SCALAR GAUGE THEORY ON A CYLINDER

by

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ABSTRACT

SCALAR GAUGE THEORY ON A CYLINDER

In this thesis we study the large N_c limit of $SU(N_c)$ gauge theory coupled to a scalar field in the fundamental and adjoint representations on a cylinder, following the ideas of Rajeev. We use a different kind of light-cone coordinate system to eliminate the non-dynamical degrees of freedom and express the theory in terms of quark and Wilson loop variables. We can formulate a classical field theory using color invariant variables, but we don't provide the solution in this thesis.

ÖZET

SİLİNDİR ÜZERİNDE SKALER AYAR TEORİSİ

Bu tez çalışmasında, Rajeev'in yöntemleri kullanılarak silindir üzerinde skaler alanların temel ve adjoint temsillerinin büyük N_c limiti çalışılmıştır. Dinamik olmayan serbestlik derecelerini elemek ve teoriyi kuarklar ve Wilson halkası değiskenleri cinsinden ifade etmek için farklı bir ışık konisi koordinat sisteminden yararlanılmıştır. Bu tezde çift terimlilerden yararlanarak klasik bir alan teorisi formüle edilmiş ancak çözümü sağlanmamıştır.

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LIST OF SYMBOLS

A_μ	Dynamical variables
a	Annihilation operator
$a^\dagger$	Creation operator
D_μ	Covariant derivative
e	Gauge momentum variable
$F^{\mu\nu}$	Field strength tensor
G_{weak}	Coupling constant for weak field
G_{Newton}	Coupling constant for gravitation field
g	Coupling constant
H	Hamiltonian
h	A variable to eliminate non-dynamical degrees of freedom
J	Complex structure
M_p	Mass of the proton
N_c	Number of color
Q	Quadratic form
q	Wilson loop variable
S	Action
$SU(N_c)$	Special unitary group of color
t^a	Generators of the Lie algebra
α_{em}	Coupling constant for electromagnetic field
α_{strong}	Coupling constant for strong field
η	Metric tensor
λ	An element of the Lie algebra
ϕ	Scalar field
ω	Symplectic form

1. INTRODUCTION

The Standard Model can describe all known fundamental forces, with the exclusion of gravity. Standard Model is not only renormalizable, but also can explain a number of results from various areas of physics, such as neutrino scattering experiments, hadronic sum rules, weak decays, current algebras, etc. More significantly, there is not a single experimental data that violates the Standard Model.This model is based on a gauge group, namely $SU(3) \bigotimes SU(2) \bigotimes U(1)$.

There are four fundamental forces in nature:
1. The electromagnetic force, successfully described by QED.
2. The strong force, holds the nucleus together.
3. The weak force, governs the properties of decaying particles, such as the beta decay of the neutron.
4. The gravitational force, described classically by Einstein's general theory of relativity.

Quantum Field Theory can describe fully only the electromagnetic force, for a very fundamental reason that we can easily see, when we look at their coupling constants:

$$\begin{aligned}
\alpha_{em} &\sim 1/137.0359895(61) \\
\alpha_{strong} &\sim 0.118 \\
G_{weak} &\sim 1.02 \times 10^{-5}/M_p^2 \\
&\sim 1.16639(2) \times 10^{-1} GeV^{-2} \\
G_{Newton} &\sim 5.9 \times 10^{-39}/M_p^2 \\
&\sim 6.67259(85) \times 10^{-1} m^3 kg^{-1} s^{-2} \qquad (1.1)
\end{aligned}$$

where M_p is the mass of the proton and the parentheses represent the uncertainties [1].

When we look at this chart and compare the coupling constant of the electromagnetic force with the others, we see that its difference lies in having the only coupling constant that works for power expansion. Since perturbation theory, based on power expansion, is used predominantly in the Quantum Field Theory, it is clear why electromagnetic force has a well-constructed theory, namely Quantum Electrodynamics.

It is believed that the fundamental fields in nature consist of the spinor fields and the gauge fields. While the former refers to leptons and quarks the latter refers to electromagnetic, weak and gluon fields, which are referred to as the mediators. The standard model is based on non-abelian gauge theories, Yang-Mills theories, for these force carriers [2]. These bosons couple to fermionic matter particles, namely the quarks and leptons. Interactions are determined by the algebraic structure of certain internal symmetry groups. The strong interactions are determined by the group $SU(3)$, and is described by a gauge theory called Quantum Chromodynamics. The electroweak interactions are determined by the group $SU(2) \times U(1)$ and is described by the gauge theory Weinberg-Salam model. We will only concentrate on the Quantum Chromodynamics (QCD) [3], [4].

The strongly interacting particles are called hadrons, stemming from the Greek word *hadros*, for "strong". There are two kinds of particles that interact strongly, which are mesons and baryons. Proton and neutron are the examples among baryons, whereas the most common example to mesons are pions. The hadrons are made up of more basic particles called quarks. Quark model came as a solution to the many "resonances" that began to be observed in particle accelerator experiments. Quarks are fermions with fractional charges. Composite combinations of the "up", "down", and "strange" quarks could explain all the hadrons discovered up to that time, when Gell-Mann, Ne'eman and Zweig constructed the $SU(3)$ quark theory. These three quarks form a representation of the Lie group $SU(3)$:

$$q_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{1.2}$$

Mesons are quark and anti-quark pairs, whereas baryons are made up of three quarks or three anti-quarks.

Quarks carry an internal symmetry fitting to the group $SU(3)$, called color. The promotion of this internal global symmetry to a local gauge principle gives us a gluon field which is self-coupled, unlike the mediator of QED which does not carry electrical charge, and this is the field responsible for the strong force. In QCD, the color group is unbroken, unlike electroweak theory, so the gluons remain massless. In a gauge theory, the gluon and quark fields have some degrees of freedom that are not observable. The true dynamical degrees of freedom, which make up the hadrons, are determined by the principle of gauge invariance.

QCD is an asymptotically free field theory [3], [4]. Thus the theory behaves as a free theory at high energy levels, and the quarks and gluons are free particles (up to logarithmic corrections). The strong force does not grow strong at short distances, nor does fall off at large distances, in contrast to electrostatic or gravitational forces. This property leads to the phenomenon of confinement which explains why the fundamental particles, quarks and gluons, have never been observed isolated. Quarks and gluons are confined inside the colorless, gauge invariant bound states of the strong interactions, namely the mesons, baryons and glueballs that are gauge-invariant bound-states of gluons (altogether referred to as hadrons).

Due to this high energy level freedom of the theory, perturbation is possible only at high energies. At ordinary energy scales it is not an appropriate method, since the coupling constant gets larger as energy gets low. The purpose should be to establish non-perturbative methods to understand QCD, and to find out the masses of the hadrons, various resonances, and the cross sections in various processes at low energies [3], [6].

Since QCD does not manifest its ordinary classical limit, it has to be understood at the quantum level. Thus there should be a quantum description at low energy scales. At high energy scales the perturbation around $\hbar \to 0$ limit works. QCD is the only known renormalizable four-dimensional theory with the correct high-energy behavior. But perturbation around this trivial limit is not applicable at long distances, thus structure functions of the bound states cannot be understood by perturbing around the free vacuum. We need another classical limit for QCD since all the observed particles are hadronic bound states. This limit is believed to be large number of colors N_c, where the structure group of the gauge theory is $SU(N_c)$. In nature three colors seem to exist. Fluctuations in gauge-invariant observables *alone* vanish in the large-N_c limit. This is important since in the limit of vanishing Planck's constant, the fluctuations in *all* observables of a gauge theory vanish.

Quarks transform as N-component vectors under color, the $SU(N_c)$ structure group, while gluons transform in the adjoint representation as $N_c \times N_c$ hermitian matrices. The components of these vectors and the matrix elements are the so-called "gauge degrees of freedom" (quarks and gluons) that carry the color quantum number, and are not directly observable. Only color invariant observables, namely the Wilson loop and meson observables are non-local. This means that we have to pass from local gauge fields to non-local loop or string-like variables.

The large-N_c limit, as an approximation for non-abelian gauge theories, was originally proposed by 't Hooft in a perturbative context [7]. As an application 't Hooft studied $1+1$-dimensional QCD and found eigenvalue equation for the meson in the large N_c. Migdal and Witten proposed that this limit should be a kind of classical mechanics [8]. Rajeev constructed a theory of mesons in two dimensions in the limit N_c, the number of colors in $SU(N_c)$ goes to infinity using only color invariant observables (corresponding to meson operators) [3]. Rajeev's theory is a classical mechanics on an infinite dimensional space, called the Grassmannian. The linearization of this theory leads to the famous 't Hooft equations. $1+1$ dimensions is a great simplification, but it is a good tool to shed light on difficult theories. Scalar two dimensional QCD was studied by Shei and Tsao in [9] following 't Hooft, and later by Tomaras using Hamitonian methods in [10]. Rajeev's method applied to this case is studied in [11], they found a non-local theory on an infinite dimensional disk. Linearization gives again the results found by Shai and Tsao. Aoki [12], showed that three types of mesons, all obeying a certain type of 't Hooft equation (see also [6]) are possible.

When we study $1+1$ dimensional QCD on $\mathbb{R} \times \mathbb{R}$ we do not have any dynamical gauge degrees of freedom. To see the effect of gauge degrees of freedom, one needs another toy model, which is not so difficult. This is the gauge theory on a cylinder. There, the gauge fields have a dynamical global degree of freedom, the so-called Wilson loop. We study this problem in detail using the methods suggested by Rajeev and Guruswamy in [13] for scalars. Unfortunately, this theory is too complicated to be understood at the same level of completeness as in the previous work of Rajeev. We do not have a closed algebra of observables in the large N_c limit. The geometric meaning of the constraint eludes us. Yet we have a well-defined theory.

2. THE METRIC

We will study non-abelian gauge theory coupled to scalars in the fundamental and adjoint representations respectively in 1 +1 dimensions. To decouple the gauge vector from the kinetic part of the boson we will use a method suggested by Rajeev in [13]. This is an essential point, since when we have coupling between two such fields, the vacuum will change and we should find the correct vacuum to start with.

We will perform our analysis in a co-ordinate system different from the Cartesian (x, t) co-ordinates used to solve pure Yang-Mills theory in Ref. [13]. We construct a kind of light-cone formalism such that the new coordinates (u, x) are:

$$u = t + \mid x \mid. \tag{2.1}$$

Then,

$$dt = du - sgn(x)dx. \tag{2.2}$$

The metric is

$$ds^2 = dt \otimes dt - dx \otimes dx = du \otimes du - sgn(x)du \otimes dx - sgn(x)dx \otimes du. \tag{2.3}$$

Thus the metric tensor becomes

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & -sgn(x) \\ -sgn(x) & 0 \end{pmatrix} \quad , \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & -sgn(x) \\ -sgn(x) & -1 \end{pmatrix} \tag{2.4}$$

The points (u, x) and $(u, x + 2L)$ are the same on the cylinder. These light-cone coordinates avoid the quadratic energy term appearing in the mass shell condition of the Cartesian co-ordinates:

$$p_0{}^2 - p_1{}^2 = m^2. \tag{2.5}$$

We don't use the conventional light-cone co-ordinates since our gauge elimination method would not be applicable. *We will use u as our evolution variable.* In our special light-cone co-ordinate system the mass-shell condition is

$$-2sgn(x)p_x p_u - p_x{}^2 = m^2 \tag{2.6}$$

from which we obtain a unique solution for p_u:

$$p_u = -\frac{1}{2}[\frac{m^2}{p_x} + p_x]sgn(x). \tag{2.7}$$

This is important, since it suggests that in second quantization the positive and negative energies will be determined by the sign of $p_x sgn(x)$.

3. BOSONS IN THE FUNDAMENTAL REPRESENTATION

3.1. Transformation to New Variables

The action of the complex bosons in 2 dimensions interacting with an $SU(N_c)$ gauge field can be written as

$$S = \int dudx \, \sqrt{-\eta} \, \left[\frac{1}{2}(D_\mu \phi)^\dagger D^\mu \phi - \frac{1}{2} m^2 \phi \dagger \phi + \frac{1}{4g^2} Tr F^{\mu\nu} F_{\mu\nu} \right] \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here $D_\mu = \partial_\mu + A_\mu$, where A_μ is the gauge potential, D_μ is the covariant derivative. Thus A_μ is antihermitian with

$$A_\mu = A_\mu^a t^a \tag{3.2}$$

$$Tr(t^a t^b) = -\delta^{ab}. \tag{3.3}$$

The commutation of the generators of the Lie algebra,

$$[t^a, t^b] = i f^{abc} t^c, \tag{3.4}$$

f^{abc} being structure constants.

If we express this in terms of components in the new system of coordinates, we find

$$\begin{aligned} S = & \int dudx \, \left[-\frac{1}{2} sgn(x) D_u \phi^\dagger D_x \phi - \frac{1}{2} sgn(x) D_x \phi^\dagger D_u \phi \right] \\ & + \int dudx \, \left[-\frac{1}{2} D_x \phi^\dagger D_x \phi - \frac{1}{2} m^2 \phi^\dagger \phi \right] \\ & + \int dudx \, \left[-\frac{1}{g^2} Tr E (\partial_u A_x - \partial_x A_u + [A_u, A_x]) + \frac{1}{2g^2} Tr E^2 \right] \end{aligned} \tag{3.5}$$

where

$$E = \partial_u A_x - \partial_x A_u + [A_u, A_x]. \tag{3.6}$$

Here we split the E^2 term, namely:

$$\frac{1}{4g^2} TrF^{\mu\nu}F_{\mu\nu} = -\frac{1}{2g^2} TrE^2 = \frac{1}{2g^2} TrE^2 - \frac{1}{g^2} TrE^2, \tag{3.7}$$

and insert the definition of E in terms of gauge fields in the second part to get,

$$-\frac{1}{g^2} TrE(\partial_u A_x - \partial_x A_u + [A_u, A_x]). \tag{3.8}$$

If we write the action explicitly:

$$\begin{aligned} S = &-\int \; dudx \; \frac{1}{2} sgn(x)(\partial_u \phi^\dagger - \phi^\dagger A_u)(\partial_x \phi + A_x \phi) \\ &-\int \; dudx \; \frac{1}{2} sgn(x)(\partial_x \phi^\dagger - \phi^\dagger A_x)(\partial_u \phi + A_u \phi) \\ &-\int \; dudx \; \left[\frac{1}{2}(\partial_x \phi^\dagger - \phi^\dagger A_x)(\partial_x \phi + A_x \phi)\right] \\ &-\int \; dudx \; \frac{1}{2} m^2 \phi^\dagger \phi \\ &-\int \; dudx \; \left[\frac{1}{g^2} TrE(\partial_u A_x - \partial_x A_u + [A_u, A_x]) + \frac{1}{2g^2} TrE^2\right]. \end{aligned} \tag{3.9}$$

Now, following [13], we will define a variable $h(u,x)$ to eliminate the Yang-Mills field *without imposing any gauge condition*:

$$\frac{\partial h}{\partial x} + A_x h = 0, \tag{3.10}$$

$$h(u, -L) = 1. \tag{3.11}$$

The solution is represented as a path ordered exponential:

$$h(u,x) = P[e^{\int_{-L}^{x} dy A_x(u,y)}] \tag{3.12}$$

The Wilson loop is then given by

$$q = h^{-1}(u,L) \tag{3.13}$$

We will define new field variables such that:

$$\begin{aligned} \phi &= h\tilde{\phi}, \\ A_u &= h\tilde{A}_u h^{-1}, \\ E &= h\tilde{E}h^{-1}. \end{aligned} \tag{3.14}$$

Our aim is to decouple the leading bosonic action from the gauge theory part and reduce the gauge theory part to its true dynamical degrees of freedom.
Since we are on a cylinder with $(u,x) \sim (u, x+2L)$, the fields are periodic with period $2L$. Thus

$$\begin{aligned} \phi(L) &= \phi(-L), \\ \phi^\dagger(L) &= \phi^\dagger(-L), \\ E(L) &= E(-L). \end{aligned} \tag{3.15}$$

In terms of the new variables, we have

$$\begin{aligned} \tilde{\phi}(L) &= q\tilde{\phi}(-L), \\ \tilde{\phi}^\dagger(L) &= \tilde{\phi}^\dagger(-L)q^{-1}, \\ \tilde{E}(L) &= q\tilde{E}(-L)q^{-1}. \end{aligned} \tag{3.16}$$

We see that now we are dealing with a complicated boundary condition. We will define $E(-L) = e$. When we insert the new definitions of our variables, we find

$$\begin{aligned} S = &-\int dudx\ \frac{1}{2}sgn(x)(\partial_u(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}h\tilde{A}_u h^{-1})(\partial_x(h\tilde{\phi}) + A_x h\tilde{\phi}) \\ &-\int dudx\ \frac{1}{2}sgn(x)(\partial_x(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_u(h\tilde{\phi}) + h\tilde{A}_u h^{-1}h\tilde{\phi}) \\ &-\int dudx\ \left[\frac{1}{2}(\partial_x(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_x(h\tilde{\phi}) + A_x h\tilde{\phi})\right] \\ &-\int dudx\ \frac{1}{2}m^2\tilde{\phi}^\dagger h^{-1}h\phi \\ &-\int dudx\ \frac{1}{g^2}Tr\left[h\tilde{E}h^{-1}(\partial_u A_x - \{\partial_x A_u + [A_u, A_x]\})\right] \\ &+\int dudx\ \frac{1}{2g^2}Tr\left[(h\tilde{E}h^{-1}h\tilde{E}h^{-1})\right]. \end{aligned} \tag{3.17}$$

Now we take a closer look at the term

$$\partial_x A_u + [A_u, A_x]. \tag{3.18}$$

We insert identities on both sides in the form of hh^{-1} and use the equation 3.10. We see that this term is the expansion of

$$h(\partial_x \tilde{A}_u)h^{-1}. \tag{3.19}$$

Thus

$$\partial_u A_x - \{\partial_x A_u + [A_u, A_x]\} = hh^{-1}\partial_u A_x hh^{-1} - h(\partial_x \tilde{A}_u)h^{-1}. \tag{3.20}$$

By using the relation below that can also be derived from 3.10,

$$h^{-1}(\partial_u A_x)h = -\partial_x(h^{-1}\partial_u h). \tag{3.21}$$

equation 3.20 becomes:

$$-h\partial_x(h^{-1}\partial_u h)h^{-1} - h\partial_x(h^{-1}A_u h)h^{-1} = -h\partial_x(h^{-1}\partial_u h + \tilde{A}_u)h^{-1}. \tag{3.22}$$

We will introduce a new variable A via

$$A = \tilde{A}_u + h^{-1}\partial_u h. \tag{3.23}$$

If we go back to our term in the action,

$$\begin{aligned} & - \frac{1}{g^2}\int dudx\ Tr\left[h\tilde{E}h^{-1}(\partial_u A_x - \{\partial_x A_u + [A_u, A_x]\})\right] \\ & = -\frac{1}{g^2}\int dudx\ Tr\left[-\tilde{E}\partial_x(h^{-1}\partial_u h + \tilde{A}_u)\right]. \end{aligned} \tag{3.24}$$

