TeV Gravity: A Review", *Int. J. Mod. Phys. A,* Vol. 18, pp. 1843-1882, 2003.

39. Feng J.L. and A. D. Shapere, "Black Hole Production by Cosmic Rays", *Phys. Rev. Lett.,* Vol. 88, pp. 021303-1-4, 2002.

40. Anchordoqui L.A., J. L. Feng, H. Goldberg and A. D. Shapere, "Black Holes From Cosmic Rays: Probes of Extra Dimensions and New Limits on TeV-Scale Gravity", *Phys. Rev. D,* Vol. 65, pp. 124027-1-17, 2002.

41. Alvarez-Muniz J., J. L. Feng, F. Halzen, T. Han and D. Hooper, "Detecting Microscopic Black Holes With Neutrino Telescopes", *Phys. Rev. D,* Vol. 65, pp. 124015-11, 2002.

42. Kanti P., " Black Holes in Theories with Large Extra Dimensions: a Review", *Int. J. Mod. Phys. A,* Vol. 19, pp. 4899-4951, 2004.

43. Landsberg G., "Black Holes at Future Colliders and in Cosmic Rays", *Eur. Phys.*

J., Vol. C33, pp. S927-S931, 2004.

44. Myers R.C. and M. J. Perry, "Black Holes In Higher Dimensional Space-Times", *Annals Phys.,* Vol. 172, pp. 304-347, 1986.

45. ArgyresP.C., S. Dimopoulos and J. March-Russell, "Black Holes and Sub-millimeter Dimensions", *Phys. Lett. B,* Vol. 441, pp. 96-104, 1998.

46. Chamblin A, S.W Hawking, and H.S Reall, "Brane-World Black Holes". *Phys. Rev.D,* Vol. 61, 065007-1-6, 2000.

47. Jordan P., "Extension of Projective Relativity", *Ann, der. Phys., Lpz.,* Vol. 1, pp. 219, 1947; *Schwerkraft und Weltall,* Viewerg, Braunschweig, 1955.

48. Thiry Y., "Les équations de la théorie unitaire de Kaluza, Comptes Rendus", *Acad. Sci. (Paris),* Vol. 226, pp. 216, (1948), (Eng. trans. in T. Appelquist, A. Chodos and P. G. O. Freund *Modern Kaluza-Klein theories*, Addison-Wesley, Menlo Park, 1987).

49. Brans C. and R. H. Dicke, "Mach's Principle and a Relativistic Theory of Gravitation ", *Phys. Rev.,* Vol. 124, pp. 925-935, 1961.

50. P. A. M. Dirac,"A New Basis for Cosmology ", *Proc. Roy. Soc. A,* Vol. 165, pp. 199-208, 1938.

51. Brans C.H., *The Roots of Scalar-tensor Theory: An Approximate History,* http://arxiv.org/abs/gr-qc/0506063, 2005.

52. Will C.M., *Theory and Experiment in Gravitational Physics,* Cambridge University Press, Cambridge, 1993, (revised in *Proceedings of the 1998 Slac Summer Institute on Particle Physics* gr-gc/9811036).

53. Romero C. and A. Barros, "Does the Brans-Dicke Theory of Gravity Go Over to General Relativity When $\omega < \infty$?" *Phys. Lett. A,* Vol. 173, pp. 243-246, 1993.

54. Banerjee N. and S. Sen, "Does Brans-Dicke Theory Always Yield General Relativity in the Infinite ω Limit? ", *Phys. Rev. D,* Vol. 56, pp. 1334-1337 1997.

55. Faraoni V., "The Omega Tends to Infinity Limit of Brans-Dicke Theory", *Phys. Lett. A,* Vol. 245, pp. 26-30, 1998.

56. Faraoni V., "Illusions of General Relativity in Brans-Dicke Gravity", *Phys. Rev. D,* Vol. 59, pp. 084021-1-7, 1999.

57. Faraoni V., *Cosmology in Scalar-Tensor Gravity,* Kluwer Academic Publishers, Dordrecht, 2004.

58. Anchordoqui L.A., S.P. Bergliaffa, M.L. Trobo and G.S. Birman, "Cylindirically Symmetric Spinning Brans-Dicke with Closed Timelike Curves", *Mod. Phys. Lett. A,* Vol. 14, No.17, pp. 1105-1111, 1999.