We try to express this in terms of total derivatives,

$$\begin{aligned} & -\frac{1}{g^2}\int dudx\ Tr\left[-\partial_x(\tilde{E}h^{-1}\partial_u h) + (\partial_x\tilde{E})h^{-1}\partial_u h - \partial_x(\tilde{E}\tilde{A}_u) + (\partial_x\tilde{E})\tilde{A}_u\right] \\ & = \frac{1}{g^2}\int du\int_{-L}^{L} dx\ Tr\left[\partial_x(\tilde{E}h^{-1}\partial_u h)\right] - \frac{1}{g^2}\int dudx\ Tr\left[(\partial_x\tilde{E})[\tilde{A}_u + h^{-1}\partial_u h]\right]. \end{aligned} \tag{3.25}$$

where the third term vanished, since the fields drop out at the boundaries. The total derivative can be expanded as,

$$\frac{1}{g^2}\int duTr[\tilde{E}(L)h^{-1}(L)\partial_u h(L) - \tilde{E}(-L)h^{-1}\partial_u h(-L)] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})\tilde{A}_u]. \tag{3.26}$$

In this expression $\partial_u h(-L)$ gives zero, since by the boundary conditions in equation 3.11, one has

$$\partial_u h(-L, u) = 0. \tag{3.27}$$

We insert our boundary conditions; $h^{-1}(L) = q$, or $h(L) = q^{-1}$, obtaining

$$\frac{1}{g^2}\int duTr[q\tilde{E}(-L)q^{-1}q\partial_u q^{-1}] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A]. \tag{3.28}$$

So the gauge field part of our equation has become

$$-\frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2. \tag{3.29}$$

Having derived the gauge field interaction part in terms of the new variables we can now go back to 3.17:

$$\begin{aligned}
S = &-\int\ dudx\ \frac{1}{2}sgn(x)(\partial_u(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}h\tilde{A}_u h^{-1})(\partial_x(h\tilde{\phi}) + A_x h\tilde{\phi}) \\
&-\int\ dudx\ \frac{1}{2}sgn(x)(\partial_x(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_u(h\tilde{\phi}) + h\tilde{A}_u h^{-1}h\tilde{\phi}) \\
&-\int\ dudx\ [\frac{1}{2}(\partial_x(\tilde{\phi}^\dagger h^{-1}) - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_x(h\tilde{\phi}) + A_x h\tilde{\phi})] \\
&-\int\ dudx\ \frac{1}{2}m^2\tilde{\phi}^\dagger\tilde{\phi} \\
&-\frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2.
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
S = &-\int\ dudx\ \frac{1}{2}sgn(x)(\partial_u\tilde{\phi}^\dagger h^{-1} + \tilde{\phi}^\dagger\partial_u h^{-1} - \tilde{\phi}^\dagger\tilde{A}_u h^{-1})(\partial_x h\tilde{\phi} + h\partial_x\tilde{\phi} + A_x h\tilde{\phi}) \\
&-\int\ dudx\ \frac{1}{2}sgn(x)(\partial_x\tilde{\phi}^\dagger h^{-1} + \tilde{\phi}^\dagger\partial_x h^{-1} - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_u h\tilde{\phi} + h\partial_u\tilde{\phi} + h\tilde{A}_u h^{-1}h\tilde{\phi})) \\
&-\int\ dudx\ [\frac{1}{2}(\partial_x\tilde{\phi}^\dagger h^{-1} + \partial_x h^{-1}\tilde{\phi}^\dagger - \tilde{\phi}^\dagger h^{-1}A_x)(\partial_x h\tilde{\phi} + h\partial_x\tilde{\phi} + A_x h\tilde{\phi})] \\
&-\int\ dudx\ \frac{1}{2}m^2\tilde{\phi}^\dagger\tilde{\phi} \\
&-\frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2.
\end{aligned} \tag{3.31}$$

Making some cancelations using 3.10,

$$
\begin{aligned}
S = &-\int dudx\, \frac{1}{2}sgn(x)(\partial_u\tilde{\phi}^\dagger\partial_x\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}\partial_u h\partial_x\tilde{\phi} - \tilde{\phi}^\dagger\tilde{A}_u\partial_x\tilde{\phi}) \\
&-\int dudx\, \frac{1}{2}sgn(x)(\partial_x\tilde{\phi}^\dagger h^{-1}\partial_u h\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}\partial_x h h^{-1}\partial_u h\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}A_x\partial_u h\tilde{\phi}) \\
&-\int dudx\, \frac{1}{2}sgn(x)(\partial_x\tilde{\phi}^\dagger\partial_u\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}\partial_x h\partial_u\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}A_x h\partial_u\tilde{\phi}) \\
&-\int dudx\, \frac{1}{2}sgn(x)(\partial_x\tilde{\phi}^\dagger\tilde{A}_u\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}\partial_x h\tilde{A}_u\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}A_x h\tilde{A}_u\tilde{\phi}) \\
&-\int dudx\, [\frac{1}{2}(\partial_x\tilde{\phi}^\dagger\partial_x\tilde{\phi} + \tilde{\phi}^\dagger\partial_x h^{-1}h\partial_x\tilde{\phi} - \tilde{\phi}^\dagger h^{-1}A_x h\partial_x\tilde{\phi})] \\
&-\int dudx\, \frac{1}{2}m^2\tilde{\phi}^\dagger\tilde{\phi} \\
&-\frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\, Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\, Tr\tilde{E}^2. \quad (3.32)
\end{aligned}
$$

We collect the appropriate terms to form A's as in 3.23 and make some cancelations using the properties of h. In terms of the new variables the action becomes

$$
\begin{aligned}
S = &\int dudx\, [-\frac{1}{2}sgn(x)[(\partial_u\tilde{\phi}^\dagger)(\partial_x\tilde{\phi}) + (\partial_x\tilde{\phi}^\dagger)(\partial_u\tilde{\phi})]] \\
&+\int dudx\, [-\frac{1}{2}sgn(x)[A(\partial_x\tilde{\phi}^\dagger)\tilde{\phi} - \tilde{\phi}^\dagger A(\partial_x\tilde{\phi})]] \\
&-\int dudx\frac{1}{2}(\partial_x\tilde{\phi}^\dagger)(\partial_x\tilde{\phi}) - \int dudx\frac{1}{2}m^2\tilde{\phi}^\dagger\tilde{\phi} \\
&-\int dudx[\frac{1}{g^2}Tr(\partial_x\tilde{E})A - \frac{1}{2g^2}TrE^2] \\
&-\frac{1}{g^2}\int du\, Tr(q^{-1}\partial_u qe). \quad (3.33)
\end{aligned}
$$

We see that A does not have a time derivative in the action, so it is just a Lagrange multiplier imposing a constraint and can be eliminated. Varying the action w.r.t A we obtain a constraint equation:

$$
-\frac{1}{2}sgn(x)[\delta A(\partial_x\tilde{\phi}^\dagger)\tilde{\phi} - \tilde{\phi}^\dagger\delta A(\partial_x\tilde{\phi})] - \frac{1}{g^2}Tr(\partial_x\tilde{E}\delta A) = 0 \quad (3.34)
$$

$$
\frac{1}{g^2}\partial_x\tilde{E}^b\delta A^a Tr[t^b t^a] = -\frac{1}{2}sgn(x)(\tilde{\phi}(\partial_x\tilde{\phi}^\dagger) - (\partial_x\tilde{\phi})\tilde{\phi}^\dagger)^{ij}\delta A^a t^a_{ji}. \quad (3.35)
$$

Since our scalar fields are in the fundamental representation, and the gauge potentials in the adjoint representation are represented as

$$A = A^a t^a,$$
$$\tilde{E} = \tilde{E}^a t^a, \tag{3.36}$$

the constraint becomes

$$-\frac{1}{g^2}\partial_x \tilde{E}^a = -\frac{1}{2} sgn(x)\{t_j^{ia}\ \tilde{\phi}^j(\partial_x\tilde{\phi}^\dagger)_i - t_j^{ia}\ (\partial_x\tilde{\phi})^j\tilde{\phi}_i^\dagger\}. \tag{3.37}$$

Thus

$$E = e + \frac{g^2}{2}\int_{-L}^{x} dy\ sgn(y)\{\tilde{\phi}(\partial_y\tilde{\phi}^\dagger) - (\partial_y\tilde{\phi})\tilde{\phi}^\dagger\}. \tag{3.38}$$

Thus the *classical* action is reduced to:

$$\begin{aligned} S = &-\frac{1}{2}\int dudx\ sgn(x)[(\partial_u\tilde{\phi}^\dagger)(\partial_x\tilde{\phi}) + (\partial_x\tilde{\phi}^\dagger)(\partial_u\tilde{\phi})] \\ &- \int dudx\ \frac{1}{2}(\partial_x\tilde{\phi}^\dagger)(\partial_x\tilde{\phi}) - \frac{1}{2}\int dudx\ m^2\tilde{\phi}^\dagger\tilde{\phi} \\ &+ \frac{1}{2g^2}\int dudx\ Tr\left[e + \frac{g^2}{2}\int_{-L}^{x} dy\ sgn(y)\{\tilde{\phi}(\partial_y\tilde{\phi}^\dagger) - (\partial_y\tilde{\phi})\tilde{\phi}^\dagger\}\right]^2 \\ &- \frac{1}{g^2}\int du\ Tr(q^{-1}\partial_u q e). \end{aligned} \tag{3.39}$$

Here, we see that the action has three main parts consisting of bosons, the term with the new variables and their coupling. The problem has decoupled, into independent bosons and gauge fields if we assume the coupling term between these two systems. We can safely assume this due to the theory of ... []. That is we can quantize the bosonic system and the gauge part independently, the interaction will not drastically alter the vacuum structure.

When we write this action in the form

$$S = \int du\ x^i \omega_{ij} \dot{x}^j - \int du\ H \tag{3.40}$$

we read off the Hamiltonian as

$$\begin{aligned} H = &\int\ dx\ \frac{1}{2}(\partial_x \tilde{\phi}^\dagger)(\partial_x \tilde{\phi}) + \frac{1}{2}\int\ dx\ m^2 \tilde{\phi}^\dagger \tilde{\phi} \\ &- \frac{1}{2g^2}\int\ dx\ Tr\Big[e + \frac{g^2}{2}\int_{-L}^{x} dy\ sgn(y)\{\tilde{\phi}(\partial_y \tilde{\phi}^\dagger) - (\partial_y \tilde{\phi})\tilde{\phi}^\dagger\}\Big]^2, \end{aligned} \tag{3.41}$$

the rest belonging to the symplectic part.

In the next two sections we will focus on the symplectic geometry of the gauge and boson fields respectively. We will focus on the normal ordering of the bosonic theory later.

3.2. Poisson Brackets of the Gauge Theory Sector

Following the suggestion in [15], we will try to bring the variation of the action to the form

$$\begin{aligned} \delta S &= \int du\ \dot{x}^A_{(ij)} \Omega^{AB}_{(ij);(kl)} \delta x^B_{(kl)} - \int du\ \delta H \\ &= \int du\ \dot{x}^A_{(ij)} \Omega^{AB}_{(ij);(kl)} \delta x^B_{(kl)} - \int du\ x^A_{ij} Q^{AB}_{(ij);(kl)} \delta x^B_{(kl)}. \end{aligned} \tag{3.42}$$

For simplicity δH is given as a quadratic form, but we do not really need its explicit form here. Here the indices A, B belong to the coordinates (e, q) and run from $1, 2$. On top of it the i, j indices refer to matrix form of the Lie algebra. The equations of motion gives

$$\dot{x}^A_{(ij)} = (\Omega^{-1})^{AB}_{(ij);(kl)} Q^{BC}_{(kl);(st)} x^C_{(st)}. \tag{3.43}$$

Since

$$\dot{x}^A_{(ij)} = \{x^A_{(ij)}, x^B_{(kl)}\}Q^{BC}_{(ij);(kl)}x^C_{(kl)}. \tag{3.44}$$

we can read off the Poisson brackets directly from this expression. We will try to express everything in terms of our new variables ξ, v and e such that:

$$\delta\xi = q^{-1}\delta q, \tag{3.45}$$

$$v = q^{-1}\dot{q}. \tag{3.46}$$

These are the correct independent variables on a group, the variation of the group variable being written as $\delta\xi$. To obtain the Poisson brackets we make a variation in the action:

$$\delta S = -\frac{1}{g^2}\int du\ tr\delta\{q^{-1}(\partial_u q)e\}. \tag{3.47}$$

This variation can be expanded as

$$\delta S = -\frac{1}{g^2}\int du\ tr\{\delta q^{-1}\dot{q}e + q^{-1}\delta\dot{q}e + q^{-1}\dot{q}\delta e\}. \tag{3.48}$$

We do an integration by parts:

$$\delta S = -\frac{1}{g^2}\int du\ tr\{\delta q^{-1}\dot{q}e + \partial_u(q^{-1}\delta q e) - \partial_u q^{-1}\delta q e - q^{-1}\delta q\dot{e} + q^{-1}\dot{q}\delta e\}. \tag{3.49}$$

The variation of q at the time boundaries give zero so the total derivative drops out, giving

$$\delta S = -\frac{1}{g^2}\int du\ tr\{-q^{-1}\delta q q^{-1}\dot{q}e + q^{-1}\dot{q}q^{-1}\delta q e - q^{-1}\delta q\dot{e} + q^{-1}\dot{q}\delta e\}. \tag{3.50}$$

We insert the definitions in equations 3.45 and 3.46 to express the action in terms of our new coordinates as

$$\delta S = -\frac{1}{g^2}\int du\ tr[-\delta\xi ve + v\delta\xi e - \delta\xi\dot{e} + v\delta e]. \tag{3.51}$$

Explicitly,

$$\begin{aligned}\delta S = -\frac{1}{g^2}&\int du\ [e_{ij}v_{jk}\delta\xi_{ki} - v_{ij}e_{jk}\delta\xi_{ki} - \dot{e}_{ij}\delta\xi_{ji} + v_{ij}\delta e_{ji}] \\ = \quad -\frac{1}{g^2}&\int du\ [\delta_{lk}e_{ij}v_{jk}\delta\xi_{li} - \delta_{li}v_{ij}e_{jk}\delta\xi_{kl} - \dot{e}_{ij}\delta\xi_{ji} + v_{ij}\delta e_{ji}].\end{aligned} \tag{3.52}$$

Now by changing the dummy indices, we obtain

$$\delta S = -\frac{1}{g^2}\int du\ [(\delta_{kj}e_{li} - \delta_{li}e_{jk})v_{ij}\delta\xi_{kl} - \dot{e}_{ij}\delta\xi_{ji} + v_{ij}\delta e_{ji}]. \tag{3.53}$$

One can easily check that this is of the form

$$-\frac{1}{g^2}\begin{pmatrix} v_{ij} & \dot{e}_{ij}\end{pmatrix}\begin{pmatrix} A_{ij;kl} & \delta_{ij;kl} \\ -\delta_{ij;kl} & 0\end{pmatrix}\begin{pmatrix}\delta\xi_{kl} \\ \delta e_{kl}\end{pmatrix} \tag{3.54}$$

where

$$A_{ij;kl} = \delta_{kj}e_{li} - \delta_{li}e_{jk} \tag{3.55}$$

and

$$\delta_{ij;kl} = \delta_{li}\delta_{jk}. \tag{3.56}$$

This matrix we found in the above formula is Ω and written in the form

$$\mathbf{\Omega} = -\frac{1}{g^2}\begin{pmatrix} A & 1 \\ -1 & 0\end{pmatrix}. \tag{3.57}$$

If we calculate the inverse we find

$$\mathbf{\Omega}^{-1} = -g^2 \begin{pmatrix} 0 & -1 \\ 1 & A \end{pmatrix}. \tag{3.58}$$

We can read off the canonical commutation relations as follows:

$$\{\xi_{ij}, e_{kl}\} = g^2 \delta_{ij;kl} \tag{3.59}$$

$$\{\xi_{ij}, \xi_{kl}\} = 0 \tag{3.60}$$

$$\{e_{ij}, e_{kl}\} = -g^2 A_{ij;kl} \tag{3.61}$$

We can write the equations of motion for our coordinate ξ as follows:

$$(q^{-1}\dot{q})_{ij} = \{\xi_{ij}, \xi_{kl}\} \frac{\delta H}{(q^{-1}\delta q)_{kl}} + \{\xi_{ij}, e_{kl}\} \frac{\delta H}{\delta e_{kl}} \tag{3.62}$$

Due to 3.60 the first term drops

$$(q^{-1})_{is'}\dot{q}_{s'j} = \{\xi_{ij}, e_{kl}\} \frac{\delta H}{\delta e_{kl}}, \tag{3.63}$$

and we can pass q^{-1} to the other side as q, and since the $\{\xi_{ij}, \xi_{kl}\}$ term is missing, we can introduce a Poisson bracket directly between the variable q and e,

$$q_{si}\{\xi_{ij}, e_{kl}\} = \{q_{sj}, e_{kl}\} = g^2 q_{si}\delta_{li}\delta_{jk} = g^2 q_{sl}\delta_{jk} \tag{3.64}$$

Contracting e by an element of the Lie algebra, namely λ, gives

$$\{e_{kl}\lambda_{lk}, q_{sj}\} = g^2 \lambda_{lk} q_{sl}\delta_{jk} = g^2 q_{sl}\lambda_{lj} = g^2 (q\lambda)_{sj}. \tag{3.65}$$

Thus,

$$\{tr\lambda e, q_{sj}\} = g^2(q\lambda)_{sj}. \tag{3.66}$$

Similarly from 3.61,

$$\{\lambda^1_{ji}e_{ij}, e_{kl}\} = -g^2(\lambda^1_{ki}e_{li} - \lambda^1_{jl}e_{jk}), \tag{3.67}$$

$$\{\lambda^1_{ji}e_{ij}, \lambda^2_{lk}e_{kl}\} = -g^2(-\lambda^2_{lk}\lambda^1_{ki}e_{il} + \lambda^1_{jl}\lambda^2_{lk}e_{kj}). \tag{3.68}$$

By some index changes we arrive at the result below:

$$\{Tr\lambda_1 e, Tr\lambda_2 e\} = -g^2 Tr[\lambda_1, \lambda_2]e \tag{3.69}$$

There is a nice geometric interpretation for this symplectic structure. It is indeed the canonical form on T^*G; the cotangent bundle on G.

3.2.1. Quantized Gauge Theory

Here, we will simply postulate Dirac's quantization condition, replacing Poisson brackets by commutators.

$$[\ ,\ ] = i\hbar\ \{\ \} \tag{3.70}$$

Ordering ambiguities here can be resolved by the Weyl ordering (or any other one.) Thus

$$\begin{aligned} e &\rightarrow \hat{e} \\ q &\rightarrow \hat{q} \end{aligned} \tag{3.71}$$

and

$$\hat{e} \rightarrow \hat{e}^{\dagger}$$
$$\hat{q} \rightarrow \hat{q}^{\dagger} = \hat{q}^{-1}. \tag{3.72}$$

The quantization can be stated as follows:

$$[\hat{q}^{i}_{j}, \hat{q}^{k}_{l}] = 0, \tag{3.73}$$

$$[Tr\lambda\hat{e}, \hat{q}^{l}_{m}] = ig^{2}(q\lambda)^{l}_{m}, \tag{3.74}$$

and

$$[Tr\lambda_1\hat{e}, Tr\lambda_2\hat{e}] = -ig^{2}Tr[\lambda_1, \lambda_2]\hat{e}. \tag{3.75}$$

3.3. Symplectic Form of the Scalar Field Theory Part

Now that we have constructed the symplectic geometry of the gauge field, we can focus on the scalar field theory part. Looking back to our action in equation 3.39, we take the symplectic form of the field theory part as:

$$-\frac{1}{2}\int dudx\ sgn(x)[(\partial_u\tilde{\phi}^{\dagger})(\partial_x\tilde{\phi}) + (\partial_x\tilde{\phi}^{\dagger})(\partial_u\tilde{\phi})] \tag{3.76}$$

Varying w.r.t $\tilde{\phi}^{\dagger}$, we get

$$-\frac{1}{2}\int dudx\ sgn(x)[(\partial_u\delta\tilde{\phi}^{\dagger})(\partial_x\tilde{\phi}) + (\partial_x\delta\tilde{\phi}^{\dagger})(\partial_u\tilde{\phi})]. \tag{3.77}$$

Integrating by parts in the first term gives

$$-\frac{1}{2}\int\ dudx\ sgn(x)[\partial_u(\delta\tilde{\phi}^\dagger\partial_x\tilde{\phi}) - \delta\tilde{\phi}^\dagger\partial_u\partial_x\tilde{\phi} + (\partial_x\delta\tilde{\phi}^\dagger)(\partial_u\tilde{\phi})]. \tag{3.78}$$

The first term drops out since $\delta\tilde{\phi}^\dagger$ is zero at the boundaries $\pm\infty$ of the time coordinate. The same can not be applied to the x coordinate due to the boundary conditions stated in equation 3.16. Thus we obtain:

$$-\frac{1}{2}\int\ dudx\ [\delta\tilde{\phi}^\dagger(\overleftarrow{\partial}_x sgn(x) - sgn(x)\overrightarrow{\partial}_x)\partial_u\tilde{\phi}] \tag{3.79}$$

The term in the parenthesis gives the symplectic form. It's inverse can be written formally as:

$$(\omega^{-1})^i_j = 2(sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}\delta^i_j. \tag{3.80}$$

This is an unusual operator, and in fact it is hard to compute with. What is the meaning of this operator? *Let us look at the equations of motion for the free theory in this coordinate system.*

We know the right inverse of the ∂ operator is $\frac{1}{2}sgn(x-y)$, since it gives a δ when it acts on the sgn function. We exploit this fact and write,

$$(\overleftarrow{\partial}_y sgn(x) - sgn(x)\overrightarrow{\partial}_y)\partial_u\phi = (-\partial_y^2 + m^2)\phi, \tag{3.81}$$

as:

$$\int dy\ \frac{1}{2}sgn(x-y)(\overleftarrow{\partial}_y sgn(y) - sgn(y)\overrightarrow{\partial}_y)\partial_u\phi = \int dy\ \frac{1}{2}sgn(x-y)(-\partial_y^2+m^2)\phi. \tag{3.82}$$

This becomes:

$$\begin{aligned}&\int dy\ \delta(x-y)sgn(y)\partial_u\phi(y) - \int dy\ \frac{1}{2}sgn(x-y)sgn(y)\partial_y\partial_u\phi \\ =\ &\frac{1}{2}\int dy\ sgn(x-y)(-\partial_y^2 + m^2)\phi.\end{aligned} \tag{3.83}$$

So the equation of motion is:

$$\partial_u\phi(x) = \frac{1}{2}\int dy\ sgn(x)sgn(x-y)sgn(y)\partial_y\partial_u\phi + \frac{1}{2}\int dy\ sgn(x)sgn(x-y)(-\partial_y^2+m^2)\phi \tag{3.84}$$

We should supply the proper boundary conditions.

3.4. Quantization

To second quantize the theory one has to know the annihilation and creation operators. Once we find these operators, we will introduce the normal ordering procedure to make sure that the products of the fields at the same point are well-defined.

Apart from the coupling with the gauge field, our problem is like the harmonic oscillator. First we will focus on a finite dimensional problem with constant symplectic and quadratic form.

We are going to make the quantization as in a real vector space. *We can do this since in our case the symplectic and quadratic form are real operators and the real and imaginary parts of the complex vectors decouple.* We can see this easily by taking our complex fields as:

$$\begin{aligned} \phi &= a + ib \\ \phi^\dagger &= a - ib \end{aligned} \tag{3.85}$$

Our action has the form

$$\int du\ (\phi^\dagger\omega\dot{\phi} - \phi^\dagger Q\phi) \tag{3.86}$$

and it can be expanded as

$$\int du\ \left[(a+ib)^\dagger\omega(\dot{a}+i\dot{b}) - (a+ib)^\dagger Q(a+ib)\right]. \tag{3.87}$$

The action is decoupled to an action of two real fields by using the anti-symmetry of ω and the symmetry of Q, yileding

$$\int du \left[a^\dagger \omega \dot{a} - a^\dagger Q a + b^\dagger \omega \dot{b} - b^\dagger Q b\right]. \tag{3.88}$$

Since a and b commute, the normal ordering of the fields can now be defined in a simple way:

$$: \phi^\dagger \phi :=: (a + ib)^\dagger (a + ib) :=: a^\dagger a : + : b^\dagger b : + i[a^\dagger b - b^\dagger a]. \tag{3.89}$$

At the end, we will subtract twice the commutator of the annihilation and creation operators from the normal ordered product, to get the correct vacuum energy, after we construct the quantized theory.

To start quantization, the first step is to look at the equations of motion:

$$\dot{x}^i = (\omega^{-1})^{ij} Q_{jk} x^k \tag{3.90}$$

where we have x^i real. For simplicity we work with a finite dimensional system, in our application the indices $(i, j, k, l...)$ will refer to both a set of continuous indices and a set of discrete indices.

The next step is to complexify the underlying vector space to get the oscillatory solutions. Indeed the oscillatory solutions are related to a complex structure. (This is not the original complex structure in our complex case). This quantization method is applied following the work of Rajeev and Bowick in [14].

The polar decomposition of tensor $\omega^{-1}Q$ will reveal a complex J and a positive K. And the equations of motion become

$$\dot{x}^i = J^i_k K^k_l x^l. \tag{3.91}$$

where

$$J^\tau J = 1. \tag{3.92}$$

Here the transpose is not the usual one but instead it is defined with respect to the metric defined by the quadratic form, Q as

$$J^\tau = Q^{-1} J^T Q. \tag{3.93}$$

And,

$$K^\tau = K, \tag{3.94}$$

$$K \geq 0. \tag{3.95}$$

It will be useful to define a new variable as

$$\tilde{\omega} = \omega^{-1} Q. \tag{3.96}$$

We can see that this operator is antisymmetric in the metric defined by the quadratic form such that:

$$\tilde{\omega}^\tau = -\tilde{\omega}, \tag{3.97}$$

since,

$$\begin{aligned} (\omega^{-1}Q)^\tau &= Q^{-1}(\omega^{-1}Q)^T Q \\ &= Q^{-1}Q^T(\omega^{-1})^T Q \\ &= Q^{-1}Q(-\omega^{-1})Q = -\omega^{-1}Q. \end{aligned} \tag{3.98}$$

If we use equation 3.97, we can solve for K and J in terms of $\tilde{\omega}$, giving

$$\begin{aligned} \tilde{\omega}^{\tau}\tilde{\omega} &= -\tilde{\omega}^2 && (3.99) \\ &= (JK)^{\tau}JK = K^{\tau}J^{\tau}JK = K^2, && (3.100) \end{aligned}$$

where we used the equalities in 3.92 and 3.94.
Thus, from equations 3.99 and 3.100 above, we get

$$K = (-\tilde{\omega}^2)^{\frac{1}{2}}, \tag{3.101}$$

and,

$$J = (-\tilde{\omega}^2)^{-\frac{1}{2}}\tilde{\omega}, \tag{3.102}$$

since,

$$JK = \tilde{\omega}. \tag{3.103}$$

One can check that

$$J^{\tau} = -(-\tilde{\omega}^2)^{-\frac{1}{2}}\tilde{\omega} = -J. \tag{3.104}$$

Thus;

$$J^2 = -1, \tag{3.105}$$

from equation 3.92.
So, J *defines a complex structure.* What follows after the construction of the complex structure is to project the coordinates into $\pm i$ eigenspaces to obtain complex coordinates. But it would be instructive to stop at this point and apply this procedure to the well-known Klein-Gordon field in the usual coordinate system.

3.4.1. Application to Klein-Gordon

The Hamiltonian of a Klein-Gordon field on $\mathbb{R}^3$ can be written as follows:

$$H = \int d^3x \, [\pi^2 + (\nabla\phi)^2 + m^2\phi^2]. \tag{3.106}$$

By partial integral we get the quadratic form,

$$\begin{aligned} H &= \int d^3x \, [\pi^2 + \phi(-\nabla^2 + m^2)\phi] && (3.107) \\ &= \int d^3x \, [\Phi^T Q \Phi]. && (3.108) \end{aligned}$$

Here we use a two component representation such that

$$\Phi = \begin{pmatrix} \phi \\ \partial_t\phi \end{pmatrix}. \tag{3.109}$$

We can read off the quadratic form as

$$Q = \begin{pmatrix} (-\nabla^2 + m^2) & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.110}$$

The symplectic form is;

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.111}$$

Thus

$$\begin{aligned}\tilde{\omega} = \omega^{-1}Q &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (-\nabla^2 + m^2) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ (-\nabla^2 + m^2) & 0 \end{pmatrix}. \end{aligned} \tag{3.112}$$

One can easily show that the transpose is

$$\tilde{\omega}^\tau = \begin{pmatrix} 0 & 1 \\ (-\nabla^2 + m^2) & 0 \end{pmatrix}. \tag{3.113}$$

The product of its transpose with itself would give

$$\tilde{\omega}^\tau \tilde{\omega} = \begin{pmatrix} (-\nabla^2 + m^2) & 0 \\ 0 & (-\nabla^2 + m^2) \end{pmatrix}. \tag{3.114}$$

This diagonal matrix helps us to have the square root in J easily, such that

$$\begin{aligned} J = (\tilde{\omega}^\tau \tilde{\omega})^{-\frac{1}{2}}\tilde{\omega} &= \begin{pmatrix} (-\nabla^2 + m^2)^{-\frac{1}{2}} & 0 \\ 0 & (-\nabla^2 + m^2)^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ (-\nabla^2 + m^2) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(-\nabla^2 + m^2)^{-\frac{1}{2}} \\ (-\nabla^2 + m^2)^{\frac{1}{2}} & 0 \end{pmatrix}. \end{aligned} \tag{3.115}$$

3.4.2. Back to Normal Ordering

We will now project our coordinates into $\pm i$ eigenspaces to obtain complex coordinates. Let us point out that, there is always a basis $\{e_\alpha\}$, and a dual basis $\{e^{*\alpha}\}$for which

$$J(e^{*\alpha}) = e^{*\alpha+n},$$
$$J(e^{*\alpha+n}) = -e^{*\alpha}. \tag{3.116}$$

where $\alpha = 1, 2, ..., n$, since J is a complex structure [16]. We may define the coordinates through

$$x^j = e^{*j}(x), \tag{3.117}$$

where x is a vector in our space. This can easily be verified, by looking at

$$e^{*j}(x) = e^{*j}(x^k e_k), \tag{3.118}$$

since the dual product gives a delta, $e^{*j}e_k = \delta^j_k$.
If we don't think about a basis, symbolically we can write the projections as

$$x^j = \frac{1}{2}(1 + iJ)^j_s x^s + \frac{1}{2}(1 - iJ)^j_s x^s. \tag{3.119}$$

We can verify that

$$\left[\frac{1}{2}(1 \pm iJ)\right]^2 = \frac{1}{2}(1 \pm iJ), \tag{3.120}$$

and

$$(1 + iJ)(1 - iJ) = 0. \tag{3.121}$$

Here the first part is the projection into the $+i$ eigenspace and the second is into $-i$, and j runs from 1, ..., 2n. Let's make a choice from the linearly independent dual basis elements as

$$\bar{f}^{*\alpha} = \frac{1}{2}(1+iJ)(e^{*\alpha}) \sim \bar{z}^{\alpha},$$
$$f^{*\alpha} = \frac{1}{2}(1-iJ)(e^{*\alpha}) \sim z^{\alpha}. \tag{3.122}$$

where $\alpha = 1, 2, ...n$.

So we think of dual basis as if they are the coordinates, this simplifies our writing. Now we should compute the Poisson brackets of these projected coordinates. We expect to have nonzero Poisson bracket only for $z, \bar{z}$ term. Let us first check that the others give zero.

$$\{\frac{1}{2}(\delta^i_k + iJ^i_k)x^k,\ \frac{1}{2}(\delta^j_l + iJ^j_l)x^l\}$$
$$= \frac{1}{4}\left[\{x^i,\ x^j\} + iJ^i_k\{x^k,\ x^j\} + iJ^j_l\{x^i,\ x^l\} - J^i_k J^j_l\{x^k,\ x^l\}\right] \tag{3.123}$$

Since the Poisson brackets give the inverse of the symplectic form, then

$$= \frac{1}{4}\left[(\omega^{-1})^{ij} + iJ^i_k(\omega^{-1})^{kj} + iJ^j_l(\omega^{-1})^{il} - J^i_k J^j_l(\omega^{-1})^{kl}\right]. \tag{3.124}$$

$$= \frac{1}{4}\left[(\omega^{-1})^{ij} + iJ^i_k J^k_r K^{rj} - iJ^j_l J^l_r K^{ri} - J^i_k J^j_l J^k_s K^{sl}\right] \tag{3.125}$$

Here $K^{ij} = K^i_s(Q^{-1})^{sj}$ and we used the equality of $\omega^{-1}Q$ to JK.

$$= \frac{1}{4}\left[(\omega^{-1})^{ij} - i\delta^i_r K^{rj} + i\delta^j_r K^{ri} + J^j_l \delta^i_s K^{sl}\right] \tag{3.126}$$
$$= \frac{1}{4}\left[(\omega^{-1})^{ij} - iK^{ij} + iK^{ji} + J^j_l K^{il}\right] \tag{3.127}$$
$$= \frac{1}{4}\left[(\omega^{-1})^{ij} - iK^{ij} + iK^{ij} + J^j_l K^{li}\right] \tag{3.128}$$
$$= \frac{1}{4}\left[(\omega^{-1})^{ij} + (\omega^{-1})^{ji}\right] \tag{3.129}$$
$$= 0 \tag{3.130}$$

where we used the symmetry property of K.

In the same manner,

$$\begin{aligned} & \{\frac{1}{2}(\delta^i_k - iJ^i_k)x^k,\ \frac{1}{2}(\delta^j_l - iJ^j_l)x^l\} \\ = & \ \frac{1}{4}\left[\{x^i,\ x^j\} - iJ^i_k\{x^k,\ x^j\} - iJ^j_l\{x^i,\ x^l\} - J^i_k J^j_l\{x^k,\ x^l\}\right] \end{aligned} \tag{3.131}$$

Since only the signs of the terms in the middle change and they cancel at the end the result will be the same.

Now we can compute $\{\bar{z}, z\}$:

$$\begin{aligned} & \{\frac{1}{2}(\delta^i_k + iJ^i_k)x^k,\ \frac{1}{2}(\delta^j_l - iJ^j_l)x^l\} \\ = & \ \frac{1}{4}\left[\{x^i,\ x^j\} + iJ^i_k\{x^k,\ x^j\} - iJ^j_l\{x^i,\ x^l\} + J^i_k J^j_l\{x^k,\ x^l\}\right] \end{aligned} \tag{3.132}$$

$$= \frac{1}{4}\left[(\omega^{-1})^{ij} + iJ^i_k(\omega^{-1})^{kj} - iJ^j_l(\omega^{-1})^{il} + J^i_k J^j_l(\omega^{-1})^{kl}\right] \tag{3.133}$$

$$= \frac{1}{4}\left[(\omega^{-1})^{ij} + iJ^i_k J^k_r K^{rj} + iJ^j_l J^l_r K^{ri} + J^i_k J^j_l J^k_s K^{sl}\right] \tag{3.134}$$

$$\begin{aligned} = & \ \frac{1}{4}\left[(\omega^{-1})^{ij} - i\delta^i_r K^{rj} - i\delta^j_r K^{ri} - J^j_l \delta^i_s K^{sl}\right] & (3.135) \\ = & \ \frac{1}{4}\left[(\omega^{-1})^{ij} - iK^{ij} - iK^{ji} - J^j_l K^{il}\right] & (3.136) \\ = & \ \frac{1}{4}\left[(\omega^{-1})^{ij} - iK^{ij} - iK^{ij} - J^j_l K^{li}\right] & (3.137) \\ = & \ \frac{1}{4}\left[2(\omega^{-1})^{ij} - 2iK^{ij}\right] = \frac{1}{2}\left[(\omega^{-1})^{ij} - iK^{ij}\right] & (3.138) \\ = & \ \frac{1}{2}\left[J^i_s K^{sj} - iK^{ij}\right] & (3.139) \\ = & \ \frac{1}{2}\left[J^i_s - i\delta^i_s\right] K^{sj} & (3.140) \\ = & \ \frac{1}{2}(-i)\left[\delta^i_s + iJ^i_s\right] K^{sj}. & (3.141) \end{aligned}$$

Alternatively we could switch the indices of the symplectic form in equation 3.138, to get

$$\frac{1}{2}\left[-(\omega^{-1})^{ji} - iK^{ij}\right] = \frac{1}{2}(-i)\left[\delta^j_s - iJ^j_s\right] K^{si}. \tag{3.142}$$

Thus

$$\frac{1}{2}(-i)\left[\delta^i_s + iJ^i_s\right] K^{sj} = \frac{1}{2}\left[-(\omega^{-1})^{ji} - iK^{ij}\right] = \frac{1}{2}(-i)\left[\delta^j_s - iJ^j_s\right] K^{si}. \tag{3.143}$$