59. Chauvineau B., "On the Limit of Brans–Dicke Theory When $\omega \rightarrow \infty$", *Class. Quantum Grav.*, Vol. 20, pp. 2617–2625, 2003.

60. Perlmutter, S., G. Aldering, M. Della Valle, et al., "Discovery of a Supernova Explosion at Half the Age of the Universe and its Cosmological Implications", *Nature,* Vol. 391, pp. 51-54, 1998.

61. Perlmutter, S., G. Aldering, G. Goldhaber, et al., "Measurements of Omega and Lambda from 42 High-Redshift Supernovae", *Astrophys. J.,* Vol. 517, pp. 565-586, 1999.

62. Garnavich P M et al, "Constraints on Cosmological Models from Hubble Space Telescope Observations of High-z Supernovae", *Astrophys. J.,* Vol. 493, pp. L53-L57, 1998.

63. Riess A G et al, "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant", *Astron. J.,* Vol. 116, pp.1009-1038, 1998.

64. Wetterich, C., "Cosmology and the Fate of Dilatation Symmetry", *Nucl. Phys. B*, Vol. 302, pp. 668-696, 1988.

65. Ratra, R., & P.J.E. Peebles, "Cosmological Consequences of a Rolling Homogeneous Scalar Field", *Phys. Rev. D*, Vol. 37, pp. 3406-3427, 1988.

66. Zlatev, I., L. Wang and P.J. Steinhardt, "Quintessence, Cosmic Coincidence, and the Cosmological Constant", *Phys. Rev. Lett.*, Vol. 82, pp. 896-899, 1999.

67. Banerjee N. and Pavón D, "Cosmic Acceleration Without Quintessence", *Phys. Rev. D,* Vol. 63**,** pp. 043504-1-5, 2001.

68. Sen S. and A.A. Sen, "Late Time Acceleration in Brans-Dicke Cosmology", *Phys. Rev. D,* Vol. 63, pp. 124006-1-8, 2001.

69. Arık M. and Çalık M, "Primordial and Asymptotic Inflation in Brans-Dicke Cosmology", *JCAP* Vol. 0501, pp.013-1-11, 2005.

70. Mohammedi N, "Dynamical Compactification, Standard Cosmology, and the Accelerating Universe", *Phys. Rev. D,* Vol. 65, pp. 104018-1-8, 2002.

71. Darabi F., "An Accelerating Universe and Dynamical Compactification of Extra Dimensions", *Class. Quantum. Grav.,* Vol. 20, pp. 3385-3402, 2003.

72. Dereli T. and Y.N. Obukhov, "Massless Scalar Fields and Topological Black Holes", *Phys.Rev.D*, Vol. 61, pp. 084015-1-7, 2000.

73. Fujii Y. and K. Maeda, *The Scalar–Tensor Theory of Gravitation*, Cambridge University Press, Cambridge, 2003.

74. Dereli T. and A. Eriş, "Conformal Scale Invariance and Scalar-Tensor Theories of Gravitation in Higher Dimensions", *Doğa A*, Vol. 9, pp. 30-35, 1985.

75. Cho Y.M, "Reinterpretation of Jordan-Brans-Dicke Theory and Kaluza-Klein cos-

mology", *Phys. Rev. Lett.,* Vol. 68, pp. 3133-3136, 1992.

76. Freund P.G.O., "Kaluza-Klein Cosmologies", *Nucl. Phys. B,* Vol. 209, pp. 146-156, 1982.

77. Qiang L., Y. Ma, M. Han and D. Yu, "Five-dimensional Brans-Dicke Theory and Cosmic Acceleration", *Phys. Rev. D,* Vol. 71, pp. 061501-1-5, 2005.