We use the fact that the operator acting on K is the projection operator $(1+iJ)$. Now we have

$$(-i)\frac{1}{2}\left[\delta^i_s + iJ^i_s\right]\frac{1}{2}\left[\delta^s_{s'} + iJ^s_{s'}\right] K^{s'j} = (-i)\frac{1}{2}\left[\delta^i_s + iJ^i_s\right] K^{sj}. \tag{3.144}$$

From the equality of 3.143 we thus write

$$= (-i)\frac{1}{2}\left[\delta^i_s + iJ^i_s\right]\frac{1}{2}\left[\delta^j_{s'} - iJ^j_{s'}\right] K^{s's}. \tag{3.145}$$

Now we can rewrite this equation as

$$\{\frac{1}{2}(1+iJ)^i_l x^l,\ \frac{1}{2}(1-iJ)^j_k x^k\} = (-i)P_{+}{}^i_s P_{-}{}^j_{s'} K^{ss'}. \tag{3.146}$$

Indeed the meaning of this equation is clear if we construct a Hermitian product, using our original real inner product as

$$\begin{aligned} & [\frac{1}{2}\,(1+iJ)\,e^{*\alpha}](x)\frac{1}{2}[(1-iJ)\,e^{*\beta}](x)K(e_\alpha, e_\beta) \\ = \ & [\frac{1}{2}(x^\alpha + ix^{\alpha+n})\frac{1}{2}(x^\beta - ix^{\beta+n})K(e_\alpha, e_\beta) = H(x,x), \end{aligned} \tag{3.147}$$

or in a dual manner

$$\begin{aligned} & [\frac{1}{2}\,(1-iJ)\,e^{*i}](e_s)[\frac{1}{2}\,(1+iJ)\,e^{*j}](e_{s'})K(e^{*s}, e^{*s'}) \\ = \ & e^{*i}[\frac{1}{2}\,(1+iJ)\,e_s]e^{*j}[\frac{1}{2}\,(1-iJ)\,e_{s'}]K(e^{*s}, e^{*s'}). \end{aligned} \tag{3.148}$$

In a compact form using $x_i e^{*i} = x^*$ this could be expressed as

$$x^*[\frac{1}{2}\,(1+iJ)\,e_s]x^*[\frac{1}{2}\,(1-iJ)\,e_{s'}]K(e^{*s},e^{*s'}). \tag{3.149}$$

If we now use our special basis, $\{e_\alpha\}$, we get

$$\begin{aligned} &= \frac{1}{4}x^*(e_\alpha - ie_{\alpha+n})x^*(e_\beta + ie_{\beta+n})K(e^{*\alpha},e^{*\beta}) \\ &= \frac{1}{4}(x_\alpha - ix_{\alpha+n})(x_\beta + ix_{\beta+n})K(e^{*\alpha},e^{*\beta}) = H(x^*,x^*). \end{aligned} \tag{3.150}$$

So, if we specialize to our choice of coordinates, $\{z^\alpha, \bar{z}^\alpha\}$, the expression on the right is indeed a Hermitian inner product. We will denote this Hermitian form as $H_{\alpha\underline{\beta}}$ and it has the symmetry

$$H^*_{\alpha\underline{\beta}} = H_{\beta\underline{\alpha}}. \tag{3.151}$$

We will find the creation and annihilation operators in terms of our special choice of coordinates. By 3.119,

$$\frac{1}{2}\,(1+iJ)\,e^{*\alpha} + \frac{1}{2}\,(1-iJ)\,e^{*\alpha} = e^{*\alpha}. \tag{3.152}$$

The same is true for the second half, so

$$\frac{1}{2}\,(1+iJ)\,e^{*\alpha+n} + \frac{1}{2}\,(1-iJ)\,e^{*\alpha+n} = e^{*\alpha+n}. \tag{3.153}$$

We can extract the J from the $e^{*\alpha+n}$ term to get $e^{*\alpha}$ and act to the left to obtain

$$\frac{1}{2}\left(J+iJ^2\right)e^{*\alpha} + \frac{1}{2}\left(J-iJ^2\right)e^{*\alpha} = e^{*\alpha}. \tag{3.154}$$

By 3.116 and 3.105,

$$\frac{1}{2}\,(J-i)\,e^{*\alpha} + \frac{1}{2}\,(J+i)\,e^{*\alpha} = e^{*\alpha+n}. \tag{3.155}$$

Then

$$(-i)[\frac{1}{2}(1+iJ)\,e^{*\alpha}-\frac{1}{2}(1-iJ)\,e^{*\alpha}]=e^{*\alpha+n}. \tag{3.156}$$

Thus, we have the usual relation, such that

$$z^{\alpha}+\bar{z}^{\alpha}=x^{\alpha}, \tag{3.157}$$

$$(-i)(z^{\alpha}-\bar{z}^{\alpha})=x^{\alpha+n}. \tag{3.158}$$

Let us assume that we have made a special choice and found

$$\{\bar{z}^{\underline{\alpha}},z^{\beta}\}=(-i)H^{\underline{\alpha}\beta}. \tag{3.159}$$

Now we use the Dirac quantization rule and set

$$[a^{\alpha},a^{\dagger\beta}]=H^{\underline{\alpha}\beta}, \tag{3.160}$$

with

$$a^{\alpha} \quad = \quad H^{\underline{\alpha}\beta}\frac{\partial}{\partial z^{\beta}} \tag{3.161}$$

$$,a^{\dagger\beta} \quad = \quad z^{\beta}. \tag{3.162}$$

And we set,

$$a^{\alpha}|0\rangle=0. \tag{3.163}$$

Equation 3.160 creates a Fock space with positive norm elements such that

$$|\lambda\rangle=\lambda_{\alpha}a^{\dagger\alpha}|0\rangle, \tag{3.164}$$

$$\langle\lambda|\lambda\rangle = \lambda^*_\beta\lambda_\alpha H^{\beta\alpha} > 0. \tag{3.165}$$

We will demonstrate an alternative way to show that this is well-defined. a^α and $a^{\dagger\alpha}$ are hermitian conjugates under the inner product of $f(z^\alpha)$ and $g(z^\alpha)$, thus

$$\langle f|g\rangle = \int \Pi_\alpha dz^\alpha d\bar{z}^{\underline{\alpha}}\, e^{-z^\beta (H^{-1})_{\beta\underline{\alpha}}\bar{z}^{\underline{\alpha}}}\overline{f(z)}g(z). \tag{3.166}$$

A calculation reveals that

$$\langle f|H^{\underline{\gamma}\lambda}\partial_\lambda g\rangle = \langle z^\gamma f|g\rangle. \tag{3.167}$$

As a result we have

$$\langle f|a^{\dagger\gamma}g\rangle = \langle a^\gamma f|g\rangle. \tag{3.168}$$

We have a well-defined normal ordering rule

$$: a^\alpha a^{\dagger\beta} := a^{\dagger\beta}a^\alpha, \tag{3.169}$$

and no change for all the other combinations.

In general it will be not practical to switch to this special basis, but $\frac{1}{2}(1-iJ)$, $\frac{1}{2}(1+iJ)$ terms are respectively projections onto these subspaces and normal ordering brings a negative commutator at the end, so

$$\begin{aligned}[\frac{1}{2}\left((1+iJ)x\right)^i, \frac{1}{2}\left((1-iJ)x\right)^j] &= i\frac{1}{2}[(\omega^{-1})^{ij} - iK^{ij}] \\ &= \frac{1}{2}[i(\omega^{-1})^{ij} + K^{ij}]. \end{aligned} \tag{3.170}$$

So we can use this idea to set a normal ordering rule as

$$: \hat{x}^i\hat{x}^j := \hat{x}^i\hat{x}^j - \frac{1}{2}[i(\omega^{-1})^{ij} + K^{ij}]. \tag{3.171}$$

The discussion on projections imply that there is no reason to look for two different relations, our projections have all the desired properties. (So the net result will be the above commutator relations.)
For complex coordinates as in our case, we can thus postulate

$$:\hat{w}^{*i}\hat{w}^{j} := \hat{w}^{*i}\hat{w}^{j} - 2\frac{1}{2}[i(\omega^{-1})^{ij} + K^{ij}]. \tag{3.172}$$

It should be noted that in this equation the indices are assigned to a discrete set, and *from now on, we will use continuous indices as well as discrete ones; the continuous indices will be shown explicitly in parenthesis or as bra-kets and i, j, k, l will imply only discrete degrees of freedom.*

If we go back to our example of Klein-Gordon field we can state our normal ordering as

$$:\Phi^{k}(x)\Phi^{j}(y) := \Phi^{k}(x)\Phi^{j}(y) - \frac{1}{2}[i(\omega^{-1})^{kj}(x,y) + K^{k}_{s}(Q^{-1})^{sj}(x,y)]. \tag{3.173}$$

$$\begin{aligned} &= \begin{pmatrix} \phi(x)\phi(y) & \phi(x)\pi(y) \\ \pi(x)\phi(y) & \phi(x)\phi(y) \end{pmatrix} \\ &- \frac{1}{2}\begin{pmatrix} 0 & -i\delta(x-y) \\ i\delta(x-y) & 0 \end{pmatrix} \\ &- \frac{1}{2}\begin{pmatrix} \langle x|(-\nabla^2+m^2)^{-\frac{1}{2}}|y\rangle & 0 \\ 0 & \langle x|(-\nabla^2+m^2)^{\frac{1}{2}}|y\rangle \end{pmatrix} \end{aligned} \tag{3.174}$$

$$= \begin{pmatrix} \phi(x)\phi(y) & \phi(x)\pi(y) \\ \pi(x)\phi(y) & \phi(x)\phi(y) \end{pmatrix} \tag{3.175}$$

$$- \frac{1}{2}\begin{pmatrix} \langle x|(-\nabla^2+m^2)^{-\frac{1}{2}}|y\rangle & -i\delta(x-y) \\ i\delta(x-y) & \langle x|(-\nabla^2+m^2)^{\frac{1}{2}}|y\rangle \end{pmatrix}. \tag{3.176}$$

Let us now write the normal ordering in terms of our operators as

$$: \phi^{\dagger}_k(x)\phi^j(y) := \phi^{\dagger}_k(x)\phi^j(y) - [i(\omega^{-1})^j_k(x,y) + K^j_k(x,y)], \tag{3.177}$$

where

$$(\omega^{-1})^j_k(x,y) = \langle x|(\overleftarrow{\partial}\, sgn - sgn\, \overrightarrow{\partial}\,)^{-1}|y\rangle\delta^j_k. \tag{3.178}$$

Here *sgn* is

$$\langle x|sgn|y\rangle = sgn(x)\delta(x-y). \tag{3.179}$$

The system is diagonal in the color indices, hence we can write it as it is

$$K^j_k(x,y) = \langle x| \left([-(\omega^{-1}Q)^2]^{\frac{1}{2}}\right)^j_k |y\rangle. \tag{3.180}$$

There is no difference between upper and lower color indices for our purpose here. Thus

$$K^j_k(x,y) = \langle x| \left(-\left[(\overleftarrow{\partial}\, sgn - sgn\, \overrightarrow{\partial}\,)^{-1}\ (-\partial^2_x + m^2)\right]^2\right)^{\frac{1}{2}} |y\rangle\delta^j_k. \tag{3.181}$$

Then, transforming our expression,

$$K^{js} \equiv K^j_k(Q^{-1})^{ks}, \tag{3.182}$$

in the finite dimensional case to the infinite dimensional case only for the continuous indices, this becomes,

$$\left(-\left[(\overleftarrow{\partial}\, sgn - sgn\, \overrightarrow{\partial}\,)^{-1}\ (-\partial^2 + m^2)\right]^2\right)^{\frac{1}{2}} (-\partial^2 + m^2)^{-1}\delta^j_s. \tag{3.183}$$

We *keep the color index* down (s) in general, since the fields have special color indices attached to them. It is not so simple to evaluate these expressions. Only as symbols of operators we have some chance. We will not attack this question in the present work. From now on we will denote this expression as the vacuum expectation value $V(x,y)$.

$$V(x,y) \equiv \langle x| \left(- \left[(\overleftarrow{\partial}\, sgn - sgn\, \overrightarrow{\partial})^{-1} \, (-\partial^2 + m^2) \right]^2 \right)^{\frac{1}{2}} (-\partial^2 + m^2)^{-1} |y\rangle \tag{3.184}$$

And we have,

$$[\tilde{\phi}_i^\dagger(x), \tilde{\phi}^j(y)] = i\, \omega^{-1}(x,y) \delta_i^j. \tag{3.185}$$

3.5. Gauge Invariance Constraint

When we quantize the theory, we get a normal ordered expression for 3.38, since we have products of the fields at the same point. Thus a reduction at the quantum level should read

$$\hat{E} = \hat{e} + \frac{g^2}{2} \int_{-L}^{x} dy \; sgn(y) : \{\tilde{\phi}(\partial_y \tilde{\phi}^\dagger) - (\partial_y \tilde{\phi})\tilde{\phi}^\dagger\} : . \tag{3.186}$$

With this expression our *Quantum* Hamiltonian can be written as

$$\begin{aligned} \hat{H} = \frac{1}{2} \int dx \; : \; (\partial_x \tilde{\phi}^\dagger)(\partial_x \tilde{\phi}) \; : \; + \frac{1}{2} \int dx \; : \; m^2 \tilde{\phi}^\dagger \tilde{\phi} \; : \\ - \frac{1}{2g^2} \int dx \; Tr \left[\hat{e} + \frac{g^2}{2} \int_{-L}^{x} dy \; sgn(y) \; : \; \{\tilde{\phi}(\partial_y \tilde{\phi}^\dagger) - (\partial_y \tilde{\phi})\tilde{\phi}^\dagger\} \; : \right]^2 \end{aligned} \tag{3.187}$$

If we express our previous boundary condition (3.16), $\tilde{E}(+L) = q\tilde{E}(-L)q^{-1}$ we end up with a constraint,

$$\hat{e} + \frac{g^2}{2} \int_{-L}^{L} dy \; sgn(y) : \{\tilde{\phi}(\partial_y \tilde{\phi}^\dagger) - (\partial_y \tilde{\phi})\tilde{\phi}^\dagger\} := \widehat{qeq}^{-1}, \tag{3.188}$$

where $\widehat{qeq}^{-1}$ is taken as Weyl ordered.

This is a too strong condition in Quantum theory. We should impose this on physical states. Its meaning is clear if we define a global color operator as

$$\hat{Q} = \hat{e} - \widehat{qeq}^{-1} + \frac{g^2}{2}\int_{-L}^{L} dy\ sgn(y) : \{\tilde{\phi}(\partial_y\tilde{\phi}^\dagger) - (\partial_y\tilde{\phi})\tilde{\phi}^\dagger\} : . \tag{3.189}$$

which gives zero acting on any color invariant physical state. Shortly we will demonstrate that $\hat{Q}$ truly generates color invariance. Before that we should form the simplest mesonic variable, i.e. a color invariant physical state, in the fundamental representation. We start with the product of two fields at different points:

$$\phi^\dagger(x)\ \phi(y). \tag{3.190}$$

This product is not gauge invariant as it stands. We should check and see that the choice below is gauge invariant

$$\phi^\dagger(x)P\{e^{\int_x^y A_x dz}\}\phi(y). \tag{3.191}$$

Here the fields at these points are brought to the same point by a Parallel transport operator. When we transform the fields according to equation 3.14, we obtain

$$(\tilde{\phi}^\dagger(x)P\{e^{\int_{-L}^x A_x dz}\})P\{e^{\int_x^y A_x dz}\}(P\{e^{-\int_{-L}^y A_x dz}\}\tilde{\phi}(y)). \tag{3.192}$$

The path integrals add up to one due to the boundary conditions of the Wilson loop in 3.11. We could as well construct a more general state by having several loops such as

$$\phi^\dagger(x)P\{e^{\int_x^{y+2KL} A_x dz}\}\phi(y). \tag{3.193}$$

Making the transformation and expanding this expression, we obtain

$$\tilde{\phi}^\dagger(x)P\{e^{\int_{-L}^x A_x dz}\}P\{e^{\int_x^{-L} A_x dz}\}(P\{e^{\int_{-L}^L A_x dz}\})^K P\{e^{\int_L^y A_x dz}\}P\{e^{-\int_{-L}^y A_x dz}\}\tilde{\phi(y)}. \tag{3.194}$$

So the more general expression for the mesonic variable is as follows

$$\tilde{\phi}^{\dagger}_{\alpha}(x)(q^K)^{\alpha}_{\beta}\tilde{\phi}^{\beta}(y). \tag{3.195}$$

We should normal order this at the quantum level. These should be the general "mesonic" observables. Nevertheless, we will use even a more general expression for possible observables in our work, as we will see later on.

Now that we have constructed a color invariant physical state we should show that it commutes with our color operator as that would imply

$$\hat{Q}\left| \begin{smallmatrix} color \\ invariant \\ state \end{smallmatrix} \right\rangle = 0. \tag{3.196}$$

since that state is acting on the vacuum state and $\hat{Q}$ acting on vacuum would already give zero. So lets take a closer look at the commutation

$$[\hat{Q}, \tilde{\phi}^{\dagger}_{l}(z)(q^K)^{l}_{m}\tilde{\phi}^{m}(w)]. \tag{3.197}$$

We will calculate it separately for the bosonic part

$$[+\frac{g^2}{2}\int_{-L}^{L} dy\ \lambda^i_j : \tilde{\phi}^{\dagger}_i(y)(\overleftarrow{\partial}_y sgn(y) - sgn(y)\overrightarrow{\partial}_y)\tilde{\phi}^j(y) : ,\ \tilde{\phi}^{\dagger}_l(z)(q^K)^l_m\tilde{\phi}^m(w)]. \tag{3.198}$$

Where we used the fact that the normal ordered product in 3.189 is actually the symplectic form. λ^i_j is an element of the Lie algebra. Hence we generate an infinitesimal action.

$$\begin{aligned}
&= +\frac{g^2}{2}\int_{-L}^{L}\int_{-L}^{L} dydy'\ \lambda^i_j\tilde{\phi}^{\dagger}_i(y')\langle y'|(\overleftarrow{\partial}\, sgn - sgn\,\overrightarrow{\partial})|y\rangle \times \\
&[\tilde{\phi}^j(y),\ \tilde{\phi}^{\dagger}_l(z)](q^K)^l_m\tilde{\phi}^m(w) \\
&+\frac{g^2}{2}\int_{-L}^{L}\int_{-L}^{L} dydy'\ \lambda^i_j\langle y'|(\overleftarrow{\partial}\, sgn - sgn\,\overrightarrow{\partial})|y\rangle \times \\
&[\tilde{\phi}^{\dagger}_i(y'),\ \tilde{\phi}^m(w)]\tilde{\phi}^j(y)(q^K)^l_m\tilde{\phi}^{\dagger}_l(z)
\end{aligned} \tag{3.199}$$

Since the inverse of the symplectic form gives the Poisson brackets of the fields,

$$
\begin{aligned}
&= i\,\frac{g^2}{2}\int_{-L}^{L}\int_{-L}^{L} dydy'\ \lambda^i_j\tilde{\phi}^\dagger_i(y')\langle y'|(\overleftarrow{\partial}\,sgn - sgn\,\overrightarrow{\partial})|y\rangle\tilde{\phi}^\dagger_i(y')\times \\
&\quad \langle y|(\overleftarrow{\partial}\,sgn - sgn\,\overrightarrow{\partial})^{-1}|z\rangle\ \delta^j_l\ (q^K)^l_m\tilde{\phi}^m(w) \\
&+ i\,\frac{g^2}{2}\int_{-L}^{L}\int_{-L}^{L} dydy'\ \lambda^i_j\langle y'|(\overleftarrow{\partial}\,sgn - sgn\,\overrightarrow{\partial})|y\rangle\times \\
&\quad (-\langle w|(\overleftarrow{\partial}\,sgn - sgn\,\overrightarrow{\partial})^{-1}|y\rangle)\ \delta^m_i\ \tilde{\phi}^j(y)(q^K)^l_m\tilde{\phi}^\dagger_l(z)
\end{aligned}
\tag{3.200}
$$

$$
\begin{aligned}
&= +i\,\frac{g^2}{2}\int_{-L}^{L} dy'\ \lambda^i_j\tilde{\phi}^\dagger_i(y')\delta(y'-z)(q^K)^j_m\tilde{\phi}^m(w) \\
&+ i\,\frac{g^2}{2}\int_{-L}^{L} dy\ \lambda^i_j(-\delta(y-w))(q^K)^l_i\tilde{\phi}^j(y)\tilde{\phi}^\dagger_l(z)
\end{aligned}
\tag{3.201}
$$

The delta functions kill the integrals, and we are left with

$$
\begin{aligned}
&= \quad i\,\frac{g^2}{2} :\ \tilde{\phi}^\dagger_i(z)\lambda^i_j(q^K)^j_m\tilde{\phi}^m(w) : -i\,\frac{g^2}{2} :\ \tilde{\phi}^\dagger_l(z)(q^K)^l_i\lambda^i_j\tilde{\phi}^j(w) : \\
&- \quad iV(z,w)\delta^m_i\lambda^i_j(q^K)^j_m + iV(z,w)\delta^j_l\lambda^i_j(q^K)^l_i
\end{aligned}
\tag{3.202}
$$

$$
= \quad +i\,\frac{g^2}{2} :\ \tilde{\phi}^\dagger_i(z)\lambda^i_j(q^K)^j_m\tilde{\phi}^m(w) : -i\,\frac{g^2}{2} :\ \tilde{\phi}^\dagger_l(z)(q^K)^l_i\lambda^i_j\tilde{\phi}^j(w) : . \tag{3.203}
$$

Now we can compute the rest:

$$
[Tr\lambda\hat{e} - Tr\lambda\widehat{qeq}^{-1},\ :\ \tilde{\phi}^\dagger q^K\tilde{\phi}\ :], \tag{3.204}
$$

where the wide hats represent Weyl quantization. So they are symmetrized and we can use cyclic property of the trace operator such that

$$
Tr\widehat{\lambda qeq}^{-1} = Tr\hat{q}^{-1}\lambda\hat{q}\hat{e}. \tag{3.205}
$$

$$
[Tr\lambda\hat{e},\ :\ \tilde{\phi}^\dagger q^K\tilde{\phi}\ :] = i\,g^2\sum_R\ :\ \tilde{\phi}^\dagger q^R(q\lambda)q^{K-R-1}\tilde{\phi}\ :. \tag{3.206}
$$

Since the operator only acts on the variable q. And,

$$[Tr\lambda\widehat{qe}q^{-1},\ :\ \tilde{\phi}^{\dagger}q^{K}\tilde{\phi}\ :] = i\ g^{2}\sum_{R}\ :\ \tilde{\phi}^{\dagger}q^{R}(qq^{-1}\lambda q)q^{K-R-1}\tilde{\phi}\ :. \tag{3.207}$$

Thus

$$[Tr\lambda\hat{e} - Tr\lambda\widehat{qe}q^{-1},\ \tilde{\phi}^{\dagger}q^{K}\tilde{\phi}] = i\ g^{2}\sum_{R}\ :\ \tilde{\phi}^{\dagger}q^{R}[q,\ \lambda]q^{K-R-1}\tilde{\phi}\ :. \tag{3.208}$$

When we add this result with 3.202 we get zero as expected.

3.6. Large N Limit

The basic idea of large N_c limit is to write everything in terms of color invariant observables. In the limit of large N_c, only the color invariant operators survive and the expectation values of color invariant operators split as a product up to order $\frac{1}{N_c}$ corrections, so

$$< AB > = < A >< B > + O(\frac{1}{N_c}). \tag{3.209}$$

The equation above implies that the set of color invariant operators becomes classical as N_c goes to ∞. Thus all color invariant operators should be representable as classical observables in the large N_c limit. This is an idea of Migdal [17], see also Witten [15].

We had found the color invariant physical states to be of the form in equation 3.195. A more general expression would be of the form

$$:\tilde{\phi}_{i}^{\dagger}(x)[\hat{q}^{K_1}\hat{e}^{S_1}\hat{q}^{K_2}\hat{e}^{S_2}...\hat{q}^{K_m}]^{i}_{j}\tilde{\phi}_{j}(y):. \tag{3.210}$$

In the large N_c limit there should be classical observables defined to be

$$N(x,\ y\mid K_1, S_1, ..., K_m)\ = \lim_{N_c\to\infty}\frac{1}{N_c}:\tilde{\phi}_{i}^{\dagger}(x)[\hat{q}^{K_1}\hat{e}^{S_1}\hat{q}^{K_2}\hat{e}^{S_2}...\hat{q}^{K_m}]^{i}_{j}\tilde{\phi}^{j}(y):. \tag{3.211}$$

We note that its complex conjugate is not an independent variable, but satisfies the relation

$$
\begin{aligned}
N^*(x,\ y \mid K_1, S_1, ..., K_m) &= \lim_{N_c\to\infty} \frac{1}{N_c}\Big(: \tilde{\phi}_i^\dagger(x)[\hat{q}^{K_1}\hat{e}^{S_1}\hat{q}^{K_2}\hat{e}^{S_2}...\hat{q}^{K_m}]_i^j\tilde{\phi}_i(y)\Big)^\dagger : \\
&= N(y,\ x \mid -K_m, S_{n-1}, ..., -K_1)(-1)^{S_1+...+S_m}, \quad (3.212)
\end{aligned}
$$

since,

$$
\hat{e}^\dagger = -\hat{e}. \quad (3.213)
$$

When we apply our dynamical boundary conditions, namely

$$
\tilde{\phi}(L) = q_j^i\tilde{\phi}^j(-L), \quad (3.214)
$$

the N observable satisfies the relation

$$
: \tilde{\phi}_i^\dagger(x)[q^{K_1}\hat{e}^{S_1}q^{K_2}\hat{e}^{S_2}...q^{K_m}]_i^j\tilde{\phi}_i(L) : = : \tilde{\phi}_i^\dagger(x)[q^{K_1}\hat{e}^{S_1}q^{K_2}\hat{e}^{S_2}...q^{K_m+1}]_i^j\tilde{\phi}_i(-L) : . \quad (3.215)
$$

In the large N_c limit this condition should be postulated as

$$
N(x,\ L \mid K_1, S_1, ..., K_m) = N(x,\ -L \mid K_1, S_1, ..., K_m+1). \quad (3.216)
$$

Color invariance implies a constraint on the bilinears of the theory, due to our requirement

$$
\hat{Q}|physical\rangle = 0. \quad (3.217)
$$

Let us now compute the following,

$$
\int dydy' \ : \tilde{\phi}_i^\dagger(x)(q^K)_j^i\tilde{\phi}^j(y') : \ \omega(y', y) \ : \tilde{\phi}_i^\dagger(y)(q^K)_j^i\tilde{\phi}^j(z) : . \quad (3.218)
$$

We expand the normal orderings by subtracting the vacuum energies

$$\int dydy' \left[\tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y') - V(x,y')\delta^j_i(q^K)^i_j\right]\omega(y',y)\times$$
$$\left[\tilde{\phi}_s^\dagger(x)(q^K)^s_p\tilde{\phi}^p(y') - V(y,z)\delta^s_p(q^L)^s_p\right]. \tag{3.219}$$

Collecting the cross terms, we obtain

$$-\int dydy'\ V(x,y')(q^K)^i_i\omega(y',y)\ :\tilde{\phi}_s^\dagger(y)(q^L)^s_p\tilde{\phi}^p(z)\ :$$
$$-\int dydy' V(x,y')\omega(y',y)V(y,z)\delta^p_s(q^L)^s_p(q^L)^i_i$$
$$-\int dydy'\ V(y,z)(q^L)^s_s\ :\tilde{\phi}_i^\dagger(y)(q^K)^i_j\tilde{\phi}^j(y')\ :\omega(y',y)$$
$$-\int dydy'\omega(y',y)V(y,z)V(x,y')\delta^j_i(q^K)^i_j(q^L)^s_s. \tag{3.220}$$

Also adding the product of the vacuum energies,

$$-\int dydy'\ (Trq^K)V(x,y')\omega(y',y)\ :\tilde{\phi}_s^\dagger(y)(q^L)^s_p\tilde{\phi}^p(z)\ :$$
$$-\int dydy'\ (Trq^L)V(y,z)\omega(y',y)\ :\tilde{\phi}_i^\dagger(y)(q^K)^i_j\tilde{\phi}^j(y')\ :$$
$$-\int dydy'\ (Trq^L)(Trq^K)V(x,y')\omega(y',y)V(y,z). \tag{3.221}$$

Now let us look at the other product coming from 3.219:

$$\int dydy'\ \tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}_s^\dagger(y)(q^L)^s_p\tilde{\phi}^p(z). \tag{3.222}$$

Here we will change the order of $\tilde{\phi}_s^\dagger(y)$ and $\tilde{\phi}^p(z)$ inserting their commutation relation which brings the symplectic form as

$$\int dydy'\ \tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\left[\tilde{\phi}^p(z)\tilde{\phi}_s^\dagger(y) - i\ \omega^{-1}(y,z)\delta^p_s\right](q^L)^s_p. \tag{3.223}$$

If we expand, it becomes

$$\begin{aligned}&\int dydy'\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^p(z)\tilde{\phi}^{\dagger}_s(y)(q^L)^s_p\\ -i&\int dydy'\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\omega^{-1}(y,z)\delta^p_s(q^L)^s_p.\end{aligned} \tag{3.224}$$

and as a result

$$\begin{aligned}&\int dydy'\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^p(z)\tilde{\phi}^{\dagger}_s(y)(q^L)^s_p\\ &\quad -i\int dydy'\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(y')\delta(y'-z)\delta^p_s(q^L)^s_p.\end{aligned} \tag{3.225}$$

Hence,

$$\begin{aligned}\int\quad & dydy'\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^p(z)\tilde{\phi}^{\dagger}_s(y)(q^L)^s_p\\ &\qquad -i\ \tilde{\phi}^{\dagger}_i(x)(q^K)^i_j\tilde{\phi}^j(z)(q^L)^s_s.\end{aligned} \tag{3.226}$$

Now the first term is

$$\tilde{\phi}^{\dagger}_i(x)(q^K)^i_j(q^L)^s_p\tilde{\phi}^p(z)\int dydy'\ \tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^{\dagger}_s(y) \tag{3.227}$$

The integral is converted to normal ordering as

$$\tilde{\phi}^{\dagger}_i(x)(q^K)^i_j(q^L)^s_p\tilde{\phi}^p(z)\left[\int dydy'\ :\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^{\dagger}_s(y): +\int dydy'\ \delta^j_s V(y',y)\omega(y,y')\right]. \tag{3.228}$$

The vacuum energy term gives $tr(V\omega)$, an infinite dimensional trace, and $\tilde{\phi}^{\dagger}_i(x)(q^K)^i_j(q^L)^j_p\tilde{\phi}^p(z)$ is of smaller order in $\frac{1}{N_c}$. Thus we drop this term as $N_c\to\infty$. The symplectic term in 3.228 gives the following on color invariant states:

$$\int dydy'\ :\tilde{\phi}^j(y')\omega(y,y')\tilde{\phi}^{\dagger}_s(y): +\frac{1}{g^2}\hat{e}^j_s-\frac{1}{g^2}(\widehat{qeq^{-1}})^j_s=0. \tag{3.229}$$

Then equation 3.227 gives

$$\tilde{\phi}_i^\dagger(x)(q^K)^i_j(q^L)^s_p\tilde{\phi}^p(z)\left[\frac{1}{g^2}\hat{e}^j_s - \frac{1}{g^2}(q\hat{e}q^{-1})^j_s\right]. \tag{3.230}$$

The $\frac{1}{g^2}$ will eat up one of the $\frac{1}{N_c}$ terms:

$$g^2N_c = \tilde{g}^2 \ as\ N_c \to \infty. \tag{3.231}$$

So,

$$-\frac{1}{\tilde{g}^2}\tilde{\phi}_i^\dagger(x)(q^K\hat{e}q^L)^i_p\tilde{\phi}^p(z) + \frac{1}{\tilde{g}^2}\tilde{\phi}_i^\dagger(x)(q^{K+1}\hat{e}q^{L-1})^i_p\tilde{\phi}^p(z). \tag{3.232}$$

Now let us normal order these terms again to obtain

$$\begin{aligned}&-\frac{1}{\tilde{g}^2}\ :\ \tilde{\phi}_i^\dagger(x)(q^K\hat{e}q^L)^i_p\tilde{\phi}^p(z)\ :\ + \frac{1}{\tilde{g}^2}\ :\ \tilde{\phi}_i^\dagger(x)(q^{K+1}\hat{e}q^{L-1})^i_p\tilde{\phi}^p(z)\ :\\ &-\frac{1}{\tilde{g}^2}\delta^p_i V(x,z)(q^K\hat{e}q^L)^i_p + \frac{1}{\tilde{g}^2}V(x,z)\delta^p_i(q^{K+1}\hat{e}q^{L-1})^i_p.\end{aligned} \tag{3.233}$$

$$\begin{aligned}= \ &-\frac{1}{\tilde{g}^2}\ :\ \tilde{\phi}_i^\dagger(x)(q^K\hat{e}q^L)^i_p\tilde{\phi}^p(z)\ :\ + \frac{1}{\tilde{g}^2}\ :\ \tilde{\phi}_i^\dagger(x)(q^{K+1}\hat{e}q^{L-1})^i_p\tilde{\phi}^p(z)\ :\\ &-\frac{1}{\tilde{g}^2}V(x,z)Tr(q^K\hat{e}q^L) + \frac{1}{\tilde{g}^2}V(x,z)Tr(q^{K+1}\hat{e}q^{L-1}).\end{aligned} \tag{3.234}$$

Let us look closer at the trace in the last term, namely

$$Tr(q^{K+1}\hat{e}q^{L-1}) = (q^K)^i_s q^s_m e^m_r (q^{L-1})^r_i. \tag{3.235}$$

We change the places of the middle terms by the commutation relation, and we have

$$(q^K)^i_s\left([q^s_m, e^m_r] + e^s_m q^m_r\right)(q^{L-1})^r_i. \tag{3.236}$$

This becomes

$$-(q^K)^i_s q^m_m \delta^s_r (q^{L-1})^r_i + (q^K)^i_s e^s_m q^m_r (q^{L-1})^r_i = -q^m_m (q^K)^i_s (q^{L-1})^s_i + (q^K)^i_s e^s_m (q^L)^m_i . \quad (3.237)$$

The last term brings $Tr(q^K \hat{e} q^L)$ and cancels with the other vacuum energy term in the equation 3.234. We are left with an extra term

$$-TrqTr(q^{K+L-1}). \quad (3.238)$$

With this term, equation 3.234 becomes

$$\begin{aligned} = \quad & -\frac{1}{\tilde{g}^2} \ : \ \tilde{\phi}^\dagger_i(x)(q^K \hat{e} q^L)^i_p \tilde{\phi}^p(z) \ : \ + \frac{1}{\tilde{g}^2} \ : \ \tilde{\phi}^\dagger_i(x)(q^{K+1} \hat{e} q^{L-1})^i_p \tilde{\phi}^p(z) \ : \\ & -\frac{1}{\tilde{g}^2} V(x,z) TrqTr(q^{K+L-1}). \end{aligned} \quad (3.239)$$

Now we are ready to write the constraint in terms of the large N_c variables, thus

$$\begin{aligned} & \int \ dy'dy \ N(x,y'|K)\omega(y',y)N(y,z|L) \\ & +\frac{1}{N_c} Trq^K \int \ dy'dy \ V(x,y')\omega(y',y)N(y,z|L) \\ & +\frac{1}{N_c} Trq^L \int \ dy'dy \ N(x,y'|K)\omega(y',y)V(y,z) \\ & +\frac{1}{N_c} Trq^L \frac{1}{N_c} Trq^K \int \ dy'dy \ V(x,y')\omega(y',y)V(y,z) \\ & +i \ \frac{1}{N_c} Tr(q^L) \ [: \tilde{\phi}^\dagger_i(x)(q^K)^i_j \tilde{\phi}^j(z) : +V(x,z)\delta^j_i (q^K)^i_j] \\ = \quad & -\frac{1}{\tilde{g}^2}\frac{1}{N_c} \ : \ \tilde{\phi}^\dagger_i(x)(q^K \hat{e} q^L)^i_p \tilde{\phi}^p(z) \ : \\ & +\frac{1}{\tilde{g}^2}\frac{1}{N_c} \ : \ \tilde{\phi}^\dagger_i(x)(q^{K+1} \hat{e} q^{L-1})^i_p \tilde{\phi}^p(z) \ : \\ & -\frac{1}{\tilde{g}^2} V(x,z) TrqTr(q^{K+L-1}). \end{aligned} \quad (3.240)$$

Here,

$$N(x,z|K,1,L) = \lim_{N_c \to \infty} \frac{1}{N_c} \ : \ \tilde{\phi}^\dagger_i(x)(q^K \hat{e} q^L)^i_p \tilde{\phi}^p(z) \ : . \quad (3.241)$$

And

$$N(x,z|K+1,1,L-1)=\lim_{N_c\to\infty}\frac{1}{N_c}\ :\ \tilde{\phi}^{\dagger}_i(x)(q^{K+1}\hat{e}q^{L-1})^i_p\tilde{\phi}^p(z)\ :. \tag{3.242}$$

And if we define

$$Q(K)=\lim_{N_c\to\infty}\frac{1}{N_c}\ Trq^K, \tag{3.243}$$

then

$$\begin{aligned}
&\int dy'dy\ N(x,y'|K)\omega(y',y)N(y,z|L)\\
&+Q(K)\int dy'dy\ V(x,y')\omega(y',y)N(y,z|L)\\
&+Q(L)\int dy'dy\ N(x,y'|k)\omega(y',y)V(y,z)\\
&+Q(K)Q(L)\int dy'dy\ V(x,y')\omega(y',y)V(y,z)\\
&+i\ Q(L)\ [N(x,z|L)+V(x,z)Q(K)]\\
&+Q(1)Q(K+L-1)V(x,z)\\
=\ &\frac{1}{\tilde{g}^2}N(x,z|K+1,1,L-1)-\frac{1}{\tilde{g}^2}N(x,z|K,1,L).
\end{aligned} \tag{3.244}$$

We see that the constraint is not a classical algebra. There are similar constraints for other variables as well, but we will not use them in this thesis.