78. Mendes L.E. and A. Mazumdar, "Brans–Dicke Brane Cosmology", *Phys. Lett. B,* Vol. 501, pp. 249-256, 2001.

79. Aliev A.N, H. Cebeci and T. Dereli, "Exact Solutions in Five-dimensional Axi-dilaton Gravity with Euler-Poincare Term", *Class. Quantum Grav*, Vol. 24, pp. 3425-3436, 2007; Aliev A.N, H. Cebeci and T. Dereli, "Plane symmetric solutions of gravitational field equations in five dimensions", *Turk.J.Phys.*, Vol. 30, pp. 401-406, 2006.

80. Bander M., "Expanding Cosmologies in Brane Geometries", *Phys. Rev. D,* Vol. 69, pp. 043505-1-4, 2004.

81. Arık M. and D. Çiftci, "Brane World Cosmology in Jordan-Brans-Dicke Theory", *Gen.Rel.Grav.*, Vol. 37, pp. 2211-2221, 2005.

82. Binetruy P., C. Deffayet, U. Ellanger and D. Langlois, "Brane Cosmological Evolution in a Bulk with Cosmological Constant", *Phys. Lett. B*, Vol. 477, pp. 285-291, 2000; Binetruy P., C. Deffayet and D. Langlois, "Non-conventional Cosmology from a Brane Universe", *Nucl. Phys. B,* Vol. 565, pp. 269-287, 2000; Carroll S.M. and L. Mersini, "Can we live in a self-tuning universe?", *Phys.Rev. D,* Vol. 64, pp. 124008-1-9, 2001; Cline J.M, C. Grojean and G. Servant, "Cosmological Expansion in the Presence of an Extra Dimension", *Phys. Rev. Lett.,* Vol. 83, pp. 4245-4248, 1999; Langlois D., "Brane Cosmology: An Introduction", *Prog.Theor.Phys.Suppl.,* Vol. 148, pp. 181-212, 2003.

83. Arık M, A. Baykal, M.C. Çalık, D. Çiftci and Ö. Delice, "Einstein Spaces in Warped Geometries in Five Dimensions", *Phys. Rev. D,* Vol. 68, pp. 123503-1-5, 2003.

84. Weinberg E.J., "Some Problems with Extended Inflation", *Phys. Rev. D,* Vol. 40, pp. 3950-3959, 1989.

85. La D. and P.J. Steinhardt, "Extended Inflationary Cosmology", *Phys. Rev. Lett.,* Vol. 62, pp. 376-378, 1989.

86. Guth A.H., "Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems", *Phys. Rev. D,* Vol. 23, pp. 347-356, 1981.

87. Reasenberg R.D. *et al.*, "Viking Relativity Experiment: Verification of Signal Retardation by Solar Gravity", *Astrophys. J.,* Vol. 234, pp. L219-L221, 1979.

88. Riess A.G. *et. al.,* "The Farthest Known Supernova: Support for an Accelerating Universe and a Glimpse of the Epoch of Deceleration", *Astrophys. J.,* Vol. 560, pp. 49-71, 2001.

89. Townsend P.k. and M.N.R. Wohlfarth, "Accelerating Cosmologies from Compactification", *Phys. Rev. Lett*, Vol. 91, pp. 061302-1-4, 2003.

90. Wohlfarth M.N.R, "Accelerating Cosmologies and a Phase Transition in M-Theory", *Phys. Lett B,* Vol. 563, pp. 1-5, 2003.

91. Ohta N., "Accelerating Cosmologies from Spacelike Branes", *Phys. Rev. Lett.,* Vol. 91, pp. 061303-1-4, 2003.

92. Kasner E., "Geometrical Theorems on Einstein's Cosmological Equations", *Am. J. Math.,* Vol. 43, pp. 217-221, 1921.

93. Frolov A.V., "Kasner–AdS Space-time and Anisotropic Brane-world Cosmology", *Phys. Lett. B,* Vol. 514, pp. 213-216, 2001.

94. Chamblin A., H.S. Reall, H.A. Shinkai, and T. Shiromizu, "Charged Brane-world Black Holes", *Phys. Rev. D,* Vol. 63, pp. 064015-1-11, 2001.

95. Gregory R., R. Whisker, K. Beckwith, and C. Done, "Observing Braneworld Black Holes", *JCAP,* Vol. 10, 013-1-20, 2004.

96. Kofinas G., "Black Hole Solutions in Braneworlds with Induced Gravity", *Phys. Rev. D,* Vol. 66, pp. 104028-1-8, 2002.

97. Myung Y.S., "Relationship Between Five-dimensional Black Holes and de Sitter Spaces", *Class. Quantum Grav.,* Vol. 21, pp. 1279-1293, 2004.

98. Nojiri S., S. D. Odinstov and S. Ogushi, "Cosmological and Black Hole Brane-world Universes in Higher Derivative Gravity", *Phys. Rev. D,* Vol. 65, pp. 023521-1-22, 2002.

99. Takayuki H. and K. Gungwon, "Stable Black Strings in Anti–de Sitter Space", *Phys. Rev. D,* Vol. 64, pp. 064010-1-8, 2001.

100. Casadio R., A. Fabbri, and L. Mazzacurati, "New Black Holes in the Brane World?", *Phys. Rev. D,* Vol. 65, pp. 084040-1-5, 2002.

101. Casadio R. and L. Mazzacurati, "Bulk Shape of Brane-world Black Holes", *Mod. Phys. Lett. A,* Vol. 18, 651-660, 2003.

102. Emparan R., G.T. Horowitz, and R.C. Myers, "Exact Description of Black Holes on Branes II: Comparison with BTZ Black Holes and Black Strings", *JHEP,* Vol. 01, pp. 021-1-28, 2000.

103. Ida D., "Brane World Cosmology", *JHEP,* Vol. 0009, pp. 014-1-4, 2000.

104. Mukohyama S., T. Shiromizu and K. Maeda, "Global Structure of Exact Cosmological Solutions in the Brane World", *Phys. Rev. D,* Vol. 62, pp. 024012-1-6, 2000.

105. Bowcock, P., C. Charmousis, and R. Gregory, "General Brane Cosmologies and Their Global Space-time Structure", *Class. Quantum Grav.,* Vol. 17, pp. 4745-4764, 2000.

106. Arık M. and D. Çiftci, "Schwarzschild-de Sitter Black Holes in 4+1 Dimensional Bulk", *Mod.Phys.Lett.A,* Vol. 22, pp. 289-296, 2007.

107. Brevik I., K. Ghoroku, S. D. Odintsov and M. Yahiro, "Localization of Gravity on a Brane Embedded in AdS5 and dS5", *Phys. Rev. D,* Vol. 66, pp. 064016-1-9, 2002.

108. Csaki C., J. Erlich, T. J. Hollowood and Y. Shirman, "Universal Aspects of Gravity Localized on Thick Branes", *Nucl. Phys. B,* Vol. 581, pp. 309-338, 2000.

109. Vazquez-Poritz J.F., "Gravity-Trapping Domain Walls From Resolved Branes", *JHEP,* Vol. 09, pp. 001-1-11, 2002.

110. Ito M., "Localized Gravity on de Sitter Brane in Five Dimensions", *Europhys. Lett.,* Vol. 64, 295-301, 2003.

111. Padmanabhan T. and S. Shankaranarayanan, "Vanishing of the Cosmological Constant in Nonfactorizable Geometry", *Phys. Rev. D,* Vol. 63, pp. 105021-1-7, 2001.

112. Garriga, J., and Tanaka, T., "Gravity in the Randall–Sundrum Brane World", *Phys. Rev. Lett.*, Vol. 84, pp. 2778–2781, 2000.