Now we will construct the classical Hamiltonian by dividing our Quantum expression in equation 3.187 by N_c, then

$$\begin{aligned}
\frac{H}{N_c}=&\frac{1}{2N_c}\int dx\ :\ \tilde{\phi}^{\dagger}(-\partial_x)^2\tilde{\phi}\ :\ +\frac{1}{2N_c}\int dx\ :\ m^2\tilde{\phi}^{\dagger}\tilde{\phi}\ :\\
&-\frac{1}{2g^2N_c}\int dx\ Tr\left[e+\frac{g^2}{2}\int_{-L}^{x}dy\ sgn(y)\ :\ \{\tilde{\phi}(\partial_y\tilde{\phi}^{\dagger})-(\partial_y\tilde{\phi})\tilde{\phi}^{\dagger}\}\ :\right]^2\\
&.
\end{aligned} \tag{3.245}$$

We will try to express it in terms of our classical N variables. The first two terms can be expressed easily but the third term should be written explicitly as

$$\frac{H}{N_c} = \frac{1}{2}\int \lim_{y\to x} dx\,[-\partial_y^2 N(x,y)] + \frac{1}{2}\int dx\; m^2 N(x,x) - \frac{1}{2g^2N_c}\int dx\; Tre^2$$
$$- \frac{1}{2g^2N_c}\int dx\; Tre\,\frac{g^2}{2}\int_{-L}^{x} dy\; sgn(y)\;:\;\{\tilde{\phi}(\partial_y\tilde{\phi}^\dagger) - (\partial_y\tilde{\phi})\tilde{\phi}^\dagger\}\;:$$
$$- \frac{1}{2g^2N_c}\int dx Tr\frac{g^4}{4}\int_{-L}^{x} dy\; sgn(y)\;:\;\{\tilde{\phi}(\partial_y\tilde{\phi}^\dagger) - (\partial_y\tilde{\phi})\tilde{\phi}^\dagger\}\;:$$
$$\int_{-L}^{x} dy'\; sgn(y')\;:\;\{\tilde{\phi}(\partial_y'\tilde{\phi}^\dagger) - (\partial_y'\tilde{\phi})\tilde{\phi}^\dagger\}\;:\;. \tag{3.246}$$

Then

$$H = \frac{1}{2}\int \lim_{y\to x} dx\,[-\partial_y^2 N(x,y)] + \frac{1}{2}\int dx\; m^2 N(x,x) - \frac{1}{2g^2N_c}\int dx\; Tre^2$$
$$- \frac{1}{2g^2N_c}\int dx\; e^i_j\,\frac{g^2}{2}\int_{-L}^{x} dy\; sgn(y)\;:\;\{\tilde{\phi}^j(\partial_y\tilde{\phi}^\dagger)_i - (\partial_y\tilde{\phi})^j\tilde{\phi}_i^\dagger\}\;:$$
$$- \frac{g^2}{8N_c}\int dx\int_{-L}^{x} dy\; sgn(y)\;:\;\{\tilde{\phi}^i(\partial_y\tilde{\phi}^\dagger)_j - (\partial_y\tilde{\phi})^i\tilde{\phi}_j^\dagger\}\;:$$
$$\int_{-L}^{x} dy'\; sgn(y')\;:\;\{\tilde{\phi}^j(\partial_y'\tilde{\phi}^\dagger)_i - (\partial_y'\tilde{\phi})^j\tilde{\phi}_i^\dagger\}\;:\;. \tag{3.247}$$

To regularize this Hamiltonian we add another normal ordering, then

$$H = \frac{1}{2}\int \lim_{y\to x} dx\,[-\partial_y^2 N(x,y)] + \frac{1}{2}\int dx\; m^2 N(x,x) - \frac{1}{2\tilde{g}^2}\int dx\; Tre^2$$
$$- \frac{1}{2g^2N_c}\int dx\,\frac{g^2}{2}\int_{-L}^{x} dy\; sgn(y)\;:\;\{\tilde{\phi}^j e^i_j(\partial_y\tilde{\phi}^\dagger)_i - (\partial_y\tilde{\phi})^j e^i_j\tilde{\phi}_i^\dagger\}\;:$$
$$- \frac{g^2}{8N_c}\int dx\int_{-L}^{x} dy\; sgn(y)\int_{-L}^{x} dy'\; sgn(y')\{\;:\;\tilde{\phi}^i(y)(\partial_y\tilde{\phi}^\dagger)_j\tilde{\phi}^j(y')(\partial_{y'}\tilde{\phi}^\dagger)_i\;:$$
$$-\;:\;\tilde{\phi}^i(y)(\partial_y\tilde{\phi}^\dagger)_j(\partial_{y'}\tilde{\phi})^j\tilde{\phi}_i^\dagger(y')\;:\;-\;:\;(\partial_y\tilde{\phi})^i\tilde{\phi}_j^\dagger(y)\tilde{\phi}^j(y')(\partial_{y'}\tilde{\phi}^\dagger)_i\;:$$
$$+\;:\;\partial_y\tilde{\phi}^i\tilde{\phi}_j^\dagger(y)\tilde{\phi}^j\partial_y\tilde{\phi}_i^\dagger\;:\}. \tag{3.248}$$

By reorganizing we obtain

$$
\begin{aligned}
H = & \frac{1}{2}\int \lim_{y\to x} dx\ [-\partial_y^2 N(x,y)] + \frac{1}{2}\int\ dx\ m^2 N(x,x) - \frac{1}{2\tilde{g}^2}\int\ dx\ Tre^2 \\
& - \frac{1}{4N_c}\int\ dx\ \int_{-L}^{x} dy\ sgn(y)\ :\ \lim_{y'\to y}\{\partial_y(\tilde{\phi}^j(y)e^i_j\tilde{\phi}^\dagger_i(y')) - \partial_{y'}(\tilde{\phi}^j(y')e^i_j\tilde{\phi}^\dagger_i(y))\}\ : \\
& - \frac{g^2}{8N_c}\int\ dx\int_{-L}^{x} dy\ sgn(y)\int_{-L}^{x} dy'\ sgn(y')\{\ :\ \partial_{y'}(\tilde{\phi}^\dagger_i(y')\tilde{\phi}^i(y))\partial_y(\tilde{\phi}^\dagger_j(y)\tilde{\phi}^j(y'))\ : \\
& -\ :\ \tilde{\phi}^\dagger_i(y')\tilde{\phi}^i(y)\partial_y\partial_{y'}(\tilde{\phi}^\dagger_j(y)\tilde{\phi}^j(y'))\ :\ -\ :\ \partial_{y'}\partial_y(\tilde{\phi}^\dagger_i(y')\tilde{\phi}^i(y))\tilde{\phi}^\dagger_j(y)\tilde{\phi}^j(y')\ : \\
& +\ :\ \partial_{y'}(\tilde{\phi}^\dagger_i(y')\tilde{\phi}^i(y))\partial_y(\tilde{\phi}^\dagger_j(y)\tilde{\phi}^j(y'))\ :\}.
\end{aligned} \tag{3.249}
$$

Eventually expressing in terms of our classical variables, our large-N_c Hamiltonian becomes

$$
\begin{aligned}
H = & \frac{1}{2}\int dx \lim_{y\to x} [-\partial_y^2 N(x,y)] + \frac{1}{2}\int\ dx\ m^2 N(x,x) - \frac{1}{2\tilde{g}^2}\int\ dx\ Tre^2 \\
& - \frac{1}{4}\int\ dx\ \int_{-L}^{x} dy\ sgn(y) \lim_{y'\to y}\{\partial_y N(y,y'|0,1) - \partial_{y'}N(y',y|0,1)\} \\
& - \frac{\tilde{g}^2}{8}\int\ dx\int_{-L}^{x} dy\ sgn(y)\int_{-L}^{x} dy'\ sgn(y')\{\partial_{y'}N(y',y)\partial_y N(y,y') \\
& - N(y',y)\partial_y\partial_{y'}N(y,y') - N(y,y')\partial_{y'}\partial_y N(y',y) \\
& + \partial_{y'}N(y',y)\partial_y N(y,y')\}.
\end{aligned} \tag{3.250}
$$

3.7. Poisson Brackets of the Classical Variables

We will check the Poisson brackets of our classical variables. We will first compute

$$
[:\tilde{\phi}^\dagger_i(x)(q^K)^i_j\tilde{\phi}^j(y):\ ,\ :\tilde{\phi}^\dagger_m(z)(q^L)^m_n\tilde{\phi}^n(w):]. \tag{3.251}
$$

Then, the Poisson bracket will follow as

$$
\{N(x,y|\ K), N(z,w|\ L)\} = -iN_c[N(x,y|\ K), N(z,w|\ L)], \tag{3.252}
$$

as $N_c \to \infty$.

We start by subtracting the vacuum energies as

$$\left[\left(\tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y) - V(x-y)\delta_i^j(q^K)^i_j\right), \left(\tilde{\phi}_m^\dagger(z)(q^L)^m_n\tilde{\phi}^n(w) - V(z-w)\delta_m^n(q^L)^m_n\right)\right]. \tag{3.253}$$

Explicitly,

$$\begin{aligned} & \left[\tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y), \tilde{\phi}_m^\dagger(z)(q^L)^m_n\tilde{\phi}^n(w)\right] \\ + & \left[V(x-y)\delta_i^j(q^K)^i_j, V(z-w)\delta_m^n(q^L)^m_n\right] \\ - & \left[\tilde{\phi}_i^\dagger(x)(q^K)^i_j\tilde{\phi}^j(y), V(z-w)\delta_m^n(q^L)^m_n\right] \\ - & \left[V(x-y)\delta_i^j(q^K)^i_j, \tilde{\phi}_m^\dagger(z)(q^L)^m_n\tilde{\phi}^n(w)\right]. \end{aligned} \tag{3.254}$$

Only the first term brings non-commuting results

$$\tilde{\phi}_m^\dagger(z)(q^L)^m_n\left[\tilde{\phi}_i^\dagger(x), \tilde{\phi}^n(w)\right](q^K)^i_j\tilde{\phi}^j(y) + \tilde{\phi}_i^\dagger(x)(q^K)^i_j\left[\tilde{\phi}^j(y), \tilde{\phi}_m^\dagger(z)\right](q^L)^m_n\tilde{\phi}^n(w) \tag{3.255}$$

The commutations give the symplectic forms, then

$$= \tilde{\phi}_m^\dagger(z)(q^L)^m_n\omega^{-1}(x,w)\delta_i^n(q^K)^i_j\tilde{\phi}^j(y) - \tilde{\phi}_i^\dagger(x)(q^K)^i_j\omega^{-1}(y,z)\delta_m^j(q^L)^m_n\tilde{\phi}^n(w) \tag{3.256}$$

$$\begin{aligned} = & :\tilde{\phi}_m^\dagger(z)(q^L)^m_n\omega^{-1}(x,w)\delta_i^n(q^K)^i_j\tilde{\phi}^j(y): \; - \; :\tilde{\phi}_i^\dagger(x)(q^K)^i_j\omega^{-1}(y,z)\delta_m^j(q^L)^m_n\tilde{\phi}^n(w): \\ + & V(z-y)\delta_m^j(q^L)^m_n\omega^{-1}(x,w)\delta_i^n(q^K)^i_j - V(x-w)\delta_i^n(q^K)^i_j\omega^{-1}(y,z)\delta_m^j(q^L)^m_n. \end{aligned} \tag{3.257}$$

Here we imposed normal ordering on the products of the fields. Then

$$\begin{aligned} = & :\tilde{\phi}_m^\dagger(z)(q^L)^m_n\omega^{-1}(x,w)(q^K)^n_j\tilde{\phi}^j(y): \; - \; :\tilde{\phi}_i^\dagger(x)(q^K)^i_j\omega^{-1}(y,z)(q^L)^j_n\tilde{\phi}^n(w): \\ + & V(z-y)(q^L)^j_n\omega^{-1}(x,w)(q^K)^n_j - V(x-w)(q^K)^n_j\omega^{-1}(y,z)(q^L)^j_n. \end{aligned} \tag{3.258}$$

$$\begin{aligned} = & :\tilde{\phi}_i^\dagger(z)(q^Lq^K)^m_j\tilde{\phi}^j(y): \; \omega^{-1}(x,w) - \; :\tilde{\phi}_i^\dagger(x)(q^Lq^K)^i_n\tilde{\phi}^n(w): \; \omega^{-1}(y,z) \\ + & tr(q^{L+K})V(z-y)\omega^{-1}(x,w) - tr(q^{L+K})V(x-w)\omega^{-1}(y,z). \end{aligned} \tag{3.259}$$

The result can be written as

$$
\begin{aligned}
& \{N(x,y|\ K), N(z,w|\ L)\} = \\
+ \quad & \left[\omega^{-1}(x,w)N(z,y|L+K) - \omega^{-1}(y,z)N(x,w|L+K)\right] \\
+ \quad & Q(q^{L+K})\left[\omega^{-1}(x,w)V(z-y) - \omega^{-1}(y,z)V(x-w)\right]. \qquad (3.260)
\end{aligned}
$$

Now let us look at the more complicated commutation of two different N variables, namely

$$
[:\tilde{\phi}^{\dagger}(x)e\tilde{\phi}(y):,:\tilde{\phi}^{\dagger}(z)q^{K}\tilde{\phi}(w):]. \qquad (3.261)
$$

Explicitly,

$$
\left[\left(\tilde{\phi}^{\dagger}_{i}(x)e^{i}_{j}\tilde{\phi}^{j}(y) - V(x-y)\delta^{j}_{i}e^{i}_{j}\right), \left(\tilde{\phi}^{\dagger}_{m}(z)(q^{K})^{m}_{n}\tilde{\phi}^{n}(w) - V(z-w)\delta^{n}_{m}(q^{K})^{m}_{n}\right)\right]. \qquad (3.262)
$$

This gives four non-commuting terms, namely:

$$
\begin{aligned}
= \quad & [\tilde{\phi}^{\dagger}_{i}(x)e^{i}_{j}\tilde{\phi}^{j}(y)\ ,\tilde{\phi}^{\dagger}_{m}(z)(q^{K})^{m}_{n}\tilde{\phi}^{n}(w)] \\
- \quad & [\tilde{\phi}^{\dagger}_{i}(x)e^{i}_{j}\tilde{\phi}^{j}(y)\ ,V(z-w)\delta^{n}_{m}(q^{K})^{m}_{n}] \\
- \quad & [V(x-y)\delta^{j}_{i}e^{i}_{j}\ ,\tilde{\phi}^{\dagger}_{m}(z)(q^{K})^{m}_{n}\tilde{\phi}^{n}(w)] \\
+ \quad & [V(x-y)\delta^{j}_{i}e^{i}_{j}\ ,V(z-w)\delta^{n}_{m}(q^{K})^{m}_{n}]. \qquad (3.263)
\end{aligned}
$$

Then

$$
\begin{aligned}
= \quad & \tilde{\phi}^{\dagger}_{m}(z)(q^{K})^{m}_{n}[\tilde{\phi}^{\dagger}_{i}(x)\ ,\tilde{\phi}^{n}(w)]e^{i}_{j}\tilde{\phi}^{j}(y) + \tilde{\phi}^{\dagger}_{i}(x)\tilde{\phi}^{\dagger}_{m}(z)[e^{i}_{j}\ ,(q^{K})^{m}_{n}]\tilde{\phi}^{j}(y)\tilde{\phi}^{n}(w) \\
+ \quad & \tilde{\phi}^{\dagger}_{i}(x)e^{i}_{j}[\tilde{\phi}^{j}(y)\ ,\tilde{\phi}^{\dagger}_{m}(z)](q^{K})^{m}_{n}\tilde{\phi}^{m}(w) - \tilde{\phi}^{\dagger}_{i}(x)V(z,w)\delta^{n}_{m}[e^{i}_{j}\ ,(q^{K})^{m}_{n}]\tilde{\phi}^{j}(y) \\
- \quad & V(x-y)\delta^{j}_{i}\tilde{\phi}^{\dagger}_{m}(z)[e^{i}_{j}\ ,(q^{K})^{m}_{n}]\tilde{\phi}^{n}(w) \\
+ \quad & V(x-y)\delta^{j}_{i}V(z-w)\delta^{n}_{m}[e^{i}_{j}\ ,(q^{K})^{m}_{n}]. \qquad (3.264)
\end{aligned}
$$

The commutation of the fields bring ω's:

$$
\begin{aligned}
= \quad & \tilde{\phi}^{\dagger}_{m}(z)(q^K)^m_n\omega^{-1}(x,w)\delta^n_i e^i_j\tilde{\phi}^j(y) + \tilde{\phi}^{\dagger}_i(x)\tilde{\phi}^{\dagger}_m(z)[e^i_j\ ,(q^K)^m_n]\tilde{\phi}^j(y)\tilde{\phi}^n(w) \\
+ \quad & \tilde{\phi}^{\dagger}_i(x)e^i_j\omega^{-1}(y,z)\delta^j_m(q^K)^m_n\tilde{\phi}^n(w) - \tilde{\phi}^{\dagger}_i(x)V(z-w)\delta^n_m[e^i_j\ ,(q^K)^m_n]\tilde{\phi}^j(y) \\
- \quad & V(x-y)\delta^j_i\tilde{\phi}^{\dagger}_m(z)[e^i_j\ ,(q^K)^m_n]\tilde{\phi}^n(w) \\
+ \quad & V(x-y)\delta^j_i V(z-w)\delta^n_m[e^i_j\ ,(q^K)^m_n]
\end{aligned}
\tag{3.265}
$$

Expanding the commutations according to equation 3.206 we obtain

$$
\begin{aligned}
= \quad & \tilde{\phi}^{\dagger}_{m}(z)(q^K)^m_n\omega^{-1}(x,w)e^n_j\tilde{\phi}^j(y) + \tilde{\phi}^{\dagger}_i(x)e^i_j\omega^{-1}(y,z)(q^K)^j_n\tilde{\phi}^n(w) \\
+ \quad & \tilde{\phi}^{\dagger}_i(x)\tilde{\phi}^{\dagger}_m(z)\sum(q^R)^m_l[e^i_j\ ,q^l_{l'}](q^{K-R-1})^{l'}_n\tilde{\phi}^j(y)\tilde{\phi}^n(w) \\
- \quad & \tilde{\phi}^{\dagger}_i(x)V(z-w)\delta^n_m\sum(q^R)^m_l[e^i_j\ ,q^l_{l'}](q^{K-R-1})^{l'}_n\tilde{\phi}^j(y) \\
- \quad & V(x-y)\delta^j_i\tilde{\phi}^{\dagger}_m(z)\sum(q^R)^m_l[e^i_j\ ,q^l_{l'}](q^{K-R-1})^{l'}_n\tilde{\phi}^n(w) \\
+ \quad & V(x-y)\delta^j_i V(z-w)\delta^n_m\sum(q^R)^m_l[e^i_j\ ,q^l_{l'}](q^{K-R-1})^{l'}_n
\end{aligned}
\tag{3.266}
$$

$$
\begin{aligned}
= \quad & \tilde{\phi}^{\dagger}_{m}(z)(q^K)^m_n\omega^{-1}(x,w)e^n_j\tilde{\phi}^j(y) + \tilde{\phi}^{\dagger}_i(x)e^i_j\omega^{-1}(y,z)(q^K)^j_n\tilde{\phi}^n(w) \\
+ \quad & \tilde{\phi}^{\dagger}_i(x)\tilde{\phi}^{\dagger}_m(z)\sum(q^R)^m_l g^2q^l_j\delta^i_{l'}(q^{K-R-1})^{l'}_n\tilde{\phi}^j(y)\tilde{\phi}^n(w) \\
- \quad & \tilde{\phi}^{\dagger}_i(x)V(z-w)\delta^n_m\sum(q^R)^m_l g^2q^l_j\delta^i_{l'}(q^{K-R-1})^{l'}_n\tilde{\phi}^j(y) \\
- \quad & V(x-y)\delta^j_i\tilde{\phi}^{\dagger}_m(z)\sum(q^R)^m_l g^2q^l_j\delta^i_{l'}(q^{K-R-1})^{l'}_n\tilde{\phi}^n(w) \\
+ \quad & V(x-y)\delta^j_i V(z-w)\delta^n_m\sum(q^R)^m_l g^2q^l_j\delta^i_{l'}(q^{K-R-1})^{l'}_n
\end{aligned}
\tag{3.267}
$$

$$
\begin{aligned}
= \quad & \tilde{\phi}^{\dagger}_{m}(z)(q^K)^m_n\omega^{-1}(x,w)e^n_j\tilde{\phi}^j(y) + \tilde{\phi}^{\dagger}_i(x)e^i_j\omega^{-1}(y,z)(q^K)^j_n\tilde{\phi}^n(w) \\
+ \quad & \tilde{\phi}^{\dagger}_m(z)\sum(q^R)^m_l g^2q^l_j\tilde{\phi}^j(y)\tilde{\phi}^{\dagger}_i(x)(q^{K-R-1})^i_n\tilde{\phi}^n(w) \\
- \quad & V(z-w)\sum\tilde{\phi}^{\dagger}_i(x)(q^{K-R-1})^i_n(q^R)^n_l g^2q^l_j\tilde{\phi}^j(y) \\
- \quad & V(x-y)\sum\tilde{\phi}^{\dagger}_m(z)(q^R)^m_l g^2q^l_j(q^{K-R-1})^j_n\tilde{\phi}^n(w) \\
+ \quad & V(x-y)V(z-w)\sum(q^R)^n_l g^2q^l_j(q^{K-R-1})^j_n.
\end{aligned}
\tag{3.268}
$$

We insert normal ordering, then

$$
\begin{aligned}
= \quad & :\tilde{\phi}^{\dagger}_m(z)(q^K)^m_n\omega^{-1}(x,w)e^n_j\tilde{\phi}^j(y): \; + \; :\tilde{\phi}^{\dagger}_i(x)e^i_j\omega^{-1}(y,z)(q^K)^j_n\tilde{\phi}^n(w): \\
+ \quad & V(z-y)\delta^j_m(q^K)^m_n\omega^{-1}(x,w)e^n_j + V(x-w)\delta^n_i e^i_j\omega^{-1}(y,z)(q^K)^j_n \\
+ \quad & g^2\sum\Big(:\tilde{\phi}^{\dagger}_m(z)(q^R)^m_l q^l_j\tilde{\phi}^j(y): + V(z-y)\delta^j_m(q^R)^m_l q^l_j\Big) \\
& \Big(:\tilde{\phi}^{\dagger}_i(x)(q^{K-R-1})^i_n\tilde{\phi}^n(w): + V(x-w)\delta^n_i(q^{K-R-1})^i_n\Big) \\
- \quad & V(z-w)\sum \; \tilde{\phi}^{\dagger}_i(x)(q^{K-R-1})^i_n(q^r)^n_l g^2 q^l_j\tilde{\phi}^j(y): \\
- \quad & V(z-w)\sum V(x-y)\delta^j_i(q^{K-R-1})^i_n(q^R)^n_l g^2 q^l_j \\
- \quad & V(x-y)\sum :\tilde{\phi}^{\dagger}_m(z)(q^R)^m_l g^2 q^l_j(q^{K-R-1})^j_n\tilde{\phi}^n(w): \\
- \quad & V(x-y)\sum V(z-w)\delta^n_m(q^R)^m_l g^2 q^l_j(q^{K-R-1})^j_n \\
+ \quad & V(x-y)V(z-w)\sum(q^R)^n_l g^2 q^l_j(q^{K-R-1})^j_n.
\end{aligned} \tag{3.269}
$$

In terms of our classical variables we have

$$
\begin{aligned}
= \quad & \omega^{-1}(x,w)N(z,y|K,1) + \omega^{-1}(y,z)N(x,w|K,1) \\
+ \quad & V(z-y)\omega^{-1}(x,w)tr\{q^K e\} + V(x-w)\omega^{-1}(y,z)tr\{eq^K\} \\
+ \quad & g^2\sum\big(N(z,y|R+1) + V(z-y)tr\{q^R q\}\big) \\
& \big(N(x,w|K-R-1,0,0) + V(x-w)tr\{q^{K-R-1})\big) \\
- \quad & V(z-w)\sum g^2 :\tilde{\phi}^{\dagger}_i(x)(q^{K-R-1}q^R q)^i_j\tilde{\phi}^j(y): \\
- \quad & \sum g^2 V(z-w)V(x-y)tr\{q^{K-R-1}q^R q\} \\
- \quad & V(x-y)\sum g^2 :\tilde{\phi}^{\dagger}_m(z)(q^R q q^{K-R-1})^m_n\tilde{\phi}^n(w): \\
- \quad & \sum g^2 V(x-y)V(z-w)tr\{q^R q q^{K-R-1}\} \\
+ \quad & g^2 V(x-y)V(z-w)\sum tr\{q^R q q^{K-R-1}\}.
\end{aligned} \tag{3.270}
$$

The result is

$$\begin{aligned}
&= \omega^{-1}(x,w)N(z,y|K,1)+\omega^{-1}(y,z)N(x,w|K,1)\\
&+ V(z-y)\omega^{-1}(x,w)Q(K,e)+V(x-w)\omega^{-1}(y,z)Q(K,e)\\
&+ g^2\sum\left(N(z,y|R+1)+V(z-y)Q(R+1)\right)\\
&\quad \left(N(x,w|K-R-1)+V(x-w)Q(K-R-1)\right)\\
&- V(z-w)\tilde{g}^2N(x,y|K)K\\
&- V(x-y)\tilde{g}^2N(z,w|K)K\\
&- \tilde{g}^2V(z-w)V(x-y)Q(K)K. \qquad (3.271)
\end{aligned}$$

We will need one more Poisson bracket which is

$$[tre^2,\ :\tilde{\phi}^\dagger(x)q^K\tilde{\phi}(y):]. \qquad (3.272)$$

More explicitly,

$$\left[e^m_n e^n_m,\left(\tilde{\phi}^\dagger_i(x)(q^K)^i_j\tilde{\phi}^j(y)-V(x,y)\delta^j_i(q^K)^i_j\right)\right]. \qquad (3.273)$$

This gives four commutations:

$$\begin{aligned}
&= e^m_n\tilde{\phi}^\dagger_i(x)[e^n_m,(q^K)^i_j]\tilde{\phi}^j(y)+\tilde{\phi}^\dagger_i(x)[e^m_n,(q^K)^i_j]e^n_m\tilde{\phi}^j(y)\\
&- e^m_nV(x,y)\delta^j_i[e^n_m,(q^K)^i_j]-V(x,y)\delta^j_i[e^m_n,(q^K)^i_j]e^n_m \qquad (3.274)
\end{aligned}$$

The same procedure follows as above, thus we have

$$\begin{aligned}
&= e^m_n\tilde{\phi}^\dagger_i(x)\sum(q^R)^i_l[e^n_m,q^l_{l'}](q^{K-R-1})^{l'}_j\tilde{\phi}^j(y)\\
&+ \tilde{\phi}^\dagger_i(x)\sum(q^R)^i_l[e^m_n,q^l_{l'}](q^{K-R-1})^{l'}_je^n_m\tilde{\phi}^j(y)\\
&- e^m_nV(x,y)\delta^j_i\sum(q^R)^i_l[e^n_m,q^l_{l'}](q^{K-R-1})^{l'}_j\\
&- V(x,y)\delta^j_i\sum(q^R)^i_l[e^m_n,q^l_{l'}](q^{K-R-1})^{l'}_je^n_m. \qquad (3.275)
\end{aligned}$$

$$\begin{aligned}
&= e^m_n \tilde{\phi}^\dagger_i(x) \sum (q^R)^i_l g^2 q^l_m \delta^n_{l'} (q^{K-R-1})^{l'}_j \tilde{\phi}^j(y) \\
&+ \tilde{\phi}^\dagger_i(x) \sum (q^R)^i_l g^2 q^l_n \delta^m_{l'} (q^{K-R-1})^{l'}_j e^n_m \tilde{\phi}^j(y) \\
&- e^m_n V(x,y) \delta^j_i \sum (q^R)^i_l g^2 q^l_m \delta^n_{l'} (q^{K-R-1})^{l'}_j \\
&- V(x,y) \delta^j_i \sum (q^R)^i_l g^2 q^l_n \delta^m_{l'} (q^{K-R-1})^{l'}_j e^n_m.
\end{aligned} \tag{3.276}$$

$$\begin{aligned}
&= g^2 \tilde{\phi}^\dagger_i(x) \sum (q^R)^i_l q^l_m e^m_n (q^{K-R-1})^n_j \tilde{\phi}^j(y) \\
&+ g^2 \tilde{\phi}^\dagger_i(x) \sum (q^R)^i_l q^l_n e^n_m (q^{K-R-1})^m_j \tilde{\phi}^j(y) \\
&- g^2 V(x,y) \sum (q^R)^j_l q^l_m e^m_n (q^{K-R-1})^n_j \\
&- g^2 V(x,y) \sum (q^R)^j_l q^l_n e^n_m (q^{K-R-1})^m_j.
\end{aligned} \tag{3.277}$$

Now we convert these products into normal orderings as

$$\begin{aligned}
&= g^2 : \tilde{\phi}^\dagger_i(x) \sum (q^{R+1})^i_m e^m_n (q^{K-R-1})^n_j \tilde{\phi}^j(y) : \\
&+ g^2 V(x,y) \sum (q^{R+1})^j_m e^m_n (q^{K-R-1})^n_j \\
&+ g^2 : \tilde{\phi}^\dagger_i(x) \sum (q^{R+1})^i_n e^n_m (q^{K-R-1})^m_j \tilde{\phi}^j(y) : \\
&+ g^2 V(x,y) \sum (q^{R+1})^j_n e^n_m (q^{K-R-1})^m_j \\
&- g^2 V(x,y) \sum (q^{R+1})^j_m e^m_n (q^{K-R-1})^n_j \\
&- g^2 V(x,y) \sum (q^{R+1})^j_n e^n_m (q^{K-R-1})^m_j.
\end{aligned} \tag{3.278}$$

In terms of our classical variables, it becomes

$$\begin{aligned}
&= \tilde{g}^2 N(x,y|R+1,1,K-R-1) \\
&+ \tilde{g}^2 V(x,y) \sum Q(R+1,1,K-R-1) \\
&+ \tilde{g}^2 N(x,y|R+1,1,K-R-1) \\
&+ \tilde{g}^2 V(x,y) \sum Q(R+1,1,K-R-1) \\
&- \tilde{g}^2 V(x,y) \sum Q(R+1,1,K-R-1) \\
&- \tilde{g}^2 V(x,y) \sum Q(R+1,1,K-R-1).
\end{aligned} \tag{3.279}$$

The result is

$$2\tilde{g}^2 N(x, y|R+1, 1, K-R-1). \tag{3.280}$$

3.8. Summary

In the previous section we computed three Poisson brackets to help us write the equations of motion of our classical variable, namely

$$\frac{\partial}{\partial u} N(x, y|K) = \{H, N(x, y|K)\}. \tag{3.281}$$

We had computed our Hamiltonian as

$$\begin{aligned}
H = & \frac{1}{2}\int dx \lim_{y\to x} [-\partial_y^2 N(x,y)] + \frac{1}{2}\int dx\ m^2 N(x,x) - \frac{1}{2\tilde{g}^2}\int dx\ Tre^2 \\
& - \frac{1}{4}\int dx \int_{-L}^{x} dy\ sgn(y) \lim_{y'\to y}\{\partial_y N(y,y'|0,1,0) - \partial_{y'} N(y',y|0,1)\} \\
& - \frac{\tilde{g}^2}{8}\int dx \int_{-L}^{x} dy\ sgn(y) \int_{-L}^{x} dy'\ sgn(y')\{\partial_{y'} N(y',y)\partial_y N(y,y') \\
& - N(y',y)\partial_y\partial_{y'} N(y,y') - N(y,y')\partial_{y'}\partial_y N(y',y) \\
& + \partial_{y'} N(y',y)\partial_y N(y,y')\}.
\end{aligned} \tag{3.282}$$

The required Poisson brackets are stated as

$$\begin{aligned}
& [N(x,y|K), N(z,w|L)] \\
= & \left[\omega^{-1}(x,w)N(z,y|L+K) - \omega^{-1}(y,z)N(x,w|L+K)\right] \\
+ & Q(L+K)\left[\omega^{-1}(x,w)V(z-y) - \omega^{-1}(y,z)V(x-w)\right],
\end{aligned} \tag{3.283}$$

$$
\begin{aligned}
& [N(x,y|0,1), N(z,w|K)] \\
= \; & \omega^{-1}(x,w)N(z,y|K,1) + \omega^{-1}(y,z)N(x,w|K,1) \\
+ \; & V(z-y)\omega^{-1}(x,w)Q(K,e) + V(x-w)\omega^{-1}(y,z)Q(K,e) \\
+ \; & \tilde{g}^2 \sum (N(z,y|R+1) + V(z-y)Q(R+1)) \\
& (N(x,w|K-R-1) + V(x-w)Q(K-R-1)) \\
- \; & V(z-w)\tilde{g}^2 N(x,y|K,0)K \\
- \; & V(x-y)\tilde{g}^2 N(z,w|K,0)K \\
- \; & \tilde{g}^2 V(z-w)V(x-y)Q(K)K,
\end{aligned} \tag{3.284}
$$

$$
[tre^2, N(x,y|K)] = 2\tilde{g}^2 N(x,y|R+1,1,K-R-1). \tag{3.285}
$$

The definition of our classical variables were

$$
N(x,\ y \mid K_1, S_1, ..., K_m) \ = \lim_{N_c \to \infty} \frac{1}{N_c} : \tilde{\phi}_i^\dagger(x)[\hat{q}^{K_1}\hat{e}^{S_1}\hat{q}^{K_2}\hat{e}^{S_2}...\hat{q}^{K_m}]^i_j \tilde{\phi}^j(y) : \tag{3.286}
$$

Its complex conjugate is written as

$$
\begin{aligned}
& N^*(x,\ y \mid K_1, S_1, ..., K_m) \\
= \; & \lim_{N_c \to \infty} \frac{1}{N_c} \Big(: \tilde{\phi}_i^\dagger(x)[\hat{q}^{K_1}\hat{e}^{S_1}\hat{q}^{K_2}\hat{e}^{S_2}...\hat{q}^{K_m}]^j_i \tilde{\phi}_i(y) \Big)^\dagger : \qquad (3.287) \\
= \; & N(y,\ x \mid -K_m, S_{n-1}, ..., -K_1)(-1)^{S_1+...+S_m}. \qquad (3.288)
\end{aligned}
$$

The boundary condition gives the relation

$$
N(x,\ L \mid K_1, S_1, ..., K_m) \ = \ N(x,\ -L \mid K_1, S_1, ..., K_m + 1). \tag{3.289}
$$

And

$$
Q(K) = \lim_{N_c \to \infty} \frac{1}{N_c}\, Trq^K. \tag{3.290}
$$

4. BOSONS IN THE ADJOINT REPRESENTATION

4.1. Transformation to New Variables

The action for the scalar bosons in 2 dimensions with a gauge interaction can be written as

$$S = \int dudx \, \sqrt{-\eta} \, [\frac{1}{2} Tr D_\mu \phi D^\mu \phi - \frac{1}{2} m^2 Tr \phi^2 + \frac{1}{4g^2} Tr F^{\mu\nu} F_{\mu\nu}]. \tag{4.1}$$

Here the covariant derivative changes to $D_\mu = \partial_\mu + [A_\mu, .]$, since ϕ is in adjoint representation. Explicitly,

$$\begin{aligned} S = \int dudx \, [-sgn(x) Tr \nabla_u \phi \nabla_x \phi - \frac{1}{2} Tr \nabla_x \phi \nabla_x \phi] - \frac{1}{2} \int dudx \, m^2 Tr \phi^2 \\ + \int dudx [-\frac{1}{g^2} Tr E(\partial_u A_x - \partial_x A_u + [A_u, A_x]) + \frac{1}{2g^2} Tr E^2], \end{aligned} \tag{4.2}$$

then

$$\begin{aligned} S = -\int & dudx \, sgn(x) Tr[(\partial_u \phi + [A_u, \phi])(\partial_x \phi + [A_x, \phi])] \\ -\int & dudx [\frac{1}{2} Tr(\partial_x \phi + [A_x, \phi])(\partial_x \phi + [A_x, \phi])] \\ -\int & dudx \frac{1}{2} m^2 Tr \phi^2 \\ -\int & dudx [\frac{1}{g^2} Tr E(\partial_u A_x - \partial_x A_u + [A_u, A_x]) + \frac{1}{2g^2} Tr E^2]. \end{aligned} \tag{4.3}$$

This time our new variables are defined according to the adjoint transformation rules:

$$\begin{aligned} \phi &= h \tilde{\phi} h^{-1}, \\ A_u &= h \tilde{A}_u h^{-1}, \\ E &= h \tilde{E} h^{-1}. \end{aligned} \tag{4.4}$$

Thus the boundary condition for the scalar field is like the gauge fields,

$$\tilde{\phi}(L) = q\tilde{\phi}(-L)q^{-1}. \tag{4.5}$$

When we insert the new variables into the action, we get

$$\begin{aligned}
S \;=\; & -\int dudx\; sgn(x)Tr[(\partial_u(h\tilde{\phi}h^{-1}) + [h\tilde{A}_u h^{-1}, h\tilde{\phi}h^{-1}]) \times \\
& \quad (\partial_x(h\tilde{\phi}h^{-1}) + [A_x, h\tilde{\phi}h^{-1}])] \\
& - \int dudx[\frac{1}{2}Tr(\partial_x(h\tilde{\phi}h^{-1}) + [A_x, h\tilde{\phi}h^{-1}])(\partial_x(h\tilde{\phi}h^{-1}) + [A_x, h\tilde{\phi}h^{-1}])] \\
& - \int dudx\frac{1}{2}m^2 Tr(h\tilde{\phi}h^{-1}h\tilde{\phi}h^{-1}) \\
& - \frac{1}{g^2}\int du Tr[q^{-1}\partial_u q e] - \frac{1}{g^2}\int dudx\; Tr[(\partial_x\tilde{E})A] \\
& + \frac{1}{2g^2}\int dudx\; Tr\tilde{E}^2,
\end{aligned} \tag{4.6}$$

where we have directly taken the result 3.28.

$$\begin{aligned}
S = & -\int dudx\; sgn(x)Tr[\partial_u h\tilde{\phi}h^{-1} + h\partial_u\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_u h^{-1} + h\tilde{A}_u\tilde{\phi}h^{-1} - h\tilde{\phi}\tilde{A}_u h^{-1}) \\
& (\partial_x h\tilde{\phi}h^{-1} + h\partial_x\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_x h^{-1} + A_x h\tilde{\phi}h^{-1} - h\tilde{\phi}h^{-1}A_x)] \\
& - \int dudx[\frac{1}{2}Tr(\partial_x h\tilde{\phi}h^{-1} + h\partial_x\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_x h^{-1} + A_x h\tilde{\phi}h^{-1} - h\tilde{\phi}h^{-1}) \\
& (\partial_x h\tilde{\phi}h^{-1} + h\partial_x\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_x h^{-1} + A_x h\tilde{\phi}h^{-1} - h\tilde{\phi}h^{-1})] \\
& - \int dudx\frac{1}{2}m^2 Tr(\tilde{\phi})^2 \\
& - \frac{1}{g^2}\int du Tr[q^{-1}\partial_u q e] - \frac{1}{g^2}\int dudx\; Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\; Tr\tilde{E}^2.
\end{aligned} \tag{4.7}$$

∇_x parts simplify, they become

$$\begin{aligned}
\nabla_x(h\tilde{\phi}h^{-1}) &= \partial_x h\tilde{\phi}h^{-1} + h\partial_x\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_x h^{-1} + A_x h\tilde{\phi}h^{-1} - h\tilde{\phi}h^{-1}A_x \\
&= -A_x h\tilde{\phi}h^{-1} + h\partial_x\tilde{\phi}h^{-1} + h\tilde{\phi}\partial_x h^{-1} + A_x h\tilde{\phi}h^{-1} - h\tilde{\phi}h^{-1}A_x \\
&= h\partial_x\tilde{\phi}h^{-1}.
\end{aligned} \tag{4.8}$$

Thus

$$
\begin{aligned}
S = -\int & dudx\ sgn(x)\ Tr[\partial_u h\tilde{\phi}h^{-1}h\partial_x\tilde{\phi}h^{-1} + h\partial_u\tilde{\phi}h^{-1}h\partial_x\tilde{\phi}h^{-1} \\
& + h\tilde{\phi}\partial_u h^{-1}h\partial_x\tilde{\phi}h^{-1} + h\tilde{A}_u\tilde{\phi}h^{-1}h\partial_x\tilde{\phi}h^{-1} - h\tilde{\phi}\tilde{A}_u h^{-1}h\partial_x\tilde{\phi}h^{-1}] \\
& - \int dudx[\frac{1}{2}Tr(h\partial_x\tilde{\phi}h^{-1})(h\partial_x\tilde{\phi}h^{-1}) \\
& - \int dudx\frac{1}{2}m^2Tr(\tilde{\phi})^2 \\
& - \frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2].
\end{aligned} \quad (4.9)
$$

$$
\begin{aligned}
S = -\int & dudx\ sgn(x)\ Tr[h^{-1}\partial_u h\tilde{\phi}\partial_x\tilde{\phi} + \partial_u\tilde{\phi}\partial_x\tilde{\phi} \\
& - \tilde{\phi}h^{-1}\partial_u hh^{-1}h\partial_x\tilde{\phi} + \tilde{A}_u\tilde{\phi}\partial_x\tilde{\phi} - \tilde{\phi}\tilde{A}_u\partial_x\tilde{\phi}] \\
& - \int dudx[\frac{1}{2}Tr(\partial_x\tilde{\phi}\partial_x\tilde{\phi}) \\
& - \int dudx\frac{1}{2}m^2Tr(\tilde{\phi})^2 \\
& - \frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2].
\end{aligned} \quad (4.10)
$$

We obtained the decoupled equation for the bosons as

$$
\begin{aligned}
S = -\int & dudx\ sgn(x)\ Tr[\partial_u\tilde{\phi}\partial_x\tilde{\phi} + [A,\tilde{\phi}]\partial_x\tilde{\phi}] \\
& - \int dudx\frac{1}{2}Tr(\partial_x\tilde{\phi}\partial_x\tilde{\phi}) \\
& - \int dudx\frac{1}{2}m^2Tr(\tilde{\phi})^2 \\
& - \frac{1}{g^2}\int duTr[q^{-1}\partial_u qe] - \frac{1}{g^2}\int dudx\ Tr[(\partial_x\tilde{E})A] + \frac{1}{2g^2}\int dudx\ Tr\tilde{E}^2.
\end{aligned} \quad (4.11)
$$

Once more we vary the action w.r.t A since it does not have a time derivative, we have

$$
sgn(x)Tr(\delta A\tilde{\phi}\partial_x\tilde{\phi} - \tilde{\phi}\delta A\partial_x\tilde{\phi}) - \frac{1}{g^2}Tr\partial_x\tilde{E}\delta A = 0. \quad (4.12)
$$

$$sgn(x)Tr(\delta A^a T^a[\tilde{\phi}, \partial_x \tilde{\phi}]^b T^b) - \frac{1}{g^2} Tr(\partial_x \tilde{E}^a T^a \delta A^b T^b) = 0. \tag{4.13}$$

$$sgn(x)\delta A^a[\tilde{\phi}, \partial_x \tilde{\phi}]^b Tr[T^a T^b] - \frac{1}{g^2} \partial_x \tilde{E}^a \delta A^b Tr[T^a T^b] = 0. \tag{4.14}$$

By 3.3,

$$-sgn(x)\delta A^a[\tilde{\phi}, \partial_x \tilde{\phi}]^a + \frac{1}{g^2} \partial_x \tilde{E}^a \delta A^a = 0. \tag{4.15}$$

We obtain

$$\tilde{E}(x) = -g^2 \int_{-L}^{x} dy \; sgn(y)[\tilde{\phi}, \partial_y \tilde{\phi}] + e. \tag{4.16}$$

We insert this result in our action to obtain

$$\begin{aligned} S = & -\int \; dudx \; sgn(x) \; Tr[\partial_u \tilde{\phi} \partial_x \tilde{\phi}] - \frac{1}{g^2} \int \; duTr[q^{-1} \partial_u q e] \\ & -\int \; dudx[\frac{1}{2} Tr(\partial_x \tilde{\phi} \partial_x \tilde{\phi}) \\ & -\int \; dudx \frac{1}{2} m^2 Tr(\tilde{\phi})^2 \\ & + \frac{1}{2g^2} \int dudx \; Tr \left[e - g^2 \int_{-L}^{x} dy \; sgn(y)[\tilde{\phi}, \partial_y \tilde{\phi}] \right]^2 . \end{aligned} \tag{4.17}$$

Here the order of the terms has changed to read off the symplectic and Hamiltonian parts separately. The first line is the symplectic part that we are going to analyze in the next section. The rest corresponds to the negative of the Hamiltonian. Thus

$$\begin{aligned} H = & \int \; dudx[\frac{1}{2} Tr(\partial_x \tilde{\phi} \partial_x \tilde{\phi}) + \int \; dudx \frac{1}{2} m^2 Tr(\tilde{\phi})^2 \\ & + \frac{1}{2g^2} \int dudx \; Tr \left[e - g^2 \int_{-L}^{x} dy \; sgn(y)[\tilde{\phi}, \partial_y \tilde{\phi}] \right]^2 . \end{aligned} \tag{4.18}$$

4.2. Symplectic Form

This time our symplectic term in the action of equation 4.17 reads

$$-\int dudx\ sgn(x)\ Tr[\partial_u\tilde{\phi}\partial_x\tilde{\phi}]. \tag{4.19}$$

A variation w.r.t $\tilde{\phi}$ would give

$$-\int dudx\ sgn(x)Tr[(\partial_u\delta\tilde{\phi})(\partial_x\tilde{\phi}) + (\partial_x\delta\tilde{\phi})(\partial_u\tilde{\phi})]. \tag{4.20}$$

Doing integration by parts in the first term, we obtain

$$-\int dudx\ sgn(x)Tr[\partial_u(\delta\tilde{\phi}\partial_x\tilde{\phi}) - \delta\tilde{\phi}\partial_u\partial_x\tilde{\phi} + (\partial_x\delta\tilde{\phi})(\partial_u\tilde{\phi})]. \tag{4.21}$$

The first term drops due to the same reasoning in the section 3.3. We obtain

$$-\int dudx\ Tr[\delta\tilde{\phi}^a(\overleftarrow{\partial}_x sgn(x) - sgn(x)\overrightarrow{\partial}_x)\partial_u\tilde{\phi}^a]. \tag{4.22}$$

We can see that the symplectic form does not change. And we write the inverse of it as

$$\omega^{-1}_{ab} = (sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}\delta_{ab}. \tag{4.23}$$

4.3. Quantization

The Poisson bracket of the scalar fields are

$$\{\phi^a(x), \phi^b(y)\} = (sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}\delta_{ab}. \tag{4.24}$$

If we want to change this into the usual index form we may use,

$$\{\phi^a(x)T^{ai}_j, \phi^b(y)T^{bk}_l\} = (sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}\sum \delta_{ab}T^{ai}_j T^{bk}_l \tag{4.25}$$

$$= (sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}[-(\delta^i_l\delta^k_j - \frac{1}{N_c}\delta^i_j\delta^k_l)]. \tag{4.26}$$

The second term drops when N_c goes to infinity, since it is of smaller order. Thus

$$\{\phi^i_j(x), \phi^k_l(y)\} = -(sgn\overrightarrow{\partial}_x - \overleftarrow{\partial}_x sgn)^{-1}\delta^i_l\delta^k_j. \tag{4.27}$$

The quantization process is done as in section 3.4. Since it was constructed already in the real space we can completely take the same results. In terms of our operators,

$$:\phi^a\phi^b := \phi^a\phi^b - [i(\omega^{-1})^{ab} + K^{ab}], \tag{4.28}$$

or in terms of the indices, by dropping the $\frac{1}{N_c}$ corrections we write

$$:\phi^i_j\phi^k_l := \phi^i_j\phi^k_l - [i(\omega^{-1})^{ik}_{jl} + K^{ik}_{jl}]. \tag{4.29}$$

4.4. Gauge Invariance Constraint

As we did in the fundamental representation, we introduce the normal ordered products following the quantization. So equation 4.16 becomes

$$\tilde{E}(x) = -g^2\int_{-L}^{x} dy\ sgn(y) : [\tilde{\phi}, \partial_y\tilde{\phi}] : +e. \tag{4.30}$$

When we impose our boundary conditions we obtain a constraint similar to 3.188

$$\hat{e} - g^2\int_{-L}^{L} dy\ sgn(y) : [\tilde{\phi}, \partial_y\tilde{\phi}] : = \widehat{qeq}^{-1}. \tag{4.31}$$

Thus the global color operator is defined to be

$$\hat{Q} = \hat{e} - \widehat{qeq}^{-1} - g^2 \int_{-L}^{L} dy\ sgn(y) : [\tilde{\phi}, \partial_y \tilde{\phi}] : . \tag{4.32}$$

Now we should construct the physical states that this operator gives zero when it acts on. Let us check that the product below is gauge invariant:

$$Tr \left\{ \left(P\{e^{\int_x^{y+2k_1 L}}\} \right)^\dagger \phi(x) P\{e^{\int_x^{y+2k_2 L}}\} \phi(y) \right\} \tag{4.33}$$

$$Tr \left\{ \left(P\{e^{\int_x^{-L}}\} \left(P\{e^{\int_{-L}^{L}}\} \right)^{k_1} P\{e^{\int_L^y}\} \right)^\dagger \phi(x) P\{e^{\int_x^{-L}}\} \left(P\{e^{\int_{-L}^{L}}\} \right)^{k_2} P\{e^{\int_L^y}\} \phi(y) \right\} \tag{4.34}$$

$$Tr \left\{ P\{e^{-\int_L^y}\} \left(P\{e^{\int_{-L}^{L}}\} \right)^{-k_1} P\{e^{-\int_x^{-L}}\} \phi(x) P\{e^{\int_x^{-L}}\} \left(P\{e^{\int_{-L}^{L}}\} \right)^{k_2} P\{e^{\int_L^y}\} \phi(y) \right\} \tag{4.35}$$

Using the cyclic property of the trace and the transformation properties of the new fields as in equation 4.4, the above expression simplifies to

$$Tr \left\{ q^{-k_1} \tilde{\phi}(x) q^{k_2} \tilde{\phi}(y) \right\} \tag{4.36}$$

Explicitly this reads

$$(q^{k_1})^{\alpha_1}_{\alpha_2}\ \tilde{\phi}^{\alpha_2}_{\alpha_3}(x)\ (q^{k_2})^{\alpha_3}_{\alpha_4}\ \tilde{\phi}^{\alpha_4}_{\alpha_5}(y). \tag{4.37}$$

Here we changed the sign of the exponential since that would only imply a change in the number of turns.

So in general we have strings of operators,

$$N(x_1, K_1, x_2, K_2 ....) = \lim_{N_c \to \infty} \frac{1}{N_c^{m+1}} : Tr \phi(x_1) q^{K_1} \phi(x_2) q^{K_2} ... \phi(x_m) q^{K_m} : . \tag{4.38}$$

We can check the commutation of this physical state in the adjoint representation with the corresponding global color operator as in the fundamental representation as

$$[\hat{Q}, (q^{K_1})^{\alpha_1}_{\alpha_2}\ \tilde{\phi}^{\alpha_2}_{\alpha_3}(x)\ (q^{K_2})^{\alpha_3}_{\alpha_4}\ \tilde{\phi}^{\alpha_4}_{\alpha_5}(y)]. \tag{4.39}$$

To calculate this, we only need

$$[\hat{Q}, \tilde{\phi}^{\alpha_1}_{\alpha_2}(x)]. \tag{4.40}$$

$$[e - qeq^{-1} - g^2 \int_{-L}^{L} dy\ \lambda : \tilde{\phi}\omega\tilde{\phi} :, \tilde{\phi}^{\alpha_1}_{\alpha_2}(x)] \tag{4.41}$$

Only the symplectic part makes a contribution, then

$$-g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ [\lambda^k_i : \tilde{\phi}^i_j(y')\omega(y', y)\tilde{\phi}^j_k(y) :, \tilde{\phi}^{\alpha_1}_{\alpha_2}(x)]. \tag{4.42}$$

$$\begin{aligned} &-g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^k_i[\tilde{\phi}^i_j(y'), \tilde{\phi}^{\alpha_1}_{\alpha_2}(x)]\omega(y', y)\tilde{\phi}^j_k(y) \\ &-g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^k_i\tilde{\phi}^i_j(y')\omega(y', y)[\tilde{\phi}^j_k(y), \tilde{\phi}^{\alpha_1}_{\alpha_2}(x)] \end{aligned} \tag{4.43}$$

$$\begin{aligned} &-i\ g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^k_i\omega^{-1}(y', x)\delta^i_{\alpha_2}\delta^{\alpha_1}_j\omega(y', y)\tilde{\phi}^j_k(y) \\ &-i\ g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^k_i\tilde{\phi}^i_j(y')\omega(y', y)\omega(y, x)\delta^j_{\alpha_2}\delta^{\alpha_1}_k \end{aligned} \tag{4.44}$$

$$\begin{aligned} &+i\ g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^k_{\alpha_2}\delta(x, y)\tilde{\phi}^{\alpha_1}_k(y) \\ &-i\ g^2 \int_{-L}^{L} dy \int_{-L}^{L} dy'\ \lambda^{\alpha_1}_i\tilde{\phi}^i_{\alpha_2}(y')\delta(y', x) \end{aligned} \tag{4.45}$$

$$
\begin{aligned}
&+i\, g^2 \int_{-L}^{L} dy'\ \tilde{\phi}_k^{\alpha_1}(x)\lambda_{\alpha_2}^k \\
&-i\, g^2 \int_{-L}^{L} dy\ \lambda_i^{\alpha_1}\tilde{\phi}_{\alpha_2}^i(x)
\end{aligned}
\tag{4.46}
$$

This gives the commutation of ϕ with the element of Lie algebra,

$$
[\hat{Q}, \tilde{\phi}] \sim [\phi, \lambda]. \tag{4.47}
$$

Thus we can state that Q is the generator of color.

The constraint equation can be obtained similarly, but we will not deal with this problem in this thesis, since the computations hereafter are more complicated in the adjoint representation.

5. CONCLUSIONS

In this thesis, we studied the large N_c-limit of $SU(N_c)$ gauge theory coupled to a scalar field both in fundamental and adjoint representations.

We used a special coordinate system and a transformation to a new set of fields proposed by Rajeev and Guruswamy to eliminate the redundant degrees of freedom in the gauge field sector. This forces us to reexamine normal ordering for a bosonic field, we can write down the result as an operator kernel but we are unable to actually compute the result explicitly.

We have derived a classical field theory using color invariant variables only for the fundamental representation. We saw that the computations of the adjoint representation was much harder than the fundamental representation. One needs to find a way to deal with complicated strings of the fields, that come out in the adjoint representation.

In principle we can compute the equations of motion for these classical variables, however we are not able to solve these equations. In the fundamental representation, the large-N theory satisfies a set of complicated constraints.The geometric meaning of these constraints is an open problem. As a conclusion, the theory constructed does not satisfy a closed algebra, thus one needs to find new approximations and variational methods.

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